Year 9 Mathematics | Topic 2 | Pythagoras and Surds [2/2]

PEN Education

2024

Contents

1 Introduction

This week we shall continue with our study of the type of numbers which are created when we take some kind of root $\sqrt{}$ of another *positive* number. These quantities we now know to be called ____________, and last week we learned the basic arithmetic of these objects.

Before we begin our review / brain warm-up let us contextualise today's theory: We will begin by inserting **surds** into the algebraic structures of our first topic ($(a+b)^2 = a^2 + 2ab + b^2$, etc...) where a and b are now surd terms.

Then we will learn about something known as *rationalising the denominator* which is an important simplification method in mathematics, and one of the earlier conventions adopted by the *Babylonians* so that they would not be dividing by incomensurable quantities - i.e. what does it mean to divide 4 apples into $\sqrt{2}$ quantities? That would be asking to divide the 4 apples into $1.41421356237\ldots$ parts, where the 'dot, dot, dot' means the denominator never terminates!

After practising the above simplifications, we return to Pythagora's Theorem and Geometry. Specifically we attempt problems in 3 dimensions where the need for visualisation and mathematical stamina increase.

The final 2 sections then deal with rationalising *binomial denominators* and converting irrational numbers to fractions respectively. The latter of these makes for a cool party trick: WHAT FRACTION IS EQUAL TO THE REPEATING DECIMAL $0.81818181\ldots$?

$$\boxed{0.\overline{81} =}$$

2 Review

2.1 Exercises:

1. S implify: $\boxed{4}$

   (a) $\sqrt{108} =$ ____________

   (b) $\sqrt{90} =$ ____________

   (c) $\sqrt{52} =$ ____________

   (d) $\sqrt{98} =$ ____________

2. A dd or Subtract: $\boxed{4}$

   (a) $3\sqrt{2} + 2\sqrt{2} =$ ____________

   (b) $\sqrt{32} - \sqrt{18} =$ ____________

   (c) $\sqrt{45} + 2\sqrt{5} - \sqrt{80} =$ ____________

   (d) $\sqrt{28} + 2\sqrt{63} - 5\sqrt{7} =$ ____________

3. M ultiply or Divide: $\boxed{4}$

(a) $2\sqrt{3} \times 5\sqrt{6} =$ ____________

(b) $3\sqrt{5} \times 2\sqrt{10} =$ ____________

(c) $14\sqrt{40} \div 7\sqrt{5} =$ ____________

(d) $3\sqrt{2} \div \sqrt{2} =$ ____________

3 Special Products

This is an extension of the things you learned in *Topic 1: Algebra*. We will be using 3 main results and then replacing the a's and b's with *surds*.

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$
$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2}$$
$$(a-b)(a+b) = a^2 - b^2 \tag{3}$$

3.1 Examples:

1. (a) 5

$$(\sqrt{5} + \sqrt{3})^2 =$$

. . . .

. . . .

. . . .

(b)

$$(3\sqrt{2} - 2\sqrt{3})^2 =$$

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(c)

$$(2\sqrt{3} + 4\sqrt{6})^2 =$$

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. . . .

. . . .

(d)

$$(\sqrt{7}-5)(\sqrt{7}+5) =$$

....

....

....

(e)

$$(5\sqrt{6}-2\sqrt{5})(5\sqrt{6}+2\sqrt{5}) =$$

....

....

....

3.2 Exercises:

1. (a) $(5+\sqrt{3})^2$ 12

   (b) $(\sqrt{2}+6)^2$

   (c) $(5\sqrt{6}+2\sqrt{3})^2$

   (d) $(\sqrt{21}+\sqrt{3})^2$

   (e) $(4+2\sqrt{5})^2$

   (f) $(\sqrt{7}-2)^2$

   (g) $(4-\sqrt{3})^2$

   (h) $(2\sqrt{35}+\sqrt{5})^2$

   (i) $(\sqrt{70}-3\sqrt{10})^2$

   (j) $(3-\sqrt{5})(3+\sqrt{5})$

   (k) $(2\sqrt{5}-1)^2$

   (l) $(\sqrt{6}-1)(\sqrt{6}+1)$

4 Rationalising the Denominator

You will want to pay attention to this part, because we will come up against a more involved version of *rationalising the denominator* in about 20 minutes. We will begin to see them not just as $\frac{1}{\sqrt{2}}$ as we see them in this section, but rather as *binomials*: $\frac{1}{\sqrt{3}-\sqrt{2}}$.
We explained the motivations behind rationalising the denominator in the introduction but a quick revision will not hurt:

