MATH3611 | Assignment 2

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§ Q1

Let $(f_n)_{n=2}^{\infty} \subset C[0,1]$ be a sequence of piecewise linear functions, where $n \in \mathbb{N}$ and $n \geq 2$, and each function f_n is defined by:

$$f_n(x) = \begin{cases} 1, & \text{for } x \in [0, \frac{1}{2} - \frac{1}{n}], \\ \text{linear}, & \text{for } x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}], \\ 0, & \text{for } x \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

(a) Show that the sequence $(f_n)_{n=2}^{\infty}$ is Cauchy in the d_2 metric, where

$$d_2(f,g) = \left(\int_0^1 |f(x) - g(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}, f, g \in C[0,1]$$

(b) Show that the sequence $(f_n)_{n=2}^{\infty}$ is not Cauchy in the d_∞ metric, where

$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|, f, g \in C[0,1]$$

Definition 1.1. A sequence $\{ x_n \}_{n=0}^{\infty}$ in a metric space (X, d) is a Cauchy sequence if for every $\epsilon > 0$, there is a $K(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ whenever $m, n > K(\epsilon)$.

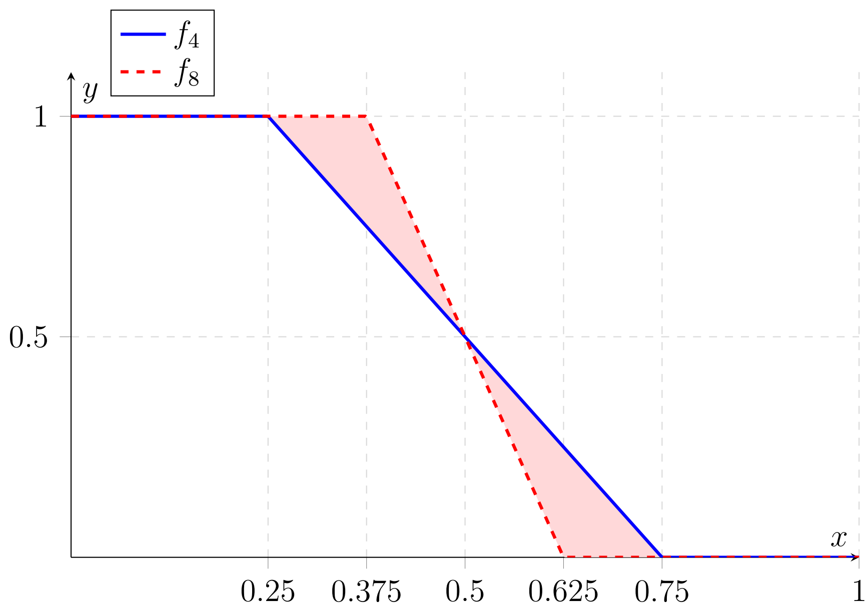

Figure 1: linear piece-wise for f_4 and f_8

§§ a)

Proof. For $m > n$[1]:

$$\begin{aligned} d_2(f_n, f_m)^2 &= \int_0^1 |f_n(x) - f_m(x)|^2 \, \mathrm{d}x && (1) \\ &= \int_{I_{m,n}} |f_n(x) - f_m(x)|^2 \, \mathrm{d}x && (2) \end{aligned}$$

where $I_{m,n} := [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}] \cup [\frac{1}{2} - \frac{1}{m}, \frac{1}{2} + \frac{1}{m}]$, i.e. the contributions when $f_n \neq f_m$.

Now, each f_k takes values of 0 and 1 along the flat regions, and remains linear in between: $0 \leq f_k \leq 1$. So $|f_n - f_m|^2 \leq 1$ and we have a bound for the integrand.

Next, by considering the maximal region of integration:

$$\begin{aligned} \lambda(I_{m,n}) &= \cancel{\frac{1}{2}} + \frac{1}{n} - (\cancel{\frac{1}{2}} - \frac{1}{n}) + \cancel{\frac{1}{2}} + \frac{1}{m} - (\cancel{\frac{1}{2}} - \frac{1}{m}) && (3) \\ &= \frac{2}{n} + \frac{2}{m} && (4) \\ &< \frac{4}{n} \quad (m > n) && (5) \end{aligned}$$

We can formulate bounds on our squared metric:

$$d_2(f_n, f_m)^2 \leq 1 \times \frac{4}{n} \implies d_2(f_n, f_m) \leq \frac{2}{\sqrt{n}} \tag{6}$$

Fixing $\epsilon > 0$ and $K(\epsilon) := \lceil \frac{8}{\epsilon^2} \rceil$ with $m, n > K(\epsilon)$:

$$\frac{2}{\sqrt{n}} < \frac{2}{\sqrt{K(\epsilon)}} < \epsilon \tag{7}$$

Thus, by the definition of a Cauchy sequence, f_n is Cauchy in $(C[0,1], d_2)$.

