Proof. Let $\mathscr{U}$ consist of those open subintervals I of U that are maximal, in the sense that the only open interval J that satisfies $I \subseteq J \subseteq U$ is I itself. Of course $\cup\mathscr{U} \subseteq U$. One can verify the reverse inclusion by noting that if $x \in U$, then the union of those open intervals that contain x and are included in U is an open interval that contains x and belongs to $\mathscr{U}$. It is easy to check (do so) that the intervals in $\mathscr{U}$ are disjoint from one another. If for each I in $\mathscr{U}$ we choose a rational number x_I that belongs to I, then (since the sets in $\mathscr{U}$ are disjoint from one another) the map $I \mapsto x_I$ is an injection; thus $\mathscr{U}$ has the same cardinality as some subset of $\mathbb{Q}$, and so is countable (see item A.6 in Appendix A). □

C.5. A sequence $\{x_n\}$ of elements of $\mathbb{R}^d$ *converges* to the element x of $\mathbb{R}^d$ if $\lim_n \|x_n - x\| = 0$; x is then called the *limit* of the sequence $\{x_n\}$ (note that here x and $x_1, x_2, \ldots$ are elements of $\mathbb{R}^d$; in particular, $x_1, x_2, \ldots$ are *not* the components of x). A sequence in $\mathbb{R}^d$ is *convergent* if it converges to some element of $\mathbb{R}^d$.

C.6. Let A be a subset of $\mathbb{R}^d$, and let x_0 belong to A. A function $f\colon A \to \mathbb{R}$ is *continuous at* x_0 if for each positive number ε there is a positive number δ such that $|f(x) - f(x_0)| < \varepsilon$ holds whenever x belongs to A and satisfies $\|x - x_0\| < \delta$; f is *continuous* if it is continuous at each point in A. The function $f\colon A \to \mathbb{R}$ is *uniformly continuous* if for each positive number ε there is a positive number δ such that $|f(x) - f(x')| < \varepsilon$ holds whenever x and x' belong to A and satisfy $\|x - x'\| < \delta$. A function $f\colon A \to \mathbb{R}$ is *continuous on* (or *uniformly continuous on*) the subset A_0 of A if the restriction of f to A_0 is continuous (or uniformly continuous).

C.7. Let A be a subset of $\mathbb{R}^d$, let f be a real- or complex-valued function on A, and let a be a limit point of A. Then $f(x)$ has *limit* L as x approaches a, written $\lim_{x\to a} f(x) = L$, if for every positive ε there is a positive δ such that $|f(x) - f(a)| < \varepsilon$ holds whenever x is a member of A that satisfies $0 < \|x - a\| < \delta$.

One can check that $\lim_{x\to a} f(x) = L$ if and only if $\lim_n f(x_n) = L$ for every sequence $\{x_n\}$ of elements of A, all different from a, such that $\lim_n x_n = a$. (Let us consider the more difficult half of that assertion, namely that if $\lim_n f(x_n) = L$ for every sequence $\{x_n\}$ of elements of A, all different from a, such that $\lim_n x_n = a$, then $\lim_{x\to a} f(x) = L$. We prove this by proving its contrapositive. So assume that $\lim_{x\to a} f(x) = L$ is not true. Then there exists a positive ε such that for every positive δ there is a value x in A such that $0 < \|x - a\| < \delta$ and $|f(x) - L| \geq \varepsilon$. If for each n we let $\delta = 1/n$ and choose an element x_n of A such that $0 < \|x_n - a\| < 1/n$ and $|f(x_n) - L| \geq \varepsilon$, we will have a sequence $\{x_n\}$ of elements of A, all different from a, that satisfy $\lim_n x_n = a$ but not $\lim_n f(x_n) = L$.)

C.8. Let A be a subset of $\mathbb{R}^d$. An *open cover* of A is a collection $\mathscr{S}$ of open subsets of $\mathbb{R}^d$ such that $A \subseteq \cup\mathscr{S}$. A *subcover* of the open cover $\mathscr{S}$ is a subfamily of $\mathscr{S}$ that is itself an open cover of A.

