Year 9 Mathematics | Topic 2 | Pythagoras and Surds [2/2]

PEN Education

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Contents

1 Introduction

This week we shall continue with our study of the type of numbers which are created when we take some kind of root $\sqrt{\ }$ of another *positive* number. These quantities we now know to be called **surds**, and last week we learned the basic arithmetic of these objects.

Before we begin our review / brain warm-up let us contextualise today's theory: We will begin by inserting **surds** into the algebraic structures of our first topic ($(a+b)^2 = a^2 + 2ab + b^2$, etc...) where a and b are now surd terms.

Then we will learn about something known as *rationalising the denominator* which is an important simplification method in mathematics, and one of the earlier conventions adopted by the *Babylonians* so that they would not be dividing by incomensurable quantities - i.e. what does it mean to divide 4 apples into $\sqrt{2}$ quantities? That would be asking to divide the 4 apples into $1.41421356237\ldots$ parts, where the 'dot, dot, dot' means the denominator never terminates!

After practising the above simplifications, we return to Pythagora's Theorem and Geometry. Specifically we attempt problems in 3 dimensions where the need for visualisation and mathematical stamina increase.

The final 2 sections then deal with rationalising *binomial denominators* and converting irrational numbers to fractions respectively. The latter of these makes for a cool party trick: WHAT FRACTION IS EQUAL TO THE REPEATING DECIMAL $0.81818181\ldots$?

$$0.\overline{81} =$$

2 Review

2.1 Exercises:

1. S implify: [4]

   (a) $\sqrt{108} = \underline{\quad 6\sqrt{3} \quad}$

   (b) $\sqrt{90} = \underline{\quad 3\sqrt{10} \quad}$

   (c) $\sqrt{52} = \underline{\quad 2\sqrt{13} \quad}$

   (d) $\sqrt{98} = \underline{\quad 7\sqrt{2} \quad}$

2. A dd or Subtract: [4]

   (a) $3\sqrt{2} + 2\sqrt{2} = \underline{\quad 5\sqrt{2} \quad}$

   (b) $\sqrt{32} - \sqrt{18} = \underline{\quad 2\sqrt{2} \quad}$

   (c) $\sqrt{45} + 2\sqrt{5} - \sqrt{80} = \underline{\quad 5\sqrt{5} \quad}$

   (d) $\sqrt{28} + 2\sqrt{63} - 5\sqrt{7} = \underline{\quad 9\sqrt{7} \quad}$

3. M ultiply or Divide: [4]

(a) $2\sqrt{3} \times 5\sqrt{6} =$ $\underline{\quad 30\sqrt{2} \quad}$

(b) $3\sqrt{5} \times 2\sqrt{10} =$ $\underline{\quad 30\sqrt{2} \quad}$

(c) $14\sqrt{40} \div 7\sqrt{5} =$ $\underline{\quad 4\sqrt{8} \quad}$

(d) $3\sqrt{2} \div \sqrt{2} =$ $\underline{\quad 3 \quad}$

3 Special Products

This is an extension of the things you learned in *Topic 1: Algebra*. We will be using 3 main results and then replacing the a's and b's with *surds*.

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$
$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2}$$
$$(a-b)(a+b) = a^2 - b^2 \tag{3}$$

3.1 Examples:

1. (a) 5

$$(\sqrt{5}+\sqrt{3})^2 =$$

Solution: $(\sqrt{5}+\sqrt{3})^2 = (\sqrt{5})^2 + 2(\sqrt{5})(\sqrt{3}) + (\sqrt{3})^2 = 5 + 2\sqrt{15} + 3 = 8 + 2\sqrt{15}$

(b)

$$(3\sqrt{2}-2\sqrt{3})^2 =$$

Solution: $(3\sqrt{2}-2\sqrt{3})^2 = (3\sqrt{2})^2 - 2(3\sqrt{2})(2\sqrt{3}) + (2\sqrt{3})^2 = 9\cdot 2 - 2\cdot 6\sqrt{6} + 4\cdot 3 =$ $18 - 12\sqrt{6} + 12$

(c)

$$(2\sqrt{3}+4\sqrt{6})^2 =$$

Solution: $(2\sqrt{3}+4\sqrt{6})^2 = (2\sqrt{3})^2 + 2(2\sqrt{3})(4\sqrt{6}) + (4\sqrt{6})^2 = 4\cdot 3 + 2\cdot 8\sqrt{18} + 16\cdot 6 = 12 + 16\sqrt{18} + 96$

(d)

$$(\sqrt{7}-5)(\sqrt{7}+5) =$$

Solution: $(\sqrt{7}-5)(\sqrt{7}+5) = (\sqrt{7})^2 - (5)^2 = 7 - 25 = -18$

(e)

$$(5\sqrt{6}-2\sqrt{5})(5\sqrt{6}+2\sqrt{5}) =$$

Solution: $(5\sqrt{6}-2\sqrt{5})(5\sqrt{6}+2\sqrt{5}) = (5\sqrt{6})^2 - (2\sqrt{5})^2 = 25\cdot 6 - 4\cdot 5 = 150 - 20 = 130$

