Chapter 4
Signed and Complex Measures

In this chapter we study signed and complex measures, which are defined to be the countably additive functions from a σ-algebra to $[-\infty, +\infty]$ or to $\mathbb{C}$ that have value 0 on the empty set. We begin in Sect. 4.1 with some basic definitions and facts. Section 4.2 is devoted to the main result of this chapter, the Radon–Nikodym theorem. Let μ be a σ-finite positive measure on a measurable space $(X, \mathscr{A})$. The Radon–Nikodym theorem characterizes those finite positive, signed, or complex measures ν whose values can be computed by integrating some μ-integrable function—in other words, it characterizes those ν for which there is a μ-integrable f such that $\nu(A) = \int_A f\,d\mu$ holds for all A in $\mathscr{A}$. The last part of the chapter is devoted to the relation of the material in the early parts of the chapter to the classical concepts of bounded variation and absolute continuity (Sect. 4.4) and to the use of the Radon–Nikodym theorem to compute the dual spaces of a number of the L^p spaces (Sect. 4.5).

4.1 Signed and Complex Measures

Let $(X, \mathscr{A})$ be a measurable space, and let μ be a function on $\mathscr{A}$ with values in $[-\infty, +\infty]$. The function μ is *finitely additive* if the identity

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

holds for each finite sequence $\{A_i\}_{i=1}^{n}$ of disjoint sets in $\mathscr{A}$ and is *countably additive* if the identity

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

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holds for each infinite sequence $\{A_i\}$ of disjoint sets in $\mathscr{A}$. If μ is countably additive and satisfies $\mu(\varnothing) = 0$, then it is a *signed measure*. Thus signed measures are the functions that result if in the definition of measures the requirement of nonnegativity is removed. This section is devoted to signed measures and complex measures (to be defined below) and to their relationship to measures.[1]

A signed measure is *finite* if neither $+\infty$ nor $-\infty$ occurs among its values.

Suppose that μ is a signed measure on the measurable space $(X, \mathscr{A})$. Then for each A in $\mathscr{A}$ the sum $\mu(A) + \mu(A^c)$ must be defined (that is, must not be of the form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$) and must equal $\mu(X)$. Hence if there is a set A in $\mathscr{A}$ for which $\mu(A) = +\infty$, then $\mu(X) = +\infty$, and if there is a set A in $\mathscr{A}$ for which $\mu(A) = -\infty$, then $\mu(X) = -\infty$. Consequently a signed measure can attain at most one of the values $+\infty$ and $-\infty$. A similar argument shows that if B is a set in $\mathscr{A}$ for which $\mu(B)$ is finite, then $\mu(A)$ is finite for each $\mathscr{A}$-measurable subset A of B.

Examples 4.1.1.

(a) Let $(X, \mathscr{A}, \mu)$ be a measure space, let f belong to $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$, and define a function ν on $\mathscr{A}$ by $\nu(A) = \int_A f\,d\mu$. Then the linearity of the integral and the dominated convergence theorem imply that ν is a signed measure on $(X, \mathscr{A})$. Note that such a signed measure is the difference of the positive measures ν_1 and ν_2 defined by $\nu_1(A) = \int_A f^+\,d\mu$ and $\nu_2(A) = \int_A f^-\,d\mu$.

(b) More generally, if ν_1 and ν_2 are positive measures on the measurable space $(X, \mathscr{A})$ and if at least one of them is finite, then $\nu_1 - \nu_2$ is a signed measure on $(X, \mathscr{A})$. We will soon see that every signed measure arises in this way. □

Lemma 4.1.2. *Let $(X, \mathscr{A})$ be a measurable space, and let μ be a signed measure on $(X, \mathscr{A})$. If $\{A_k\}$ is an increasing sequence of sets in $\mathscr{A}$, then*

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \lim_k \mu(A_k),$$

and if $\{A_k\}$ is a decreasing sequence of sets in $\mathscr{A}$ such that $\mu(A_n)$ is finite for some n, then

$$\mu\Big(\bigcap_{k=1}^{\infty} A_k\Big) = \lim_k \mu(A_k).$$

Lemma 4.1.3. *Suppose that $(X, \mathscr{A})$ is a measurable space and that μ is an extended real-valued function on $\mathscr{A}$ that is finitely additive and satisfies $\mu(\varnothing) = 0$.*

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[1] We will try not to abbreviate the phrases "signed measure" and "complex measure" with the word "measure"; thus the word "measure" by itself will continue to mean a nonnegative countably additive function whose value at $\varnothing$ is 0. However, for clarity and emphasis, we will sometimes refer to a measure as a positive measure.

If $\mu(\cup_{k=1}^{\infty} A_k) = \lim_k \mu(A_k)$ *holds for each increasing sequence* $\{A_k\}$ *of sets in* $\mathscr{A}$ *or if* $\lim_k \mu(A_k) = 0$ *holds for each decreasing sequence* $\{A_k\}$ *of sets in* $\mathscr{A}$ *for which* $\cap_{k=1}^{\infty} A_k = \varnothing$, *then* μ *is a signed measure.*

The proofs of these lemmas are very similar to those of Propositions 1.2.5 and 1.2.6 and so are omitted.

