E.2. (Proposition) *Let $(X, \mathscr{A})$ be a measurable space, let E be a real or complex Banach space, and let $f: X \to E$ be strongly measurable. Then there is a sequence $\{f_n\}$ of strongly measurable simple functions such that*

$$f(x) = \lim_n f_n(x)$$

and

$$|f_n(x)| \le |f(x)|, \textit{ for } n = 1, 2, \ldots,$$

hold at each x in X.

Proof. We can certainly assume that $f(X)$ contains at least one nonzero element of E. Let C be a countable dense subset of $f(X)$, let $C^{\sim}$ be the set of rational multiples of elements of C, and let $\{y_n\}$ be an enumeration of $C^{\sim}$. We can assume that $y_1 = 0$. It is easy to check (do so) that

$$\text{for each } y \text{ in } f(X) \text{ and each positive number } \varepsilon \text{ there is a term } y_m \text{ of } \{y_n\} \text{ that satisfies } |y_m| \le |y| \text{ and } |y_m - y| < \varepsilon. \tag{1}$$

For each x in X and each positive integer n define a subset $A_n(x)$ of E by

$$A_n(x) = \{y_j : j \le n \text{ and } |y_j| \le |f(x)|\}.$$

Since $y_1 = 0$, each $A_n(x)$ is nonempty.

We now construct the required sequence $\{f_n\}$ by letting $f_n(x)$ be the element of $A_n(x)$ that lies closest to $f(x)$ (in case

$$|f(x) - y_j| = \inf\{|f(x) - y_i| : y_i \in A_n(x)\} \tag{2}$$

holds for several elements y_j of $A_n(x)$, let $f_n(x)$ be y_{j_0}, where j_0 is the smallest value of j for which y_j belongs to $A_n(x)$ and satisfies (2)). It is clear that each f_n is a simple function and that $|f_n(x)| \le |f(x)|$ holds for each n and x. Since the sets $\{x \in X : f_n(x) = y_j\}$ can be described by means of inequalities involving $|f(x)|$, $|y_i|$, $i = 1, \ldots, n$, and $|f(x) - y_i|$, $i = 1, \ldots, n$, each f_n is strongly measurable. Finally, observation (1) implies that $\{f_n\}$ converges pointwise to f (if y_m satisfies the inequalities $|y_m| \le |f(x)|$ and $|y_m - f(x)| < \varepsilon$, then $|f_n(x) - f(x)| < \varepsilon$ holds whenever $n \ge m$). □

Let us note two consequences of Propositions E.1 and E.2. The first is immediate: a function from X to E is strongly measurable if and only if it is the pointwise limit of a sequence of Borel (or strongly) measurable simple functions. The second is given by the following corollary (see, however, Exercise 2).

E.3. (Corollary) *Let $(X, \mathscr{A})$ be a measurable space, and let E be a real or complex Banach space. Then the set of all strongly measurable functions from X to E is a vector space.*

Proof. Suppose that f and g are strongly measurable and that a and b are real (or complex) numbers. Choose sequences $\{f_n\}$ and $\{g_n\}$ of strongly measurable simple functions that converge pointwise to f and g respectively (Proposition E.2). Since $\{af_n + bg_n\}$ converges pointwise to $af + bg$, and since each $af_n + bg_n$ is strongly measurable (it is simple and each of its values is attained on a measurable set), Proposition E.1 implies that $af + bg$ is strongly measurable. $\square$

We turn to the integration of functions with values in a Banach space. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let E be a real or complex Banach space. A function $f\colon X \to E$ is *integrable* (or *strongly integrable*, or *Bochner integrable*) if it is strongly measurable and the function $x \to |f(x)|$ is integrable.[1]