3.2 Exercises:

1. (a) $(5+\sqrt{3})^2$ 12

Solution: $(5+\sqrt{3})^2 = 5^2 + 2(5)(\sqrt{3}) + (\sqrt{3})^2 = 25 + 10\sqrt{3} + 3 = 28 + 10\sqrt{3}$

(b) $(\sqrt{2}+6)^2$

Solution: $(\sqrt{2}+6)^2 = (\sqrt{2})^2 + 2(\sqrt{2})(6) + 6^2 = 2 + 12\sqrt{2} + 36 = 38 + 12\sqrt{2}$

(c) $(5\sqrt{6}+2\sqrt{3})^2$

Solution: $(5\sqrt{6}+2\sqrt{3})^2 = (5\sqrt{6})^2 + 2(5\sqrt{6})(2\sqrt{3}) + (2\sqrt{3})^2 = 25\cdot 6 + 2\cdot 10\sqrt{18} + 4\cdot 3 = 150 + 20\sqrt{18} + 12$

(d) $(\sqrt{21}+\sqrt{3})^2$

Solution: $(\sqrt{21}+\sqrt{3})^2 = (\sqrt{21})^2 + 2(\sqrt{21})(\sqrt{3}) + (\sqrt{3})^2 = 21 + 2\sqrt{63} + 3 = 24 + 2\sqrt{63}$

(e) $(4+2\sqrt{5})^2$

Solution: $(4+2\sqrt{5})^2 = 4^2 + 2(4)(2\sqrt{5}) + (2\sqrt{5})^2 = 16 + 16\sqrt{5} + 4\cdot 5 = 16 + 16\sqrt{5} + 20 = 3616\sqrt{5}$

(f) $(\sqrt{7}-2)^2$

Solution: $(\sqrt{7}-2)^2 = (\sqrt{7})^2 - 2(\sqrt{7})(2) + 2^2 = 7 - 4\sqrt{7} + 4 = 11 - 4\sqrt{7}$

(g) $(4-\sqrt{3})^2$

Solution: $(4-\sqrt{3})^2 = 4^2 - 2(4)(\sqrt{3}) + (\sqrt{3})^2 = 16 - 8\sqrt{3} + 3 = 19 - 8\sqrt{3}$

(h) $(2\sqrt{35}+\sqrt{5})^2$

Solution: $(2\sqrt{35}+\sqrt{5})^2 = (2\sqrt{35})^2 + 2(2\sqrt{35})(\sqrt{5}) + (\sqrt{5})^2 = 4\cdot 35 + 2\cdot 2\sqrt{1755} = 140 + 4\sqrt{175} + 5$

(i) $(\sqrt{70}-3\sqrt{10})^2$

Solution: $(\sqrt{70}-3\sqrt{10})^2 = (\sqrt{70})^2 - 2(\sqrt{70})(3\sqrt{10}) + (3\sqrt{10})^2 = 70 - 2\cdot 3\sqrt{700} + 9\cdot 10 = 70 - 6\sqrt{700} + 90$

(j) $(3-\sqrt{5})(3+\sqrt{5})$

Solution: $(3-\sqrt{5})(3+\sqrt{5}) = 3^2 - (\sqrt{5})^2 = 9 - 5 = 4$

(k) $(2\sqrt{5}-1)^2$

Solution: $(2\sqrt{5}-1)^2 = (2\sqrt{5})^2 - 2(2\sqrt{5})(1) + 1^2 = 4\cdot 5 - 2\cdot 2\sqrt{5} + 1 = 20 - 4\sqrt{5} + 1 = 2 - 4\sqrt{5}$

(l) $(\sqrt{6}-1)(\sqrt{6}+1)$

Solution: $(\sqrt{6}-1)(\sqrt{6}+1) = (\sqrt{6})^2 - 1^2 = 6 - 1 = 5$

4 Rationalising the Denominator

You will want to pay attention to this part, because we will come up against a more involved version of *rationalising the denominator* in about 20 minutes. We will begin to see them not just as $\frac{1}{\sqrt{2}}$ as we see them in this section, but rather as *binomials*: $\frac{1}{\sqrt{3}-\sqrt{2}}$.