Proofs of the following results can be found in almost any text on advanced calculus or basic analysis (see, for example, Bartle [4], Hoffman [60], Rudin [104], or Thomson et al. [117]).

C.9. (Theorem) *Let A be a closed bounded subset of $\mathbb{R}^d$. Then every open cover of A has a finite subcover.*

Theorem C.9 is often called the *Heine–Borel* theorem.

C.10. (Theorem) *Let A be a closed bounded subset of $\mathbb{R}^d$. Then every sequence of elements of A has a subsequence that converges to an element of A.*

C.11. It is easy to check that the converses of Theorems C.9 and C.10 hold: if A satisfies the conclusion of Theorem C.9 or of Theorem C.10, then A is closed and bounded. The subsets of $\mathbb{R}^d$ that satisfy the conclusion of Theorem C.9 (hence the closed bounded subsets of $\mathbb{R}^d$) are often called *compact*. See also Appendix D.

C.12. (Theorem) *Let C be a nonempty closed bounded subset of $\mathbb{R}^d$, and let $f: C \to \mathbb{R}$ be continuous. Then*

(a) *f is uniformly continuous on C, and*
(b) *f is bounded on C. Moreover, there are elements x_0 and x_1 of C such that $f(x_0) \le f(x) \le f(x_1)$ holds at each x in C.*

C.13. (The Intermediate Value Theorem) *Let A be a subset of $\mathbb{R}$, and let $f: A \to \mathbb{R}$ be continuous. If the interval $[x_0, x_1]$ is included in A, then for each real number y between $f(x_0)$ and $f(x_1)$ there is an element x of $[x_0, x_1]$ such that $y = f(x)$.*

C.14. (The Mean Value Theorem) *Let a and b be real numbers such that $a < b$. If $f: [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$ and differentiable at each point in the open interval (a,b), then there is a number c in (a,b) such that $f(b) - f(a) = f'(c)(b-a)$.*

Appendix D
Topological Spaces and Metric Spaces

A number of the results in this appendix are stated without proof. For additional details, the reader should consult a text on point-set topology (for example, Kelley [69], Munkres [91], or Simmons [109]).

D.1. Let X be a set. A *topology* on X is a family $\mathscr{O}$ of subsets of X such that

(a) $X \in \mathscr{O}$,
(b) $\varnothing \in \mathscr{O}$,
(c) if $\mathscr{S}$ is an arbitrary collection of sets that belong to $\mathscr{O}$, then $\cup\mathscr{S} \in \mathscr{O}$, and
(d) if $\mathscr{S}$ is a finite collection of sets that belong to $\mathscr{O}$, then $\cap\mathscr{S} \in \mathscr{O}$.

A *topological space* is a pair $(X, \mathscr{O})$, where X is a set and $\mathscr{O}$ is a topology on X (we will generally abbreviate the notation and simply call X a topological space). The *open* subsets of X are those that belong to $\mathscr{O}$. An *open neighborhood* of a point x in X is an open set that contains x.

The collection of all open subsets of $\mathbb{R}^d$ (as defined in Appendix C) is a topology on $\mathbb{R}^d$; it is sometimes called the *usual* topology on $\mathbb{R}^d$.

D.2. Let $(X, \mathscr{O})$ be a topological space. A subset F of X is *closed* if F^c is open. The union of a finite collection of closed sets is closed, as is the intersection of an arbitrary collection of closed sets (use De Morgan's laws and parts (c) and (d) of the definition of a topology). It follows that if $A \subseteq X$, then there is a smallest closed set that includes A, namely the intersection of all the closed subsets of X that include A; this set is called the *closure* of A and is denoted by $\overline{A}$ or by A^-. A point x in X is a *limit point* of A if each open neighborhood of x contains at least one point of A other than x (the point x itself may or may not belong to A). A set is closed if and only if it contains each of its limit points. The closure of the set A consists of the points in A, together with the limit points of A.

D.3. Let $(X, \mathscr{O})$ be a topological space, and let A be a subset of X. The *interior* of A, written A^o, is the union of the open subsets of X that are included in A; thus A^o is the largest open subset of A. It is easy to check that $A^o = ((A^c)^-)^c$.

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