Appendix B
Algebra and Basic Facts About $\mathbb{R}$ and $\mathbb{C}$

B.1. A *field* is a set F, together with binary operations $+$ and $\cdot$ on F such that

(a) $(x+y)+z = x+(y+z)$ holds for all x, y, z in F,
(b) $x+y = y+x$ holds for all x, y in F,
(c) there is an element 0 of F such that $x+0 = x$ holds for all x in F,
(d) for each x in F there is an element $-x$ of F such that $x+(-x) = 0$,
(e) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds for all x, y, z in F,
(f) $x \cdot y = y \cdot x$ holds for all x, y in F,
(g) there is an element 1 of F, distinct from 0, such that $1 \cdot x = x$ holds for all x in F,
(h) for each nonzero x in F there is an element x^{-1} of F such that $x \cdot x^{-1} = 1$, and
(i) $x \cdot (y+z) = x \cdot y + x \cdot z$ holds for all x, y, z in F.

Of course, one usually writes xy in place of $x \cdot y$.

B.2. An *ordered field* is a field F, together with a linear order $\leq$ (see A.11) on F such that

(a) if x, y, and z belong to F and if $x \leq y$, then $x+z \leq y+z$, and
(b) if x and y belong to F and satisfy $x > 0$ and $y > 0$, then $x \cdot y > 0$.

Let F be an ordered field, and let A be a subset of F. An *upper bound* of A is an element x of F such that $a \leq x$ holds for each a in A; a *least upper bound* (or *supremum*) of A is an upper bound of A that is smaller than all other upper bounds of A. *Lower bounds* and *greatest lower bounds* (or *infima*) are defined analogously. An ordered field F is *complete* if each nonempty subset of F that has an upper bound in F has a least upper bound in F.

B.3. The field $\mathbb{R}$ of real numbers is a complete ordered field; it is essentially the only complete ordered field (see Birkhoff and MacLane [9, Chapter 4], Gleason [49, Chapters 8 and 9], or Spivak [111, Chapters 28 and 29] for a precise statement and proof of this assertion).

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B.4. The *extended real numbers* consist of the real numbers, together with $+\infty$ and $-\infty$. We will use $\overline{\mathbb{R}}$ or $[-\infty,+\infty]$ to denote the set of all extended real numbers. The relations $-\infty < x$ and $x < +\infty$ are declared to hold for each real number x (of course $-\infty < +\infty$). We define arithmetic operations on $\overline{\mathbb{R}}$ by declaring that

$$x+(+\infty) = (+\infty)+x = +\infty$$

and

$$x+(-\infty) = (-\infty)+x = -\infty$$

hold for each real x, that

$$x\cdot(+\infty) = (+\infty)\cdot x = +\infty$$

and

$$x\cdot(-\infty) = (-\infty)\cdot x = -\infty$$

hold for each positive real x, and that

$$x\cdot(+\infty) = (+\infty)\cdot x = -\infty$$

and

$$x\cdot(-\infty) = (-\infty)\cdot x = +\infty$$

hold for each negative real x; we also declare that

$$(+\infty)+(+\infty) = +\infty,$$

$$(-\infty)+(-\infty) = -\infty,$$

$$(+\infty)\cdot(+\infty) = (-\infty)\cdot(-\infty) = +\infty,$$

$$(+\infty)\cdot(-\infty) = (-\infty)\cdot(+\infty) = -\infty,$$

and

$$0\cdot(+\infty) = (+\infty)\cdot 0 = 0\cdot(-\infty) = (-\infty)\cdot 0 = 0.$$

The sums $(+\infty)+(-\infty)$ and $(-\infty)+(+\infty)$ are left undefined. (The products $0\cdot(+\infty)$, $(+\infty)\cdot 0$, $(-\infty)\cdot 0$, and $0\cdot(-\infty)$, even though left undefined in many other areas of mathematics, are defined to be 0 in the study of measure theory; this simplifies the definition of the Lebesgue integral.)

The absolute values of $+\infty$ and of $-\infty$ are defined by

$$|+\infty| = |-\infty| = +\infty.$$

The maximum and minimum of the extended real numbers x and y are often denoted by $x\vee y$ and $x\wedge y$.

B.5. Each subset of $\overline{\mathbb{R}}$ has a least upper bound, or supremum, and a greatest lower bound, or infimum, in $\overline{\mathbb{R}}$. The supremum and infimum of a subset A of $\overline{\mathbb{R}}$ are often denoted by $\sup(A)$ and $\inf(A)$. Note that the set under consideration here may be empty: each element of $\overline{\mathbb{R}}$ is an upper bound and a lower bound of $\varnothing$; hence $\sup(\varnothing) = -\infty$ and $\inf(\varnothing) = +\infty$. Note also that $\sup(A)$ is a real number (rather than $+\infty$ or $-\infty$) if and only if A is nonempty and bounded above; a similar remark applies to infima.