We explained the motivations behind rationalising the denominator in the introduction but a quick revision will not hurt:

(1) Historical Convention

(2) Simplicity

(3) Aesthetics

(4) Clarity

Now having said this, what is the mathematical method to rationalise a denominator? Simple, just multiply the denominator with whatever you need to such that the surd cancels out:

$$\begin{aligned}\frac{2}{\sqrt{2}} &= \frac{2\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} \\ &= \frac{\cancel{2}\sqrt{2}}{\cancel{2}} \\ &= \sqrt{2}\end{aligned}$$

Recall that: you may multiply or divide a fraction by any number as long as you *do the top what you do to the bottom.*

4.1 Examples:

1. E xpress the following with a *rational* denominator 2

   (a) $\frac{4}{\sqrt{2}} =$

   Solution: $\frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

   (b) $\frac{9}{4\sqrt{3}} =$

   Solution: $\frac{9}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{12} = \frac{3\sqrt{3}}{4}$

   (c)

   $$\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2} =$$

Solution: $\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{3}}{2} \times \frac{2}{2} = \frac{2\sqrt{3}}{3} \frac{2\sqrt{3}}{4} = \frac{4\sqrt{3}+3\sqrt{3}}{6} = \frac{7\sqrt{3}}{6}$

Formulae:

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}, \text{ where } a \text{ is positive,}$$

4.2 Exercises:

1. (a) [6]

$$\frac{\sqrt{3}}{\sqrt{7}} =$$

Solution: $\frac{\sqrt{3}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{21}}{7}$

(b)

$$\frac{2\sqrt{7}}{\sqrt{3}} =$$

Solution: $\frac{2\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{21}}{3}$

(c)

$$\frac{\sqrt{2}}{3\sqrt{10}} =$$

Solution: $\frac{\sqrt{2}}{3\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{20}}{30} = \frac{\sqrt{5}}{15}$

(d)

$$\frac{\sqrt{15}}{3\sqrt{5}} =$$

Solution: $\frac{\sqrt{15}}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{75}}{15} = \frac{5}{3}$

(e)

$$\frac{2}{\sqrt{3}} + \frac{3}{2\sqrt{3}} =$$

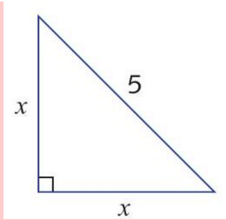
Solution: $\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} + \frac{3}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} + \frac{3\sqrt{3}}{6} = \frac{4\sqrt{3}+3\sqrt{3}}{6} = \frac{7\sqrt{3}}{6}$

(f)

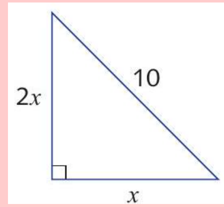
$$\frac{5\sqrt{2}}{3} - \frac{1}{\sqrt{3}} =$$

Solution: $\frac{5\sqrt{2}}{3} - \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{2}}{3} - \frac{\sqrt{3}}{3} = \frac{5\sqrt{2}-\sqrt{3}}{3}$

2. F ind the value of x. Express your answer with a rational denominator. [4]

(a) 

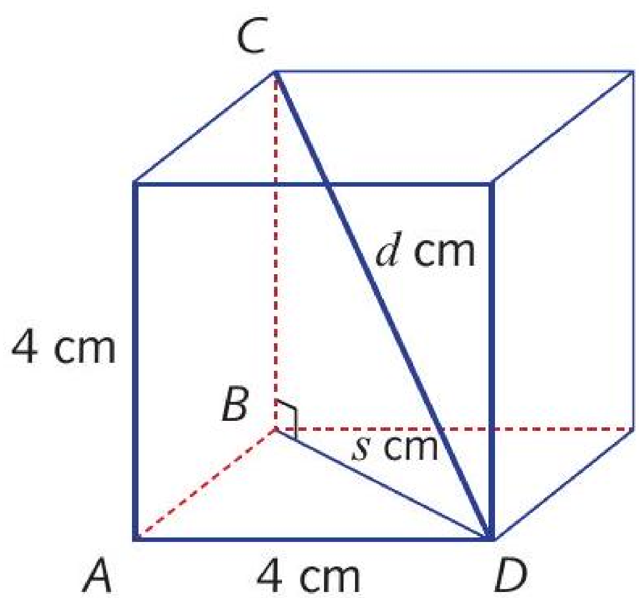
Solution:

(b) 

Solution:

5 Pythagoras in 3-Dimensions!

Let us consider a problem:



How can we find the length of the diagonal CD of this cube with side lengths of 4cm?
We apply Pythagoras' Theorem!

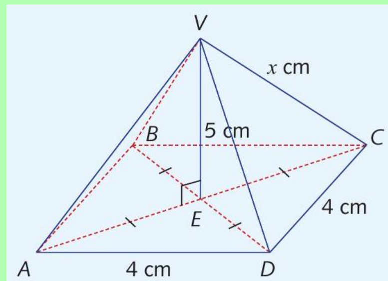
Recall that: $c^2 = a^2 + b^2$

And so here we first find the length of s to be $\underline{\sqrt{32} = 4\sqrt{2}}$ and thus we can use this length of s along with the side CB to form another triangle and find $d = \underline{\sqrt{32 + 16} = \sqrt{48}} = 4\sqrt{3} = \approx 6.93$.

