D.4. Let $(X, \mathscr{O})$ be a topological space, let Y be a subset of X, and let $\mathscr{O}_Y$ be the collection of all subsets of Y that have the form $Y \cap U$ for some U in $\mathscr{O}$. Then $\mathscr{O}_Y$ is a topology on Y; it is said to be *inherited from* X, or to be *induced* by $\mathscr{O}$. The space $(Y, \mathscr{O}_Y)$ (or simply Y) is called a *subspace* of $(X, \mathscr{O})$ (or of X).

Note that if Y is an open subset of X, then the members of $\mathscr{O}_Y$ are exactly the subsets of Y that are open as subsets of X. Likewise, if Y is a closed subset of X, then the closed subsets of the topological space $(Y, \mathscr{O}_Y)$ are exactly the subsets of Y that are closed as subsets of $(X, \mathscr{O}_X)$.

D.5. Let X and Y be topological spaces. A function $f \colon X \to Y$ is *continuous* if $f^{-1}(U)$ is an open subset of X whenever U is an open subset of Y. It is easy to check that f is continuous if and only if $f^{-1}(C)$ is closed whenever C is a closed subset of Y. A function $f \colon X \to Y$ is a *homeomorphism* if it is a bijection such that f and f^{-1} are both continuous. Equivalently, f is a homeomorphism if it is a bijection such that $f^{-1}(U)$ is open exactly when U is open. The spaces X and Y are *homeomorphic* if there is a homeomorphism of X onto Y.

D.6. We will on occasion need the following techniques for verifying the continuity of a function. Let X and Y be topological spaces, and let f be a function from X to Y. If $\mathscr{S}$ is a collection of open subsets of X such that $X = \cup\mathscr{S}$, and if for each U in $\mathscr{S}$ the restriction f_U of f to U is continuous (as a function from U to Y), then f is continuous (to prove this, note that if V is an open subset of Y, then $f^{-1}(V)$ is the union of the sets $f_U^{-1}(V)$, and so is open). Likewise, if $\mathscr{S}$ is a *finite* collection of closed sets such that $X = \cup\mathscr{S}$, and if for each C in $\mathscr{S}$ the restriction of f to C is continuous, then f is continuous.

D.7. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are topologies on the set X, and if $\mathscr{O}_1 \subseteq \mathscr{O}_2$, then $\mathscr{O}_1$ is said to be *weaker* than $\mathscr{O}_2$.

Now suppose that $\mathscr{A}$ is an arbitrary collection of subsets of the set X. There exist topologies on X that include $\mathscr{A}$ (for instance, the collection of all subsets of X). The intersection of all such topologies on X is a topology; it is the weakest topology on X that includes $\mathscr{A}$ and is said to be *generated* by $\mathscr{A}$.

We also need to consider topologies generated by sets of functions. Suppose that X is a set and that $\{f_i\}$ is a collection of functions, where for each i the function f_i maps X to some topological space Y_i. A topology on X makes all these functions continuous if and only if $f_i^{-1}(U)$ is open (in X) for each index i and each open subset U of Y_i. The topology *generated* by the family $\{f_i\}$ is the weakest topology on X that makes each f_i continuous, or equivalently, the topology generated by the sets $f_i^{-1}(U)$.

D.8. A subset A of a topological space X is *dense* in X if $\overline{A} = X$. The space X is *separable* if it has a countable dense subset.

D.9. Let $(X, \mathscr{O})$ be a topological space. A collection $\mathscr{U}$ of open subsets of X is a *base* for $(X, \mathscr{O})$ if for each V in $\mathscr{O}$ and each x in V there is a set U that belongs to $\mathscr{U}$ and satisfies $x \in U \subseteq V$. Equivalently, $\mathscr{U}$ is a base for X if the open subsets of

X are exactly the unions of (possibly empty) collections of sets in $\mathscr{U}$. A topological space is said to be *second countable*, or to *have a countable base*, if it has a base that contains only countably many sets.