B.6. Let $\{x_n\}$ be a sequence of elements of $\overline{\mathbb{R}}$. The *limit superior* of $\{x_n\}$, written $\overline{\lim}_n x_n$ or $\limsup_n x_n$, is defined by

$$\overline{\lim_n} x_n = \inf_k \sup_{n \ge k} x_n.$$

Likewise, the *limit inferior* of $\{x_n\}$, written $\underline{\lim}_n x_n$ or $\liminf_n x_n$, is defined by

$$\underline{\lim_n} x_n = \sup_k \inf_{n \ge k} x_n.$$

The relation $\underline{\lim}_n x_n \le \overline{\lim}_n x_n$ holds for each sequence $\{x_n\}$. The sequence $\{x_n\}$ has a *limit* (in $\overline{\mathbb{R}}$) if $\overline{\lim}_n x_n = \underline{\lim}_n x_n$; the limit of $\{x_n\}$ is then defined by

$$\lim_n x_n = \overline{\lim_n} x_n = \underline{\lim_n} x_n$$

(note that $\lim_n x_n$ can be $+\infty$ or $-\infty$).

In cases where each x_n, along with $\lim_n x_n$, is finite, the definition of limit given above is equivalent to the usual ε–δ (or ε–N) definition: $x = \lim_n x_n$ if and only if for every ε there is a positive integer N such that $|x_n - x| < \varepsilon$ holds for each n larger than N. (We need our definition of limits in $\overline{\mathbb{R}}$, involving lim sups and lim infs, because we need to handle infinite limits and sums, and sums some of whose terms may include $+\infty$ or $-\infty$.)

B.7. We will occasionally need the fact that if a and a_n, $n = 1, 2, \ldots$, are real (or complex) numbers such that $a = \lim_n a_n$, then $a = \lim_n (a_1 + \cdots + a_n)/n$. To verify this, note that if $1 \le M < n$, then

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i - a \right| \le \frac{1}{n} \sum_{i=1}^{M} |a_i - a| + \frac{1}{n} \sum_{i=M+1}^{n} |a_i - a|.$$

If we first make M so large that $|a_i - a| < \varepsilon$ if $i > M$ and then choose N so large that $(1/n)\sum_{i=1}^{M} |a_i - a|$ is less than ε if $n > N$, then $(1/n)\sum_{i=1}^{n} a_i$ is within 2ε of a if $n > \max(M,N)$.

B.8. Let $\sum_{k=1}^{\infty} x_k$ be an infinite series whose terms belong to $\overline{\mathbb{R}}$. This series *has a sum* if

(a) $+\infty$ and $-\infty$ do not both occur among the terms of $\sum_{k=1}^{\infty} x_k$, and
(b) the sequence $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ of partial sums of $\sum_{k=1}^{\infty} x_k$ has a limit in $\overline{\mathbb{R}}$.

The sum of the series $\sum_{k=1}^{\infty} x_k$ is then defined to be $\lim_n \sum_{k=1}^{n} x_k$ and is denoted by $\sum_{k=1}^{\infty} x_k$. (Note that condition (a) above is needed to guarantee that each of the partial sums $\sum_{k=1}^{n} x_k$ is defined.)

The reader can check that the sum of the series $\sum_{k=1}^{\infty} x_k$ exists *and belongs to* $\mathbb{R}$ if and only if

(a) each term of $\sum_{k=1}^{\infty} x_k$ belongs to $\mathbb{R}$, and
(b) the series $\sum_{k=1}^{\infty} x_k$ is convergent (in the sense of elementary calculus).

Suppose that $\sum_{k=1}^{\infty} x_k$ is an infinite series whose terms belong to $[0,+\infty]$. It is easy to see that the sum of the series $\sum_{k=1}^{\infty} x_k$ exists and is the supremum of the set of sums $\sum_{k\in F} x_k$, where F ranges over the set of finite subsets of $\mathbb{N}$.

