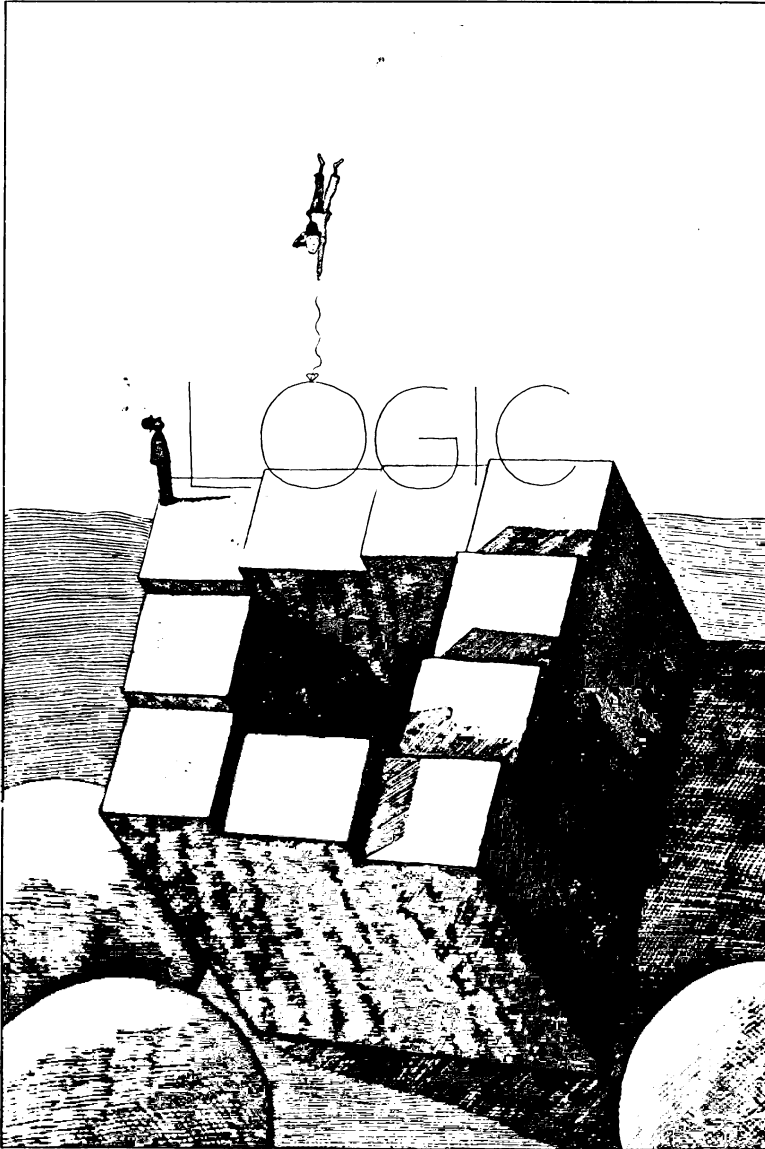


LOGIC

WILFRID HODGES



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LOGIC

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Wilfrid Hodges

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Wilfrid Hodges
Bedford College
March 1976

Introduction

This book was written for people who want to learn some elementary logic, regardless of whether they are taking a course in it.

The book is written as a conversation between you (the reader) and me (the author). To keep the conversation from being too one-sided, I put in fairly frequent exercises. You are strongly urged to try to answer these as you reach them. Correct answers are given at the end of the book.

If you have a phobia of symbols, you should leave out the seven sections marked '+' in the text. These cover the branch of logic known as Formal Logic; they contain the worst of the mathematics.

English words and phrases which are being discussed are usually printed bold; most of these are listed in the index. A word is printed in quotes ' ' when we are chiefly interested in the way it occurs in some phrase.

Consistency

Logic can be defined as the study of consistent sets of beliefs; this will be our starting-point. Some people prefer to define logic as the study of valid arguments. Between them and us there is no real disagreement, as section 11 will show. But consistency makes an easier beginning.

1. Consistent Sets of Beliefs

Logic is about consistency – but not about all types of consistency. For example, if a man supports Arsenal one day and Spurs the next, then he is fickle but not necessarily illogical. If the legal system makes divorce easy for the rich but hard and humiliating for the poor, then it is unjust but not illogical. If a woman slaps her children for telling lies, and then tells lies herself, she may be two-faced but she need not be illogical.

The type of consistency which concerns logicians is not loyalty or justice or sincerity; it is *compatibility of beliefs*. A set of beliefs is consistent if the beliefs are compatible with each other. To give a slightly more precise definition, which will guide us through the rest of this book: a set of beliefs is called *consistent* if these beliefs could all be true together in some possible situation. The set of beliefs is called *inconsistent* if there is no possible situation in which all the beliefs are true.

For example, suppose a man believes:

It would be wrong to censor violent programmes on television, because people's behaviour isn't really affected by what they

1.1

see on the screen. All the same it would be a good idea to have more programmes showing the good sides of our national way of life, because it would straighten out some of the people who are always knocking our country.

These beliefs are inconsistent: if it's really true that people's behaviour is not affected by what they see on the television screen, then it can't also be true that critics of a country will be reformed by what they see on the television screen. There is no possible situation in which (1.1) could all be true.

Inconsistency of beliefs is not at all the same thing as stupidity or unreasonableness. Take for example the man who believes:

During the last five years I have been involved in three major accidents and several minor ones, while driving my car. After two of the major accidents, courts held me responsible. But basically I'm a thoroughly safe driver; I've simply had a run of bad luck. 1.2

This man is almost certainly deceiving himself when he says he is a safe driver. His views are unreasonable. But they are not inconsistent: there is a possible (but unlikely) situation in which all of (1.2) would be true.

Or consider the man who believes:

The surface of the earth is flat (apart from mountains, oceans and other relatively small bumps and dips). When people think they have sailed round the earth, all they have really done is to set out from one place and finish up in another place exactly like the one they started from, but several thousand miles away. 1.3

This could all have been true, if the universe had been different from the way we know it is. A flat earth like the one described is a possibility. What is *not* a possibility is a flat earth with

properties like the real earth as we know it. The beliefs in (1.3) are consistent in themselves, even though they are not consistent with the known facts.

Our definition of consistency can be applied to beliefs one at a time too. A single belief is called *consistent* if it could be true in some possible situation. An inconsistent belief is said to be *self-contradictory* and a *contradiction*.

For example, suppose a man tells you:

I have invented an amazing new sedative which 1.4
makes people faster and more excited.

Then you can tell him – if you feel it would help – that he believes something self-contradictory. There is no possible situation in which a thing that made people faster and more excited could also be a sedative.

Is consistency of beliefs a virtue? Is it something we should spend time trying to achieve?

To some extent this is a question in a vacuum. Nobody is *deliberately* inconsistent in his beliefs. It is simply impossible to believe, fully and without reservation, two things which you know are inconsistent with each other.

Exercise 1A. You know that human beings normally have two legs. Try to convince yourself that they normally have five. (Allow yourself at least a minute.)

It seems we are obliged to believe only what we think is consistent, without having any real choice in the matter. In this way we are all logicians, simply because we are human. When we study logic, we are teaching ourselves to do deliberately, by rule, something we have been doing semi-consciously, by hunch, ever since the age of four.

Exercise 1B. Test your hunches by deciding which of the following sets of beliefs are consistent. (At this stage, don't take any of the examples too solemnly.)

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1. I've never drawn anything in my life. But if I sat down to it now, it would take me two minutes to produce a drawing worth as much as anything Picasso has produced.
 2. I knew I would never get pregnant. But somehow it just happened.
 3. There is no housing shortage in Lincoln today – just a rumour that is put about by people who have nowhere to live.
 4. Walter joined the friendly club two years ago, and has been one of its most loyal members ever since. Last year he paid for the holidays of precisely those club members who didn't pay for their own holidays.
 5. The handless hold the hoe.
A pedestrian walks, riding on a water buffalo.
A man passes over the bridge;
The bridge but not the water flows.
-

Expressing Beliefs in Sentences

Logic, then, is about beliefs and about when they are consistent with each other. But beliefs are hard to study directly: they are invisible, inaudible, weightless and without perceptible odour. To ease our task, we shall think of beliefs as being expressed by *written sentences*. This is not as straightforward as one might think. Two people can write down the same sentence and mean entirely different things by it. T. S. Eliot's despairing cry is often quoted:

Words strain,
Crack and sometimes break, under the burden,
Under the tension, slip, slide, perish,
Decay with imprecision, will not stay in place,
Will not stay still.

'Burnt Norton', V

This is why we must spend a few sections examining how words are related to beliefs.

2. Beliefs and Words

When people want to let other people know what their beliefs are, they put them into words. We shall take a cue from this: instead of studying beliefs directly, we shall study the sentences which are used to state them. There are hazards in this.

In the first place, *many sentences do not naturally state beliefs*. For example, questions and commands don't express beliefs, at least not directly. (True, a person may betray his beliefs by the questions he asks.)

We shall limit ourselves to a class of sentences which are particularly suited to stating beliefs. These sentences are the so-called *declarative* sentences, and section 3 will explain what they are.

Second, *one sentence may have two different meanings*; in other words, it may be *ambiguous*. An ambiguous sentence may express either of two quite different beliefs. In section 4 we shall consider ambiguity, its types and its implications.

Third, *one sentence may be used on different occasions to talk about different things*. The weather forecaster who says 'It's going to rain tomorrow' is talking about the day after the day on which he makes his prediction. If he makes the same prophecy on two hundred different days in one year, he is using the same sentence to express two hundred different beliefs. A sentence may express a true belief when it is used in one situation, and an incorrect belief in another situation.

In section 5 we shall examine how the situation in which a sentence is used determines what it is about. This forms the theory of *reference*, which is a vital part of logic – unlike ambiguity, which is just a nuisance.

Fourth, *it may not always be clear whether a given sentence does correctly state a given belief*. This happens with confused or inchoate beliefs. It also happens because there is a nasty borderline between telling the truth in a misleading way, and telling downright lies. Like ambiguity, this is a tiresome nuisance; section 7 will consider it.

Fifth, since this is a book, all sentences in it will be written down. But *the written word is not a perfect substitute for the spoken word* – all the nuances of stress and intonation are lost.

One final difference between beliefs and sentences is that *sentences have to be in a language*. In this book, nearly all sentences are in English. Beliefs themselves are not in any language – in fact animals which speak no language can still believe things. It has been maintained that logicians have been unduly influenced by the grammar of English and other European languages. If Eskimos had invented the study of logic, it is said, the whole subject would have been quite differ-

ent. This may be so, but it seems unlikely. For example, Japanese logicians do very much the same kind of work as white American ones.

3. Declarative Sentences

A *declarative sentence* of English is defined to be a grammatical English sentence which can be put in place of 'x' in

Is it true that x ? 3.1

so as to yield a grammatical English question. (Usually we shall omit the reference to English, since there is no danger of confusion with any other language.)

For example,

The price of beef has fallen. 3.2

is a declarative sentence, since we can form the grammatical English question

Is it true that the price of beef has fallen? 3.3

To apply the definition we have just given, we have to be able to tell the difference between grammatical and ungrammatical strings of words. This difference is important for logic, and in some ways very perplexing. So before we look for any further examples of declarative sentences, we shall make a detour to decide what 'grammatical' means.

The roots of grammar lie in the feelings, which every speaker of the language has, that certain strings of words are 'correctly put together' and others are not. These feelings are hard to describe and explain; but, for the most part, two speakers of the same language will agree about what feels right and what feels wrong. We may call a string of words *grammatical* if most speakers of the language would accept it as correctly formed.

Strings of words can feel wrong in different ways and to different extents. We shall taste a few samples.

SAMPLE I

These pages must not be removed or defaced. 3.4

Your breast will not lie by the breast 3.5
Of your beloved in sleep.

These are two perfectly grammatical sentences; it's hard to see anyone raising an objection to either of them.

SAMPLE II

*The very so not was, wasn't it? 3.6

*echo foxtrot golf hotel 3.7

(3.6) and (3.7) are disastrously wrong. By any reckoning they are ungrammatical. (Linguists put * at the beginning of a phrase to indicate that it's ungrammatical.)

SAMPLE III

*Did you be angry with Sally? 3.8

*Her train was decorated with silver gorgeous 3.9
lace.

*Government hastening collapse of economy. 3.10

*By swaggering could I never thrive. 3.11

None of (3.8)–(3.11) are completely acceptable in standard English. If a foreigner said one of them, we would know at once what he meant, but we might well correct him. The natural corrections are, respectively,

Were you angry with Sally? 3.12

Her train was decorated with gorgeous silver 3.13
lace.

The government is hastening the collapse of 3.14
the economy.

I could never thrive by swaggering. 3.15

Of course there are many situations where a slightly ungrammatical sentence is more appropriate than a strictly grammatical one. For example, newspaper headlines (such as (3.10)) are meant to catch the eye and give a rapid indication of what's in

the print below them. It would be silly to expect them to contain all the 'of's and 'the's. Likewise only a pedant with time to kill would think of correcting the grammar of people's telegrams. Again, everyone allows poets to break the rules (for emphasis, for special effect, or just to make it scan).

It will be useful to have a word for the kind of mistake which occurs in (3.8)–(3.11). We shall say that one sentence is a *perturbation* of another if the first sentence is grammatically wrong, but so nearly right that if it was used by someone whose English was imperfect we could safely correct it to the second sentence. Thus (3.8) is a perturbation of (3.12).

SAMPLE IV

?The pianist then played a red hat topped with geraniums and wisdom.	3.16
?He ate a slice of boredom.	3.17
?Nothing spoke her more than a hot bath.	3.18
?The civilization of the ancient Persians fervently knew at least two inches.	3.19

(3.16)–(3.19) put concepts together in impossible ways, but otherwise they are sound. Some logicians regard sentences like these as contradictions, while others take the view that a sentence like (3.16) is too nonsensical to be regarded as stating any belief. Grammarians also differ among themselves about whether these sentences should be counted as grammatical. (The ? at the beginning indicates doubtful grammaticality.)

The kind of mistake which occurs in sentences like (3.16)–(3.19) is called a *selection mistake*. (The term is based on a grammatical theory of Noam Chomsky.†) Selection mistakes are easy to recognize by their bizarre and poetic feel. In fact they play an important role in poetic or metaphorical writing. By committing a selection mistake deliberately, a writer can force his prosaic readers to forget the literal sense of what he says; since they can make nothing of his words if they take them literally, they have to notice the colours and the overtones.

†Expounded in *Aspects of the Theory of Syntax*, M.I.T. Press, Cambridge, Mass., 1965.

Exercise 3A. Examine each of the sentences below, and decide whether it is

- (a) grammatically correct,
 - (b) a perturbation of a grammatically correct sentence (say which),
 - (c) a sentence committing a selection mistake, or
 - (d) hopelessly ungrammatical.
1. Colourless green ideas sleep furiously.
 2. Furiously sleep ideas green colourless.
 3. Please pass me a butter.
 4. I don't care who you invite; I won't be there.
 5. singing each morning out of each night my father moved through depths of height
 6. My father moved through theys of we
 7. I cannot bear the thought of him going away.
 8. His face was calm and relaxed, like the face of an asleep child.
 9. Not, Father, further do prolong
Our necessary defeat.
 10. Time and the bell have buried the day.

Now we can come back to our definition of declarative sentences. A string of words will be called a *declarative sentence* if it is a grammatical sentence which can be put in place of 'x' in

Is it true that x ? 3.20

so as to yield a question which is grammatical in the fullest sense – i.e. not a perturbation, and not a sentence with a selection mistake.

Thus a grammatical sentence telling part of a story or stating a scientific or domestic fact will normally be declarative:

- The accused entered the hall by the front door. 3.21
- The current through a capacitance becomes 3.22
greater when the frequency is increased.
- It could do with a spot more salt. 3.23

In contrast to these, *questions* are not declarative sentences:

When did you last see your father? 3.24
yielding

*Is it true that when did you last see your
 father?

Neither are *commands*; neither are *invitations*:

Let us go then, you and I, 3.25
 When the evening is spread out against the
 sky.

Neither, by our definition, are sentences containing selection mistakes:

?The heart consists of three syllables. 3.26

The following exercises are not just drill. They illustrate some more points about declarative sentences, and at least two of the answers are controversial.

Exercise 3B. Which of the following are declarative sentences?

1. Twice two is four.
2. Please write a specimen of your signature in the space provided.
3. Would you believe you're standing where Cromwell once stood?
4. That's true.
5. I promise not to peep.
6. Adultery is wicked.

4. Ambiguity

A string of words is said to be *ambiguous* if it can be understood as a meaningful sentence in two or more different ways.

Broadly speaking, there are two kinds of ambiguity – though they often occur together. The first kind is *lexical ambiguity*;

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this occurs when a single word in the string can be understood in more than one way. For example, the string

I thought it was rum. 4.1

contains a lexical ambiguity, because the word **rum** can mean either 'queer' or 'a drink made from fermented molasses'. (The context would usually make clear which is meant, of course: but the string itself is ambiguous.)

The cobbler who advertised

Our shoes are guaranteed to give you a fit. 4.2

committed another lexical ambiguity.

The second type of ambiguity is *structural ambiguity*; this occurs when the words in the string can be grouped together in different ways. For example, the string

I heard about him at school. 4.3

is structurally ambiguous: does it say that I was at school when I heard about him, or that I heard about what he did when he was at school? A similar structural ambiguity occurs in

Bert was a fat stock breeder. 4.4

One kind of structural ambiguity deserves a special name. This is *ambiguity of cross-reference*: it occurs when a word or phrase in the string refers back to something mentioned elsewhere, but it isn't clear which thing. In the following example, the word **it** is the culprit – does it refer back to 'word' or to 'voice'?

As he uttered the all-important word he 4.5
dropped his voice, but she just managed to
catch it.

Exercise 4A. What kinds of ambiguity occur in the following?

1. Launching the ship with impressive ceremony, the Admiral's lovely daughter smashed a bottle of champagne over her stern as she slid gracefully down the slipways.

2. Miss Crichton pluckily extinguished the blaze while Herr Eckold pulled the orchestra through a difficult passage.
 3. A seventeen-year-old Copnor youth was remanded in custody to Portsmouth Quarter Sessions by Portsmouth Magistrates yesterday after he had admitted stealing three bicycles, a record player, thirty-one records, a National Insurance Card, and two cases of false pretences.
 4. The font so generously presented by Mrs Smith will be set in position at the East end of the Church. Babies may now be baptized at both ends.
 5. The visit has been planned in the atmosphere of the general economy going on in the country. The lunch has been cut to the bare bones.
 6. The Government were strongly urged to take steps to put a stop to the growing evil of methylated spirit drinking by the Liverpool justices at their quarterly meetings.
-

There are two ways of talking about ambiguity: we can talk of one string representing two different sentences, or we can say that one sentence has two different meanings. The first style is probably preferable, but no harm will accrue if we sometimes talk in the second way.

It is easy to generate bogus inconsistencies by ignoring lexical ambiguities. For example, some people said that J. M. Keynes's *General Theory of Employment, Interest and Money* was inconsistent, because it maintained that the level of savings is always equal to the level of investment, although Keynes admitted that if people saved all their money rather than spending it, there would be no investment. The ambiguity lies in the word *save*: in common parlance it means hoarding, but it has quite a different sense in Keynes's theory.

In cases like this, it may help to distinguish two different words which happen to be written alike. Thus we may distinguish *saving*₁, which means hoarding, from Keynes's technical term *saving*₂, which means something more subtle.

Structural ambiguities may also lead to spurious inconsistencies. The best defence is to rewrite the offending sentences so as

When is a Sentence True?

Most contemporary logicians believe that there are two fundamental links between words and things. The first link is that declarative sentences are *true in certain situations*, and *not true in other situations*. The second link is that certain phrases *refer to* things in certain situations. Of course science advances, and it may be that some Einstein of logic will appear in A.D. 2000 and convince us that some quite different and hitherto unimagined notion is the key to the relations between words and things. But at present there is no sign of this.

Perhaps the first thinkers to take seriously the questions we now consider were the Heraclitean philosophers of ancient Greece, who maintained that 'It is impossible to say anything true about things which change.' One of them, Cratylus, found the whole matter so distressing that he thought it best to stop talking altogether, and simply waggle his finger.

5. Truth and References

The weather forecaster, we recall, makes the same prediction on many different days. Sometimes he is right, sometimes he is wrong. This illustrates an important point: *one and the same sentence can be used to make a true statement in one situation and an untrue statement in another situation.*

When a person utters a sentence, various parts of the sentence refer to various things in the world. The things which are referred to will normally depend on who uttered the sentence, and where and when and how he uttered it – in short they will depend on the *situation* in which the sentence was

uttered. There are rules which determine what things are referred to by which parts of the sentence. As we shall see, these rules are complicated, and it takes children several years to learn them.

The examples which follow are somewhat oversimplified. The thing referred to by a phrase in a situation is called the *reference* of that phrase.

(1) There are some words and phrases whose reference depends entirely on where and when they are uttered. For instance:

I, now, here, the future, this month, next door. 5.1

We can also put words like *will* and *was* in this group, because they refer to the times after or before the time of utterance. (Words of this group are said to be *token-reflexive*.)

(2) Some words and phrases require the speaker to give some special indication of what is referred to. For example, if I say 'You' when there are several people in the room, then the usual convention is that 'You' refers to the person I'm looking at or pointing to. Some other phrases which need a pointing finger are

this chair, the house over there, thus. 5.2

(3) A proper name, such as *Arthur*, is only used to refer to people called *Arthur*. But of course thousands of people are called *Arthur*. The convention is that if you and I are in a situation where only one person called *Arthur* is likely to come to mind, then I can use the name *Arthur* to refer to that one person.

(4) Some phrases, such as *the latter reason* or *the aforementioned reindeer*, have a reference which depends entirely on the reference of some previously used phrase. This is a phenomenon called *cross-referencing*; we shall meet it many times again. (Recall the ambiguities of cross-reference which we discussed in section 4.) A simple example is:

Brutus killed Caesar by stabbing him. 5.3

Here **him** refers to whatever the word **Caesar** refers to; i.e. it refers to the man **Caesar**. Pronouns such as **he**, **him**, **it** are often used in this way.

(5) Some phrases have the same reference whatever the situation; for example

sulphur, spherical, eighty-two, angrily. 5.4

As these examples show, the rules are complicated. Note that **I** comes under (1), **you** comes under (2) and **he** comes under (4); this indicates that a lot more than grammar is involved in the rules. Also one word may change its reference several times during a conversation.

Exercise 5. The following passage is about a situation in which Jacob wrestled with a man. Find every occurrence of **he**, **him**, **I**, **me**, **thou**, **thee** in the passage, and say who (Jacob or the man) is the reference at each occurrence.

And when he saw that he prevailed not against him, he touched the hollow of his thigh; and the hollow of Jacob's thigh was out of joint, as he wrestled with him. And he said, Let me go, for the day breaketh. And he said, I will not let thee go, except thou bless me. And he said unto him, What is thy name? And he said, Jacob. And he said, Thy name shall be called no more Jacob, but Israel: for as a prince hast thou power with God and with men, and hast prevailed. And Jacob asked him, and said, Tell me, I pray thee, thy name. And he said, Wherefore is it that thou dost ask after my name? And he blessed him there.

Genesis 32: 25-9, Authorized Version

The truth or untruth of a sentence commonly depends on the references of its parts; these references may change from one

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situation to another, and this is why the sentence may be true in one situation but not in another.

Rules are sometimes broken. If I tell you

The key is in the box by the phone. 5.5

when in fact there is no box by the phone, then the phrase **the box by the phone** has no reference. The sentence (5.5) is being used improperly, because a phrase which ought to have a reference has none; in such cases we shall say that *referential failure* occurs.

Carelessness about referential failure easily leads to mistakes in logic. A particularly glaring example occurs in an argument which St Anselm of Canterbury used in order to prove the existence of God. Anselm reckoned it would be enough if he could prove that 'that than which no greater thing can be conceived' must exist. In order to prove that this thing does exist, he argued:

Suppose that than which no greater thing can
be conceived doesn't exist outside our minds. 5.6
Then it is not as great as it would have been if
it had existed. Therefore we can conceive
something greater than that than which no
greater thing can be conceived; which is
impossible. Therefore our original supposi-
tion is incorrect.

Now in the second sentence of (5.6), Anselm is assuming that there is something for 'it' to refer to. In view of the cross-referencing, this means he is assuming that the expression **that than which no greater thing can be conceived** refers to something. But he is not entitled to assume this, because it's precisely what he was supposedly proving in this argument. This makes the argument completely unconvincing.

Anselm's disregard for referential failure is rather extreme. But people often use words in ways which fail to make clear what they are talking about. The psychologist Jean Piaget

noticed that young children frequently do this; for example he heard an eight-year-old child explaining the workings of a tap as follows:

That and that is that and that because there it is 5.7
for the water to run through, and that you see
them inside because the water can't run out. The
water is there and cannot run.†

This should be counted as an example of referential failure; although the child himself knows what he means, he uses words in such a way that the conventions of language fail to provide the needed references. Piaget thought that young children believe that references pass magically from their minds into the minds of the people they are talking to. An alternative explanation is that young children have not yet learned all the complex linguistic and social rules which determine references.

6. Borderline Cases and Bizarre Situations

If a declarative sentence is not true in a certain situation, we shall say that it is *false* in that situation. For example, the sentence

The King of France is bald. 6.1

is not true in the present state of affairs, because there is a referential failure – France has no King. Therefore we shall count (6.1) as being false in the present situation. The sentence

The King of France is hairy. 6.2

is also false, for the same reason.

Logicians say that the *truth-value* of a declarative sentence is Truth when the sentence is true, and Falsehood when it is false. In any situation, a declarative sentence has just one truth-value:

†Jean Piaget, *The Language and Thought of the Child*, Routledge & Kegan Paul, 1959, p. 104f.

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either Truth or Falsehood. There is something in this to catch the imagination. Life seems full of half-truths, grey areas, borderline cases, but Logic stands with sword uplifted to divide the world cleanly into the True and the False.

Let life, waned, ah let life wind
Off her once skeined stained veined variety upon, all on two spools;
part, pen, pack
Now her all in two flocks, two folds – black, white; right, wrong;
reckon but, reck but, mind
But these two; ware of a world where but these two tell, each off the
other . . .

from Gerard Manley Hopkins, 'Spelt from Sibyl's Leaves'

Many people have been attracted to logic by some such feeling. But an honest thinker must ask himself whether this clean and absolute division into Truth and Falsehood is perhaps no more than a verbal illusion. Maybe Truth itself has degrees and blurred edges?

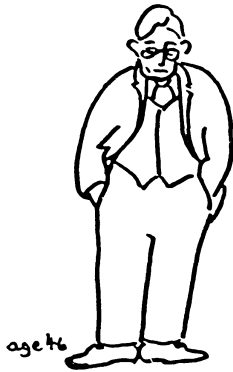
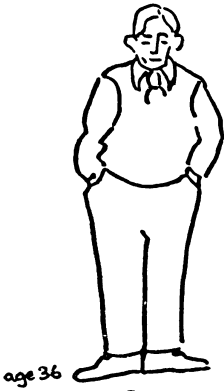
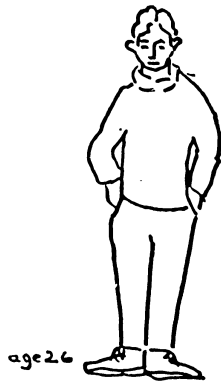
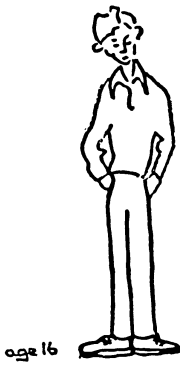
We shall try to answer this question. But first spare a thought for Ted Bartlett, who got fatter as he grew older – as Figure 1 illustrates.

Notice that even when Ted was definitely fat, it was possible for him to get still fatter. There can be two really fat people, one of them fatter than the other. In the same way it may be that my jokes are really funny, and your jokes are really funny too, but yours are much funnier than mine. Adjectives like *fat* and *funny* which have this feature will be called *scaling adjectives*. All of the following adjectives are scaling:

happy, expensive, heavy, unpleasant. 6.3

The adjective *straight* is not a scaling adjective. If you and I draw lines, and your line is straighter than mine, then mine can't really be straight. Likewise *silent* is not a scaling adjective: if your machine is more silent than mine, then mine isn't really silent. Some other adjectives which are not scaling are

square, perfect, smooth, daily. 6.4



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Figure 1

Exercise 6. Which of the following adjectives are scaling?

- | | |
|---------------|-------------|
| 1. cold | 6. red |
| 2. fast | 7. free |
| 3. circular | 8. accurate |
| 4. two-legged | 9. generous |
| 5. old | 10. full |
-

Is **true** a scaling adjective? It is not: if your statement is truer than mine, then mine is not wholly true. **More true** can only mean 'more nearly true' or 'nearer the truth'. In this sense there are no degrees of truth. Truth is absolute.

But this is not the end of the matter. There is another way in which truth can be inexact. Ted Bartlett is the key once more; the thing to notice this time is that there was no exact time at which he became fat. At twenty-six he definitely wasn't fat, at fifty-six he definitely was. But there was a period in the middle when one could only describe him by roundabout phrases like **not all that fat, really**. There is no precise cut-off point between fat and not fat. We express this by saying that fat has *borderline cases*.

Most adjectives have borderline cases. Obviously **funny** and the adjectives in (6.3) have them. But so do some non-scaling adjectives like **silent**. There are situations which could be described as silent or as not silent, depending on what you count as a noise. (In Norse mythology, Heimdallr can hear the wool grow on a sheep's back.) On the other hand it seems that **daily** has no borderline cases; there is a clear and exact dividing line between daily and not daily.

How does **true** fare? Unfortunately it fares as badly as it possibly could. To see this, consider Ted Bartlett yet again. This time, instead of asking whether he is fat, ask whether it is *true* that he is fat. The borderline area comes just where it did before.

Thus we see that although truth doesn't have degrees, it does have many borderline cases.

There is a paradox here. In a given situation a declarative

sentence must be either true or false, and not both, as we saw; nevertheless there are situations in which a declarative sentence may be not definitely true and not definitely false, in fact indeterminate. The dividing sword cuts somewhere, but there may be no definite place where it cuts.

The implications for logic are quite serious. If true has borderline cases, then so does consistent. Consider the pair of sentences

Ted Bartlett is not fat.

6.5

Ted Bartlett's vital statistics are XYZ.

(6.5) may or may not be consistent, depending on what we put for XYZ. But we have seen that there are some XYZ which make it quite indeterminate whether (6.5) is consistent. We are forced to admit that *where borderline cases may arise, logic is not an exact science.*

The absoluteness of Truth receives some hard knocks from yet another quarter, in the shape of *bizarre situations*. The issue is this. When a person learns a language, he learns that certain words are appropriate for certain situations, and inappropriate for others. Sometimes he meets a new situation, which is so different from the ones in which he learned the use of a word that he simply can't say, on the basis of his previous experience with the language, whether the word is correct to use in this new situation. A child who knows that *bone* is the appropriate word for the hard parts of a roast chicken may well be unsure whether he can use it for the hard part of a plum.

An adult can usually find the answer by asking someone who knows the language better than he does. 'In French, can one use "compétent" of a butler?' But it sometimes happens that a situation is so new and unusual that no speaker of the language is equipped to say what words are appropriate for it. We shall call such situations *bizarre*.

Here is an example of a *bizarre situation*. The two cerebral hemispheres of the human brain are joined by a structure called the corpus callosum. Surgeons sometimes cut through the corpus callosum in order to control epilepsy. People whose

corpus callosum has been severed are quite normal in most ways, but they show one or two very strange symptoms. Suppose we show an object to such a person, and then ask him to indicate what we showed him by writing a mark on a piece of paper. If the object fell in the left half of his field of vision, then he can answer our question with his left hand but not with his right. If the object fell in the right half, then he can answer with his right hand but not with his left. Does a person with severed callosum, who can see a mouse to the left of him, *know* that he is seeing a mouse? It seems that he knows with one half of his brain but not with the other; but in this case is it appropriate to say simply that he *knows*?

In a bizarre situation it may be impossible to say whether or not a sentence is true – not because we are stupid or we lack the facts, but simply because our language is not sufficiently articulated.

In real life, bizarre situations are the exception. But we have already seen (in section 1, (1.3)) that in logic it may be necessary to consider imaginary situations, and anybody with a creative imagination can dream up any number of bizarre situations.

There is a moral for logicians. If you want to get definite answers, then avoid the bizarre. As far as possible, stick to matter-of-fact notions, and leave the flights of fancy to the philosophers.

7. Misleading Statements

There are some cases of doubtful truth-value which revolve round the interpretation of common English words like *and* or *all*. Since it would be quite impractical to try to avoid these words, we shall have to settle these cases by some rough and ready convention.

Four examples follow. In each case a person makes a misleading statement.

(1) A witness in the case of *Thumtman v. Thumper* states that

Mr Thumper hit Mr Thumptomann three 7.1
times with the camera tripod, and Mr
Thumptomann fell to the floor.

What the witness actually saw was that Mr Thumptomann fell to the floor just before Mr Thumper came into the room, and Mr Thumper hit him three times with the camera tripod before he could get up.

(2) After the office party, a man admits to his wife

I did kiss some of the girls. 7.2

In fact he kissed all nineteen of them.

(3) After another office party, another man boasts to his wife

All the girls kissed me. 7.3

In fact there were no girls at the party.

(4) Shortly before his retirement presentation, a man says to his wife

They're going to give me either a watch or a 7.4
silver pot.

In fact he knows he is going to be given both, but he hasn't yet made up his mind which to pawn.

Everybody will agree that these four statements are misleading. But are they true or not? One view is that none of these statements are true. On this view, the word *and* in a narrative implies *and then*, the word *some* implies *not all*, the word *all* implies *at least one*, and either . . . or implies *not both*. We shall say that this view puts the *strong reading* on the sentences (7.1)–(7.4).

There is another view. We may say that each of these men has told the truth but not the whole truth. On this view, each man has misled by omitting to mention something which any honest person would have mentioned, but not by saying something untrue. If we take this view, then we say that (7.1)

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strictly says nothing about the order of the two events described, even though people normally describe the earlier event before the later. Likewise we say that *some* is quite compatible with *all*, even though people don't normally say *some* when they could have said *all*. Similarly with the other two examples. We shall say that this latter view puts the *weak reading* on (7.1)–(7.4).

The question at issue here is not a practical or moral one; the four men are equally dishonest on either reading. But it is a question which affects logic.

To see this, imagine that the speaker in example (2) decides he had better make a clean breast of it:

I did kiss some of the girls. In fact I kissed all 7.5
of them – nineteen there were.

Has the speaker of (7.5) contradicted himself? On the strong reading, he has; the second sentence is a correction of the first. On the weak reading he has not; the second sentence merely amplifies the first.

A choice is called for: must we adopt the weak or the strong reading? Arguments have been advanced on both sides, based on various theories about meaning. In this book we shall normally opt for the weak reading. We shall do so mainly because this is the habit among twentieth-century logicians. The weak reading is usually much easier to describe than the strong one.

8. Possible Situations and Meanings

What situations are possible?

In common parlance, a thing is possible only if it is consistent with the known facts. A thing is possible if it could be so, given what we know; otherwise it is impossible. It is possible that Queen Elizabeth I was a virgin till her dying day; it is possible that there is a monster in Loch Ness; it is possible that I shall catch a cold tomorrow. On the other hand it is impossible for

me to hold my breath for ten minutes, and it is not possible that the earth is flat.

But as we saw in section 1, this is not quite what a logician means when he talks of a possible situation. In logic, a situation is described as possible if it *could have been* the actual situation, forgetting what we know about the world. If things had worked out differently, I could have been a multimillionaire, I could have become a famous and successful disc-jockey – these are possible situations.

Again, the world could have been the way it actually was in 1066; the state of the world in 1066 is a possible situation.

Again, if evolution had worked out differently, I could have had compound eyes and six arms. If pigs had wings, maybe I could fly one over the Atlantic. All these are possible situations.

There are limits to what is possible. For example, there is no possible situation in which two plus two is anything but four. True, people could *count* differently; they could count

one, two, four, three, five, six, . . . 8.1

But if this was so, then two plus two would still be four, although it would be *called* three. In just the same way, two plus three is still five in Arabic countries, although Arab script writes five in a way which looks like our 0. We must remember that we are describing possible situations in *our* language, and not in the possible languages which people might adopt in those situations.

In logic, we are only interested in possible situations from the point of view of examining what is true in them. As we saw in section 6, if a situation is bizarre, it may be impossible to say whether or not a certain sentence is true in it. Even among possible situations, there are some which are too peculiar to serve any useful purpose in logic.

It is worth noting that we can refer to situations which are possible but not actual, without thereby committing a referential failure. For example, I can refer to the state of Europe in the mid-seventeenth century, although that state no longer obtains. I can also refer to things and people that don't actually exist,

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provided I say something to indicate that I am talking about a possible situation in which they do exist. For example, I can tell you that

In 1650 the Holy Roman Emperor still had 8.2
certain powers of taxation.

without falling into referential failure, because the first words of (8.2) have shifted the situation back to 1650, when there was a Holy Roman Emperor. Phrases such as **in 1650** will be known as *situation-shifters*; we shall meet them again.

Philosophers, who like to be precise in their use of words, often use possible situations in order to explain subtle differences between the meanings of words. The method is to describe a possible situation in which one sentence would be true but another similar sentence would be false. For example, J. L. Austin explained the difference between **mistake** and **accident** by telling a short story:

You have a donkey, so have I, and they graze in 8.3
the same field. The day comes when I conceive a
dislike for mine. I go to shoot it, draw a bead on
it, fire . . . but as I do so, the beasts move, and to
my horror yours falls.†

In the situation described by (8.3), I have shot your beast by accident, I have not shot it by mistake; which shows that shooting by mistake and shooting by accident are not the same thing.

The same method may be used to point up differences between legal terms. For example, to explain the difference between ownership and possession, I may point out that if I steal your watch and put it in my pocket, then I possess it but I do not own it.

A parent or a teacher uses this same method when he explains the difference between cynical and sarcastic by describing how one can be sarcastic without being cynical.

†J.L. Austin, *Philosophical Papers*, Oxford University Press, 1961, p. 133.

Exercise 8. For each of the following pairs of sentences, show that the two sentences mean different things, by describing a possible situation in which one is true and the other is false.

1. The alderman was lying.
What the alderman said was untrue.
 2. It's in your best interests to go to Corsica.
It would do you good to go to Corsica.
 3. He knows I'm at home.
He thinks I'm at home, and I am.
 4. Brutus killed Caesar.
Brutus caused Caesar to die.
 5. A crow is a kind of bird.
The word 'crow' is used to denote a kind of bird.
-

In the last few examples, we have begun to talk of sentences being true in situations where the sentences are not even uttered. Obviously this makes sense; for example the sentence

No human beings exist yet.

was true a hundred million years ago, when there was nobody around even to think it. In such cases, we use a sentence to describe a possible situation as if it was actual.

Many logicians have felt deeply unhappy about possible situations. The whole notion seemed much too speculative and metaphysical to them. They hoped that logic would guide us to greater certainties, not to perplexing questions about imaginary states of affairs. Alas, perplexing questions do not go away if we ignore them. There may be some better approach to the links between language and the world; but is it likely that Cratylus (see p. 27) and his followers will lead us there?

On a lighter note, I can report that on 1 November 1975 the television time-traveller Dr Who briefly transported himself and his police-box into the situation which would have obtained in 1980 if he had not been about to trap an Egyptian god in a time-corridor. If time-travellers can travel to situations which never have occurred nor ever will, some very interesting possibilities open up.

Testing for Consistency and Validity

Our task is to determine when a set of declarative sentences is consistent. If the sentences are all short enough, then close inspection will probably give the answer. When the sentences are longer and more complex, we need rules to guide us. The tableau method, which we follow in this book, works by breaking the sentences down into smaller ones. Although the idea seems very obvious, it was apparently first invented in the 1930s by the German mathematician Gerhard Gentzen.

We shall see that the same method can be used to test the validity of arguments.

9. Consistent Sets of Short Sentences

We began this book by defining logic as the study of the consistency of sets of beliefs; we then saw that beliefs can be expressed by declarative sentences. Just as with beliefs, a set of declarative sentences is called *consistent* if there is some possible situation in which all the sentences are true. Henceforth our task is to determine when a set of declarative sentences is consistent.

Ideally, we should like a method which could be applied to any finite set of declarative sentences, and which was guaranteed to tell us whether or not the set is consistent. Such a method would constitute a *decision procedure* for consistency. In fact one can show mathematically that no such decision procedure could possibly exist. The best we can hope for is a method which will work efficiently in most of the cases we are likely to meet.

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If the sentences in the set are all reasonably short – say five or six words apiece – then close inspection is probably the best method. A set of short sentences is not usually inconsistent unless it contains an inconsistent pair of sentences, and inconsistent pairs are for the most part easy to recognize.

For example, each of the following pairs is obviously inconsistent:

Schizophrenia is curable. 9.1

It's not true that schizophrenia is curable.

Little is known about Heimdalargaldr. 9.2

Plenty is known about Heimdalargaldr.

Hazel's dress was bright red all over. 9.3

Hazel's dress had broad blue stripes.

Occasionally we meet sets of short sentences which are inconsistent but contain no inconsistent pairs. The following example was devised by Lewis Carroll:

All puddings are nice. 9.4

This dish is a pudding.

No nice things are wholesome.

This dish is wholesome.

Note that if a set of sentences is inconsistent, then it remains inconsistent if we add more sentences to the set; this is the so-called *monotonicity* property of consistency. We can never remove a contradiction by adding a few irrelevant remarks (though we may be able to stop people noticing it).

Exercise 9. Which of the following sets of sentences are consistent? (This exercise is extremely easy, and is chiefly meant to illustrate some ways of being inconsistent.)

1. That was hardly an adequate payment. That was an entirely adequate payment.
2. Matilda is a hen. Matilda has four legs. Julia is a hen.

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3. Few people speak or understand Cornish. Cornish is similar to Breton. Cornish is very musical. Cornish is a widespread language.
 4. His hat is quite different from yours. Your hat is just like mine. My hat is unique.
 5. Alf is taller than Bernard. Bernard is taller than David. Alf is shorter than David.
 6. Angela is younger than Chris. Diana is older than Brenda. Brenda is younger than Chris. Diana is older than Chris.
-

There are the inevitable controversial examples too. We have already seen that borderline cases and bizarre situations may cause trouble. To close this section we shall consider two problem cases which have perplexed philosophers for centuries.

The first is the problem of *moral conflict*. Consider the two sentences

I ought to do it. 9.5
I ought not to do it.

It might seem obvious that (9.5) is inconsistent. But some philosophers maintain that there are situations in which both sentences of (9.5) are true at once. For example, suppose I have arranged to baby-sit for some friends, but just as I am setting out, my mother starts to feel ill. I ought to go to baby-sit, because my friends are relying on me; I ought not to go, because I should stay at home in case my mother needs help.

The second is the *problem of evil*: this involves three short sentences:

God is loving. 9.6
God is omnipotent.
God allows people to get hurt.

According to traditional Christian teaching, these three sentences are not merely consistent: they are all actually true. But can they be? If God is omnipotent and allows people to get

hurt, we have to suppose that God doesn't care whether people get hurt, since otherwise (being omnipotent) he could have prevented it. If God doesn't care whether people get hurt, can he really be described as 'loving'? Humans who don't care whether people get hurt are not normally described as loving.

It could be argued that God is so different from humans that the use of words like love and care in human situations is no guide to how they should be used of God – situations involving God are bizarre. This has often been suggested. But it is unconvincing, for two reasons. The first is that the problem of evil has been discussed for several centuries longer than the English language has existed; bizarre situations occur only when the speakers of a language have not yet had a chance to determine how they wish to use certain words to describe unusual circumstances. The second reason is that, far from there being nothing to guide us in using words like loving of God, God's reported or observed behaviour might seem to make it impossible to describe him as straightforwardly and constantly loving.

10. The Tableau Technique

Semantic tableaux, or *tableaux* as we shall call them for short, are a method for testing the consistency of sets of complex sentences. In outline, the method is as follows.

Suppose X is a set of complex sentences; we wish to test whether X is consistent. If it is consistent, then the sentences of X are true together in some situation. We try to describe such a situation, using sentences which are as short as possible. X itself forms a first attempt at a description, so we start by writing down X . This forms the beginning of the tableau.

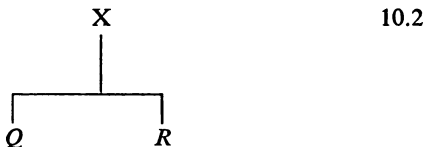
We then take any complex sentence P in X , and we try to describe, using only sentences shorter than P , those situations in which P would be true. If, for example, P is true precisely

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when two shorter sentences Q and R are both true, then we add Q and R to the tableau:

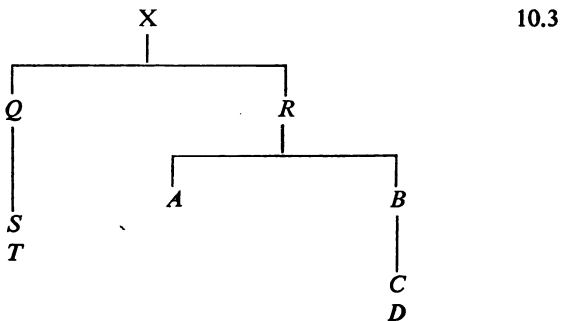


On the other hand we may only be able to find two shorter sentences Q and R , such that P is true in precisely those situations where *at least one* of Q and R is true. There are then two possible ways of finding a situation in which X is all true; so we add the sentences Q and R to the tableau, but we make them branch out in different directions:

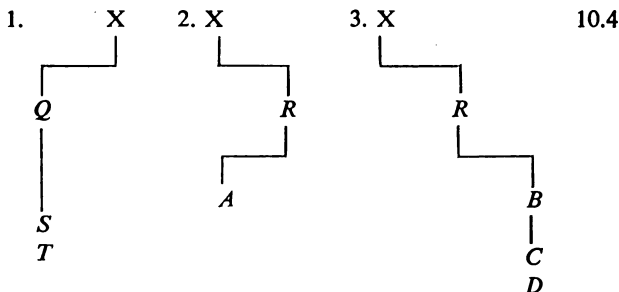


We then repeat the operation with another complex sentence, which may be Q or R or another sentence from X . We continue in this way for as long as we can.

At a stage half-way through the construction of the tableau, it may look something like this:



(Q , R , etc., are sentences.) We think of (10.3) as a tree with branches pointing downwards. In fact there are three branches in (10.3), namely



Each of these branches is an attempt to describe a possible situation. It succeeds in describing one if and only if the sentences in the branch form a consistent set; so we may extend each branch separately in just the same way as we extended the original set X .

When a branch contains an obvious inconsistency, it represents a failed attempt to describe a situation; we therefore *close* it by drawing a line across the bottom, and turn our attention to the other branches instead.

Eventually we shall reach a stage when it is impossible to extend any branch by adding new shorter sentences, and it is impossible to close off any more branches. The tableau is then finished. One of two things may happen.

(i) It may be that every branch of the finished tableau is closed; in this case we say that the tableau itself is *closed*. This means that every attempt to describe a situation in which X is true has led to contradictions. We can deduce that X is *inconsistent*.

(ii) It may be that there are some branches of the finished tableau which are not closed. Take just one of these branches – it doesn't matter which. Since the branch is not closed, the short sentences in it are not obviously inconsistent; in most cases we can see that they are in fact consistent. They then

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describe for us a situation in which X is true. Therefore X is *consistent*.

In case (i), the tableau gives us a proof that X is inconsistent; in case (ii), it usually shows that X is consistent.

We shall illustrate the tableau technique with two examples.

First example: is the following set of sentences (taken from the report of a chemical analysis) consistent?

If cobalt but no nickel is present, a brown colour appears. 10.5

Nickel and manganese are absent.

Cobalt is present but only a green colour appears.

We begin the tableau by writing down (10.5). We choose any sentence of (10.5); say the second. This sentence is true precisely if 'Nickel is absent' and 'Manganese is absent' are both true. We therefore extend the tableau by writing

If cobalt but no nickel is present, a brown colour appears. 10.6

✓ Nickel and manganese are absent.

Cobalt is present but only a green colour appears.

|
Nickel is absent.

Manganese is absent.

(The tick shows that the second sentence has been dealt with.)

Now we do the same, say with the third sentence:

If cobalt but no nickel is present, a brown colour appears. 10.7

✓ Nickel and manganese are absent.

✓ Cobalt is present but only a green colour appears.

|

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Nickel is absent.

Manganese is absent.

Cobalt is present.

Only a green colour appears.

No contradiction has appeared yet among the shorter sentences; so we proceed to break down the first sentence. Now this sentence is true precisely if *either* a brown colour appears, *or* it's not true that 'Cobalt but no nickel is present'.† We therefore split the tableau into two branches:

✓ If cobalt but no nickel is present, 10.8
a brown colour appears.

✓ Nickel and manganese are absent.

✓ Cobalt is present but only a
green colour appears.

Nickel is absent.

Manganese is absent.

Cobalt is present.

Only a green colour appears.

It's not true that cobalt
but no nickel is present.

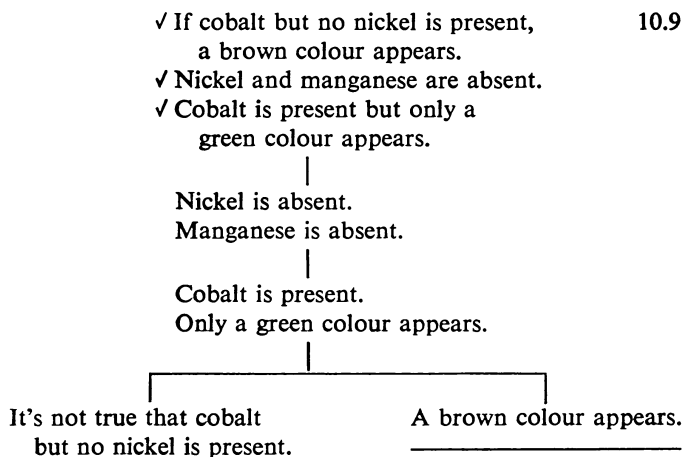
A brown colour appears.

At this stage the right-hand branch contains two sentences, 'Only a green colour appears' and 'A brown colour appears',

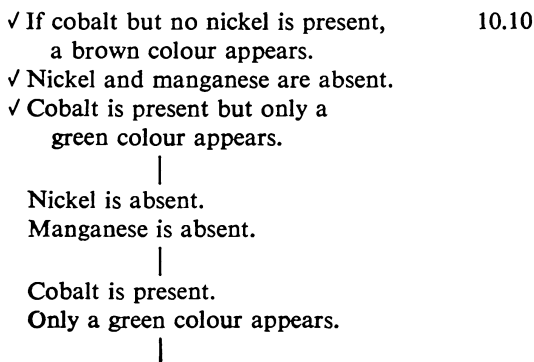
† In a situation where cobalt but no nickel is present, yet no brown colour appears, the sentence is false; also this is the only kind of situation which could make the sentence false. This point needs further discussion, and we shall come back to it in section 17 below.

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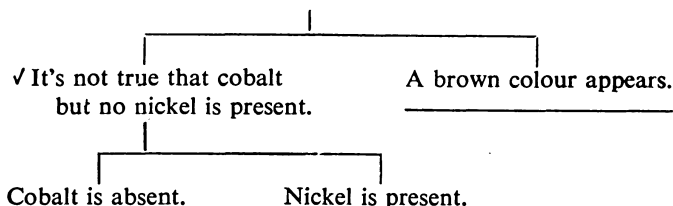
which obviously contradict each other. We may therefore close the right-hand branch:



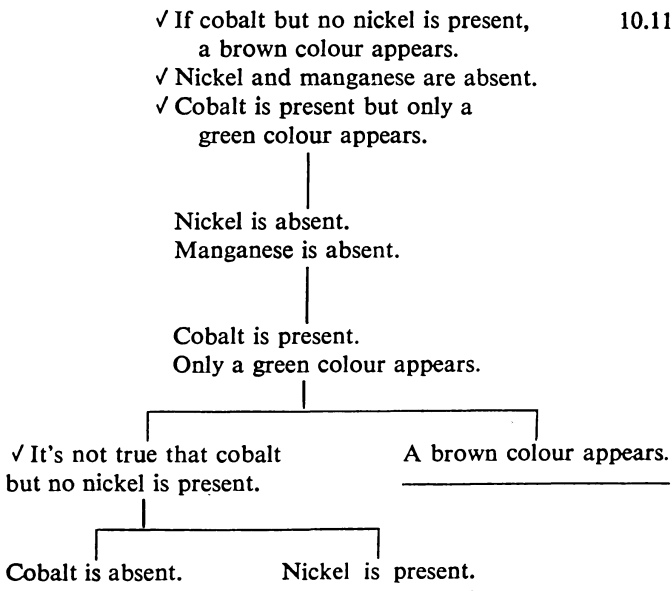
The longest unanalysed sentence in (10.9) is the one at the foot of the left-hand branch. This sentence is true precisely if *either* cobalt is absent *or* nickel is present. We therefore extend the tableau below the left-hand branch:



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The left-hand branch of (10.10) contains the two inconsistent sentences 'Cobalt is present' and 'Cobalt is absent'; we may therefore close it. The middle branch can be closed too, since it contains the inconsistent sentences 'Nickel is absent' and 'Nickel is present'. The resulting tableau is:



(10.11) is a finished tableau. All three branches are closed. Therefore (10.5) is *inconsistent*.

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Second example: is the following set of sentences consistent?

If cobalt but no nickel is present, a brown colour appears. 10.12

Either nickel or manganese is absent.

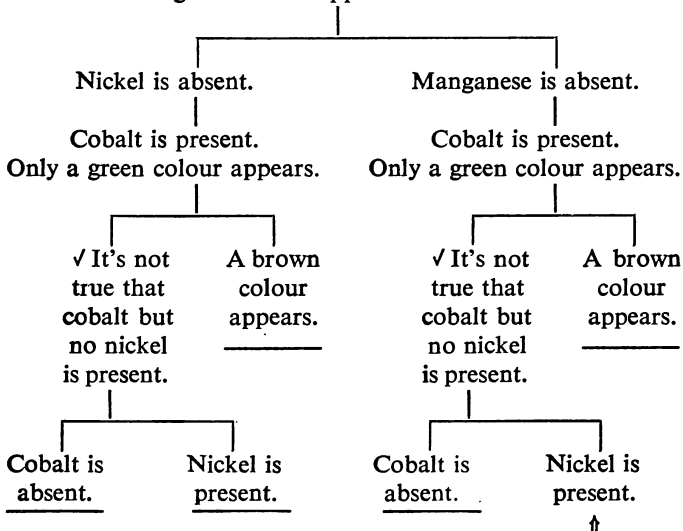
Cobalt is present but only a green colour appears.