Solution:

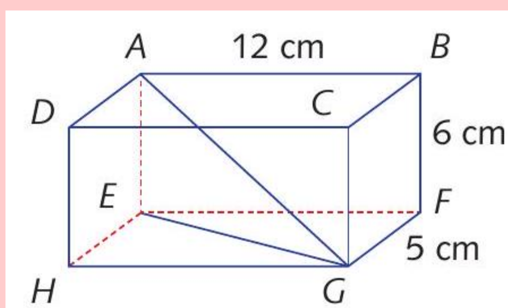
5.1 Example:

A square pyramid has height 5 cm and square base with side length 4 cm. Find the length of the edge VC in this diagram.



Solution: To find the length of the edge VC, we can use the Pythagorean theorem on the triangle formed by the height of the pyramid, half the base, and the slant height (edge VC). The half of the base is 2 cm (half of 4 cm). So, $VC^2 = 5^2 + 2^2 = 25 + 4 = 29$, thus $VC = \sqrt{29} \approx 5.39$ cm.

Exercises:

1. 

The rectangular prism in the diagram has a length of 12 cm, a width of 5 cm and a height of 6 cm.

(a) Consider triangle EFG. Find:

i. EF

Solution: $EF = 12$ cm (given as the length of the prism).

ii. the size of EFG

Solution: $\angle EFG$ is a right angle since EF and FG are edges of the rectangular prism meeting at a right angle.

(b) Find EG.

Solution: To find EG, we use Pythagoras' theorem on triangle EFG: $EG^2 = EF^2 + FG^2 = 12^2 + 6^2 = 144 + 36 = 180$, $EG = \sqrt{180} = 6\sqrt{5} \approx 13.42$ cm.

(c) Find AG, correct to 1 decimal place.

Solution: To find AG, we use Pythagoras' theorem on triangle AFG: $AG^2 = AF^2 + FG^2 = 5^2 + 6^2 = 25 + 36 = 61$, so $AG = \sqrt{61} \approx 7.8$ cm.

2. Note: AG is called the space diagonal of the rectangular prism. □

3. F ind the length of the longest pencil that can fit inside a cylindrical pencil case of length 3
15 cm and radius 2 cm.

Solution: The longest pencil that can fit inside the cylindrical pencil case would be the length of the diagonal of the cylinder. This can be found using Pythagoras' theorem with the length and diameter of the cylinder: $pencil^2 = length^2 + diameter^2 = 15^2 + (2 \times 2)^2 = 225 + 16 = 241$, so the pencil length is $\sqrt{241} \approx 15.52$ cm.

6 Binomial Denominators

As promised, here lie the *binomial denominators*. There are a couple of prerequisites to understanding this mathematics, the first is that you must totally grasp $(a+b)(a-b) = a^2 - b^2$. The significance of this result is that if we make a and b surds like so: $(\sqrt{a}+\sqrt{b})$, then multiplying by $(\sqrt{a}-\sqrt{b})$ yields a fantastic result, namely:

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 \quad = a - b \tag{4}$$

This act of starting with $\sqrt{a}+\sqrt{b}$, and multiplying it with the sign flipped: $\sqrt{a}-\sqrt{b}$ is called *multiplying by the reciprocal*. Let's practise this method:

6.1 Examples:

1. 1

$$\frac{2\sqrt{5}}{2\sqrt{5}-2} =$$

Solution: Multiply by the conjugate: $\frac{2\sqrt{5}}{2\sqrt{5}-2} \cdot \frac{2\sqrt{5}+2}{2\sqrt{5}+2} = \frac{4\cdot+4\sqrt{5}}{20-4} = \frac{20+4\sqrt{5}}{16} = \frac{5}{4} + \sqrt{5}$

2. $$\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}}$$ [1]

Solution: Multiply by the conjugate: $\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} \cdot \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} = \frac{3\cdot 2-2\cdot 3}{6-4\sqrt{6}+4\sqrt{6}-12} = \frac{6-6}{-6} = 0$

6.2 Exercises:

1. (a) $\frac{3}{\sqrt{2}-1}$ [6]

   Solution: Multiply by the conjugate: $\frac{3}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{3(\sqrt{2}+1)}{2-1} 3\sqrt{2}+3$

   (b) $\frac{2}{3+\sqrt{5}}$

   Solution: Multiply by the conjugate: $\frac{2}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{6-2\sqrt{5}}{9} - = \frac{6-2\sqrt{5}}{4} = \frac{3}{2} - \frac{\sqrt{5}}{2}$

   (c) $\frac{4}{\sqrt{5}+\sqrt{2}}$

   Solution: Multiply by the conjugate: $\frac{4}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{4\sqrt{5}-4\sqrt{2}}{5-2} = \frac{4\sqrt{5}-4\sqrt{2}}{3}$

   (d) $\frac{1}{\sqrt{7}-\sqrt{5}}$

   Solution: Multiply by the conjugate: $\frac{1}{\sqrt{7}-\sqrt{5}} \cdot \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}} = \frac{\sqrt{7}+\sqrt{5}}{7-5} = \frac{\sqrt{7}+\sqrt{5}}{2}$

   (e) $\frac{1}{\sqrt{2}-1}$

   Solution: Multiply by the conjugate: $\frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$

   (f) $\frac{1}{\sqrt{3}-\sqrt{2}}$

   Solution: Multiply by the conjugate: $\frac{1}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{3-2} = \sqrt{3}+\sqrt{2}$

7 Irrational Numbers and Surds

Welcome to the final section. This is your party trick:
Here are 2 worked examples and then a number of exercises for you to mimic the method with:

7.1 Worked Examples

Question 1

Write $0.\dot{2}$ as a fraction.