(1) Historical Convention

(2) Simplicity

(3) Aesthetics

(4) Clarity

Now having said this, what is the mathematical method to rationalise a denominator? Simple, just multiply the denominator with whatever you need to such that the surd cancels out:

$$\begin{aligned}\frac{2}{\sqrt{2}} &= \frac{2\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\\ &= \frac{\cancel{2}\sqrt{2}}{\cancel{2}}\\ &= \sqrt{2}\end{aligned}$$

Recall that: you may multiply or divide a fraction by any number as long as you *do the top what you do to the bottom.*

4.1 Examples:

1. E xpress the following with a *rational* denominator 2

   (a) $\frac{4}{\sqrt{2}} =$

   ....

   ....

   ....

   (b) $\frac{9}{4\sqrt{3}} =$

   ....

   ....

   ....

   (c)

   $$\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2} =$$

   ....

   ....

   ....

Formulae:

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}, \text{ where } a \text{ is positive,}$$

4.2 Exercises:

1. (a) $$\frac{\sqrt{3}}{\sqrt{7}} =$$ 6

   ....

   ....

   ....

   (b) $$\frac{2\sqrt{7}}{\sqrt{3}} =$$

   ....

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   ....

   (c) $$\frac{\sqrt{2}}{3\sqrt{10}} =$$

   ....

   ....

   ....

   (d) $$\frac{\sqrt{15}}{3\sqrt{5}} =$$

   ....

   ....

   ....

   (e) $$\frac{2}{\sqrt{3}} + \frac{3}{2\sqrt{3}} =$$

   ....

   ....

   ....

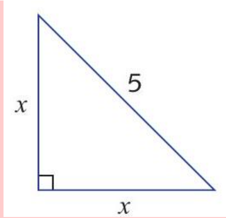
   (f) $$\frac{5\sqrt{2}}{3} - \frac{1}{\sqrt{3}} =$$

   ....

   ....

   ....

2. F ind the value of x. Express your answer with a rational denominator. 4

(a) 

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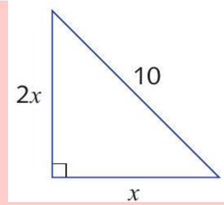
(b) 

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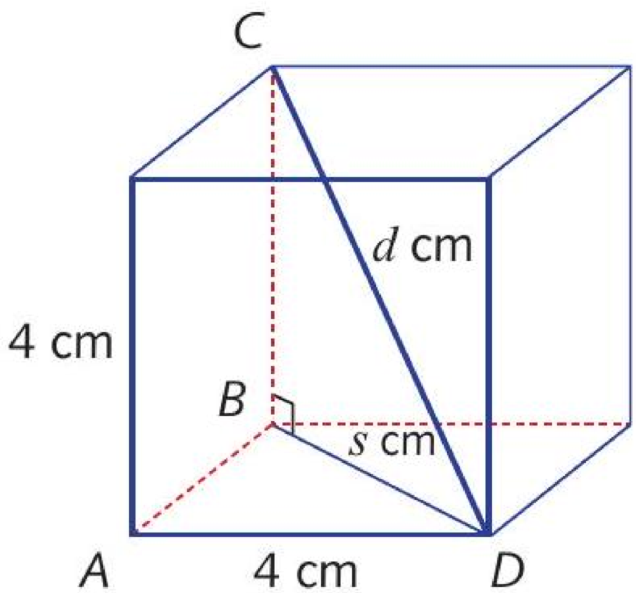
5 Pythagoras in 3-Dimensions!

Let us consider a problem:



How can we find the length of the diagonal CD of this cube with side lengths of 4cm?
We apply Pythagoras' Theorem!