D.10. It is easy to see that if X is second countable, then X is separable (if $\mathscr{U}$ is a countable base for X, then we can form a countable dense subset of X by choosing one point from each nonempty set in $\mathscr{U}$). The converse is not true. (Construct a topological space $(X,\mathscr{O})$ by letting $X=\mathbb{R}$ and letting $\mathscr{O}$ consist of those subsets A of X such that either $A=\varnothing$ or $0\in A$. Then $\{0\}$ is dense in X, and so X is separable; however, X is not second countable. Exercise 7.1.8 contains a more interesting example.)

D.11. If X is a second countable topological space, and if $\mathscr{V}$ is a collection of open subsets of X, then there is a countable subset $\mathscr{V}_0$ of $\mathscr{V}$ such that $\cup\mathscr{V}_0=\cup\mathscr{V}$. (Let $\mathscr{U}$ be a countable base for X, and let $\mathscr{U}_0$ be the collection of those elements U of $\mathscr{U}$ for which there is a set in $\mathscr{V}$ that includes U. For each U in $\mathscr{U}_0$ choose an element of $\mathscr{V}$ that includes U. The collection of sets chosen is the required subset of $\mathscr{V}$.)

D.12. A topological space X is *Hausdorff* if for each pair x,y of distinct points in X there are open sets U,V such that $x\in U$, $y\in V$, and $U\cap V=\varnothing$.

D.13. Let A be a subset of the topological space X. An *open cover* of A is a collection $\mathscr{S}$ of open subsets of X such that $A\subseteq\cup\mathscr{S}$. A *subcover* of the open cover $\mathscr{S}$ is a subfamily of $\mathscr{S}$ that is itself an open cover of A. The set A is *compact* if each open cover of A has a finite subcover. A topological space X is *compact* if X, when viewed as a subset of the space X, is compact.

D.14. A collection $\mathscr{C}$ of subsets of a set X satisfies the *finite intersection property* if each finite subcollection of $\mathscr{C}$ has a nonempty intersection. It follows from De Morgan's laws that a topological space X is compact if and only if each collection of closed subsets of X that satisfies the finite intersection property has a nonempty intersection.

D.15. If X and Y are topological spaces, if $f\colon X\to Y$ is continuous, and if K is a compact subset of X, then $f(K)$ is a compact subset of Y.

D.16. Every closed subset of a compact set is compact. Conversely, every compact subset of a Hausdorff space is closed (this is a consequence of Proposition 7.1.2; in fact, the first half of the proof of that proposition is all that is needed in the current situation).

D.17. It follows from D.15 and D.16 that if X is a compact space, if Y is a Hausdorff space, and if $f\colon X\to Y$ is a continuous bijection, then f is a homeomorphism.

D.18. If X is a nonempty compact space, and if $f\colon X\to\mathbb{R}$ is continuous, then f is bounded and attains its supremum and infimum: there are points x_0 and x_1 in X such that $f(x_0)\le f(x)\le f(x_1)$ holds at each x in X.

D.19. Let $\{(X_\alpha, \mathscr{O}_\alpha)\}$ be an indexed family of topological spaces, and let $\prod_\alpha X_\alpha$ be the product of the corresponding indexed family of sets $\{X_\alpha\}$ (see A.5). The *product topology* on $\prod_\alpha X_\alpha$ is the weakest topology on $\prod_\alpha X_\alpha$ that makes each of the coordinate projections $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ continuous (the projection π_β is defined by $\pi_\beta(x) = x_\beta$); see D.7. If $\mathscr{U}$ is the collection of sets that have the form $\prod_\alpha U_\alpha$ for some family $\{U_\alpha\}$ for which

(a) $U_\alpha \in \mathscr{O}_\alpha$ holds for each α and
(b) $U_\alpha = X_\alpha$ holds for all but finitely many values of α,

then $\mathscr{U}$ is a base for the product topology on $\prod_\alpha X_\alpha$.