Let μ be a signed measure on the measurable space $(X, \mathscr{A})$. A subset A of X is a *positive set* for μ if $A \in \mathscr{A}$ and each $\mathscr{A}$-measurable subset E of A satisfies $\mu(E) \geq 0$. Likewise A is a *negative set* for μ if $A \in \mathscr{A}$ and each $\mathscr{A}$-measurable subset E of A satisfies $\mu(E) \leq 0$.

The role of positive and negative sets is explained by Theorem 4.1.5 and Corollary 4.1.6 below. For the proofs of these results, we will need the following construction.

Lemma 4.1.4. *Let* μ *be a signed measure on the measurable space* $(X, \mathscr{A})$, *and let* A *be a subset of* X *that belongs to* $\mathscr{A}$ *and satisfies* $-\infty < \mu(A) < 0$. *Then there is a negative set* B *that is included in* A *and satisfies*

$$\mu(B) \leq \mu(A). \tag{1}$$

Proof. We will remove a suitable sequence of subsets from A and then let B consist of the points of A that remain. To begin, let

$$\delta_1 = \sup\{\mu(E) : E \in \mathscr{A} \text{ and } E \subseteq A\}, \tag{2}$$

and choose an $\mathscr{A}$-measurable subset A_1 of A that satisfies[2]

$$\mu(A_1) \geq \min\left(\frac{1}{2}\delta_1, 1\right).$$

Then δ_1 and $\mu(A_1)$ are nonnegative (note that (2) implies that $\delta_1 \geq \mu(\varnothing) = 0$). We proceed by induction, constructing sequences $\{\delta_n\}$ and $\{A_n\}$ by letting

$$\delta_n = \sup\left\{\mu(E) : E \in \mathscr{A} \text{ and } E \subseteq \left(A - \bigcup_{i=1}^{n-1} A_i\right)\right\},$$

and then choosing an $\mathscr{A}$-measurable subset A_n of $A - \bigcup_{i=1}^{n-1} A_i$ that satisfies

$$\mu(A_n) \geq \min\left(\frac{1}{2}\delta_n, 1\right).$$

Now define A_∞ and B by $A_\infty = \cup_{n=1}^{\infty} A_n$ and $B = A - A_\infty$.

[2] We require that $\mu(A_1)$ be at least $\min(\delta_1/2, 1)$, rather than at least $\delta_1/2$, because we have not yet proved that δ_1 is finite (see Exercise 4).

Let us check that B has the required properties. Since the sets A_n are disjoint and satisfy $\mu(A_n) \ge 0$, it follows that $\mu(A_\infty) \ge 0$ and hence that

$$\mu(A) = \mu(A_\infty) + \mu(B) \ge \mu(B).$$

Thus B satisfies (1).

We turn to the negativity of B. The finiteness of $\mu(A)$ implies the finiteness of $\mu(A_\infty)$ and so, since $\mu(A_\infty) = \sum_n \mu(A_n)$, implies that $\lim_n \mu(A_n) = 0$. Consequently $\lim_n \delta_n = 0$. Since an arbitrary $\mathscr{A}$-measurable subset E of B satisfies $\mu(E) \le \delta_n$ for each n and so satisfies $\mu(E) \le 0$, B must be a negative set for μ. □

The following theorem and its corollary give the standard decomposition of signed measures.

Theorem 4.1.5 (Hahn Decomposition Theorem). *Let $(X, \mathscr{A})$ be a measurable space, and let μ be a signed measure on $(X, \mathscr{A})$. Then there are disjoint subsets P and N of X such that P is a positive set for μ, N is a negative set for μ, and $X = P \cup N$.*

Proof. Since the signed measure μ cannot include both $+\infty$ and $-\infty$ among its values, we can for definiteness assume that $-\infty$ is not included. Let

$$L = \inf\{\mu(A) : A \text{ is a negative set for } \mu\} \tag{3}$$

(the set on the right side of (3) is nonempty, since $\varnothing$ is a negative set for μ). Choose a sequence $\{A_n\}$ of negative sets for μ for which $L = \lim_n \mu(A_n)$, and let $N = \cup_{n=1}^{\infty} A_n$. It is easy to check that N is a negative set for μ (each $\mathscr{A}$-measurable subset of N is the union of a sequence of disjoint $\mathscr{A}$-measurable sets, each of which is included in some A_n). Hence $L \le \mu(N) \le \mu(A_n)$ holds for each n, and so $L = \mu(N)$. Furthermore, since μ does not attain the value $-\infty$, $\mu(N)$ must be finite.

Let $P = N^c$. Our only remaining task is to check that P is a positive set for μ. If P included an $\mathscr{A}$-measurable set A such that $\mu(A) < 0$, then it would include a negative set B such that $\mu(B) < 0$ (Lemma 4.1.4), and $N \cup B$ would be a negative set such that

$$\mu(N \cup B) = \mu(N) + \mu(B) < \mu(N) = L$$

(recall that $\mu(N)$ is finite). However this contradicts (3), and so P must be a positive set for μ. □

A *Hahn decomposition* of a signed measure μ is a pair (P, N) of disjoint subsets of X such that P is a positive set for μ, N is a negative set for μ, and $X = P \cup N$. Note that a signed measure can have several Hahn decompositions. For example, if X is the interval $[-1, 1]$, if $\mathscr{A}$ is the σ-algebra of Borel subsets of $[-1, 1]$, and if μ is defined by $\mu(A) = \int_A x\lambda(dx)$, then $([0, 1], [-1, 0))$ and $((0, 1], [-1, 0])$ are both Hahn decompositions of μ. On the other hand, if μ is an arbitrary signed measure on a measurable space $(X, \mathscr{A})$ and if (P_1, N_1) and (P_2, N_2) are Hahn decompositions of μ, then $P_1 \cap N_2$ is both a positive set and a negative set for μ, and so each $\mathscr{A}$-measurable subset of $P_1 \cap N_2$ has measure zero under μ. Likewise, each $\mathscr{A}$-measurable subset of $P_2 \cap N_1$ has measure zero under μ. Thus the Hahn decomposition of μ is essentially unique.

