

# 21 Problems

Aayush Bajaj

December 26, 2022

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Welcome! Today is the 26<sup>th</sup> of December, and it is my birthday :D.

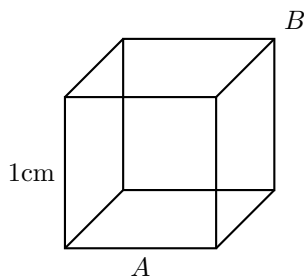
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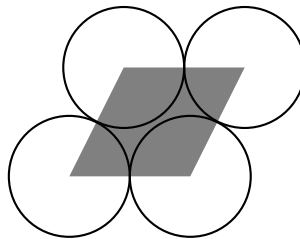
## 1 Rules

1. Solutions must be written on a piece of **WHITE** paper in **BLACK** pen.
  - 1.1. White paper can be found attached to the board in the study. Black pens are beside the board.
2. To create a submission:
  - 2.1. Fold the piece of paper so that your solution is **not** visible, and
  - 2.2. Attach it to the board in the study with a magnet.
3. Each submission must contain:
  - 3.1. Your name;
  - 3.2. The question number;
4. Submissions will not be accepted after **11:59PM** on the 26th of December, 2022
5. You may not use the *internet*, but you may use any *book*.

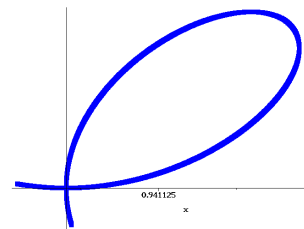
## 2 Diagrams



Q4. Sugar Cube



Q5. Hexagonal Packing



Q18. Implicit Curve

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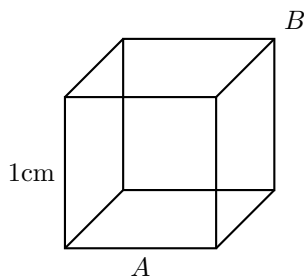
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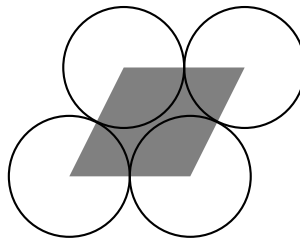
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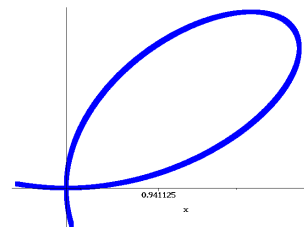
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Q4. Sugar Cube



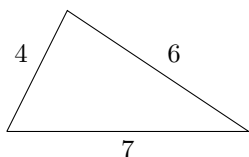
Q5. Hexagonal Packing



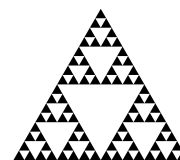
Q18. Implicit Curve

### 3 Problems

1. Prove that  $\frac{1}{0}$  is undefined. (2 marks)
2. Derive the identity  $\sin^2(\theta) + \cos^2(\theta) = 1$ . (2 marks)
  - 2.1. Hence, and not otherwise, show that  $1 + \cot^2(\theta) = \csc^2(\theta)$ . (1 marks)
3. Find the sum of the first 1,000 positive integers. (2 marks)
4. An ant sits on point A of  $1\text{cm} \times 1\text{cm}$  sugar cube. She wants to get to point B. What is the shortest distance she can take? (3 marks)
5. What fraction of total area do the circles cover if the circles have a radius of 1. (3.5 marks)
6. What is the dimension of Sierpinski's triangle? (4 marks)
7. Prove that  $\sqrt{2}$  is irrational. (3 marks)
8. Derive the quadratic formula. (3.5 marks)
9. Find the equation of the tangent and the equation of the normal to the function  $f(x) = x^3 - 3x$  at the point  $x = 2$ . (4 marks)
10. Solve  $p(x) = 2x^3 - 11x^2 + 14x + 10$  if  $p(3 + i) = 0$ . (3 marks)
11.  $\int (e^{t^2} + 16)te^{t^2} dt$ . (2.5 marks)
12.  $\int \tan(t) \sec^2(t) dt$ . (4 marks)
13. Sketch  $\frac{1}{(x-3)(x-4)}$ . (4 marks)
14. Balance the following chemical equations:
  - 14.1.  $C_3H_8O_2 \rightarrow CO_2 + H_2O$  (combustion of propane!) (1 marks)
  - 14.2.  $CO_2 + H_2O \rightarrow C_6H_{12}O_6$  (photosynthesis) (1 marks)
  - 14.3.  $HCl + Na_3PO_4 \rightarrow H_3PO_4 + NaCl$  (1 marks)
15. How many *distinct* arrangements are there of the word **BANANA**? (3 marks)
16. Find the exact area of the following triangle. (4 marks)
17.  $\int_{-1}^1 \cos(2x) + x^2 + 2^x + \frac{2}{x} dx$ . (3.5 marks)
18. Find the equations of the tangents to  $2x^3 + 2y^3 = 9xy$  at  $x = 1$ . (4.5 marks)
19. Glenn, the fast bowler runs in to bowl and releases the ball 2.4 metres above the ground with a speed of 144 km/h at an angle of  $7^\circ$  below the horizontal. Take  $g = 10\text{m/s}^2$  and find how long before the ball hits the pitch. (5 marks)
20. Let  $\vec{u} = (4, -1)$ ,  $\vec{v} = (0, 5)$ ,  $\vec{w} = (-3, -3)$ : Find:
  - 20.1.  $\vec{u} + \vec{w}$  (1 marks)
  - 20.2.  $|\vec{u} + \vec{w}|$  (1 marks)
  - 20.3.  $3\vec{v} - 2\vec{u} + \vec{v}$  (2 marks)
21. Solve the values of  $x$  which satisfy the equation  $23x \equiv 11 \pmod{30}$ . (3 marks)



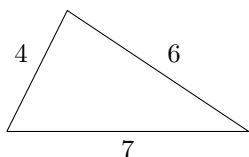
Q16. Hero's Triangle



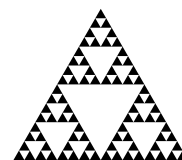
Q6. Sierpinski's Triangle

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Q16. Hero's Triangle



Q6. Sierpinski's Triangle