D.20. (Tychonoff's Theorem) *Let $\{(X_\alpha, \mathscr{O}_\alpha)\}$ be an indexed collection of topological spaces. If each $(X_\alpha, \mathscr{O}_\alpha)$ is compact, then $\prod_\alpha X_\alpha$, with the product topology, is compact.*

D.21. Let X be a set. A collection $\mathscr{F}$ of functions on X *separates the points of* X if for each pair x, y of distinct points in X there is a function f in $\mathscr{F}$ such that $f(x) \neq f(y)$. A vector space $\mathscr{F}$ of real-valued functions on X is an *algebra* if fg belongs to $\mathscr{F}$ whenever f and g belong to $\mathscr{F}$ (here fg is the product of f and g, defined by $(fg)(x) = f(x)g(x)$). Now suppose that $\mathscr{F}$ is a vector space of bounded real-valued functions on X. A subset of $\mathscr{F}$ is *uniformly dense* in $\mathscr{F}$ if it is dense in $\mathscr{F}$ when $\mathscr{F}$ is given the topology induced by the uniform norm (see Example 3.2.1(f) in Sect. 3.2).

D.22. (Stone–Weierstrass Theorem) *Let X be a compact Hausdorff space. If A is an algebra of continuous real-valued functions on X that contains the constant functions and separates the points of X, then A is uniformly dense in the space $C(X)$ of continuous real-valued functions on X.*

D.23. (Stone–Weierstrass Theorem) *Let X be a locally compact*[1] *Hausdorff space, and let A be a subalgebra of $C_0(X)$ such that*

(a) *A separates the points of X, and*
(b) *for each x in X there is a function in A that does not vanish at x.*

Then A is uniformly dense in $C_0(X)$.

Theorem D.23 can be proved by applying Theorem D.22 to the one-point compactification of X.

D.24. Suppose that X is a set and that $\leq$ is a linear order on X. For each x in X define intervals $(-\infty, x)$ and $(x, +\infty)$ by

$$(-\infty, x) = \{z \in X : z < x\}$$

and

$$(x, +\infty) = \{z \in X : x < z\}.$$

[1] Locally compact spaces are defined in Sect. 7.1, and $C_0(X)$ is defined in Sect. 7.3.

The *order topology* on X is the weakest topology on X that contains all of these intervals. The set that consists of these intervals, the intervals of the form $\{z \in X : x < z < y\}$, and the set X, is a base for the order topology on X.

D.25. Let X be a set. A *metric* on X is a function $d\colon X \times X \to \mathbb{R}$ that satisfies

(a) $d(x,y) \geq 0$,
(b) $d(x,y) = 0$ if and only if $x = y$,
(c) $d(x,y) = d(y,x)$, and
(d) $d(x,z) \leq d(x,y) + d(y,z)$

for all x, y, and z in X. A *metric space* is a pair (X,d), where X is a set and d is a metric on X (of course, X itself is often called a metric space).

Let (X,d) be a metric space. If $x \in X$ and if r is a positive number, then the set $B(x,r)$ defined by

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

is called the *open ball* with center x and radius r; the *closed ball* with center x and radius r is the set

$$\{y \in X : d(x,y) \leq r\}.$$

A subset U of X is *open* if for each x in U there is a positive number r such that $B(x,r) \subseteq U$. The collection of all open subsets of X is a topology on X; it is called the topology *induced* or *generated* by d.[2] The open balls form a base for this topology.

D.26. A topological space $(X,\mathscr{O})$ (or a topology $\mathscr{O}$) is *metrizable* if there is a metric d on X that generates the topology $\mathscr{O}$; the metric d is then said to *metrize* X (or $(X,\mathscr{O})$).

D.27. Let X be a metric space. The *diameter* of the subset A of X, written $\operatorname{diam}(A)$, is defined by

$$\operatorname{diam}(A) = \sup\{d(x,y) : x,y \in A\}.$$

The set A is *bounded* if $\operatorname{diam}(A)$ is not equal to $+\infty$. The *distance* between the point x and the nonempty subset A of X is defined by

$$d(x,A) = \inf\{d(x,y) : y \in A\}.$$

Note that if x_1 and x_2 are points in X, then

$$d(x_1,A) \leq d(x_1,x_2) + d(x_2,A).$$

Since we can interchange the points x_1 and x_2 in the formula above, we find that

$$|d(x_1,A) - d(x_2,A)| \leq d(x_1,x_2),$$

[2] When dealing with a metric space (X,d), we will often implicitly assume that X has been given the topology induced by d.

from which it follows that $x \mapsto d(x,A)$ is continuous (and, in fact, uniformly continuous).