The full tableau is as follows. You should work through this tableau step by step, to see why each sentence was added. Note in particular that where a sentence is in two branches which are not closed, then the shorter sentences got by analysing this sentence must be put at the foot of both branches. (Where does this apply in the tableau (10.13)?)

✓ If cobalt but no nickel is present, a brown colour appears. 10.13

✓ Either nickel or manganese is absent.

✓ Cobalt is present but only a green colour appears.



The tableau (10.13) is finished, but not all its branches are closed. We therefore pick one which is not closed; in fact there is just one unclosed branch, which is marked with an arrow. The short sentences in this branch are:

Manganese is absent. Cobalt is present. Only 10.14
a green colour appears. Nickel is present.

The sentences in (10.14) describe a situation, and in this situation the sentences (10.12) are true. Therefore (10.12) is *consistent*.

Exercise 10. Use tableaux to determine which of the following sets of sentences are consistent.

1. Mr Zak is a Russian spy.
Mr Zak is not both a C.I.A. spy and a Russian spy.
Mr Zak is a C.I.A. spy and a cad.
2. At least one of Auguste and Bruno lives in Bootle.
At least one of Bruno and Chaim is an estate agent.
Bruno is not an estate agent, and doesn't live in Bootle.
3. Either Yvonne or Zoe gave me this book last Tuesday.
If Yvonne gave me this book, then I was in Oslo at Tuesday lunchtime.
I was miles away from Oslo all Tuesday, and Zoe has never given me anything.

11. Arguments

Logic is sometimes defined as the study of valid arguments. What is an argument, and when is it valid?

An *argument*, in the sense that concerns us here, is what a person produces when he makes a statement and gives reasons for believing the statement. The statement itself is called the *conclusion* of the argument (though it can perfectly well come at the beginning); the stated reasons for believing the conclusion

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are called the *premises*. A person who presents or accepts an argument is said to *deduce* or *infer* its conclusion from its premises.

In logic books it's usual to write an argument with the premises first, then '*Therefore*', then the conclusion. For example:

The car is making a devil of a noise, and it won't overtake properly. *Therefore* a gasket has blown. 11.1

Outside logic books, arguments occur in all sorts of forms. To pick an example off the shelf at random:

There can, in fact, be no enduring solution to wretched living conditions unless the houses in which they are found are either altered or replaced. Until then, they will simply fill up again each time a family moves to something better. 11.2

In (11.2) the conclusion comprises the first sentence, and the second sentence indicates the premise. In logic book style:

If houses in which wretched living conditions are found are neither altered nor replaced, they will simply fill up again each time a family moves to something better. *Therefore* there can be no enduring solution to wretched living conditions unless the houses in which they are found are either altered or replaced. 11.3

Notice that we had to repeat part of the conclusion in order to state the premise in full. People often present their arguments in an abbreviated form. In the next example, the conclusion is not stated outright at all:

I think you ought to give it a rest. How would you like it if someone kept making jokes about *your* accent? 11.4

In logic book style, this might be:

It's not pleasant to have people making constant jokes about one's accent. *Therefore* you ought to stop making jokes about so-and-so's accent. 11.5

Exercise 11. Rewrite each of the following as an argument in logic book style. (As you do them, notice the words and phrases like *in view of*, which serve to mark out the premise and the conclusion.)

1. Help is needed urgently, in view of the fact that two hundred people are dying every day.
2. When Communists operate as a minority group within unions, settlements by the established officials must be denounced as sellouts. It follows that strikes are unlikely to wither away in any democratic country so long as Communists have strong minority influence.
3. The nests of the verdin are surprisingly conspicuous, for they are usually placed at or near the end of a low branch.
4. The effect of ACTH on gout is not due to the increased renal uric acid clearance alone, since the effect of salicylates on this clearance is greater.
5. Some contribution to the magnetic field comes from electric currents in the upper atmosphere; otherwise we cannot account for the relation between the variations in the magnetic elements and the radiation received from the sun.

An argument is said to be *valid* if there is no possible situation in which its premises are all true and its conclusion is not true. An argument which is not valid is called *invalid*. When an argument is valid, its premises are said to *entail* its conclusion.

For example, this is a valid argument, so that its premises entail its conclusion:

Single persons earning less than £9 a week are eligible for the maximum rate rebate. I earn 11.6

only £8 a week. I am a single person. *Therefore* I am eligible for the maximum rate rebate.

If the premises of (11.6) are true, then the conclusion must be true too; there is no other possibility. On the other hand the argument (11.1) is certainly not valid; a car could make a noise and lose power without having a broken gasket, even if a broken gasket is the most likely cause.

Notice that a valid argument need not have true premises. In fact I earn more than £8 a week; nevertheless (11.6) is still valid. People are notoriously prone to accept an argument as valid if they believe its premises and conclusion, and reject it as invalid if they disbelieve them. Take care.

If an argument consists of declarative sentences, then we can transform it into a certain set of declarative sentences which will be known as the *counterexample set*, as follows. The counterexample set consists of the premises, and the conclusion with the words 'It is not true that' tacked on to the front. For example, the counterexample set of (11.6) is

Single persons earning less than £9 a week are eligible for the maximum rate rebate. I earn only £8 a week. I am a single person. It is not true that I am eligible for the maximum rate rebate. 11.7

From the way in which we defined the validity of an argument, it's clear that *an argument is valid precisely if its counterexample set is inconsistent*.

This method of proving that an argument is valid – by showing that there is no possible situation in which its premises are true and its conclusion is false – is known as the method of *reductio ad absurdum* (the Latin for 'reduction to an absurdity'). For example, to prove (11.6) by *reductio ad absurdum*, we might argue:

Assume that the premises are true and the conclusion is false. Since the first premise is 11.8

true and the conclusion false, I can't be a single person earning less than £9 a week. But by the other two premises, I am a single person and I earn £8 a week, which has to be less than £9. This is absurd. Hence the argument (11.6) is valid.

This style of argument has caused some puzzlement, because it seems ridiculous to start the discussion by *assuming* something which is palpably untrue. The answer, of course, is that when we say 'Assume the premises are true and the conclusion is false,' we are not asking anybody to *believe* any such thing; rather we are setting out to try to describe a possible situation in which the premises are true and the conclusion is false. When we reach an absurdity, this shows that there is no such possible situation.

All this was misunderstood rather badly by a nineteenth-century mathematician called James Smith, who thought he had proved that a certain number π is precisely $25/8$. (It isn't.) He maintained that by an adapted version of *reductio ad absurdum*, it would be enough if he first assumed that π is $25/8$, and then failed to deduce a contradiction from this assumption. Other mathematicians quickly pointed out that they could quite easily deduce contradictions from this assumption, but he simply took this as proof that they were incompetent.

If an argument expressed in declarative sentences is not valid, then we can show this by describing a possible situation in which its counterexample set is all true; such a situation is called a *counterexample* to the argument. For example, I can show that (11.1) is invalid by describing a car which makes a noise because its exhaust-pipe has a hole, and which won't overtake properly because a big end has gone, though the gaskets are intact. There could be such a car; it provides a counterexample to (11.1).

We may test the validity of an argument by using a tableau to check the consistency of its counterexample set. If the tableau closes, then the counterexample set is inconsistent and

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so the argument was valid. If the tableau refuses to close, then some unclosed branch should describe for us a situation which is a counterexample to the argument.

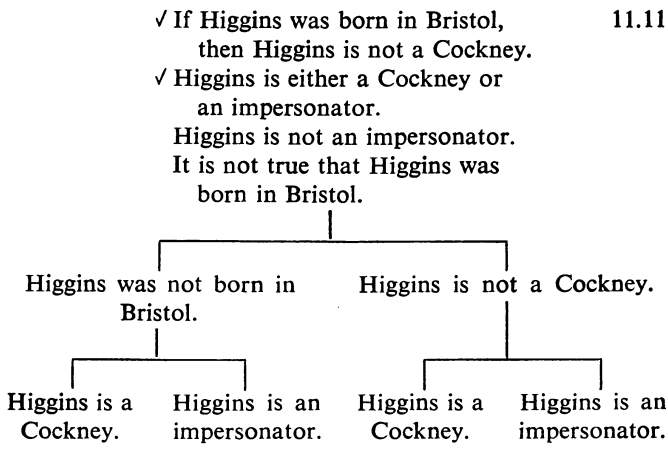
For example, we test the argument:

If Higgins was born in Bristol, then Higgins is not a Cockney. Higgins is either a Cockney or an impersonator. Higgins is not an impersonator. *Therefore* Higgins was born in Bristol. 11.9

The counterexample set of (11.9) is:

If Higgins was born in Bristol, then Higgins is not a Cockney. Higgins is either a Cockney or an impersonator. Higgins is not an impersonator. It is not true that Higgins was born in Bristol. 11.10

We test the consistency of (11.10) by a tableau:



The left-most branch of (11.11) is not closed; it describes the situation

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Higgins is not an impersonator; he was not born in Bristol and he is a Cockney. 11.12

In the situation described by (11.12), the premises of the argument (11.9) are true and its conclusion is false. Therefore (11.12) forms a counterexample to the argument, and so the argument is *invalid*.

Closely related to valid arguments are *necessary truths*; these are declarative sentences which are true in every possible situation. Poem 449 in the *Oxford Book of Twentieth-Century English Verse* begins with a fine specimen:

As we get older we do not get any younger. . 11.13

Later we shall meet some less poetic examples.

Arguments can be good without being valid. We may call an argument *rational* if its premises provide good reason for believing the conclusion, even if the reason is not absolutely decisive. For example, on the face of it (11.1) is quite rational – a blown gasket may be the most likely cause of noise and a loss of power, even if it is not the only possible cause.

If we can see that an argument is valid and has true premises, then we can see that its conclusion must be true too. Thus a *demonstrably valid argument whose premises are known to be true is rational*.

It would be pleasant to have some tests for the rationality of arguments, but this seems a vain hope. One difficulty is that the rationality of an argument depends on more than the stated premises; the evidence offered for the conclusion may become less convincing when further facts are pointed out. (With valid arguments this is not so: a valid argument remains valid even when new facts come to light.)

For example, a Government report on corporal punishment attacked those who believe that judicial corporal punishment would act as a deterrent:

[These people] attribute the great increase in offences of violence against the person to the abolition of this penalty, but this argument 11.14

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overlooks the fact that there were similar increases before and after 1948, and that judicial corporal punishment was not available before 1948 for crimes of violence generally.†

The argument under attack here could be stated as follows:

When judicial corporal punishment was abolished, offences of violence against the person increased. *Therefore* judicial corporal punishment acts as a deterrent against offences of violence against the person. 11.15

The argument (11.15) is attacked, not because it is unconvincing as it stands, but because it ceases to be convincing once we point out the known fact that offences of violence have increased in certain other circumstances too.

It seems, then, that in order to assess the rationality of an argument, we need to take into account all the known facts, and not just the stated premises. An argument is normally deployed against a background of known facts and agreed beliefs, and the rationality of the argument depends on what these facts and beliefs are. All this makes it hard to see how one could devise a simple and practical test of rationality.

Perhaps also we are to some extent free to choose for ourselves what we count as an adequate reason for believing a thing. By nature some people are more sceptical than others. In 1670 Chief Justice Vaughan put it rather well:

A man cannot see by anothers eye, nor hear by anothers ear, no more can a man conclude or infer the thing to be resolved by anothers understanding or reasoning.

†Para. 46 of *Corporal Punishment*, H.M.S.O., 1960.

How are Complex Sentences Built Up ?

Since our approach is to analyse sentences into their component parts, we ought to be sure we understand what the parts of a sentence really are. This demands a few sections on grammar.

There are two other reasons for having some grammar in a book about logic. First, the notions of structural ambiguity and scope, which a logician ought to know about, can only be understood with the help of a little grammar. Second, one of the techniques of modern logic is translation into certain formal languages; these languages are easier to set in motion if we gather up suitable grammatical ideas first.

12. Phrase-classes

Anybody who speaks a language can distinguish fairly accurately between the grammatical and the ungrammatical sentences of the language. In fact he can do something more: he can say which words within a grammatical sentence go together to form *natural groups*.

For example, any English speaker can tell you that in the sentence

You can do anything but don't step on my 12.1
blue suede shoes.

the following are all natural groups:

You can do anything 12.2
blue suede shoes
don't step on my blue suede shoes

whereas the following are not:

do anything but 12.3
 You can do anything but don't step on my
 my blue

This feeling for natural groupings is part of the raw data which a grammarian can use in constructing a grammar for a language. It's even less tangible than our feelings of what is and what is not grammatical. But it's a real feeling nevertheless, and people do agree to a remarkable extent about what the natural groups are, even if they can't say why.

The natural groups of words that occur in a language are called the *grammatical phrases* of the language. This is not a term with a precise meaning, because people do disagree about what groups are natural. But probably everybody will agree that the strings in (12.2) are grammatical phrases, while those in (12.3) are not. It will be convenient to count single words as grammatical phrases.

When a grammatical phrase occurs as a natural group in a sentence, it is called a *constituent* of the sentence. This is an important notion, and we shall use it frequently.

If a grammatical phrase occurs twice in a sentence, then the two occurrences are counted as *different constituents* of the sentence. For example, the phrase *blue suede shoes* forms two constituents of the following sentence:

My brother wanted blue suede shoes for 12.4
 Christmas, but I can't find a shop that sells
 blue suede shoes.

It sometimes happens that a grammatical phrase occurs within a sentence, but doesn't form a natural group within the sentence. In this case the phrase does not count as a constituent of the sentence. For example, in

He wants a pair of dark blue suede shoes. 12.5

'blue suede shoes' is not a natural group, since **dark blue** hangs closely together. The phrase *blue suede shoes* does *not* occur as a constituent of (12.5).

HOW ARE COMPLEX SENTENCES BUILT UP?

By convention, a whole sentence is counted as one of its own constituents.

Traditional English grammars group the grammatical phrases of English into *phrase-classes* (also known as *parts of speech*). Although the definitions given for these classes have often been absurd, these grammars agree very closely about what the main phrase-classes are. Four of the classes are called *noun*, *adjective*, *adverb* and *verb* respectively, containing words such as the following:

<i>noun</i> – John, room, answer, play	12.6
<i>adjective</i> – happy, steady, new, large, round	
<i>adverb</i> – steadily, completely, really, very, then	
<i>verb</i> – search, grow, play, be, have, do†	

Many modern linguists believe that phrase-classes are an outdated concept that can never give us more than a rough approximation to the truth about English grammar. Nevertheless they do give us a rough approximation at least. It is therefore worth while to have a test which determines when two phrases belong to the same class. The test we shall use is called the *frame test*. Don't regard it as any more than a rule of thumb.

By a *frame* we mean a string of English words, among which the symbol 'x' occurs once. 'x' must occur only once, either between words or at the beginning or the end of the string. For example, here are three frames:

You really x Smith, don't you?	12.7
x people have heard of Xerxes.	12.8
Is it true that x?	12.9

The 'x' serves to mark a hole in the frame, where other phrases can be put in.

If we take a frame and a grammatical phrase, and put the phrase into the frame in place of the 'x', the result is a string of words. We say the frame *accepts* the phrase if this string is a grammatical sentence which has the introduced phrase as a

†Randolph Quirk and Sidney Greenbaum, *A University Grammar of English*, Longman, 1973, p. 18.

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constituent; if the frame doesn't accept the phrase, we say it *rejects* the phrase.

For example, the frame (12.7) accepts **hate**, but it rejects **hate Jack** and it rejects **my friend**. It accepts **hate**, because the sentence

You really hate Smith, don't you? 12.10

is grammatical and has **hate** as a constituent. (A single word occurring in a sentence always counts as a constituent of the sentence.) It rejects **hate Jack**, because although

You really hate Jack Smith, don't you? 12.11

is grammatical, **hate Jack** doesn't form a constituent of it. Finally it rejects **my friend** because

*You really my friend Smith, don't you? 12.12

is not grammatical.

Exercise 12. Write ticks and crosses in the following chart, to show which of the frames on the left accept which of the phrases at the top:

	health	a wife	wife	girls	two girls
He wanted to have <i>x</i> .					
He wanted to have the <i>x</i> .					
He wanted to have more <i>x</i> .					

Each frame picks out for us a class of grammatical phrases, namely those which it accepts. For example, the class of grammatical phrases accepted by the frame

Is it true that *x*? 12.13

consists of the declarative sentences (see p. 19).

HOW ARE COMPLEX SENTENCES BUILT UP?

It often happens that two frames accept very nearly the same grammatical phrases, but not quite. For example, take the two frames

x will be mentioned afterwards.	12.14
There will be a chance to mention x .	12.15

A large number of grammatical phrases are accepted by both. These include the following:

Mary Morgan	12.16
Mary Morgan and her sister	
some ways of cooking omelettes	
the fire hazard	
the threat which he made last Saturday	
a few rats	

But some grammatical phrases are accepted by (12.14) and rejected by (12.15):

I	12.17
we	
?nothing	

(The question-mark means that it is debatable whether (12.15) accepts nothing; this may be a matter of dialect.) Some other phrases are accepted by (12.15) but rejected by (12.14):

me	12.18
us	
?how to improve one's tomato crop	

There are also a number of frames which accept roughly but not exactly the same phrases as (12.14) or (12.15). For example

He mentioned x .	12.19
--------------------	-------

accepts very nearly the same phrases as (12.15); but

himself	12.20
---------	-------

is accepted by (12.19) and rejected by (12.15).

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In the cases where a grammatical phrase is accepted by one of the frames above but rejected by another, it usually turns out that the rejection is a fairly gentle one: the ungrammatical sentence is a perturbation of a grammatical one. For example,

*Me will be mentioned afterwards. 12.21

is a perturbation of

I will be mentioned afterwards. 12.22

Likewise

?There will be a chance to mention nothing. 12.23

is (if not grammatical already) a perturbation of

There won't be a chance to mention anything. 12.24

Thus we have a cluster of frames which accept or nearly accept the same phrases.

In such cases, we can say that the grammatical phrases which are accepted or nearly accepted by every frame in the cluster form a *phrase-class*.

For example, the frames (12.14), (12.15), (12.19) and similar frames form a cluster which defines the phrase-class of *noun phrases*.

Similarly we can define *nouns* to be those words which are accepted or nearly accepted by frames in the cluster

I saw the *x*. 12.25
A *x* will be needed.
(etc.)

Likewise we can define *adjectives* to be those words which are accepted or nearly accepted by frames in the cluster

He produced a *x* book 12.26
The *x* things were another matter.
(etc.)

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Thus we can use clusters of frames to define classes of words or phrases. Some of the traditional parts of speech can be seen as two or more such classes put together. For instance the class of verbs is got by combining the class of transitive verbs with the class of intransitive verbs, and each of these smaller classes can be defined by a cluster of frames. We need not pursue this here.

Instead of combining classes, we can cut down to smaller classes by eliminating all the phrases which are rejected by some particular frame. For example the phrase-class of *singular noun phrases* consists of the noun phrases which are accepted by

x was mentioned afterwards. 12.27

in standard English. Looking through (12.16), we see that the following are singular noun phrases:

Mary Morgan 12.28
the fire hazard
the threat which he made last Saturday

while the following are not:

Mary Morgan and her sister 12.29
some ways of cooking omelettes
a few rats

Actually this is not quite right: on the usual definition, *me*, *him* and *her* are singular noun phrases too, though they are rejected by (12.27). This illustrates the frailty of frame tests.

Why is it that frames give only a rough-and-ready classification of grammatical phrases? Could something better be found?

This is a question of linguistics rather than logic, so only a brief answer is called for here. According to the *transformational syntax* of Noam Chomsky, every sentence has an underlying structure which is different from the written or spoken 'surface structure' of the sentence; in fact there may be several different levels of underlying structure for one sentence. The

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frame test may give perfect classifications at one of the underlying levels, but these classifications are tangled up as we pass to the surface structure. For example, the words **I** and **me** are not distinguished at the deeper levels; neither are the words **we** and **us**. In this way the main discrepancy between (12.14) and (12.15) disappears at the lower levels. Some features of Chomsky's theory are highly technical – such as his notion that there is a deepest underlying structure, the so-called *deep structure*. But most English speakers would agree that it's perfectly natural to think of **I** and **me** as being fundamentally the same word.

We shall see some more examples of underlying structures in later sections.

One last point. English words have a disarming way of hopping about from one phrase-class to another.

Please clear the room.	12.30
The bell rang the hour loud and clear.	12.31
I didn't get a clear view of him as he ran past.	12.32

Clear is verb in (12.30), adverb in (12.31) and adjective in (12.32). It will be convenient and natural to adopt the same convention as in our treatment of lexical ambiguity in section 4, and regard **clear** as being three different words which happen to be written alike. **Clear** in (12.30) is a verb, a different word from the adjective **clear** in (12.32).

13. Phrase-markers

Phrase-markers are a handy way of labelling the constituents of a sentence.

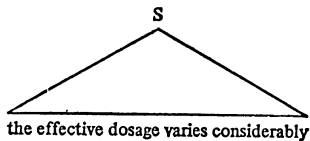
We begin with an example. Take the string

The effective dosage varies considerably. 13.1

(13.1) forms a grammatical sentence. To express this, we first write

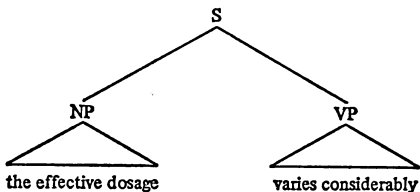
HOW ARE COMPLEX SENTENCES BUILT UP?

13.2



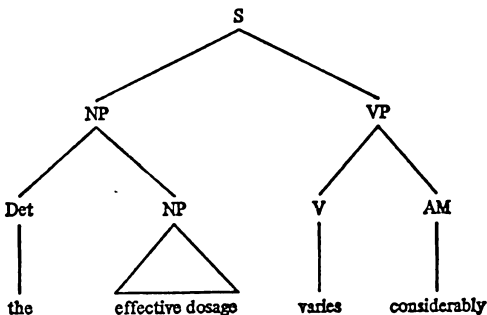
(S means Sentence.) Is there a main break in the sentence? Most people would say that 'the effective dosage' forms a unit, and 'varies considerably' forms another. The first of these units is a noun phrase (=NP); the second is of the type usually called a *verb phrase* (=VP). So we now write

13.3



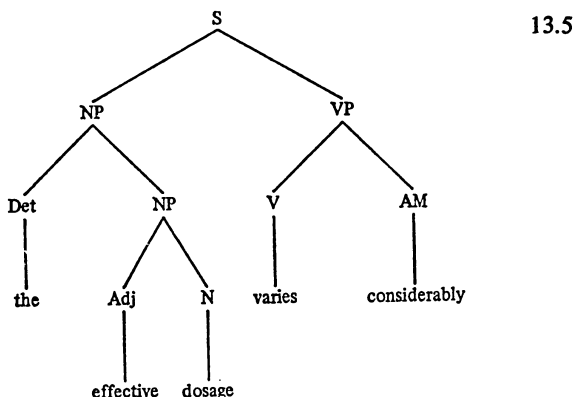
On the right, 'varies' is a verb (=V), and 'considerably' belongs to a phrase-class sometimes called adverbs of manner (=AM). On the left, most people would feel that 'effective dosage' forms a unit; it's another noun phrase. 'the' is a determiner (=Det). Thus:

13.4



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Finally, 'effective' is an adjective (=Adj), and 'dosage' is a noun (=N):



The analysis is complete. Apart from the names of the phrase-classes, which are quite unimportant, probably most speakers of English would make this analysis of (13.1).

(13.5) is an upside-down tree, with the words of the sentence (13.1) strung along the tips of the branches. We call a diagram like (13.5) a *phrase-marker*. The words at the tips of the branches of a phrase-marker are called the *terminal symbols* of the phrase-marker; read from left to right, they form the *terminal string* of the phrase-marker. Thus (13.1) is the terminal string of (13.5). Higher up in a phrase-marker are symbols representing phrase-classes; these symbols are called the *non-terminal symbols* of the phrase-marker. Thus the non-terminal symbols of (13.5) are

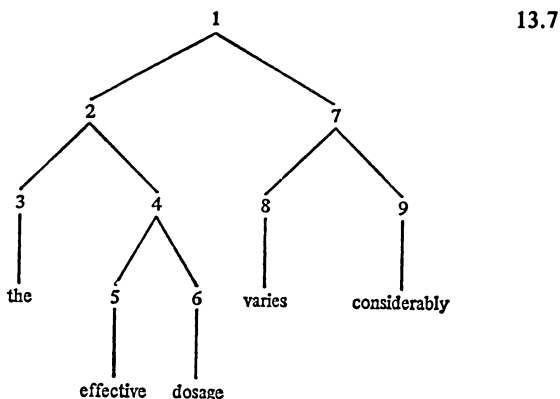
S, NP, VP, Det, V, AM, Adj, N. 13.6

Wherever a non-terminal symbol appears in a phrase-marker, it tells us that a certain part of the terminal string belongs in the

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stated phrase-class. Any part of the terminal string which is assigned to a phrase-class in this way is a *constituent* of the terminal string. (This is a refinement of the definition of constituents on page 62.)

For example, we can number the appearances of non-terminal symbols in (13.5) as follows:

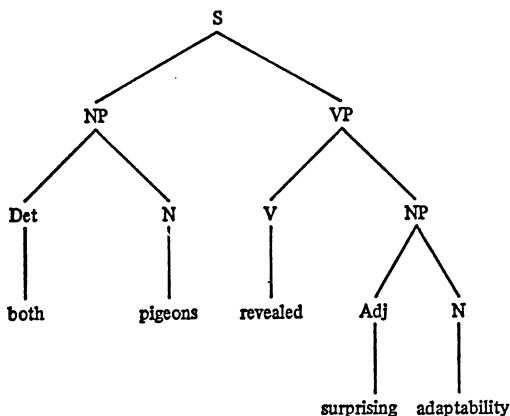


There are nine occurrences of non-terminal symbols, and nine constituents corresponding to these, namely:

- | | |
|--|------|
| 1 [the effective dosage varies considerably] | 13.8 |
| 2 [the effective dosage] varies considerably | |
| 3 [the] effective dosage varies considerably | |
| 4 the [effective dosage] varies considerably | |
| 5 the [effective] dosage varies considerably | |
| 6 the effective [dosage] varies considerably | |
| 7 the effective dosage [varies considerably] | |
| 8 the effective dosage [varies] considerably | |
| 9 the effective dosage varies [considerably] | |

Exercise 13A. Consider the following phrase-marker:

13.8



1. What is the terminal string of this phrase-marker?
2. What are the non-terminal symbols of this phrase-marker?
3. List all the constituents of the terminal string.

Exercise 13B. Construct your own phrase-marker for the sentence

The flame melted the wire.

Then list the constituents of this sentence.

So far so good. Probably no speaker of English would object to either of the analyses (13.5) and Exercise 13A. Does this mean that every grammatical English sentence has one correct phrase-marker, which we can find by following our intuitions?

Alas, no. It is very easy to find grammatical English sentences which different people would be inclined to analyse in quite different ways. Often people find they have no feelings at all

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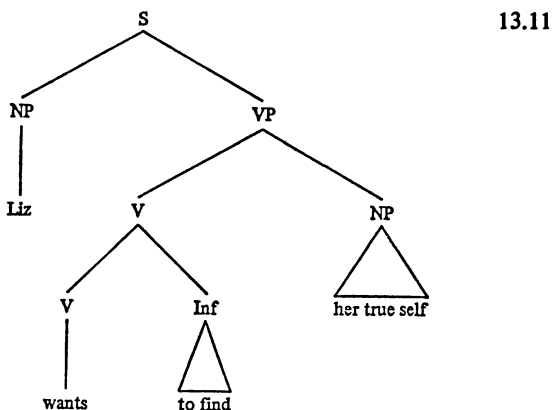
about how a particular sentence should be divided up. Here is an example of a difficult case:

Liz wants to find her true self. 13.9

How should the verb phrase 'wants to find her true self' be split up? One argument is that 'wants to find' is a unit, because it could be rephrased as 'seeks':

Liz seeks her true self. 13.10

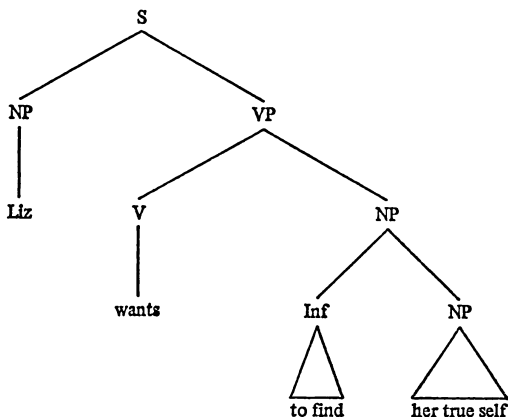
This would lead to the phrase-marker



(Inf = Infinitive; the name is not important.) But another argument says that 'to find her true self' is a unit, because it means 'self-knowledge':

Liz wants self-knowledge. 13.12

This argument supports the quite different phrase-marker



It is self-deception to think that in cases like (13.9) we can find the right phrase-marker simply by sharpening and training our intuitions. The only way to make progress is to formulate some theory about language, some theory which explains the reasons for the intuitions we do have, and which relates them to the rules of grammar. At this point we touch on current research in linguistics, which is a dangerous field to touch on – the unwary lose their foothold easily.

Nevertheless, one further step into linguistics is worth making. Most modern grammarians agree with Noam Chomsky that we ought to think of certain grammatical sentences as having been derived from other underlying strings by *transformations*. These transformations may distort the ‘true’ constituents, by altering the order of the items in the string, or leaving out or repeating or altering parts of the string.

For example, take this sentence:

Bob seized it and broke it up. 13.14

We do surely feel that ‘broke up’ is a unit here, although the word ‘it’ has squatted in the middle. Compare

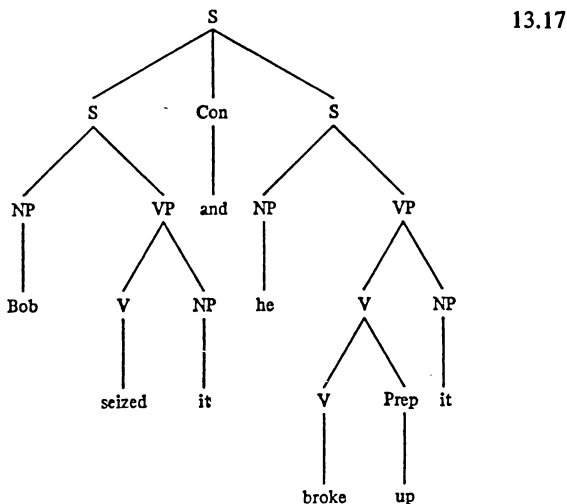
Bob broke up the lump. 13.15

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It's natural to suggest that (13.14) is derived from an underlying string such as

Bob seized it and he broke up it. 13.16

This underlying string, which is not a grammatical sentence in English, would presumably have some phrase-marker like



(Con = Conjunction.) (13.17) would then be an *underlying phrase-marker* of the sentence (3.14). 'broke up' is a constituent of (13.16), according to (13.17); we can express this by saying that 'broke up' is an *underlying constituent* of (13.14).

There are plenty of examples of sentences where we would all feel that some underlying phrase-marker would naturally express the 'real' units.

Exercise 13C. Construct underlying phrase-markers to bring out the underlying constituents of the following sentences:

1. I can only see Leila. (i.e., I can see Leila and nobody else.)
2. Did you hear the thunder?

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3. I came because I was told to.
4. She will certainly faint.

In some modern theories, the terminal symbols of the underlying phrase-markers need not even be words. For example I and me might be represented by the same symbol in the underlying phrase-marker (see p. 68). Indeed many contemporary linguists are happy to postulate underlying phrase-markers which seem to have virtually nothing in common with the sentences they are thought to underlie. This will warn us off any deeper probing in this area.

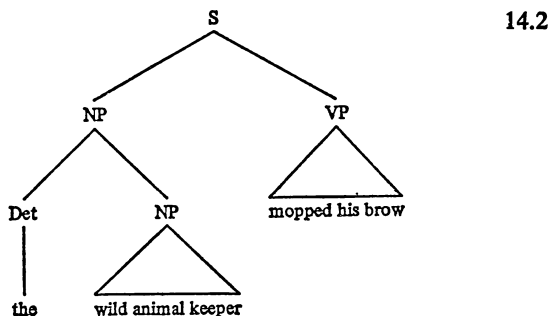
14. Scope

In this section we shall see how phrase-markers can be used for uncovering and curing structural ambiguity (see section 4).

Consider the sentence

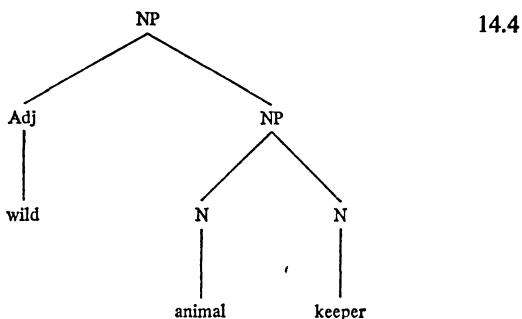
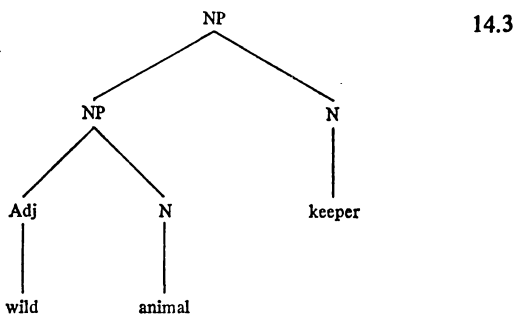
The wild animal keeper mopped his brow. 14.1

Part of the phrase-marker for (14.1) is quite straightforward:



HOW ARE COMPLEX SENTENCES BUILT UP?

The noun phrase 'wild animal keeper' can be split apart in two ways:



This sentence is totally unlike the problem example (13.9) which we considered earlier. In the present case, the phrases analysed in (14.3) and (14.4) obviously mean different things. In (14.3), 'wild animal' hangs together, and the whole phrase means

keeper of the wild animals. 14.5

But in (14.4), 'animal keeper' is a unit, and the whole phrase must mean

animal keeper who is wild. 14.6

The two different phrase-markers reveal a structural ambiguity.

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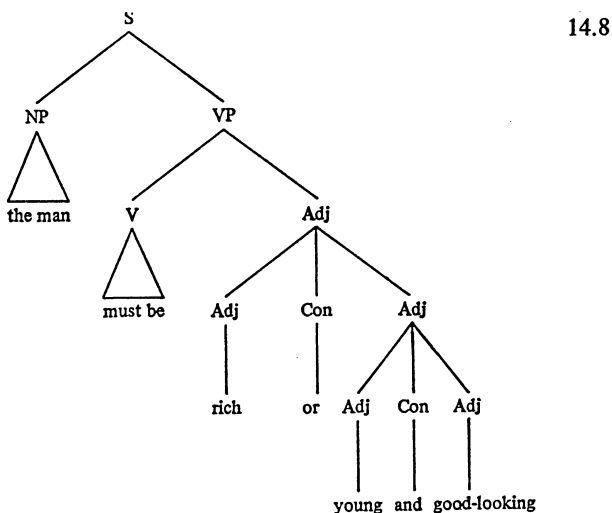
The notion of *scope* is helpful here. If S is some sentence with a phrase-marker supplied, and P is a part of S consisting of one or more words, but not the whole of S, then we define the *scope* of P to be the smallest constituent of S which contains both P and something else besides.

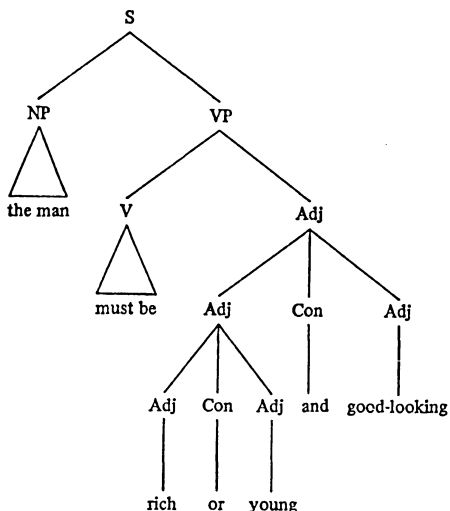
For example, if (14.3) is fitted into a phrase-marker for (14.1), then the scope of 'wild' is 'wild animal'; this is a precise way of expressing that 'wild' goes with 'animal'. But if (14.4) is put in place of (14.3), then the scope of 'wild' becomes 'wild animal keeper'; which is a precise way of expressing that 'wild' goes with the whole of 'animal keeper'.

Here is another example, which is important in logic.

The man must be rich or young and good-looking. 14.7

There are two possible phrase-markers (we abbreviate the irrelevant bits):





Clearly the terminal strings of (14.8) and (14.9) mean different things. The man in (14.9) has got to be good-looking, but in (14.8) he can be ugly so long as he is rich. This can be expressed by saying that in (14.9) the scope of 'and' is the whole of

rich or young and good-looking 14.10

whereas in (14.8) the scope of 'and' is only

young and good-looking. 14.11

In many cases, as we saw in the last section, the 'true' constituents of a sentence can only be found in some underlying phrase-marker. In these cases we must distinguish between the *surface scope* determined by the phrase-marker of the given sentence, and an *underlying scope* determined by an underlying phrase-marker. For example, the surface scope of 'broke' in (13.14) is presumably 'broke it up', whereas according to the underlying phrase-marker (13.17), 'broke' has underlying scope 'broke up'.

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When we write phrase-markers in order to clear up structural ambiguities, it is usually unnecessary to present a whole phrase-marker; an abbreviated one using the triangle notation (as (14.8) and (14.9)) is usually quite adequate.

Exercise 14. Write abbreviated phrase-markers (underlying if necessary) to explain the structural ambiguities in the following sentences:

1. The ambassador was ordered to leave in the morning.
 2. Amos shaved and played with the cat.
 3. I won't vote as a protest.
 4. He only relaxes on Sundays.
-

15. Context-free Grammars⁺

A grammar for a language is basically a set of rules which tell us how to construct the grammatical sentences of the language. The rules should produce all the grammatical sentences of the language, and they should not produce anything else. It is reasonable to expect a grammar to tell us what the constituents of a sentence are as well.

The simplest grammars which are any use at all are the so-called *context-free grammars*, or *CF grammars* for short. We shall define them in a moment. They are hopelessly inadequate to deal with a full-grown language like English. But they are competent to handle the basic formal languages of logic, as we shall see later. Also there is some reason to think that the language spoken by a child under the age of about two and a half can be best described by a CF grammar; see Exercise 15C.

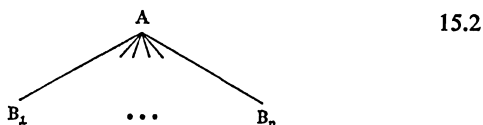
The main notion in CF grammars is that of a *CF rewriting rule*. By a *CF rewriting rule* we mean an expression of the form

$$A \Rightarrow B_1 \dots B_n \qquad 15.1$$

⁺This section is not used later, except in mathematical sections.

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where A, B_1, \dots, B_n are written expressions, called the *symbols* of this rule. Exactly one symbol must appear on the left of ' \Rightarrow ', while one or more symbols may appear on the right of ' \Rightarrow '. (15.1) should be thought of as standing for an upside-down tree



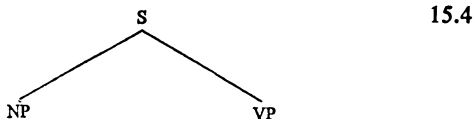
A CF grammar is defined to be a list of CF rewriting rules.

For example, here is a CF grammar:

- | | | | | |
|----|-----|---------------|----------|------|
| 1 | S | \Rightarrow | NP VP | 15.3 |
| 2 | NP | \Rightarrow | Det N | |
| 3 | N | \Rightarrow | Adj N | |
| 4 | VP | \Rightarrow | V NP | |
| 5 | Det | \Rightarrow | the | |
| 6 | Det | \Rightarrow | this | |
| 7 | Adj | \Rightarrow | old | |
| 8 | N | \Rightarrow | fool | |
| 9 | N | \Rightarrow | sentence | |
| 10 | V | \Rightarrow | ignored | |

The numbers on the left in (15.3) are not part of the rules; they simply mark the place of each rule in the list.

Rule number 1 of (15.3) could also be written in the form



which should look familiar.

If C is a CF grammar, then we refer to the left-hand symbol of the first rule in C as the *initial symbol* of C . If a symbol occurs on the left-hand side of some rule in C , then we say the symbol is a *non-terminal symbol* of C ; symbols which occur only

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on the right-hand side of rules in C are called *terminal symbols* of C .

For example, the initial symbol of the CF grammar (15.3) is 'S', and its non-terminal symbols are

S, NP, N, VP, Det, Adj, V. 15.5

The terminal symbols of this CF grammar are

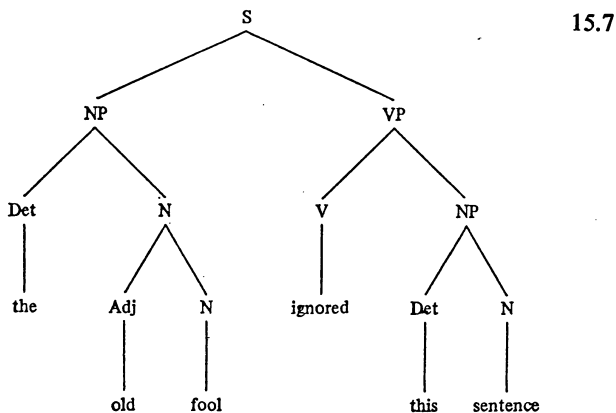
the, this, old, fool, sentence, ignored. 15.6

If C is a CF grammar and P is a phrase-marker, then we say that C *generates* P if the following three things are true:

- a. the symbol at the top of P is the initial symbol of C ;
- b. every terminal symbol of P is a terminal symbol of C ;
- c. every step between the top and the bottom of P is one of the CF rewriting rules of C .

We say that C *generates* a string if it generates a phrase-marker with the string in question as its terminal string.

For example, the CF grammar (15.3) generates the following phrase-marker:

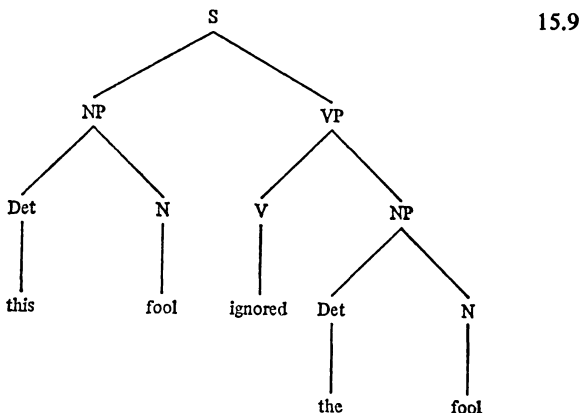


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It follows that (15.3) generates the terminal string of (15.7), which is

The old fool ignored this sentence. 15.8

The CF grammar (15.3) also generates the phrase-marker



and so this grammar also generates the sentence

This fool ignored the fool. 15.10

The same CF grammar also generates

This fool ignored this fool. 15.11
 The old old sentence ignored the fool.
 The sentence ignored the sentence.
 (etc.)

Exercise 15A. Here is a CF grammar:

- 1 S ⇒ NP VP
- 2 VP ⇒ V NP
- 3 NP ⇒ Det N
- 4 N ⇒ N that VP
- 5 Det ⇒ the
- 6 N ⇒ Adj N

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- 7 Adj \Rightarrow white
- 8 N \Rightarrow man
- 9 N \Rightarrow cat
- 10 V \Rightarrow resembles

What are the initial symbol, the non-terminal symbols, and the terminal symbols of this CF grammar?

Exercise 15B. Give phrase-markers to show that the CF grammar of Exercise 15A generates each of the following strings:

1. the man resembles the man
 2. the white man resembles the cat
 3. the cat resembles the man that resembles the cat
 4. the man resembles the white white cat
 5. the man that resembles the cat resembles the white cat that resembles the man
-

The CF grammar (15.3) and the CF grammar of Exercise 15A each generate certain grammatical English sentences, and they provide these sentences with phrase-markers that seem to fit our natural intuitions. These two CF grammars are therefore reasonable grammars for two fragments of English. (They might well seem less reasonable if we tried to extend them to cover larger classes of grammatical sentences.)

It's interesting to note that both these two CF grammars generate infinitely many different strings. In (15.3), rule 3 can be applied over and over again, as many times as we like; each time we apply it, the effect is to add another 'old'. Thus the grammar generates sentences such as

The old old old old old fool ignored this 15.12
sentence.

In the CF grammar of Exercise 15A, rule 4 has a similar effect.

Exercise 15C. A one-year-old child is overheard making the following remarks:

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boy. sock. mommy. allgone boy. allgone
mommy. byebye sock. byebye mommy. boy
off. sock off. sock on. mommy on. mommy
fall.

Construct an appropriate CF grammar for the language of this child. (Remember that these are only a sample of what the child can produce; you have to work out what the implicit rules are.)

Logical Analysis

In the next four sections we encounter one of the chief skills of logic: that of logical analysis. We first select a small number of ways of combining short sentences into longer sentences. Then we show that very many sentences, if they are not already built up in these ways, mean the same as certain other sentences which are so built up. *Logical analysis* consists in finding these other sentences; it stands somewhere between translating and paraphrasing.

Though logicians agree about how to analyse, they disagree about the purpose. Some regard themselves as uncovering the 'real form' of the sentence they analyse, while others see logical analysis as part of an enterprise to replace English by a new and more rational language. We need not enter the controversy; for us the justification is that a tableau for logically analysed sentences can be constructed quite mechanically, according to strict mathematical rules.

16. Sentence-functors and Truth-functors

Some of the sentences in the examples of section 10 were particularly easy to handle, because they had shorter constituents which were sentences. For example,

[Cobalt is present] but [only a green colour appears]. 16.1

(The constituent sentences are marked with square brackets.)
We can analyse such a sentence into its constituent sentences

Cobalt is present. 16.2
 Only a green colour appears.

together with the matrix which contained them:

ϕ but ψ . 16.3

The Greek letters ' ϕ ' and ' ψ ' are called *sentence variables*, which means that they are symbols standing for sentences. Here they mark the holes where the constituent sentences should go. We shall also use the Greek letter ' χ ' as a sentence variable; we may add subscripts too, as in ' ϕ_1 ', ' ϕ_2 ', etc.†

There are several different senses in which a symbol can be used to 'stand for' expressions. Rather than catalogue these senses, we shall introduce them as and when they are needed; the context should always make clear what is meant. Where a variable serves to mark a hole that can be filled with an expression of a certain sort, the variable is said to have a *free* occurrence; in (16.3) both the occurrences of variables are free.

The matrix (16.3) is an example of a *sentence-functor*. More precisely, a *sentence-functor* is defined to be a string of English words and sentence variables, such that if the sentence variables are replaced by declarative sentences, then the whole becomes a declarative sentence with the inserted sentences as constituents. Here is a selection of sentence-functors:

It's a lie that ϕ .	16.4
Many authorities have noted that ψ .	16.5
She went and bought some fish, then ϕ .	16.6
If ϕ , then ψ .	16.7
ϕ because ψ , unless χ .	16.8
Since he swears that ψ , we can take it that ψ .	16.9

Every occurrence of a sentence variable in a sentence-functor is free.

Sentence-functors are classified by the number of different sentence variables they contain. The three examples (16.4)–

† ϕ pronounced *fie*; χ pronounced *khi*; ψ pronounced *psi*; all rhyming with *sky*.

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(16.6) each contain just one sentence variable, so that they are described as *1-place* sentence-functors. (16.3) and (16.7) are *2-place*, while (16.8) is *3-place*.

Note that (16.9) is *1-place*, since only one sentence variable occurs in it, even though it occurs twice. When a sentence variable is repeated in a sentence-functor, this is understood to mean that the sentence variable must be replaced by *the same sentence at each occurrence*. For example the holes in (16.9) may be filled to form the sentence

Since he swears that he was at home, we can 16.10
take it that he was at home.

They cannot be filled to produce

Since he swears that he was at home, we can 16.11
take it that he is not guilty.

Exercise 16A. Analyse the following sentence into one 4-place sentence-functor (with sentence variables ' ϕ_1 ', ' ϕ_2 ', ' ϕ_3 ' and ' ϕ_4 ') and four constituent sentences:

I scattered the strong warriors of Hadadezer,
and then at once I pushed the remnants of
his troops into the Orontes, so that they
dispersed to save their lives; Hadadezer
himself perished.

Returning to example (16.1), we see that this sentence is true precisely when both the constituent sentences (16.2) are true. In fact if we replace ' ϕ ' and ' ψ ' in (16.3) by declarative sentences, then the whole resulting sentence will be true precisely if both the added sentences are true. We may express this in a chart, as follows:

ϕ	ψ	ϕ but ψ .	16.12
T	T	T	
T	F	F	
F	T	F	
F	F	F	

Here T = True and F = False; thus the third row of the table (16.12) indicates that if in ' ϕ but ψ ' we replace ' ϕ ' by a false sentence and ' ψ ' by a true one, then the whole resulting sentence is false. This chart (16.12) is called a *truth-table* for the sentence-functor (16.3).

Likewise we may write down a truth-table for the sentence-functor

It's true that ϕ . 16.13

as follows:

ϕ	It's true that ϕ .	16.14
T	T	
F	F	

Similarly the sentence-functor

Either it's true that ϕ or it's not true that ϕ . 16.15

has a truth-table. The sentence variable ' ϕ ' must be replaced by the same declarative sentence at both places, so as to yield sentences like

Either it's true that high taxes are inflationary 16.16
or it's not true that high taxes are inflationary.

High taxes may be inflationary, or they may not be; (16.16) is equally true in either case. The truth-table is accordingly:

ϕ	Either it's true that ϕ or it's not true that ϕ .	16.17
T	T	
F	T	

However, not every sentence-functor has a truth-table. Sometimes the truth-values of the constituent sentences are not enough by themselves to determine the truth-value of the whole. For example, the sentence-functor

I know that ϕ . 16.18

has only the partial truth-table:

ϕ	I know that ϕ .	16.19
T	-	
F	F	

Nobody can know something which is false; hence the second row of (16.19) shows Falsehood. But there are truths which I know and truths which I don't know, and this compels us to leave the first row blank. For the sentence-functor

It is often asserted that ϕ . 16.20

the partial truth-table is even more sparse:

ϕ	It is often asserted that ϕ .	16.21
T	-	
F	-	

A sentence-functor which has a truth-table is called a *truth-functor*; thus (16.3), (16.13) and (16.15) are truth-functors, while (16.18) and (16.20) are not. The next few sections will be entirely concerned with truth-functors. The logical theory of sentence-functors which are not truth-functors is not well developed; we shall consider one example in section 42.

Exercise 16B. Write out a truth-table or a partial truth-table (as appropriate) for each of the following sentence-functors:

1. It's a lie that ϕ .
2. ϕ because ψ .
3. ϕ whenever ψ .
4. If ϕ , then ϕ .
5. Whether or not ϕ , what will be will be.
6. Whether or not ϕ , smoking causes cancer.

To make a logician happy, find him some constituent sentences. Very often a sentence will yield us more constituent sentences if we allow ourselves to *paraphrase* it, i.e., to replace it by another sentence which means the same thing. If two declarative sentences mean the same thing, then they have the same truth-value in all situations; so there is no harm in replacing one by the other in an argument which is being tested for validity, or in a set of sentences which is being tested for consistency.

To illustrate the powers of paraphrase, consider the following morsel of seventeenth-century legal prose:

If the process be legal, and in a right Court, yet I conceive that His Majesty alone, without assistance of the Judges of the Court, cannot give judgment. 16.22

The following paraphrase of (16.22) contains four constituent short sentences, which we mark with brackets:

I conceive that, if [the process is legal] and [the process is in a right court], yet if [the Judges of the Court are not assisting His Majesty], [His Majesty cannot give judgment]. 16.23

The sentence (16.23) can be got by filling the gaps in the sentence-functor

I conceive that, if ϕ_1 and ϕ_2 , yet if ϕ_3 , ϕ_4 . 16.24

Many sentences can be paraphrased in a simple way to elicit short constituent sentences. (In quite a few cases, linguists would regard such a paraphrase as bringing to the surface an underlying constituent.)

Exercise 16C. Paraphrase the following so as to find constituent sentences:

1. I am aware of your intention to sue.
2. He regrets not having married Suzy.
3. He completed his task before the end of the week.

4. An accident to the train resulted from the failure of its brakes.
 5. Your Majesty may be pleased to take notice of the great mischiefs which may fall upon this kingdom if the intentions which have been credibly reported, of bringing in Irish and foreign forces, shall take effect.
-

17. Some Basic Truth-functors

In this section we shall introduce the five truth-functors most commonly used in logic; each of them has a special symbol to represent it. At the same time we shall mention some English expressions which can be paraphrased by means of these truth-functors.

(i) The *negation* truth-functor 'It is not true that ϕ .'

In symbols, this truth-functor is written ' $\neg\phi$ '. It yields true sentences precisely when false sentences are put for ' ϕ ', so that its truth-table is

ϕ	$\neg\phi$	17.1
T	F	
F	T	

' $\neg\phi$ ' is called the *negation* of the sentence ϕ .† ' \neg ' is pronounced 'not'.

Here are some other ways in which English expresses the sense of ' \neg ':

I am not a Dutchman. 17.2
 \neg I am a Dutchman.

She didn't say anything. 17.3
 \neg she said something.

†In this sentence, the symbol ' ϕ ' is being used to talk about sentences, not to mark a hole where a sentence can be put. Thus the two occurrences of ' ϕ ' here are not free.

- George Washington never told a lie. 17.4
 \neg George Washington sometimes told a lie.
- I hardly think he will reach Athens in that old bus. 17.5
 \neg I think he will reach Athens in that old bus.
- It isn't as if he needs the money. 17.6
 \neg he needs the money.

