# 1 Questions

**Q1:** Let $\{f_n\}_{n=2}^{\infty} \subset C[0,1]$ be a sequence of piecewise linear functions, where $n \in \mathbb{N}$ and $n \geq 2$, and each function $f_n$ is defined by:

$$f_n(x) = \begin{cases} 1, & \text{for } x \in \left[0, \frac{1}{2} - \frac{1}{n}\right], \\ \text{linear}, & \text{for } x \in \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right], \\ 0, & \text{for } x \in \left[\frac{1}{2} + \frac{1}{n}, 1\right] \end{cases}$$

- Show that the sequence $\{f_n\}_{n=2}^{\infty}$ is Cauchy in the $d_2$ metric, where

$$d_2(f,g) = \left( \int_0^1 |f(x) - g(x)|^2 \, \mathrm{d}x \right)^{1/2}, \quad f, g \in C[0,1].$$

- Show that the sequence $\{f_n\}_{n=2}^{\infty}$ is not Cauchy in the $d_\infty$ metric, where

$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|, \quad f, g \in C[0,1].$$

**Q2:** Show that $\ell^2$ is a vector space; that is, if $x, y \in \ell^2$, then $x + y \in \ell^2$ and $\lambda x \in \ell^2$ for any $\lambda \in \mathbb{R}$. You may assume, without proof, the triangle inequality for the norm $\|\cdot\|_2$ on $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

**Q3:** Show that the subset $c_{00}$ is dense in the metric space $\ell^2$.