D.28. Each closed subset of a metric space is a G_δ, and each open subset is an F_σ. To check the first of these claims, note that if C is a nonempty closed subset of the metric space X, then

$$C = \bigcap_n \left\{ x \in X : d(x,C) < \frac{1}{n} \right\},$$

and so C is the intersection of a sequence of open sets. Now use De Morgan's laws (see Sect. A.1) to check that each open set is an F_σ.

D.29. Let x and $x_1, x_2, \ldots$ belong to the metric space X. The sequence $\{x_n\}$ *converges* to x if $\lim_n d(x_n, x) = 0$; if $\{x_n\}$ converges to x, we say that x is the *limit* of $\{x_n\}$, and we write $x = \lim_n x_n$.

D.30. Let X be a metric space. It is easy to check that a point x in X belongs to the closure of the subset A of X if and only if there is a sequence in A that converges to x.

D.31. Let (X,d) and (Y,d') be metric spaces, and give X and Y the topologies induced by d and d' respectively. Then a function $f\colon X \to Y$ is continuous (in the sense of D.5) if and only if for each x_0 in X and each positive number ε there is a positive number δ such that $d'(f(x), f(x_0)) < \varepsilon$ holds whenever x belongs to X and satisfies $d(x,x_0) < \delta$. The observation at the end of C.7 generalizes to metric spaces, and a small modification of the argument given there yields the following characterization of continuity in terms of sequences: the function f is continuous if and only if $f(x) = \lim_n f(x_n)$ holds whenever x and $x_1, x_2, \ldots$ are points in X such that $x = \lim_n x_n$.

D.32. We noted in D.10 that every second countable topological space is separable. The converse holds for metrizable spaces: if d metrizes the topology of X, and if D is a countable dense subset of X, then the collection consisting of those open balls $B(x,r)$ for which $x \in D$ and r is rational is a countable base for X.

D.33. If X is a second countable topological space, and if Y is a subspace of X, then Y is second countable (if $\mathscr{U}$ is a countable base for X, then $\{U \cap Y : U \in \mathscr{U}\}$ is a countable base for Y). It follows from this, together with D.10 and D.32, that every subspace of a separable metrizable space is separable.

D.34. Let (X,d) be a metric space. A sequence $\{x_n\}$ of elements of X is a *Cauchy sequence* if for each positive number ε there is a positive integer N such that $d(x_m, x_n) < \varepsilon$ holds whenever $m \geq N$ and $n \geq N$. The metric space X is *complete* if every Cauchy sequence in X converges to an element of X.

D.35. (Cantor's Nested Set Theorem) *Let X be a complete metric space. If $\{A_n\}$ is a decreasing sequence of nonempty closed sets of X such that* $\lim_n \operatorname{diam}(A_n) = 0$, *then $\cap_{n=1}^{\infty} A_n$ contains exactly one point.*

Proof. For each positive integer n choose an element x_n of A_n. Then $\{x_n\}$ is a Cauchy sequence whose limit belongs to $\cap_{n=1}^{\infty} A_n$. Thus $\cap_{n=1}^{\infty} A_n$ is not empty. Since $\lim_n \operatorname{diam}(A_n) = 0$, the set $\cap_{n=1}^{\infty} A_n$ cannot contain more than one point. □

D.36. A subset A of a topological space X is *nowhere dense* if the interior of $\overline{A}$ is empty.