Corollary 4.1.6 (Jordan Decomposition Theorem). *Every signed measure is the difference of two positive measures, at least one of which is finite.*

Proof. Let μ be a signed measure on $(X, \mathscr{A})$. Choose a Hahn decomposition (P, N) for μ (see Theorem 4.1.5), and then define functions μ^+ and μ^- on $\mathscr{A}$ by

$$\mu^+(A) = \mu(A \cap P)$$

and

$$\mu^-(A) = -\mu(A \cap N).$$

It is clear that μ^+ and μ^- are positive measures such that $\mu = \mu^+ - \mu^-$. Since $+\infty$ and $-\infty$ cannot both occur among the values of μ, at least one of the values $\mu(P)$ and $\mu(N)$, and hence at least one of the measures μ^+ and μ^-, must be finite. □

Let (P, N) be a Hahn decomposition of the signed measure μ, let μ^+ and μ^- be the measures constructed from (P, N) in the proof of Corollary 4.1.6, and suppose that A belongs to $\mathscr{A}$. Then each $\mathscr{A}$-measurable subset B of A satisfies

$$\mu(B) = \mu^+(B) - \mu^-(B) \le \mu^+(B) \le \mu^+(A).$$

Since in addition $\mu^+(A) = \mu(A \cap P)$, it follows that

$$\mu^+(A) = \sup\{\mu(B) : B \in \mathscr{A} \text{ and } B \subseteq A\}.$$

Likewise the measure μ^- satisfies

$$\mu^-(A) = \sup\{-\mu(B) : B \in \mathscr{A} \text{ and } B \subseteq A\}.$$

Thus μ^+ and μ^- do not depend on the particular Hahn decomposition used in their construction. The measures μ^+ and μ^- are called the *positive part* and the *negative part* of μ, and the representation $\mu = \mu^+ - \mu^-$ is called the *Jordan decomposition* of μ.

The *variation* of the signed measure μ is the positive measure $|\mu|$ defined by $|\mu| = \mu^+ + \mu^-$. It is easy to check that

$$|\mu(A)| \le |\mu|(A)$$

holds for each A in $\mathscr{A}$ and in fact that $|\mu|$ is the smallest of those positive measures ν that satisfy $|\mu(A)| \le \nu(A)$ for each A in $\mathscr{A}$ (see Exercise 5). The *total variation* $\|\mu\|$ of the signed measure μ is defined by $\|\mu\| = |\mu|(X)$.

Let $(X,\mathscr{A})$ be a measurable space. A *complex measure* on $(X,\mathscr{A})$ is a function μ from $\mathscr{A}$ to $\mathbb{C}$ that satisfies $\mu(\varnothing)=0$ and is *countably additive*, in the sense that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

holds for each infinite sequence $\{A_n\}$ of disjoint sets in $\mathscr{A}$. Note that by definition a complex measure has only complex values and so has no infinite values.

Each complex measure μ on $(X,\mathscr{A})$ can of course be written in the form $\mu = \mu' + i\mu''$, where μ' and μ'' are finite signed measures on $(X,\mathscr{A})$. Hence the Jordan decomposition theorem implies that each complex measure μ can be written in the form

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4, \tag{4}$$

where μ_1, μ_2, μ_3, and μ_4 are finite positive measures on $(X,\mathscr{A})$. Such a representation is called the *Jordan decomposition* of μ if $\mu' = \mu_1 - \mu_2$ and $\mu'' = \mu_3 - \mu_4$ are the Jordan decompositions of the real and imaginary parts of μ.

We turn to the *variation* $|\mu|$ of the complex measure μ. For each A in $\mathscr{A}$ let $|\mu|(A)$ be the supremum of the numbers $\sum_{j=1}^{n} |\mu(A_j)|$, where $\{A_j\}_{j=1}^{n}$ ranges over all finite partitions of A into $\mathscr{A}$-measurable sets.

Proposition 4.1.7. *Let $(X,\mathscr{A})$ be a measurable space, and let μ be a complex measure on $(X,\mathscr{A})$. Then the variation $|\mu|$ of μ is a finite measure on $(X,\mathscr{A})$.*

Proof. The relation $|\mu|(\varnothing)=0$ is immediate.

We can check the finite additivity of $|\mu|$ by showing that if B_1 and B_2 are disjoint sets that belong to $\mathscr{A}$, then $|\mu|(B_1\cup B_2) = |\mu|(B_1) + |\mu|(B_2)$. For this, note that if $\{A_j\}_{j=1}^{n}$ is a finite partition of $B_1 \cup B_2$ into $\mathscr{A}$-measurable sets, then

$$\begin{aligned}\sum_j |\mu(A_j)| &\le \sum_j |\mu(A_j \cap B_1)| + \sum_j |\mu(A_j\cap B_2)| \\ &\le |\mu|(B_1) + |\mu|(B_2).\end{aligned}$$

Since $|\mu|(B_1\cup B_2)$ is the supremum of the numbers that can appear on the left side of the inequality, it follows that

$$|\mu|(B_1\cup B_2) \le |\mu|(B_1) + |\mu|(B_2).$$

A similar argument, based on partitioning B_1 and B_2, shows that

$$|\mu|(B_1) + |\mu|(B_2) \le |\mu|(B_1\cup B_2).$$

Thus $|\mu|(B_1\cup B_2) = |\mu|(B_1) + |\mu|(B_2)$, and the finite additivity of $|\mu|$ is proved.