Recall that: $c^2 = a^2 + b^2$

And so here we first find the length of s to be ____________ and thus we can use this length of s along with the side CB to form another triangle and find $d =$ ____________.

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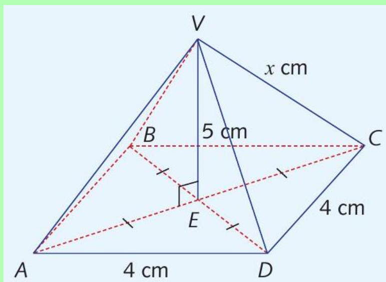
5.1 Example:

A square pyramid has height 5 cm and square base with side length 4 cm. Find the length of the edge VC in this diagram.



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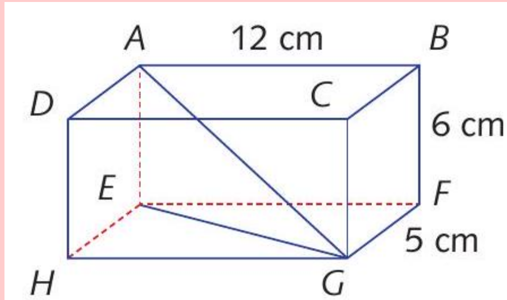
Exercises:

1. 

The rectangular prism in the diagram has a length of 12 cm, a width of 5 cm and a height of 6 cm.

(a) Consider triangle EFG. Find:

i. EF

. . . .

. . . .

. . . .

ii. the size of EFG

. . . .

. . . .

. . . .

(b) Find EG.

. . . .

. . . .

. . . .

(c) Find AG, correct to 1 decimal place.

. . . .

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2. Note: AG is called the space diagonal of the rectangular prism. □

3. F ind the length of the longest pencil that can fit inside a cylindrical pencil case of length 15 cm and radius 2 cm. $\boxed{3}$

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6 Binomial Denominators

As promised, here lie the *binomial denominators*. There are a couple of prerequisites to understanding this mathematics, the first is that you must totally grasp $(a+b)(a-b) = a^2 - b^2$. The significance of this result is that if we make a and b surds like so: $(\sqrt{a}+\sqrt{b})$, then multiplying by $(\sqrt{a}-\sqrt{b})$ yields a fantastic result, namely:

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 \quad = a - b \tag{4}$$

This act of starting with $\sqrt{a}+\sqrt{b}$, and multiplying it with the sign flipped: $\sqrt{a}-\sqrt{b}$ is called *multiplying by the reciprocal*. Let's practise this method:

6.1 Examples:

1. $\boxed{1}$

$$\frac{2\sqrt{5}}{2\sqrt{5}-2} =$$

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. . . .

. . . .

. . . .

2. $\boxed{1}$

$$\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}}$$

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. . . .

6.2 Exercises:

1. (a) $\frac{3}{\sqrt{2}-1}$ 6

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. . . .
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. . . .

(b) $\frac{2}{3+\sqrt{5}}$

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(c) $\frac{4}{\sqrt{5}+\sqrt{2}}$

. . . .
. . . .
. . . .
. . . .

(d) $\frac{1}{\sqrt{7}-\sqrt{5}}$

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(e) $\frac{1}{\sqrt{2}-1}$

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. . . .
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(f) $\frac{1}{\sqrt{3}-\sqrt{2}}$

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7 Irrational Numbers and Surds

Welcome to the final section. This is your party trick:
Here are 2 worked examples and then a number of exercises for you to mimic the method with:

7.1 Worked Examples

Question 1

Write $0.\dot{2}$ as a fraction.

Solution 1

Let $\quad S = 0.\dot{2}$

So $\quad S = 0.2222\ldots$

Then $\quad 10S = 2.2222\ldots \quad$ (Multiply by 10.)

$10S = 2 + 0.222\ldots$

Therefore $10S = S + 2$

Hence $\quad 9S = 2$

So $\quad S = \frac{2}{9}$

Thus $\quad 0.\dot{2} = \frac{2}{9}$

Question 2

Write 0.12 as a fraction.