None of these paraphrases is perfect. For example, the first sentence of (17.3) implies that there is some woman under discussion – for otherwise we have referential failure. But 'She said something' is false if there is no woman to be referred to by 'she', so that in such a situation the second sentence of (17.3) is true.

Similarly if there is no old bus in the situation, then the first sentence of (17.5) is false while the second is true.

In fact, when one adds words such as *not* or *never* to an English sentence, this cancels some of the implications of the sentence, but it usually leaves other implications intact. A famous example is

I have not stopped beating my wife. 17.7

By contrast the symbol ' \neg ' cancels all the implications of the original sentence.

Since they are not completely accurate, paraphrases like (17.2)–(17.6) may lead to mistakes in logic. In practice this happens very rarely, and the experienced logician knows when to take care. (We shall return to this in section 28.)

(ii) The *conjunction* truth-functor ' ϕ and ψ '.

This truth-functor is written ' $[\phi \wedge \psi]$ '. It yields a false sentence unless truths are put for both ' ϕ ' and ' ψ ', in which case it yields a truth. (Comparing with (7.1), one can see that we are assuming the weak reading of *and*.) Hence the truth-table is

ϕ	ψ	$[\phi \wedge \psi]$	17.8
T	T	T	
T	F	F	
F	T	F	
F	F	F	

' $[\phi \wedge \psi]$ ' is called the *conjunction* of the sentences ϕ and ψ ; ϕ and ψ are its *conjuncts*. It is pronounced ' ϕ and ψ '.

Here are some other ways in which English expresses the sense of this truth-functor:

- 17.9
Although it was raining, he ran out in his vest.
 [it was raining \wedge he ran out in his vest]
- 17.10
 The powder contains sulphur **and** magnesium.
 [the powder contains sulphur \wedge the powder contains magnesium]
- 17.11
 The method is simple **but** effective.
 [the method is simple \wedge the method is effective]
- 17.12
 Neither I nor my wife speak German.
 [I don't speak German \wedge my wife doesn't speak German]
- 17.13
 The damping is effected by the water roller, **which** can be found above the plate cylinder.
 [the damping is effected by the water roller \wedge the water roller can be found above the plate cylinder]

Notice that in (17.10) **and** is between nouns instead of sentences. The word **and** can occur between adjectives too, and other parts of speech. We can usually rephrase the whole sentence so that **and** occurs just between sentences, allowing a translation by ' \wedge ':

- 17.14
 Your review of my book was **both** insulting **and** inaccurate.
 [your review of my book was insulting \wedge your review of my book was inaccurate]

He coughs often and loudly. 17.15
 [he coughs often \wedge he coughs loudly]

But sometimes this kind of rephrasing is quite wrong:

New York and Cairo are over a thousand miles apart. 17.16

NOT: [New York is over a thousand miles apart \wedge Cairo is over a thousand miles apart]

My sister wants a black and white cat. 17.17

NOT: [my sister wants a black cat \wedge my sister wants a white cat]

Twenty people were rounded up and shot. 17.18

NOT: [twenty people were rounded up \wedge twenty people were shot]

(iii) The *disjunction* truth-functor 'Either ϕ or ψ , or both'.

This truth-functor is written ' $[\phi \vee \psi]$ '; its truth-table is

ϕ	ψ	$[\phi \vee \psi]$
T	T	T
T	F	T
F	T	T
F	F	F

17.19

The table shows that ' $[\phi \vee \psi]$ ' yields a truth in every case but one; the one case is where ' ϕ ' and ' ψ ' are both replaced by false sentences. ' $[\phi \vee \psi]$ ' is called the *disjunction* of the sentences ϕ and ψ , and ϕ and ψ are called its *disjuncts*. It is pronounced ' ϕ or ψ '.

Here are some ways in which English expresses the sense of this truth-functor:

There will be a stiff wages policy, or we shall see massive unemployment. 17.20

[there will be a stiff wages policy \vee we shall see massive unemployment]

He is a fool or a liar. 17.21
 [he is a fool \vee he is a liar]

We'll go to the seaside unless it rains. 17.22
 [we'll go to the seaside \vee it will rain]

We take (17.20) to be true (but possibly misleading) if there will be *both* a stiff wages policy *and* massive unemployment – this is the top line of the truth-table (17.19). Some people would regard it as false in this situation. They take the strong reading, while we take the weak one (see section 7).

In (17.22), note that the underlying constituent is 'it will rain', not 'it rains'. After *unless*, English uses present tense instead of future; we have to remember to put the verb back into future form when we introduce ' \vee '.

There are a few unusual sentences in which *or* means something more like *and*, and it should be translated to ' \wedge ':

Uri Geller can read your mind, or he can bend 17.23
 your spoons.

[Uri Geller can read your mind \wedge he can bend
 your spoons]

(iv) The *arrow* truth-functor ' $[\phi \rightarrow \psi]$ ' (sometimes called *material implication*).

There is no neat and exact way of expressing this truth-functor in ordinary English, though several English expressions come close to it. The best way to define it is by its truth-table, which is

ϕ	ψ	$[\phi \rightarrow \psi]$
T	T	T
T	F	F
F	T	T
F	F	T

17.24

As the table shows, the only way to make a false sentence out of ' $[\phi \rightarrow \psi]$ ' is to put a true sentence for ' ϕ ' and a false sentence for ' ψ '. ' \rightarrow ' is best read as 'arrow'.

Here are some English phrases which can be expressed with this truth-functor:

If the paper turns red, then the solution is acid.	17.25
[the paper will turn red \rightarrow the solution is acid]	
The paper will only turn red if the solution is acid.	17.26
(or: The paper will turn red only if the solution is acid.)	
[the paper will turn red \rightarrow the solution is acid]	
You will get a room provided you have no pets.	17.27
[you have no pets \rightarrow you will get a room]	
Assuming that the timer is correctly set, the relay will close after two minutes.	17.28
[the timer is correctly set \rightarrow the relay will close after two minutes]	
If there are any more patients, I shall be home late.	17.29
[there are some more patients \rightarrow I shall be home late]	
NOT: [there are any more patients \rightarrow I shall be home late]	

The first sentence of (17.25) excludes just one possible state of affairs, namely that the paper will turn red and the solution is not acid. As we see from the table (17.24), this is precisely the case which is ruled out by the second sentence of (17.25) too. We can test the accuracy of the other translations (17.26)–(17.29) in the same way. Note the dramatic effect of adding **only** in (17.26) compared with (17.25) – it shifts the **if** to the other clause.

In some of these examples, we have to seek out the underlying constituents. Thus in (17.25) the underlying constituent

sentence is 'the paper will turn red'; English drops the future tense after *if*. In (17.29) we have to paraphrase to remove *any*.

Some of these translations may raise doubts. For example, surely the first sentence of (17.25) implies there is some kind of connection between the redness and the acidity? And surely it suggests that if the paper does not turn red, then the solution is not acid? Neither of these things is conveyed by our translation. We take the view that although somebody might well assume these things if he heard the first sentence of (17.25), they are not actually stated in that sentence. In the terminology of section 7, we adopt the weak reading.

There are some cases where *if* should definitely not be translated by ' \rightarrow '. Here are three examples; more are given in section 18.

The choir was sensitive, *if* a little strained. 17.30
 [the choir was sensitive \wedge the choir was a little strained]

If you want to wash your hands, the bathroom is first on the left. 17.31

I won't sing, *even if* you pay me £1000. 17.32

In fact (17.32) is true in just the same situations as

I won't sing. 17.33

The difference lies in the emphasis alone.

(v) The *biconditional* truth-functor ' ϕ if and only if ψ '.

This truth-functor is written ' $[\phi \leftrightarrow \psi]$ '. Its truth-table is

ϕ	ψ	$[\phi \leftrightarrow \psi]$	17.34
T	T	T	
T	F	F	
F	T	F	
F	F	T	

Note the third line, which distinguishes ' \leftrightarrow ' from ' \rightarrow '. ' \leftrightarrow ' is pronounced 'if and only if'.

Two other phrases which carry the sense of ' \leftrightarrow ' are **precisely if and just if**:

The number is even **precisely if** it's divisible by two. 17.35
 [the number is even \leftrightarrow it's divisible by two]

The company has to be registered **just if** its annual turnover is above £5000. 17.36
 [the company has to be registered \leftrightarrow its annual turnover is above £5000]

In America, but not normally in Britain, **just in case** is used in the same way.

This completes our list of basic truth-functors. There is one important point to bear in mind before we face any exercises. Our object is to analyse complex sentences into shorter ones, so that we can extract the shorter sentences in a tableau. Now if a complex sentence contains cross-referencing, the references of some of its parts may change when we extract the constituent sentences; we may even face referential failure. For example, in the first sentence of (17.35), the pronoun 'it' refers back to the number mentioned at the beginning. But if we take the second constituent sentence on its own:

It's divisible by two. 17.37

there's nothing to determine what 'it' refers to. To avoid this, you should always *try to eliminate cross-referencing* when you translate. In the example just given, you should replace the translation in (17.35) by

[the number is even \leftrightarrow the number is divisible by two] 17.38

The same applies to (17.23) and (17.36). There are some more examples in the exercises below.

Exercise 17. Express each of the following sentences as faithfully as possible, using the truth-functors introduced in this section; remove cross-referencing where possible.

1. No dogs will be admitted.
 2. The brain is bisected, yet the character remains intact.
 3. Unless the safety conditions are tightened, there is going to be a nasty accident.
 4. Supposing you're right, I stand to lose a lot of money.
 5. You broke the law if and only if the agreement formed a contract.
 6. If anybody calls, I shall pretend I am designing St Paul's.
 7. Schubert is terrific, and so is Hindemith.
 8. This is Bert Bogg, who taught me that limerick I was quoting yesterday.
 9. You can only claim the allowance if you earn less than £16 a week.
 10. Liszt is horrible, and the same goes for Vivaldi.
 11. She needs all the help she can get, being a single parent.
 12. The elder son was highly intelligent, while the younger was an imbecile.
 13. Her performance lacked zest.
 14. If he gets anything right at all, he'll pass.
 15. Either the metal will stretch, or it will snap.
-

18. Special Problems with ' \rightarrow ' and ' \wedge '

The truth-functor symbols ' \rightarrow ' and ' \wedge ' raise some special problems, which deserve a few words of warning. The points at issue are fairly subtle; this section can be left out without breaking the continuity.

We begin with ' \rightarrow '.

In a sentence of form 'If A then B', we often find cross-referencing from B to A, and it may be impossible to para-

phrase so as to remove the cross-referencing. Here's a fairly mild example:

If the tonsils are removed, the adenoids are often cut out too. 18.1

If we ask 'Which adenoids?', the answer is: the adenoids of whoever has his tonsils cut out. But thousands of people have their tonsils cut out, so that there is no question of finding a phrase which pins us down to just one set of adenoids. Even if we could find one, we should hardly interpret (18.1) as saying that one particular person's adenoids are often cut out! So we must not analyse (18.1) as

[the tonsils are removed \rightarrow the adenoids are often cut out]. 18.2

For similar reasons we must avoid such translations as the following:

If you poured in the sulphuric, the solution would turn muddy. 18.3

NOT: [you poured in the sulphuric \rightarrow the solution would turn muddy]

If you had poured in the sulphuric, the solution would have turned muddy. 18.4

NOT: [you had poured in the sulphuric \rightarrow the solution would have turned muddy]

The English sentences in (18.3) and (18.4) are examples of *subjunctive conditionals*; they say what would happen in hypothetical states of affairs. The second clause refers to the hypothetical state of affairs described by the first clause, so that there is a cross-reference.

Exercise 18A. Which of these sentences (with apologies to Dr Spock) can be translated by means of ' \rightarrow ' without cross-referencing?

1. If the nappies are becoming hard, you can soften them by using a water conditioner.

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2. If it contains soap, this helps in removing stains.
 3. If an injured child has not already built his own protection from toxoid inoculations, it is sometimes hard to decide whether horse serum is necessary.
 4. If most of his former protection has worn off, his new vaccination develops much like the previous one.
 5. If a vaccination doesn't take, it doesn't mean that the person is immune.
 6. If your baby is colicky, he may be soothed when you first pick him up.
-

We turn to '∧'.

The symbol '∧' can sometimes be used to eliminate the relative pronouns *which* and *who*. For example,

The policeman, who was watching through binoculars, ducked just in time. 18.5

can be analysed as

[the policeman was watching through binoculars ∧ the policeman ducked just in time] 18.6

See (17.13) for another example.

However, there is another sentence very like (18.5), which must not be analysed in this way. The sentence is

The policeman who was watching through binoculars ducked just in time. 18.7

(Note the commas.) As it occurs in (18.7), the phrase 'who was watching through binoculars' serves to indicate which of several policemen is being talked about; in this use it is said to be *restrictive*. The same phrase in (18.5) serves, not to pick out one policeman from several, but to say something about a policeman who has already been picked out; we say it is *non-restrictive* in (18.5). The truth-functor symbol '∧' *must not be used to replace which or who in restrictive phrases*.

We can make the same distinction with adjectives. When **clever** is used non-restrictively, we may be able to split it off with the help of '∧':

My clever husband has found a tax loophole 18.8
that saves us £500.

[my husband is clever ∧ my husband has
found a tax loophole that saves us £500]

Karel is a clever boy. 18.9

[Karel is clever ∧ Karel is a boy]

When it is used restrictively, no such analysis is possible:

The clever twin was always teasing her dim- 18.10
witted sister.

NOT: [the twin was clever ∧ the twin was
always teasing her dim-witted sister]

Even when we find an adjective in a non-restrictive posture, we may be unable to split it off with '∧', because the meaning requires it to stay attached to a particular noun. For example:

Arturo is a famous pianist. 18.11

NOT: [Arturo is famous ∧ Arturo is a pianist]

(The second sentence is true if, for example, the pianist Arturo is famous only as a female impersonator.)

You are a perfect stranger. 18.12

NOT: [you are perfect ∧ you are a stranger]

The Rolls Royce was a small compensation. 18.13

NOT: [the Rolls Royce was small ∧ the Rolls
Royce was a compensation]

Amaryllis is my daughter. 18.14

NOT: [Amaryllis is mine ∧ Amaryllis is a
daughter]

Exercise 18B. Which of these sentences can be analysed by means of '∧'?

1. Stavros is a so-called radical.
 2. Stavros is a former radical.
 3. The thoroughly pleasant evening concluded with a waltz.
 4. Their next encounter was more restrained.
 5. Her aunt, who from her earliest youth
Had kept a strict regard for Truth,
Attempted to believe Matilda.
 6. The animal which you saw was probably a fox.
 7. Marianne is a teacher, who should have known better.
 8. Britain, once a superpower, is now seeking a new role.
 9. I can see Don growing into a bespectacled pedant.
-

19. Analysis of Complex Sentences

We wish to rewrite the following sentence using truth-functors:

The female bearded reedling has no black marks underneath, and its head is tawny. 19.1

There are an *and* and a *no* to contend with. Two translations suggest themselves:

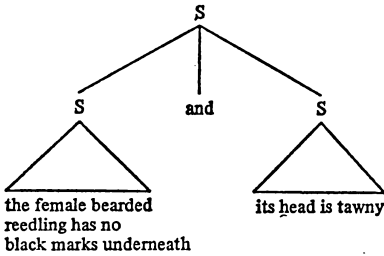
[¬the female bearded reedling has black marks underneath ∧ the female bearded reedling has a tawny head] 19.2

¬[the female bearded reedling has black marks underneath ∧ the female bearded reedling has a tawny head] 19.3

(19.2) is right and (19.3) is wrong. Why?

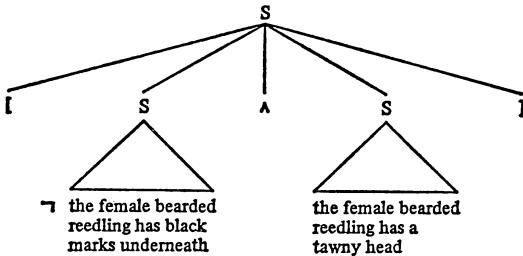
The answer is a matter of scope. In (19.1) the scope of 'and' is the whole sentence:

19.4



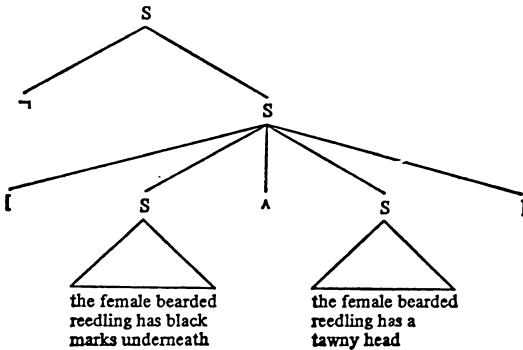
We shall express this by saying that the *overall form* of (19.1) is ' ϕ and ψ '. The correct translation (19.2) likewise has the overall form ' $[\phi \wedge \psi]$ ':

19.5



But in the incorrect translation, the scope of ' \wedge ' is only part of the sentence, and the overall form is ' $\neg \phi$ ':

19.6



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This example shows why we included the two brackets in our notation; without them it would be impossible to tell whether the scope of ' \wedge ' included ' \neg ' or not.

To translate a complex sentence with more than one truth-functor, *start with the truth-functor of largest scope, as shown by the overall form of the sentence; then work inwards.*

The following example will show how. We shall translate:

If the battery is flat, then the starter will be dead, and you won't get the car started unless we push it. 19.7

We first look for the overall form of (19.7). Does it say 'If ϕ then ψ ' or ' ϕ and ψ ', or 'It's not true that ϕ ', or what? On the natural reading, the part after 'then' forms a unit, so the answer is 'If ϕ then ψ '. The truth-functor of largest scope will be an arrow, and we can write

[the battery is flat \rightarrow the starter will be dead, and you won't get the car started unless we push it] 19.8

Next we find the largest constituent sentence which has no truth-functor symbols in it:

The starter will be dead, and you won't get the car started unless we push it. 19.9

This has overall form ' ϕ and ψ ', so that the truth-functor of largest scope in the translation of (19.9) will be conjunction:

[the starter will be dead \wedge you won't get the car started unless we push it] 19.10

The longest unanalysed constituent sentence of (19.10) is

You won't get the car started unless we push it. 19.11

The overall form of (19.11) is ' ϕ unless ψ '. Remembering to remove the cross-referencing as we introduce ' \vee ', we write

[you won't get the car started \vee we'll push the car] 19.12

There remains

You won't get the car started. 19.13
 \neg you'll get the car started.

The remaining sentences leave little opportunity to introduce more truth-functors; so at this point we call a halt and fit the pieces back together:

[the battery is flat \rightarrow [the starter will be dead \wedge 19.14
 [\neg you'll get the car started \vee we'll push the car]]]

(19.14) is the correct analysis of (19.7).

After some practice, you will find you can write down an analysis such as (19.14) as soon as you see the original sentence. But for the moment you should go step by step, for safety.

You may find that you have to paraphrase quite freely in order to introduce truth-functors, just as in section 17. In some cases this may lead you to change the scope of a word. Two examples will make the point.

He will be coming down by the 8.15 or the 9.15; if the former, then he will be in time to see the opening. 19.15

[[he will be coming down by the 8.15 \vee he will be coming down by the 9.15] \wedge [he will be coming down by the 8.15 \rightarrow he will be in time to see the opening]]

Here we repeat part of the first half, to eliminate the cross-referencing ('the former'); this brings the part about 8.15 into the scope of 'if'.

If you cut the party, then Jane, who is none too fond of you anyway, will just make life hell for you. 19.16

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[Jane is none too fond of you \wedge [you will cut the party \rightarrow Jane will just make life hell for you]]

Here we take the clause after 'who' outside the scope of 'if', in order to turn the 'who' into ' \wedge '.

Exercise 19. Analyse each of the following sentences as faithfully as possible, using truth-functor symbols:

1. I shall not fail to write to you.
 2. He was gassed, not shot.
 3. He was neither gassed nor shot.
 4. Nobody will get any chocolate if Tracey screams again.
 5. If the State schools lack adequate space, then the private schools, providing as they do an excellent education, ease the burden on the State's facilities.
 6. If the private schools are socially top-heavy, then they are perpetuating social injustice; but in that case they cannot reasonably demand charitable status.
-

Sentence Tableaux

As promised on page 86, we shall now show that logical analysis leads to simpler tableaux. You should re-read section 10 quickly, to recall the main features of tableaux.

20. Sentence Tableaux

In order to construct a tableau, one has to be able to do two things, both of which demand concentration:

(i) given a sentence S , to describe the situations in which S is true, using only sentences shorter than S ;

(ii) given a set of short sentences, to determine whether the set is consistent.

We shall see that when truth-functors are used, (i) becomes completely automatic, so that a machine could do it.

Consider first the truth-table for ' \wedge ', which we gave in section 17:

ϕ	ψ	$[\phi \wedge \psi]$	20.1
T	T	T	
T	F	F	
F	T	F	
F	F	F	

We may think of ' ϕ ' and ' ψ ' as standing for two sentences. As the table shows, there is precisely one case in which ' $[\phi \wedge \psi]$ ' is true, namely the case where ϕ is true and ψ is true. Hence if

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' $[\phi \wedge \psi]$ ' occurs in a tableau, we may tick it and add ϕ and ψ at the bottom of each unclosed branch containing it:

$$\begin{array}{c} \checkmark [\phi \wedge \psi] \\ | \\ \phi \\ \psi \end{array} \quad 20.2$$

(20.2) will be called the *derivation rule* for ' $[\phi \wedge \psi]$ '.

Next we consider the sentence ' $\neg[\phi \wedge \psi]$ '. According to the truth-table for ' \neg ' in section 17, this sentence is true just if ' $[\phi \wedge \psi]$ ' is false. According to (20.1), there are just three kinds of situation in which ' $[\phi \wedge \psi]$ ' is false; we can group these situations into two overlapping groups, namely those where ϕ is false and those where ψ is false. But of course ϕ is false just if ' $\neg\phi$ ' is true, and likewise with ψ . Drawing all this together, we see that ' $\neg[\phi \wedge \psi]$ ' is true precisely if either ' $\neg\phi$ ' is true or ' $\neg\psi$ ' is true. The *derivation rule* for ' $\neg[\phi \wedge \psi]$ ' in tableaux is therefore a branching rule, as follows:

$$\begin{array}{c} \checkmark \neg[\phi \wedge \psi] \\ | \\ \neg\phi \quad \neg\psi \end{array} \quad 20.3$$

The derivation rules for the other types of complex sentence can be worked out from the truth-tables in just the same way. In fact this is so easy that you should try it at once, as an exercise.

Exercise 20A. Using the truth-tables given in section 17, and taking (20.2) and (20.3) as guides, work out what the derivation rules must be for the following forms of sentence:

1. $\neg\neg\phi$
2. $[\phi \vee \psi]$
3. $\neg[\phi \vee \psi]$
4. $[\phi \rightarrow \psi]$

5. $\neg[\phi \rightarrow \psi]$
6. $[\phi \leftrightarrow \psi]$
7. $\neg[\phi \leftrightarrow \psi]$

(You will find all the derivation rules, including the answers to this exercise, listed on pp. 227–8 and 321–2.)

To test the consistency of a set of sentences, we can now proceed as follows. First, analyse the sentences by means of symbolic truth-functors, just as in section 19. Then apply the tableau technique, using the derivation rules we have just described. A tableau which uses just these derivation rules is called a *sentence tableau*.

For instance, let's see how the example on p. 48 fares under this treatment. The set of sentences was

- If cobalt but no nickel is present, a brown colour appears. 20.4
 Nickel and manganese are absent.
 Cobalt is present but only a green colour appears.

After analysis, (20.4) becomes

- $[[\text{cobalt is present} \wedge \neg \text{nickel is present}] \rightarrow \text{a brown colour appears}]$ 20.5
 $[\neg \text{nickel is present} \wedge \neg \text{manganese is present}]$
 $[\text{cobalt is present} \wedge \text{only a green colour appears}]$

Using the derivation rules, we construct the tableau

- $\checkmark [[\text{cobalt is present} \wedge \neg \text{nickel is present}] \rightarrow \text{a brown colour appears}]$ 20.6
 $\checkmark [\neg \text{nickel is present} \wedge \neg \text{manganese is present}]$
 $\checkmark [\text{cobalt is present} \wedge \text{only a green colour appears}]$

|

is valid if and only if its counterexample set

$P_1 \dots P_n$. 'It's not true that C .' 20.8

is inconsistent. This means that sentence tableaux can be used to test the validity of arguments.

Here is an example. We shall test the validity of the argument

The mother will die unless the doctor kills the child. 20.9

If the doctor kills the child, the doctor will be taking life. If the mother dies, the doctor will be taking life. *Therefore* either way, the doctor will be taking life.

We first analyse:

[the mother will die \vee the doctor will kill the child] 20.10

[the doctor will kill the child \rightarrow the doctor will be taking life]

[the mother will die \rightarrow the doctor will be taking life]

Therefore the doctor will be taking life.

The counterexample set is:

[the mother will die \vee the doctor will kill the child] 20.11

[the doctor will kill the child \rightarrow the doctor will be taking life]

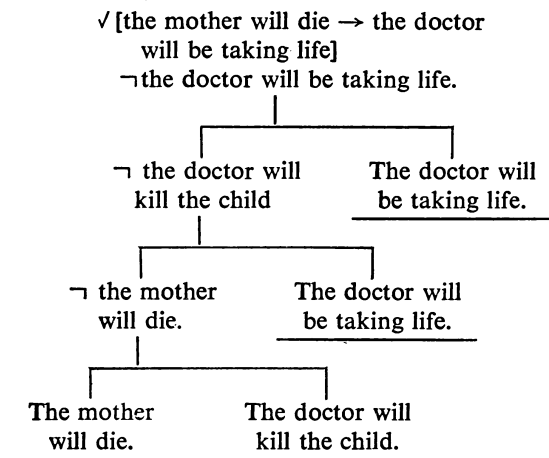
[the mother will die \rightarrow the doctor will be taking life]

\neg the doctor will be taking life.

We test the consistency of (20.11) by a sentence tableau:

\checkmark [the mother will die \vee the doctor will kill the child] 20.12

\checkmark [the doctor will kill the child \rightarrow the doctor will be taking life]



The tableau (20.12) is closed; therefore (20.11) was inconsistent, and so the argument (20.9) is *valid*.

Exercise 20C. Use truth-functors to analyse the sentences in the following arguments:

1. If the gunmen are tired, then they are on edge. If the gunmen are armed and on edge, then the hostages are in danger. The gunmen are armed and tired. *Therefore* the hostages are in danger.
2. If the driver was in control, then he passed the signal if and only if it was green. The driver passed the signal, although he was in control. Only if the electronics were faulty was the signal green. *Therefore* the electronics were faulty.
3. If the boy has spots in his mouth, then he has measles. If the boy has a rash on his back, then he has heat-spots. The boy has a rash on his back. *Therefore* the boy hasn't got measles.
4. Either the vicar or the butler shot the earl. If the butler shot the earl, then the butler wasn't drunk at nine o'clock. Unless the vicar is a liar, the butler was drunk at nine o'clock. *Therefore* either the vicar is a liar, or he shot the earl.

Exercise 20D. Using your answers to Exercise 20C, test the validity of these four arguments by means of sentence tableaux.

21. Interpretations

Section 20 introduced one labour-saving device; in this section we meet another.

The most irksome feature of sentence tableaux is that we have to keep writing out the same sentences time after time in different combinations. It would be sensible to use single letters as abbreviations for short sentences. This leads to the notion of an *interpretation*, which is defined as follows.

By an *interpretation* we mean a list of capital letters, in which each letter has a declarative sentence assigned to it. The same declarative sentence may be assigned to two different letters, but only one declarative sentence must be assigned to each letter. For example, here is an interpretation, written as we shall usually write them:

- | | | |
|------------|------------------------------|------|
| <i>P</i> : | Cobalt is present. | 21.1 |
| <i>Q</i> : | Nickel is present. | |
| <i>R</i> : | Manganese is present. | |
| <i>S</i> : | A brown colour appears. | |
| <i>T</i> : | Only a green colour appears. | |

Using an interpretation, we can translate a complex sentence completely into symbols. The resulting string of symbols is called a *formula*.

A good procedure for translating into formulae is to put in the truth-functor symbols first, as in section 19, and then to use the interpretation to remove the remaining pieces of English. For example, we can first express

- | | |
|---|------|
| If cobalt but no nickel is present, a brown | 21.2 |
| colour appears. | |

[[cobalt is present \wedge \neg nickel is present] \rightarrow 21.3
a brown colour appears].

Then we may use the interpretation (21.1) to contract (21.3) into the formula

[[$P \wedge \neg Q$] $\rightarrow S$] 21.4

If a set of sentences has been completely symbolized, we can test it for consistency by means of sentence tableaux, as in section 20; but there is one difference. Some types of inconsistency become hidden from view when the sentences are all symbolized. For example the two sentences

Schubert died at the age of thirty-one. 21.5
Schubert died at the age of sixty-eight.

are inconsistent. But symbolized, they would look something like this:

$P. Q.$ 21.6

There is nothing in (21.6) to indicate that it is inconsistent.

We shall count a set of formulae as being 'obviously inconsistent' if and only if the set contains both a formula and its negation. Accordingly, in a completely symbolic tableau, *we close a branch if and only if there is a formula ϕ such that both ϕ and its negation ' $\neg\phi$ ' are in the branch.*

To locate the 'hidden' inconsistencies in a finished tableau, we must inspect one by one the branches which are not closed. In each such branch we must take all the formulae which are either a single letter or the negation of a single letter, and we must use the interpretation to translate them back into English sentences; the resulting set of sentences can be checked directly for inconsistencies. Of course it may happen that the finished tableau is closed. In this case the original set of sentences is certainly inconsistent - there is no need to hunt for hidden inconsistencies.

To illustrate all this, we return to the example on page 52:

If cobalt but no nickel is present, a brown colour appears. 21.7

Either nickel or manganese is absent.

Cobalt is present but only a green colour appears.

Analysing, we have

$[[\text{cobalt is present} \wedge \neg \text{nickel is present}] \rightarrow$ 21.8
 $\text{a brown colour appears}]$

$[\neg \text{nickel is present} \vee \neg \text{manganese is present}]$

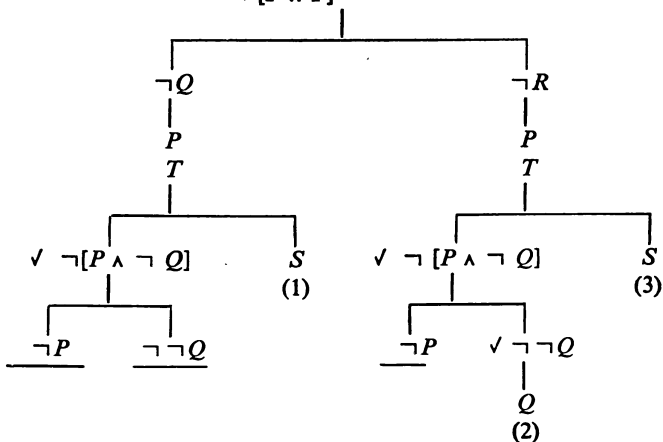
$[\text{cobalt is present} \wedge \text{only a green colour appears}]$

Translating into symbols by means of the interpretation (21.1), we reach the formulae

$[[P \wedge \neg Q] \rightarrow S], [\neg Q \vee \neg R], [P \wedge T]$ 21.9

We test the consistency of (21.9) by means of a sentence tableau:

$\checkmark [[P \wedge \neg Q] \rightarrow S]$ 21.10
 $\checkmark [\neg Q \vee \neg R]$
 $\checkmark [P \wedge T]$



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Three branches have failed to close; we have numbered them (1), (2), (3). If we translate the unticked formulae in each of these branches back into English by means of the interpretation (21.1), we find:

- (1) Nickel is absent. Cobalt is present. Only a green colour appears. A brown colour appears. 21.11
- (2) Manganese is absent. Cobalt is present. Only a green colour appears. Nickel is present.
- (3) Manganese is absent. Cobalt is present. Only a green colour appears. A brown colour appears.

(1) and (3) are clearly inconsistent, although the branches are not closed; we must therefore discount them. There remains (2), which is consistent and so describes a situation in which (21.7) is true. Thus (21.7) has been proved *consistent*.

Exercise 21A. Using the following interpretation:

P: Tax is levied on this income under Schedule E.

Q: The source of this income is employment.

R: The source of this income is an annuity taxed under Schedule C.

S: The source of this income is an annuity not taxed under Schedule C.

T: This income is attributable to Government stocks.

translate each of the following sets of sentences into symbols:

1. The source of this income is not both employment and an annuity taxed under Schedule C.

If tax is levied on this income under Schedule E, then the source of this income is employment.

The source of this income is an annuity taxed under Schedule C.

2. This income is attributable to Government stocks if and only if its source is an annuity taxed under Schedule C.

The source of this income is either employment or an annuity not taxed under Schedule C.

Employment is not the source of this income, which is in fact attributable to Government stocks.

3. Tax is levied on this income under Schedule E if the source of this income is employment or an annuity not taxed under Schedule C.

If the source of this income is an annuity taxed under Schedule C, then the income is attributable to Government stocks.

The source of this income is an annuity, but the income is not attributable to Government stocks.

Tax is not levied on this income under Schedule E.

Exercise 21B. Which of the sets of sentences in Exercise 21A are consistent? Test by symbolic tableaux.

As in section 20, we can use these methods to test the validity of arguments. Here is an example.

If the soil is suitable for carrots, then it is deep, sandy and free of stones. The soil is not suitable for linseed if it is sandy or a heavy clay. *Therefore* the soil is not suitable for both carrots and linseed. 21.12

A suitable interpretation would be:

P: The soil is suitable for carrots. 21.13

Q: The soil is suitable for linseed.

R: The soil is deep and free of stones.

S: The soil is a heavy clay.

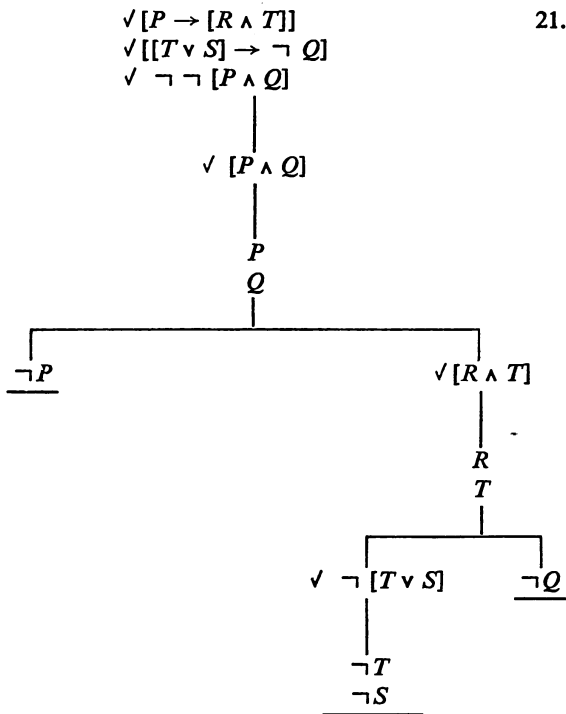
T: The soil is sandy.

(Notice that in (21.13) we have saved ourselves a letter by taking 'deep' and 'free of stones' together; they never occur apart in (21.12).) By means of (21.13), we can translate (21.12) into

$[P \rightarrow [R \wedge T]]$, $[[T \vee S] \rightarrow \neg Q]$. *Therefore* 21.14
 $\neg [P \wedge Q]$

Testing the validity of (21.14) by a tableau, we have

21.15



The tableau is closed. Therefore (21.12) was *valid*.

Exercise 21C. Test each of the following arguments for validity, using the interpretation provided. (For the origin of these examples, see the Answers.)

P: Uhha-muwas has bitten off Pissuwattis' nose.

Q: Pissuwattis is a female slave.

R: Pissuwattis is a free woman.

S: Uhha-muwas is liable to a 1 mina fine.

T: Uhha-muwas is liable to a 30 shekel fine.

1. If Uhha-muwas has bitten off Pissuwattis' nose, then he is liable to a 30 shekel fine. Uhha-muwas has not bitten off Pissuwattis' nose. *Therefore* Uhha-muwas is not liable to a 30 shekel fine.
 2. If Uhha-muwas has bitten off Pissuwattis' nose, then, unless Pissuwattis is a female slave, Uhha-muwas is liable to a 1 mina fine. But Pissuwattis is not a female slave. *Therefore* either Uhha-muwas is liable to a 1 mina fine, or he has not bitten off Pissuwattis' nose.
 3. Only if Pissuwattis is a free woman is Uhha-muwas, who has bitten off her nose, liable to a 1 mina fine. Pissuwattis is a female slave. *Therefore* Pissuwattis is not a free woman, and so Uhha-muwas is not liable to a 1 mina fine, even though he has bitten off her nose.
 4. If Uhha-muwas has bitten off Pissuwattis' nose, then Uhha-muwas is liable to a 1 mina fine if Pissuwattis is a free woman, or a 30 shekel fine if Pissuwattis is a female slave. Uhha-muwas has bitten off Pissuwattis' nose, but is not liable to a 30 shekel fine. Pissuwattis is either a free woman or a female slave. *Therefore* Uhha-muwas is liable to a 1 mina fine.
-

Propositional Calculus

The task we now approach is that of *formalizing* logic. To formalize is to strip away the concepts which give meaning and application to the subject, so that nothing remains but bare symbols. Translation will disappear first, situations will fade away next, and finally even truth will make its exit.

On the negative side, this surgery pleases folk who regard possible situations as suspect anyhow. On the positive side, it exposes those parts of logic which can be developed as a mathematical theory. The mathematical theory we study in sections 22–25 is known as *propositional calculus*.

Unmathematical readers may prefer to leave out sections 22–25, and pass straight to section 26.

22. A Formal Language⁺

The first step in formalizing logic is to describe those strings of symbols which can occur as the translations of English declarative sentences. Our description must be purely formal, so that it must not mention English or translations. The simplest approach is to think of these strings as forming a language L_1 , and to present a CF grammar for this language.

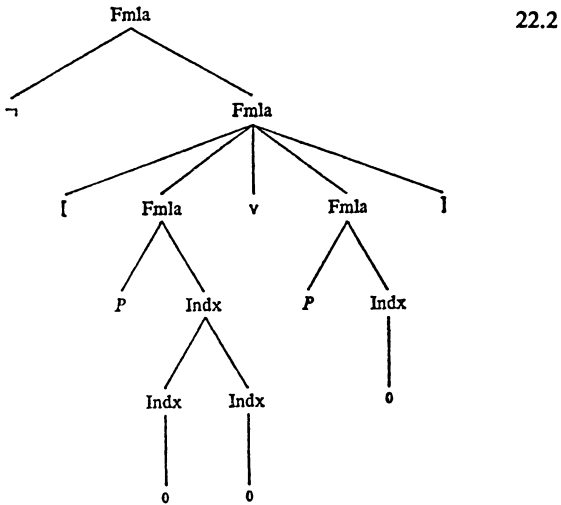
The CF grammar for L_1 shall be as follows:

- | | | |
|---|---|------|
| 1 | $Fmla \Rightarrow \neg Fmla$ | 22.1 |
| 2 | $Fmla \Rightarrow [Fmla \wedge Fmla]$ | |
| 3 | $Fmla \Rightarrow [Fmla \vee Fmla]$ | |

⁺Readers who dislike mathematics may omit this section.

- 4 Fmla \Rightarrow [Fmla \rightarrow Fmla]
- 5 Fmla \Rightarrow [Fmla \leftrightarrow Fmla]
- 6 Fmla \Rightarrow P Indx
- 7 Indx \Rightarrow Indx Indx
- 8 Indx \Rightarrow 0

The terminal symbols of this grammar are the brackets '[', ']', the truth-functor symbols ' \neg ', ' \wedge ', ' \vee ', ' \rightarrow ', ' \leftrightarrow ', the letter ' P ' and the subscript ' 0 '; the initial symbol is 'Fmla'. A typical phrase-marker is



with terminal string ' $\neg[P_{00} \vee P_0]$ '.

The grammatical sentences of L_1 will be known as *formulae*. When a constituent of a formula ϕ is itself a formula, we say that the constituent is a *subformula* of ϕ . In particular every formula is a subformula of itself, since every formula is a constituent of itself (see p. 63). There are four subformulae of ' $\neg[P_{00} \vee P_0]$ ', as shown in (22.3):

$$\begin{array}{l}
 \neg [P_{00} \vee P_0] \\
 \hline
 (1) \\
 \hline
 (2) \\
 \hline
 (3) \\
 \hline
 (4) \quad \quad \quad -
 \end{array}$$

The simplest formulae are those of form ‘ P_0 ’, ‘ P_{00} ’, ‘ P_{000} ’, etc.; these are called *sentence letters*. Since long strings of subscripts are tiresome to write and read, we shall often allow ourselves to pretend that the sentence letters include ‘ P ’, ‘ Q ’, ‘ R ’, etc.

Exercise 22. Which of the following are formulae of L_1 ? Write phrase-markers for those which are, and indicate their subformulae.

1. P_0
2. P_{00}
3. $[P_0 \rightarrow P_0]$
4. $[P_0 \vee P_{00} P_{000}]$
5. $[P_0 \wedge P_0 \wedge P_0]$
6. $\neg[\neg P_0]$
7. $P_0 \leftrightarrow P_{00}$
8. $[[\neg P_0 \rightarrow P_{00}] \rightarrow \neg \neg P_0]$

23. Truth-tables⁺

Having formalized the language, we now proceed to formalize the notions of situation and truth-value. This part of the subject is known as *logical semantics*.

Suppose a sentence of English has been translated completely into symbols by means of truth-functors and an interpretation. The result is a formula of L_1 ; we may wish to determine the truth-value of this formula in some situation. The truth-values to be assigned to the sentence letters can be worked out by comparing the interpretation with the situation. Once the

⁺Readers who dislike mathematics may omit this section.

sentence letters have been tagged with truth-values, it's easy to calculate the truth-value of the whole sentence, by using the truth-tables of section 17. *The interpretation and the situation are needed only for assigning truth-values to sentence letters; truth-tables take care of the rest.*

An assignment of truth-values to sentence letters is called a *structure*. We shall use bold capital letters **A**, **B**, etc., to stand for structures. Here is an example of a structure:

P	Q	R	23.1
T	F	T	

(23.1) is the structure which assigns Truth to ' P ' and ' R ', and assigns Falsehood to ' Q '. We can list all the structures which make assignments to just the letters ' P ', ' Q ' and ' R '; there are eight of them:

P	Q	R	23.2
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

(23.2) shows the normal style for listing structures; 'T' comes above 'F', and the right-hand columns change faster than the left-hand ones. For one sentence letter there are two structures, for two sentence letters there are four, for four letters there are sixteen, and so on.

A formula ϕ of L_1 is said to be *defined in* a structure **A** if every constituent sentence letter of ϕ has a truth-value assigned to it by **A**. As we noted, a formula which is defined in **A** has a truth-value determined by **A** together with the truth-tables of section 17; we say that the formula is *true in A* if this truth-value is Truth, and *false in A* if the truth-value is Falsehood.

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The truth-value of a formula in a structure can easily be calculated, by working step by step from shorter subformulae to longer ones. We write 'T' or 'F' as appropriate under each occurrence of a sentence letter, and for each complex subformula we record its truth-value under the truth-functor symbol of largest scope in the subformula. For example, we calculate the truth-value of the formula ' $[P \rightarrow [Q \wedge \neg R]]$ ' in the structure (23.1) as follows. First we write

P	Q	R	$[P \rightarrow [Q \wedge \neg R]]$	23.3
T	F	T	T F T	

The shortest complex subformula is ' $\neg R$ '; since ' R ' is true, the truth-table for ' \neg ' on page 92 shows that ' $\neg R$ ' must be false. We record this:

P	Q	R	$[P \rightarrow [Q \wedge \neg R]]$	23.4
T	F	T	T F FT	

The next shortest subformula is ' $[Q \wedge \neg R]$ '; the truth-table for ' \wedge ' on page 94 shows that this must be false, since its two conjuncts are both false. We record this:

P	Q	R	$[P \rightarrow [Q \wedge \neg R]]$	23.5
T	F	T	T F F FT	

The only remaining subformula is the whole formula itself. Using the truth-table for ' \rightarrow ' on page 96, we record its truth-value Falsehood beneath the truth-functor symbol ' \rightarrow ':

P	Q	R	$[P \rightarrow [Q \wedge \neg R]]$	23.6
T	F	T	T F F F FT	

We can repeat this calculation for each of the structures listed in (23.2); each line of the following table represents one such calculation:

P	Q	R	$[P \rightarrow [Q \wedge \neg R]]$	23.7
T	T	T	T F T F FT	
T	T	F	T T T T TF	
T	F	T	T F F F FT	
T	F	F	T F F F TF	
F	T	T	F T T F FT	
F	T	F	F T T T TF	
F	F	T	F T F F FT	
F	F	F	F T F F TF	

The column written beneath ' \rightarrow ' shows the truth-value of the formula in each structure.

The table (23.7) is known as a *truth-table* for the formula ' $[P \rightarrow [Q \wedge \neg R]]$ '. It tells us the truth-value of this formula in every structure which makes assignments to just the sentence letters used in the formula. If a structure made assignments to some other letters as well, these would play no role in calculating the truth-value of our formula. For this reason, (23.7) in fact tells us the truth-value of the formula in every structure in which it is defined.

Exercise 23A. Give truth-tables for each of the following formulae. (Keep your answers; you will need them for Exercise 23B).

1. $[P \vee \neg P]$
2. $[[P \wedge Q] \vee [P \wedge \neg Q]]$
3. $[[P \rightarrow Q] \rightarrow [P \leftrightarrow Q]]$
4. $[[[P \rightarrow Q] \rightarrow P] \rightarrow P]$
5. $[[P \wedge Q] \wedge [\neg P \vee \neg Q]]$
6. $[[[P \rightarrow Q] \wedge [Q \rightarrow R]] \rightarrow [P \rightarrow R]]$

We can now formalize consistency. If X is a finite set of formulae of L_1 , we write

$X \vDash$ 23.8

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when there is no structure in which all the formulae of X are defined and true. (23.8) is read ' X is *semantically inconsistent*'; X is said to be *semantically consistent* when (23.8) is not true. The symbol ' \vDash ' can be pronounced 'semantic turnstile'.

Similarly we can formalize validity, by using the idea of counterexample sets as in section 11. If X is a finite set of formulae, and ψ a formula, we write

$$X \vDash \psi \qquad 23.9$$

to mean that there is no structure in which ψ and all the formulae of X are defined, and all the formulae of X are true while ψ is false. (23.9) is read ' X *semantically entails* ψ '. Obviously

$$X \vDash \psi \text{ if and only if } X, \neg \psi \vDash. \qquad 23.10$$

If X is empty, (23.9) boils down to

$$\vDash \psi \qquad 23.11$$

which says that ψ is true in every structure in which it is defined; this formalizes the notion of a necessary truth. (23.11) is read ' ψ is a *semantic theorem*' or ' ψ is a *tautology*'.

Expressions of the forms (23.8), (23.9) and (23.11) are known as *semantic sequents*. Observe that semantic sequents are not themselves part of the language L_1 , because ' \vDash ' is not a symbol of L_1 . Rather, semantic sequents make statements about formulae of L_1 . A semantic sequent is correct or incorrect, just as any statement is correct or incorrect. If a semantic sequent is incorrect, then there is a structure in which all the formulae of the sequent are defined, those to the left of ' \vDash ' are true, and any to the right of ' \vDash ' is false; such a structure is said to be a *counterexample* to the sequent. A sequent is correct if and only if it has no counterexample.

Truth-tables provide an easy check of the correctness of sequents. For example we shall test whether or not the following sequent is correct:

$$[P \rightarrow \neg Q], [P \leftrightarrow Q] \vDash \neg Q. \qquad 23.12$$

To do this, we write out simultaneous truth-tables for these formulae. Recording only the columns which show the truth-values of the whole formulae, we have

P	Q	$[P \rightarrow \neg Q]$	$[P \leftrightarrow Q]$	\vDash	$\neg Q$	23.13
T	T	F	T		F	
T	F	T	F		T	
F	T	T	F		F	
F	F	T	T		T	

In the first structure listed in (23.13), ' $[P \rightarrow \neg Q]$ ' is false. ' $[P \leftrightarrow Q]$ ' is false in the second and third, while ' $\neg Q$ ' is true in the fourth. Therefore (23.12) is *correct*.

We can check likewise whether

$$\vDash [P \rightarrow [Q \wedge \neg R]] \qquad 23.14$$

The truth-table for this formula has already been calculated in (23.7). In fact this truth-table shows that the formula is false in the structure

P	Q	R	23.15
T	T	T	

so it is not a tautology, and (23.14) is *incorrect*.

Exercise 23B. Which of the formulae of Exercise 23A are tautologies?

Exercise 23C. Use truth-tables to check which of the following semantic sequents are correct; indicate a counterexample to each incorrect one.

1. $P, [P \rightarrow Q] \vDash$.
2. $P \vDash [Q \rightarrow P]$.
3. $P, \neg P \vDash Q$.
4. $[Q \rightarrow P], [Q \rightarrow \neg P] \vDash \neg Q$.
5. $[P \rightarrow Q], [Q \rightarrow R] \vDash [R \rightarrow P]$.
6. $[R \leftrightarrow [P \vee Q]] \vDash [[R \wedge \neg P] \rightarrow Q]$.

We may summarize the connection between this section and earlier sections as follows.

Suppose a set of English sentences is translated, by means of a suitable interpretation, into a set X of formulae of L_1 . If $X \vDash$, then the set of English sentences is inconsistent. For otherwise the sentences would be true together in some situation, and this situation together with the interpretation would give us a structure in which all the formulae of X are true.

Similarly, an argument which can be symbolized as a correct semantic sequent, with '⊢' for 'Therefore', must be a valid argument; likewise a sentence which can be symbolized as a tautology must be a necessary truth.

The converses fail. Obviously a set of sentences can be inconsistent for reasons which have nothing to do with truth-functores. Therefore the fact that such a set can't be symbolized as a semantically inconsistent set of formulae is no guarantee that the sentences are consistent. We shall return to this matter on page 149.

A List of Tautologies

- 1 $\neg [P \wedge \neg P]$
- 2 $[P \vee \neg P]$
- 3 $[P \leftrightarrow \neg \neg P]$
- 4 $[[P \vee Q] \leftrightarrow \neg [\neg P \wedge \neg Q]]$
- 5 $[[P \vee Q] \leftrightarrow [\neg P \rightarrow Q]]$
- 6 $[[P \vee Q] \leftrightarrow [Q \vee P]]$
- 7 $[[[P \vee Q] \vee R] \leftrightarrow [P \vee [Q \vee R]]]$
- 8 $[[P \vee P] \leftrightarrow P]$
- 9 $[P \rightarrow [P \vee Q]]$
- 10 $[P \rightarrow [Q \vee P]]$
- 11 $[[P \rightarrow R] \rightarrow [[Q \rightarrow R] \rightarrow [[P \vee Q] \rightarrow R]]]$
- 12 $[\neg P \rightarrow [[P \vee Q] \leftrightarrow Q]]$
- 13 $[[P \wedge Q] \leftrightarrow \neg [\neg P \vee \neg Q]]$
- 14 $[[P \wedge Q] \leftrightarrow \neg [P \rightarrow \neg Q]]$
- 15 $[[P \wedge Q] \leftrightarrow [Q \wedge P]]$

- 16 $[[[P \wedge Q] \wedge R] \leftrightarrow [P \wedge [Q \wedge R]]]$
 17 $[[P \wedge P] \leftrightarrow P]$
 18 $[[P \wedge Q] \rightarrow P]$
 19 $[[P \wedge Q] \rightarrow Q]$
 20 $[P \rightarrow [Q \rightarrow [P \wedge Q]]]$
 21 $[[P \rightarrow Q] \rightarrow [[P \rightarrow R] \rightarrow [P \rightarrow [Q \wedge R]]]]$
 22 $[P \rightarrow [[P \wedge Q] \leftrightarrow Q]]$
 23 $[[P \wedge [Q \vee P]] \leftrightarrow P]$
 24 $[[P \vee [Q \wedge P]] \leftrightarrow P]$
 25 $[[P \wedge [Q \vee R]] \leftrightarrow [[P \wedge Q] \vee [P \wedge R]]]$
 26 $[[P \vee [Q \wedge R]] \leftrightarrow [[P \vee Q] \wedge [P \vee R]]]$
 27 $[[[P \vee Q] \wedge \neg P] \rightarrow Q]$
 28 $[P \leftrightarrow [[P \wedge Q] \vee [P \wedge \neg Q]]]$
 29 $[P \leftrightarrow [[P \vee Q] \wedge [P \vee \neg Q]]]$
 30 $[[P \rightarrow Q] \leftrightarrow [\neg P \vee Q]]$
 31 $[[P \rightarrow Q] \leftrightarrow \neg [P \wedge \neg Q]]$
 32 $[P \rightarrow P]$
 33 $[P \rightarrow [Q \rightarrow P]]$
 34 $[[[P \rightarrow Q] \rightarrow P] \rightarrow P]$
 35 $[[P \rightarrow [Q \rightarrow R]] \rightarrow [[P \rightarrow Q] \rightarrow [P \rightarrow R]]]$
 36 $[[P \rightarrow Q] \rightarrow [[Q \rightarrow R] \rightarrow [P \rightarrow R]]]$
 37 $[\neg P \rightarrow [P \rightarrow Q]]$
 38 $[[\neg Q \rightarrow \neg P] \rightarrow [P \rightarrow Q]]$
 39 $[[P \rightarrow [Q \rightarrow R]] \leftrightarrow [[P \wedge Q] \rightarrow R]]$
 40 $[[P \leftrightarrow Q] \leftrightarrow [[P \wedge Q] \vee [\neg P \wedge \neg Q]]]$
 41 $[[P \leftrightarrow Q] \leftrightarrow [\neg [P \wedge \neg Q] \wedge \neg [\neg P \wedge Q]]]$
 42 $[[P \leftrightarrow Q] \leftrightarrow [[P \rightarrow Q] \wedge [Q \rightarrow P]]]$
 43 $[P \leftrightarrow P]$
 44 $[[P \leftrightarrow Q] \rightarrow [P \rightarrow Q]]$
 45 $[[P \leftrightarrow Q] \leftrightarrow [Q \leftrightarrow P]]$
 46 $[[P \leftrightarrow Q] \rightarrow [[Q \leftrightarrow R] \rightarrow [P \leftrightarrow R]]]$
 47 $[[P \leftrightarrow Q] \leftrightarrow [\neg P \leftrightarrow \neg Q]]$
 48 $[[P \leftrightarrow Q] \leftrightarrow \neg [P \leftrightarrow \neg Q]]$

24. Properties of Semantic Entailment[†]

When the shrubs are chopped away, the broader shapes of the landscape begin to appear.