If $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ is the Jordan decomposition of μ, then

$$|\mu|(A) \le \mu_1(A) + \mu_2(A) + \mu_3(A) + \mu_4(A) \tag{5}$$

holds for each A in $\mathscr{A}$. Since the measures μ_1, μ_2, μ_3, and μ_4 are finite, the finiteness of $|\mu|$ follows. Furthermore, if $\{A_n\}$ is a decreasing sequence of $\mathscr{A}$-measurable sets such that $\cap_n A_n = \varnothing$, then $\lim_n \mu_k(A_n) = 0$ holds for $k = 1$, $\dots$, 4, and so (5) implies that $\lim_n |\mu|(A_n) = 0$. Thus $|\mu|$ is countably additive (Proposition 1.2.6). □

It is easy to check that if μ is a complex measure on $(X,\mathscr{A})$, then $|\mu|$ is the smallest of the positive measures ν that satisfy $|\mu(A)| \le \nu(A)$ for all A in $\mathscr{A}$ (see Exercise 5). Note that if μ is a finite signed measure, then μ is also a complex measure; it is easy to check that in this case the variation of μ as a signed measure is the same as its variation as a complex measure (Exercise 6).

The *total variation* $\|\mu\|$ of the complex measure μ is defined by $\|\mu\| = |\mu|(X)$.

Suppose that $(X,\mathscr{A})$ is a measurable space. Let $M(X,\mathscr{A},\mathbb{R})$ be the collection of all finite signed measures on $(X,\mathscr{A})$, and let $M(X,\mathscr{A},\mathbb{C})$ be the collection of all complex measures on $(X,\mathscr{A})$. It is easy to check that $M(X,\mathscr{A},\mathbb{R})$ and $M(X,\mathscr{A},\mathbb{C})$ are vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively, and that the total variation gives a norm on each of them.

Proposition 4.1.8. *Let $(X,\mathscr{A})$ be a measurable space. Then the spaces $M(X,\mathscr{A},\mathbb{R})$ and $M(X,\mathscr{A},\mathbb{C})$ are complete under the total variation norm.*

Proof. Let $\{\mu_n\}$ be a Cauchy sequence in $M(X,\mathscr{A},\mathbb{R})$ or in $M(X,\mathscr{A},\mathbb{C})$. The inequality $|\mu_m(A) - \mu_n(A)| \le \|\mu_m - \mu_n\|$ implies that for each A in $\mathscr{A}$ the sequence $\{\mu_n(A)\}$ is a Cauchy sequence of real or complex numbers and hence is convergent. Define a real- or complex-valued function μ on $\mathscr{A}$ by letting $\mu(A) = \lim_n \mu_n(A)$ hold at each A in $\mathscr{A}$. We need to check that μ is a signed or complex measure and that $\lim_n \|\mu_n - \mu\| = 0$.

It is clear that $\mu(\varnothing) = 0$ and that μ is at least finitely additive.

As preparation for checking the countable additivity of μ, we will show that the convergence of $\mu_n(A)$ to $\mu(A)$ is uniform in A. If ε is a positive number and if N is a positive integer such that $\|\mu_m - \mu_n\| < \varepsilon$ holds whenever $m \ge N$ and $n \ge N$, then

$$|\mu_m(A) - \mu_n(A)| < \varepsilon \tag{6}$$

holds whenever $A \in \mathscr{A}$, $m \ge N$, and $n \ge N$, and so

$$|\mu(A) - \mu_n(A)| \le \varepsilon$$

holds whenever $A \in \mathscr{A}$ and $n \ge N$ (let m approach infinity in (6)). Since ε is arbitrary, the uniformity of the convergence of $\mu_n(A)$ to $\mu(A)$ follows.

We now use Lemmas 4.1.2 and 4.1.3 (and their extensions to complex measures) to prove the countable additivity of μ. Let $\{A_k\}$ be a decreasing sequence of sets in $\mathscr{A}$ such that $\cap_k A_k = \varnothing$, and let ε be a positive number. Use the uniformity of the convergence of $\mu_n(A)$ to $\mu(A)$ to choose N so that $|\mu(A) - \mu_n(A)| < \varepsilon/2$ holds whenever $A \in \mathscr{A}$ and $n \geq N$, and then use Lemma 4.1.2 to choose K such that $|\mu_N(A_k)| < \varepsilon/2$ holds whenever $k \geq K$. It follows that if $k \geq K$ then

$$|\mu(A_k)| \leq |\mu(A_k) - \mu_N(A_k)| + |\mu_N(A_k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\lim_k \mu(A_k) = 0$, and the countable additivity of μ follows.