The integral of such functions is defined as follows. First suppose that $f\colon X \to E$ is simple and integrable. Let $a_1, \ldots, a_n$ be the nonzero values of f, and suppose that these values are attained on the sets $A_1, \ldots, A_n$. Then Proposition 2.3.10, applied to the real-valued function $x \mapsto |f(x)|$, implies that each A_i has finite measure under μ. Thus the expression $\sum_{i=1}^n a_i \mu(A_i)$ makes sense; we define the *integral* of f, written $\int f\, d\mu$, to be this sum. It is easy to see that

$$\left| \int f\, d\mu \right| \le \int |f|\, d\mu. \tag{3}$$

It is also easy to see that if f and g are simple integrable functions and a and b are real (or complex) numbers, then $af + bg$ is a simple integrable function, and

$$\int (af + bg)\, d\mu = a \int f\, d\mu + b \int g\, d\mu. \tag{4}$$

Now suppose that f is an arbitrary integrable function. Choose a sequence $\{f_n\}$ of simple integrable functions such that $f(x) = \lim_n f_n(x)$ holds at each x in X and such that the function $x \mapsto \sup_n |f_n(x)|$ is integrable (see Proposition E.2). The dominated convergence theorem for real-valued functions (Theorem 2.4.5) implies that $\lim_n \int |f_n - f|\, d\mu = 0$, and hence that $\lim_{m,n} \int |f_m - f_n|\, d\mu = 0$. Thus (see (3) and (4)) $\{\int f_n\, d\mu\}$ is a Cauchy sequence in E, and so is convergent. The *integral* (or *Bochner integral*) of f, written $\int f\, d\mu$, is defined to be the limit of the sequence $\{\int f_n\, d\mu\}$. (It is easy to check that the value of $\int f\, d\mu$ does not depend on the choice of the sequence $\{f_n\}$: if $\{g_n\}$ is another sequence having the properties required of $\{f_n\}$, then $\lim_n \int |f_n - g_n|\, d\mu = 0$, from which it follows that $\lim_n \int (f_n - g_n)\, d\mu = 0$ and hence that $\lim_n \int f_n\, d\mu = \lim_n \int g_n\, d\mu$.)

Let us note a few basic properties of the Bochner integral.

E.4. (Proposition) *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let E be a real or complex Banach space. Suppose that $f, g\colon X \to E$ are integrable and that a and b are real (or complex) numbers. Then $af + bg$ is integrable, and*

[1] See Exercise 4 for an indication of another standard definition of Bochner integrability.

$$\int (af+bg)\,d\mu = a\int f\,d\mu + b\int g\,d\mu. \tag{5}$$

Proof. The integrability of $af+bg$ follows from Corollary E.3 and the inequality $|(af+bg)(x)| \le |a|\,|f(x)| + |b|\,|g(x)|$. Let $\{f_n\}$ and $\{g_n\}$ be sequences of simple integrable functions that converge pointwise to f and g respectively and are such that $x \mapsto \sup_n |f_n(x)|$ and $x \mapsto \sup_n |g_n(x)|$ are integrable. Then the functions $af_n + bg_n$ are simple and integrable, and they satisfy

$$\int (af_n+bg_n)\,d\mu = a\int f_n\,d\mu + b\int g_n\,d\mu \tag{6}$$

(see (4)). Furthermore $x \mapsto \sup_n |(af_n+bg_n)(x)|$ is integrable, and so according to the definition of the integral, we can take limits in (6), obtaining (5). □

E.5. (Proposition) *Let $(X,\mathscr{A},\mu)$ be a measure space, and let E be a real or complex Banach space. If $f\colon X \to E$ is integrable, then $|\int f\,d\mu| \le \int |f|\,d\mu$.*

Proof. Let f be an integrable function, and let $\{f_n\}$ be a sequence of simple integrable functions such that $\sup_n |f_n(x)| \le |f(x)|$ and $f(x) = \lim_n f_n(x)$ hold at each x in X (Proposition E.2). Then

$$\left|\int f_n\,d\mu\right| \le \int |f_n|\,d\mu \le \int |f|\,d\mu$$

(see (3)); since $\int f\,d\mu = \lim_n \int f_n\,d\mu$, the proposition follows. □

The dominated convergence theorem can be formulated as follows for E-valued functions.