Things which mathematicians prove are called *theorems*. Accordingly we shall list some theorems about semantic entailment and truth-tables, and see how they can be used, and what they tell us about validity of arguments in English.

I. *Extension Theorem*. If X and Y are finite sets of formulae, possibly empty, and ψ is a formula, then

$$\text{if } X \vDash \psi \text{ then } X, Y \vDash \psi. \quad 24.1$$

This is easy to see. If there is no structure in which all the formulae of X are true and ψ is false, then there can be no structure in which all the formulae of X and Y together are true and ψ is false. (The Extension Theorem tells us that a valid argument can't be made invalid by adding new premises – this is the so-called *monotonicity* property of validity.)

II. *Repetition Theorem*. If X is a finite set of formulae and ϕ is a formula in X , then $X \vDash \phi$.

This is obvious too; if every sentence of X is true in some structure A , and ϕ is in X , then ϕ must also be true in A . (As a statement about validity, this tells us that if the conclusion of an argument is one of the premises, then the argument is valid even if useless.)

III. *Cut Theorem*. If X is a finite set of formulae and ϕ and ψ are formulae, then

$$\text{if } X \vDash \phi \text{ and } X, \phi \vDash \psi, \text{ then } X \vDash \psi. \quad 24.2$$

The name refers to the fact that ϕ is 'cut out'. To see that the theorem is true, suppose that $X \vDash \phi$ and $X, \phi \vDash \psi$, and let A be a structure in which ψ is defined and all the formulae of X are

[†]Readers who dislike mathematics may omit this section.

true. We must show that ψ is true in A . ϕ may not be defined in A , but we can always add truth-value assignments to A so as to get a structure B in which ϕ is defined. Since $X \vDash \phi$, ϕ is true in B . Therefore, since $X, \phi \vDash \psi$, the formula ψ must also be true in B . But then ψ is true in A too, since B is only A with pieces added.

IV. *Transitivity of \vDash* . If ϕ , ψ and χ are formulae, then

$$\text{if } \phi \vDash \psi \text{ and } \psi \vDash \chi, \text{ then } \phi \vDash \chi. \quad 24.3$$

Suppose $\phi \vDash \psi$ and $\psi \vDash \chi$. By the Extension Theorem, the latter implies that $\phi, \psi \vDash \chi$; this and the former imply that $\phi \vDash \chi$ by the Cut Theorem.

The next theorem needs some preparation. The idea is that the correctness of a semantic sequent never depends on the exact choice of letters in the sequent; for example we could replace ' P ' by ' Q ' throughout, without destroying correctness.

A *substitution scheme* is defined to be a list of sentence letters, to each of which a formula is assigned. For example, here is a substitution scheme, written as we shall write them:

$$\begin{array}{l} P \mapsto [R \rightarrow Q] \\ R \mapsto Q \end{array} \quad 24.4$$

(24.4) lists the sentence letters ' P ' and ' R '; it assigns ' $[R \rightarrow Q]$ ' to ' P ', and ' Q ' to ' R '.

We apply a substitution scheme to a semantic sequent by taking each occurrence, in any formula of the sequent, of a sentence letter listed in the scheme, and replacing this occurrence by the formula assigned to this letter by the scheme. We do this simultaneously for all such occurrences throughout the semantic sequent. For example, if we apply (24.4) to the sequent

$$[P \rightarrow \neg Q], [P \leftrightarrow Q] \vDash \neg Q. \quad 24.5$$

the result is

$$[[R \rightarrow Q] \rightarrow \neg Q], [[R \rightarrow Q] \leftrightarrow Q] \vDash \neg Q. \quad 24.6$$

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(The second part of (24.4) is not used, since R doesn't occur in (24.5).) When a substitution scheme is applied to a semantic sequent to produce a second semantic sequent, we say that the second sequent is a *substitution instance* of the first; thus (24.6) is a substitution instance of (24.5).

V. *Substitution Theorem.* Every substitution instance of a correct semantic sequent is again correct.

For suppose S_2 is the substitution instance of the sequent S_1 which comes from applying the substitution scheme T . If S_2 is not a correct sequent, then it has a counterexample A . Let B be the structure which is just like A , except that where a sentence letter is listed in T , B assigns to this letter the same truth-value as A assigned to the formula which T allots to the letter. Then B is a counterexample to S_1 , so that S_1 is incorrect too.

For example, (24.5) happens to be a correct sequent, as we proved in (23.13). Therefore, by the Substitution Theorem, (24.6) is also correct. For another example, consider tautology 37 in the list on p. 131:

$$\vdash [\neg P \rightarrow [P \rightarrow Q]] \quad 24.7$$

The Substitution Theorem tells us that ' P ' can be replaced in (24.7) by any formula, and ' Q ' by any formula. Thus the following are also tautologies:

$$\begin{aligned} &[\neg [R \rightarrow Q] \rightarrow [[R \rightarrow Q] \rightarrow [P \vee S]]] && 24.8 \\ &[\neg Q \rightarrow [Q \rightarrow P]] \\ &[\neg Q \rightarrow [Q \rightarrow Q]] \\ &\text{etc.} \end{aligned}$$

Exercise 24A. Each of the following can be shown to be a tautology by applying a suitable substitution scheme to one of the tautologies in the list on pp. 130–31; say which tautology in the list, and give the substitution scheme.

1. $[[Q \rightarrow R] \vee \neg [Q \rightarrow R]]$
 2. $[P \rightarrow [[P \rightarrow P] \rightarrow P]]$
 3. $[[P \rightarrow [[P \rightarrow P] \rightarrow P]] \rightarrow [[P \rightarrow [P \rightarrow P]] \rightarrow [P \rightarrow P]]]$
 4. $[[[Q \leftrightarrow R] \vee [P \wedge R]] \leftrightarrow [[[Q \leftrightarrow R] \vee P] \wedge [[Q \leftrightarrow R] \vee R]]]$
-

At first sight, the Substitution Theorem might seem to suggest that we can take any valid argument, and replace each occurrence of some sentence in the argument by another sentence (the same throughout), without destroying the validity. *This is true for those arguments which can be symbolized as correct semantic sequents of formulae of L_1 .* For example, the valid argument

I am a mole and I live in a hole. *Therefore* I live in a hole. 24.9

can be symbolized as

$$[P \wedge Q] \vDash Q \quad 24.10$$

which is obviously a correct sequent. In this case we can replace 'I live in a hole' by any other declarative sentence, and the result will still be valid:

I am a mole and diamonds consist of carbon. 24.11
Therefore diamonds consist of carbon.

However, there are arguments which are valid for reasons not expressible in L_1 . For these arguments, even a slight substitution may damage the validity beyond repair. For example, here is a valid argument:

Suilven has always been unclimbable. *Therefore* nobody has ever climbed Suilven. 24.12

Slight changes yield the following argument:

Indecent exposure has always been undesirable. *Therefore* nobody has ever desired indecent exposure. 24.13

(24.13) is hardly valid – indeed most people regard its premise as true and its conclusion as false.

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Concentration on examples like (24.9), at the expense of examples like (24.12), has led many people to the quite false conclusion that there is something essentially formal about logic. The truth is rather that the formal parts of logic are formal, and the Substitution Theorem is more a statement about formalization than about logic itself.

We say that the formula ϕ is *logically equivalent* to the formula ψ if $\phi \vDash \psi$ and $\psi \vDash \phi$. The following properties of logical equivalence are obvious, using the Repetition Theorem for VI and transitivity of ' \vDash ' for VIII:

VI. *Reflexiveness of Logical Equivalence.* Every formula is logically equivalent to itself.

VII. *Symmetry of Logical Equivalence.* If a formula ϕ is logically equivalent to a formula ψ , then ψ is logically equivalent to ϕ .

VIII. *Transitivity of Logical Equivalence.* If ϕ , ψ and χ are formulae, ϕ is logically equivalent to ψ and ψ is logically equivalent to χ , then ϕ is logically equivalent to χ .

By the definition, two formulae are logically equivalent if and only if they have the same truth-value in every structure in which they are defined. It follows that if two sentences can be symbolized as logically equivalent formulae, using the same interpretation, then the two sentences are true in exactly the same situations. The two sentences therefore play the same role as each other in any questions of consistency or validity. We should expect two logically equivalent formulae to behave the same way in all matters of logic. The next theorem illustrates this.

IX. *Congruence of Logical Equivalence.* If, in a correct semantic sequent, every formula is replaced by a formula logically equivalent to it, then the resulting sequent is also correct.

This is obvious from the truth-table test of the correctness of sequents. When the formulae are replaced by other formulae with the same truth-tables, the tables themselves are not altered.

For example, any formula logically equivalent to a tautology is itself a tautology.

If ϕ and ψ are formulae, and A is a structure in which both ϕ and ψ are defined, then ' $[\phi \leftrightarrow \psi]$ ' is true in A precisely if ϕ and ψ have the same truth-value in A . This tells us that ϕ and ψ are logically equivalent if and only if

$$\vDash [\phi \leftrightarrow \psi]. \quad 24.14$$

Many of the tautologies listed on pp. 130–31 have the form ' $[\phi \leftrightarrow \psi]$ '; each such tautology provides us with a logical equivalence. For example, by tautology 16,

$$[[P \wedge Q] \wedge R] \text{ is logically equivalent to } [P \wedge [Q \wedge R]]. \quad 24.15$$

By the Substitution Theorem, the letters ' P ', ' Q ' and ' R ' in tautology 16 can be replaced by any other formulae, so that

$$\text{for any formulae } \phi, \psi \text{ and } \chi, '[[\phi \wedge \psi] \wedge \chi]' \text{ is } \quad 24.16 \\ \text{logically equivalent to } '[\phi \wedge [\psi \wedge \chi]]'.$$

X. Equivalence Theorem. If ψ is any formula, ϕ a formula which occurs as a subformula of ψ , ϕ' a formula logically equivalent to ϕ , and ψ' a formula got from ψ by replacing one or more occurrences of ϕ in ψ by ϕ' , then ψ' is logically equivalent to ψ .

This is most easily seen by imagining oneself working out the truth-tables of ψ and ψ' . From a certain point on, the calculations will be identical for the two formulae.

For example, according to tautologies 4, 31 and 41,

$$\begin{aligned} '[\phi \vee \psi]' \text{ is logically equivalent to } & \quad 24.17 \\ '& \neg [\neg \phi \wedge \neg \psi]', \\ '[\phi \rightarrow \psi]' \text{ is logically equivalent to } & \\ '& \neg [\phi \wedge \neg \psi]', \end{aligned}$$

' $[\phi \leftrightarrow \psi]$ ' is logically equivalent to
' $[\neg [\phi \wedge \neg \psi] \wedge \neg [\neg \phi \wedge \psi]]$ '.

Using (24.17) and the Equivalence Theorem, we can turn any formula of L_1 into a logically equivalent formula in which no truth-functor symbols occur except ' \neg ' and ' \wedge '. To illustrate this, we take the formula

$$\neg [P \rightarrow [Q \vee R]]. \quad 24.18$$

Since ' $[Q \vee R]$ ' is logically equivalent to ' $\neg [\neg Q \wedge \neg R]$ ', the Equivalence Theorem tells us that (24.18) is logically equivalent to

$$\neg [P \rightarrow \neg [\neg Q \wedge \neg R]]. \quad 24.19$$

Since ' $[P \rightarrow \neg [\neg Q \wedge \neg R]]$ ' is logically equivalent to ' $\neg [P \wedge \neg \neg [\neg Q \wedge \neg R]]$ ', the Equivalence Theorem tells us that (24.19) is logically equivalent to

$$\neg \neg [P \wedge \neg \neg [\neg Q \wedge \neg R]]. \quad 24.20$$

(24.20) has the desired form. We can improve it by using the fact (tautology 3) that ' $\neg \neg \phi$ ' is logically equivalent to ϕ . Invoking this, we can bring (24.20) to the following logical equivalent of (24.18):

$$[P \wedge [\neg Q \wedge \neg R]]. \quad 24.21$$

Exercise 24B. For each of the following formulae, find a logically equivalent formula in which ' \vee ', ' \rightarrow ' and ' \leftrightarrow ' do not occur.

1. $\neg [P \leftrightarrow Q]$
2. $[[P \vee Q] \vee R]$
3. $\neg [[P \vee Q] \rightarrow P]$

The fact that every formula of L_1 is logically equivalent to one in which ' \vee ', ' \rightarrow ' and ' \leftrightarrow ' don't occur has led some logicians to suggest that these three symbols should be dropped from the language, and only reintroduced as shorthands for the more complicated expressions given in (24.17). This would

simplify the language L_1 . Equally, we could use tautologies 5, 14 and a variant of 42 to drop 'v', '∧' and '↔' in favour of '¬' and '→'.

One might ask how far we can go in the other direction. Can we find new truth-functors which can't be expressed in terms of the present ones? The answer is no.

XI. Functional Completeness Theorem. Any possible truth-functor which could be added to the language L_1 would only yield formulae logically equivalent to formulae which are already in the language.

This is proved as follows. Suppose we have an n -place truth-functor, which we may write $f(\phi_1, \dots, \phi_n)$. Since it is a truth-functor, it has a truth-table:

ϕ_1	...	ϕ_n	$f(\phi_1, \dots, \phi_n)$	24.22
T	...	T	.	
T	...	F	.	
.		.	.	

where the column on the right lists the truth-values in all the structures listed on the left. We must find a formula of L_1 which has precisely the same truth-table as (24.22). We consider the possibilities. If there are no 'T's at all in the truth-table, then the formula ' $[\phi_1 \wedge \neg \phi_1]$ ', which is always false, will serve. If there is precisely one 'T' in the column, then the following formula will serve:

$$[\dots [\psi_1 \wedge \psi_2] \wedge \psi_3] \dots \wedge \psi_n \tag{24.23}$$

where each ψ_i is ϕ_i or ' $\neg \phi_i$ ' according as ϕ_i is true or false in the one structure which makes $f(\phi_1, \dots, \phi_n)$ true. Finally if there is more than one 'T' in the column, then the formula

$$[\dots [\chi_1 \vee \chi_2] \vee \chi_3] \dots \vee \chi_k \tag{24.24}$$

will serve, where χ_1, χ_2, \dots , are respectively the formulae (24.23) corresponding to the structures which make $f(\phi_1, \dots, \phi_n)$ true.

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For example, if a mad scientist invents the new truth-functor ' $[\phi + \psi]$ ', with the truth-table

ϕ	ψ	$[\phi + \psi]$	24.25'
T	T	F	
T	F	T	
F	T	T	
F	F	F	

then we may challenge his originality by producing a formula logically equivalent to ' $[\phi + \psi]$ ', as follows. There are two 'T's in the column on the right of (24.25). The first is against the structure making ϕ true and ψ false; the corresponding formula (24.23) is

$$[\phi \wedge \neg \psi]. \tag{24.26}$$

The second 'T' is against the third structure, which has the corresponding formula

$$[\neg \phi \wedge \psi]. \tag{24.27}$$

Putting these together as in (24.24), we reach the desired formula

$$[[\phi \wedge \neg \psi] \vee [\neg \phi \wedge \psi]]. \tag{24.28}$$

Exercise 24C. Construct a formula logically equivalent to the new formula $f(\phi, \psi, \chi)$ whose truth-table is

ϕ	ψ	χ	$f(\phi, \psi, \chi)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

XII. *Interpolation Theorem.* If ϕ and ψ are formulae such that $\phi \vDash \psi$, and at least one sentence letter occurs in both ϕ and ψ , then there is a formula χ such that

$$\phi \vDash \chi, \quad \chi \vDash \psi \tag{24.29}$$

and every sentence letter in χ is in both ϕ and ψ . (χ is known as an *interpolant* for the sequent ' $\phi \vDash \psi$ '.)

This is a little harder to prove. We say a structure **B** is an *expansion* of **A** if **B** is **A**, or consists of **A** together with assignments to other sentence letters not mentioned in **A**. To prove the Interpolation Theorem, one first lists all the structures which make assignments to just those sentence letters which are in both ϕ and ψ . Against each structure **A** one writes 'T' if **A** has an expansion in which ϕ is true, and 'F' if **A** has an expansion in which ψ is false. (This is possible since $\phi \vDash \psi$.) Any remaining structures are tagged 'T'. Using the procedure we employed to prove XI, one can construct a formula χ which has the truth-table thus defined. χ is the wanted interpolant.

For example,

$$[[P \rightarrow \neg Q] \wedge [Q \rightarrow P]] \vDash [\neg R \rightarrow [Q \leftrightarrow R]] \tag{24.30}$$

is a correct sequent. We write out the truth-tables:

P	Q	$[[P \rightarrow \neg Q] \wedge [Q \rightarrow P]]$	24.31
T	T	F	
T	F	T	
F	T	F	
F	F	T	

Q	R	$[\neg R \rightarrow [Q \leftrightarrow R]]$	24.32
T	T	T	
T	F	F	
F	T	T	
F	F	T	

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There is one sentence letter used in both the formulae, namely 'Q'. According to the second line of (24.32), there is a structure in which 'Q' is true and ' $[\neg R \rightarrow [Q \leftrightarrow R]]$ ' is false. According to the second or the fourth line of (24.31) there is a structure in which 'Q' is false and ' $[[P \rightarrow \neg Q] \wedge [Q \rightarrow P]]$ ' is true. Therefore we set down

Q	*	24.33
T	F	
F	T	

Clearly the needed interpolant is ' $\neg Q$ '.

Exercise 24D. Find an interpolant for the sequent

$$[\neg [P \vee Q] \wedge [P \leftrightarrow R]] \vDash [[R \rightarrow Q] \wedge \neg [S \wedge R]]$$

The Interpolation Theorem says, roughly, that if a valid argument has a single premise which mentions sage but not onions, and a conclusion which mentions onions but not sage, then the argument can be split into two steps, and the half-way stage mentions neither onions nor sage. Perhaps this is obvious; but there is mathematical interest in the fact that if the argument can be symbolized as a correct semantic sequent of formulae of L_1 , then so can the two half-way steps.

25. Formal Tableaux⁺

The next step in formalization is to remove the notion of truth altogether. We have seen that tableaux can be used to test consistency. Our procedure will be to redefine closed tableaux without any reference at all to truth, situations or structures. Once the definition has been given, we may compare closed tableaux with truth-tables and verify that a set of formulae generates a closed tableau if and only if the set is semantically

⁺Readers who dislike mathematics may omit this section.

inconsistent. But the definition itself will be quite independent of section 23.

Accordingly, we now define what we mean by an L_1 -tableau generated by a finite set of formulae of L_1 .

Let X be a finite set of formulae of L_1 . Then by an L_1 -tableau generated by X , we mean an array of formulae of L_1 arranged as an upside-down tree, with some branches possibly closed by having a line drawn below their bottom formula; so that (1) for each occurrence of a formula in the tree, either the formula itself is in X , or the occurrence is derived from another formula occurring higher in the branch according to one of the sentence derivation rules (as listed on pp. 321–2); (2) a branch is closed only if it contains an occurrence of some formula ϕ and an occurrence of its negation ' $\neg \phi$ '; (3) the top part of the tree consists of a list of all the formulae of X .

L_1 -tableaux will be called simply *tableaux* for the rest of section 25. Note that we make no mention of ticks against the formulae; the ticks we used in earlier sections should be regarded as an aid to constructing a tableau, but not as a part of the tableau itself.

A tableau is said to be *closed* if all its branches are closed. If there is a closed tableau generated by the finite set X of formulae, then we say that X is *syntactically inconsistent*, and we express this by writing

$$X \vdash \quad 25.1$$

(' \vdash ' is called the *syntactic entailment* symbol; it may be read 'syntactic turnstile'). By analogy with ' \vdash ', we also write

$$X \vdash \psi \quad 25.2$$

to mean that $X, \neg \psi \vdash$ (where X is a finite set of formulae and ψ is a formula). (25.2) may be read as ' X *syntactically entails* ψ '. When X is empty, (25.2) contracts to

$$\vdash \psi \quad 25.3$$

which may be read ' ψ is a *syntactic theorem*'.

Expressions of the forms (25.1), (25.2) and (25.3) are known as *syntactic sequents*. Like semantic sequents, they are used for

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making statements about formulae of L_1 ; they are not themselves formulae of L_1 .

It can be shown – without appealing to extraneous notions such as truth or consistency – that if there is a closed tableau generated by X , then *every* tableau generated by X can be extended to a closed one. Because of this, we can check the correctness of a syntactic sequent simply by writing out a tableau generated by the appropriate formulae; the sequent is correct if and only if the tableau can be persuaded to close. For example we verify that

$$\vdash [P \rightarrow [Q \rightarrow P]] \quad 25.4$$

as follows, by constructing a closed tableau generated by ' $\neg [P \rightarrow [Q \rightarrow P]]$ ':

$$\begin{array}{l} \neg [P \rightarrow [Q \rightarrow P]] \quad 25.5 \\ | \\ P \\ | \\ \neg [Q \rightarrow P] \\ | \\ Q \\ | \\ \neg P \\ \hline \end{array}$$

On the other hand we may show that

$$\text{it is not correct that } \vdash [P \rightarrow [P \rightarrow Q]] \quad 25.6$$

as follows:

$$\begin{array}{l} \neg [P \rightarrow [P \rightarrow Q]] \quad 25.7 \\ | \\ P \\ | \\ \neg [P \rightarrow Q] \\ | \\ P \\ | \\ \neg Q \end{array}$$

There is no way of closing (25.7) or extending it with new formulae. This proves (25.6).

Exercise 25. Show that tautologies 9, 11, 16, 26 and 48 in the list on pp. 130–31 are syntactic theorems.

We shall now prove that a syntactic sequent is correct if and only if the corresponding semantic sequent is correct. Since

$$X \vDash \psi \text{ if and only if } X, \neg \psi \vDash \quad 25.8$$

and likewise

$$X \vdash \psi \text{ if and only if } X, \neg \psi \vdash \quad 25.9$$

we only need to prove that a finite set of formulae is syntactically inconsistent precisely if it is semantically inconsistent.

Accordingly we begin by proving that for every finite set X of formulae of L_1 ,

$$\text{if } X \vDash \text{ then } X \vdash. \quad 25.10$$

(This is one of a cluster of theorems which go by the name of *Completeness Theorem*; the drift of all these theorems is that semantic inconsistencies can be detected by methods which were defined independently of the notion of truth.) We shall prove (25.10) by assuming that it is not true that $X \vdash$, and deducing that it is not true that $X \vDash$.

Assume then that X is a finite set of formulae of L_1 , and it is not true that $X \vdash$. We can construct a tableau generated by X . Since no tableau generated by X is closed, there will certainly be a branch in our tableau which is not closed, and we can try to extend this branch downwards by applying the derivation rules and closing branches. Now as we keep closing branches and applying the derivation rules, we reach shorter and shorter formulae. This can't go on for ever; so we must eventually reach a point where any branch which can be closed is closed, and any formula which is added to the tableau by a derivation rule must already have occurred in the branch to which it is added. At this point we say the tableau is *finished*.

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There must then be a finished but not closed tableau generated by X . We take such a tableau, and in it we choose a branch which is not closed. We construct a structure A as follows: A makes assignments to precisely those sentence letters which occur in formulae in the tree, and it assigns Truth to any sentence letter ϕ precisely if ϕ occurs as a formula in the chosen branch. We claim that every formula in the branch is true in A . This claim is proved by starting with the sentence letters and working up through longer and longer formulae. For example, if ' $\neg P$ ' occurs in the branch, then ' P ' certainly doesn't occur in it, since otherwise the branch would have been closed; therefore ' P ' is false in A , and so ' $\neg P$ ' is true in A by the truth-table for ' \neg '. As another example, suppose ' $[P \rightarrow Q]$ ' occurs in the branch; then since the tableau is finished, the derivation rule for such formulae shows that either ' $\neg P$ ' or ' Q ' must occur in the branch. These formulae are shorter than ' $[P \rightarrow Q]$ ', so we may assume that if either of them is in the branch, it must be true in A . If ' $\neg P$ ' is true in A , then ' P ' is false in A , and so ' $[P \rightarrow Q]$ ' is true in A by the truth-table for ' \rightarrow '; similarly if ' Q ' is true in A then ' $[P \rightarrow Q]$ ' is true in A . Either way, the formula ' $[P \rightarrow Q]$ ' is true in A . The arguments for the other types of formulae are similar.

We therefore have a tableau generated by X , a structure A , and a branch of the tableau which consists of formulae true in A . But the formulae of X are listed at the top of the tableau, so that they are in every branch. Hence all the formulae of X are true in A . This shows that it is not true that $X \vDash$. We have thus proved (25.10).

The whole proof above is simply a mathematical description of the way we used unclosed branches in section 10 to show that a set of sentences was consistent.

The converse of (25.10) must now be proved: if X is a finite set of formulae of L_1 , then

$$\text{if } X \vDash \text{ then } X \vDash. \quad 25.11$$

To prove this, we assume that X is semantically consistent, and we use this fact to enable us to find an unclosed branch in

any tableau generated by X . The method is as follows. Since X is semantically consistent, there is a structure A in which all the formulae of X are defined and true. We shall try to walk downward through the tableau, starting at the top, so that every formula we pass through is true in A . When we reach the bottom edge of the tableau, we shall have paced out a branch consisting of formulae which are all true in A . If a formula ϕ is in the branch, then ϕ is true in A , so ' $\neg \phi$ ' is not true in A , and hence is not in the branch. Therefore the branch is not closed.

It has to be checked that we can always keep going down the tableau, taking appropriate roads at the branching-points, until we reach the bottom edge of the tree. The first part of the journey is easy: at the top of the tree is a list of the formulae in X , and we know these are all true in A . From the bottom of the list onwards, we have to examine the next derivation rule which is used in the tree. For example, suppose the formulae traversed so far constitute a set Y , which is all true in A , Y includes the formula ' $\neg [\phi \rightarrow \psi]$ ', and the next step in the tableau is to apply the derivation rule for this formula; then the next formulae in the tableau are ϕ and ' $\neg \psi$ '. But if ' $\neg [\phi \rightarrow \psi]$ ' is in Y , then it's true in A , and so by the truth-table for ' \rightarrow ', ϕ is true and ' $\neg \psi$ ' is false in A . We can therefore extend our downward journey to take in these two formulae. For another example, suppose the set Y of formulae traversed so far contains the formula ' $[\phi \vee \psi]$ ', and the derivation rule for this formula is the next one to be applied. Then since ' $[\phi \vee \psi]$ ' is in Y , it must be true in A , and so by the truth-table for ' \vee ', at least one of ϕ and ψ is true in A . We may therefore continue our journey to take in the appropriate one of these two formulae.

This proves (25.11). (25.10) and (25.11) together show that semantic entailment and syntactic entailment amount to the same thing, although they were defined quite differently. With this our formalization is complete.

It must be admitted that there is an easier way to formalize this part of logic. One can simply regard truth-tables themselves as meaningless arrays of symbols constructed according to certain formal rules. There are several reasons why logicians

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prefer to formalize in terms of tableaux or some similar proof-calculus. One reason is that tableaux can be adapted to other parts of logic where the analogues of truth-tables can no longer be formalized. (We shall return to this in section 41.)

Some logicians prefer to avoid truth-tables for another reason which has nothing to do with formalization. According to these logicians, the notion of truth leads to too many philosophical perplexities – we have seen a few in sections 5–8. Instead, they say, we should take as basic the notions of *proof* or *entailment*, and build our logical apparatus on these. This is broadly the position of the Dutch school of *Intuitionists*, who reject truth-tables as a general method of logic, and accept tableaux only in a modified form.

Designators and Identity

There are inconsistent sets of sentences whose inconsistency can't be proved by symbolizing them and then applying sentence tableaux; likewise there are valid arguments whose validity is not demonstrable by this method. One example is:

The boy's father is liable for the damage caused. I am the boy's father. *Therefore* I am liable for the damage caused.

When we have analysed how a sentence may be built up from constituents such as nouns and verbs, we shall be able to see what rules govern the validity of arguments such as the example above. In due course these rules will be added to the rules of sentence tableaux.

26. Designators and Predicates

The time has come to look closer inside sentences, and find constituents which are smaller than sentences.

In section 5 we saw that the truth-value of a sentence in a situation depends closely on the references of parts of the sentence. There are certain types of phrase which are specially suitable for referring to things; we shall pick these phrases out and call them *designators*.

Four kinds of phrase will count as designators. These are proper names, non-count nouns, singular personal pronouns and definite descriptions – we shall define them in a moment. It may seem arbitrary to select just these four kinds of phrase; in fact the idea is simply to pick out a class which will suit our purposes in later sections.

The fourth type of designator is *definite descriptions*. These are singular noun phrases beginning with any one of

the	my	Henry's	26.9
this	your	Birmingham's	
that	his	Tuesday's	
	her	(etc.)	
	its		
	our		
	their		

Examples are:

the largest marrow of the season	26.10
this bottle of distilled water	
that thing you were telling me about Mary my aunt in Surbiton	
your lumbago	
his performance of Schubert's last sonata	
Birmingham's drainage system	
Odysseus' homecoming	

Phrases beginning with words from the second or third columns of (26.9) could all be regarded as short for phrases beginning with **the**:

*the aunt of me in Surbiton	26.11
*the lumbago of you	
the homecoming of Odysseus	

Phrases of these four types are commonly used to name things. They have something else in common: they are all noun phrases. This is not an accident. It will be highly convenient to have all designators in one phrase-class, and there are at least two reasons why the class of noun phrases is the right one to seek them in. The first reason is that all proper names are noun phrases. (This seems to be a pure accident of language. One can easily imagine a dialect in which all proper names are adjectives: the Hodges household, yon Birminghamly town.) The second reason is that English is very adept at making

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noun phrases out of other kinds of phrase. For instance you may say, in a smelly place,

It reeks horribly in here. 26.12

The word 'reeks' refers to the smell, although reeks is not a designator. But you could have expressed exactly the same thought by turning 'reeks' into a definite description:

The reek in here is horrible. 26.13

What kinds of noun phrase have we excluded from being designators? There are two main kinds. First, we have excluded plural noun phrases, such as **Sacco and Vanzetti** or **my knees**. The objection to these is that they are normally used to indicate more than one thing at once.

Second, we have excluded singular noun phrases beginning with any of the following:

some	any	every	no	26.14
something	anything	everything	nothing	
somebody	anybody	everybody	nobody	
someone	anyone	everyone	no one	
each	a	much	either	
all	one	little	neither	

Thus we have excluded

every dog 26.15
anything you say
each time he knocks
all the world
a stitch
some mustard
much ado
one way to cure wind
no self-respecting citizen

Some of these phrases could be used in settings where one might reasonably say they referred to something. But usually

they serve a quite different purpose, as we shall see in section 34 below. For example, consider

I have never, in spite of much searching, 26.16
found one way to cure wind.

Obviously the bold phrase is not referring to one way to cure wind!

Exercise 26. Find all the designators which occur as constituents in the following sentences; say which of the four types each designator belongs to.

1. Ordinarily I think he does this simply because the ox, which has suffered from parasites throughout the day, gets relief from its back being rubbed, but I was told that he may occasionally as he does so speak a few words to God or to the ghosts.
2. To this rule, Dr Jekyll was no exception; and as he now sat on the opposite side of the fire – a large, well-made, smooth-faced man of fifty, with something of a slyish cast perhaps, but every mark of capacity and kindness – you could see by his looks that he cherished for Mr Utterson a sincere and warm affection.

Here is a sentence in which two designators occur as constituents; the designators are bracketed:

[Jeeves] poured [the sherry]. 26.17

We can analyse such a sentence into its constituent designators

Jeeves, the sherry 26.18

together with the matrix which contained them:

x poured y . 26.19

The letters ' x ' and ' y ' which occur in (26.19) are called *individual variables*. They mark the holes where the constituent designators should go; in the terminology of page 87, the

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occurrences of these variables in (26.19) are *free*. The letters 'z', ' x_1 ', ' x_2 ', etc., will also be used as individual variables.

The matrix (26.19) is an example of a *predicate*. More precisely, a *predicate* is defined to be a string of English words and individual variables, such that if the individual variables are replaced by appropriate designators, then the whole becomes a declarative sentence with these designators as constituents.

Here are some predicates. As you read them you should think of appropriate designators for them; remember that you only have to make a declarative sentence, not necessarily a true one.

x loves a bit of night-life.	26.20
Angela was somewhat lacking in y .	26.21
Thiamine pyrophosphate is required for the prevention of x in birds.	26.22
x loves y .	26.23
James Thurber wrote x while he was staying at y .	26.24
x_1 added to x_2 makes x_3 .	26.25
x gave it to z , who promptly gave it back to x .	26.26

Notice that we don't expect every choice of designators to turn a predicate into a declarative sentence. Inappropriate designators can easily lead to selection mistakes, as in

?The sherry poured Jeeves.	26.27
?Angela was somewhat lacking in John.	26.28

In later sections we shall be obliged to give truth-values to sentences such as (26.27) and (26.28); we shall take them to be false.

Predicates are classified by the number of different individual variables which have free occurrences in them. (Occurrences which are not free will occur for the first time in section 35; we may ignore them for the moment.) Thus (26.20)–(26.22) are *1-place* predicates. (26.19), (26.23) and (26.24) are *2-place*, while (26.25) is *3-place*.

Note that (26.26) is *2-place*, since only two individual variables have free occurrences in it, though one occurs twice. When an individual variable has two or more free occurrences in a predicate, this is understood to mean that the individual variable must be replaced by *the same designator at each free occurrence*. For example the holes in (26.26) may be filled to produce

The plumber gave it to the clerk, who promptly gave it back to the plumber. 26.29

They cannot be filled to produce

The plumber gave it to the clerk, who promptly gave it back to Father McCloskey. 26.30

Cross-referencing by pronouns (see p. 28) can often have the same effect as repetition of variables. For instance, if we replace the second occurrence of 'x' in (26.26) by 'him', we have the predicate

x gave it to y, who promptly gave it back to him. 26.31

If, as in (26.29), we now replace 'x' in (26.31) by 'the plumber' and 'y' by 'the clerk', then we reach the sentence

The plumber gave it to the clerk, who promptly gave it back to him. 26.32

(26.32) is not quite a paraphrase of (26.29), because there are female plumbers; but the difference in meaning is slight.

It's often convenient to think of declarative sentences as being 0-place predicates. In line with this idea, we shall sometimes use sentence variables to stand for predicates as well as for declarative sentences.

For convenience we shall also allow ourselves to say simply that a designator *occurs* in a sentence, when we mean that it *occurs as a constituent*.

The word *predicate* is often used by grammarians and philosophers in ways which are at variance with the definition

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we have given. For example, some people use the word to mean property or quality.

27. Purely Referential Occurrences⁺

Most designators can be used on their own, and not as part of a sentence. Thus we find the label **Turps substitute** on a bottle, the tag **J. Smith** on a coat, the title **The accumulation of capital** on a book, or the sign announcing **Anne Hathaway's cottage**. When a designator can be used *on its own* in a situation, so as to refer to something, we call that thing the *primary reference* of the designator.

One might think that designators have a straightforward role to play in language: namely to refer to their primary references. One might think, for example, that

the Queen 27.1

refers to the Queen, whether it occurs on its own or in a sentence. But not so. For instance, the historian G.M. Trevelyan wrote in 1922 that

The Queen had in 1840 married Prince Albert 27.2
of Saxe-Coburg-Gotha.

Queen Victoria, to whom Trevelyan's sentence referred, was dead in 1901. She is not now the primary reference of the designator (27.1), nor was she when Trevelyan wrote. The primary reference of (27.1), now and in England, is Queen Elizabeth II.

In fact there are many roles a designator may play in a sentence, besides referring to its primary reference. Sometimes, as in (27.2), it refers to something else; sometimes it doesn't refer to anything at all. We shall want to distinguish those cases where a designator does straightforwardly refer to its primary

⁺ The topics in this section and the next, though important, are rather subtle. You may find it best to read these two sections fairly fast, and come back to them again later.

reference. This demands a test. The test we shall use is by no means perfect, because it sometimes gives rather obscure results; but it will point in the right direction.

The test is as follows. Suppose a designator *D* occurs in a sentence. We try to rewrite the sentence in the form

D is a person (thing) who (which, such that) 27.3
 . . . he (she, it) . . .

with the designator brought out to the front. If there is such a sentence (27.3) which expresses the same as the original sentence, granted the assumption that *D* has a primary reference, then we say that the occurrence of *D* in the original sentence is *purely referential*. The purpose of this test is to bring the designator to a position where it must refer to its primary reference.

For example, the sentence

Mr Hashimoto is Japanese. 27.4

can be paraphrased as

Mr Hashimoto is a person who is Japanese. 27.5

Therefore the occurrence of 'Mr Hashimoto' in (27.4) is purely referential. This fits our intuition that in (27.4) this designator serves only to refer to its primary reference, namely Mr Hashimoto.

Similarly, the sentence

Mr Hashimoto is the Managing Director of 27.6
 Kiki Products Inc.

can be paraphrased (not very beautifully) as

The Managing Director of Kiki Products 27.7
 Inc. is a person such that Mr Hashimoto is
 him.

This shows that the occurrence of 'the Managing Director of Kiki Products Inc.' in (27.6) is purely referential.

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For a third example, consider the sentence

It isn't true that Mr Hashimoto is divorced. 27.8

This doesn't mean the same as

Mr Hashimoto is a person such that it isn't true that he is divorced. 27.9

because if there is no such person as Mr Hashimoto, then (27.8) is true while (27.9) is false. But if we take it as granted that there is such a person as Mr Hashimoto, then (27.8) and (27.9) do express the same thing. Therefore the occurrence of 'Mr Hashimoto' in (27.8) is purely referential.

Here are some examples of occurrences which are not purely referential. Such occurrences fall broadly into four main groups.

(i) *Temporal contexts.*

Here the past or future reference of a phrase is relevant. An example is

It's unprecedented for the Managing Director of Kiki Products to be Japanese. 27.10

Any attempt to paraphrase this as

The Managing Director of Kiki Products is a person such that it's unprecedented for him to be Japanese. 27.11

would be most unhappy; the present Managing Director maybe always was Japanese, long before he took this post on. To make (27.11) mean the same as (27.10), we should have to invent a fictitious person, who is now the present Managing Director, was the previous Managing Director, and will eventually become the next Managing Director. No such shifty figure is involved in (27.10). Thus the occurrence of 'the Managing Director of Kiki Products' in (27.10) is not purely referential.

(ii) Modal contexts.

These talk about possibilities or necessities. An example is

Mr Hashimoto can't help being the Managing Director. 27.12

There is no fair paraphrase of (27.12) along the lines of

The Managing Director is a person such that Mr Hashimoto can't help being him. 27.13

(OR: The Managing Director is a person whom Mr Hashimoto can't help being.)

For instance, the Managing Director actually is Mr Hashimoto, and obviously Mr Hashimoto can't help being himself, so (27.13) is true; but Mr Hashimoto could and should have resigned years ago, so (27.12) is false. Thus the occurrence of 'the Managing Director' in (27.12) is not purely referential.

Another example, involving a subjunctive conditional, is:

If the Managing Director had been chosen by ballot, he would have been some flashy whizz-kid. 27.14

The sentence

The Managing Director is a person who, if he had been chosen by ballot, would have been some flashy whizz-kid. 27.15

is hopeless as a paraphrase of (27.14). Mr Hashimoto is far too sage and unassuming to be a flashy whizz-kid, even if he had been chosen by ballot.

(iii) Intentional contexts.

These talk about beliefs, hopes, knowledge, and other facets of people's minds. (The name is from the Latin *intentio animae*, 'mental concept'.) An example is:

Smith never realized that Mr Hashimoto was the Managing Director. 27.16

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The attempted paraphrase

The Managing Director is a person such that Smith never realized that Mr Hashimoto was him. 27.17

fails, because it implies that Smith failed to realize that Mr Hashimoto was Mr Hashimoto. Thus the occurrence of 'the Managing Director' in (27.16) is not purely referential.

(iv) *Quotational contexts.*

These talk about the actual words used by some speaker or writer. An example is:

Mr Hashimoto likes to be referred to as the Managing Director. 27.18

The sentence

?The Managing Director is a person such that Mr Hashimoto likes to be referred to as him. 27.19

is of doubtful grammaticality. If it means anything at all, it hardly means the same as (27.18). Hence again, the occurrence of 'the Managing Director' in (27.18) is not purely referential.

Sometimes an ambiguity prevents us from saying whether an occurrence of a designator in a sentence is purely referential. For example, a woman who has divorced her first husband and then remarried may say

My husband used to play in the town band. 27.20

The primary reference of her phrase 'my husband' is her present husband. If she means him, then the occurrence of this phrase in (27.20) is purely referential. If she means her previous husband, the occurrence is not purely referential. Like all structural ambiguities, this ambiguity can be overcome by rewriting the sentence – for example, with the words **my present husband** or **my then husband**.

Exercise 27. In each of the following sentences, is the occurrence of the bold phrase a purely referential occurrence? If it is not, classify the context as temporal, modal, intentional or quotational.

1. We suspect Abe is her latest lover.
2. If Abe is her latest lover, she's in trouble.
3. They accused Abe of being her latest lover.
4. Abe may be her only lover.
5. Abe will soon be her only lover.
6. Abe is her latest lover.
7. Cyanide is composed of carbon and nitrogen.
8. Siam changed its name to Thailand.
9. George wants to be the first person to swim the Atlantic.
10. It's impossible to see Oxford from Bletchley.
11. When I was at school, the Bishop of Sodor and Man came and preached us a sermon.

When an occurrence of a designator is not purely referential, what is its function? We can begin to answer this if we follow the four-part classification given above.

In temporal contexts, the designator points to what was or will be its primary reference in one or more past or future situations. In modal contexts, the topic is the primary reference which the designator has in some hypothetical situations. To use the language of section 8, these two types of context serve as *situation-shifters*.

In quotational contexts, a designator usually refers not to its primary reference, but to itself – or sometimes to itself as written or spoken in some particular way. This is clearest in such examples as

THE SACRED RIVER is printed in capitals. 27.21

For the moment we must leave on one side the intentional contexts; neither of the above treatments will account for them satisfactorily. We shall return to them, tentatively, in section 44.

28. Two Policies on Reference

Referential failure makes problems for logic; so do designator occurrences which are not purely referential. To leave a problem unsolved is hurtful. But there is a limit to the number of questions we can tackle at once, and frankly these problems about reference are not the most pressing. We shall therefore pass a Self-Denying Ordinance, and restrict ourselves (at least until the end of section 41) to arguments and sets of sentences which do not raise these problems.

Specifically, we shall make two assumptions.

First Assumption. We shall assume that if a designator has a purely referential occurrence in a sentence (from an argument or a set of sentences under discussion), then *the designator is understood to have a primary reference.*

For example, when the set of sentences or the argument under review contains a sentence such as

Cinnamic acid is made by Perkins's reaction. 28.1

we shall leave out of account any possible situations in which there is no such thing as cinnamic acid. The effect is just the same as if we had added the sentence

There is such a thing as cinnamic acid. 28.2

to the set of sentences, or to the premises of the argument. (On the other hand we would allow situations in which there is no Perkins. Perkins's reaction is fairly regarded as an unanalysable proper name, which applies to the reaction in question, whether or not there ever was such a person as Perkins.)

The restriction imposed by this First Assumption is not very arduous. Poets and St Anselm apart, people don't often use designators in purely referential occurrences, unless they take it as understood that the designators do have something to refer to.

Here is an example of the kind of argument which our First Assumption forbids us to consider:

Irene never had any daughters. *Therefore* it's not true that Sigmund is the father of Irene's eldest daughter. 28.3

Here the point at issue is whether or not the designator Irene's eldest daughter does have a primary reference. We make utter nonsense of the argument if we begin by assuming that it does.

The argument (28.3) is clearly valid. Logicians have various ways of taking it into account. One approach is to paraphrase the conclusion in such a way that the offending designator no longer occurs; we shall describe how in section 38.

Second Assumption. We shall assume that every occurrence of a designator in a set of sentences or an argument is purely referential. We can allow exceptions where a designator is obviously irrelevant to questions of consistency or validity.

For example, we shall leave out of account such arguments as

I once dreamed I was the mean annual rainfall in Mombasa. *Therefore* dreams do not consist of assorted memories. 28.4

The occurrence of 'the mean annual rainfall in Mombasa' in the premise is not purely referential. (The mean annual rainfall in Mombasa is forty-seven inches; I never dreamed I was forty-seven inches.) The argument (28.4) seems valid, though this would need careful justification.

Probably no general approach will codify more than a fraction of the valid arguments in which designators have occurrences which are not purely referential. We shall reconsider this question in sections 42-4.

29. Identity

One particularly important predicate is the 2-place predicate

x_1 is one and the same thing as x_2 . 29.1

This predicate is called *identity*; in symbols it's written ' $x_1 = x_2$ ', and the symbol '=' is read 'equals'. A sentence got by

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putting designators in place of ' x_1 ' and ' x_2 ' in (29.1) is called an *equation*.

Various English phrases can be paraphrased by means of identity. For example:

Everest is the highest mountain in the world. 29.2

Everest = the highest mountain in the world.

Cassius Clay and Muhammad Ali are the same person. 29.3

Cassius Clay = Muhammad Ali.

This is none other than the lost city. 29.4

This = the lost city.

Two plus two equals four. 29.5

Two plus two = four.

The word *identical* is normally used in English to express close similarity rather than identity. For example, identical twins are not one and the same twin, and two women who are wearing identical dresses are not wearing one and the same dress.

Everything equals itself. This undeniable truth is sometimes known as the *Law of Identity*; according to taste, it is either the supreme metaphysical truth or the utmost banality. Since it is true always and everywhere, we can't deny it with consistency; any set of sentences which does deny it must be inconsistent.

From the Law of Identity we deduce the *Identity Rule*: if \bar{D} is a designator, then any set of sentences containing the sentence

' $\neg D = D$ ' 29.6

is *inconsistent*.

The Identity Rule is based on our First Assumption from section 28, as follows. The designator D has a purely referential occurrence in (29.6), so it is assumed to have a primary reference. The Law of Identity says that this primary reference is equal to itself, so that the sentence ' $D = D$ ' must be true, contradicting (29.6).

The First Assumption tacitly adds

‘There is such a thing as D.’ 29.7

to the set of sentences. Without the assumption, the Identity Rule would actually be incorrect. For example, in a situation where there is no devil, the sentence

It’s not true that the devil = the devil 29.8

is true; so in the absence of our assumption, (29.8) is entirely consistent.

There is a second law governing identity: if $b = c$, then anything which is true of b is also true of c . This obvious truth is known as *Leibniz’s Law*, in honour of the seventeenth-century logician and mathematician G. W. Leibniz, who made a study of identity.

Leibniz’s Law tells us that certain arguments are valid. The arguments in question can be described as follows. *Let D and E be designators, and ϕ a declarative sentence, and suppose ψ is got from ϕ by replacing one or more occurrences of D in ϕ by occurrences of E. Then both the arguments*

ϕ , ‘D = E’. Therefore ψ . 29.9

ϕ , ‘E = D’. Therefore ψ . 29.10

are valid. We may call this *Leibniz’s Rule*.

For example, let D be ‘Mr Helly’, let E be ‘the treasurer’, and let ϕ be

Mr Helly has lost the petty cash. 29.11

Then ψ must be

The treasurer has lost the petty cash. 29.12

and (29.9) is illustrated by the argument

Mr Helly has lost the petty cash. Mr Helly 29.13
is the treasurer. Therefore the treasurer has
lost the petty cash.

This argument is obviously valid.

Leibniz's Rule rests on our Second Assumption from section 28. Suppose S is a situation in which the two premises of (29.9) are true; then since 'D = E' is true in S, these two designators must have the same primary reference in S. Since the occurrences of D and E in ϕ and ψ respectively are assumed to be purely referential, ϕ must say the same thing about the primary reference of D as ψ says about the primary reference of E. These are one and the same primary reference; so by Leibniz's Law, if ϕ is true in S, then so is ψ .

What would happen if we dropped the Second Assumption? The following argument will demonstrate; it is invalid, although it looks very much like (29.13):

Hamish can name Hamish's brother. 29.14
 Hamish's brother is the man in the distance.
Therefore Hamish can name the man in the distance.

(Say, Hamish lives with his brother Tavish, but at this moment Tavish is impossible to recognize in the gloomy mountain mist.) In the conclusion of (29.14), the occurrence of the designator 'the man in the distance' is not purely referential. In fact it is clear that

Hamish can name the man in the distance. 29.15

can't be paraphrased as

The man in the distance is a person whom 29.16
 Hamish can name.

since (29.15) is false and (29.16) is true in the situation we imagined. (29.15) could more fairly be paraphrased as

Hamish can answer the question 'Who is the 29.17
 man in the distance?'

which is a quotational context – as the quotation marks show. The Second Assumption is violated in (29.14), and so the application of Leibniz's Rule is fallacious.

(29.14) might well be called the *fallacy of education* – the

assumption that students who have learned something under classroom conditions will recognize the same thing when they meet it again in the hustle of the outside world.

Similar fallacious applications of Leibniz's Rule have sometimes been used to discredit Leibniz's Law. Often the fallacy is hard to winkle out. For example, Figure 2 is a picture of me as

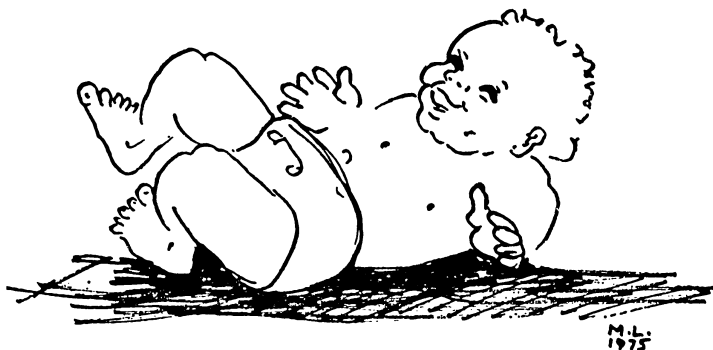


Figure 2

I was a few decades ago. The baby in the picture is me. Nevertheless the baby has no beard, whereas I have a beard. This seems to contradict Leibniz's Rule. Further reflection shows that this paradox is just the result of careless statement. The person portrayed in the picture really *does now* have a beard, because it's me; however, he *did not* have a beard at the stage of his life which the picture portrays. Straighten the tenses and the problem disappears.

Exercise 29. Use Leibniz's Rule to find valid arguments with the following as their premises. In some cases there are two or more correct answers: give as many as you can.

1. Six is greater than five. Three plus three equals six.
2. I am thy father's spirit. I scent the morning air.
3. Parasurama is Rama. Rama is Krishna.
4. Rama is Parasurama. Rama is Krishna.
5. The daughters of Lear disown Lear. Lear is the king.

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6. Errol owns the gun that fired this bullet. This bullet is the bullet that killed Cheryl. The gun that fired the bullet that killed Cheryl is the murder weapon.
 7. June is the third month after March. March is the ninth month after June.
 8. The Son suffers. The Father is the same God as the Son.
-

Relations

There are many valid arguments which revolve around such phrases as **longer than** or **the same length as**. To bring these arguments under control, we need to find an appropriate analogue of truth-tables. Such an analogue exists in *relations*, which serve to tell us when we get a true sentence by filling the blank spaces in a predicate. Relations are more complicated than truth-tables, mainly because there are only two truth-values for a sentence to take, while there are indefinitely many different things available to be named by designators.

30. Satisfaction

As we saw in section 16, a truth-table tells us what sentences we can put into the holes of a truth-functor, so as to make it into a true sentence. There was no need to list the constituent sentences themselves – it sufficed to say what their truth-values must be. In the same way we now seek to determine what designators can be put for the individual variables of a predicate, so as to produce a true sentence. It would be pleasant if the answer could be given in terms of the primary references of the designators, and we shall see that by and large this is possible.

The key notion is that of *satisfaction*, which is a relationship between things and predicates. Since the full definition of satisfaction is indigestible, we shall postpone it and begin with some typical examples instead.

Consider the 1-place predicate

x is at least two years old.

30.1

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Things which are at least two years old are said to *satisfy* this predicate. Naturally this depends on the situation. A thing starts by being no years old, and failing to satisfy (30.1). In later situations it becomes one year old, and still fails to satisfy (30.1). Only in those situations where the thing is two years old or more does it satisfy (30.1).

Similarly, in any situation, a thing is said to *satisfy* the 1-place predicate

Angela was somewhat lacking in y . 30.2

if, in the given situation, it is true that Angela was somewhat lacking in the thing. For example, the situation might be that you are making a report about Angela's performance in a business negotiation. Suppose that Angela showed enthusiasm but no tact during the venture. Then in this situation, tact satisfies (30.2) but enthusiasm fails to satisfy it.

You can now answer Exercise 30A by analogy with (30.1) and (30.2).

Exercise 30A. The following chart claims to record the number of houses built per year per thousand of population, in each country and each year shown:

	1954	1960	1966
U.S.A.	9	7	6
U.S.S.R.	7	12	10
Sweden	8	9	12

Which of the three countries satisfied which of the following predicates in 1960?

1. x builds at least ten houses per year per thousand of population.
2. Six years ago, x built fewer than nine houses per year per thousand of population.
3. In six years' time, x will be building more than ten houses per year per thousand of population.

There is a similar notion for 2-or-more-place predicates, but now it becomes important to check that the right objects are steered to the right holes in the predicate. For this a convention is needed, and we shall adopt the following one. If n is a number greater than 1, and ϕ is an n -place predicate, then we do not define satisfaction of ϕ unless the individual variables in ϕ are numbered ' x_1 ', ' x_2 ', ..., ' x_n '. A list of n things T_1, T_2, \dots, T_n will be written thus:

$$\langle T_1, T_2, \dots, T_n \rangle \quad 30.3$$

and described as an *ordered n-tuple*; the things T_1, T_2 , etc., are described as the *terms* of this ordered n -tuple. Ordered 2-tuples $\langle T_1, T_2 \rangle$ are known as *ordered pairs*.

