Proof. Let $\epsilon > 0$. If $y = 0$ then $x_n y_n = 0 = xy$ for all n and the result is immediate. Now suppose $y \neq 0$. Since $x_n \to x$, the sequence (x_n) is bounded, so there exists $M > 0$ such that $|x_n| \leq M$ for all n. Since $\lim_n x_n = x$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|x_n - x| < \frac{\epsilon}{2|y|}$. Since $\lim_n y_n = y$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|y_n - y| < \frac{\epsilon}{2M}$. Let $N = \max(N_1, N_2)$. Then for $n \geq N$, by the add-and-subtract identity and the triangle inequality,

$$\begin{aligned} |x_n y_n - xy| &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n||y_n - y| + |y||x_n - x| \\ &< M \cdot \frac{\epsilon}{2M} + |y| \cdot \frac{\epsilon}{2|y|} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

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