Solution 2

Let $\quad S = 0.12$

So $\quad S = 0.12121212\ldots$

Then $\quad 100S = 12121212\ldots$

Therefore $100S = 12 + S$

Hence $\quad 99S = 12$

So $\quad S = \frac{12}{99} = \frac{4}{33}$

Thus $\quad 0.\dot{1}\dot{2} = \frac{4}{33}$

7.2 Exercises

1. (a) $0.\dot{5} = \frac{5}{9}$ 10
   (b) $0.\dot{7} = \frac{7}{9}$
   (c) $0.\dot{9} = \frac{9}{9} = 1$
   (d) $0.1\dot{4} = \frac{14}{99}$
   (e) $0.2\dot{3} = \frac{23}{99}$
   (f) $0.6\dot{2} = \frac{62}{99}$
   (g) $0.\dot{1}\dot{3} = \frac{13}{99}$
   (h) $0.\dot{0}7 = \frac{7}{90}$
   (i) $0.\dot{9}\dot{1} = \frac{91}{99}$
   (j) $0.\dot{2}4\dot{1} = \frac{239}{990}$

   Which of these numbers are irrational?
   (k) ☑ $\sqrt{7}$ ☐ $\sqrt{25}$ ☑ $\frac{\pi}{3}$ ☐ $0.\dot{6}$

8 Homework

1. S implify the following: 6

(a) $(2\sqrt{3}+\sqrt{2})^2$

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....
....

(b) $(2\sqrt{5}+4\sqrt{3})^2$

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....
....

(c) $(\sqrt{xy}+1)^2$

....
....
....

(d) $(4\sqrt{2}-3\sqrt{7})^2$

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(e) $(\sqrt{x}-\sqrt{y})^2$

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....
....

(f) $(\sqrt{11}-2\sqrt{22})^2$

....
....
....

(g) If $x=\sqrt{2}-1$ and $y=\sqrt{2}+1$, find:

i. xy

....
....
....

ii. x^2y

....
....
....

iii. y^2x

....
....
....

iv. $\frac{1}{x}+\frac{1}{y}$

....
....
....

(h)

$$(\sqrt{5}+\sqrt{3})^2-(\sqrt{5}-\sqrt{3})^2$$

....
....
....

(i)

$$(\sqrt{3}+\sqrt{2})^2-(\sqrt{3}-\sqrt{2})^2$$

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....

2. S implify $\boxed{2}$

$$(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d})$$

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3. F or each of the figures shown, find: $\boxed{9}$

(i) the value of x

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(ii) the area of the triangle

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(iii) the perimeter of the triangle

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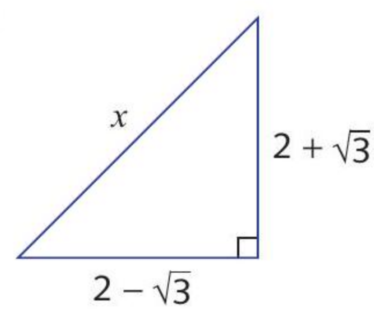
(a)



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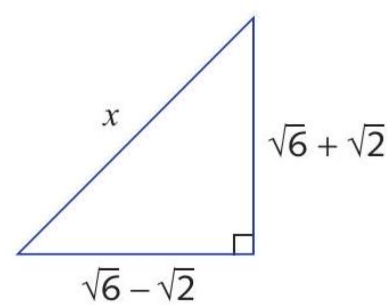
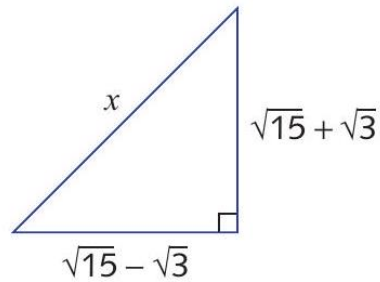


(b)

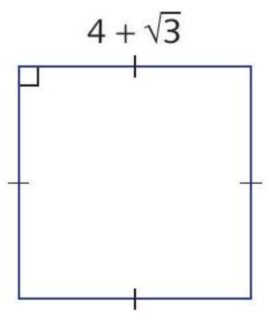
(c)



4. F or the 2 figures below, find: $\boxed{4}$

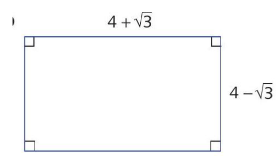
   (i) the perimeter

   (ii) the area



(a)

(b)



5. R ationalise the following: $\boxed{8}$

(a)

$$\frac{14}{\sqrt{7}}$$

(b)

$$\frac{\sqrt{14}}{\sqrt{7}}$$

(c)

$$\frac{\sqrt{5}}{3\sqrt{7}}$$

. . . .
. . . .
. . . .