We turn to the relation $\lim_n \|\mu - \mu_n\| = 0$. Let ε be a positive number, and use the fact that $\{\mu_n\}$ is a Cauchy sequence to choose N so that $\|\mu_m - \mu_n\| < \varepsilon$ holds whenever $m \geq N$ and $n \geq N$. Note that if $m \geq N$ and $n \geq N$, then each partition of X into $\mathscr{A}$-measurable sets A_j, $j = 1, \dots, k$, satisfies

$$\sum_{j=1}^{k} |\mu_m(A_j) - \mu_n(A_j)| \leq \|\mu_m - \mu_n\| < \varepsilon,$$

and hence satisfies

$$\sum_{j=1}^{k} |\mu(A_j) - \mu_n(A_j)| = \lim_m \sum_{j=1}^{k} |\mu_m(A_j) - \mu_n(A_j)| \leq \varepsilon.$$

Since $\|\mu - \mu_n\|$ is the supremum of the numbers that can appear on the left side of this inequality, it follows that $\|\mu - \mu_n\| \leq \varepsilon$ holds whenever $n \geq N$. Consequently $\lim_n \|\mu - \mu_n\| = 0$. Thus $M(X, \mathscr{A}, \mathbb{R})$ and $M(X, \mathscr{A}, \mathbb{C})$ are complete. □

Let us deal briefly with integration with respect to a finite signed or complex measure.

Suppose that $(X, \mathscr{A})$ is a measurable space. We will denote by $B(X, \mathscr{A}, \mathbb{R})$ the vector space of bounded real-valued $\mathscr{A}$-measurable functions on X and by $B(X, \mathscr{A}, \mathbb{C})$ the vector space of bounded complex-valued $\mathscr{A}$-measurable functions on X. If μ is a finite signed measure on $(X, \mathscr{A})$, if $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ, and if f belongs to $B(X, \mathscr{A}, \mathbb{R})$, then the *integral* of f with respect to μ is defined by

$$\int f\, d\mu = \int f\, d\mu^+ - \int f\, d\mu^-.$$

It is clear that $f \mapsto \int f\, d\mu$ defines a linear functional on $B(X, \mathscr{A}, \mathbb{R})$.

If $A \in \mathscr{A}$, then $\int \chi_A\, d\mu = \mu(A)$ holds for each μ in $M(X, \mathscr{A}, \mathbb{R})$. Thus the formula

$$\mu \mapsto \int f\, d\mu$$

defines a linear functional on $M(X, \mathscr{A}, \mathbb{R})$ if f is an $\mathscr{A}$-measurable characteristic function and hence if f is an arbitrary function in $B(X, \mathscr{A}, \mathbb{R})$ (use the linearity of the integral and the dominated convergence theorem).

Similarly, if μ is a complex measure on $(X, \mathscr{A})$, then we can use the Jordan decomposition of μ to define the integral with respect to μ of a function in $B(X, \mathscr{A}, \mathbb{C})$. The expressions $f \mapsto \int f\,d\mu$ and $\mu \mapsto \int f\,d\mu$ define linear functionals on $B(X, \mathscr{A}, \mathbb{C})$ and on $M(X, \mathscr{A}, \mathbb{C})$, respectively.

Now use the formula

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

to define norms on $B(X, \mathscr{A}, \mathbb{R})$ and $B(X, \mathscr{A}, \mathbb{C})$ (see Example 3.2.1(f)). If μ is a finite signed or complex measure on $(X, \mathscr{A})$ and if f is a simple $\mathscr{A}$-measurable function on X, say with values $a_1, \ldots, a_k$, attained on the sets $A_1, \ldots, A_k$, then

$$\left| \int f\,d\mu \right| = \left| \sum_{j=1}^{k} a_j \mu(A_j) \right| \le \sum_{j=1}^{k} |a_j||\mu(A_j)| \le \sum_{j=1}^{k} \|f\|_\infty |\mu(A_j)|,$$

and so

$$\left| \int f\,d\mu \right| \le \|f\|_\infty \|\mu\|. \tag{7}$$

Since each function in $B(X, \mathscr{A}, \mathbb{R})$ or in $B(X, \mathscr{A}, \mathbb{C})$ is the uniform limit of a sequence of simple $\mathscr{A}$-measurable functions, it follows that (7) holds whenever f belongs to $B(X, \mathscr{A}, \mathbb{R})$ or $B(X, \mathscr{A}, \mathbb{C})$.

Exercises

1. Let μ be a signed or complex measure on $(X, \mathscr{A})$, and let A belong to $\mathscr{A}$.
   (a) Show that $|\mu|(A) = 0$ holds if and only if each $\mathscr{A}$-measurable subset B of A satisfies $\mu(B) = 0$.
   (b) Show that in general the relation $\mu(A) = 0$ does not imply the relation $|\mu|(A) = 0$.
2. Let μ be a signed measure on $(X, \mathscr{A})$, and let ν_1 and ν_2 be positive measures on $(X, \mathscr{A})$ such that $\mu = \nu_1 - \nu_2$. Show that $\nu_1(A) \ge \mu^+(A)$ and $\nu_2(A) \ge \mu^-(A)$ hold for each A in $\mathscr{A}$.
3. Let μ_1 and μ_2 be finite signed measures on the measurable space $(X, \mathscr{A})$. Define signed measures $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ on $(X, \mathscr{A})$ by $\mu_1 \vee \mu_2 = \mu_1 + (\mu_2 - \mu_1)^+$ and $\mu_1 \wedge \mu_2 = \mu_1 - (\mu_1 - \mu_2)^+$.
   (a) Show that $\mu_1 \vee \mu_2$ is the smallest of those finite signed measures ν that satisfy $\nu(A) \ge \mu_1(A)$ and $\nu(A) \ge \mu_2(A)$ for all A in $\mathscr{A}$.
   (b) Find and prove an analogous characterization of $\mu_1 \wedge \mu_2$.
4. Show that the quantities δ_1, δ_2, ... defined in the proof of Lemma 4.1.4 are finite. (Hint: Use Theorem 4.1.5; this is legitimate, since Lemma 4.1.4 and Theorem 4.1.5 were proved without using the finiteness of the δ_n's.)