□

[1]the same argument applies for $n > m$ by symmetry

§§ b)

Proof. Contrariwise, we show that there exists an $\epsilon > 0$ such that the Cauchy condition fails. We accomplish this by picking $m = 2n$ such that

$$\begin{aligned} z &= \frac{1}{2} - \frac{1}{m} \\ &= \frac{1}{2} - \frac{1}{2n} \\ \implies f_m(z) &= 1 \\ f_n(z) &= \text{linear} \end{aligned}$$

The gradient[2] of the linear section is $-\frac{n}{2}$ and the equation for this segment becomes:

$$f_n(z) = 1 - \frac{n}{2}\left(z - \left(\frac{1}{2} - \frac{1}{n}\right)\right) = 1 - \frac{n}{2}\left(\frac{1}{n} - \frac{1}{2n}\right) = \frac{3}{4}$$

Consequently, $|f_m(z) - f_n(z)| = 1 - \frac{3}{4} = \frac{1}{4}$ and

$$d_\infty(f_n, f_m) = \sup_{x\in[0,1]} |f_n(x) - f_{2n}(x)| \geq \frac{1}{4}$$

But then with $\epsilon := \frac{1}{8}$ there is **no** choice of an index K that can satisfy our requirement $d_\infty(f_m, f_n) < \frac{1}{8}$ for $m, n > K(\epsilon)$ because $d_\infty(f_m, f_n)$ is already greater than $\frac{1}{4}$!
Thus (f_n) is not Cauchy in $(C[0,1], d_\infty)$. □

[2] by $\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{1/2 - 1/n - (1/2 + 1/n)}$

§ Question 2

Show that ℓ^2 is a vector space; that is, if $x, y \in \ell^2$, then $x + y \in \ell^2$ and $\lambda x \in \ell^2$ for any $\lambda \in \mathbb{R}$. You may assume, without proof, the triangle inequality for the norm $|| \cdot ||_2$ on $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Lemma 2.1.

$$\begin{aligned} (a-b)^2 &\geq 0 \\ \implies a^2 - 2ab + b^2 &\geq 0 \\ \implies 2ab &\leq a^2 + b^2 \\ \implies a^2 + 2ab + b^2 &\leq 2a^2 + 2b^2 \quad \textit{by adding } a^2 + b^2 \textit{ to both sides} \\ \implies (a+b)^2 &\leq 2a^2 + 2b^2 \end{aligned}$$

Lemma 2.2 (Triangle Inequality for Absolute Value)**.** *Proof.*

$$\begin{aligned} |a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 &\leq a^2 + 2|ab| + b^2 = (|a| + |b|)^2 \\ \implies |a+b| &\leq |a| + |b| \end{aligned}$$

□

Corollary 2.2.1. *By 2.1 we have:*

$$\begin{aligned} (a+b)^2 &\leq 2a^2 + 2b^2 \\ \implies (|a| + |b|)^2 &\leq 2|a|^2 + 2|b|^2 \end{aligned}$$

Which we can combine with 2.2:

$$\begin{aligned} |a+b| &\leq |a| + |b| \\ |a+b|^2 &\leq (|a| + |b|)^2 \end{aligned}$$

To produce

$$|a+b|^2 \leq (|a| + |b|)^2 \leq 2|a|^2 + 2|b|^2 \tag{8}$$

Notation.

$$\ell^2 = \{ \{ x_n \}_{n=1}^{\infty} \subset \mathbb{R} \mid \sum_{n=1}^{\infty} | x_n|^2 < \infty \} \tag{9}$$

$$\|x\|_2 = \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} \tag{10}$$

Proof. **Closure under addition:**
Let $x, y \in \ell^2$ such that $x = \{ x_n \}_{n=1}^{\infty}$ and $y = \{ y_n \}_{n=1}^{\infty}$.
Then we apply 8 termwise:

