Monty Hall Problem

Bayes' Theorem

Let $\{B_1, B_2, \ldots, B_n\}$ be a partition of the sample space such that $P(B_k) > 0$ for each k, and let A be an event with $P(A) > 0$. Then for any k,

$$P(B_k \mid A) = \frac{P(A \mid B_k) \cdot P(B_k)}{\sum_{j=1}^{n} P(A \mid B_j) \cdot P(B_j)}$$

The Classic Problem

You know the story, 3 doors (one car, two goats), pick a door and the host (knowing the location of the car) opens another door revealing a goat.

1. Let the car be at Door 1 (C_1), if the player picks Door 1 then the host reveals one of the the other doors at random and **switching loses**. 1/3
2. Let the car be at Door 1 (C_1), if the player picks Door 2 then the host must reveal a goat and **switching wins**. 1/3
3. Let the car be at Door 1 (C_1), if the player picks Door 3 then the host must reveal a goat and **switching wins**. 1/3

We can see here clearly that switching wins 2/3 times, as the host knows which door and is constrained by this knowledge, but what does that look like formally?

Formal Derivation

Let:

- C_1: event that the car is behind Door 1.
- A: event that the host opens Door 2, revealing a goat.
- Assume the player initially chooses Door 1.

We are going to have the probability of C_1 given A, where $P(C_1)$ being 1/3 (as prior, player was free in their choice). This will, according to Bayes' Theorem, be calculated by the probability of A given C_1 times $P(C_1)$, over the probability of A given C_1 times $P(C_1)$ plus the probability of A given C_2 times $P(C_2)$ plus the probability of A given C_3 times $P(C_3)$. We know that if the player chose correctly C_1, then the host picks randomly between doors 2 and 3 as they both have goats and thus $P(A \mid C_1) = \frac{1}{2}$. If the player was incorrect and the car was behind C_2, then the host cannot pick Door 2 and thus this conditional probability drops to zero $P(A \mid C_2) = 0$. If the player was incorrect and the car was behind C_3, then the host must pick Door 2 and thus this conditional probability is $P(A \mid C_3) = 1$. Thus we have:

$$P(C_1 \mid A) = \frac{P(A \mid C_1) \cdot P(C_1)}{P(A \mid C_1) \cdot P(C_1) + P(A \mid C_2) \cdot P(C_2) + P(A \mid C_3) \cdot P(C_3)}$$

and with substitution:

$$P(C_1 \mid A) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

So what does this mean? It means that if you pick correctly, then your odds never change and you have a probability of doing so one in three times. This means that swapping to the other option, given that the host **knows** which door hides the car, has different conditional probability than your choice - *it is affected by the knowledge of the host.* At any time, the total probability must add up to 1, so the probability of switching $= 1 - \frac{1}{3} = \frac{2}{3}$. You could go on and show this even further but I believe this communicates the point.

Let:

- C_1: event that the car is behind Door 1.
- A: event that the host opens Door 3, revealing a goat.
- Assume the player initially chooses Door 2.

We are going to calculate the probability of C_1 given A, where $P(C_1) = \frac{1}{3}$ (as prior, the player was free in their choice). This will, according to Bayes' Theorem, be calculated by the probability of A given C_1 times $P(C_1)$, over the probability of A given C_1 times $P(C_1)$ plus the probability of A given C_2 times $P(C_2)$ plus the probability of A given C_3 times $P(C_3)$.

We know that if the player was incorrect and the car is at C_1, then the host must avoid the car and reveal the only remaining goat. Since the player chose Door 2 and the car is at Door 1, the host is forced to open Door 3 (the only available goat). Thus $P(A \mid C_1) = 1$.

If the player chose correctly and the car is at C_2, then the host chooses randomly between Doors 1 and 3 (both goats), and so $P(A \mid C_2) = \frac{1}{2}$.

If the car is at C_3, then the host cannot open Door 3 (it hides the car), so $P(A \mid C_3) = 0$.

Thus we have:

$$P(C_1 \mid A) = \frac{P(A \mid C_1) \cdot P(C_1)}{P(A \mid C_1) \cdot P(C_1) + P(A \mid C_2) \cdot P(C_2) + P(A \mid C_3) \cdot P(C_3)}$$

and with substitution:

$$P(C_1 \mid A) = \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

So what does this mean? It means that in this case, the probability that the car is behind Door 1 has *increased* based on what the host does — because the host had no choice but to open Door 3. That action was fully determined by the game state. The conditional probability of C_1 given A is higher than it was before the host acted. This is a case where the host's behavior **is informative**, because the host's action was constrained. In contrast, had the player picked correctly (as in the previous case), the host would have had a choice. Here, the forced action reveals information. The host's knowledge affects the outcome. I hope that you can see that this will hold for the identical situation where the car was behind Door 1, the player picks Door 3 and the host is forced to open Door 2 (**swapping to Door 1 has odds of two in three**).

PS8 Variant

So what about the case where the host **DOES NOT KNOW WHICH DOOR HIDES THE CAR**, but opens a door and it is a goat? What is the **posterior probability of winning the car if you switch**?

Let:

- C_1: event that the car is behind Door 1.
- A: event that the host opens Door 3, revealing a goat.
- Assume the player initially chooses Door 2 (so that switching to Door 1 results in winning the car).

We now consider a variant where the host **does not know** which door hides the car. The host simply picks one of the other two doors at random. If it reveals a goat, the player is then offered the chance to switch.

We are going to calculate the probability of C_1 given A, where $P(C_1) = \frac{1}{3}$ (as prior, the player was free in their choice). This will, according to Bayes' Theorem, be calculated by the probability of A given C_1 times $P(C_1)$, over the probability of A given C_1 times $P(C_1)$ plus the probability of A given C_2 times $P(C_2)$ plus the probability of A given C_3 times $P(C_3)$.

We now compute each conditional:

- If the car is behind Door 1 (C_1), and the player picks Door 2, then the host randomly chooses between Doors 1 and 3. Since Door 1 hides the car, **the host must have opened Door 3**, which contains a goat. So $P(A \mid C_1) = 1$.
- If the car is behind Door 2 (C_2), the player has picked the car. The host randomly chooses between Doors 1 and 3. Since Door 1 hides the car, **the host must have opened Door 3**, which contains a goat. So $P(A \mid C_2) = 1$.
- If the car is behind Door 3 (C_3), and the player picks Door 2, the host opens Door 3 and accidentally reveals the car. This contradicts the observed event, so $P(A \mid C_3) = 0$.

Thus we have:

$$P(C_1 \mid A) = \frac{P(A \mid C_1) \cdot P(C_1)}{P(A \mid C_1) \cdot P(C_1) + P(A \mid C_2) \cdot P(C_2) + P(A \mid C_3) \cdot P(C_3)}$$

and with substitution:

$$P(C_1 \mid A) = \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

So what does this mean? It means that under this variant — where the host does not know where the car is and opens one of the remaining two doors at random — the host's action is only partially informative and only in the particular context of the question where a door has been opened and it was **NOT** the car. The situation has arisen where your initial odds were as usual one in three (from free choice), but now in this scenario where the host has picked at random and **NOT** revealed the car, some information has still been gained (more from the scenario that the random choice made by the host) and it is still better to switch ($\frac{1}{2} > \frac{1}{3}$). Hopefully you can see that this would hold if you picked Door 1 and the car was at Door 2 and the host revealed Door 3. Hopefully you can see that if you picked Door 1 and the car was actually at Door 1, then switching would be a bad outcome for you but on average it is still better odds to switch.