5. Let μ be a signed or complex measure on $(X,\mathscr{A})$, and let ν be a positive measure on $(X,\mathscr{A})$ such that $|\mu(A)| \le \nu(A)$ holds for each A in $\mathscr{A}$. Show that $|\mu|(A) \le \nu(A)$ holds for each A in $\mathscr{A}$.
6. Note that if μ is a finite signed measure, then μ is both a signed measure and a complex measure. Show that in this case the two definitions of $|\mu|$ yield the same result.
7. Let μ_1 and μ_2 be finite signed measures, and let ν be the complex measure defined by $\nu = \mu_1 + i\mu_2$. Show that $|\mu_1| \le |\nu|$, $|\mu_2| \le |\nu|$ and $|\nu| \le |\mu_1| + |\mu_2|$. Is it necessarily true that $\|\nu\| \le \sqrt{\|\mu_1\|^2 + \|\mu_2\|^2}$?
8. Let μ and μ_1, μ_2, ... be finite signed or complex measures on $(X,\mathscr{A})$. Show that $\lim_n \|\mu_n - \mu\| = 0$ holds if and only if $\mu_n(A)$ converges to $\mu(A)$ uniformly in A as n approaches infinity.
9. Use Proposition 3.2.5, Exercise 1.2.6, and the Jordan decomposition to give another proof of Proposition 4.1.8.
10. Check that the spaces $B(X,\mathscr{A},\mathbb{R})$ and $B(X,\mathscr{A},\mathbb{C})$ are complete under the norm $\|\cdot\|_\infty$.
11. Let μ be a finite signed or complex measure on $(X,\mathscr{A})$, and let $\{f_n\}$ be a uniformly bounded sequence of real- or complex-valued $\mathscr{A}$-measurable functions on X (thus there is a positive number B such that $|f_n(x)| \le B$ holds for each x and n). Show that if $f(x) = \lim_n f_n(x)$ holds at each x in X, then $\int f\,d\mu = \lim_n \int f_n\,d\mu$.

4.2 Absolute Continuity

Let $(X,\mathscr{A})$ be a measurable space, and let μ and ν be positive measures on $(X,\mathscr{A})$. Then ν is *absolutely continuous with respect to* μ if each set A that belongs to $\mathscr{A}$ and satisfies $\mu(A) = 0$ also satisfies $\nu(A) = 0$. One sometimes writes $\nu \ll \mu$ to indicate that ν is absolutely continuous with respect to μ. A measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is simply called *absolutely continuous* if it is absolutely continuous with respect to d-dimensional Lebesgue measure.

Suppose that $(X,\mathscr{A},\mu)$ is a measure space and that f is a nonnegative function in $\mathscr{L}^1(X,\mathscr{A},\mu,\mathbb{R})$. We have seen (in Sect. 2.4) that the formula $\nu(A) = \int_A f\,d\mu$ defines a finite positive measure ν on $\mathscr{A}$. If $\mu(A) = 0$, then $f\chi_A$ vanishes μ-almost everywhere, and so $\nu(A) = 0$. Thus ν is absolutely continuous with respect to μ. We will see that if μ is σ-finite, then every finite measure on $(X,\mathscr{A})$ that is absolutely continuous with respect to μ arises in this way.

The following lemma characterizes those finite positive measures that are absolutely continuous with respect to an arbitrary positive measure; this characterization is useful in the classical theory of functions of a real variable (see Sect. 4.4).

Lemma 4.2.1. *Let $(X,\mathscr{A})$ be a measurable space, let μ be a positive measure on $(X,\mathscr{A})$, and let ν be a finite positive measure on $(X,\mathscr{A})$. Then $\nu \ll \mu$ if and only if for each positive ε there is a positive δ such that each $\mathscr{A}$-measurable set A that satisfies $\mu(A) < \delta$ also satisfies $\nu(A) < \varepsilon$.*

Proof. First suppose that for each positive ε there is a corresponding δ. Let A be an $\mathscr{A}$-measurable set that satisfies $\mu(A) = 0$. Then $\mu(A) < \delta$ holds for each δ, and so $\nu(A) < \varepsilon$ holds for each ε; hence A satisfies $\nu(A) = 0$. Thus ν is absolutely continuous with respect to μ.

Next suppose that there is a positive number ε (which we will hold fixed) for which there is no suitable δ. Then for each positive integer k we can (and do) choose an $\mathscr{A}$-measurable set A_k that satisfies $\mu(A_k) < 1/2^k$ and $\nu(A_k) \ge \varepsilon$. Then the inequalities $\mu(\cup_{k=n}^{\infty} A_k) \le \sum_{k=n}^{\infty} \mu(A_k) < 1/2^{n-1}$ and $\nu(\cup_{k=n}^{\infty} A_k) \ge \nu(A_n) \ge \varepsilon$ hold for each n, and so the set A defined by $A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ satisfies $\mu(A) = 0$ and $\nu(A) \ge \varepsilon$ (see Proposition 1.2.5). Thus A satisfies $\mu(A) = 0$ but not $\nu(A) = 0$, and so ν is not absolutely continuous with respect to μ. □

We turn to the main result of this section.