Consider the 2-place predicate

$$x_1 \text{ is larger than } x_2. \quad 30.4$$

An ordered pair is said to *satisfy* (30.4) if its first term is larger than its second term. For example the following ordered pairs satisfy it (in the present situation):

$$\begin{aligned} &\langle \text{Canada, Luxemburg} \rangle, && 30.5 \\ &\langle \text{Los Angeles, Oslo} \rangle, \\ &\langle \text{the standard metre, the standard yard} \rangle, \\ &\langle 50, 4\frac{1}{2} \rangle. \end{aligned}$$

The following ordered pairs do not satisfy it (again in the present situation):

$$\begin{aligned} &\langle \text{Canada, Canada} \rangle, && 30.6 \\ &\langle \text{Luxemburg, Canada} \rangle, \\ &\langle \text{Oslo, Los Angeles} \rangle, \\ &\langle \text{the standard yard, the standard metre} \rangle, \\ &\langle 4\frac{1}{2}, 50 \rangle \end{aligned}$$

Similarly, consider the 3-place predicate

$$x_1 \text{ is composed of } x_2 \text{ and } x_3. \quad 30.7$$

An ordered 3-tuple is said to *satisfy* (30.7) if its first term is composed of its second term and its third term. Thus

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<water, hydrogen, oxygen> 30.8
<salt, sodium, chlorine>

both satisfy (30.7) (in the present situation), whereas

<bread, sweat, sorrow> 30.9

does not.

Exercise 30B. There are nine different ordered pairs whose terms are taken from the set of countries

U.S.A., U.S.S.R., Sweden

Write down all nine.

Exercise 30C. According to the chart in Exercise 30A, which ordered pairs of countries satisfied which of the following predicates in 1960?

1. x_1 builds more houses per year per thousand of population than x_2 does.
 2. Six years ago, x_1 built at least as many houses per year per thousand of population as x_2 did.
 3. x_1 will build fewer houses per year per thousand of population in six years' time than x_2 did six years ago.
-

A collection of things or people which is under discussion – for any reason – will be called a *domain*; the people or things in the domain will be called *individuals* (as in ‘individual variable’). A collection of ordered n -tuples of individuals is called an *n -place relation* in the domain. For example, in Exercise 30C there was a domain consisting of the three countries U.S.A., U.S.S.R. and Sweden, and the answers to the three parts of Exercise 30C were 2-place relations in this domain. 2-place relations are commonly called *binary relations*.

It often proves convenient to regard a single individual as an ordered 1-tuple, so that a collection of individuals counts as a 1-place relation. The answers to the three parts of Exercise 30A

were 1-place relations. (1-place relations are also known as *classes*.)

There are two main ways to describe a relation. The first is by listing the n-tuples which are in it.

For example, suppose we have invited Anne, Brenda, Clothilde, Ashok, Brian and Carl to a party, and we want to arrange them in pairs for a game. Then the domain consists of these six people. To pair them off, we can write down a collection of ordered pairs, say:

$$\{\langle \text{Anne, Brian} \rangle, \langle \text{Brenda, Ashok} \rangle, \langle \text{Clothilde, Carl} \rangle\}. \quad 30.10$$

(30.10) is a binary relation. (It's normal to write curly brackets '{', '}' at the ends when a relation is specified by listing the ordered n-tuples in it.) The order in which we write the three ordered pairs of (30.10) is not significant; in fact (30.10) is just the same binary relation as

$$\{\langle \text{Clothilde, Carl} \rangle, \langle \text{Brenda, Ashok} \rangle, \langle \text{Anne, Brian} \rangle\}. \quad 30.11$$

However, (30.12) below is a different relation, because the individuals are changed around within the ordered pairs:

$$\{\langle \text{Brian, Anne} \rangle, \langle \text{Ashok, Brenda} \rangle, \langle \text{Carl, Clothilde} \rangle\}. \quad 30.12$$

Exercise 30D. Let the domain consist of the numbers 1 and 2; there are then four ordered pairs of individuals, namely $\langle 1, 1 \rangle$, $\langle 1, 2 \rangle$, $\langle 2, 1 \rangle$, and $\langle 2, 2 \rangle$. There are sixteen different binary relations in this domain; write down all of them. (They include the *empty* relation $\{ \}$, which has no ordered pairs in it.)

The second way to describe an n-place relation is by naming an n-place predicate ϕ and a situation S; the relation is to consist of all the ordered n-tuples of individuals which satisfy ϕ in A. The relation is said to be *expressed* by the predicate in the situation.

For example, let the situation be as things actually are, and let the domain be the collection of all towns in the world. The predicate

$$x_1 \text{ is further east than } x_2. \quad 30.13$$

expresses a binary relation in this situation; the binary relation contains such ordered pairs as

$$\begin{aligned} &\langle \text{New Delhi, Boston} \rangle, \langle \text{Moscow, Paris} \rangle, & 30.14 \\ &\langle \text{Kyoto, Nairobi} \rangle. \end{aligned}$$

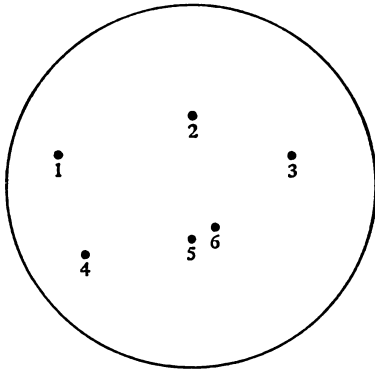
Of course it contains an extremely large number of other ordered pairs too. This illustrates one of the main advantages of describing a relation by a predicate which expresses it: many relations are far too large to be listed in full.

Before we close this section, we should attempt a proper definition of satisfaction. The following will serve. Suppose ϕ is an n -place predicate in which the individual variables ' x_1 ', ..., ' x_n ' have free occurrences. Then the ordered n -tuple $\langle T_1, \dots, T_n \rangle$ is said to *satisfy* ϕ in a situation S if it is possible to turn ϕ into a declarative sentence which is true in S , by replacing each free occurrence of ' x_1 ' by a purely referential occurrence of a designator whose primary reference in S is T_1 , and similarly with ' x_2 ' and T_2 , etc., to ' x_n ' and T_n . The condition that the occurrences of designators should be purely referential is needed to ensure that the truth-value of the whole sentence depends only on the individuals T_1, \dots, T_n , and not on the designators which are chosen to name them.

31. Binary Relations

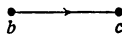
Binary relations are so many and so diverse that one might despair of finding any pattern in them. We shall not despair. In this section we shall describe some of the simpler patterns one might hope to find among binary relations. We shall then see, in section 32, that these simple patterns do emerge in the relations expressed by a wide range of 2-place predicates.

There is a handy way to picture binary relations. Suppose we have a domain D and a binary relation R in D . Then we draw one dot for each individual in D , and we enclose all these dots in a curve to represent D itself. For example, if D is the class of integers from 1 to 6 inclusive, then we draw



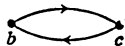
31.1

Now for each ordered pair $\langle b, c \rangle$ in R we draw an arrow



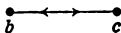
31.2

from b 's dot to c 's dot. If there are arrows going both ways:



31.3

we conflate them into one double arrow:



31.4

If R contains an ordered pair $\langle b, b \rangle$ whose first and second terms are the same individual, then there is an arrow from b to b , and conversely an arrow from b to b . So we draw a double arrow from b to itself (or rather from b 's dot to itself):



31.5

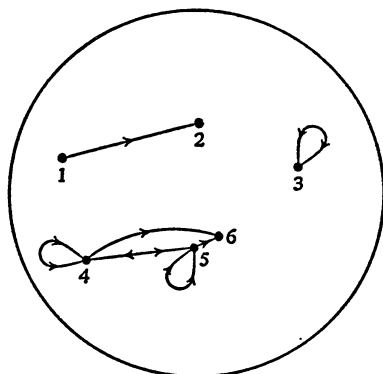
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This configuration (31.5) is called a *loop*. The whole diagram, loops and arrows and all, is called a *graph* of R .

To continue the example (31.1), suppose D is as before, and R is the relation

$$\{\langle 1, 2 \rangle, \langle 3, 3 \rangle, \langle 4, 4 \rangle, \langle 4, 5 \rangle, \langle 4, 6 \rangle, \langle 5, 4 \rangle, \langle 5, 5 \rangle, \langle 5, 6 \rangle\}. \quad 31.6$$

Then this is a graph of R :



31.7

Of course this way of picturing binary relations is only feasible if D is finite, and fairly small at that. But if D is infinite, we can still imagine how an old Norse god with time on his hands might construct an infinite graph. So we have some excuse for classifying all binary relations in terms of their graphs.

(i) A binary relation is called

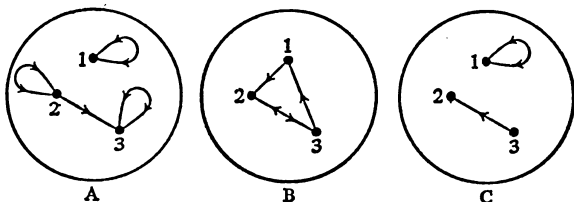
- reflexive* if every dot in its graph has a loop attached;
- irreflexive* if no dot in its graph has a loop attached;
- non-reflexive* if it's neither reflexive nor irreflexive.

For example, let the domain consist of the numbers 1, 2 and 3, and consider these three relations and their graphs:

$$A = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\};$$

$$B = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle\};$$

$$C = \{\langle 1,1 \rangle, \langle 3,2 \rangle\}.$$



31.8

Here A is reflexive, B is irreflexive and C is non-reflexive.

We may apply this classification to predicates. For example, picking a word at random from the dictionary, we may consider the relation expressed by the predicate

$$x_1 \text{ dispurveys } x_2. \quad 31.9$$

(in a given situation and domain). Regardless of what *dispurveys* means, this relation is reflexive if every individual *dispurveys* itself, and irreflexive if no individual *dispurveys* itself.

Exercise 31A. Let the domain be the class of all people now living, and let the situation be the present. Consider the binary relations expressed by the following predicates, and classify them as reflexive, irreflexive or non-reflexive:

1. x_1 is the same height as x_2 .
2. x_1 is x_2 's father.
3. x_1 laughs at x_2 's jokes.
4. x_1 is taller than x_2 .
5. x_1 is no taller than x_2 .
6. x_1 is a conservationist, and so is x_2 .

(ii) A binary relation is called

$\left\{ \begin{array}{l} \textit{symmetric} \text{ if no arrow in its graph is single;} \\ \textit{asymmetric} \text{ if no arrow in its graph is} \\ \quad \text{double;} \\ \textit{non-symmetric} \text{ if it's neither symmetric nor} \\ \quad \text{asymmetric.} \end{array} \right.$

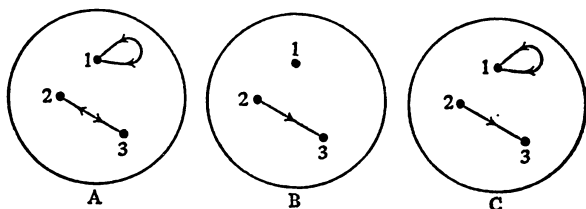
LOGIC

For example, let the domain consist of the numbers 1, 2 and 3, and consider these three relations and their graphs:

$$A = \{\langle 1, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle\};$$

$$B = \{\langle 2, 3 \rangle\};$$

$$C = \{\langle 1, 1 \rangle, \langle 2, 3 \rangle\}.$$



Then A is symmetric, B is asymmetric, and C is non-symmetric.

The relation expressed by (31.9) is symmetric if whenever an individual b dispurveys an individual c , then it is also true that c dispurveys b . The relation is asymmetric if whenever b dispurveys c , it is false that c dispurveys b .

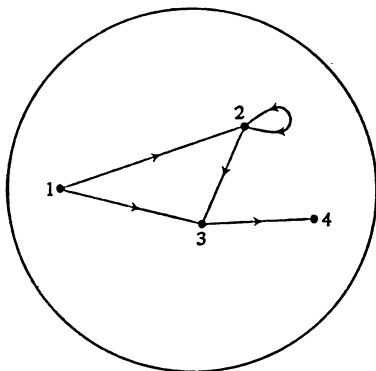
Exercise 31B. Let the domain and situation be as in Exercise 31A. Consider the binary relations expressed by the following predicates, and classify them as symmetric, asymmetric or non-symmetric:

1. x_1 is younger than x_2 .
2. x_1 is married to x_2 .
3. x_1 is x_2 's wife.
4. x_1 was x_2 's best man.
5. x_1 and x_2 went to the same school.
6. x_1 is no younger than x_2 .

Before we give our third classification, we must examine graphs once more. Suppose that there are dots representing b , c and d (not necessarily distinct individuals). We shall say there is a *broken journey* from b to d via c if there is an arrow from b 's dot to c 's dot, and an arrow from c 's dot to d 's dot. If there is a broken journey from b to d , we shall say this broken

journey has a *short cut* if there is also an arrow direct from b 's dot to d 's dot. For example in the graph

31.11



there are just six broken journeys:

from 1 to 2 via 2	}	all with short cuts
from 1 to 3 via 2		
from 2 to 2 via 2		
from 2 to 3 via 2	}	both without short cuts
from 1 to 4 via 3		
from 2 to 4 via 3		

(iii) A binary relation is called

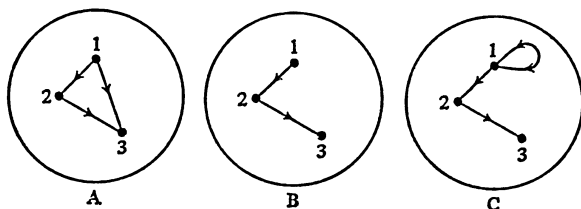
{	<i>transitive</i> if its graph contains no broken journey without a short cut;
	<i>intransitive</i> if its graph contains no broken journey with a short cut;
	<i>non-transitive</i> if it's neither transitive nor intransitive.

For example, let the domain consist of the numbers 1, 2 and 3, and consider these three relations and their graphs:

$$A = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle\};$$

$$B = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle\};$$

$$C = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle\}.$$



A is transitive, B is intransitive, C is non-transitive.

The relation expressed by (31.9) is transitive if whenever b , c and d are individuals such that b dispurveys c and c dispurveys d , then b dispurveys d . The relation is intransitive if whenever b , c and d are individuals such that b dispurveys c and c dispurveys d , then b doesn't dispurvey d .

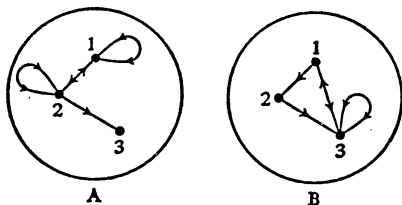
Exercise 31C. Let the domain and situation be as in Exercise 31A. Consider the binary relations expressed by the following predicates, and classify them as transitive, intransitive or non-transitive:

1. x_1 knows x_2 .
2. x_1 is descended from x_2 .
3. x_1 is the father of x_2 .
4. x_1 is no taller than x_2 .
5. x_1 is married to x_2 .
6. x_1 is a parent of x_2 .

(iv) A binary relation is called *connected* if in its graph, any two dots are connected by an arrow in one direction or the other (or both). For example, let the domain consist of the numbers 1, 2 and 3, and consider these two relations and their graphs:

$$A = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle\};$$

$$B = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 3 \rangle\}.$$



A is not connected, B is connected.

The relation expressed by (31.9) is connected if given any two different individuals, at least one of them dispurveys the other.

Exercise 31D. Let the domain and situation be as in Exercise 31A. Consider the binary relations expressed by the following predicates, and classify them as connected or not connected:

1. x_1 has the same surname as x_2 .
2. x_1 is not the same person as x_2 .
3. x_1 is no older than x_2 .
4. x_1 is a farmer, while x_2 is not.

32. Same, at least and more

In this section we shall examine some kinds of predicate that have a logic of their own. Most adults find this logic all rather obvious once it is pointed out; but Jean Piaget and his collaborators have shown convincingly that the ideas involved are not usually understood by children until age eleven or older.

(i) Same

Here are some typical 2-place predicates using the word same:

x_1 is the same height as x_2	32.1
x_1 was born in the same town as x_2 .	32.2
x_1 has exactly the same I.Q. as x_2 .	32.3
x_1 has the same number of pages as x_2 .	32.4
x_1 is one and the same thing as x_2 . ($x_1 = x_2$.)	32.5

We shall refer to predicates like (32.1)–(32.5) as *sameness* predicates. In any situation, a sameness predicate carries with it a natural domain: the natural domain for (32.1) is the class of things or people which have a height, the natural domain for (32.4) is the class of books (i.e. things which have a certain number of pages), and so on. The natural domain for (32.5) is everything.

Every sameness predicate expresses a binary relation in its natural domain, in any situation. *The relation is always reflexive, symmetric and transitive*, apart from some few exceptions which we shall come to in a moment.

For example, if Chen was born in a town, then he was born in the same town as himself. If Chen was born in the same town as Ping, then Ping was born in the same town as Chen. If Chen was born in the same town as Ping, and Ping was born in the same town as Tsu, then Chen was born in the same town as Tsu. These sentences are true in any possible situation in which Chen, Ping and Tsu can be referred to – they are necessary truths (granted the existence of these people).

Exercise 32A. *n*-Propyl alcohol (NPA) has the same empirical formula as *iso*-propyl alcohol (IPA). *Iso*-Propyl alcohol has the same empirical formula as methyl ethyl ether (MEE). What seven other facts of the same kind can you deduce immediately from these, without having to know any organic chemistry?

The exceptions we referred to are cases such as

x_1 went to the same school as x_2 .	32.6
x_1 uses the same bank as x_2 .	32.7

A person can have been to two different schools, and this allows (32.6) to express a relation which isn't transitive. If Jenny went with Albert to the Bellevue Infant School when she was five, and later went to the Crackworth High School with Roderick when she was fourteen, then we can't deduce that Albert went to the same school as Roderick, even though Albert went to the same school as Jenny and Jenny went to the same school as Roderick. In the same way a man can use two banks. This

difficulty doesn't arise with most sameness predicates. For example, nobody has two different I.Q.s (unless something has gone wrong with the test).

(ii) At least as, at most as

x_1 is at least as red as x_2 .	32.8
x_1 has at least as many children as x_2 .	32.9
The half-life of x_1 is at most as long as the half-life of x_2 .	32.10

We shall call predicates such as (32.8)–(32.10) *reflexive comparative* predicates. Like sameness predicates, they each have in any situation a natural domain; the domain for (32.8) consists of those things which it makes sense to describe as red; the domain for (32.10) consists of radioactive elements (i.e., things that have a half-life). The domain for (32.9) consists of those things or people that have some number of children – the number can be none.

Every reflexive comparative predicate expresses a binary relation in its natural domain, in any situation. *This relation is always reflexive, transitive and connected.*

For example, Pavel has at least as many children as Pavel has. If Pavel has at least as many children as Bengt, and Bengt has at least as many children as Giuseppe, then Pavel has at least as many children as Giuseppe. If Pavel and Giuseppe are two people, then either Pavel has at least as many children as Giuseppe, or Giuseppe has at least as many children as Pavel.

If we state a reflexive comparative predicate both ways round, the expressed relation is still reflexive and transitive; *but it becomes symmetric too.* For example, to symmetrize (32.8) we write

x_1 is at least as red as x_2 , and x_2 is at least as red as x_1 .	32.11
---	-------

Likewise for (32.9):

x_1 has at least as many children as x_2 , and x_2 has at least as many children as x_1 .	32.12
---	-------

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These both-ways predicates can be shortened by using **exactly**:

x_1 is exactly as red as x_2 . 32.13

x_1 has exactly as many children as x_2 . 32.14

(32.13) and (32.14) can also be paraphrased as sameness predicates, by using words such as **number**, **amount** or **degree**:

x_1 has the same degree of redness as x_2 . 32.15

x_1 has the same number of children as x_2 . 32.16

Is this a general pattern? Can the symmetrized version of any reflexive comparative predicate be paraphrased as a sameness predicate? This would be surprising, because different nouns have to be used in different cases; the following would both be wrong:

* x_1 has the same number of redness as x_2 . 32.17

* x_1 has the same degree of children as x_2 . 32.18

We shall come back to this in section 33.

Exercise 32B. Try to find sameness predicates which paraphrase the symmetrized versions of the following reflexive comparative predicates:

1. x_1 is at least as small as x_2 .
2. x_1 is at least as young as x_2 .
3. x_1 is at most as far as x_2 .
4. x_1 does at least as much as x_2 .
5. x_1 breaks at least as often as x_2 .
6. x_1 's father is at least as affable as x_2 's father.

(iii) more, less, -er

x_1 is cooler than x_2 . 32.19

x_1 has more children than x_2 . 32.20

x_1 's bite is less dangerous than x_2 's bite. 32.21

We shall call predicates such as (32.19)–(32.21) *irreflexive comparative* predicates. Each has a natural domain in any situation, just as reflexive comparative predicates. In this domain

it expresses a relation which is always *irreflexive, asymmetric and transitive*.

For example, the bite of the black widow spider (if it has one) is not less dangerous than the bite of the black widow. If the bite of the black widow is less dangerous than the bite of the tsetse fly, then the bite of the tsetse fly is not less dangerous than the bite of the black widow. If the bite of the black widow is less dangerous than the bite of the tsetse fly, and the bite of the tsetse fly is less dangerous than that of the lion, then the bite of the black widow is less dangerous than that of the lion.

Irreflexive comparative predicates are simply the back halves of reflexive comparative predicates, in the following sense. If we add **not** in a suitable way, we can always turn a predicate of the one sort into something which can be paraphrased as a predicate of the other sort. For example:

x_1 is not at least as red as x_2 . 32.22

x_1 is less red than x_2 .

x_1 is not at most as far as x_2 . 32.23

x_1 is further than x_2 .

x_1 is no cooler than x_2 . 32.24

x_1 is at most as cool as x_2 .

x_1 's bite is not less dangerous than x_2 's bite. 32.25

x_1 's bite is at least as dangerous as x_2 's bite.

There are several other connections between these types of predicate, as the following exercise illustrates.

Exercise 32C. Paraphrase each of the following predicates as a predicate of the type stated:

- x_1 is at least as greedy as x_2 but x_2 is not at least as greedy as x_1 . (*as an irreflexive comparative predicate*)
- x_1 has at most as many antennae as x_2 , but not the same number of antennae as x_2 . (*as an irreflexive comparative predicate*)
- x_1 is either knobblier than x_2 , or precisely as knobblily as x_2 . (*as a reflexive comparative predicate*)

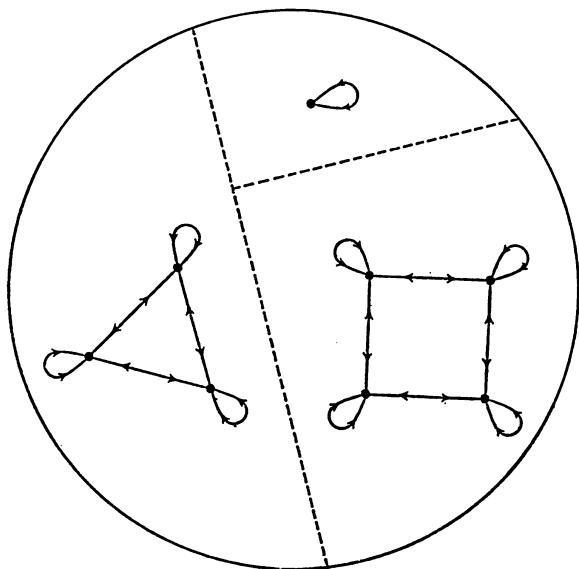
4. x_1 is neither hotter than x_2 , nor less hot than x_2 . (as a same-ness predicate)
-

33. Equivalence Relations

In this one section we shall intrude on the corner of logic known as the *theory of definition*, which studies the ways in which one concept can go proxy for others. I wish there were space for more; this is an unjustly neglected field.

A relation which is reflexive, symmetric and transitive (in a given domain) is called an *equivalence relation*.

An equivalence relation is easily recognized from its graph: the domain is split up into parts, and there is an arrow from b to c precisely when b and c are in the same part. (33.1) is an example:



33.1

The parts are known as the *equivalence classes* of the relation; thus (33.1) has three equivalence classes.

In section 32 we saw that every sameness predicate expresses an equivalence relation (in a given situation). For example the predicate

x_1 has the same number of children as x_2 33.2

expresses an equivalence relation. Two people are in the same equivalence class of this relation precisely if they have the same number of children. There is one equivalence class of people who have no children, another equivalence class of people who have just one child, and so on.

The converse is also true: not only does every sameness predicate express an equivalence relation, but *every equivalence relation is expressed by some sameness predicate*.

For suppose R is an equivalence relation; to express R, we simply write the predicate

x_1 has the same equivalence class of R as x_2 . 33.3

(33.3) is a sameness predicate, and clearly it expresses R.

All this might seem a triviality, but in fact it is not. It tells us that we can introduce a certain kind of new noun into the language. If ϕ is any 2-place predicate which expresses an equivalence relation, then we may introduce into the language a new noun N, which can be taken to mean

equivalence class of the relation expressed 33.4
by ϕ .

Then in place of ϕ , we may say simply

‘ x_1 has the same N as x_2 .’ 33.5

To describe this process, we may call N an *abstraction* of ϕ .

For example, let ϕ be the predicate

x_1 's mother was the same age when x_1 was 33.6
born as x_2 's mother was when x_2 was born.

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This predicate expresses an equivalence relation in the domain of all human beings. Choosing an N to correspond, say *squirch*, we may translate (33.6) into

$$x_1 \text{ has the same } \textit{squirch} \text{ as } x_2. \quad 33.7$$

Then *squirch* is an abstraction of the predicate (33.6). (Serious scientists would doubtless invent some more sober term, such as *maternal birth-age*.)

What do we gain by being able to abstract in this way?

Besides the easy point that N may be much shorter and simpler than ϕ was, the gains are two. First, it is obvious at a glance that (33.5) and (33.7) are sameness predicates, and hence that they express equivalence relations. This can serve as a mental guide in areas where arguments matter, for example in science. It may be much less obvious that ϕ itself expresses an equivalence relation.

For example, let ϕ be the predicate

$$x_1 \text{ is } x_2, \text{ or could be exchanged for } x_2 \text{ in the} \quad 33.8 \\ \text{open market.}$$

Under certain assumptions, which economists study, (33.8) expresses an equivalence relation. We may therefore abstract it to form the noun *exchange-value*. It's obvious that if *b* has the same exchange-value as *c*, and *c* has the same exchange-value as *d*, then *b* has the same exchange-value as *d*. The corresponding statement of transitivity using (33.8) is much less obvious.

The second gain is that other jobs may lie waiting for the new noun phrase N. For example, consider the predicate

$$x_1 \text{ holds at least as much as } x_2. \quad 33.9$$

This is a reflexive comparative predicate. Its symmetrized version

$$x_1 \text{ holds exactly as much as } x_2. \quad 33.10$$

expresses an equivalence relation. There's no need to invent an

abstraction for (33.10); one already exists, namely **capacity**. But the word **capacity** is not only used to form the predicate

x_1 has the same capacity as x_2 . 33.11

It also goes to work in such contexts as

This jug has a greater **capacity** than that one. 33.12

We want to increase the **capacity** of the stores. 33.13

The **capacity** of the tube is 5 c.c. 33.14

Note that each of these uses of **capacity** is closely related to the predicate (33.9) which first gave rise to the equivalence relation; the further uses of an abstraction noun rarely wander far from home. One doesn't debate whether a capacity is fluffy, or how old it is.

Incidentally this last example answers a question we posed on p. 184: can the symmetrized version of a reflexive comparative predicate always be paraphrased as a sameness predicate? The answer is that it can, because it expresses an equivalence relation, but a new noun may need to be invented for the purpose. We may well marvel that English has not yet evolved a uniform way of forming the required abstraction.

Science is riddled with abstractions of predicates which express equivalence relations. One example is **hardness** of minerals, as defined by the mineralogist F. Mohs: two minerals are said to have the same **hardness** if neither will scratch the other. Further examples are **species**, **blood-group**, **genotype**, **personality**, **temperature** and **electrovalency**, to name a few.

There are dangers in abstraction. Gobbledygook is not the least. A more insidious danger is that the abstraction may be applied to things outside the original equivalence relation. For example, we noted that (33.8) expresses an equivalence relation under certain assumptions: these assumptions have to do with such things as the rationality of the merchants and the marketability of the goods. Even in a situation where the merchants are rational, highly perishable goods and goods of sentimental value may have to be excluded from the domain if (33.8) is to

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express an equivalence relation. If these goods are not in the domain, they do not have an exchange-value, and we land ourselves in contradiction if we pretend that they do have one. To insist on putting money values on everything is not only bad for the soul; it is also bad logic.

Quantifiers

According to the story, there was a man with a headache, who saw the advertisement

NOTHING ACTS FASTER THAN *****

– so at once he went and took nothing. This man failed to understand *quantifiers*. In the next few sections we hope to set him right.

34. Quantification

To *quantify* a predicate is to alter it so as to form a declarative sentence or predicate in which fewer variables have free occurrences.

In English, the simplest way to quantify a predicate is to put a noun phrase in place of one of its individual variables. The noun phrase need not be a designator. In fact it can even be a plural phrase; but in this case we usually have to make some other adjustments to the predicate.

For example, the predicate

x is Japanese. 34.1

can be quantified to form

Somebody is Japanese. 34.2

Anyone who lives in Shimonoseki is Japanese. 34.3

Nothing marked 'Made in Britain' is Japanese. 34.4

One of the best contemporary poets is Japanese. 34.5

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Most of the people who live in Kyoto are Japanese.	34.6
Three of these books are Japanese.	34.7
A few of the characters are Japanese.	34.8

Comparing (34.2)–(34.8), we can see that the added noun phrases all have a common purpose. Their purpose is to tell us something about *how many things of a certain type do or do not satisfy the predicate* (34.1). For example, in (34.5) we are told that at least one outstanding contemporary poet satisfies it; in (34.3) we are told that there are no people living in Shimonoseki who fail to satisfy it; and similarly with the rest.

Quantification is a complex matter. Piaget found that children are not usually able to handle even simple examples correctly before age seven. To understand a sentence in which quantification occurs, we normally have to appreciate three things, namely the *profile* of the quantification, the *domain of quantification* and the *degree of exaggeration*. We shall discuss these in turn, starting with the profile.

Most English noun phrases can be written schematically in some such form as

the S, the Ss, an S, several Ss, a few Ss, half of the Ss, thousands of Ss, no S, (etc.)	34.9
---	------

For example anyone who lives in Shimonoseki can be written as any S, with 'person who lives in Shimonoseki' for S. Likewise something can be written as some S, with 'thing' for S.

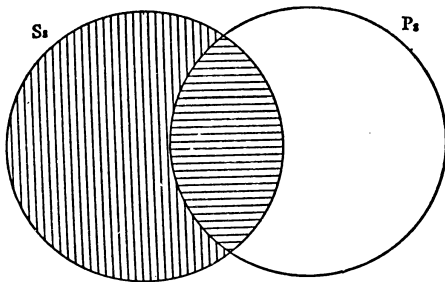
We shall take as a schematic 1-place predicate:

x is a P.	34.10
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If we put one of our schematic noun phrases (34.9) into the predicate (34.10), we get a sentence

the S is a P, the Ss are Ps, an S is a P, (etc.)	34.11
--	-------

The sentence (34.11) then tells us something about the numbers of Ss which are or are not Ps (the hatched areas in (34.12)):



The type of information given about these numbers is called the *profile* of the quantification; it depends chiefly on the form of the noun phrase as shown in (34.9). In English there are four main types of profile.

In the first type of profile, the sentence (34.11) tells us only about how many Ss *are* Ps (the horizontal hatching in (34.12)), but nothing about the Ss which are not Ps. The main examples are

- | | | | | |
|-------------|-----------------|----------------|--------|-------|
| no Ss | | | | 34.13 |
| just one S | at least one S | at most one S | one S | |
| just two Ss | at least two Ss | at most two Ss | two Ss | |
| (etc.) | (etc.) | (etc.) | (etc.) | |
- some Ss, several Ss, many Ss, lots of Ss, a few Ss,
 some of the Ss, several of the Ss, (etc.)

Two Ss is ambiguous; sometimes it's used to mean just two Ss, and sometimes at least two Ss.

In the second type of profile, the sentence (34.11) tells us only about how many Ss *are not* Ps (the vertical hatching in (34.12)), and nothing about the Ss which are Ps. For example,

All Boy Scouts are honest. 34.14

tells us how many dishonest Boy Scouts there are – i.e. none. But it tells us nothing about how many honest Boy Scouts there are. (See section 7; we are assuming the weak reading.) Here are some other phrases which are used to indicate that no Ss are not Ps:

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every S, each S, all the Ss, every one of the Ss, 34.15
each of the Ss, the Ss, any S.

Some other noun phrases with profiles of this second type are
all but one (two, three, etc.) of the Ss. 34.16

In the third type of profile, the sentence (34.11) tells us *what proportion* of the Ss are Ps (relating the horizontal to the vertical hatching in (34.12)). Examples are:

half the Ss	most Ss	34.17
two thirds of the Ss	nearly all Ss	
a quarter of the Ss	few Ss	
(etc.)	a majority of Ss	

Note that a few Ss had the first type of profile, unlike few Ss.

The fourth type of profile is shown by definite descriptions, which are normally used only in situations where there is just one S. Thus it is wrong to say

John is the tallest boy in the class. 34.18

if there are two other boys exactly as tall as John in the class, even if all the rest of the boys are shorter.

Exercise 34. Both the following kinds of phrase occur sometimes with a profile of the first type, and sometimes with one of the second type. Give examples to show this.

1. An S.
2. Ss.

We turn to the *domain of quantification*, and we begin with an example. Imagine this situation: the fog at London Airport is heavy and is not expected to lift for twenty-four hours. All planes have been grounded, and passengers with urgent business are advised to fly from Manchester instead. The staff at London Airport have put up notices which say

ALL FLIGHTS ARE CANCELLED 34.19

In this situation, (34.19) is true. But why is it true, if in fact planes are flying from Manchester? Don't flights from Manchester count?

The answer is that they don't, for purposes of a notice such as (34.19) in London Airport. 'All flights' in this notice refers only to flights from London Airport, and flights from any other airport simply do not count. The flights which count as relevant to the truth of (34.19) are said to be *in the domain of quantification*; those which don't count are not.

For another example, consider the headmaster who is showing off his school to a visitor; he says

Most of the boys study Latin. 34.20

The only boys who count as relevant are the boys in his school. These boys are in the domain of quantification; the boys in the Secondary School down the road are not.

In both these examples we can say that certain things definitely are in the domain of quantification, while certain other things definitely are not. For all the rest of the things in the world, it is quite arbitrary whether we choose to put them in the domain or not. In example (34.19) we can put the airport clocks into the domain or leave them out as we please, since they make no difference to the truth of (34.19) in the given situation. In example (34.20) we can toss a coin over the question whether the headmaster's dog is in the domain; it is not a boy, so it has no effect on the truth-value of the given sentence. Because the domain of quantification is arbitrary to this extent, we have to admit that it is something of an artificial construct.

The domain of quantification depends on the situation; it also depends on the precise wording of the noun phrase. To see this, contrast the following two sentences:

Each boy has a favourite teacher. 34.21

All boys have a favourite teacher. 34.22

One would use (34.21) when a particular group of boys was under discussion; only these boys are in the domain of quantification. With (34.22) on the other hand, all boys whatever must

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be included in the domain of quantification, even if the topic of conversation is one small group of boys. The sentence

Every boy has a favourite teacher. 34.23

leaves it in doubt whether the domain is large or small – the context must decide.

The word **the** is often used to contract the domain to a group of people or things which have been already mentioned. This is the role of **the** in the sentence

All the boys have a favourite teacher. 34.24

which could be used to mean the same as (34.21). To contract the domain still further, one deploys phrases such as **all these boys** or **all those boys**.

Finally we must consider *exaggeration*; this is something which chiefly infects phrases such as (34.15), with the second type of profile. In daily discourse, everybody uses these phrases in ways which he doesn't expect to be taken literally; since everybody does it, nobody is deceived.

For example, it is common practice to say 'every S', when one means 'every S, leaving aside some exceptions which make no difference to the point at issue':

Everybody has heard of Mao Tse-Tung. 34.25

(Infants? recluses?) Likewise one says 'Ss' when one means 'normal Ss', or even 'typical Ss':

Sparrows have a cheerful grey crown. 34.26

(Even filthy ones?)

Working-class women are less interested in sport than their men-folk. 34.27

(All of them??)

Some writers on logical themes have taken a curiously strong moral stand against this kind of exaggeration. For example, Susan Stebbing, complaining about the exaggerated use of **everybody**, writes:

There are serious dangers in indulging in such a habit . . . It encourages us to turn aside from contrary evidence, to oversimplify important issues, to attribute to other people an unwarranted extension of what they have been asserting.†

She goes on to argue that this type of verbal usage is a threat to political moderation. It seems to me true that politicians are particularly prone to exaggerated quantifications; they do often make broad statements which ignore the exceptions. But unlike Miss Stebbing, I believe it is entirely proper for them to do so. Men in practical affairs need to have a strong sense of what is important, and they need to be able to direct their influence to the major problem of the moment. This is the difference between practical people and pedants.

35. All and some

Logicians cultivate three main methods of quantification. The first is to put a designator in place of a variable; this is called *instantiation*. The remaining two are called *universal quantification* and *existential quantification*. They are both a little different from anything which occurs in ordinary English; but we shall see that they can be used to express the sense of many different English noun phrases.

(i) *Universal Quantification*

Suppose we have a predicate, written schematically as

$$- - - - x * * * *, \quad 35.1$$

in which the individual variable 'x' has one or more free occurrences. Then

$$\forall x - - - - x * * * * \quad 35.2$$

means

$$\begin{array}{l} \text{Everything has the property that } - - - - \text{ it} \\ * * * * . \end{array} \quad 35.3$$

†*Thinking To Some Purpose*, Penguin, 1939, p. 125.

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' $\forall x$ ' is called a *universal quantifier*, and is pronounced 'for all x '. The same applies if ' x ' is replaced throughout by another individual variable; for example

$$\forall z \text{ --- } z \text{ ****} \quad 35.4$$

means just the same as (35.3).

Thus the sentence

$$\forall x \text{ } x \text{ is identical with } x. \quad 35.5$$

means

$$\text{Everything has the property that it is identical with it.} \quad 35.6$$

or in more idiomatic English

$$\text{Everything is identical with itself.} \quad 35.7$$

Here are some other ways in which English expresses the sense of the universal quantifier; note how it combines with ' \rightarrow ':

$$\text{Every cloud has a silver lining.} \quad 35.8$$

$$\forall x \text{ if } x \text{ is a cloud then } x \text{ has a silver lining.}$$

$$\forall x [x \text{ is a cloud } \rightarrow x \text{ has a silver lining}]$$

$$\text{All the bells in heaven shall ring.} \quad 35.9$$

$$\forall x [x \text{ is a bell in heaven } \rightarrow x \text{ shall ring}]$$

$$\text{Each student must hand in homework.} \quad 35.10$$

$$\forall x [x \text{ is a student } \rightarrow x \text{ must hand in homework}]$$

$$\text{Nobody knows the trouble I seen.} \quad 35.11$$

$$\forall y [y \text{ is a person } \rightarrow y \text{ doesn't know the trouble I seen}]$$

$$\text{Roses are red.} \quad 35.12$$

$$\forall z [z \text{ is a rose } \rightarrow z \text{ is red}]$$

In the next two examples, we quantify a 2-place predicate:

$$\text{Nothing will upset } x. \quad 35.13$$

$$\forall y \text{ } y \text{ won't upset } x.$$

y doesn't know anybody who can sign his application for a passport. 35.14

$\forall x$ [y knows $x \rightarrow x$ can't sign y 's application for a passport]

Note that in (35.13), the variable chosen for the quantifier must be different from 'x'; otherwise we should have

$\forall x$ x won't upset x . 35.15

which means 'Nothing will upset itself.' For the same reason, the variable chosen for the quantifier in (35.14) must be different from 'y'.

In (35.2), the variable 'x' is no longer serving to mark the place where a designator can be put; it is now simply part of a complex expression which means (35.3). In the terminology of p. 87, this implies that the occurrences of 'x' in (35.2) are not free. Instead we say that the occurrences of 'x' in (35.2) are *bound*. Likewise the two occurrences of 'y' in

$\forall y$ y won't upset x . 35.16

are bound. However, the occurrence of 'x' in (35.16) is free, since it was not involved in the quantification. (35.16) is therefore a 1-place predicate, formed by quantifying the 2-place predicate

y won't upset x . 35.17

In (35.15), all three occurrences of 'x' are bound, and there are no other individual variables; (35.15) is a declarative sentence.

(ii) *Existential quantification*

Suppose we have a predicate, written schematically as

--- x ****, 35.18

in which the individual variable 'x' has one or more free occurrences. Then

$\exists x$ --- x **** 35.19

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means

At least one thing has the property that 35.20
 - - - - it ****.

' $\exists x$ ' is called an *existential quantifier*, and is pronounced 'there is x such that'. As with universal quantification, the same definition applies if ' x ' in (35.19) is everywhere replaced by another individual variable.

For example,

$\exists x$ x has got into the tank 35.21

means

At least one thing has the property that it has 35.22
 got into the tank.

or in more idiomatic English

Something's got into the tank. 35.23

Here are some other ways in which English expresses the sense of the existential quantifier; note how it combines with ' \wedge ':

There is a tavern in the town. 35.24

$\exists x$ x is a tavern and x is in the town.

$\exists x$ [x is a tavern \wedge x is in the town]

I heard it from one of your girlfriends. 35.25

$\exists y$ [y is a girlfriend of yours \wedge I heard it from
 y]

A mad dog has bitten x . 35.26

$\exists z$ [z is a mad dog \wedge z has bitten x]

Some people prefer z . 35.27

$\exists x$ [x is a person \wedge x prefers z]

The translation in (35.27) is not perfect, since some people often implies more than one person. But it's not too bad.

The variable chosen for the quantifier in (35.26) must be different from 'x', for the same reason as in (35.13); the sentence

$\exists x [x \text{ is a mad dog} \wedge x \text{ has bitten } x]$ 35.28

means 'Some mad dog has bitten itself.'

Just as with universal quantification, the occurrences of 'x' in (35.19) are not free, and they are said to be *bound*. Existential quantification binds variables, and thus reduces the number of places in a predicate.

Exercise 35. Express each of the following (from Shakespeare's *Macbeth*) as faithfully as possible, using a sentence or predicate which starts with a universal or existential quantifier.

1. Every noise appals me.
2. Something wicked this way comes.
3. I have a strange infirmity.
4. Their candles are all out.
5. He has no children.
6. Murders have been performed.
7. x is a tale told by an idiot.
8. None of woman born shall harm x.

In the terminology of section 34, the universal quantifier has the second type of profile, while the existential quantifier has the first type. What about domain of quantification?

We leave the answer open: the domain can be whatever is convenient for the purpose in hand. It's usually convenient to stipulate the same domain of quantification for all the quantifiers which occur in the set of sentences under discussion; for technical simplicity we shall expect the domain of quantification to include the primary reference of any designator which occurs in the set of sentences.

36. Quantifier Rules

A sentence in which no universal or existential quantifiers occur is said to be *quantifier-free*. In this section we shall see that the inconsistency of a set X of sentences can often be demonstrated by proving the inconsistency of certain other sentences which are quantifier-free. These other sentences are called *Herbrand sentences* of the set X (after Jacques Herbrand, who suggested this approach in 1930).

The Herbrand sentences of the set X are constructed from X by certain rules, which we shall describe in a moment. But here is an example of what to expect. Suppose X is the set

$\forall x$ x doesn't excite me. Gertie excites me. 36.1

Then we shall find that X has three Herbrand sentences, which are

Gertie doesn't excite me. I don't excite myself. 36.2
Gertie excites me.

(36.2) is blatantly inconsistent; we may infer that X is inconsistent too.

The Herbrand sentences of a set X come from three sources. First, *every quantifier-free sentence in X is a Herbrand sentence of X* . The second and third kinds of Herbrand sentence are found by dropping universal and existential quantifiers respectively; this is to be done in ways which are justified by two rules, the $\forall x\phi$ Rule and the $\exists x\phi$ Rule, which we shall now describe.

(i) *The $\forall x\phi$ Rule*

Intuitively, this rule asserts that if everything has some property, then any one named thing has the property. For example, if everyone has a skeleton in his cupboard, then even the well-known public benefactor Sir Jasper Virtue has a skeleton in his cupboard. More precisely stated, the rule is as follows:

Suppose X is a set of declarative sentences, among which is a sentence of form ' $\forall x\phi$ ', and suppose D is a designator which

occurs in a sentence in X . Suppose that ψ is the sentence got by putting D in place of every free occurrence of ' x ' in ϕ . Then

X . Therefore ψ .

is a valid argument.

The variable ' x ' in this rule should be understood as a typical individual variable; the rule holds if ' x ' is replaced by any other individual variable throughout.

For example, suppose X is the set

$\forall z$ [z is an element $\rightarrow z$ has an atomic number]. 36.3
Rhenium is an element.

We may take ϕ to be

[z is an element $\rightarrow z$ has an atomic number] 36.4

and D to be 'Rhenium', so that ψ is

[Rhenium is an element \rightarrow Rhenium has an atomic number]. 36.5

The $\forall x\phi$ Rule then tells us that (36.3) entails (36.5).

The rule is justified by our First and Second Assumptions in section 28. Suppose S is a situation in which all of X is true. If D occurs in a sentence in X , then by the Second Assumption its occurrence is purely referential, so by the First Assumption it has a primary reference in S ; this primary reference is assumed to lie in the domain of quantification, by p. 201. By the Second Assumption the occurrences of D in ψ are purely referential, so that ψ does express that ϕ is satisfied by the primary reference of D . Since ' $\forall x\phi$ ' is true in S , ψ must also be true in S .

Exercise 36A. What sentences does the $\forall x\phi$ Rule allow us to deduce from the following set?

$\forall y$ [y is an older boy than John $\rightarrow y$ has
learned to read].
The boy in the corner is dyslexic.

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Now suppose X and ψ are as stated in the $\forall x\phi$ Rule, and ψ is quantifier-free; then ψ will be a Herbrand sentence of X . With a refinement to be added in a moment, this is our second source of Herbrand sentences.

We can see that all the Herbrand sentences of X described so far are actually entailed by X , so that if these Herbrand sentences are inconsistent, X itself must certainly be inconsistent. We may therefore prove the inconsistency of X by producing an inconsistent set of Herbrand sentences of X , as in (36.2).

Here is another example. We consider the set of sentences

$$\begin{aligned} \forall x [x \text{ is a chimpanzee} \rightarrow x \text{ can't use language}]. & \quad 36.6 \\ [\text{Washoe is a chimpanzee} \wedge \text{Washoe can use} & \\ \text{language}]. & \end{aligned}$$

To show that (36.6) is inconsistent, it's enough to write down two of its Herbrand sentences:

$$\begin{aligned} [\text{Washoe is a chimpanzee} \rightarrow \text{Washoe can't use} & \quad 36.7 \\ \text{language}]. & \\ [\text{Washoe is a chimpanzee} \wedge \text{Washoe can use} & \\ \text{language}]. & \end{aligned}$$

(36.7) is obviously inconsistent. (Check this if necessary, by a sentence tableau.) Thus (36.6) is proved *inconsistent*.

In proving inconsistency by this method, it is not always necessary to write down every single one of the Herbrand sentences. Thus in (36.2) we could have omitted 'I don't excite myself.'

Exercise 36B. Prove the inconsistency of each of the following sets of sentences by writing down an inconsistent set of Herbrand sentences; where necessary, check the inconsistency of the Herbrand sentences by a tableau.

1. $\forall x [x \text{ is odd} \vee x \text{ is even}]$.
The number $1/2$ is neither odd nor even.
2. $\forall x [x \text{ is a person} \rightarrow x\text{'s I.Q. never varies by more than 15}]$.
Case 946 is a girl whose I.Q. has varied between 142 and 87.

3. $\forall x [x \text{ is Austrian} \rightarrow x \text{ is of Alpine type}]$.
 $\forall x [x \text{ is of Alpine type} \rightarrow x \text{ has a broad head}]$.
 Horst is Austrian, but doesn't have a broad head.
4. $\forall y [[y \text{ is a cat} \wedge y \text{ has two ginger parents}] \rightarrow y \text{ is ginger}]$.
 $\forall x [x \text{ is a female cat} \rightarrow x \text{ is not ginger}]$.
 The female cat over the road has two ginger parents.
-

We allow still more Herbrand sentences of X to be produced by applying the $\forall x\phi$ Rule to X *together with any Herbrand sentences already constructed*. For example, let X be the set of sentences

$\forall x [x \text{ is a widow} \rightarrow \text{the spouse of } x \text{ is dead}]$. 36.8
 $\forall x x \text{ is not dead}$.
 Maxine is a widow.

As a first step, we can write down two Herbrand sentences of X:

Maxine is a widow. 36.9
 $[\text{Maxine is a widow} \rightarrow \text{the spouse of Maxine is dead}]$.

Now (36.9) presents us with a new designator: 'the spouse of Maxine'. Applying the $\forall x\phi$ Rule with this designator to the second sentence of (36.8), we may add the further Herbrand sentence of X:

The spouse of Maxine is not dead. 36.10

(36.9) and (36.10) together are inconsistent, proving that (36.8) is *inconsistent*.

Exercise 36C. By repeating the process just described as many times as we like, we can produce infinitely many different Herbrand sentences of (36.8). Write down three more besides those in (36.9) and (36.10). (Use the first sentence of (36.8) again with the new designator.)

(ii) *The $\exists x\phi$ Rule*

Intuitively, this rule says that one can never create an inconsistency by naming a thing, provided that the name is not already in use for something else. More precisely:

Suppose X is a set of declarative sentences, among which is a sentence of form ' $\exists x\phi$ ' and suppose D is a proper name which occurs nowhere in X. Suppose that ψ is the sentence got by putting D in place of every free occurrence of ' x ' in ϕ . Then ψ can be added to X without creating an inconsistency.

As with the $\forall x\phi$ Rule, ' x ' can be replaced throughout by any other individual variable. The rule is justified much as the $\forall x\phi$ Rule; if it's true in situation S that something satisfies ϕ , then there is a situation just like S except that some named thing satisfies ϕ .

It's important that the proper name should be a new one, lest we baptize two different objects with the same name. A large supply of unused proper names would be convenient. You may use your imagination for this. But the custom is to take as proper names the letters ' b ', ' c ', ' d ', ' b_1 ', ' b_2 ', etc.; these letters are known as *individual constants*.

For example, suppose X is the set

- $$\begin{aligned} &\forall x [x \text{ is an anarchist} \rightarrow x \text{ belongs to the Left}] && 36.11 \\ &\forall x [x \text{ belongs to the Left} \rightarrow x \text{ favours state} \\ &\quad \text{control}]. \\ &\exists x [x \text{ is an anarchist} \wedge x \text{ doesn't favour state} \\ &\quad \text{control}]. \end{aligned}$$

The $\exists x\phi$ Rule tells us that if X is consistent, then so is the set Y consisting of X together with the sentence

- $$[b \text{ is an anarchist} \wedge b \text{ doesn't favour state control}]. \quad 36.12$$

In fact, of course, Y is inconsistent. We can see this by writing down three of its Herbrand sentences, which are visibly inconsistent:

- [b is an anarchist $\wedge b$ doesn't favour state control]. 36.13
 [b is an anarchist $\rightarrow b$ belongs to the Left].
 [b belongs to the Left $\rightarrow b$ favours state control].

Since Y is inconsistent, we must conclude that X was already inconsistent. Thus is (36.11) proved *inconsistent*.

If X and ψ are as stated in the $\exists x\phi$ Rule, and ψ is quantifier-free, then ψ will be a Herbrand sentence of X . This forms our third and final source of Herbrand sentences.

Thus (36.12) was a Herbrand sentence of (36.11), and hence so were (36.13). Note that in finding the Herbrand sentences (36.13) we used the $\exists x\phi$ Rule first, and then the $\forall x\phi$ Rule with the new proper name; this is usually the best strategy when one is faced with both universal and existential quantifiers.

Exercise 36D. Prove the inconsistency of each of the following sets of sentences by writing down an inconsistent set of Herbrand sentences. (Satisfy yourself that the set you write down is inconsistent.)

- $\forall x [x \text{ is female} \rightarrow x \text{ is not a jockey}]$.
 $\exists x [x \text{ is a jockey} \wedge x \text{ is female}]$.
- $\forall x [x \text{ has a conscience} \rightarrow x \text{ has experienced the castration complex}]$.
 $\forall x [x \text{ is female} \rightarrow x \text{ has not experienced the castration complex}]$.
 $\exists x [x \text{ is female} \wedge x \text{ has a conscience}]$.
- $\forall x [x \text{ is male} \vee x \text{ is female}]$.
 $\forall x [x \text{ is male} \rightarrow x \text{ has just forty-six chromosomes}]$.
 $\forall x [x \text{ is female} \rightarrow x \text{ has just forty-six chromosomes}]$.
 $\exists x x \text{ has forty-seven chromosomes}$.
- $\forall x [x \text{ is male} \rightarrow \text{the mate of } x \text{ is female}]$.
 $\forall x [x \text{ is female} \rightarrow x \text{ is not male}]$.
 $\exists x [x \text{ is male} \wedge \text{the mate of } x \text{ is male}]$.

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Herbrand sentences can also be used for showing the validity of arguments, once we can deal with a '¬' at the beginning of a sentence. Fortunately this is easy, by virtue of the following two rules:

(iii) *The $\neg\forall x\phi$ Rule: ' $\neg\forall x\phi$ ' is true in exactly the same situations as ' $\exists x\neg\phi$ '.*

(iv) *The $\neg\exists x\phi$ Rule: ' $\neg\exists x\phi$ ' is true in exactly the same situations as ' $\forall x\neg\phi$ '.*

For example,

Not all has been lost. 36.14
 $\neg\forall x x$ has been lost.

is true in precisely the same circumstances as

Something has not been lost. 36.15
 $\exists x\neg x$ has been lost.

We shall illustrate the use of these rules by proving the validity of the following argument:

Bank-notes all carry a metal strip. Anything 36.16
with a metal strip can be detected by X-rays.
Therefore bank-notes can be detected by
X-rays.