D.37. (Baire Category Theorem) *Let X be a nonempty complete metric space* (*or a nonempty topological space that can be metrized with a complete metric*). *Then X cannot be written as the union of a sequence of nowhere dense sets. Moreover, if $\{A_n\}$ is a sequence of nowhere dense subsets of X, then $(\cup_n A_n)^c$ is dense in X.*

D.38. The metric space (X,d) is *totally bounded* if for each positive ε there is a finite subset S of X such that

$$X = \bigcup \{B(x,\varepsilon) : x \in S\}.$$

D.39. (Theorem) *Let X be a metric space. Then the conditions*

(a) *the space X is compact,*
(b) *the space X is complete and totally bounded, and*
(c) *each sequence of elements of X has a subsequence that converges to an element of X*

are equivalent.

D.40. (Corollary) *Each compact metric space is separable.*

Proof. Let X be a compact metric space. Theorem D.39 implies that X is totally bounded, and so for each positive integer n we can choose a finite set S_n such that $X = \cup\{B(x,1/n) : x \in S_n\}$. The set $\cup_n S_n$ is then a countable dense subset of X. □

D.41. Note, however, that a compact Hausdorff space can fail to be second countable and can even fail to be separable (see Exercises 7.1.7, 7.1.8, and 7.1.10).

D.42. Let $\{X_n\}$ be a sequence of nonempty metrizable spaces, and for each n let d_n be a metric that metrizes X_n. Let $\mathbf{x}$ and $\mathbf{y}$ denote the points $\{x_n\}$ and $\{y_n\}$ of the product space $\prod_n X_n$. Then the formula

$$d(\mathbf{x},\mathbf{y}) = \sum_n \frac{1}{2^n} \min(1, d_n(x_n,y_n))$$

defines a metric on $\prod_n X_n$ that metrizes the product topology. This fact, together with Theorem D.39, can be used to give a fairly easy proof of Tychonoff's theorem for *countable* families of compact *metrizable* spaces.

Appendix E
The Bochner Integral

Let $(X, \mathscr{A})$ be a measurable space, let E be a real or complex Banach space (that is, a Banach space over $\mathbb{R}$ or $\mathbb{C}$), and let $\mathscr{B}(E)$ be the σ-algebra of Borel subsets of E (that is, let $\mathscr{B}(E)$ be the σ-algebra on E generated by the open subsets of E). We will sometimes denote the norm on E by $|\cdot|$, rather than by the more customary $\|\cdot\|$. This will allow us to use $\|\cdot\|$ for the norm of elements of certain spaces of E-valued functions; see, for example, formula (7) below. A function $f\colon X \to E$ is *Borel measurable* if it is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E)$, and is *strongly measurable* if it is Borel measurable and has a separable range (here by the range of f we mean the subset $f(X)$ of E). The function f is *simple* if it has only finitely many values. Of course, a simple function is strongly measurable if and only if it is Borel measurable.

It is easy to see that if f is Borel measurable, then $x \mapsto |f(x)|$ is $\mathscr{A}$-measurable (use Lemma 7.2.1 and Proposition 2.6.1).

Note that if E is separable, then every E-valued Borel measurable function is strongly measurable. On the other hand, if E is not separable and if $(X, \mathscr{A}) = (E, \mathscr{B}(E))$, then the identity map from X to E is Borel measurable, but is not strongly measurable.

E.1. (Proposition) *Let $(X, \mathscr{A})$ be a measurable space, and let E be a real or complex Banach space. Then*

(a) *the collection of Borel measurable functions from X to E is closed under the formation of pointwise limits, and*
(b) *the collection of strongly measurable functions from X to E is closed under the formation of pointwise limits.*

Proof. Part (a) is a special case of Proposition 8.1.10, and so we can turn to part (b).

Let $\{f_n\}$ be a sequence of strongly measurable functions from X to E, and suppose that $\{f_n\}$ converges pointwise to f. It follows from the separability of the sets $f_n(X)$, $n = 1, 2, \ldots$, that $\cup_n f_n(X)$ is separable, that the closure of $\cup_n f_n(X)$ is separable, and finally that $f(X)$ is separable (see D.33). Since f is Borel measurable (part (a)), the proof is complete. □

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