(d)

$$\frac{\sqrt{3}}{4\sqrt{6}}$$

. . . .
. . . .
. . . .

(e)

$$\frac{5}{\sqrt{3}}$$

. . . .
. . . .
. . . .

(f)

$$\frac{1}{\sqrt{18}}$$

. . . .
. . . .
. . . .

(g)

$$\frac{1}{\sqrt{2}} + \sqrt{2}$$

. . . .
. . . .
. . . .

(h)

$$\frac{\sqrt{72}}{\sqrt{3}} + \frac{3}{\sqrt{2}} - \frac{2}{2\sqrt{2}}$$

. . . .
. . . .
. . . .

(i)

$$\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{7}{2\sqrt{3}}$$

. . . .
. . . .
. . . .

6. I f $x = 2\sqrt{14}$ and $y = 4\sqrt{2}$, find and rationalise the denominator. $\boxed{6}$

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $\frac{\sqrt{2}x}{\sqrt{3}y}$

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7. A bowl in the shape of a hemisphere of radius length 5 cm is partially filled with water. The surface of the water is a circle of radius 4 cm when the rim of the bowl is horizontal. Find the depth of the water. $\boxed{3}$

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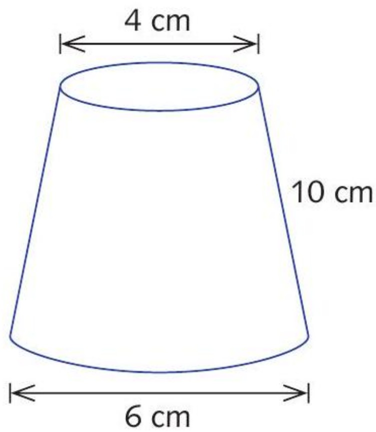
8. A bobbin for an industrial knitting machine is in the shape of a truncated cone. The diameter $\boxed{3}$

of the top is 4 cm, t diameter of the base is 6 cm and the length of the slant is 10 cm. Find the height of the bobbin.



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9. R ationalise the following: $\boxed{4}$

(a)
$$\frac{1}{\sqrt{3}+2}$$

(b)
$$\frac{1}{\sqrt{3}+\sqrt{2}}$$

. . . .
. . . .
. . . .

10. F ind the integers p and q such that $\boxed{3}$

$$\frac{\sqrt{5}}{\sqrt{5}-2}=p+q\sqrt{5}$$

.

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. . . .

11. S implify the following: $\boxed{9}$

(a) $\frac{3}{\sqrt{5}-2}+\frac{2}{\sqrt{5}+2}$

. . . .
. . . .
. . . .

(b) $\frac{5}{(\sqrt{7}-\sqrt{2})^2}$

(c) $0.0\dot{1}\dot{6}$

(d) $0.3\dot{2}\dot{4}$

(e) $0.51\dot{2}\dot{6}$

(f) $0.001\dot{1}$

12. **Challenge:** $\boxed{6}$

(a) Show that

$$\frac{138}{19} = 7 + \frac{1}{3 + \frac{1}{1+\frac{1}{4}}}$$

(b) Express

$$\frac{153}{11}$$

as a continued fraction with all numerators 1 .

9 Marks

Marker's use only.

SECTION	1	2	3	4	5	6	7	HW	Total
MARKS	$\frac{\quad}{0}$	$\frac{\quad}{12}$	$\frac{\quad}{17}$	$\frac{\quad}{12}$	$\frac{\quad}{9}$	$\frac{\quad}{8}$	$\frac{\quad}{10}$	$\frac{\quad}{63}$	$\frac{\quad}{131}$