Solution 1

Let $S = 0.\dot{2}$

So $S = 0.2222\ldots$

Then $10S = 2.2222\ldots$ (Multiply by 10.)

$10S = 2 + 0.222\ldots$

Therefore $10S = S + 2$

Hence $9S = 2$

So $S = \frac{2}{9}$

Thus $0.\dot{2} = \frac{2}{9}$

Question 2

Write 0.12 as a fraction.

Solution 2

Let $S = 0.12$

So $S = 0.12121212\ldots$

Then $100S = 12121212\ldots$

Therefore $100S = 12 + S$

Hence $99S = 12$

So $S = \frac{12}{99} = \frac{4}{33}$

Thus $0.\dot{1}\dot{2} = \frac{4}{33}$

7.2 Exercises

1. (a) $0.\dot{5} = \frac{5}{9}$ [10]
   (b) $0.\dot{7} = \frac{7}{9}$
   (c) $0.\dot{9} = \frac{9}{9} = 1$
   (d) $0.1\dot{4} = \frac{14}{99}$
   (e) $0.2\dot{3} = \frac{23}{99}$
   (f) $0.6\dot{2} = \frac{62}{99}$
   (g) $0.\dot{1}\dot{3} = \frac{13}{99}$
   (h) $0.\dot{0}7 = \frac{7}{90}$
   (i) $0.\dot{9}\dot{1} = \frac{91}{99}$
   (j) $0.\dot{2}4\dot{1} = \frac{239}{990}$
   (k) Which of these numbers are irrational?
   ☑ $\sqrt{7}$
   ☐ $\sqrt{25}$
   ☐ $0.\dot{6}$
   ☑ $\frac{\pi}{3}$

8 Homework

1. S implify the following: [6]

(a) $(2\sqrt{3}+\sqrt{2})^2$

Solution: $(2\sqrt{3}+\sqrt{2})^2 = 4\cdot 3 + 4\sqrt{3}\sqrt{2}+2 = 12+4\sqrt{6}+2 = 14+4\sqrt{6}$

(b) $(2\sqrt{5}+4\sqrt{3})^2$

Solution: $(2\sqrt{5}+4\sqrt{3})^2 = 4\cdot5+16\cdot3+2\cdot8\sqrt{15} = 20+48+16\sqrt{15} = 68+16\sqrt{15}$

(c) $(\sqrt{xy}+1)^2$

Solution: $(\sqrt{xy}+1)^2 = xy+2\sqrt{xy}+1$

(d) $(4\sqrt{2}-3\sqrt{7})^2$

Solution: $(4\sqrt{2}-3\sqrt{7})^2 = 16\cdot 2+9\cdot 7-2\cdot 12\sqrt{14} = 32+63-24\sqrt{14} = 95-24\sqrt{14}$

(e) $(\sqrt{x}-\sqrt{y})^2$

Solution: $(\sqrt{x}-\sqrt{y})^2 = x-2\sqrt{xy}+y$

(f) $(\sqrt{11}-2\sqrt{22})^2$

Solution: $(\sqrt{11}-2\sqrt{22})^2 = 11-4\cdot22+4\cdot 11 = 11-88+44 = -33+44 = 11$

(g) If $x=\sqrt{2}-1$ and $y=\sqrt{2}+1$, find:

i. xy

Solution: $xy = (\sqrt{2}-1)(\sqrt{2}+1) = (\sqrt{2})^2-(1)^2 = 2-1 = 1$

ii. x^2y

Solution: $x^2y = (\sqrt{2}-1)^2(\sqrt{2}+1) = (2-2\sqrt{2}+1)(\sqrt{2}+1) = (3-2\sqrt{2})(\sqrt{2}+1) = 3\sqrt{2}+3-2\cdot2-2\sqrt{2} = 3\sqrt{2}+3-4-2\sqrt{2} = \sqrt{2}-1$

iii. y^2x

Solution: $y^2x = (\sqrt{2}+1)^2(\sqrt{2}-1) = (2+2\sqrt{2}+1)(\sqrt{2}-1) = (3+2\sqrt{2})(\sqrt{2}-1) = 3\sqrt{2}-3+2\cdot2-2\sqrt{2} = 3\sqrt{2}-3+4-2\sqrt{2} = \sqrt{2}+1$

iv. $\frac{1}{x}+\frac{1}{y}$

Solution: $\frac{1}{x}+\frac{1}{y} = \frac{1}{\sqrt{2}-1}+\frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2-1}+\frac{\sqrt{2}-1}{2-1} = \sqrt{2}+1+\sqrt{2}-1 = 2\sqrt{2}$