E.6. (Theorem) *Let $(X,\mathscr{A},\mu)$ be a measure space, let E be a real or complex Banach space, and let g be a $[0,+\infty]$-valued integrable function on X. Suppose that f and f_1, f_2, … are strongly measurable E-valued functions on X such that the relations*

$$f(x) = \lim_n f_n(x)$$

and

$$|f_n(x)| \le g(x), \text{ for } n = 1, 2, \ldots,$$

hold at almost every x in X. Then f and f_1, f_2, … are integrable, and $\int f\,d\mu = \lim_n \int f_n\,d\mu$.

Proof. The integrability of f and $f_1, f_2, \ldots$ is immediate. Since $|f_n - f| \le 2g$ holds almost everywhere, the dominated convergence theorem for real-valued functions (Theorem 2.4.5) implies that $\lim_n \int |f_n - f|\,d\mu = 0$. In view of Propositions E.4 and E.5, this implies that $\int f\,d\mu = \lim_n \int f_n\,d\mu$. □

Let $\mathscr{L}^1(X,\mathscr{A},\mu,E)$ be the set of all E-valued integrable functions on X. Then $\mathscr{L}^1(X,\mathscr{A},\mu,E)$ is a vector space (see Proposition E.4). It is easy to check that the collection $L^1(X,\mathscr{A},\mu,E)$ of equivalence classes (under almost everywhere equality) of elements of $\mathscr{L}^1(X,\mathscr{A},\mu,E)$ can be made into a vector space in the natural way, and that the formula

$$\|f\|_1 = \int |f|\,d\mu \tag{7}$$

induces a norm on $L^1(X,\mathscr{A},\mu,E)$ (and, of course, a seminorm on $\mathscr{L}^1(X,\mathscr{A},\mu,E)$). The proof of Theorem 3.4.1 can be modified so as to show that $L^1(X,\mathscr{A},\mu,E)$ is complete under $\|\cdot\|_1$.

One often finds it useful to be able to deal with vector-valued functions in terms of real- (or complex-) valued functions. For this we need to recall the Hahn–Banach theorem.

E.7. (Hahn–Banach Theorem) *Let E be a real or complex normed linear space, let F be a linear subspace of E, and let φ_0 be a continuous linear functional on F. Then there is a continuous linear functional φ on E such that $\|\varphi\| = \|\varphi_0\|$ and such that φ_0 is the restriction of φ to F. In other words, φ_0 can be extended to a continuous linear functional on all of E without increasing its norm.*

A proof of the Hahn–Banach theorem can be found in almost any basic text on functional analysis (see, for example, Conway [31], Kolmogorov and Fomin [73], Royden [102], or Simmons [109]).

We also need the following consequence of the Hahn–Banach theorem.

E.8. (Corollary) *Let E be a real or complex normed linear space that does not consist of 0 alone. Then for each y in E there is a continuous linear functional φ on E such that $\|\varphi\| = 1$ and $\varphi(y) = \|y\|$.*

Proof. Let y be a nonzero element of E, let F be the subspace of E consisting of all scalar multiples of y, and let φ_0 be the linear functional on F defined by $\varphi_0(ty) = t\|y\|$. Then φ_0 satisfies $\|\varphi_0\| = 1$ and $\varphi_0(y) = \|y\|$, and we can produce the required functional φ by applying Theorem E.7 to φ_0. (In case $y = 0$, let φ be an arbitrary linear functional on E that satisfies $\|\varphi\| = 1$.) □

Let us now apply Theorem E.7 and Corollary E.8 to the study of vector-valued functions.