Theorem 4.2.2 (Radon–Nikodym Theorem). *Let $(X, \mathscr{A})$ be a measurable space, and let μ and ν be σ-finite positive measures on $(X, \mathscr{A})$. If ν is absolutely continuous with respect to μ, then there is an $\mathscr{A}$-measurable function $g \colon X \to [0, +\infty)$ such that $\nu(A) = \int_A g\, d\mu$ holds for each A in $\mathscr{A}$. The function g is unique up to μ-almost everywhere equality.*

Proof. First consider the case where μ and ν are both finite. Let $\mathscr{F}$ be the set consisting of those $\mathscr{A}$-measurable functions $f \colon X \to [0, +\infty]$ that satisfy $\int_A f\, d\mu \le \nu(A)$ for each A in $\mathscr{A}$. We will show first that $\mathscr{F}$ contains a function g such that

$$\int g\, d\mu = \sup\left\{\int f\, d\mu : f \in \mathscr{F}\right\} \tag{1}$$

and then that this function g satisfies $\nu(A) = \int_A g\, d\mu$ for each A in $\mathscr{A}$. Finally, we will show that g can be modified so as to have only finite values.

We begin by checking that if f_1 and f_2 belong to $\mathscr{F}$, then $f_1 \vee f_2$ belongs to $\mathscr{F}$; to see this note that if A is an arbitrary set in $\mathscr{A}$, if $A_1 = \{x \in A : f_1(x) > f_2(x)\}$, and if $A_2 = \{x \in A : f_2(x) \ge f_1(x)\}$, then

$$\int_A (f_1 \vee f_2)\, d\mu = \int_{A_1} f_1\, d\mu + \int_{A_2} f_2\, d\mu \le \nu(A_1) + \nu(A_2) = \nu(A).$$

Furthermore, $\mathscr{F}$ is not empty (the constant 0 belongs to it). Now choose a sequence $\{f_n\}$ of functions in $\mathscr{F}$ for which

$$\lim_n \int f_n\, d\mu = \sup\left\{\int f\, d\mu : f \in \mathscr{F}\right\}.$$

By replacing f_n with $f_1 \vee \cdots \vee f_n$, we can assume that the sequence $\{f_n\}$ is increasing. Let $g = \lim_n f_n$. The monotone convergence theorem implies that the relation

$$\int_A g\, d\mu = \lim_n \int_A f_n\, d\mu \le \nu(A)$$

holds for each A and hence that g belongs to $\mathscr{F}$. It also implies that $\int g\,d\mu = \sup\{\int f\,d\mu : f \in \mathscr{F}\}$. Thus g has the first of the properties claimed for it.

We turn to the proof that $\nu(A) = \int_A g\,d\mu$ holds for each A in $\mathscr{A}$. Since g belongs to $\mathscr{F}$, the formula $\nu_0(A) = \nu(A) - \int_A g\,d\mu$ defines a positive measure on $\mathscr{A}$. We need only show that $\nu_0 = 0$. Assume the contrary. Then, since μ is finite, there is a positive number ε such that

$$\nu_0(X) > \varepsilon\mu(X). \tag{2}$$

Let (P,N) be a Hahn decomposition (see Sect. 4.1) for the signed measure $\nu_0 - \varepsilon\mu$. Note that for each A in $\mathscr{A}$ we have $\nu_0(A \cap P) \geq \varepsilon\mu(A \cap P)$, and hence we have

$$\begin{aligned} \nu(A) = \int_A g\,d\mu + \nu_0(A) &\geq \int_A g\,d\mu + \nu_0(A \cap P) \\ &\geq \int_A g\,d\mu + \varepsilon\mu(A \cap P) = \int_A (g + \varepsilon\chi_P)\,d\mu. \end{aligned} \tag{3}$$

Note also that $\mu(P) > 0$, since if $\mu(P) = 0$, then[3] $\nu_0(P) = 0$, and so

$$\nu_0(X) - \varepsilon\mu(X) = (\nu_0 - \varepsilon\mu)(N) \leq 0,$$

contradicting (2). It follows from this, the relation $\int g\,d\mu \leq \nu(X) < +\infty$, and (3) that $g + \varepsilon\chi_P$ belongs to $\mathscr{F}$ and satisfies $\int (g + \varepsilon\chi_P)\,d\mu > \int g\,d\mu$. This, however, contradicts (1) and so implies that $\nu_0 = 0$. Hence $\nu(A) = \int_A g\,d\mu$ holds for each A in $\mathscr{A}$. Since g can have an infinite value only on a μ-null set (Corollary 2.3.14), it can be redefined so as to have only finite values. With this we have constructed the required function in the case where μ and ν are finite.

Now suppose that μ and ν are σ-finite. Then X is the union of a sequence $\{B_n\}$ of disjoint $\mathscr{A}$-measurable sets, each of which has finite measure under μ and under ν. For each n the first part of this proof provides an $\mathscr{A}$-measurable function $g_n : B_n \to [0,+\infty)$ such that $\nu(A) = \int_A g_n\,d\mu$ holds for each $\mathscr{A}$-measurable subset A of B_n. The function $g : X \to [0,+\infty)$ that agrees on each B_n with g_n is then the required function.

