Informal Measure Theory Notes

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1 Warm-up

To start I think it might be worth-while to prove some basic limits:

1.1 Epsilons and Limits

Definition 1.1.1. The statement $\lim_{x\to\infty} x_n = x$ means that for any real $\epsilon > 0$, there is a number $N > 0$ for which $n \geq N$ implies $|x_n - x| < \epsilon$.

Theorem 1.1.1 (Archimedean Property)**.** *If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that*

$$nx > y \tag{1}$$

Proof. Let A be the set of all nx, where n runs through the positive integers. If 1 were false ($nx \leq y$), then y would be an upper-bound of A. But then A has a least upper bound in $\mathbb{R}$. Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A. Hence $\alpha - x < mx$ for some positive integer m. But then $\alpha < (m+1)x \in A$, which is impossible, since α is an upper bound of A.

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Proposition 1.1.1.

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Proof. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ (By the Archimedean Property). Then let $n \geq N$ such that:

$$\begin{aligned} n \geq N &> \tfrac{1}{\epsilon} \\ \tfrac{1}{n} \leq \tfrac{1}{N} &< \epsilon \qquad \text{taking reciprocals} \\ |\tfrac{1}{n} - 0| = |\tfrac{1}{n}| &= \tfrac{1}{n} \qquad \text{since } \tfrac{1}{n} \text{ is always positive for } n \in \mathbb{N} \\ &< \epsilon \end{aligned}$$

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Proposition 1.1.2.

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n^k} = 0 \qquad &\textit{for any } k > 0. \\ \iff |\frac{1}{n^k} - 0| &< \epsilon, \epsilon > 0, n > N \in \mathbb{N} \\ \iff \frac{1}{n^k} &< \epsilon \qquad \textit{since } n^k > 0 \end{aligned}$$

Proof. Let $N > \epsilon^{-\frac{1}{k}}$ (by Archimedean Property of Natural Numbers) and $n \geq N$. Then

$$\begin{aligned} n &> \left(\frac{1}{\epsilon}\right)^{\frac{1}{k}} \\ \Rightarrow n^k &> \left(\frac{1}{\epsilon}\right) \\ \Rightarrow \frac{1}{n^k} &< \epsilon \end{aligned}$$

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Proposition 1.1.3.

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = 1$$

Proof. Let $\epsilon > 0$. Since $2^{1/n} > 1$ for all $n \in \mathbb{N}$, we have $|2^{1/n} - 1| = 2^{1/n} - 1$. Since $\epsilon > 0$, we have $\epsilon + 1 > 1$, so $\log(\epsilon + 1) > 0$. Let $n \geq N > \log_{\epsilon+1} 2 = \frac{\log 2}{\log(\epsilon+1)}$ (by Archimedean Property). Then

$$\begin{aligned} n &> \frac{\log 2}{\log(\epsilon+1)} \\ \Rightarrow \frac{1}{n} &< \frac{\log(\epsilon+1)}{\log 2} \\ \Rightarrow \log\left(2^{\frac{1}{n}}\right) &< \log(\epsilon+1) \\ \Rightarrow 2^{\frac{1}{n}} &< \epsilon + 1 \\ \Rightarrow \left|2^{\frac{1}{n}} - 1\right| &< \epsilon. \end{aligned}$$

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Proposition 1.1.4. *If* $\lim_{n\to\infty} x_n = x$ *and* $a \in \mathbb{R}$*, then* $\lim_{n\to\infty} ax_n = ax$.

Proof. If $a = 0$ then $ax_n = 0 = ax$ for all n and the result is immediate. Now suppose $a \neq 0$ and let $\epsilon > 0$. Since $\lim_n x_n = x$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies, $|x_n - x| < \frac{\epsilon}{|a|}$. Then for $n \geq N$,

$$|ax_n - ax| = |a||x_n - x| < |a| \cdot \frac{\epsilon}{|a|} = \epsilon.$$

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Proposition 1.1.5. *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$*, then* $\lim_{n\to\infty}(x_n + y_n) = x + y$.

Proof. Let $\epsilon > 0$. Since $\lim_n x_n = x$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|x_n - x| < \frac{\epsilon}{2}$. Since $\lim_n y_n = y$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|y_n - y| < \frac{\epsilon}{2}$. Let $N = \max(N_1, N_2)$. Then for $n \geq N$, by the triangle inequality,

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

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Proposition 1.1.6. *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$*, then* $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$.

Proof. Let $\epsilon > 0$. Consider

$$\begin{aligned} x_n y_n - xy &= x_n y_n - x_n y + x_n y - xy \\ &= x_n(y_n - y) + y(x_n - x) \end{aligned}$$

Then by the Triangle inequality and by the boundedness of convergent sequences we obtain:

$$\begin{aligned} |x_n y_n - xy| &\leq |x_n||y_n - y| + |y||x_n - x| \\ &< M\frac{\epsilon}{2M} + |y|\frac{\epsilon}{2|y|} \\ &= \epsilon \end{aligned}$$

where $M > 0$. ■

Proof. Let $\epsilon > 0$. If $y = 0$ then $x_n y_n = 0 = xy$ for all n and the result is immediate. Now suppose $y \neq 0$. Since $x_n \to x$, the sequence (x_n) is bounded, so there exists $M > 0$ such that $|x_n| \leq M$ for all n. Since $\lim_n x_n = x$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|x_n - x| < \frac{\epsilon}{2|y|}$. Since $\lim_n y_n = y$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|y_n - y| < \frac{\epsilon}{2M}$. Let $N = \max(N_1, N_2)$. Then for $n \geq N$, by the add-and-subtract identity and the triangle inequality,

$$\begin{aligned} |x_n y_n - xy| &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq |x_n||y_n - y| + |y||x_n - x| \\ &< M \cdot \frac{\epsilon}{2M} + |y| \cdot \frac{\epsilon}{2|y|} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

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Proposition 1.1.7. *If* $\lim_{n\to\infty} x_n = x$ *and* $x \neq 0$*, then* $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$*.*

Proof. Let $\epsilon > 0$. Since $x_n \to x$ and $|x| > 0$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|x_n - x| < \frac{|x|}{2}$. By the reverse triangle inequality this gives $|x_n| \geq |x| - |x_n - x| > \frac{|x|}{2}$, so $\frac{1}{|x_n|} < \frac{2}{|x|}$ for all $n \geq N_1$. There also exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|x_n - x| < \frac{|x|^2\epsilon}{2}$. Let $N = \max(N_1, N_2)$. Then for $n \geq N$,

$$\begin{aligned} \left| \frac{1}{x_n} - \frac{1}{x} \right| &= \frac{|x_n - x|}{|x_n||x|} \\ &< \frac{2}{|x|^2}|x_n - x| \\ &< \frac{2}{|x|^2} \cdot \frac{|x|^2\epsilon}{2} = \epsilon. \end{aligned}$$

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