The first step is to translate into quantifier notation, as in section 35:

$\forall x [x \text{ is a bank-note} \rightarrow x \text{ has a metal strip}]$. 36.17
 $\forall x [x \text{ has a metal strip} \rightarrow x \text{ can be detected}$
by X-rays].
Therefore $\forall x [x \text{ is a bank-note} \rightarrow x \text{ can be}$
detected by X-rays].

When we form the counterexample set of (36.17), '*Therefore*' is replaced by '¬'. We may then use the $\neg\forall x\phi$ Rule to

replace ' $\neg\forall x$ ' by ' $\exists x\neg$ ', and paraphrase the counterexample set as

- $\forall x$ [x is a bank-note $\rightarrow x$ has a metal strip]. 36.18
 $\forall x$ [x has a metal strip $\rightarrow x$ can be detected by
 X-rays].
 $\exists x \neg$ [x is a bank-note $\rightarrow x$ can be detected
 by X-rays].

(36.19) has the following Herbrand sentences, which are inconsistent (as a sentence tableau would show):

- \neg [b is a bank-note $\rightarrow b$ can be detected by 36.19
 X-rays].
 [b is a bank-note $\rightarrow b$ has a metal strip].
 [b has a metal strip $\rightarrow b$ can be detected by
 X-rays].

Thus (36.16) is proved *valid*.

Exercise 36E. Prove the validity of each of the following arguments, by writing down inconsistent sets of Herbrand sentences. (Satisfy yourself that the sets are inconsistent.)

1. No ground-feeding birds are brightly coloured. Dunnocks are ground-feeding birds. *Therefore* no dunnocks are brightly coloured.
2. Some finches crack cherry seeds. All finches are birds. All birds which crack cherry seeds have massive beaks. *Therefore* some finches have massive beaks.
3. All the birds are either chiff-chaffs or willow warblers. The birds are singing near the ground. Chiff-chaffs don't sing near the ground. *Therefore* the birds are all willow-warblers.
4. Trumpeter bullfinches can sing two notes at once. Trumpeter bullfinches are birds. There are some trumpeter bullfinches. *Therefore* there are some birds that can sing two notes at once.

Predicate Logic

In the next few sections we shall amalgamate nearly everything we have done so far. The result is *first-order predicate logic*, the crowning achievement of modern logic. (In fact it was first isolated as an amenable portion of logic by two mathematicians, David Hilbert and Wilhelm Ackermann, in 1928.) Curiously there is a flourishing branch of mathematics – called *abstract model theory* – which is largely devoted to explaining why first-order predicate logic works so much better than any other kind of formal logic which has been proposed.

37. Logical Scope

When we first encountered the logical analyst, in sections 16–19, he was busy translating English sentences into truth-functor notation. On our second visit we find that he knows about quantifiers too.

Translating into quantifier notation is a fairly subtle business; unlike truth-functor analyses, it nearly always calls for a major reorganization of the sentence. According to section 35, a sentence or predicate can only have ‘ \forall ’ or ‘ \exists ’ put at its head if it can be paraphrased

Everything has the property that . . . 37.1

or

At least one thing has the property that . . . 37.2

Neither (37.1) nor (37.2) is exactly a common turn of phrase in English, so we must expect wholesale paraphrasing.

If a sentence or predicate can be paraphrased as (37.1), we say it has *overall form* \forall . If it can be paraphrased as (37.2), we say it has *overall form* \exists . Analysis of complex sentences proceeds just as in section 19: we look for the overall form of the whole sentence, and then for the overall forms of the parts inside it, moving from larger phrases to smaller.

For example, suppose we wish to analyse

Brand X doesn't remove all kinds of stain. 37.3

The overall form of (37.3) is 'It's not true that ϕ '; so we first put

\neg Brand X removes all kinds of stain. 37.4

The part after the ' \neg ' sign in (37.4) has overall form \forall , and is easily rendered as

$\forall y$ [y is a kind of stain \rightarrow Brand X removes y]. 37.5

Adding the ' \neg ', we have the final analysis:

$\neg \forall y$ [y is a kind of stain \rightarrow Brand X removes y]. 37.6

For our next example, we analyse

Brand X doesn't remove any kinds of stain. 37.7

(37.7) has overall form \forall , and we may begin by paraphrasing it as

$\forall y$ [y is a kind of stain \rightarrow Brand X doesn't remove y]. 37.8

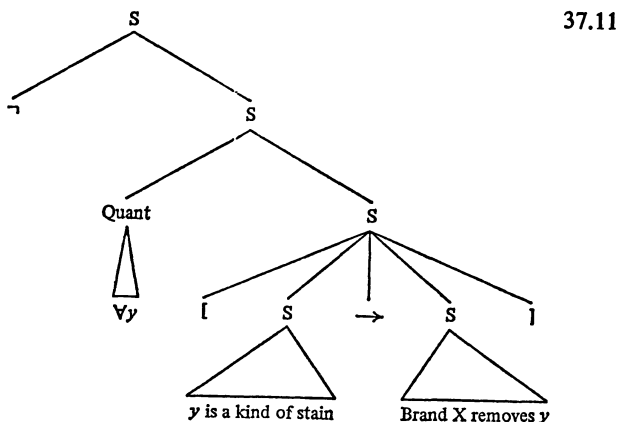
The phrase 'Brand X doesn't remove y ' has overall form ' $\neg \phi$ ', and can be translated into

\neg Brand X removes y . 37.9

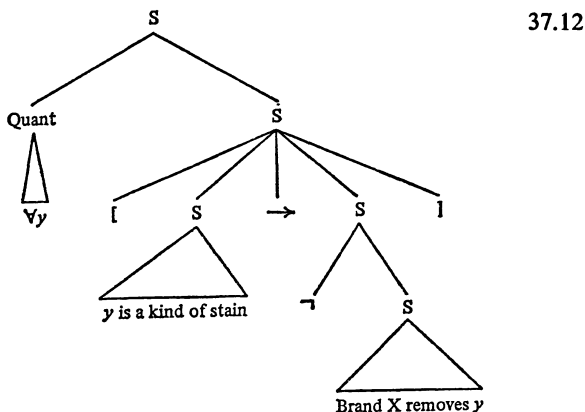
Incorporating this back into (37.8), we have the final paraphrase

$\forall y$ [y is a kind of stain $\rightarrow \neg$ Brand X removes y]. 37.10

The two English sentences which we have just analysed seem very similar at first glance, yet their analyses are quite different. Why? Let us contrast the phrase-markers of the two analyses. That for (37.6) is:



The phrase-marker for (37.10) is:



In (37.11) the scope of '¬' is the whole sentence, and includes that of '∀y'; but in (37.12) the scope of '∀y' includes that of

' \neg '. From a purely grammatical point of view this is surprising, because the two English sentences (37.3) and (37.7) must have very much the same phrase-markers as each other, and there is no suggestion that 'all' and 'any' have different scopes. It seems that the grammatical scope of a word such as 'any' is a very poor guide to the symbolic translation.

However, (37.3) and (37.7) do certainly mean different things and the difference between the two phrase-markers above records this difference of meaning.

Logicians often talk of the *logical scope* of an occurrence of any (or similar words), meaning the scope of a symbolic quantifier which would be used to translate it. Thus a logician might say that the *logical scope* of 'any' in (37.7) is the whole sentence while the *logical scope* of 'all' in (37.3) doesn't include the negation. The logical scope of a word is an important guide to the meaning of the sentence.

In fact we have just uncovered an important difference between all and any: any *tends to have larger logical scope than all*. Here is another pair of examples to illustrate this:

If any of the brakes hold, the train will halt. 37.13
 $\forall x$ [x is a brake \wedge x will hold] \rightarrow the train will
halt].

If all of the brakes hold, the train will halt. 37.14
 $[\forall x$ [x is a brake \wedge x will hold] \rightarrow the train will
halt].

Every is like all; it tends to have a small logical scope, as in the following examples:

I don't know anything. 37.15
 $\forall x \neg$ I know x .

I don't know everything. 37.16
 $\neg \forall x$ I know x .

Exercise 37A. Analyse with the aid of ' \forall ' or ' \exists ':

1. The room isn't heated at all times.
2. The room is unheated at all times.

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3. The room isn't heated at any time.
 4. There are times at which the room isn't heated.
 5. There aren't any times at which the room is heated.
-

So far, we have not considered any sentences in which two quantifiers occur. We shall remedy this at once, by analysing

Some girl won all the prizes. 37.17

This has overall form \exists , and we can start by writing

$\exists x [x \text{ is a girl} \wedge x \text{ won all the prizes}].$ 37.18

Then 'x won all the prizes' has overall form \forall :

$\forall y [y \text{ was a prize} \rightarrow x \text{ won } y].$ 37.19

Fitting (37.19) back into (37.18), we have the final analysis

$\exists x [x \text{ is a girl} \wedge \forall y [y \text{ was a prize} \rightarrow x \text{ won } y]].$ 37.20

Another example:

Each of the prizes was won by a girl. 37.21

Here the overall form is \forall :

$\forall y [y \text{ was a prize} \rightarrow y \text{ was won by a girl}].$ 37.22

The second phrase inside (37.22) has overall form \exists :

$y \text{ was won by a girl}$ 37.23
 $\exists x [x \text{ is a girl} \wedge x \text{ won } y]$

so that the final analysis is:

$\forall y [y \text{ was a prize} \rightarrow \exists x [x \text{ is a girl} \wedge x \text{ won } y]].$ 37.24

Contrast (37.17) with (37.21). In (37.17) a single girl takes all, while (37.21) allows each prize to have been won by a different girl. The quantifier analyses (37.20) and (37.24) bring out the precise difference: they show that in (37.17) 'some' has greater logical scope than 'all', whereas in (37.21), 'each' has greater logical scope than 'a'.

In English, logical scopes are determined in a complicated and obscure way. It has something to do with the exact choice of words (some/a; all/each) and something to do with the order of the words in the sentence. The rules are so tangled that different people often find they interpret one and the same sentence in quite different ways. For this reason, you should regard the next exercise as a piece of research into your own brand of English; there are no universal right answers.

Exercise 37B. In each of the following sentences, which of the two bold words has the greater logical scope?

1. A girl won **all** the prizes.
2. **All** the prizes were won by **some** girl.
3. A girl won **every** prize.
4. **Every** prize was won by **some** girl.
5. **Every** one of the prizes was won by a girl.
6. **Some** girl won **each** prize.

When Phineas Barnum fooled all the people some of the time, did he fool everybody at once, or did he merely fool each person at some time or other?

Apparently no natural language is altogether happy with complicated arrays of quantifiers. It is true that in favourable cases, English can handle a fair amount of quantificational complexity:

Each of us must admit that there have been 37.25
 times in our lives when we have felt that
 everyone round us has posed some kind of
 threat to all our values.

But some other assertions, which are theoretically no more complex than (37.25), become almost unintelligible when they are set out in brute English:

For every person and every age, and every 37.26
 positive number, there is a second positive
 number such that at any age which differs

from the first-mentioned age by fewer days than the latter positive number, the person's height differs from his height at the first-mentioned age by less than the former positive number of inches.

Mathematicians will recognize (37.26) as the statement that a person's height is a continuous function of his age. Slight changes in logical scope would lead to the quite different statements that a person's height was a continuous function of his age uniformly with respect to people, or with respect to age. These are examples of vitally important mathematical notions. It should be no surprise that universal and existential quantifiers were first discovered by two mathematicians (Gottlob Frege 1879, C. S. Peirce 1883) at just the time when the notions of continuity and uniform continuity were being absorbed into the corpus of mathematics.

When two or more quantifiers occur in a sentence, we must be careful about our choice of individual variables. We shall make it a rule that *no quantifier should occur within the scope of another occurrence of a quantifier with the same variable.*

For example, we analyse:

Every house has a deep freeze and a colour television. 37.27

The first steps are clear:

$\forall x [x \text{ is a house} \rightarrow [x \text{ has a deep freeze} \wedge x \text{ has a colour television}]]$. 37.28

The variable of the quantifier used in 'x has a deep freeze' must be different from 'x', because this phrase lies within the scope of ' $\forall x$ ' at the front; we may choose 'y':

$\forall x [x \text{ is a house} \rightarrow [\exists y [y \text{ is a deep freeze} \wedge x \text{ has } y] \wedge x \text{ has a colour television}]]$. 37.29

Now 'x has a colour television' lies within the scope of ' $\forall x$ ', but not within that of ' $\exists y$ '. So we must avoid adding a quantifier with 'x', but there is no objection to another 'y':

$\forall x [x \text{ is a house} \rightarrow [\exists y [y \text{ is a deep freeze} \wedge x \text{ has } y] \wedge \exists y [y \text{ is a colour television} \wedge x \text{ has } y]]]$ 37.30

Exercise 37C. Express each of the following sentences as faithfully as possible, using truth-functor symbols and universal and existential quantifiers.

1. If there are any starlings nesting here, then I'll shoot them.
2. If there are any starlings nesting here, then that bird is a starling.
3. That bird has a longer bill than any finch.
4. That bird has a bill no longer than some finches.
5. That bird has a bill no longer than any starling.
6. Not all starlings have longer bills than any finch.
7. There are finches with longer bills than any starling.
8. For any finch, there is a starling with a longer bill.
9. All the birds nesting here, except possibly finches, have long bills.
10. Among the birds nesting here, only the starlings have long bills.

The notion of logical scope is not always wholly determinate, because it may be possible to analyse one English sentence in two quite different ways, both equally correct. Exercise 37C.2 is an example; see the answers. Even the logician's tools may break if you lean too hard on them.

38. Analyses Using Identity

In section 35 we chose to adopt symbols for just two types of quantifier, the universal and the existential. Why not for other kinds of phrase too, such as *at least two* or *most*? Part of the answer is that many of these other quantifier phrases can be paraphrased by means of the two quantifiers we already have, together with '='.

(i) At least, at most *and* exactly

We can express the senses of at least one, at least two and so on, by combining '∃' with '=':

I know at least one pop-star. 38.1

$\exists x [x \text{ is a pop-star} \wedge \text{I know } x]$

I know at least two pop-stars. 38.2

$\exists x_1 \exists x_2 [\neg x_1 = x_2 \wedge [[x_1 \text{ is a pop-star} \wedge \text{I know } x_1] \wedge [x_2 \text{ is a pop-star} \wedge \text{I know } x_2]]]$

I know at least three pop-stars. 38.3

$\exists x_1 \exists x_2 \exists x_3 [[[\neg x_1 = x_2 \wedge \neg x_1 = x_3] \wedge \neg x_2 = x_3] \wedge [[x_1 \text{ is a pop-star} \wedge \text{I know } x_1] \wedge [x_2 \text{ is a pop-star} \wedge \text{I know } x_2]] \wedge [x_3 \text{ is a pop-star} \wedge \text{I know } x_3]]]$.

The exact arrangement of the brackets is not important.

Then at most one, at most two and so on can be expressed by saying 'not at least two', 'not at least three', etc.:

I know at most one prince. 38.4

\neg I know at least two princes.

I know at most two princes. 38.5

\neg I know at least three princes.

Exactly means 'at most and at least':

I know **exactly** one name-dropper. 38.6

$[\text{I know at least one name-dropper} \wedge \text{I know at most one name-dropper}]$.

Written out in full, the paraphrase of (38.6) is quite lengthy. There is a neater way of expressing **exactly** one:

I know **exactly** one name-dropper. 38.7

$\exists x \forall y [x = y \leftrightarrow [y \text{ is a name-dropper} \wedge \text{I know } y]]$.

Exercise 38A. Write out paraphrases of the following, using '∀', '∃' and '='.

1. There are at least two mistakes.
2. There are at least four mistakes.
3. More than one person has pointed out the mistakes.
4. There are precisely two hemispheres.
5. Only Sir Henry is allowed to use that bath.

(ii) The

A sentence of form

The S is a P. 38.8

is true when there is precisely one S under discussion, and every S under discussion is a P. This suggests the paraphrase

There is exactly one S, and every S is a P. 38.9

As it stands, the paraphrase (38.9) is clearly wrong; for example the following two sentences obviously mean quite different things:

The boy is a genius. 38.10

There is exactly one boy, and every boy is a
genius. 38.11

But the reason why (38.11) fails to paraphrase (38.10) is simply that the domain of quantification demanded by (38.11) is far too large. As we noted on p. 196, every allows a large domain of quantification, while the is used to narrow the domain to those things which are immediately in hand. If we use '∀', '∃' and '=' to symbolize (38.11), and then stipulate a small enough domain of quantification, the resulting paraphrase of (38.10) will be quite adequate.

With this caution on domains of quantification, we therefore paraphrase

The boy is a genius. 38.12

[$\exists x \forall y [x = y \leftrightarrow y \text{ is a boy}] \wedge \forall x [x \text{ is a boy}$
 $\rightarrow x \text{ is a genius}]$].

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The same style may be used to eliminate other definite descriptions:

Ahmed loves his wife dearly. 38.13
[$\exists x \forall y [x = y \leftrightarrow y \text{ is a wife of Ahmed}] \wedge$
 $\forall x [x \text{ is a wife of Ahmed} \rightarrow \text{Ahmed loves } x \text{ dearly}]$].

Sentences of form 'A is the B' can be paraphrased more simply:

Today is the day of liberation. 38.14
 $\forall y [\text{today} = y \leftrightarrow y \text{ is a day of liberation}]$.

There are also special styles of analysis which can be used for superlatives such as the **biggest**, the **most embarrassing**, and so forth:

Harrow United is the best team. 38.15
Harrow United is a team and is better than
any other team.
[Harrow United is a team $\wedge \forall y [(y \text{ is a team} \wedge$
 $\neg y = \text{Harrow United}) \rightarrow \text{Harrow United}$
is better than $y]$].

Chicago has the worst crime-rate. 38.16
 $\forall y [\neg y = \text{Chicago} \rightarrow \text{Chicago has a worse}$
crime-rate than $y]$.

Occasionally, definite descriptions are used in a way which is not meant to imply that just one thing fits the description; this occurs most often after *is* and *was*. In such cases the paraphrases suggested above should not be used. Thus:

Bob is the proud owner of a Rolls Royce. 38.17
NOT: $\forall y [\text{Bob} = y \leftrightarrow y \text{ is a proud owner of a}$
Rolls Royce].

(There may be other proud owners of Rolls Royces.) Non-count nouns (see p. 150) lead to definite descriptions which don't admit the question 'How many?'; here again our paraphrases are not appropriate. For example:

The music of Schubert is more popular than ever. 38.18

NOT: $[\exists x \forall y [x = y \leftrightarrow y \text{ is a music of Schubert}] \wedge \forall x [x \text{ is a music of Schubert} \rightarrow x \text{ is more popular than ever}]]$.

Naturally you will also avoid using the above paraphrases where an occurrence of a definite description is not purely referential:

Disraeli became the Prime Minister. 38.19

NOT: $[\exists x \forall y [x = y \leftrightarrow y \text{ is Prime Minister}] \wedge \forall x [x \text{ is Prime Minister} \rightarrow \text{Disraeli became } x]]$.

Exercise 38B. Where possible, eliminate the bold definite descriptions in the following sentences by paraphrasing as above. Where it's impossible, say so.

1. **The book** is bound in vellum.
2. We've avoided **her stern**.
3. **Joseph** was the son of **Jacob**.
4. **The woman in blue** is my mother.
5. I hope to be **the first to congratulate you**.
6. The shepherds were seated on **the ground**.
7. **Cassius** is **the greatest**.

The paraphrases above raise some severe problems of scope when we try to apply them to complex sentences. These problems are not of our making – they reflect the great subtlety of the use of definite descriptions in English. For example, compare

I'm not **the man you lent £10 to**. 38.20

I'm not **the tallest man in the room**. 38.21

(38.20) would normally be understood to imply that there is some one man you lent £10 to; it could be paraphrased in the style above, as

$[\exists x \forall y [x = y \leftrightarrow y \text{ is a man you lent } \pounds 10 \text{ to}] \wedge \forall x [x \text{ is a man you lent } \pounds 10 \text{ to} \rightarrow \neg x = \text{me}]]$ 38.22

But (38.21) doesn't carry a similar implication; it could well be true because there is no tallest man in the room – say, if there are three men of the same height, all taller than anybody else in the room. The correct paraphrase is

It's not true that I'm the tallest man in the room. 38.23
 $\neg [I'm \text{ in the room} \wedge \forall y [(y \text{ is a man in the room} \wedge \neg y = \text{me}) \rightarrow I'm \text{ taller than } y]]$.

We may say that the logical scope of 'the' includes that of 'not' in (38.20), but not in (38.21).

In section 34 we grouped English quantifications into four main types. Most quantifications of types (1), (2) and (4) can be handled by '∀', '∃' and '='. Not so for those of type (3); this is one of the failings of first-order predicate logic. There are plenty of logical connections between type (3) phrases, if one wanted to construct a theory around them. For example, here is an inconsistent set of sentences:

More than half the people in the room are girls. 38.24
 More than half the girls in the room have dark hair.
 Less than a quarter of the people in the room have dark hair.

39. Predicate Interpretations

As in section 21, we wish to abbreviate. Our abbreviation schemes will be called *interpretations*, or *predicate interpretations* when we want to distinguish them from the earlier sort. These interpretations must cater for designators and predicates as

well as sentences. As before, a sentence or predicate which is completely translated into symbols is called a *formula*.

An example will show what is needed. Here is a predicate interpretation, written as we shall write them.

- J : Sauvignon is the juiciest of the Bordeaux grapes. 39.1
- Gx : x is a species of grape.
- Mxy : x is made out of y .
- Sxy : y is sweeter than x .
- b : Blue Burgundy
- c : Pinot Noir
- d : Grenache

Capital letters (such as ' J ') have declarative sentences assigned to them. A capital letter followed by one or more individual variables without repetitions (such as ' Gx ', ' Mxy ' or ' Sxy ') has assigned to it a predicate in which just the same variables have free occurrences. An individual constant (such as ' b ', ' c ', ' d ') has a designator assigned to it.

Here are some sample abbreviations by the interpretation (39.1):

- Sauvignon is the juiciest of the Bordeaux grapes. 39.2
 J
- Pinot Noir is a species of grape. 39.3
 Gc
- Blue Burgundy is made out of Pinot Noir. 39.4
 Mbc
- Grenache is sweeter than y . 39.5
 Syd

Note the order of the symbols in (39.5), compared with:

- y is sweeter than Grenache. 39.6
 Sdy

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More complex sentences may also be abbreviated. For example, the sentence

There is a species of grape which is sweeter than Pinot Noir. 39.7

can first be analysed as

$\exists x [x \text{ is a species of grape} \wedge x \text{ is sweeter than Pinot Noir}]$; 39.8

then the interpretation (39.1) can be used to abbreviate this to the formula

$\exists x [Gx \wedge Scx]$. 39.9

A little paraphrasing will help us to symbolize more sentences:

Grenache is at least as sweet as Pinot Noir. 39.10
Pinot Noir is not sweeter than Grenache.
 $\neg Sdc$

Exercise 39A. Using the following interpretation:

Ex : x is an epic.
 Sxy : x is shorter than y .
 Wxy : x wrote y .
 b : Beowulf
 c : the Odyssey
 d : Homer

translate each of the following into a symbolic formula:

1. The Odyssey is an epic.
2. Homer didn't write Beowulf.
3. The Odyssey and Beowulf are not the same length.
4. Homer wrote an epic which is longer than Beowulf.
5. Homer didn't write just one epic.
6. Beowulf is not the shortest epic.
7. The Odyssey and Beowulf are not by the same author.

8. Any epics there may be that are shorter than Beowulf weren't written by Homer.
9. Whatever epic we consider, Homer wrote a longer one.
10. There are just two epics which are longer than the Odyssey, and Homer wrote one of them but not the other.

The classification of binary relations in section 31 can be recast in terms of formulae. For example, to say that the relation expressed by the predicate ' Rxy ' in the domain of quantification is reflexive, we write simply

$$\forall xRxx. \quad 39.11$$

Similarly, we may convey that this relation is transitive by writing

$$\forall x\forall y\forall z[(Rxy \wedge Ryz] \rightarrow Rxz]. \quad 39.12$$

Exercise 39B. Write down formulae which say that the relation expressed by ' Rxy ' in the domain of quantification has the following properties:

1. Irreflexive.
2. Symmetric.
3. Asymmetric.
4. Intransitive.
5. Non-reflexive.
6. Connected. (This needs '='.)

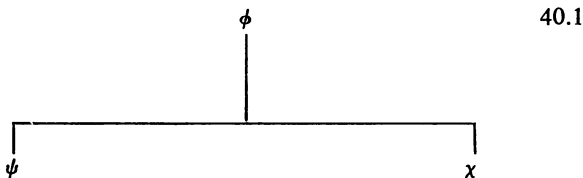
40. Predicate Tableaux

We must reconsider tableaux. As an instrument for testing consistency, they served us faithfully up to section 25. Unfortunately there is no hope of redesigning them to test the consistency of sets of sentences with quantifiers, since Alonzo

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Church proved in 1936 that there cannot be a systematic method for testing consistency of such sentences. However, we may still employ closed tableaux to *prove inconsistency*.

The method is an ingenious combination of sentence tableaux and Herbrand sentences. First we reinterpret the derivation rules of sentence tableaux. The derivation rules of form



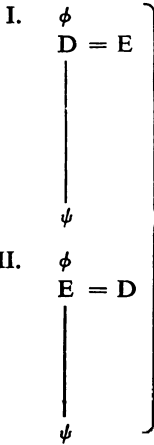
will now be interpreted as saying: *if a set of sentences containing ϕ is consistent, then either ψ can be added to the set without creating an inconsistency, or χ can*. Likewise the rules of form



will now be read as saying: *if a set of sentences containing ϕ is consistent, then ψ and χ can be added to the set without creating an inconsistency*.

On our earlier reading (as in section 10), (40.1) would have said also that if either ψ or χ was true in a situation, then ϕ was true in it; and similarly with (40.2). Interpreted in the new way, *a closed tableau still proves that a set of sentences is inconsistent*; but an unclosed tableau proves nothing at all. Ticks, which were used in earlier sections to check that an unclosed tableau was finished, may now be omitted – we have lost interest in unclosed tableaux.

When the derivation rules are read in the new way, we can immediately add six to their number, as follows:



where the designator D occurs in ϕ , and ψ is the result of replacing one or more occurrences of D in ϕ by occurrences of E .



where there is a designator D which has already occurred in the branch to which ψ is added, and ψ is the result of replacing each free occurrence of ' x ' in ϕ by an occurrence of D .



where there is a proper name D which has not occurred anywhere in the branch to which ψ is added, and ψ is the result of replacing each free occurrence of ' x ' in ϕ by an occurrence of D .



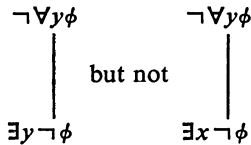
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VI. $\neg\exists x\phi$



Rules I and II are easily recognized as tableau forms of Leibniz's Rule from section 29. If the two upper sentences of one of these rules occur in a branch, and the branch is consistent, then the sentence ψ can certainly be added without creating an inconsistency, since the other sentences already entail it.

Rules III–VI likewise are the tableau forms of the quantifier rules which we studied in section 36. The 'x' can be replaced by any other individual variable, provided we don't confuse two variables within one rule. For example Rule V should be read as including



The Identity Rule from section 29 is not forgotten; it now tells us that we can close any branch in which a sentence of form ' $\neg D = D$ ' occurs.

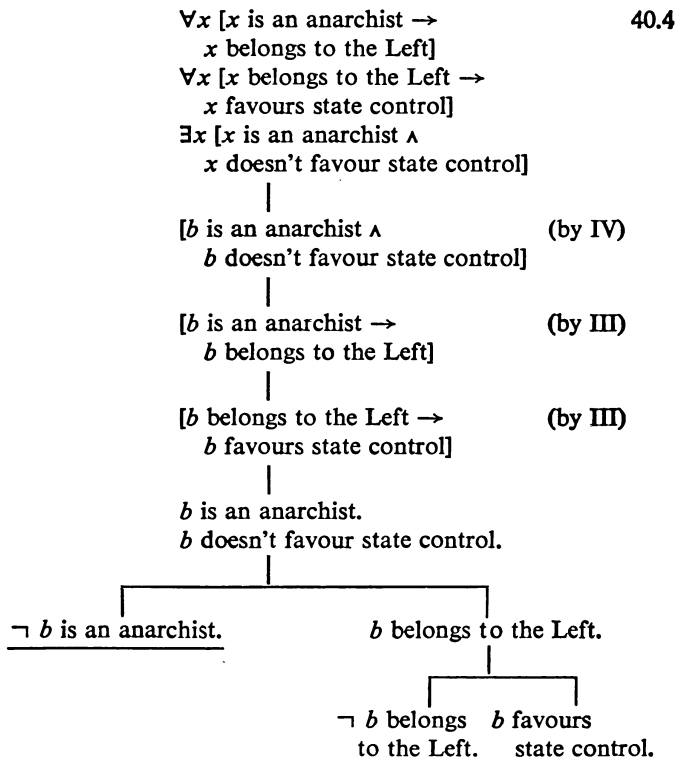
Tableaux which use the rules above, together with the rules of sentence tableaux, may be described as *predicate tableaux*.

Many logicians add one further derivation rule, which has the effect of decreeing that every domain of quantification has at least one individual in it. From our point of view this would be a mistake. A domain of quantification can perfectly well be empty – as it well may be if the conversation turns to discussing intelligent life on other planets.

To illustrate predicate tableaux, let us recast our proof of the inconsistency of the following set, which was (36.11) in section 36:

- $\forall x [x \text{ is an anarchist} \rightarrow \text{belongs to the Left}].$ 40.3
 $\forall x [x \text{ belongs to the Left} \rightarrow x \text{ favours state control}].$
 $\exists x [x \text{ is an anarchist} \wedge x \text{ doesn't favour state control}].$

The new proof is a closed tableau:



All the examples of section 36 can be recast in tableau terms. In section 36, no sentence contained more than one

occurrence of a quantifier. There was therefore no need to worry about which occurrences of individual variables were free in the sentence ϕ of the $\forall x\phi$ and $\exists x\phi$ Rules – all occurrences were automatically free. Theoretically we may now have difficulties in telling the bound from the free occurrences when we apply Rules III and IV. Since section 37, we have allowed a sentence to contain several quantifiers, and some quantifier might survive to bind an ‘ x ’ after the initial ‘ $\forall x$ ’ or ‘ $\exists x$ ’ was lopped off. But in fact this problem never arises, provided we stick to the translating procedure of section 37; when ‘ $\forall x$ ’ or ‘ $\exists x$ ’ is removed from the front, all remaining occurrences of ‘ x ’ become free. (Of course if ‘ $\forall x$ ’ or ‘ $\exists x$ ’ is taken off the front, there may be some occurrences of other individual variables such as ‘ y ’ or ‘ z ’ still in the sentence. But these will all be bound, and should be left sleeping until their respective quantifiers are reached later in the tableau.)

Since tableaux are bulky, our remaining examples will be carried out in symbols, using suitable interpretations. In symbolic formulae there is no difference between designators and proper names – both are represented by individual constants. From now on, we shall omit the connecting lines in tableaux when no branching occurs; this will reduce the clutter.

Our next example is one which, but for its length, could have been handled by the method of section 36. The following argument is on loan from Sigmund Freud:

Criticisms which stem from some psychological need of those making them don't deserve a rational answer. When people complain that psychoanalysis makes wild and arbitrary assertions about infantile sexuality, this criticism stems from certain psychological needs of these people. <i>Therefore</i> the criticism that psychoanalysis makes wild and arbitrary assertions about infantile sexuality doesn't deserve a rational answer.†	40.5
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† Sigmund Freud, *New Introductory Lectures on Psychoanalysis*, Hogarth Press, 1933; see his Lecture XXXIV. Also available in Pelican Freud Library, Penguin Books, 1973; vol. 2, Lecture 34.

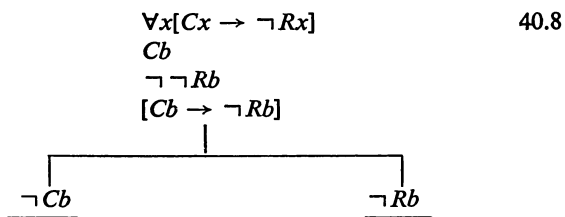
We take as interpretation:

- Cx : x is a criticism which stems from some psychological need of those making it. 40.6
 Rx : x deserves a rational answer.
 b : the complaint that psychoanalysis makes wild and arbitrary assertions about infantile sexuality.

Translated into formulae by (40.6), the argument (40.5) becomes

$$\forall x[Cx \rightarrow \neg Rx]. Cb. \text{ Therefore } \neg Rb. \quad 40.7$$

The tableau proof of (40.7) is very brief; it uses Rule III once:



Exercise 40A. Using the interpretation given, translate the following argument (partly taken from an old American state constitution) into symbols. Prove the validity of the symbolic argument by a closed tableau.

- M : No man may be beaten with above forty stripes.
 Gx : x is a true gentleman.
 Cx : x 's crime is very shameful.
 Lx : x 's course of life is vitious and profligate.
 Sx : x may be punished with shipping.
 d : John Doe Jr.

No man may be beaten with above forty stripes; nor may any true gentleman be

punished with shipping, unless his crime be very shamefull, and his course of life vicious and profligate. John Doe Jr is a true gentleman, and, though his crime is in truth very shamefull, his course of life is in no manner vicious and profligate. *Therefore* John Doe Jr may not be punished with shipping.

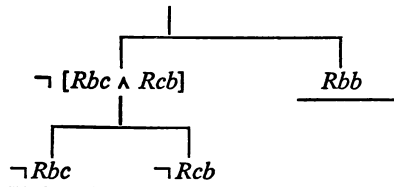
For our next example, we show that every irreflexive and transitive binary relation is asymmetric. In view of page 225, this may be expressed as follows:

$$\forall x \neg Rxx. \forall x \forall y \forall z [(Rxy \wedge Ryz) \rightarrow Rxz]. \quad 40.9$$

Therefore $\forall x \forall y [Rxy \rightarrow \neg Ryx].$

The tableau runs:

$$\begin{array}{l} \forall x \neg Rxx \quad 40.10 \\ \forall x \forall y \forall z [(Rxy \wedge Ryz) \rightarrow Rxz] \\ \neg \forall x \forall y [Rxy \rightarrow \neg Ryx] \\ \exists x \neg \forall y [Rxy \rightarrow \neg Ryx] \quad (\text{by V}) \\ \neg \forall y [Rby \rightarrow \neg Ryb] \quad (\text{by IV}) \\ \exists y \neg [Rby \rightarrow \neg Ryb] \quad (\text{by V}) \\ \neg [Rbc \rightarrow \neg Rcb] \quad (\text{by IV with } c \text{ new}) \\ Rbc \\ \neg \neg Rcb \\ \forall y \forall z [(Rby \wedge Ryz) \rightarrow Rbz] \quad (\text{by III}) \\ \forall z [(Rbc \wedge Rcz) \rightarrow Rbz] \quad (\text{by III}) \\ [[Rbc \wedge Rcb] \rightarrow Rbb] \quad (\text{by III}) \\ \neg Rbb \quad (\text{by III}) \end{array}$$



Exercise 40B. Show that every asymmetric binary relation is irreflexive. (It's easier than (40.10).)

In our next example, the placing of the quantifiers is more complicated:

There's someone who's going to pay for all the breakages. *Therefore* each of the breakages is going to be paid for by someone. 40.11

For an interpretation, we take:

Px : x is a person. 40.12

Bx : x is a breakage.

Gxy : x is going to pay for y .

Symbolized, (40.11) becomes:

$\exists x[Px \wedge \forall y[By \rightarrow Gxy]]$. 40.13

Therefore $\forall y[By \rightarrow \exists x[Px \wedge Gxy]]$.

The tableau to prove (40.13) is:

$\exists x[Px \wedge \forall y[By \rightarrow Gxy]]$ 40.14

$\neg \forall y[By \rightarrow \exists x[Px \wedge Gxy]]$

$\exists y \neg [By \rightarrow \exists x[Px \wedge Gxy]]$

$\neg [Bb \rightarrow \exists x[Px \wedge Gxb]]$

Bb

$\neg \exists x[Px \wedge Gxb]$

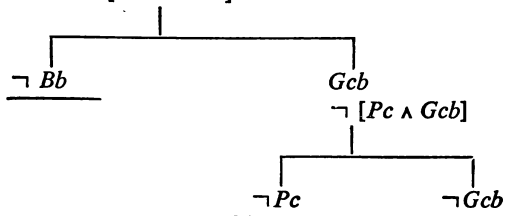
$\forall x \neg [Px \wedge Gxb]$

$[Pc \wedge \forall y[By \rightarrow Gcy]]$

Pc

$\forall y[By \rightarrow Gcy]$

$[Bb \rightarrow Gcb]$



Exercise 40C. Show that any sentence which can be symbolized as follows is inconsistent:

$$\exists x[Rx \wedge \forall y[Ry \rightarrow [Pxy \leftrightarrow \neg Pyy]]]$$

(See Exercise 1B.4 on p. 16 for an example.)

So far, none of our examples have used the rules for identity. A well-known children's riddle will illustrate these rules:

Brother and sister have I none; 40.15
 But that man's father is my father's son.
 Therefore I am that man's father.

An interpretation:

Fxy : x is the father of y . 40.16
 b : me
 c : that man's father

With enough accuracy, we may symbolize (40.15) as

$$\forall y[\exists x[Fxy \wedge Fxb] \rightarrow y = b]. \exists x[Fxc \wedge Fxb]. \quad 40.17$$

Therefore $b = c$.

The tableau:

$$\begin{array}{l} \forall y[\exists x[Fxy \wedge Fxb] \rightarrow y = b] \\ \exists x[Fxc \wedge Fxb] \\ \neg b = c \\ [\exists x[Fxc \wedge Fxb] \rightarrow c = b] \end{array} \quad 40.18$$

$\neg \exists x[Fxc \wedge Fxb]$	$c = b$ $\neg b = b$ (by I)
----------------------------------	--------------------------------

At the bottom right, ' $c = b$ ' is used to replace ' c ' by ' b ' in ' $\neg b = c$ ', reaching the inconsistent formula ' $\neg b = b$ '.

Exercise 40D. Use tableaux to show that any argument which can be symbolized in one of the following ways must be valid.

1. *Pb. Therefore* $\forall x[x = b \rightarrow Px]$.
2. *Pb. $\forall x\forall y[(Px \wedge Py) \rightarrow x = y]$.*
Therefore $\forall x[x = b \leftrightarrow Px]$.
3. $\forall x[x = b \leftrightarrow Px]$.
Therefore $[Pb \wedge \forall x\forall y[(Px \wedge Py) \rightarrow x = y]]$.

The last few exercises of this section are a varied bunch; some are quite hard. You might be well advised to check your symbolizations before proceeding with the tableaux.

Exercise 40E. Prove the validity of each of the following arguments by translating them into symbols with the aid of the interpretation given, and then using tableaux.

- Cx* : *x* is a chimpanzee.
Sxy : *x* can solve *y*.
Px : *x* is a problem.
Txy : *x* is trying harder than *y*.
Bx : *x* will get a banana.
Mx : *x* is male.
b : Sultan
c : Chica

1. All the male chimpanzees can solve every problem. There's at least one problem. Any chimpanzee who can solve a problem will get a banana. Sultan is a male chimpanzee. *Therefore* Sultan will get a banana.
2. Sultan and Chica can solve exactly the same problems. If Sultan can solve any of the problems, then he will get a banana. Sultan will not get a banana. *Therefore* Chica can't solve any of the problems.
3. Not all the chimpanzees are trying equally hard. No chimpanzee is trying harder than himself. *Therefore* there are at least two chimpanzees.
4. Sultan is not Chica. Sultan won't get a banana unless he can solve all the problems. If the chimpanzee Chica is trying

harder than Sultan, then Chica can solve a problem which Sultan can't. All the chimpanzees other than Sultan are trying harder than Sultan. *Therefore* Sultan won't get a banana.

5. Among all the chimpanzees, only Sultan is male. The chimpanzees who will get a banana are the males. *Therefore* Sultan is the chimpanzee who will get a banana.
-

41. Formalization Again[†]

From this point on, formalization of predicate logic can proceed much as in sections 22–25.

One can define a formal language L_2 whose grammatical sentences are those strings which could be got by analysing and symbolizing English predicates. The difference between free and bound occurrences of variables can also be described quite formally, and we can then pick out as *closed formulae* those grammatical sentences of L_2 which have no free occurrences of individual variables.

Having formalized the language, one tries to formalize situations and truth. Success is less complete here. It is possible to define L_2 -structures by analogy with structures in section 23, so that a situation and an interpretation together determine an L_2 -structure, and the L_2 -structure in turn contains everything necessary for determining the truth-value of any closed formula in the situation and under the interpretation. However, there are some snags. We may note three.

First, there are too many L_2 -structures. Often one meets two quite different L_2 -structures in which exactly the same closed formulae are true; this could never occur with the structures of section 23.

Second, L_2 -structures can be too large. An L_2 -structure may have infinitely many ingredients. This makes it impossible to regard L_2 -structures as arrays of symbols which might be written on a page, and it thus frustrates those who wish to reduce truth to a formal calculus.

[†]This section presupposes the mathematical sections 22–25.

Third, L_2 -structures can be too complex. Although L_2 -structures do determine the truth-values of closed formulae, they do it in a way which tends to rule out direct calculation. In fact one can easily describe an L_2 -structure which resists all systematic methods for calculating truth-values in it; if there were a method for computing truth-values in this structure, it would solve in one blow some arithmetical problems which have held out against three hundred years' battering by the best mathematicians in Europe.

For these three reasons, L_2 -structures will disappoint those who formalize in order to avoid problems. Nevertheless they have proved their worth in other ways, and the mathematical theory of models has given them a place in the armoury of modern mathematics.

Of course one may give a formal account of predicate tableaux, just as we did for sentence tableaux in section 25. One can go on to prove that if a set X of closed formulae fails to generate a closed tableau, then there is an L_2 -structure in which X is all true. This shows that the rules of predicate tableaux were adequate for the job they had to do – something which was hardly obvious from mere inspection. The fanatical formalizer can take refuge in formal predicate tableaux if he wishes, since they at least are arrays of symbols which can be manipulated without any thought of meaning or interpretation. But he should reflect that unless he is prepared to handle infinite L_2 -structures, he will be quite unable to justify using the particular assembly of tableau rules which we have described.

It seems to be a principle, at all levels of logic, that to justify a procedure (say, tableaux or truth-tables), one has to use methods which are themselves harder to justify than the procedure which they are justifying. This looks crazy, but it isn't. For example, to justify the tableau method is to justify at once all the separate arguments which tableaux are used to justify. Nobody need be astonished if something stronger than a tableau is needed to justify in one step all tableau proofs. The struggle for yet more global justifications is one of the main-springs of logical research.

Horizons of Logic

Having conquered not, and, or, if, all, some, more and equals, what next?

42. Likelihood⁺

It'll probably be a boy. 42.1
It'll probably be a girl.

These two sentences can't both be true at once – if it's probably a boy, then it's not likely to be a girl. So we have an inconsistency. Why?

One's first instinct is to blame the inconsistency of (42.1) on to the fact that boys can't be girls. But this can only be part of the reason; for consider the sentences

It'll probably be a boy. 42.2
It'll be a girl.

The sentences of (42.2) are not inconsistent, because they may both be true at once. Improbable things do happen. It can happen that everything points to the child being a boy, when in fact it's going to be a girl; in such a situation (42.2) would all be true. (Of course we could never know at the time that we were in such a situation: if we knew the child was going to be a girl, it wouldn't any longer be probable that the child would be a boy.)

It appears that the word **probably** must share the blame for the inconsistency in (42.1). Shall we try to create a logic of **probably**?

⁺Parts of this section presuppose the mathematical section 23.

Perhaps the best starting-place is the 2-place sentence-functor

It's at least as likely that ϕ as that ψ . 42.3

In symbols, we shall write (42.3) as ' $[\phi \geq \psi]$ '. Several English turns of phrase can be rendered fairly accurately with the help of (42.3). Here are some examples:

It's more likely that ϕ than that ψ . 42.4
 $\neg [\psi \geq \phi]$

More likely than not, ϕ . 42.5
 It's likely that ϕ .
 Probably ϕ .
 $\neg [\neg \phi \geq \phi]$

Certainly ϕ . 42.6
 $[\phi \geq [\phi \vee \neg \phi]]$

There's no chance that ϕ . 42.7
 $[[\phi \wedge \neg \phi] \geq \phi]$

It's fifty-fifty whether or not ϕ . 42.8
 $[[\phi \geq \neg \phi] \wedge [\neg \phi \geq \phi]]$

We want to determine what sets of sentences using the sentence-functor (42.3) are inconsistent. Unfortunately tableaux are no use to us here, since (42.3) is not a truth-functor. But the basic idea behind tableaux, namely to find conditions under which the sentences are true, may still serve.

Imagine the shuffled pack of cards in front of us. There are fifty-two cards arranged in four suits. Since the cards have been shuffled, they may be in any order. We may therefore be in any one of n different situations, where n is the number of ways a pack of cards can be ordered. Let us call these n situations the *allowed situations*. Now it is surely true that

The top card is more likely to be a diamond 42.9
 a nine.

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Why is (42.9) true? Because the top card is a diamond in a quarter of the allowed situations, but it is a nine in only a thirteenth of these situations.

More generally, if the topic is the position of cards in the pack, then

It's at least as likely that ϕ as that ψ . 42.10

is true precisely if

There are at least as many allowed situations 42.11
in which it's true that ϕ as there are allowed
situations in which it's true that ψ .

If (42.10) can be interpreted by (42.11) in this example, why not in others too? If so, then a formal calculus is easy to construct, assuming some ideas from the mathematical section 23 above.

We begin by constructing charts such as:

<i>P</i>	<i>Q</i>	<i>R</i>	measure	42.12
T	T	T	.2	
T	T	F	.1	
T	F	T	.2	
T	F	F	0	
F	T	T	.1	
F	T	F	0	
F	F	T	.1	
F	F	F	.3	

The chart (42.12) is to be interpreted as saying that among the allowed situations, the fraction in which '*P*', '*Q*' and '*R*' are true is .2, the fraction in which '*P*' and '*Q*' are true while '*R*' is false is .1, and so on. (The numbers on the right-hand side must add up to a total of 1.) A chart such as (42.12) is said to give a *probability measure*.

Using the chart (42.12), we can calculate the truth-value of any statement built up out of '*P*', '*Q*', '*R*', ' \geq ' and truth-functors, in each of the structures listed in (42.12). For example

we calculate the truth-value of ' $[P \rightarrow [Q \geq R]]$ ' in each structure as follows.

P	Q	R	$[P \rightarrow [Q \geq R]]$	42.13
T	T	T	T F T F T	
T	T	F	T F T F F	
T	F	T	T F F F T	
T	F	F	T F F F F	
F	T	T	F T T F T	
F	T	F	F T T F F	
F	F	T	F T F F T	
F	F	F	F T F F F	
			(2) (1)	

Column (1) is calculated thus: if we add up the total fraction of allowed situations in which ' Q ' is true according to the probability measure (42.12), the answer comes to

$$\cdot 2 + \cdot 1 + \cdot 1 + 0 = \cdot 4. \quad 42.14$$

Likewise the fraction in which ' R ' is true comes to

$$\cdot 2 + \cdot 2 + \cdot 1 + \cdot 1 = \cdot 6. \quad 42.15$$

Hence ' Q ' is true in fewer allowed situations than ' R ', and so ' $[Q \geq R]$ ' is false, according to (42.11). Column (2) is then calculated by the truth-table for ' \rightarrow '.

A second example:

P	Q	R	$[Q \geq [P \wedge R]]$	42.16
T	T	T	T T T T T	
T	T	F	T T T F F	
T	F	T	F T T T T	
T	F	F	F T T F F	
F	T	T	T T F F T	
F	T	F	T T F F F	
F	F	T	F T F F T	
F	F	F	F T F F F	

To calculate the column below ' \geq ', we first recall from (42.14) that ' Q ' is true in $\cdot 4$ of the allowed situations. Then we calculate the fraction in which ' $[P \wedge R]$ ' is true, again from (42.12):

$$\cdot 2 + \cdot 2 = \cdot 4. \quad 42.17$$

Since $\cdot 4$ is at least as great as $\cdot 4$, the whole is true.

For convenience we may refer to (42.13) and (42.16) as *truth-tables*. But there are two significant differences between them and the truth-tables of section 23. First, the truth-value of a formula may now depend on the choice of some probability measure which has to be supplied separately. Second, the truth-value of a formula in one structure may now depend on the truth-values of certain formulae in other structures; we have caught a situation-shifter red-handed.

Exercise 42. Using the probability measure (42.12), give truth-tables for each of the following formulae:

1. $[R \geq \neg R]$
2. $\neg [[Q \wedge P] \geq [Q \wedge \neg P]]$
3. $[R \geq [P \geq Q]]$

We may call a formula *valid* if it's true in every structure and for every probability measure. If we are on the right lines so far, the valid formulae will be the formal counterparts of the necessary truths involving the sentence-functor (42.3). It is not hard to establish which formulae are valid, though it needs more mathematics than we have space for.† One can show for example that all the following formulae are valid:

$$\begin{aligned} & [P \geq P] && 42.18 \\ & [[P \geq Q] \wedge [Q \geq R]] \rightarrow [P \geq R] \\ & [[P \geq Q] \vee [Q \geq P]] \\ & [[P \geq Q] \rightarrow [\neg Q \geq \neg P]] \end{aligned}$$

† See Peter Gärdenfors, 'Qualitative Probability as an Intensional Logic', *Journal of Philosophical Logic*, 4 (1975), pp. 171–85.

The second formula of (42.18) assures us that when 'it' has been specified, the following sentence must necessarily be true:

If it's at least as likely to be a cold as flu, and 42.19
 at least as likely to be flu as measles, then it's
 at least as likely to be a cold as measles.

The other formulae can be similarly interpreted. Pursuing the analogy with section 23, one may also devise correct semantic sequents and interpret them into valid arguments.

This logic of likelihood is elegant and convincing; it meshes well with the mathematical theory of probability. Nevertheless our account of it has one major flaw which we must correct.

The flaw is a matter of interpretation. At the beginning of our discussion we imagined a shuffled pack of cards, and we noted that as far as the order of the pack was concerned, we could be in any one of several possible situations. Without saying so explicitly, we assumed that *every one of these situations was equally likely to be the actual situation*. If some orderings of the pack had been more likely than others – if, say, there was a suspicion that a clever shuffler had brought all the nines to the top – then (42.9) need no longer be true. In a similar way, the restriction to 'allowed' situations merely excluded those situations which we thought were too unlikely to take seriously. For example, if there was a chance that someone had quietly removed the diamonds from the pack, we should have to take this into account when we assessed the truth of (42.9).

For reasons such as these, we are forced to reinterpret probability measures as showing, not what proportion of the allowed situations assign the indicated truth-values, but *how likely it is* that the actual situation is one which assigns them. Once this reinterpretation is made, our calculations can no longer be justified by an appeal to (42.11). Instead we have to base them on our intuitive understanding of probability.

In other words, our calculus only tells us how to deduce likelihoods from other likelihoods. It is not clear how far likelihoods can be deduced from anything else. Certainly we all do estimate

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likelihoods all the time – what chance I'll reach the shops before closing time? Might James take offence? Will another drink make me feel sick? Is the ladder safe? Apparently we do it on the basis of the facts we know. But nobody has yet provided a complete and convincing account of how to deduce a likelihood from brute facts alone. Maybe it can't be done, and estimating likelihoods is fundamentally different from deducing them. Maybe it can, but only by arguments which are too long to set down on paper.

Here is an example of the difficulties. It used to be thought that if every S so far discovered is a P, then the discovery of one more S which was a P would make it more likely that every S is a P. But in fact this is not so. For instance, take the question whether recognizably human animals existed three and a half million years ago. Every human being discovered so far is less than three and a half million years old. On the earlier view, the discovery of another skull less than three and a half million years old should make it *less likely* that humans go back three and a half million years. But in fact if a decidedly human skull was found to be three and a quarter million years old, this would make it *more likely* that our human ancestors go back still further.

Likewise, if both the known sufferers from a rare bone disease are called John, then the discovery of a third John with the disease would make it more likely that there are people not called John who have the disease.

43. Intension

The attempt to form a logic of likelihood presented us with some difficulties, but there was a useful practical hint among them. Although the truth-value of a sentence of form ' $[\phi \geq \psi]$ ' in a given situation couldn't be calculated from the truth-values of ϕ and ψ in that one situation, it could be worked out once we knew the truth-values of ϕ and ψ in each and every situation. (We did also have to be given a probability measure to indicate

how likely the various situations were; but this was independent of the sentence under discussion.)

By the *intension* of a declarative sentence, we mean a correlation between situations and truth-values, which correlates to each situation the truth-value which the given sentence has in that situation. In particular, two sentences have the same intension precisely if they have the same truth-value as each other in every situation. All necessary truths have the truth-value Truth in all situations, so that all necessary truths have the same intension. (In essentially this sense, the term comes from Rudolf Carnap, 1947; any resemblance to intentionality with a 't' is an unfortunate coincidence.)

We can say, then, that the truth-value of ' $[\phi \supseteq \psi]$ ' in a situation is determined by the intensions of ϕ and ψ . It may be possible to represent the intensions of ϕ and ψ by columns of 'T's and 'F's in a truth-table; the truth-value of ' $[\phi \supseteq \psi]$ ' then becomes a matter of straightforward calculation, as we saw in section 42.

Analogy suggests a similar definition for designators. The *intension* of a designator D is a correlation between situations and things; to each situation in which D has a primary reference, the intension correlates this primary reference, while it correlates nothing to the remaining situations. To describe the intension of a designator, we can list, for each situation, whether the designator has a primary reference in the situation, and if so, what. For example, the intension of the designator the monarch of England would need entries such as the following, with the dates made more precise:

situation	primary reference	43.1
1603–25	James I of England	
1625–49	Charles I	
1649–60	none – no monarch	
1660–85	Charles II	
1685–88	James II	
1688–95	none – two monarchs	

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One can define *intension* in the same spirit for other kinds of expression too. For example, the *intension* of a predicate correlates to each situation (which is here supposed to include a domain) the relation which is expressed in that domain and situation by that predicate.

Because the truth-value of a sentence about likelihoods was determined by the intensions of constituent sentences, we were able to make a logic of likelihood. We should expect the same to hold quite generally: *if the truth-value of a sentence in a situation is determined in a describable way by the intensions of its constituents, then a system of logic is there to be discovered.*

Shall we look around and see what we can find?