(h)

$$(\sqrt{5}+\sqrt{3})^2-(\sqrt{5}-\sqrt{3})^2$$

Solution: $(\sqrt{5}+\sqrt{3})^2-(\sqrt{5}-\sqrt{3})^2 = (5+2\sqrt{15}+3)-(5-2\sqrt{15}+3) = 8+2\sqrt{15}8+2\sqrt{15} = 4\sqrt{15}$

(i)

$$(\sqrt{3}+\sqrt{2})^2-(\sqrt{3}-\sqrt{2})^2$$

Solution: $(\sqrt{3}+\sqrt{2})^2-(\sqrt{3}-\sqrt{2})^2 = (3+2\sqrt{6}+2)-(3-2\sqrt{6}+2) = 5+2\sqrt{6}-52\sqrt{6} = 4\sqrt{6}$

2. S implify

$$(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d})$$

.

Solution: $(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d}) = (a\sqrt{b})^2-(c\sqrt{d})^2 = a^2b-c^2d$

2

3. F or each of the figures shown, find:

9

   (i) the value of x

   Solution:

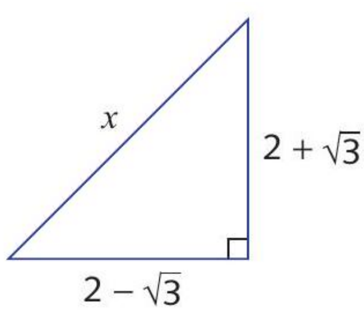
   (ii) the area of the triangle

   Solution:

   (iii) the perimeter of the triangle

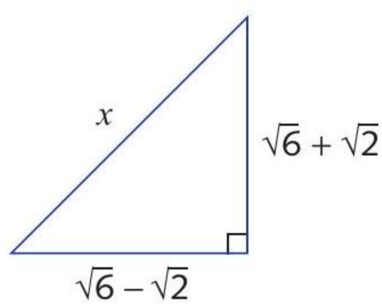
   Solution:

(a)



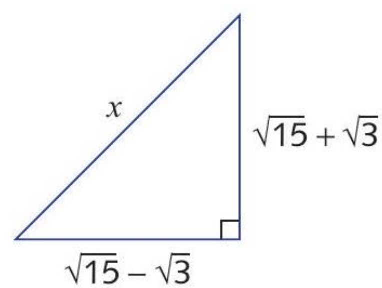
Solution:

(b)



Solution:

(c)



Solution:

4. F or the 2 figures below, find:

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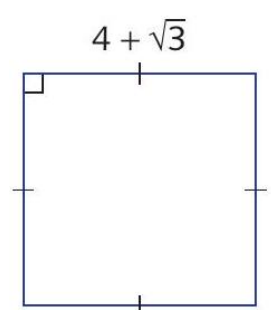
   (i) the perimeter
   (ii) the area

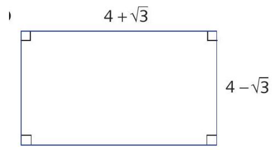


(a)

Solution:

Solution:

(b)



5. R ationalise the following: 8

(a)

$$\frac{14}{\sqrt{7}}$$

Solution: $\frac{14}{\sqrt{7}} = \frac{14}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{14\sqrt{7}}{7} = 2\sqrt{7}$

(b)

$$\frac{\sqrt{14}}{\sqrt{7}}$$

Solution: $\frac{\sqrt{14}}{\sqrt{7}} = \sqrt{\frac{14}{7}} = \sqrt{2}$

(c)

$$\frac{\sqrt{5}}{3\sqrt{7}}$$

Solution: $\frac{\sqrt{5}}{3\sqrt{7}} = \frac{\sqrt{5}}{3\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{35}}{21}$

(d)

$$\frac{\sqrt{3}}{4\sqrt{6}}$$

Solution: $\frac{\sqrt{3}}{4\sqrt{6}} = \frac{\sqrt{3}}{4\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{18}}{24} = \frac{\sqrt{2}}{8}$

(e)

$$\frac{5}{\sqrt{3}}$$

Solution: $\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$

(f)

$$\frac{1}{\sqrt{18}}$$

Solution: $\frac{1}{\sqrt{18}} = \frac{1}{\sqrt{18}} \cdot \frac{\sqrt{18}}{\sqrt{18}} = \frac{\sqrt{18}}{18} = \frac{\sqrt{2}}{6}$

(g)

$$\frac{1}{\sqrt{2}} + \sqrt{2}$$

Solution: $\frac{1}{\sqrt{2}} + \sqrt{2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} + \sqrt{2} = \frac{\sqrt{2}}{2} + \sqrt{2} = \frac{3\sqrt{2}}{2}$