E.9. (Theorem) *Let $(X,\mathscr{A})$ be a measurable space, and let E be a real or complex Banach space. A function $f\colon X \to E$ is strongly measurable if and only if*

(a) *the image $f(X)$ of X under f is separable, and*
(b) *for each φ in E^* the function $\varphi \circ f$ is $\mathscr{A}$-measurable.*

We will use the following lemma in our proof of Theorem E.9.

E.10. (Lemma) *Let E be a separable normed linear space over $\mathbb{R}$ or $\mathbb{C}$. Then there is a sequence $\{\varphi_n\}$ of elements of E^* such that*

$$\|y\| = \sup\{|\varphi_n(y)| : n = 1, 2, \dots\} \tag{8}$$

holds for each y in E.

Proof. We can assume that E does not consist of 0 alone. Choose a sequence $\{y_n\}$ whose terms form a dense subset of E. According to Corollary E.8, we can choose, for each n, an element φ_n of E^* that satisfies $\|\varphi_n\| = 1$ and $\varphi_n(y_n) = \|y_n\|$. Let us check that the sequence $\{\varphi_n\}$ meets the requirements of the lemma. Since each φ_n satisfies $\|\varphi_n\| = 1$, it follows that

$$\sup\{|\varphi_n(y)| : n = 1, 2, \dots\} \le \|y\|$$

holds for each y in E. For an arbitrary y in E we can find terms in the sequence $\{y_n\}$ that lie arbitrarily close to y, and so the calculations

$$\varphi_n(y) = \varphi_n(y - y_n) + \varphi_n(y_n) = \varphi_n(y - y_n) + \|y_n\|$$

and $|\varphi_n(y - y_n)| \le \|\varphi_n\| \|y - y_n\| = \|y - y_n\|$ imply that

$$\|y\| = \sup\{|\varphi_n(y)| : n = 1, 2, \dots\}.$$

Relation (8) follows. □

Proof of Theorem E.9. Let us assume that we are dealing with Banach spaces over $\mathbb{R}$; the case of Banach spaces over $\mathbb{C}$ is similar.

If f is strongly measurable, then (a) is immediate and (b) follows from Lemma 7.2.1 and Proposition 2.6.1.

Now suppose that f satisfies (a) and (b). In view of (a), it suffices to show that f is Borel measurable. Let E_0 be the smallest closed linear subspace of E that includes $f(X)$. Then E_0 is separable (if C is a countable dense subset of $f(X)$, then E_0 is the closure of the set of finite sums of rational multiples of elements of C). We can show that f is Borel measurable (that is, measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E)$) by showing that it is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E_0)$ (Lemma 7.2.2).

Let $\{\varphi_n\}$ be a sequence in $(E_0)^*$ such that

$$\|y\| = \sup\{|\varphi_n(y)| : n = 1, 2, \dots\} \tag{9}$$

holds for each y in E_0 (Lemma E.10). Since each continuous linear functional on E_0 is the restriction to E_0 of an element of E^* (Theorem E.7), condition (b) implies that for each n the function $\varphi_n \circ f$ is $\mathscr{A}$-measurable. If B is a closed ball in E_0, say with center y_0 and radius r, then $f^{-1}(B)$ is equal to

$$\bigcap_n \{x : |\varphi_n(f(x)) - \varphi_n(y_0)| \le r\},$$

and so belongs to $\mathscr{A}$. Since each open ball in E_0 is the union of a countable collection of closed balls, and since each open subset of E_0 is the union of a countable collection of open balls (recall that E_0 is separable), the collection of closed balls generates $\mathscr{B}(E_0)$. It now follows from Proposition 2.6.2 that f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E_0)$ and the proof is complete. □

E.11. (Proposition) *Let $(X, \mathscr{A}, \mu)$ be a measure space, let E be a real or complex Banach space, and let $f\colon X \to E$ be integrable. Then*

$$\int \varphi \circ f \, d\mu = \varphi\left(\int f \, d\mu\right) \tag{10}$$

holds for each φ in E^*.

The reader should see Exercise 3 for a strengthened form of Proposition E.11.