B.9. A *dyadic rational* is a number that can be written in the form $i/2^n$ for some integer i and some nonnegative integer n. If x is a dyadic rational that belongs to the interval $(0,1)$, then x can be written in the form $i/2^n$, where n is a positive integer and i is an odd integer such that $0 < i < 2^n$. Such an x has a binary expansion $0.b_1b_2\dots b_n$, where there are exactly n bits to the right of the binary point and where b_n, the rightmost of these bits, is equal to 1. Such an x also has an unending binary expansion, where $b_n = 0$ and all the later bits (b_{n+1}, b_{n+2}, ...) are equal to 1. These dyadic rationals are the only values in the interval $(0,1)$ that have more than one binary expansion; to see this, suppose that x has binary expansions $0.b_1b_2\dots$ and $0.c_1c_2\dots$, let n_0 be the smallest n such that $b_n \neq c_n$ (for definiteness, suppose that $b_{n_0} = 0$ and $c_{n_0} = 1$), and check that this can happen only if $b_{n_0+1} = b_{n_0+2} = \cdots = 1$ and $c_{n_0+1} = c_{n_0+2} = \cdots = 0$.

B.10. Roughly speaking, the *complex numbers* are those of the form $x+iy$, where x and y are real numbers and i satisfies $i^2 = -1$. They form a field. More precisely, the set $\mathbb{C}$ of complex numbers can be represented by the set of all ordered pairs (x,y) of real numbers; addition and multiplication are then defined on $\mathbb{C}$ by

$$(x,y)+(u,v) = (x+u,y+v)$$

and

$$(x,y)\cdot(u,v) = (xu-yv,xv+yu).$$

It is not hard to check that with these operations

(a) $\mathbb{C}$ is a field, and
(b) $(0,1)\cdot(0,1) = (-1,0)$.

If we return to the usual informal notation and write $x+iy$ in place of (x,y), then assertions (a) and (b) above provide justification for the first two sentences of this paragraph.

If z is a complex number, then the real numbers x and y that satisfy $z = x+iy$ are called the *real* and *imaginary parts* of z; they are sometimes denoted by $\Re(z)$ and $\Im(z)$.

The *absolute value*, or *modulus*, of the complex number z (where $z = x + iy$) is defined by

$$|z| = \sqrt{x^2 + y^2}.$$

It is easy to check that $|z_1 z_2| = |z_1||z_2|$ and $|z_1 + z_2| \le |z_1| + |z_2|$ hold for all z_1, z_2 in $\mathbb{C}$.

Limits of sequences of complex numbers and sums of infinite series whose terms are complex are defined in the expected way. The exponential function is defined on $\mathbb{C}$ by the usual infinite series:

$$e^z = \sum_{n=0}^{\infty} z^n/n!.$$

With some elementary manipulations of this series, one can check that

(a) $e^0 = 1$,
(b) $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for all complex z_1 and z_2, and
(c) $e^{it} = \cos t + i \sin t$ for all real t.

B.11. Let F be a field (in this book it will generally be $\mathbb{R}$ or $\mathbb{C}$). A *vector space* over F is a set V, together with operations $(v_1, v_2) \mapsto v_1 + v_2$ from $V \times V$ to V and $(\alpha, v) \mapsto \alpha \cdot v$ from $F \times V$ to V such that

(a) $(x+y)+z = x+(y+z)$ holds for all x, y, z in V,
(b) $x+y = y+x$ holds for all x, y in V,
(c) there is an element 0 of V such that $x+0 = x$ holds for all x in V,
(d) for each x in V there is an element $-x$ of V such that $x+(-x) = 0$,
(e) $1 \cdot x = x$ holds for all x in V,
(f) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$ holds for all α, β in F and all x in V,
(g) $(\alpha+\beta) \cdot x = \alpha \cdot x + \beta \cdot x$ holds for all α, β in F and all x in V, and
(h) $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$ holds for all α in F and all x, y in V.

(We will, of course, usually write αx in place of $\alpha \cdot x$.)

Note that $\mathbb{R}^d$ is a vector space over $\mathbb{R}$ and that $\mathbb{C}^d$ is a vector space over $\mathbb{C}$ (it is also a vector space over $\mathbb{R}$). Note also that if F is a field, then F is a vector space over F.

A *subspace* (or a *linear subspace*) of a vector space V over F is a subset V_0 of V that is a vector space when the operations $+$ and $\cdot$ are restricted to $V_0 \times V_0$ and $F \times V_0$.

B.12. Let V_1 and V_2 be vector spaces over the same field F. A function $L\colon V_1 \to V_2$ is *linear* if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

holds for all α, β in F and all x, y in V_1. A bijective linear map is a *linear isomorphism*. It is easy to check that the inverse of a linear isomorphism is linear.

Let V be a vector space over the field F. A *linear functional* on V is a linear map from V to the field F.

B.13. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. For each pair x, y of elements of V, the *line segment* connecting x and y is the set of points that can be written in the form $tx + (1-t)y$ for some t in the interval $[0,1]$. A subset C of V is *convex* if for each pair x, y of points in C the line segment connecting x and y is included in C.