(i) *Likelihood*

We have already discussed this.

(ii) *Necessity and possibility*

We may consider together a number of kinds of sentence, which are true in a situation precisely if some shorter sentence is true in *every one* of certain allowed situations. (These allowed situations are usually said to be *accessible*.) Examples are:

It's a **necessary truth** that one plus one is two. 43.2

(43.2) is true precisely if 'One plus one is two' is true in every possible situation.

Bob **couldn't** stop himself laughing. 43.3

(43.3) is true precisely if 'Bob is laughing' is true in every situation which Bob was capable of bringing about.

The needle **must be** somewhere in this room. 43.4

(43.4) is true precisely if 'The needle is in this room' is true in every situation which is compatible with what I know about the needle and the room.

The screw is **too big** to fit through the hole. 43.5

(43.5) is true precisely if in every possible situation in which the

hole has its present size, it's true that no screw as big as the present one is going through the hole.

Each of these sentences has a negative back half:

It's possible that one plus one is not two.	43.6
Bob was able to stop himself laughing.	43.7
The needle needn't be anywhere in this room.	43.8
The screw is small enough to fit through the hole.	43.9

The logic of necessity is called *modal logic*, and it has been widely studied.† You can discover some of it for yourself now if you try to rewrite section 42, replacing the sentence-functor (42.3) by the sentence-functor

It's a necessary truth that ϕ . 43.10

(The sentence-functor (43.10) is commonly symbolized ' $\Box\phi$ '.)

(iii) *Subjunctive conditionals*

These, as we saw on p. 101, are statements which talk of what would happen if something were to be the case or had been the case.

If I was Prime Minister I'd shoot all strikers on sight. 43.11

(43.11) is true in a situation S if in every possible situation which is like S in as many ways as possible, except that 'I am Prime Minister' is true in it, it is also true that I shoot all strikers on sight.

It is difficult to establish just when a possible situation should count as being like the present one in as many ways as possible except for some known falsehood being true in it. For this reason among others, politicians refuse to answer hypothetical questions. Nevertheless, a logic of subjunctive conditionals has been developed along the lines we have sketched out for likelihood and necessity.††

†J. Jay Zeman, *Modal Logic: The Lewis-Modal Systems*, Oxford University Press, 1973.

††David K. Lewis, *Counterfactuals*, Blackwell, 1973.

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Subjunctive conditionals are well worth studying, because they or their paraphrases play many important roles in reasoning. For example, many scientific concepts are defined in terms of subjunctive conditionals. To say that a substance is a gas is to say that it would contract in a certain way if increased pressure were applied to it. To say that a substance is poisonous is to say that it would cause harm if it were administered to organisms in certain ways.

As another example, English Law has been known to decide questions of reasonableness by appealing to what the man on the Clapham omnibus would have thought if he had been a witness. This legal criterion is admittedly something of a joke – there are after all many different men on Clapham omnibuses. But courts have used it for several decades as a rough and ready test.

Subjunctive conditionals appear also in moral reasoning. We can recognize them in the question ‘How would you like it if someone did that to you?’, and in the question ‘What if everybody went round doing that?’ It’s puzzling that so many people should be content to base their morality on wildly improbable hypothetical situations instead of the actual one; maybe future generations will regard it as a sign of lingering superstition.

Finally, it has been suggested that statements about likelihoods should be regarded as abbreviations for subjunctive conditionals. For example, on this view the statement

This coin is equally likely to fall heads or tails. 43.12

should be understood to mean

If this coin, or one just like it, were tossed an 43.13
infinite number of times, then the proportion
of heads would tend towards $1/2$.

(iv) *Tensed statements*

Some sentences about the past or future are true in the present if and only if certain other sentences were or will be true at certain other times. For example:

I have always been an idealist. 43.14

is true now if and only if the sentence 'I am an idealist' was true in every past situation in which I was anything.

I shall be in Oswestry tomorrow. 43.15

is true now if there is a time tomorrow at which the sentence 'I am in Oswestry' will be true.

I often get cramp. 43.16

is true now if at several times in the recent past the sentence 'I've got cramp' was true.

Henry was a prude until he married. 43.17

is true now if and only if there is a past situation in which 'Henry is just married' was true, such that in every earlier situation involving Henry, 'Henry is a prude' was true.

These four examples suggest that all tenses should be thought of as situation-shifters which change the relevant situation from the present to past or future ones. On this basis one can build up a logic of tenses, again very much as the logic of likelihood in section 42. But here is an example which shows that unexpected subtleties may be involved.

Consider the sentence

I have read Gibbon's *Decline and Fall*. 43.18

There is no present-tense English sentence ϕ such that (43.18) is true now if and only if ϕ was true at some time in the past. The sentence

I am reading Gibbon's *Decline and Fall*. 43.19

will not do, because it carries no implication that I shall finish the book, while (43.18) definitely implies that I have finished it. If we wish to regard (43.18) as the past-tense version of some present-tense sentence, we shall have to invent a new kind of present-tense sentence in order to do it.

44. The Quest for Ingredient X

Analysing more closely what we did in sections 42 and 43, we can see that we used the following facts about the intensions of the phrases under discussion:

The truth-value of a sentence in a situation is determined by the phrase-marker (underlying if need be) of the sentence together with the intensions of its constituent phrases; the truth-value can be calculated, at least in principle, by working out the intensions of the short constituent phrases first, then the longer ones in terms of the shorter ones, and so on up the phrase-marker, until the intension of the whole is reached; the intension of the whole sentence immediately tells us its truth-value in a given situation. 44.1

I say 'in principle' because the calculations might be inordinately long. Indeed they might be infinite, so that only an angel or a retired god could complete them. (In section 42, of course, they were all finite.)

In a way, (44.1) summarizes what logic is all about. Right from the start, we have sought rules which tell us *in what situations a set of sentences are all true*. Our logic is limited only by our knowledge of the rules which determine when a sentence is true in a situation. To lay down the general form of these rules is to draw the boundaries of logic.

What, then, are the rules which determine the truth-value of a sentence in a situation? What is their most general form?

There is no doubt, first, that the rules exist. For we all use them constantly. Whenever we judge that a sentence is true, we follow certain rules – unless you choose to believe in magic. Second, there seems no doubt that (44.1) is approximately the right general form for the rules. For the truth-value of a sentence in a situation is determined by the meaning of the sentence, and it is quite obvious that the meaning of a sentence is deter-

mined by the meanings of its constituent parts and the way they are put together. Also we understand the meaning of a complex sentence by understanding the meanings of its component words, hence the meanings of the shorter phrases, then the longer ones, and so on up to the whole. Introspection confirms this (see Exercise 44 below), and frankly no reasonable alternative procedure has ever been suggested.

Exercise 44. Watch yourself working out what the following sentence means:

His Majesty may by Order in Council transfer to, or make exercisable by, the Minister any of the functions of the Charity Commissioners in matters appearing to His Majesty to relate to education, and any such Order may make such provision as appears to His Majesty to be necessary for applying to the exercise of those functions by the Minister any enactments relating to the Charity Commissioners.

The only point in (44.1) which strikes a false note is the presence of intension, for two reasons. First, we can work out in our heads whether sentences are true, but it is utterly implausible that we can carry in our heads a complete specification of all the possible situations in which a sentence is true, or all the primary references which a designator takes in all possible situations. In general, intensions are far too complex and inscrutable to play any part in the rules by which we mortals work out truth-values.

The second reason why intension will have to be replaced is this. (44.1) is no longer true if we consider intentional contexts. (Alas for the infelicities of logical terminology!) For example, the following two sentences are necessary truths, and hence they have the same intension:

$$1 + 1 = 2. \quad 44.2$$

$$7,536,429 + 8,897,698 = 16,434,127.$$

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Nevertheless, one of the following sentences is true and the other is false:

It's obvious that $1 + 1 = 2$. 44.3

It's obvious that $7,536,429 + 8,897,698 =$
 $16,434,127$.

We see at once that some other feature of the sentences (44.2) besides their intension must be weaving an influence.

Similarly it is quite implausible to suggest that

Smith believes that ϕ . 44.4

is true precisely if it's true that ϕ in certain possible situations. Many people believe inconsistent things – perhaps we all do.

Intension is not only too complex, it is also too shallow to play the part we have auditioned it for.

The search is on: what Ingredient X can serve in place of intension, to make (44.1) into a prescription of the general form of the rules for determining truth-values?

The answer 'meaning' is easy; but it is vacuous, because we do not know how to write down a meaning, except by providing a word or phrase which has it. What is needed is a *formal representation* of meaning, so that the process of combining the meanings of constituent phrases into the meaning of the whole can be described in terms of an *explicit manipulation of the formal representations*. At least we had this in the case of intensions, as we saw in section 42 with likelihood. Without formal representations and explicit rules for manipulating them, nothing has been illuminated.

Having no satisfactory Ingredient X to report, let us speculate for one. Language is a feature of human beings. To represent human behaviour in formal terms, let us imagine ourselves constructing a robot who is to behave exactly like a human being. Being a robot, he does exactly what his internal instructions (known as his *programs*) tell him to do. The guts of a program may consist of a magnetic record on a tape, or an assembly of transistors; at any rate they are something that we can symbolize and set down on paper.

Suppose we give our robot a program which makes him use a certain proper name correctly – for example, it makes him point out the bearer of the proper name when he is asked to, and it makes him say ‘Hullo, Mr Sprott’ on the right occasions. We may then take this program to be the Ingredient X for the proper name.

Verbs, such as *stinks*, are not normally used on their own. The program which makes our robot use *stinks* correctly will tell him how to combine it with other words, including proper names, so as to make true statements. For example, the program will contain an instruction allowing him to refer to the program for a proper name such as *Rover*, so that he will be able to use correctly the sentence *Rover stinks*, and assent to it when it is true. Again we may take the program for *stinks* to be the Ingredient X for this verb. The robot himself combines this program and the program for *Rover* to form a complex program for *Rover stinks*, and we may take the complex program to be the Ingredient X for this declarative sentence.

Similarly each phrase will have its own program. The programs for the words which the robot knows are stored permanently in the robot, while the programs for complex phrases are built up by him out of the stored programs, as and when he needs to use the phrases. We propose to replace ‘intension’ by ‘program’ in (44.1), and offer it as the general form of the rules determining truth.

It might be objected that any such program would have to be hideously complicated. The answer is that the program would need to be no more or less complicated than whatever it is in a human being which records the meaning of a word which he understands; for a general theory, this seems to be exactly the right level of complexity.

It might be objected that two different programs may produce the same behaviour, so that our choice of Ingredient X is ambiguous. This is true, and a possible answer is to count two programs as interchangeable if they contain ‘essentially the same steps’. This is vague, but perhaps no more vague than the notion of two phrases ‘having the same meaning’.

We may now reconsider intentional contexts. First,

It's obvious that ϕ . 44.5

is true if and only if the method for determining whether it's true that ϕ , following the appropriate program for this sentence, leads to a positive answer in a very short time.

Next, we ask what features have to be built into our robot in order to give him beliefs. The beliefs of a human being constitute a kind of internal register which he uses to guide his behaviour. The register records YES or NO against various sentences, or more likely against some representation of the meanings of sentences. In computer terms, it's plausible that the appropriate YES or NO is called out from the register by the program for the sentence in question. Our robot, then, would have a mechanism which would recognize when he was about to work on the program for a particular sentence, and would promptly summon up the needed YES or NO from the internal register; the program itself would serve as the cue. In these terms,

Smith believes that ϕ . 44.6

is true precisely if YES is released from Smith's internal register by the program for this sentence.

It has been disputed whether or not sentences of the following form are necessary truths:

If one believes that ϕ , then one believes that 44.7
one believes that ϕ .

On the view I have just suggested, (44.7) is not a necessary truth. A robot could be made to summon up YES in response to the program for ϕ , but not to summon up anything at all in response to the program for 'I believe that ϕ .' Such a robot would have beliefs, but he would lack the concept of belief, just as very young children have beliefs but not the concept of belief.

It may be that further progress in these remote and mysterious steppes of logic must wait for a fundamental advance in our understanding of human psychology.

Answers to Exercises

Section 1

- 1A. If you think you succeeded, you probably overlooked the difference between believing a thing and imagining it.
- 1B.
1. Consistent, but highly improbable.
 2. As it stands, inconsistent; you can't literally know a thing that isn't true. Maybe the speaker only meant that she felt certain she would never get pregnant.
 3. As it stands, consistent; the people with nowhere to live might be people in Glasgow or Jersey City. Nevertheless there seems to be some serious confusion in the mind of the Mayor of Lincoln who made this ripe remark to the annual dinner and dance of Lincoln and District Association of Building Trades' Employers (as quoted on p. 70 of *This England*, New Statesman, 1960).
 4. Inconsistent. Did Walter pay for his own holiday? If he did, then he was a club member who did pay for his own holiday; so he didn't pay for his holiday. If he didn't, then he was a club member who didn't pay for his own holiday; so he did pay for his holiday. We reach an absurdity either way. (This is a version of Bertrand Russell's paradoxical barber, who shaved the villagers who didn't shave themselves.)
 5. This gāthā of the Ch'an Master Fu Ta Shih (quoted from p. 143 of Lu K'uan Yü, *Ch'an and Zen Teaching (Series One)*, Rider & Co., 1960) was surely meant to be inconsistent, and I think it succeeds. Followers of

Ch'an, or its Japanese version Zen, regard sayings like this as fingers pointing to a reality which is normally obscured by the daily round of emotions and calculations.

Section 3

- 3A. 1. Selection mistake.
2. Hopeless. Noam Chomsky thought up the two sentences 1 and 2 in his book *Syntactic Structures*, Mouton, The Hague, 1957, to illustrate a point about degrees of grammaticality. They have been frequently quoted since, for many purposes.
3. Perturbation of 'Please pass me some butter' or 'Please pass me the butter.'
4. Usually reckoned grammatical but informal.
5. This charming selection mistake is from e. e. cummings, *my father moved through dooms of love*.
6. From the same poem; hopeless, I'd say.
7. As 4.
8. Perturbation of 'His face was calm and relaxed, like the face of a sleeping child' or '... a child asleep'.
9. Perturbation of 'Father, do not prolong further our necessary defeat.' This is the opening couplet of a poem of W. H. Auden, presumably in mock imitation of the hymn-writers Tate and Brady, who in 1696 published verse translations of the Psalms. These two authors made grammatical perturbations their trademark. Specimen:

Deliverance he affords to all
Who on his succour trust.

10. Selection mistake, from *Burnt Norton*, T. S. Eliot.

- 3B. 1. Declarative sentence, as are all mathematical statements. Here and below, we shall overlook the fact that mathematical and scientific statements often contain

symbols and technical terms which are not in normal use in English.

2. Not declarative. Requests are not declarative.
3. Not declarative, although it expresses very much the same as the declarative sentence 'You're standing where Cromwell once stood.'
4. Declarative.
5. This is controversial. Those who believe it is *not declarative* say that the sentence is not used to state a belief but to make a promise. (The same would apply to sentences beginning 'I swear ...', 'I congratulate ...', 'I forbid ...', etc.) On this view, 'Is it true that I promise not to peep?' would presumably contain some kind of selection mistake, or worse. Those who believe 5 is *declarative* would argue that 'He promises not to peep' is undoubtedly declarative, and 5 is hardly a different kind of sentence.
6. Controversial, rather like 5. Most people would regard it as declarative. But some philosophers have argued that if a person says it, he isn't really making a statement at all; rather he is giving vent to strong feelings of disapproval of adultery. This view tends to diminish the relevance of logic to moral thinking.

Section 4

- 4A.
1. Cross-reference: '... *her* stern as *she* slid gracefully down the slipways'.
 2. Lexical: '*pulled ... through a difficult passage*' meant literally or metaphorically.
 3. Both lexical and structural. (1) cases = instances, and he admitted two cases. (2) cases = containers, and he admitted stealing two cases.
 4. Cross-reference: both ends of what?
 5. Lexical: (1) bare bones = essentials, (2) bare bones = uncovered pieces of skeleton.
 6. Structural: (1) were strongly urged, by the Liverpool

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justices, to take steps; (2) were strongly urged to put a stop to the evil, namely, Liverpool justices drinking methylated spirits.

(The items in this exercise were taken from Denys Parsons, *Funny Amusing and Funny Amazing*, Pan, 1969, with one slight alteration.)

- 4B. 1. a. I shall wear no clothes that would distinguish me from my Christian brethren.
b. To distinguish me from my Christian brethren, I shall wear no clothes.
2. a. He never relaxes except on Sundays.
b. On Sundays he does nothing but relax.
3. a. He gave each guest either a glass of rum or a glass of gin and tonic.
b. He gave each guest either a glass of rum and tonic or a glass of gin and tonic.
4. a. There is one person who has in his hands most of the nation's assets.
b. Most of the nation's assets are in the hands of individuals. (Not perfect, but hard to improve in less than five lines.)

Section 5

5. he = the man; he = the man; him = Jacob; he = the man; he = Jacob?; him = the man?; he = the man; me = the man; he = Jacob; I = Jacob; thee = the man; thou = the man; me = Jacob; he = the man; him = Jacob; he = Jacob; he = the man; thou = Jacob; him = the man; me = Jacob; I = Jacob; thee = the man; he = the man; thou = Jacob; he = the man; him = Jacob.

Section 6

6. 1. scaling. 2. scaling. 3. not scaling. 4. not scaling. 5. scaling.

6. scaling. 7. not scaling. 8. not scaling. 9. scaling. 10. not scaling. (Some may be controversial.)

Section 8

- 8.1. The alderman had been asked how much it cost to buy the land for the new school. He answered in good faith by reporting what he himself had been told, namely that it cost half a million pounds. However, his informant had misread the records, and the true cost was three-quarters of a million. (First sentence of exercise false, second true.)
2. If your creditors catch up with you they will ruin you. Never mind; a friend of mine has a place in Corsica where you can go underground for a year or so, and I can fix it up for you. It's a shame that the Corsican food doesn't suit your digestion. (First sentence true, second false.)
3. I'm not normally at home at this hour, because it's my evening for Yoga. But they cancelled the class, so I stayed at home. The man watching the house had been told I was at the class; but he saw Tim walking into the house just now, and he thought it was me coming home from the class. Tim and I are often mistaken for each other. (First sentence false, second true.)
4. Brutus mentioned to Servilius Casca that Tillius Cimber had sworn to pay a large sum of money to anybody who would kill Caesar. This inspired Casca – who had gambling debts – to kill Caesar by stabbing him through the heart. (First sentence false, second true.)
5. We anthropologists are learning the local Bushman language fast. We have just learned that 'crow' is their word for a daddy longlegs. Incidentally I showed some of them a photo of a carrion crow, and they said the bird must have been painted. (First sentence true, second false.)

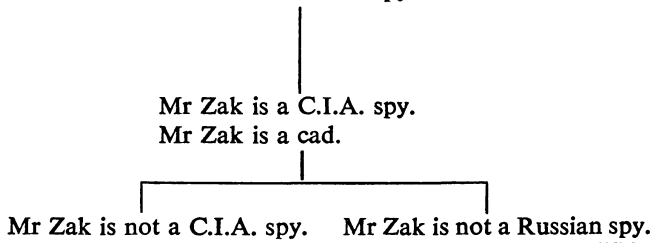
Section 9

9. All inconsistent except 2 and 6.

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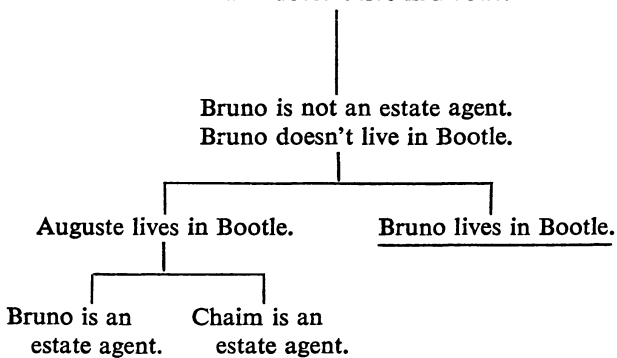
Section 10

- 10.1. Mr Zak is a Russian spy.
 ✓ Mr Zak is not both a C.I.A.
 spy and a Russian spy.
 ✓ Mr Zak is a C.I.A. spy and a cad.



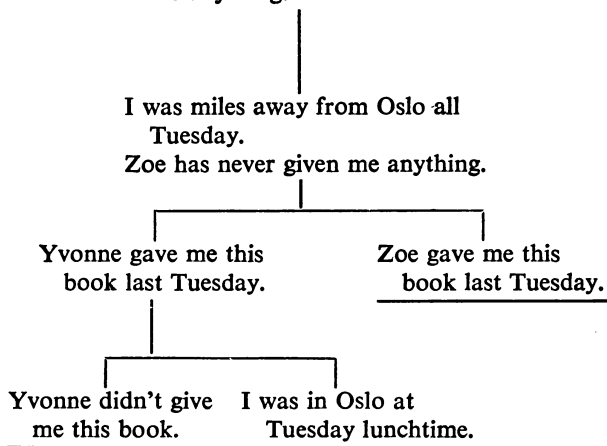
A closed tableau; therefore *inconsistent*.

2. ✓ At least one of Auguste
 and Bruno lives in Bootle.
 ✓ At least one of Bruno and
 Chaim is an estate agent.
 ✓ Bruno is not an estate agent,
 and doesn't live in Bootle.



There is an unclosed branch; it describes the possible situation that Auguste lives in Bootle, Bruno doesn't live in Bootle and is not an estate agent, and Chaim is an estate agent. In this situation the three sentences of the exercise are all true. Therefore they are *consistent*.

3. ✓ Either Yvonne or Zoe gave me
 this book last Tuesday.
 ✓ If Yvonne gave me this book then
 I was in Oslo at Tuesday lunchtime.
 ✓ I was miles away from Oslo all
 Tuesday, and Zoe has never given
 me anything.



A closed tableau; therefore *inconsistent*.

Section 11

- 11.1. Two hundred people are dying every day. *Therefore* help is needed urgently.
2. When Communists operate as a minority group within unions, settlements by the established officials must be

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- denounced as sellouts. *Therefore* strikes are unlikely to wither away in any democratic country so long as Communists have strong minority influence.
- The nests of the verdin are usually placed at or near the end of a low branch. *Therefore* the nests of the verdin are surprisingly conspicuous.
 - The effect of salicylates on renal uric acid clearance is greater than the effect which ACTH has on it. *Therefore* the effect of ACTH on gout is not due to the increased renal uric acid clearance.
 - If no contribution to the magnetic field comes from electric currents in the upper atmosphere, then we cannot account for the relation between the variations in the magnetic elements and the radiation received from the sun. *Therefore* some contribution to the magnetic field comes from electric currents in the upper atmosphere.

Section 12

12.	health	a wife	wife	girls	two girls
He wanted to have <i>x</i> .	✓	✓	×	✓	✓
He wanted to have the <i>x</i> .	×	×	✓	✓	?
He wanted to have more <i>x</i> .	✓	×	×	✓	×

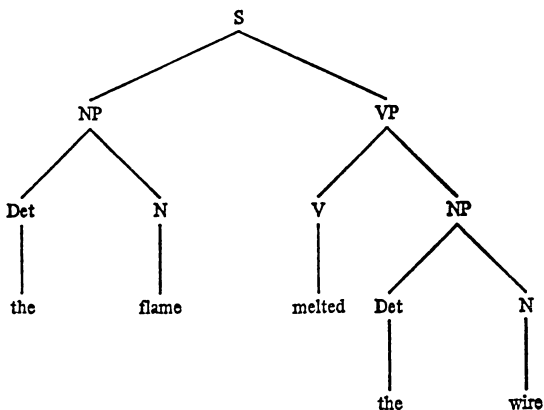
The question-mark in the last column reflects my doubt about whether 'two girls' is a constituent of the grammatical sentence 'He wanted to have the two girls.'

Section 13

- 13A. 1. both pigeons revealed surprising adaptability
 2. S, NP, VP, Det, N, V, Adj.

3. 1 [both pigeons revealed surprising adaptability]
- 2 [both pigeons] revealed surprising adaptability
- 3 [both] pigeons revealed surprising adaptability
- 4 both [pigeons] revealed surprising adaptability
- 5 both pigeons [revealed surprising adaptability]
- 6 both pigeons [revealed] surprising adaptability
- 7 both pigeons revealed [surprising adaptability]
- 8 both pigeons revealed [surprising] adaptability
- 9 both pigeons revealed surprising [adaptability]

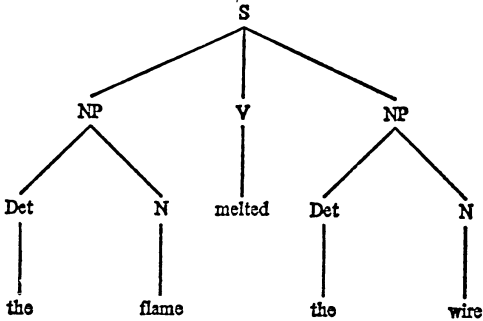
13B. Most linguists would give the phrase-marker



The constituents are then as follows:

- 1 [the flame melted the wire]
- 2 [the flame] melted the wire
- 3 [the] flame melted the wire
- 4 the [flame] melted the wire
- 5 the flame [melted the wire]
- 6 the flame [melted] the wire
- 7 the flame melted [the wire]
- 8 the flame melted [the] wire
- 9 the flame melted the [wire]

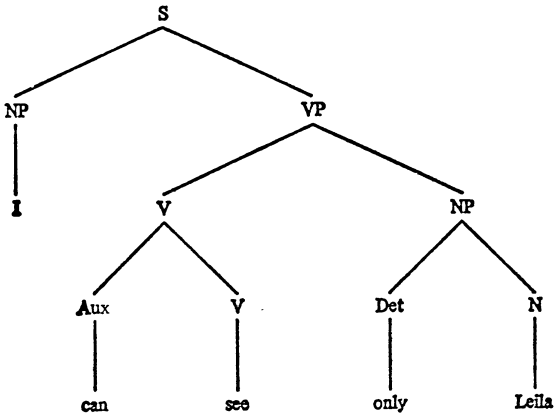
But the following phrase-marker is not definitely wrong:



The constituents are then the same as before, except that number 5 disappears.

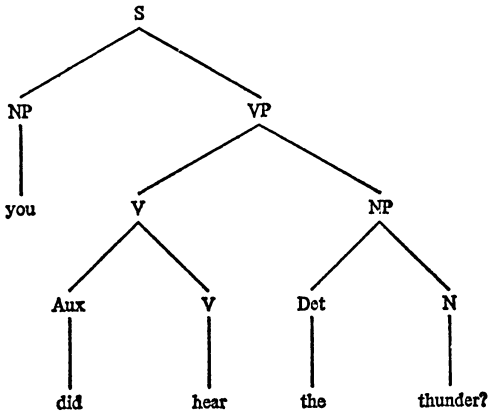
13C. Several details of the phrase-markers given below could well be different. But your answers should have the stated constituents.

1.



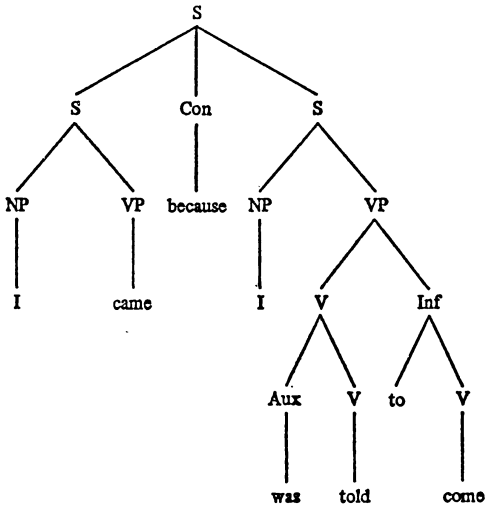
'only Leila' is thus an underlying constituent. (Aux = auxiliary.)

2. As a first approximation,



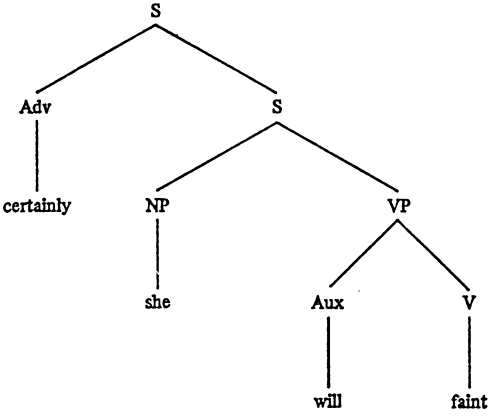
'did hear' is thus an underlying constituent. The question-mark is sometimes taken as a separate constituent too.

3.



'to come' is thus an underlying constituent.

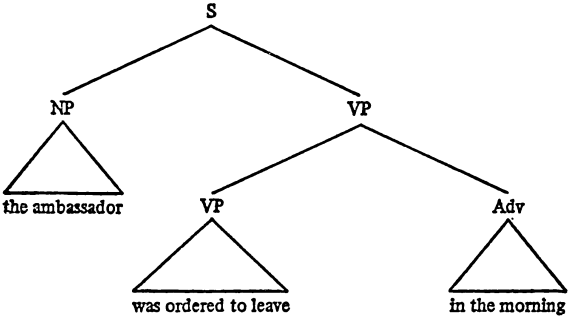
4.

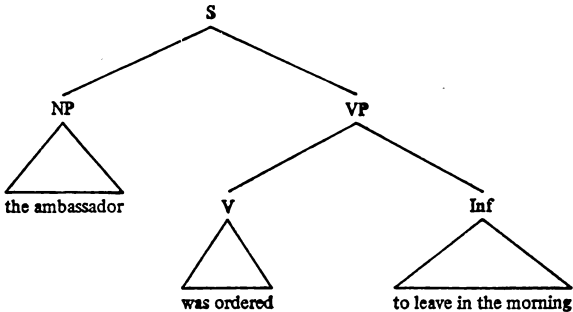


'will faint' is thus an underlying constituent.

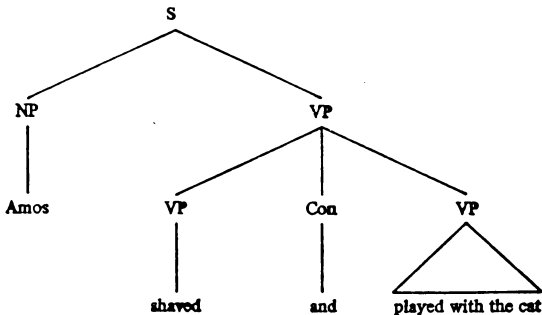
Section 14

14.1. First reading:



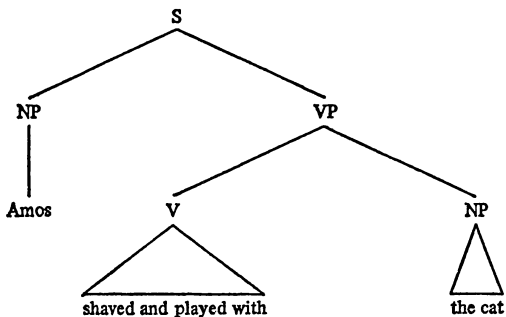
Second reading:

Alternatively you could attach 'in the morning' to the whole sentence 'the ambassador was ordered to leave' in the first reading. The central point is that in the second reading, the scope of 'in the morning' is 'to leave in the morning', while on the first reading the scope of this phrase includes 'was ordered'.

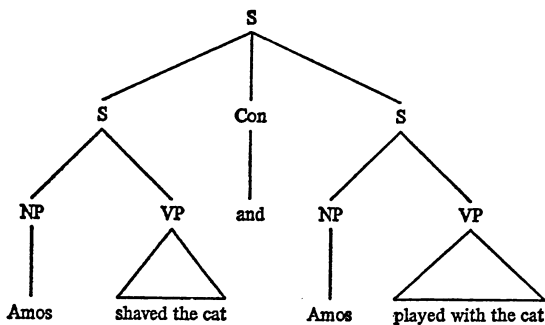
2. First reading:

LOGIC

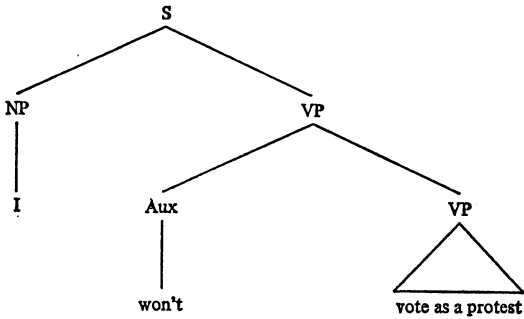
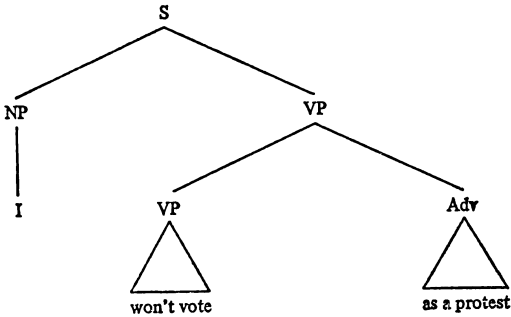
Second reading:



It seems likely that the second reading has an underlying phrase-marker

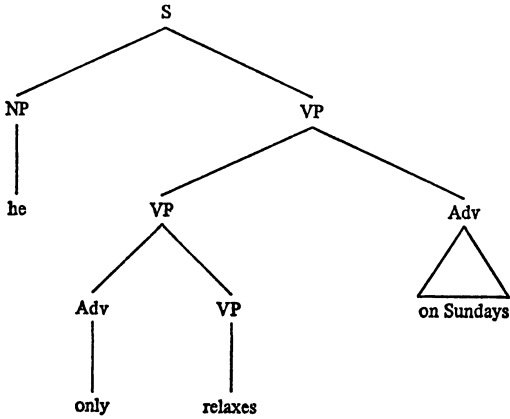


or something similar; note that 'shaved the cat' is a constituent in this underlying phrase-marker.

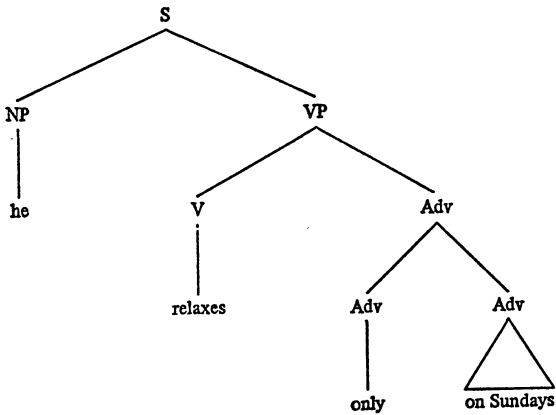
3. *First reading:**Second reading:*

Note that the scope of 'won't' is 'won't vote as a protest' in the first reading, but only 'won't vote' in the second.

4. *First reading*:



The *second reading* needs an underlying phrase-marker with 'only' shifted along so that its scope is 'only on Sundays':



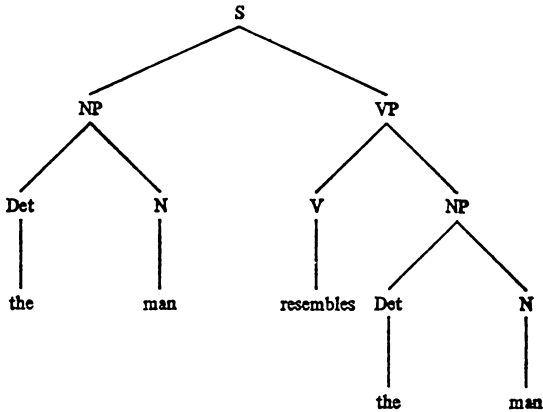
Section 15

15A. Initial symbol: S

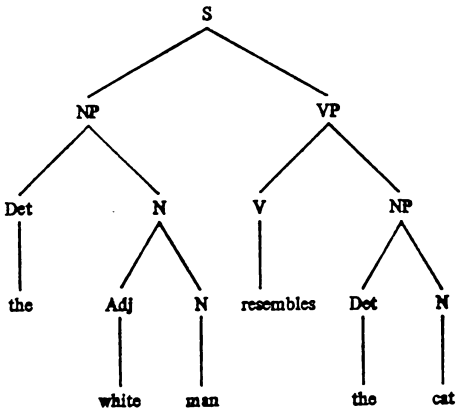
Non-terminal symbols: S, VP, NP, N, Det, Adj, V.

Terminal symbols: that, the, white, man, cat, resembles.

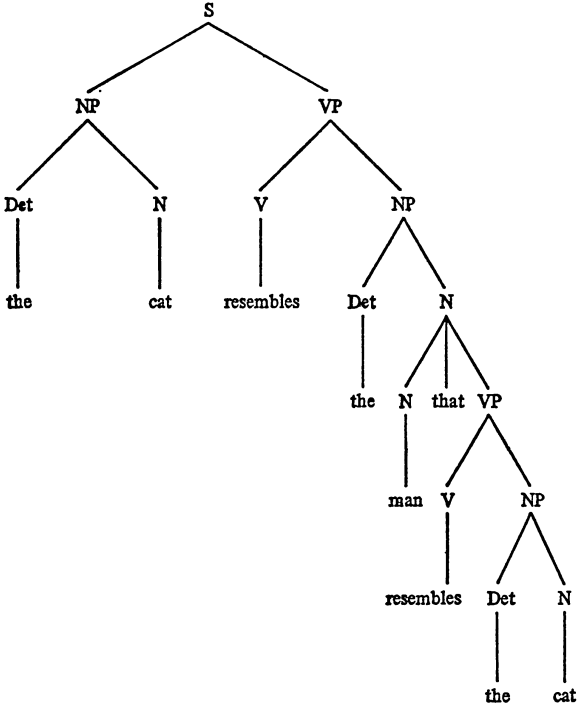
15B.1.



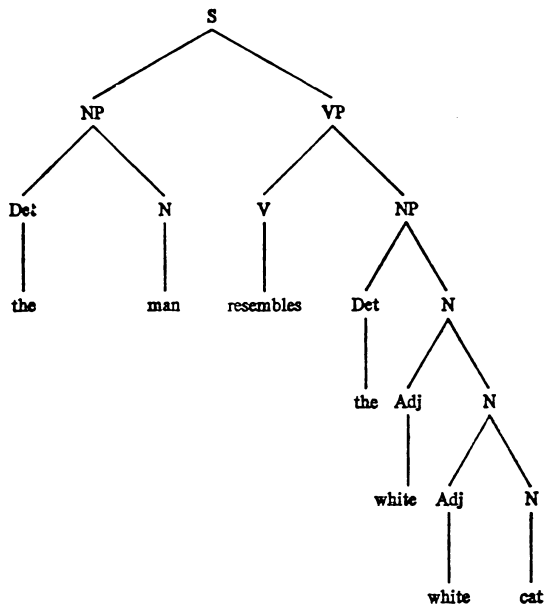
2.



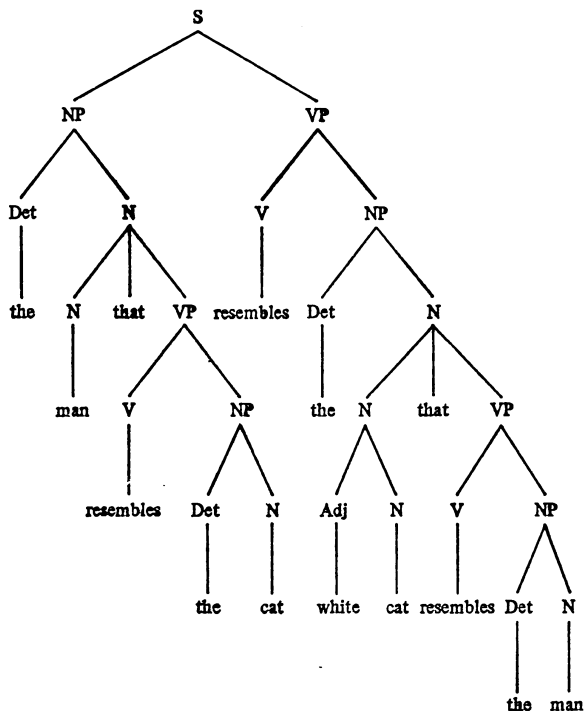
3.



4.



5.



- 15C
- S ⇒ O
 - S ⇒ LP O
 - S ⇒ O RP
 - O ⇒ boy
 - O ⇒ sock
 - O ⇒ mommy
 - LP ⇒ allgone
 - LP ⇒ byebye
 - RP ⇒ off
 - RP ⇒ on
 - RP ⇒ fall

The child has three sorts of word. (i) There are the open-class words (classed as O in the CF grammar) which can occur at any position in a sentence, provided the sentence has at most one of them. (ii) There are the left pivot words (classed as LP) which can only occur as the first word of a two-word sentence. (iii) There are the right pivot words (RP), like the left pivots but coming second. A right pivot is never combined with a left pivot. This grammar apparently represents a stage that children all over the world pass through. I took the example from P. S. Dale *Language Development*, Dryden Press, Hinsdale, Ill., 1972, p. 41.

Section 16

- 16A. ϕ_1 , and then at once ϕ_2 , so that ϕ_3 ; ϕ_4 .
 I scattered the strong warriors of Hadadezer.
 I pushed the remnants of his troops into the Orontes.
 They dispersed to save their lives.
 Hadadezer himself perished.

16B. 1.	ϕ	It's a lie that ϕ
	T	F
	F	—

2.	ϕ	ψ	ϕ because ψ .
	T	T	—
	T	F	F
	F	T	F
	F	F	F

3.	ϕ	ψ	ϕ whenever ψ .
	T	T	—
	T	F	—
	F	T	F
	F	F	—

(The truth-values at just one time of the sentences put for ' ϕ ' and ' ψ ' can never suffice to make the whole sentence true, since the whole sentence talks about all the times at which the second constituent is true.)

4. ϕ	If ϕ then ϕ .
T	T
F	T

5. ϕ	Whether or not ϕ , what will be will be.
T	T
F	T

6. ϕ	Whether or not ϕ , smoking causes cancer.
T	—
F	—

(The truth-value in 6 is the same as that of 'Smoking causes cancer', which is true in some possible situations and false in others.)

- 16C. 1. I am aware that [you intend to sue].
 2. He regrets that [he didn't marry Suzy].
 3. [He completed his task] before [the week ended].
 4. [The train had an accident] because [its brakes failed].
 5. *One possible answer:* Your Majesty may be pleased to notice that [great mischiefs may fall upon this kingdom] if [the intentions of bringing in Irish and foreign forces shall take effect]; it has been credibly reported that [there are intentions of bringing in Irish and foreign forces].

Section 17

17. 1. \neg Some dogs will be admitted. (NOT ' \neg Dogs will be admitted'; think of the case where some dogs will be admitted and others won't.)

2. [the brain is bisected \wedge the character remains intact]
3. [the safety conditions will be tightened \vee there is going to be a nasty accident]; *or* [there is going to be a nasty accident \vee the safety conditions will be tightened]
4. [you're right \rightarrow I stand to lose a lot of money]
5. [you broke the law \leftrightarrow the agreement formed a contract]
6. [somebody will call \rightarrow I shall pretend I am designing St Paul's]
7. [Schubert is terrific \wedge Hindemith is terrific]. ('So' carries a cross-reference.)
8. [this is Bert Bogg \wedge Bert Bogg taught me that limerick I was quoting yesterday]
9. [you can claim the allowance \rightarrow you earn less than £16 a week]. (' \leftrightarrow ' would be wrong; there may be other conditions you have to satisfy besides earning less than £16 a week.)
10. [Liszt is horrible \wedge Vivaldi is horrible]
11. [she needs all the help she can get \wedge she's a single parent]
12. [the elder son was highly intelligent \wedge the younger son was an imbecile]. (The second underlying constituent sentence has 'younger son'.)
13. \neg Her performance had zest.
14. [he'll get something right \rightarrow he'll pass]
15. [the metal will stretch \vee the metal will snap]. (Another cross-reference.)

Section 18

- 18A. 1. [the nappies are becoming hard \rightarrow you can soften the nappies by using a water conditioner]
2. *Impossible.*
 3. *Impossible.*
 4. [most of his former protection has worn off \rightarrow his new vaccination develops much like the previous one]
 5. *Impossible.*
 6. [your baby is colicky \rightarrow your baby may be soothed when you first pick him up]

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- 18B. 1. *Impossible.*
2. *Impossible.*
3. [the evening was thoroughly pleasant \wedge the evening concluded with a waltz]
4. *Impossible.*
5. [Matilda's aunt had from earliest youth kept a strict regard for Truth \wedge Matilda's aunt attempted to believe her]
6. *Impossible.*
7. [Marianne is a teacher \wedge Marianne should have known better]
8. [Britain was once a superpower \wedge Britain is now seeking a new role]
9. *Impossible.* (Don might grow into a spectacle-free pedant, then into a bespectacled non-pedant.)

Section 19

- 19.1. $\neg \neg$ I shall write to you.
or just I shall write to you.
2. [he was gassed \wedge \neg he was shot]
3. [\neg he was gassed \wedge \neg he was shot]
or \neg [he was gassed \vee he was shot]
4. [Tracey will scream again \rightarrow \neg somebody will get some chocolate]
5. [the private schools provide an excellent education \wedge [\neg the State schools have adequate space \rightarrow the private schools ease the burden on the State's facilities]]
6. [[the private schools are socially top-heavy \rightarrow the private schools are perpetuating social injustice] \wedge [the private schools are perpetuating social injustice \rightarrow \neg the private schools can reasonably demand charitable status]]

Section 20

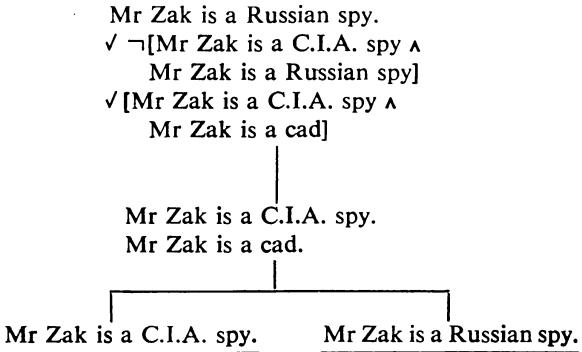
20A. See pp. 321–2 for the tableau derivation rules.

20B. 1. *Set of sentences:*

Mr Zak is a Russian spy. \neg [Mr Zak is a C.I.A. spy \wedge

Mr Zak is a Russian spy]. [Mr Zak is a C.I.A. spy \wedge
Mr Zak is a cad]

Tableau:



Inconsistent.

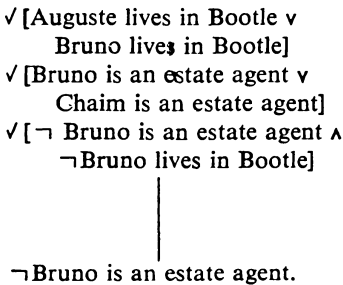
2. *Set of sentences:*

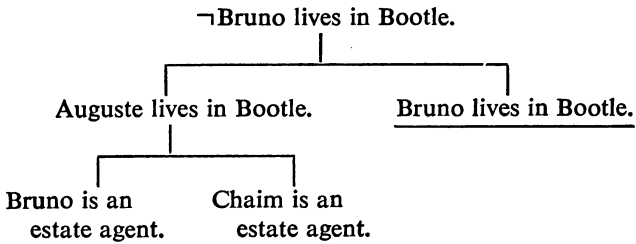
[Auguste lives in Bootle \vee Bruno lives in
Bootle]

[Bruno is an estate agent \vee Chaim is an estate
agent]

[\neg Bruno is an estate agent \wedge \neg Bruno lives
in Bootle]

Tableau:





Consistent: middle branch won't close.

3. *Set of sentences*:

[Yvonne gave me this book last Tuesday \vee Zoe gave me this book last Tuesday]. [Yvonne gave me this book \rightarrow I was in Oslo at Tuesday lunchtime]. [I was miles away from Oslo all Tuesday \wedge Zoe has never given me anything]

Tableau:

- \checkmark [Yvonne gave me this book last Tuesday \vee Zoe gave me this book last Tuesday]
- \checkmark [Yvonne gave me this book \rightarrow I was in Oslo at Tuesday lunchtime]
- \checkmark [I was miles away from Oslo all Tuesday \wedge Zoe has never given me anything]

I was miles away from Oslo all Tuesday.
Zoe has never given me anything.

├──────────┬──────────┤

Yvonne gave me this book last Tuesday. Zoe gave me this book last Tuesday.

├──────────┬──────────┤

\neg Yvonne gave me this book. I was in Oslo at Tuesday lunchtime.

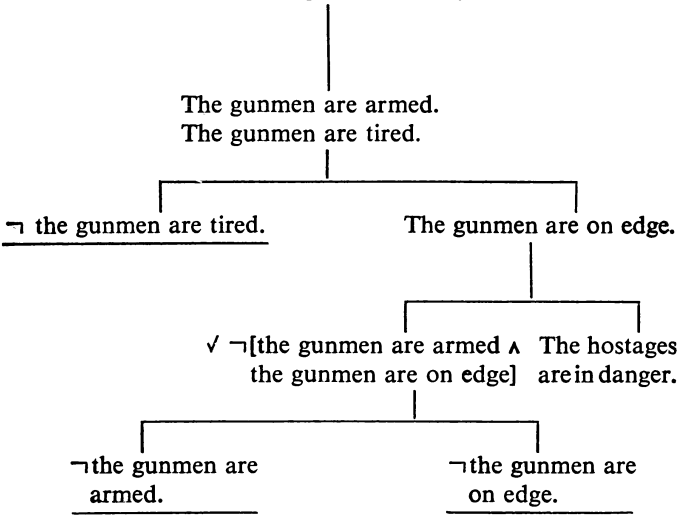
Inconsistent.

- 20C. 1. [the gunmen are tired \rightarrow the gunmen are on edge].
 [[the gunmen are armed \wedge the gunmen are on edge] \rightarrow
 the hostages are in danger]. [the gunmen are armed \wedge
 the gunmen are tired]. *Therefore* the hostages are in
 danger.
2. [the driver was in control \rightarrow [the driver passed the
 signal \leftrightarrow the signal was green]]. [the driver passed the
 signal \wedge the driver was in control]. [the signal was
 green \rightarrow the electronics were faulty]. *Therefore* the
 electronics were faulty.
3. [the boy has spots in his mouth \rightarrow the boy has measles].
 [the boy has a rash on his back \rightarrow the boy has heat-
 spots]. The boy has a rash on his back. *Therefore* \neg the
 boy has measles.
4. [the vicar shot the earl \vee the butler shot the earl].
 [the butler shot the earl $\rightarrow \neg$ the butler was drunk at
 nine o'clock]. [the vicar is a liar \vee the butler was drunk
 at nine o'clock]. *Therefore* [the vicar is a liar \vee the vicar
 shot the earl].

LOGIC

20D. 1.

\checkmark [the gunmen are tired \rightarrow
 the gunmen are on edge]
 \checkmark [[the gunmen are armed \wedge the
 gunmen are on edge] \rightarrow the
 hostages are in danger]
 \checkmark [the gunmen are armed \wedge
 the gunmen are tired]
 \neg the hostages are in danger.



Valid.

2.

- \checkmark [the driver was in control \rightarrow
 [the driver passed the signal
 \leftrightarrow the signal was green]]
 \checkmark [the driver passed the signal \wedge
 the driver was in control]
 \checkmark [the signal was green \rightarrow
 the electronics were faulty]
 \neg the electronics were faulty.

The driver passed the signal.
 The driver was in control.

\neg the driver was in control.

\checkmark [the driver passed the signal
 \leftrightarrow the signal was green]

\neg the signal
 was green.

The electronics
 were faulty.

The driver passed
 the signal.
 The signal was green.

\neg the driver passed
 the signal.
 \neg the signal was green.

Valid.

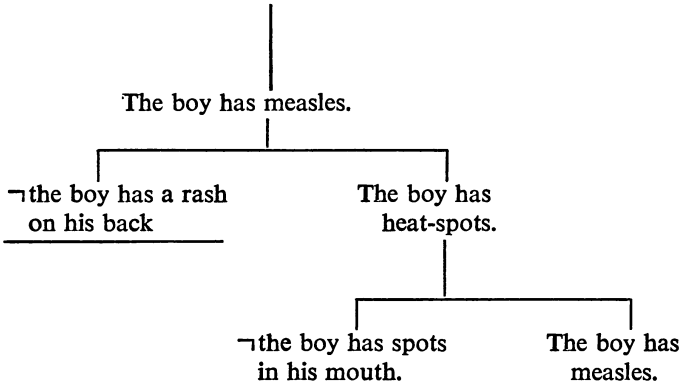
3.

✓ [the boy has spots in his mouth
→the boy has measles]

✓ [the boy has a rash on his back
→the boy has heat-spots]

The boy has a rash on his back.

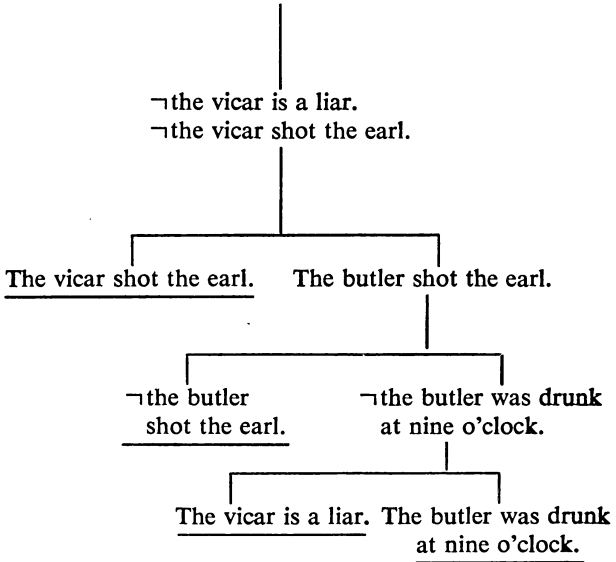
✓ $\neg\neg$ the boy has measles.



Invalid. A counterexample is the situation where the boy has both measles and heat-spots, and a rash on his back (see the right-hand branch).

4.