(h)

$$\frac{\sqrt{72}}{\sqrt{3}} + \frac{3}{\sqrt{2}} - \frac{2}{2\sqrt{2}}$$

Solution: $\frac{\sqrt{72}}{\sqrt{3}} + \frac{3}{\sqrt{2}} - \frac{2}{2\sqrt{2}} = \sqrt{24} + \frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 2\sqrt{6} + \frac{2\sqrt{2}}{2} = 2\sqrt{6} + \sqrt{2}$

(i)

$$\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{7}{2\sqrt{3}}$$

Solution: $\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{7}{2\sqrt{3}} = \frac{1+2+\frac{7}{2}}{\sqrt{3}} = \frac{\frac{2+4+7}{2}}{\sqrt{3}} = \frac{\frac{13}{2}}{\sqrt{3}} = \frac{13}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{13\sqrt{3}}{6}$

6. I f $x = 2\sqrt{14}$ and $y = 4\sqrt{2}$, find and rationalise the denominator. 6

(a) $\frac{x}{y}$

Solution: $\frac{x}{y} = \frac{2\sqrt{14}}{4\sqrt{2}} = \frac{\sqrt{14}}{2\sqrt{2}} = \frac{\sqrt{14}}{2\sqrt{2}} \cdots \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{28}}{4} = \frac{2\sqrt{7}}{4} = \frac{\sqrt{7}}{2}$

(b) $\frac{y}{x}$

Solution: $\frac{y}{x} = \frac{4\sqrt{2}}{2\sqrt{14}} = \frac{2\sqrt{2}}{\sqrt{14}} = \frac{2\sqrt{2}}{\sqrt{14}} \cdots \frac{\sqrt{14}}{\sqrt{14}} = \frac{2\sqrt{28}}{14} = \frac{2 \cdot 2\sqrt{7}}{14} = \frac{\sqrt{7}}{7}$

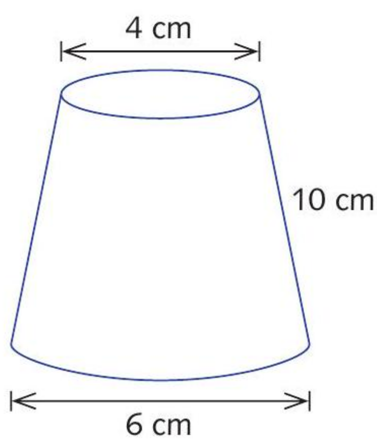
(c) $\frac{\sqrt{2}x}{\sqrt{3}y}$

Solution: $\frac{\sqrt{2}x}{\sqrt{3}y} = \frac{\sqrt{2} \cdot 2\sqrt{14}}{\sqrt{3} \cdot 4\sqrt{2}} = \frac{2\sqrt{28}}{4\sqrt{6}} = \frac{\sqrt{28}}{2\sqrt{6}} = \frac{\sqrt{28}}{2\sqrt{6}} \cdots \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{168}}{12} = \frac{2\sqrt{42}}{12} = \frac{\sqrt{42}}{6}$

7. A bowl in the shape of a hemisphere of radius length 5 cm is partially filled with water. The surface of the water is a circle of radius 4 cm when the rim of the bowl is horizontal. Find the depth of the water. 3

Solution: Using the Pythagorean theorem for the right triangle formed by the radius of the water's surface, the radius of the bowl, and the depth of the water: $5^2 = 4^2 + d^2$ $25 = 16 + d^2$ $d^2 = 25 - 16$ $d^2 = 9$ $d = 3$ cm The depth of the water is 3 cm.

8. A bobbin for an industrial knitting machine is in the shape of a truncated cone. The diameter of the top is 4 cm, t diameter of the base is 6 cm and the length of the slant is 10 cm. Find the height of the bobbin. 3



Solution: Let the height of the bobbin be h, the radius of the top be $r_1 = 2$ cm, and the radius of the base be $r_2 = 3$ cm. By the Pythagorean theorem, we have:

$$l^2 = h^2 + (r_2 - r_1)^2$$

Substituting the given values:

$$10^2 = h^2 + (3 - 2)^2$$

$$100 = h^2 + 1$$

$$h^2 = 99$$
$$h = \sqrt{99}$$
$$h \approx 9.95 \text{ cm}$$
The height of the bobbin is approximately 9.95 cm.

9. R ationalise the following: 4

(a)
$$\frac{1}{\sqrt{3}+2}$$

(b)
$$\frac{1}{\sqrt{3}+\sqrt{2}}$$

Solution: Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{1}{\sqrt{3}+2}\cdot\frac{\sqrt{3}-2}{\sqrt{3}-2} = \frac{\sqrt{3}-2}{(\sqrt{3}+2)(\sqrt{3}-2)} = \frac{\sqrt{3}-2}{3-4} = 2-\sqrt{3}$$

Solution: Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{1}{\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{3-2} = \sqrt{3}-\sqrt{2}$$

10. F ind the integers p and q such that 3

$$\frac{\sqrt{5}}{\sqrt{5}-2} = p + q\sqrt{5}$$

.