Proof. It is easy to check (do so) that the integrability of $\varphi \circ f$ follows from that of f. If f is a simple integrable function, attaining the nonzero values $a_1, \ldots, a_k$ on the sets $A_1, \ldots, A_k$, then each side of (10) is equal to $\sum_{i=1}^{k} \varphi(a_i)\mu(A_i)$; hence (10) holds for simple integrable functions. Next suppose that f is an arbitrary integrable function and that $\{f_n\}$ is a sequence of simple integrable functions such that $f(x) = \lim_n f_n(x)$ and $\sup_n |f_n(x)| \le |f(x)|$ hold at each x in X (Proposition E.2). Then Theorems E.6 and 2.4.5 enable us to take limits in the relation $\int \varphi \circ f_n \, d\mu = \varphi(\int f_n \, d\mu)$, and (10) follows for arbitrary integrable functions. □

The reader should note Exercises 5 and 7, which show some difficulties that arise in the extension of integration theory to vector-valued functions. The issues hinted at in these exercises have been the subject of much research over the years; see Diestel and Uhl [37] for a summary and for further references.

Exercises

1. Show that a simpler proof of Proposition E.2 could be given if the f_n's were not required to satisfy the inequality $|f_n(x)| \le |f(x)|$.
2. Suppose that $(X, \mathscr{A})$ is a measurable space and that E is a Banach space. Show by example that the set of Borel measurable functions from X to E can fail to be a vector space. (Hint: Let E be a Banach space with cardinality greater than that of the continuum, and let $(X, \mathscr{A})$ be $(E \times E, \mathscr{B}(E) \times \mathscr{B}(E))$. See Exercise 5.1.8.)
3. Let $(X, \mathscr{A}, \mu)$ be a measure space, let E be a Banach space, and let $f\colon X \to E$ be Bochner integrable. Show that $\int f \, d\mu$ is the *only* element x_0 of E that satisfies $\varphi(x_0) = \int \varphi \circ f \, d\mu$ for each φ in E^*. (Hint: Use Corollary E.8.)
4. (This exercise hints at another, rather common, way to define strong measurability and Bochner measurability.) Suppose that $(X, \mathscr{A}, \mu)$ is a measure space and that E is a Banach space. Let $f\colon X \to E$ be a function for which there is a sequence $\{f_n\}$ of strongly measurable simple functions such that $f(x) = \lim_n f_n(x)$ holds at μ-almost every x in X.

   (a) Show by example that f need not have a separable range.
   (b) Show that there is a strongly measurable function $g\colon X \to E$ that agrees with f μ-almost everywhere.

(c) Show that $x \mapsto |f(x)|$ is measurable with respect to the completion $\mathscr{A}_\mu$ of $\mathscr{A}$ under μ.

(d) How should $\int f\,d\mu$ be defined if $\int |f|\,d\overline{\mu}$ is finite? (Of course $\overline{\mu}$ is the completion of μ.)

5. Let $(X, \mathscr{A})$ be a measurable space, and let E be a Banach space. An *E-valued measure* on $(X, \mathscr{A})$ is a function $\nu \colon \mathscr{A} \to E$ such that $\nu(\varnothing) = 0$ and such that $\nu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \nu(A_i)$ holds for each infinite sequence $\{A_i\}$ of disjoint sets in $\mathscr{A}$. The *variation* $|\nu| \colon \mathscr{A} \to [0, +\infty]$ of the E-valued measure ν is defined by letting $|\nu|(A)$ be the supremum of the sums $\sum_{i=1}^n |\nu(A_i)|$, where $\{A_i\}_{i=1}^n$ ranges over all finite partitions of A into $\mathscr{A}$-measurable sets.

(a) Show that the variation of an E-valued measure on $(X, \mathscr{A})$ is a positive measure on $(X, \mathscr{A})$.