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n + y_n|^2 &\leq 2\sum_{n=1}^{\infty} |x_n|^2 + 2\sum_{n=1}^{\infty} |y_n|^2 \\ &= 2||x||_2^2 + 2||y||_2^2\end{aligned}$$

By the comparison test, we have found $||x + y||_2^2 \leq 2||x||_2^2 + 2||y||_2^2$ and we know that $||x||_2^2 + ||y||_2^2$ converges by our assumption of both being in ℓ^2 and hence being the sum of two finite real numbers.

Closure under scalar multiplication:
Let $\lambda \in \mathbb{R}$ and $x \in \ell^2$:

$$\begin{aligned}\sum_{n=1}^{\infty} |\lambda x_n|^2 &= \lambda^2 \sum_{n=1}^{\infty} |x_n|^2 \\ &= \lambda^2 \|x\|_2^2\end{aligned}$$

Which is finite because $\|x\|_2^2 < \infty$. Thus $\lambda x \in \ell^2$.

□

§ Question 3

Show that the subset c_{00} is dense in the metric space ℓ^2.[3]

Proof. Let ℓ^2 be the space of square-summable sequences equipped with the 2-norm.

Let $c_{00} = \{\{x_n\}_{n=1}^{\infty} \in \ell^2 \mid x_n = 0 \text{ for all but finitely many } n\}$ be the subset of sequences with only finitely many non-zero terms.

Fix $x = \{\, x_n \,\}_{n=1}^{\infty} \in \ell^2$ and $\epsilon > 0$. Since $x \in \ell^2$, the series $\sum_{n=1}^{\infty} |x_n|^2$ converges to a finite value. By the definition of convergence for an infinite series[4]:

$$\sum_{n=p+1}^{\infty} |x_n|^2 < \epsilon^2 \quad \text{with } \delta \text{ as } \epsilon^2$$

Construct $x_\epsilon = \{\, x_n^{(\epsilon)} \,\}_{n=1}^{\infty} \in c_{00}$ by defining

$$x_n^{(\epsilon)} = \begin{cases} x_n & \text{if } 1 \le n \le p, \\ 0 & \text{if } n \ge p+1. \end{cases}$$

Thus, $x_\epsilon = (x_1, x_2, \ldots, x_p, 0, 0, \ldots)$, which has finitely many non-zero terms and belongs to c_{00}. Computing the difference $x - x_\epsilon = \{x_n - x_n^{(\epsilon)}\}_{n=1}^{\infty}$ for each index n yields:

$$x_n - x_n^{(\epsilon)} = \begin{cases} x_n - x_n = 0 & \text{if } 1 \le n \le p, \\ x_n - 0 = x_n & \text{if } n \ge p+1. \end{cases}$$

So, $x - x_\epsilon = (0, 0, \ldots, 0, x_{p+1}, x_{p+2}, \ldots)$, where the first p terms are zero. Thus the ℓ^2-norm of the difference becomes

$$\|x - x_\epsilon\|_2 = \left(\sum_{n=1}^{\infty} |x_n - x_n^{(\epsilon)}|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=p+1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}} < (\epsilon^2)^{\frac{1}{2}} = \epsilon,$$

since $\sum_{n=p+1}^{\infty} |x_n|^2 < \epsilon^2$ by the choice of p.
In conclusion, for any $x \in \ell^2$ and $\epsilon > 0$, there exists $x_\epsilon \in c_{00}$ such that $\|x - x_\epsilon\|_2 < \epsilon \implies c_{00}$ is dense in ℓ^2.

□

[3] i.e. we wish to show that for $x \in \ell^2$ and $\epsilon > 0$, $\exists x_\epsilon \in c_{00}$ such that $\|x - x_\epsilon\|_2 < \epsilon$.

[4] we can make the tail of the series arbitrarily small. this follows because the partial sums $s_p = \sum_{n=1}^{p} |x_n|^2$ converge to $S = \|x\|_2^2$ and the remainder $\sum_{n=p+1}^{\infty} |x_n|^2 = S - s_p$ can be made arbitrarily small for sufficiently large p.

$$\sum_{n=p+1}^{\infty} |x_n|^2 < \delta$$