Solution: Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{\sqrt{5}}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{5+2\sqrt{5}}{5-4} = \frac{5+2\sqrt{5}}{1} = 5+2\sqrt{5}$$

Thus, $p = 5$ and $q = 2$.

11. S implify the following: 9

(a) $\frac{3}{\sqrt{5}-2} + \frac{2}{\sqrt{5}+2}$

Solution: Multiply each term by the conjugate of its denominator:

$$\frac{3}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2} + \frac{2}{\sqrt{5}+2}\cdot\frac{\sqrt{5}-2}{\sqrt{5}-2}$$
$$= \frac{3(\sqrt{5}+2)}{5-4} + \frac{2(\sqrt{5}-2)}{5-4}$$
$$= 3\sqrt{5}+6+2\sqrt{5}-4$$
$$= 5\sqrt{5}+2$$

(b) $\frac{5}{(\sqrt{7}-\sqrt{2})^2}$

Solution: Expand the denominator and simplify:

$$\frac{5}{7-2\sqrt{14}+2}$$
$$=\frac{5}{9-2\sqrt{14}}$$

Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{5}{9-2\sqrt{14}}\cdot\frac{9+2\sqrt{14}}{9+2\sqrt{14}}$$
$$=\frac{45+10\sqrt{14}}{81-56}$$
$$=\frac{45+10\sqrt{14}}{25}$$
$$=\frac{9}{5}+2\sqrt{14}$$

(c) $0.0\mathrm{i}\dot{6}$

Solution: This is a repeating decimal. Let $x = 0.0\dot{6}$, then:

$$10x = 0.\dot{6}$$
$$10x - x = 0.\dot{6} - 0.0\dot{6}$$
$$9x = 0.6$$
$$x = \frac{0.6}{9}$$
$$x = \frac{2}{30}$$
$$x = \frac{1}{15}$$

(d) $0.3\dot{2}\dot{4}$

Solution: This is a repeating decimal. Let $x = 0.3\dot{2}\dot{4}$, then:

$$1000x = 324.\dot{2}\dot{4}$$
$$1000x - x = 324.\dot{2}\dot{4} - 0.3\dot{2}\dot{4}$$
$$999x = 324$$
$$x = \frac{324}{999}$$
$$x = \frac{108}{333}$$
$$x = \frac{36}{111}$$
$$x = \frac{12}{37}$$

(e) $0.51\dot{2}\dot{6}$

Solution: This is a repeating decimal. Let $x = 0.51\dot{2}\dot{6}$, then:

$$1000x = 512.\dot{2}\dot{6}$$
$$1000x - 10x = 512.\dot{2}\dot{6} - 5.1\dot{2}\dot{6}$$
$$990x = 507.1$$
$$x = \frac{507.1}{990}$$
$$x = \frac{5071}{9900}$$
$$x = \frac{5071}{9900}$$

Simplifying the fraction, we get:

$$x = \frac{5071}{9900}$$
$$x = \frac{563}{1100}$$
$$x = \frac{563}{1100}$$

(f) $0.001\dot{1}$

Solution: This is a repeating decimal. Let $x = 0.001\dot{1}$, then:

$$1000x = 1.\dot{1}$$

$$1000x - x = 1.\dot{1} - 0.001\dot{1}$$

$$999x = 1.1$$

$$x = \frac{1.1}{999}$$

$$x = \frac{11}{9990}$$
$$x = \frac{1}{909}$$

12. **Challenge:** [6]

(a) Show that

$$\frac{138}{19} = 7 + \frac{1}{3 + \frac{1}{1+\frac{1}{4}}}$$

Solution: Start by evaluating the continued fraction from the innermost fraction outward:

$$1 + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}$$
$$3 + \frac{1}{\frac{5}{4}} = 3 + \frac{4}{5} = \frac{15}{5} + \frac{4}{5} = \frac{19}{5}$$
$$7 + \frac{1}{\frac{19}{5}} = 7 + \frac{5}{19} = \frac{133}{19} + \frac{5}{19} = \frac{138}{19}$$

Therefore, the continued fraction equals $\frac{138}{19}$.

(b) Express

$$\frac{153}{11}$$

as a continued fraction with all numerators 1 .

Solution: Perform the Euclidean algorithm to find the continued fraction:

$$\frac{153}{11} = 13 + \frac{10}{11}$$
$$\frac{11}{10} = 1 + \frac{1}{10}$$
$$\frac{10}{1} = 10$$

So,

$$\frac{153}{11} = 13 + \frac{1}{1 + \frac{1}{10}}$$

Therefore, the continued fraction is $13 + \frac{1}{1+\frac{1}{10}}$.

9 Marks

Marker's use only.

SECTION	1	2	3	4	5	6	7	HW	**Total**
MARKS	$\overline{}$ 0	___ 12	___ 17	___ 12	_ 9	_ 8	___ 10	___ 63	___ 131