B.14. (We will need this and Sect. B.15 only for the discussion of the Banach–Tarski paradox in Appendix G.) Let V be a vector space over $\mathbb{R}$, and let $T: V \to V$ be a linear operator. If x is a nonzero vector and λ is a real number such that $T(x) = \lambda x$, then x is an *eigenvector* of T and λ is an *eigenvalue* of T.

Note that if λ is an eigenvalue of T and if x is a corresponding eigenvector, then $(T - \lambda I)(x) = 0$, and so $T - \lambda I$ is not invertible. If the vector space V is finite dimensional, the converse holds: λ is an eigenvalue of T if and only if the operator $T - \lambda I$ is not invertible.

Let T be a linear operator on the finite-dimensional vector space V, let $\{e_i\}$ be a basis for V, and let A be the matrix of T with respect to $\{e_i\}$. Define $p: \mathbb{R} \to \mathbb{R}$ by $p(\lambda) = \det(A - \lambda I)$. Then $p(\lambda)$ is a polynomial in λ, called the *characteristic polynomial* of A (or of T). The eigenvalues of T are exactly the roots of the polynomial $p(\lambda)$.

B.15. The *transpose* of a matrix A (with components a_{ij}) is the matrix A^t whose components are given by $a^t_{ij} = a_{ji}$. Note that if A is a d by d matrix, if $x, y \in \mathbb{R}^d$, with x and y viewed as column vectors, and if $(\cdot,\cdot)$ is the usual inner product function on $\mathbb{R}^d$, then $(Ax, y) = (x, A^t y)$.

B.16. A *group* is a set G, together with a binary operation $\cdot$ on G such that

(a) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds for all x, y, z in G,
(b) there is an element e of G such that $e \cdot x = x \cdot e = x$ holds for all x in G, and
(c) for each x in G there is an element x^{-1} of G such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

A group G is *commutative* (or *abelian*) if $x \cdot y = y \cdot x$ holds for all x, y in G. One often uses $+$, rather than $\cdot$, to denote the operation in a commutative group. A *subgroup* of the group G is a subset G_0 of G that is a group when the operation $\cdot$ is restricted to $G_0 \times G_0$.

B.17. Let G_1 and G_2 be groups. A function $f: G_1 \to G_2$ is a *homomorphism* if $f(x \cdot y) = f(x) \cdot f(y)$ holds for all x, y in G_1. A bijective function $f: G_1 \to G_2$ is an *isomorphism* if both f and f^{-1} are homomorphisms.

Appendix C
Calculus and Topology in $\mathbb{R}^d$

C.1. Recall that $\mathbb{R}^d$ is the set of all d-tuples of real numbers; it is a vector space over $\mathbb{R}$. (The d in $\mathbb{R}^d$ is for dimension; we write $\mathbb{R}^d$, rather than $\mathbb{R}^n$, in order to have n available for use as a subscript.) Let $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ be elements of $\mathbb{R}^d$. The *norm* of x is defined by

$$\|x\| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2}$$

and the *distance* between x and y is defined to be $\|x - y\|$.

C.2. If $x \in \mathbb{R}^d$ and if r is a positive number, then the *open ball* $B(x,r)$ with center x and radius r is defined by

$$B(x,r) = \{y \in \mathbb{R}^d : \|y - x\| < r\}.$$

A subset U of $\mathbb{R}^d$ is *open* if for each x in U there is a positive number r such that $B(x,r) \subseteq U$. A subset of $\mathbb{R}^d$ is *closed* if its complement is open. A point x in $\mathbb{R}^d$ is a *limit point* of the subset A of $\mathbb{R}^d$ if for each positive r the open ball $B(x,r)$ contains infinitely many points of A (this is equivalent to requiring that for each positive r the ball $B(x,r)$ contain at least one point of A distinct from x). It is easy to check that a subset of $\mathbb{R}^d$ is closed if and only if it contains all of its limit points.

If A is a subset of $\mathbb{R}^d$, then the *closure* of A is the set $\overline{A}$ (or A^-) that consists of the points in A, together with the limit points of A; $\overline{A}$ is closed and is, in fact, the smallest closed subset of $\mathbb{R}^d$ that includes A.

C.3. A subset A of $\mathbb{R}^d$ is *bounded* if there is a real number M such that $\|x\| \leq M$ holds for each x in A.

C.4. (Proposition) *Let U be an open subset of $\mathbb{R}$. Then there is a countable collection $\mathscr{U}$ of disjoint open intervals such that $U = \cup \mathscr{U}$.*

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