(b) Show by example that the variation of an E-valued measure may not be finite. (Hint: Let X be $\mathbb{N}$, let $\mathscr{A}$ be $\mathscr{P}(\mathbb{N})$, let E be ℓ^2, and define $\nu \colon \mathscr{A} \to E$ by letting $\nu(A)$ be the sequence

$$n \mapsto \begin{cases} \frac{1}{n} & \text{if } n \in A, \\ 0 & \text{if } n \notin A.) \end{cases}$$

6. Let $(X, \mathscr{A}, \mu)$ be a measure space, let E be a Banach space, and let $f \colon X \to E$ be Bochner integrable. Define $\nu \colon \mathscr{A} \to E$ by $\nu(A) = \int \chi_A f\,d\mu$.

(a) Show that ν is an E-valued measure on $(X, \mathscr{A})$.

(b) Show that the variation $|\nu|$ of ν is finite.

7. Let λ be Lebesgue measure on $([0,1], \mathscr{B}([0,1]))$, and let E be the Banach space $L^1([0,1], \mathscr{B}([0,1]), \lambda, \mathbb{R})$. Define $\nu \colon \mathscr{B}([0,1]) \to E$ by letting $\nu(A)$ be the element of E determined by the characteristic function χ_A of A.

(a) Show that ν is an E-valued measure on $([0,1], \mathscr{B}([0,1]))$.

(b) Show that $|\nu|$ is finite.

(c) Show that ν is absolutely continuous with respect to λ (in other words, show that $\nu(A) = 0$ holds whenever A satisfies $\lambda(A) = 0$).

(d) Show that there is no Bochner integrable function $f \colon [0,1] \to E$ that satisfies $\nu(A) = \int \chi_A f\,d\lambda$ for each A in $\mathscr{B}([0,1])$. Thus the Radon–Nikodym theorem fails for the Bochner integral. (Hint: Use Proposition E.11.)

Appendix F
Liftings

Let $(X, \mathscr{A}, \mu)$ be a measure space. Throughout this appendix we will assume that the measure μ is finite but not the zero measure (see Exercise 2). Recall that $\mathscr{L}^\infty(X, \mathscr{A}, \mu, \mathbb{R})$ is the vector space of all bounded real-valued $\mathscr{A}$-measurable functions on X and that $L^\infty(X, \mathscr{A}, \mu, \mathbb{R})$ is the vector space of equivalence classes of functions in $\mathscr{L}^\infty(X, \mathscr{A}, \mu, \mathbb{R})$, where two functions are considered equivalent if they are equal μ-almost everywhere.[1] For simplicity, we will generally write $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$, instead of $\mathscr{L}^\infty(X, \mathscr{A}, \mu, \mathbb{R})$. We will occasionally use the norm $\|\cdot\|_\infty$ on $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

Note that for this version of the norm $\|\cdot\|_\infty$ a function f satisfies $\|f\|_\infty = 0$ only if f vanishes *everywhere* on X; it is not enough for it to vanish almost everywhere.

It is natural to ask whether a function in $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ can be chosen from each equivalence class in $L^\infty(X, \mathscr{A}, \mu)$ in such a way the choice is linear and multiplicative. Since notation involving functions is simpler than notation involving equivalence classes, one generally deals with functions and makes the following definitions. A *lifting* of $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ is a function $\rho : \mathscr{L}^\infty(X, \mathscr{A}, \mu) \to \mathscr{L}^\infty(X, \mathscr{A}, \mu)$ such that for all f, g in $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ and all real numbers a and b we have

[1] In the present context (i.e., in cases where the measure μ is finite), it is the same to say that two functions agree almost everywhere as to say that they agree locally almost everywhere. Thus, for our current discussion the definition of $\mathscr{L}^\infty(X, \mathscr{A}, \mu, \mathbb{R})$ given here is consistent with the one in Chap. 4. We will use the current definition since it makes the exposition that follows simpler. If we were looking at liftings on very large measure spaces, we would speak of locally null sets and of equality locally almost everywhere; see [65].

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