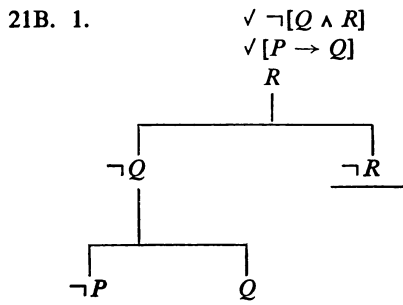
- \checkmark [the vicar shot the earl \vee
the butler shot the earl]
 \checkmark [the butler shot the earl \rightarrow
 \neg the butler was drunk at nine o'clock]
 \checkmark [the vicar is a liar \vee
the butler was drunk at nine o'clock]
 \checkmark \neg [the vicar is a liar \vee
the vicar shot the earl]

*Valid.*

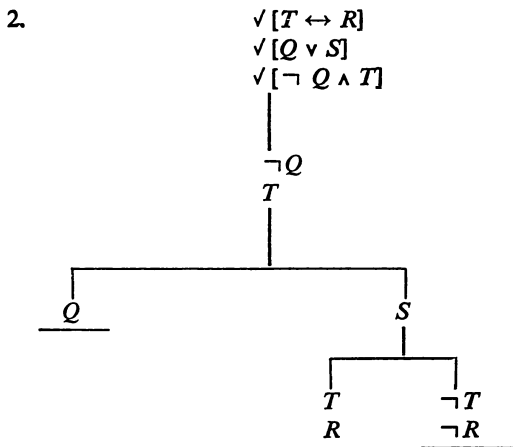
LOGIC

Section 21

- 21A. 1. $\neg[Q \wedge R]$. $[P \rightarrow Q]$. R .
 2. $[T \leftrightarrow R]$. $[Q \vee S]$. $[\neg Q \wedge T]$.
 3. $[[Q \vee S] \rightarrow P]$. $[R \rightarrow T]$. $[[R \vee S] \wedge \neg T]$. $\neg P$.

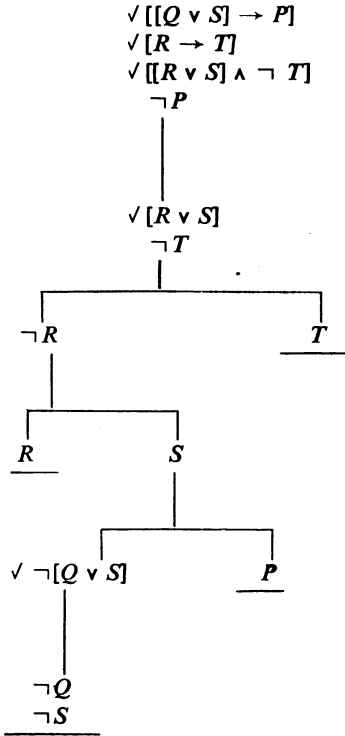


The left-hand branch describes a situation in which this income derives from an annuity taxed under Schedule C and not from employment, and tax is not levied on this income under Schedule E. There is no contradiction here; so the set is *consistent*.



ANSWERS TO EXERCISES

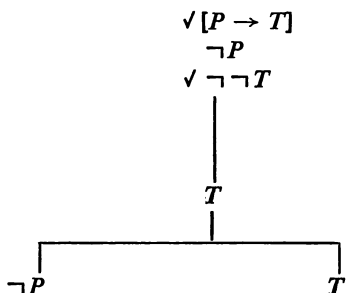
The central branch fails to close; but it describes a situation in which the source of this income is an annuity which both is and is not taxed under Schedule C - impossible. The other two branches are closed. Therefore the set is *inconsistent*.



Inconsistent.

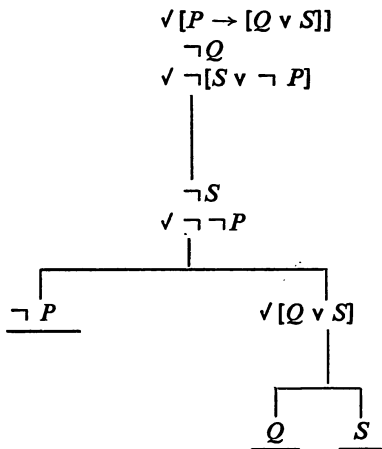
LOGIC

21C. 1. $[P \rightarrow T]. \neg P. \text{ Therefore } \neg T.$



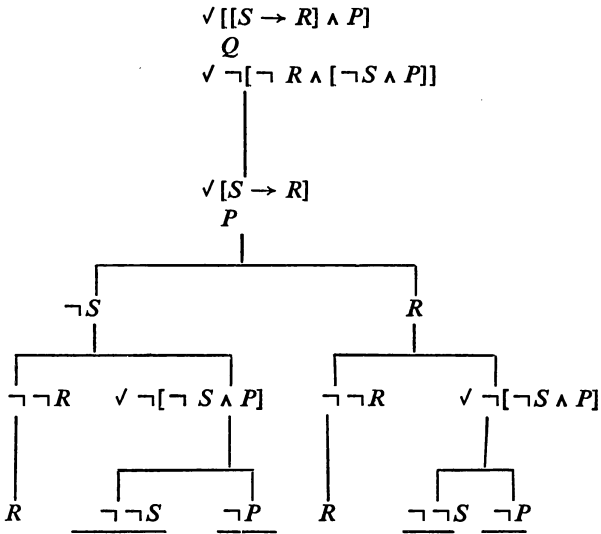
Invalid. The left branch gives the counterexample: Uhha-muwas has not bitten off Pissuwattis' nose, but is liable to a 30 shekel fine (say, for a traffic offence). The right branch also gives the same counterexample.

2. $[P \rightarrow [Q \vee S]]. \neg Q. \text{ Therefore } [S \vee \neg P].$



Valid. Tableau closed.

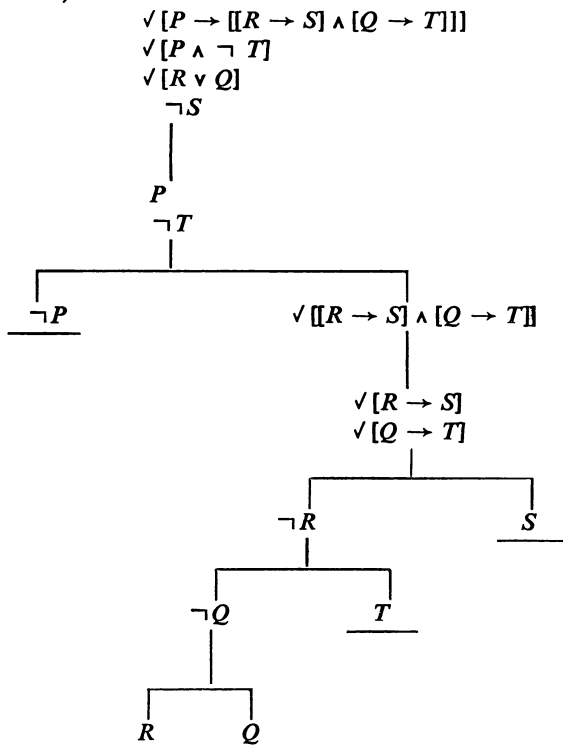
3. $[[S \rightarrow R] \wedge P]. Q. \text{ Therefore } [\neg R \wedge [\neg S \wedge P]].$



Valid. The tableau is not closed, but each of the unclosed branches describes a situation where Pissu-wattis is both a female slave (Q) and a free woman (R), which is impossible.

4. $[P \rightarrow [(R \rightarrow S) \wedge (Q \rightarrow T)]]$. $[P \wedge \neg T]$. $[R \vee Q]$.

Therefore S . (Note that 'or' in the first sentence means ' \wedge '.)

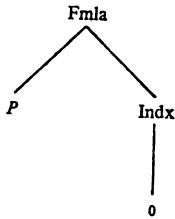


Valid. Tableau closed.

(According to the laws of the ancient Hittites, 'If anyone bites off a free man's nose, he shall give 1 mina of silver and pledge his estate as security. If anyone bites off the nose of a male or female slave, he shall give 30 shekels of silver and pledge his estate as security.')

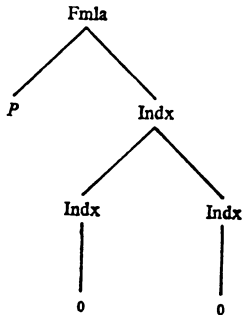
Section 22

22.1. Formula.



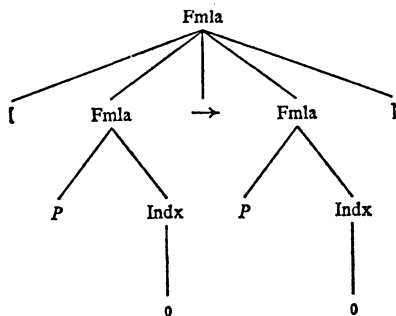
The only subformula is ' P_0 ' itself.

2. Formula.



The only subformula is ' P_{00} ' itself. (' P_0 ' is a formula but not a constituent of ' P_{00} ', as the phrase-marker shows.)

3. *Formula.*



There are three subformulae:

- | | |
|-----|-------------------------|
| | $[P_0 \rightarrow P_0]$ |
| (1) | — |
| (2) | — |
| (3) | — |

4. *Not a formula.*
5. *Not a formula* (needs more brackets).
6. *Not a formula* (shouldn't have brackets).
7. *Not a formula.*

LOGIC

2. P	Q	$[[P \wedge Q] \vee [P \wedge \neg Q]]$
T	T	T T T T T T F F T
T	F	T F F T T T T F
F	T	F F T F F F F T
F	F	F F F F F F F T F

3. P	Q	$[[P \rightarrow Q] \rightarrow [P \leftrightarrow Q]]$
T	T	T T T T T T T
T	F	T F F T T F F
F	T	F T T F F F T
F	F	F T F T F T F

4. P	Q	$[[[P \rightarrow Q] \rightarrow P] \rightarrow P]$
T	T	T T T T T T T
T	F	T F F T T T T
F	T	F T T F F T F
F	F	F T F F F T F

5. P	Q	$[[P \wedge Q] \wedge [\neg P \vee \neg Q]]$
T	T	T T T F F T F F T
T	F	T F F F F T T F
F	T	F F T F T F T F T
F	F	F F F F T F T F

6. P	Q	R	$[[[P \rightarrow Q] \wedge [Q \rightarrow R]] \rightarrow [P \rightarrow R]]$
T	T	T	T T T T T T T T T
T	T	F	T T T F T F F T T F F
T	F	T	T F F F F T T T T T T
T	F	F	T F F F F T F T T F F
F	T	T	F T T T T T T T F T T
F	T	F	F T T F T F F T F T F
F	F	T	F T F T F T T T F T T
F	F	F	F T F T F T F T F T F

23B. 1, 4 and 6 are tautologies; the rest are not.

23C. 1. $P \quad Q \quad | \quad P. [P \rightarrow Q] \vDash$

T	T	T	T
T	F	T	F
F	T	F	T
F	F	F	T

Incorrect. Counterexample

P	Q
T	T

2. $P \quad Q \quad | \quad P \vDash [Q \rightarrow P]$

T	T	T	T
T	F	T	T
F	T	F	F
F	F	F	T

Correct.

3. $P \quad Q \quad | \quad P. \neg P \vDash Q$

T	T	T	F	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

Correct: from a contradiction anything follows.

4. $P \quad Q \quad | \quad [Q \rightarrow P]. \quad [Q \rightarrow \neg P] \vDash \neg Q$

T	T	T	F	F
T	F	T	T	T
F	T	F	T	F
F	F	T	T	T

Correct.

LOGIC

5. P	Q	R	$[P \rightarrow Q]$	$[Q \rightarrow R]$	$\vdash [R \rightarrow P]$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	F
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	T

Incorrect. Counterexample $\begin{array}{ccc} P & Q & R \\ \hline F & T & T \end{array}$

6. P	Q	R	$[R \leftrightarrow [P \vee Q]]$	$\vdash [[R \wedge \neg P] \rightarrow Q]$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	F	F
F	F	F	T	T

Correct.

Section 24

- 24A. 1. Tautology 2. $P \mapsto [Q \rightarrow R]$
 2. Tautology 33. $Q \mapsto [P \rightarrow P]$
 3. Tautology 35. $Q \mapsto [P \rightarrow P]$
 $R \mapsto P$
 4. Tautology 26. $P \mapsto [Q \leftrightarrow R]$
 $Q \mapsto P$

- 24B. 1. $\neg [\neg [P \wedge \neg Q] \wedge \neg [\neg P \wedge Q]]$
 2. $\neg [[\neg P \wedge \neg Q] \wedge \neg R]$
 3. $[\neg [\neg P \wedge \neg Q] \wedge \neg P]$

- 24C. $[[[\phi \wedge \psi] \wedge \chi] \vee [[\phi \wedge \neg \psi] \wedge \neg \chi]] \vee [[\neg \phi \wedge \psi] \wedge \chi]$

- 24D. ' $\neg R$ ' is the simplest interpolant. The method described in the text would lead to the interpolant ' $[(Q \wedge \neg R) \vee (\neg Q \wedge \neg R)]$ '.

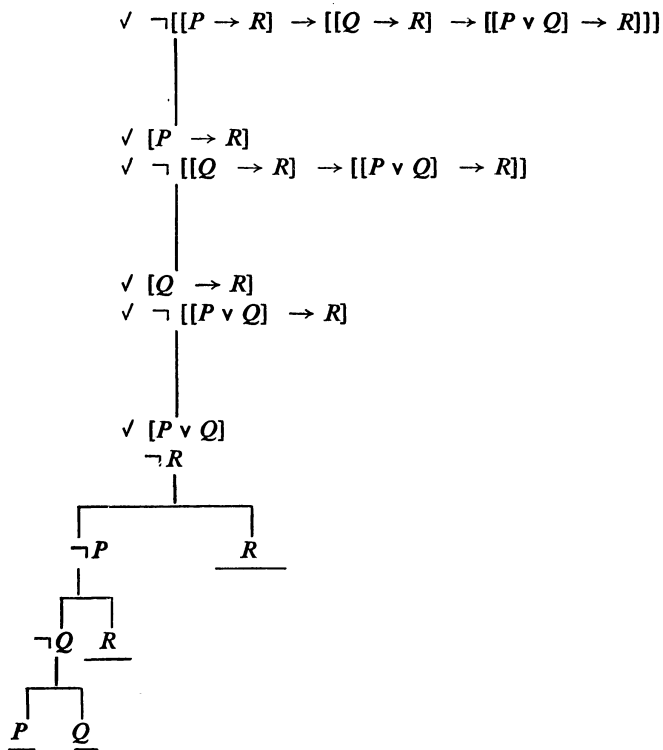
Section 25

25. Tautology 9.

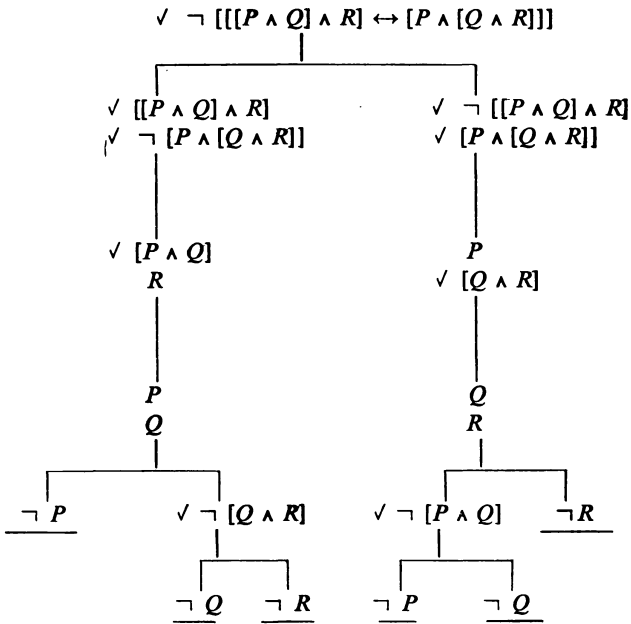
$$\begin{array}{c}
 \checkmark \neg [P \rightarrow [P \vee Q]] \\
 | \\
 P \\
 \checkmark \neg [P \vee Q] \\
 | \\
 \neg P \\
 \neg Q \\
 \hline
 \end{array}$$

LOGIC

Tautology 11.

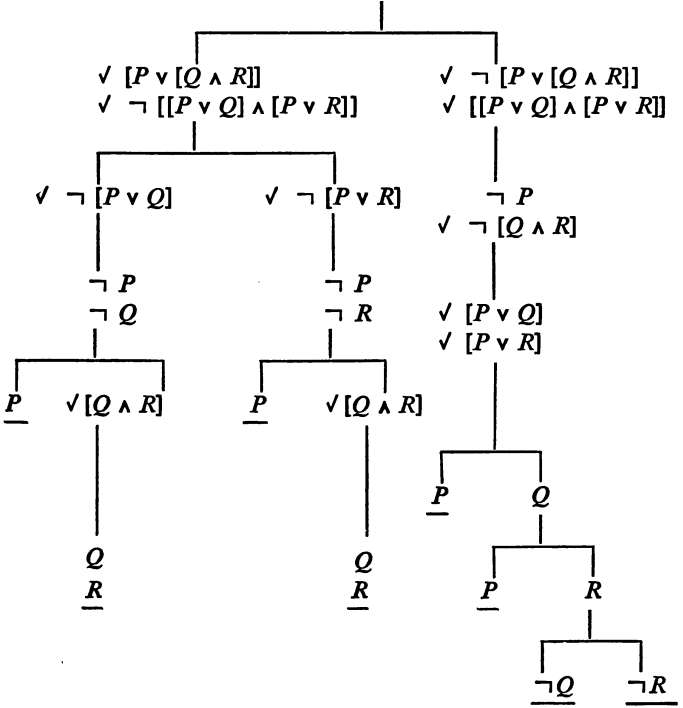


Tautology 16.

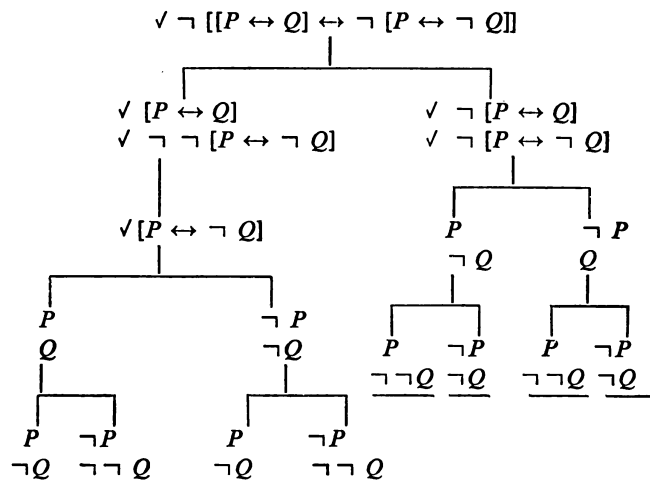


Tautology 26.

$$\checkmark \neg [[P \vee [Q \wedge R]] \leftrightarrow [[P \vee Q] \wedge [P \vee R]]]$$



Tautology 48.



Section 26

26. 1. I (singular personal pronoun); he (spp); this (definite description); the ox (dd); the day (dd); relief (non-count noun); its back (dd); I (spp); he (spp); he (spp); God (proper name). If 'its back being rubbed' is counted as a singular noun phrase, then this is a definite description too.
2. this rule (dd); Dr Jekyll (pn); he (spp); the opposite side of the fire (dd); the opposite side (dd); the fire (dd); capacity (ncn); kindness (ncn); you (spp); he (spp); Mr Utterson (pn). Although affection is normally a non-count noun, the 'a' in front of it shows that it is not a non-count noun here.

Section 27

27. 1. No: intentional.
2. Yes. (Remember that our paraphrase is allowed to assume that she does have a lover.)

LOGIC

3. No: quotational.
4. No: modal.
5. No: temporal.
6. Yes.
7. Yes.
8. No: quotational.
9. No: intentional.
10. Yes (in spite of the word 'impossible').
11. No: temporal (unless the speaker means 'the present Bishop of Sodor and Man').

Section 29

29. 1. Three plus three is greater than five.
2. Thy father's spirit scents the morning air. (Change 'scent' to 'scents', to correct the perturbation.)
3. Parasurama is Krishna.
4. (i) Parasurama is Krishna. (ii) Krishna is Parasurama.
5. (i) The daughters of the king disown Lear. (ii) The daughters of Lear disown the king. (iii) The daughters of the king disown the king.
6. (i) Errol owns the gun that fired the bullet that killed Cheryl. (ii) The gun that fired this bullet is the murder weapon. (iii) (Using Leibniz's Rule twice) Errol owns the murder weapon.
7. (i) June is the third month after the ninth month after June. (ii) March is the ninth month after the third month after March. (iii) June is the third month after the ninth month after the third month after March. (etc., etc.)
8. The Father suffers. (This is a time-honoured problem for Christian theologians, who must accept the premises but deny the conclusion, on pain of heresy.)

Section 30

30A. 1. U.S.S.R. 2. U.S.S.R., Sweden. 3. Sweden.

30B. <U.S.A., U.S.A.>, <U.S.A., U.S.S.R.>, <U.S.A., Sweden>, <U.S.S.R., U.S.A.>, <U.S.S.R., U.S.S.R.>,

$\langle \text{U.S.S.R., Sweden} \rangle$, $\langle \text{Sweden, U.S.A.} \rangle$, $\langle \text{Sweden, U.S.S.R.} \rangle$, $\langle \text{Sweden, Sweden} \rangle$.

- 30C. 1. $\langle \text{U.S.S.R., Sweden} \rangle$, $\langle \text{U.S.S.R., U.S.A.} \rangle$, $\langle \text{Sweden, U.S.A.} \rangle$.
 2. $\langle \text{U.S.A., U.S.A.} \rangle$, $\langle \text{U.S.A., U.S.S.R.} \rangle$, $\langle \text{U.S.A., Sweden} \rangle$, $\langle \text{U.S.S.R., U.S.S.R.} \rangle$, $\langle \text{Sweden, Sweden} \rangle$, $\langle \text{Sweden, U.S.S.R.} \rangle$.
 3. $\langle \text{U.S.A., U.S.A.} \rangle$, $\langle \text{U.S.A., U.S.S.R.} \rangle$, $\langle \text{U.S.A., Sweden} \rangle$.

- 30D. $\{ \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle \}$, $\{ \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle \}$,
 $\{ \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle \}$, $\{ \langle 1, 1 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle \}$,
 $\{ \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle \}$, $\{ \langle 1, 1 \rangle, \langle 1, 2 \rangle \}$, $\{ \langle 1, 1 \rangle, \langle 2, 1 \rangle \}$,
 $\{ \langle 1, 1 \rangle, \langle 2, 2 \rangle \}$, $\{ \langle 1, 2 \rangle, \langle 2, 1 \rangle \}$, $\{ \langle 1, 2 \rangle, \langle 2, 2 \rangle \}$,
 $\{ \langle 2, 1 \rangle, \langle 2, 2 \rangle \}$, $\{ \langle 1, 1 \rangle \}$, $\{ \langle 1, 2 \rangle \}$, $\{ \langle 2, 1 \rangle \}$, $\{ \langle 2, 2 \rangle \}$, $\{ \}$.

Section 31

- 31A. 1. reflexive. 2. irreflexive. 3. non-reflexive.
 4. irreflexive. 5. reflexive. 6. non-reflexive.
- 31B. 1. asymmetric. 2. symmetric. 3. asymmetric.
 4. non-symmetric. 5. symmetric. 6. non-symmetric.
- 31C. 1. non-transitive. 2. transitive. 3. intransitive.
 4. transitive. 5. intransitive, if people married to each other must be of opposite sex. 6. intransitive or non-transitive; the latter if the world contains an Oedipus (who married his mother, and had the daughter Antigone by her).
- 31D. 1. not connected. 2. connected. 3. connected. 4. not connected.

Section 32

- 32A. NPA has the same empirical formula as NPA.
 IPA has the same empirical formula as IPA.
 MEE has the same empirical formula as MEE.

LOGIC

IPA has the same empirical formula as NPA.
MEE has the same empirical formula as IPA.
NPA has the same empirical formula as MEE.
MEE has the same empirical formula as NPA.

- 32B. 1. x_1 is the same size as x_2 .
2. x_1 is the same age as x_2 .
3. x_1 is the same distance as x_2 .
4. x_1 does the same amount as x_2 .
5. x_1 has the same frequency of breaking (or breakage rate) as x_2 .
6. *See section 33.*
- 32C. 1. x_1 is greedier than x_2 .
2. x_1 has fewer antennae than x_2 .
3. x_1 is at least as knobbly as x_2 .
4. x_1 has the same temperature as x_2 .

Section 34

- 34.1. *First type:* A man came to see me this afternoon.
Second type: A gentleman doesn't pick his nose.
2. *First type:* Drinks will be provided.
Second type: Cats have an acute sense of smell.

Section 35

- 35.1. $\forall x$ [x is a noise $\rightarrow x$ appals me]
2. $\exists x$ [x is wicked $\wedge x$ comes this way]
3. $\exists x$ [x is a strange infirmity \wedge I have x]
4. $\forall x$ [x is a candle of theirs $\rightarrow x$ is out]
5. $\forall x$ x is not a child of his.
6. $\exists x$ [x is a murder $\wedge x$ has been performed]
7. $\exists y$ [y is an idiot $\wedge x$ is a tale told by y]
8. $\forall y$ [y is a person born of woman $\rightarrow y$ shall not harm x]

Section 36

- 36A. [John is an older boy than John \rightarrow John has learned to read].

[the boy in the corner is an older boy than John → the boy in the corner has learned to read].

[the corner is an older boy than John → the corner has learned to read].

(Presumably only the second of these sentences will be of any interest; but the first and third will be true too, since their first halves will be false.)

36B. 1. [the number $1/2$ is odd \vee the number $1/2$ is even].

The number $1/2$ is neither odd nor even.

2. [Case 946 is a person → Case 946's I.Q. never varies by more than 15].

Case 946 is a girl whose I.Q. has varied between 142 and 87.

3. [Horst is Austrian → Horst is of Alpine type].

[Horst is of Alpine type → Horst has a broad head].

Horst is Austrian, but doesn't have a broad head.

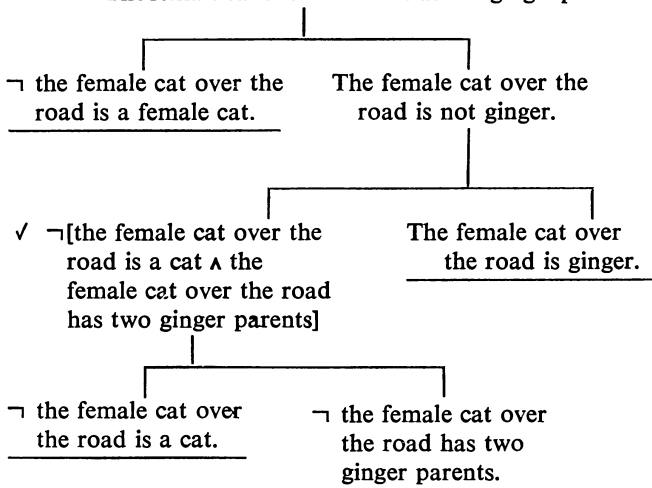
4. [[the female cat over the road is a cat \wedge the female cat over the road has two ginger parents] → the female cat over the road is ginger].

[the female cat over the road is a female cat → the female cat over the road is not ginger].

The female cat over the road has two ginger parents.

A tableau for 4:

- ✓ [[the female cat over the road is a cat \wedge the female cat over the road has two ginger parents] \rightarrow the female cat over the road is ginger]
 - ✓ [the female cat over the road is a female cat \rightarrow the female cat over the road is not ginger]
- The female cat over the road has two ginger parents.



36C. Maxine is not dead.

[the spouse of Maxine is a widow \rightarrow the spouse of the spouse of Maxine is dead].

The spouse of the spouse of Maxine is not dead.

36D. 1. [b is a jockey \wedge b is female].

[b is female \rightarrow b is not a jockey].

2. [b is female \wedge b has a conscience].

[b has a conscience \rightarrow b has experienced the castration complex].

[b is female \rightarrow b has not experienced the castration complex].

3. b has forty-seven chromosomes.
 [b is male $\vee b$ is female].
 [b is male $\rightarrow b$ has just forty-six chromosomes].
 [b is female $\rightarrow b$ has just forty-six chromosomes].
4. [b is male \wedge the mate of b is male].
 [b is male \rightarrow the mate of b is female].
 [the mate of b is female \rightarrow the mate of b is not male].

- 36E. 1. \neg [b is a dunnock $\rightarrow b$ is not brightly coloured].
 [b is a ground-feeding bird $\rightarrow b$ is not brightly coloured].
 [b is a dunnock $\rightarrow b$ is a ground-feeding bird].
2. [b is a finch $\wedge b$ cracks cherry seeds].
 [b is a finch $\rightarrow b$ is a bird].
 [[b is a bird $\wedge b$ cracks cherry seeds] $\rightarrow b$ has a massive beak].
 \neg [b is a finch $\wedge b$ has a massive beak].
3. \neg [b is one of the birds $\rightarrow b$ is a willow warbler].
 [b is one of the birds \rightarrow [b is a chiff-chaff $\vee b$ is a willow warbler]].
 [b is one of the birds $\rightarrow b$ is singing near the ground].
 [b is a chiff-chaff $\rightarrow b$ doesn't sing near the ground].
4. b is a trumpeter bullfinch.
 [b is a trumpeter bullfinch $\rightarrow b$ can sing two notes at once].
 [b is a trumpeter bullfinch $\rightarrow b$ is a bird].
 \neg [b is a bird $\wedge b$ can sing two notes at once].

Section 37

- 37A. 1. $\neg \forall x$ [x is a time \rightarrow the room is heated at x]
 2. $\forall x$ [x is a time $\rightarrow \neg$ the room is heated at x]
 3. As 2.
 4. $\exists x$ [x is a time $\wedge \neg$ the room is heated at x]
 5. $\neg \exists x$ [x is a time \wedge the room is heated at x]
 (4 is a paraphrase of 1, and 5 is a paraphrase of 2.)
- 37B. As I read them, 1 and 2 imply one girl got the lot, 4-6 don't, and 3 is doubtful.

LOGIC

- 37C. 1. $\forall x [(x \text{ is a starling} \wedge x \text{ is nesting here}) \rightarrow \text{I'll shoot } x]$
 2. $\forall x [(x \text{ is a starling} \wedge x \text{ is nesting here}) \rightarrow \text{that bird is a starling}]$
 OR: $[\exists x [x \text{ is a starling} \wedge x \text{ is nesting here}] \rightarrow \text{that bird is a starling}]$
 3. $\forall x [x \text{ is a finch} \rightarrow \text{that bird has a longer bill than } x]$
 4. $\exists x [x \text{ is a finch} \wedge \neg \text{that bird has a longer bill than } x]$
 5. $\forall x [x \text{ is a starling} \rightarrow \neg \text{that bird has a longer bill than } x]$
 6. $\neg \forall x [x \text{ is a starling} \rightarrow \forall y [y \text{ is a finch} \rightarrow x \text{ has a longer bill than } y]]$
 7. $\exists x [x \text{ is a finch} \wedge \forall y [y \text{ is a starling} \rightarrow x \text{ has a longer bill than } y]]$
 8. $\forall x [x \text{ is a finch} \rightarrow \exists y [y \text{ is a starling} \wedge y \text{ has a longer bill than } x]]$
 9. $\forall x [[(x \text{ is a bird} \wedge x \text{ is nesting here}) \wedge \neg x \text{ is a finch}] \rightarrow x \text{ has a long bill}]$
 10. $\forall x [(x \text{ is a bird} \wedge x \text{ is nesting here}) \rightarrow [x \text{ has a long bill} \leftrightarrow x \text{ is a starling}]]$

Section 38

- 38A. 1. $\exists x_1 \exists x_2 [\neg x_1 = x_2 \wedge [x_1 \text{ is a mistake} \wedge x_2 \text{ is a mistake}]]$
 2. $\exists x_1 \exists x_2 \exists x_3 \exists x_4 [([(\neg x_1 = x_2 \wedge \neg x_1 = x_3) \wedge \neg x_1 = x_4) \wedge \neg x_2 = x_3] \wedge \neg x_2 = x_4) \wedge x_3 = x_4) \wedge [([x_1 \text{ is a mistake} \wedge x_2 \text{ is a mistake}] \wedge x_3 \text{ is a mistake}) \wedge x_4 \text{ is a mistake}]]$
 3. $\exists x_1 \exists x_2 [\neg x_1 = x_2 \wedge [(x_1 \text{ is a person} \wedge x_1 \text{ has pointed out the mistakes}) \wedge (x_2 \text{ is a person} \wedge x_2 \text{ has pointed out the mistakes})]]$
 4. $[\exists x_1 \exists x_2 [\neg x_1 = x_2 \wedge [x_1 \text{ is a hemisphere} \wedge x_2 \text{ is a hemisphere}]]] \wedge \neg \exists x_1 \exists x_2 \exists x_3 [([\neg x_1 = x_2 \wedge \neg x_1 = x_3] \wedge \neg x_2 = x_3) \wedge [(x_1 \text{ is a hemisphere} \wedge x_2 \text{ is a hemisphere}) \wedge x_3 \text{ is a hemisphere}]]]$
 5. $\forall x [x = \text{Sir Henry} \leftrightarrow x \text{ is allowed to use that bath}]$
- 38B. 1. $[\exists x \forall y [x = y \leftrightarrow y \text{ is a book}]] \wedge \forall x [x \text{ is a book} \rightarrow x \text{ is bound in vellum}]$

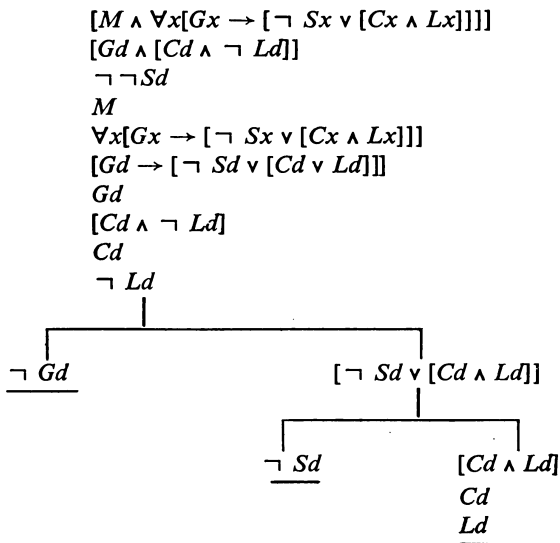
2. $[\exists x \forall y [x = y \leftrightarrow y \text{ is a stern of hers}] \wedge \forall x [x \text{ is a stern of hers} \rightarrow \text{we've avoided } x]]$
3. Impossible – there may be other sons.
4. $\forall x [x = \text{my mother} \leftrightarrow x \text{ is a woman in blue}]$
5. Impossible – not purely referential.
6. Impossible – non-count noun.
7. $\forall x [\neg x = \text{Cassius} \rightarrow \text{Cassius is greater than } x]$

Section 39

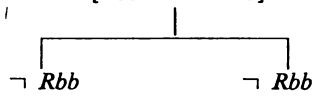
- 39A.
1. Ec
 2. $\neg Wdb$
 3. $[Sbc \vee Scb]$
 4. $\exists x [(Ex \wedge Wdx) \wedge Sbx]$
 5. $\exists x \exists y [\neg x = y \wedge [(Ex \wedge Wdx) \wedge (Ey \wedge Wdy)]]$
(It seems to me that 5 implies that Homer wrote at least one epic, unlike ‘ \neg Homer wrote just one epic’. See p. 93.)
 6. $[Eb \wedge \exists x [Ex \wedge \neg Sbx]]$
 7. $\forall x [Wxb \rightarrow \neg Wxc]$
 8. $\forall x [(Ex \wedge Sxb) \rightarrow \neg Wdx]$
 9. $\forall x [Ex \rightarrow \exists y [Ey \wedge [Wdy \wedge Sxy]]]$
 10. $\exists x \exists y [([(Ex \wedge Scx) \wedge (Ey \wedge Scy)] \wedge \forall z [(Ez \wedge Scz) \rightarrow [z = x \vee z = y]]) \wedge [Wdx \wedge \neg Wdy]]$
- 39B.
1. $\forall x \neg Rxx$
 2. $\forall x \forall y [Rxy \rightarrow Ryx]$
 3. $\forall x \forall y [Rxy \rightarrow \neg Ryx]$
 4. $\forall x \forall y \forall z [(Rxy \wedge Ryz) \rightarrow \neg Rxz]$
 5. $[\neg \forall x Rxx \wedge \neg \forall x \neg Rxx]$
 6. $\forall x \forall y [\neg x = y \rightarrow [Rxy \wedge Ryx]]$

Section 40

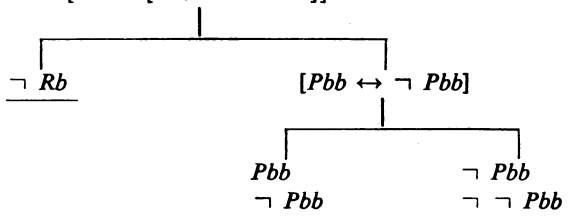
40A. $[M \wedge \forall x[Gx \rightarrow [\neg Sx \vee [Cx \wedge Lx]]]]$. $[Gd \wedge [Cd \wedge \neg Ld]]$. Therefore $\neg Sd$.



40B.

$$\begin{aligned} & \forall x \forall y [Rxy \rightarrow \neg Ryx] \\ & \neg \forall x \neg Rxx \\ & \exists x \neg \neg Rxx \\ & \neg \neg Rbb \\ & \forall x [Rby \rightarrow \neg Ryb] \\ & [Rbb \rightarrow \neg Rbb] \end{aligned}$$


40C.

$$\begin{aligned} & \exists x [Rx \wedge \forall y [Ry \rightarrow [Pxy \leftrightarrow \neg Pyy]]] \\ & [Rb \wedge \forall y [Ry \rightarrow [Pby \leftrightarrow \neg Pyy]]] \\ & Rb \\ & \forall y [Ry \rightarrow [Pby \leftrightarrow \neg Pyy]] \\ & [Rb \rightarrow [Pbb \leftrightarrow \neg Pbb]] \end{aligned}$$


40D. 1.

$$\begin{aligned} & Pb \\ & \neg \forall x [x = b \rightarrow Px] \\ & \exists x \neg [x = b \rightarrow Px] \\ & \neg [c = b \rightarrow Pc] \\ & c = b \\ & \neg Pc \\ & \neg Pb \end{aligned}$$

LOGIC

2.

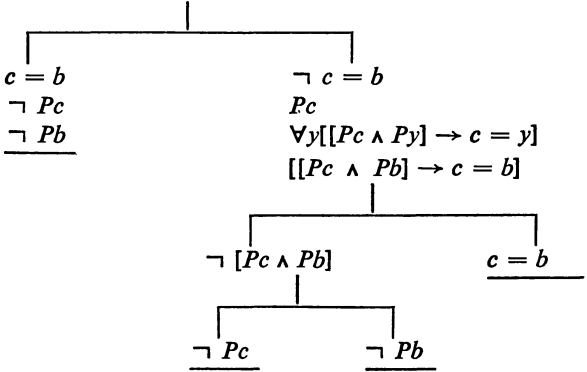
Pb

$\forall x \forall y [[Px \vee Py] \rightarrow x = y]$

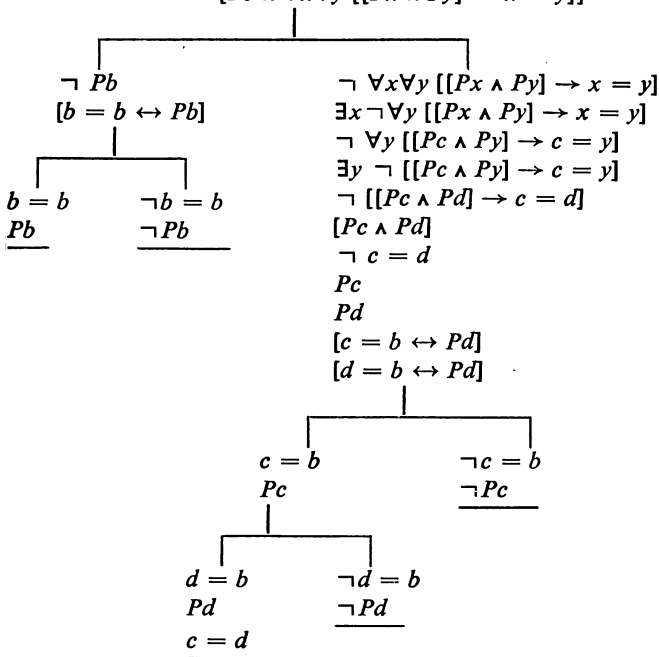
$\neg \forall x [x = b \leftrightarrow Px]$

$\exists x \neg [x = b \leftrightarrow Px]$

$\neg [c = b \leftrightarrow Pc]$



3. $\forall x[x = b \leftrightarrow Px]$
 $\neg [Pb \wedge \forall x\forall y [[Px \wedge Py] \rightarrow x = y]]$



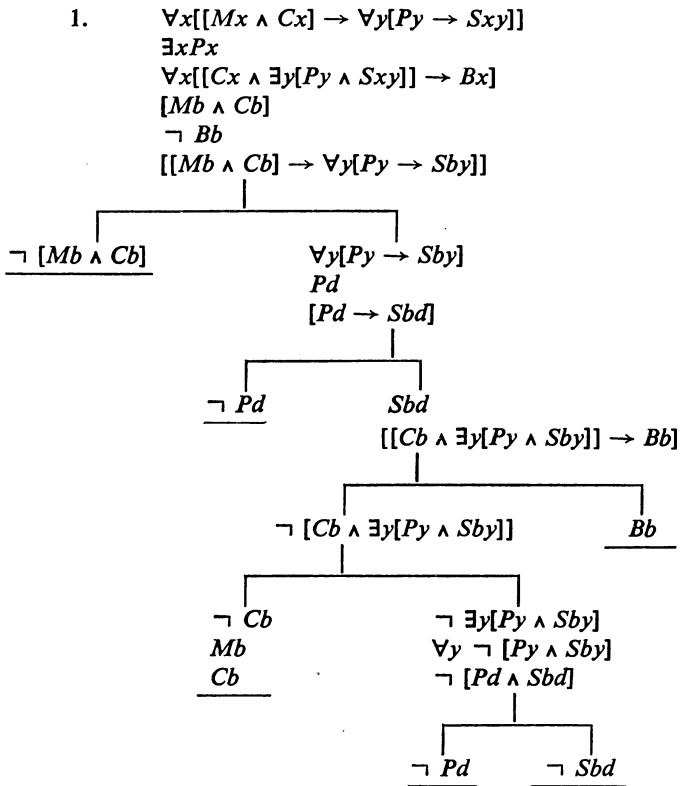
40E. The symbolizations first:

1. $\forall x[[Mx \wedge Cx] \rightarrow \forall y[Py \rightarrow Sxy]]. \exists xPx.$
 $\forall x[[Cx \wedge \exists y[Py \wedge Sxy]] \rightarrow Bx]. [Mb \wedge Cb].$
Therefore Bb.
2. $\forall x[Px \rightarrow [Sbx \leftrightarrow Scx]]. \forall x[[Px \wedge Sbx] \rightarrow Bb]. \neg Bb.$
Therefore $\forall x[Px \rightarrow \neg Scx].$
3. $\exists x\exists y[[Cx \wedge Cy] \wedge [Txy \vee Tyx]]. \forall x[Cx \rightarrow \neg Txx].$
Therefore $\exists x\exists y[\neg x = y \wedge [Cx \wedge Cy]].$
4. $\neg b = c. [\neg Bb \vee \forall x[Px \rightarrow Sbx]].$
 $[Cc \wedge [Tcb \rightarrow \exists x[Px \wedge [Scx \wedge \neg Sbx]]]].$
 $\forall x[[Cx \wedge \neg b = x] \rightarrow Txb].$ *Therefore $\neg Bb.$*

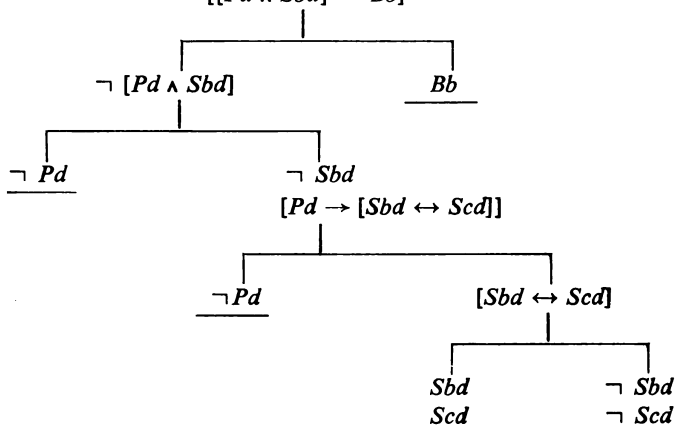
LOGIC

5. $\forall x[x = b \leftrightarrow [Mx \wedge Cx]]. \forall x[Cx \rightarrow [Bx \leftrightarrow Mx]].$
 Therefore $\forall x[x = b \leftrightarrow [Cx \wedge Bx]].$

Next the tableaux:

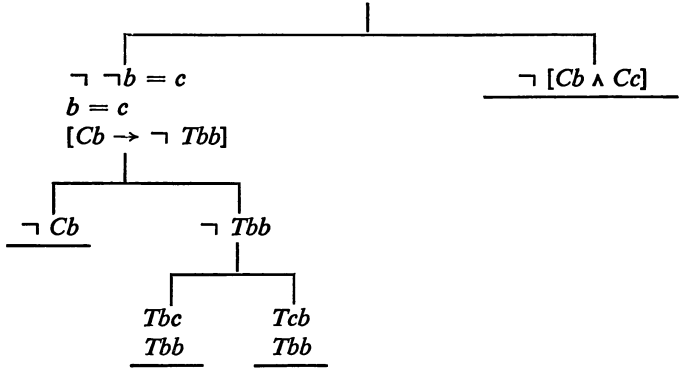


2.

 $\forall x[Px \rightarrow [Sbx \leftrightarrow Scx]]$ $\forall x[[Px \wedge Sbx] \rightarrow Bb]$ $\neg Bb$ $\neg \forall x[Px \rightarrow \neg Scx]$ $\exists x \neg [Px \rightarrow \neg Scx]$ $\neg [Pd \rightarrow \neg Scd]$ Pd $\neg \neg Scd$ $[[Pd \wedge Sbd] \rightarrow Bb]$ 

LOGIC

3. $\exists x \exists y [(Cx \wedge Cy) \wedge [Txy \vee Tyx]]$
 $\forall x [Cx \rightarrow \neg Txx]$
 $\neg \exists x \exists y [\neg x = y \wedge [Cx \wedge Cy]]$
 $\exists y [(Cb \wedge Cy) \wedge [Tby \vee Tyb]]$
 $[(Cb \wedge Cc) \wedge [Tbc \vee Tcb]]$
 $[Cb \wedge Cc]$
 $[Tbc \vee Tcb]$
 Cb
 Cc
 $\forall x \neg \exists y [\neg x = y \wedge [Cx \wedge Cy]]$
 $\neg \exists y [\neg b = y \wedge [Cb \wedge Cy]]$
 $\forall y \neg [\neg b = y \wedge [Cb \wedge Cy]]$
 $\neg [\neg b = c \wedge [Cb \wedge Cc]]$



4.

$$\neg b = c$$

$$[\neg Bb \vee \forall x[Px \rightarrow Sbx]]$$

$$[Cc \wedge [Tcb \rightarrow \exists x[Px \wedge [Scx \wedge \neg Sbx]]]]$$

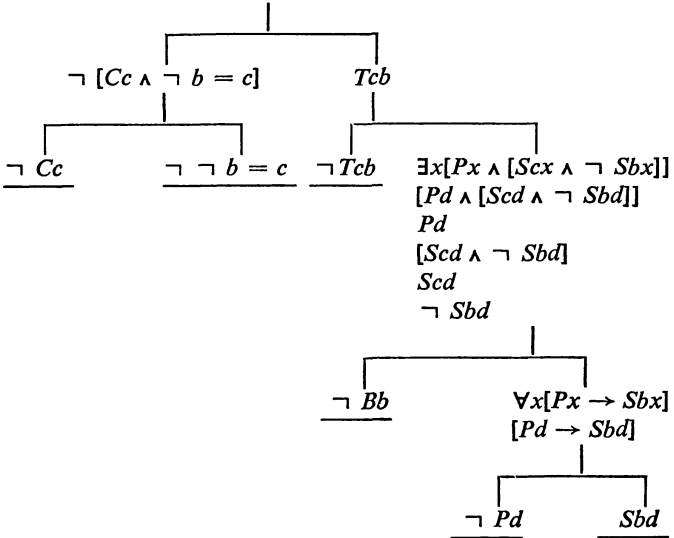
$$\forall x[[Cx \wedge \neg b = x] \rightarrow Txb]$$

$$\neg \neg Bb$$

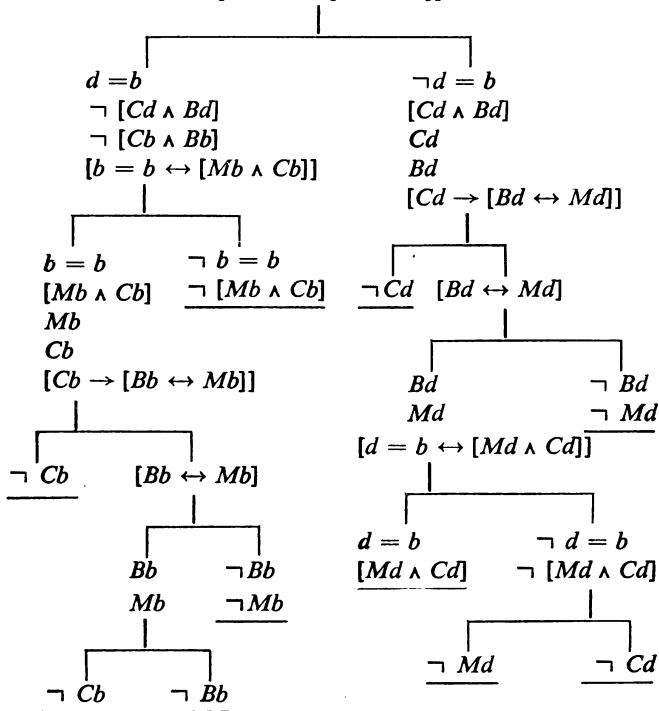
$$Cc$$

$$[Tcb \rightarrow \exists x[Px \wedge [Scx \wedge \neg Sbx]]]$$

$$[[Cc \wedge \neg b = c] \rightarrow Tcb]$$



5. $\forall x[x = b \leftrightarrow [Mx \wedge Cx]]$
 $\forall x[Cx \rightarrow [Bx \leftrightarrow Mx]]$
 $\neg \forall x[x = b \leftrightarrow [Cx \wedge Bx]]$
 $\exists x \neg [x = b \leftrightarrow [Cx \wedge Bx]]$
 $\neg [d = b \leftrightarrow [Cd \wedge Bd]]$



‘Such another proof will make me cry “Baa”.’
The Two Gentlemen of Verona

Section 42

42.1. R	$[R \supseteq \neg R]$
T	T T FT
F	F T TF

2. P Q	$\neg [[Q \wedge P] \supseteq [Q \wedge \neg P]]$
T T	F T T T T T F FT
T F	F F FT T F FFT
F T	F T FF T T TTF
F F	F F FF T F FTF

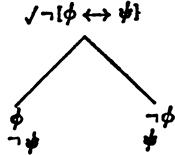
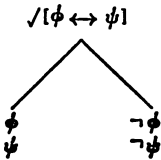
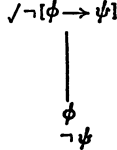
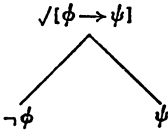
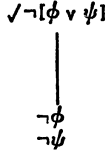
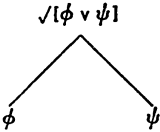
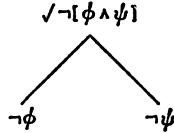
3. P Q R	$[R \supseteq [P \supseteq Q]]$
T T T	T F T T T
T T F	F F T T T
T F T	T F T T F
T F F	F F T T F
F T T	T F F T T
F T F	F F F T T
F F T	T F F T F
F F F	F F F T F

Section 44

44. My own experience is that most of the effort goes into working out the grammatical structure of the sentence. It is very hard to see by introspection alone what else may be involved. Controlled experiments yield a very little more information: see Judith Greene, *Psycholinguistics*, Penguin, 1972.

Tableau Rules

The derivation rules of sentence tableaux are as follows:



LOGIC

The six further derivation rules of predicate tableaux are listed on pp. 227–8.

A branch of a symbolic sentence tableau may be closed if there is a formula ϕ such that both ϕ and ' $\neg \phi$ ' occur as formulae in the branch. A branch of a symbolic predicate tableau may be closed in the same circumstances; it may also be closed if there is an individual constant D such that ' $\neg D = D$ ' occurs as a formula in the branch. See pp. 116 and 228.

For further information on tableaux, see the book by Richard C. Jeffrey mentioned on p. 324.

A Note on Notation

Some logicians use different symbols from those in this book. The main variants are:

\sim	for	\neg
\cdot	for	\wedge
\supset	for	\rightarrow
\equiv	for	\leftrightarrow
(x)	for	$\forall x$

There are several different conventions for use of brackets. Many logicians use parentheses ‘(,’)’ instead of square brackets. Some old-fashioned books replace brackets by a comical system of little square dots.

Further Reading

Richard C. Jeffrey, *Formal Logic: Its Scope and Limits*, McGraw-Hill, New York, 1967.

A readable textbook, the first to present tableaux in tree form.

Herbert B. Enderton, *A Mathematical Introduction to Logic*, Academic Press, New York, 1972.

What the title implies.

W. V. Quine, *Philosophy of Logic*, Prentice-Hall, Englewood Cliffs, N.J., 1970.

A rather puritan account of the nature of logic, by a stimulating and ingenious writer.

Michael Dummett, *Frege, Philosophy of Language*, Duckworth, 1973.

Goes deep into many of the questions we have skimmed over.

Bärbel Inhelder and Jean Piaget, *The Early Growth of Logic in the Child: Classification and Seriation*, Harper & Row, New York, 1964.

The authors describe their experiments on children's ability to think logically.

Noam Chomsky, *Aspects of the Theory of Syntax*, M.I.T. Press, Cambridge, Mass., 1965.

The definitive statement of Chomsky's view of grammar.

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