We turn to the uniqueness of g. Let $g, h : X \to [0,+\infty)$ be $\mathscr{A}$-measurable functions that satisfy

$$\nu(A) = \int_A g\,d\mu = \int_A h\,d\mu$$

for each A in $\mathscr{A}$. First consider the case where ν is finite. Then $g - h$ is integrable and

$$\int_A (g - h)\,d\mu = 0$$

[3] This is where we use the absolute continuity of ν.

holds for each A in $\mathscr{A}$; since in this equation A can be the set where $g > h$ or the set where $g < h$, it follows that $\int (g-h)^+ \, d\mu = 0$ and $\int (g-h)^- \, d\mu = 0$ and hence that $(g-h)^+$ and $(g-h)^-$ vanish μ-almost everywhere (Corollary 2.3.12). Thus g and h agree μ-almost everywhere. If ν is σ-finite and if $\{B_n\}$ is a sequence of $\mathscr{A}$-measurable sets that have finite measure under ν and satisfy $X = \cup_n B_n$, then the preceding argument shows that g and h agree μ-almost everywhere on each B_n and hence μ-almost everywhere on X. □

Example 4.2.3. The assumption that μ is σ-finite cannot simply be omitted from Theorem 4.2.2. To see that, let X be the interval $[0,1]$, let $\mathscr{A}$ be the σ-algebra of Borel subsets of $[0,1]$, let μ be counting measure on $(X, \mathscr{A})$, and let ν be Lebesgue measure on $(X, \mathscr{A})$. Then $\nu \ll \mu$, but there is no measurable function f such that $\nu(A) = \int_A f \, d\mu$ holds for all A. (Concerning the possibility of not requiring that ν be σ-finite, see Exercise 6.) □

Now suppose that $(X, \mathscr{A})$ is a measurable space, that μ is a positive measure on $(X, \mathscr{A})$, and that ν is a signed or complex measure on $(X, \mathscr{A})$. Then ν is *absolutely continuous with respect to* μ, written $\nu \ll \mu$, if its variation $|\nu|$ is absolutely continuous with respect to μ. It is easy to check that a signed measure ν is absolutely continuous with respect to μ if and only if ν^+ and ν^- are absolutely continuous with respect to μ and that a complex measure ν is absolutely continuous with respect to μ if and only if the measures ν_1, ν_2, ν_3, and ν_4 appearing in its Jordan decomposition $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ are absolutely continuous with respect to μ. It is also easy to check that a signed or complex measure ν is absolutely continuous with respect to μ if and only if each A in $\mathscr{A}$ that satisfies $\mu(A) = 0$ also satisfies $\nu(A) = 0$ (be careful: $\nu(A) = 0$ is not equivalent to $|\nu|(A) = 0$; see Exercise 4.1.1).

The Radon–Nikodym theorem can be formulated for signed and complex measures as follows.

Theorem 4.2.4 (Radon–Nikodym Theorem). *Let $(X, \mathscr{A})$ be a measurable space, let μ be a σ-finite positive measure on $(X, \mathscr{A})$, and let ν be a finite signed or complex measure on $(X, \mathscr{A})$. If ν is absolutely continuous with respect to μ, then there is a function g that belongs to $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$ or to $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{C})$ and satisfies $\nu(A) = \int_A g \, d\mu$ for each A in $\mathscr{A}$. The function g is unique up to μ-almost everywhere equality.*

Proof. If ν is a complex measure that is absolutely continuous with respect to μ, then it can be written in the form $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, where ν_1, ν_2, ν_3, and ν_4 are finite positive measures that are absolutely continuous with respect to μ. Then Theorem 4.2.2 yields functions g_j, $j = 1, \ldots, 4$, that satisfy $\nu_j(A) = \int_A g_j \, d\mu$ for each A in $\mathscr{A}$. The required function g is now given by $g = g_1 - g_2 + ig_3 - ig_4$. The case of a finite signed measure is similar.

The uniqueness of g can be proved with the method used in the proof of Theorem 4.2.2; in case ν is a complex measure, the real and imaginary parts of g should be considered separately. □

Let $(X,\mathscr{A})$ be a measurable space, let μ be a σ-finite positive measure on $(X,\mathscr{A})$, and let ν be a finite signed, complex, or σ-finite positive measure on $(X,\mathscr{A})$. Suppose that ν is absolutely continuous with respect to μ. An $\mathscr{A}$-measurable function g on X that satisfies $\nu(A) = \int_A g\,d\mu$ for each A in $\mathscr{A}$ is called a *Radon–Nikodym derivative* of ν with respect to μ or, in view of its uniqueness up to μ-null sets, *the* Radon–Nikodym derivative of ν with respect to μ. A Radon–Nikodym derivative of ν with respect to μ is sometimes denoted by $\frac{d\nu}{d\mu}$.

We close this section with a few facts about the relationship of a finite signed or complex measure to its variation.

Proposition 4.2.5. *Suppose that $(X,\mathscr{A},\mu)$ is a measure space, that f belongs to $\mathscr{L}^1(X,\mathscr{A},\mu,\mathbb{R})$ or to $\mathscr{L}^1(X,\mathscr{A},\mu,\mathbb{C})$, and that ν is the finite signed or complex measure defined by $\nu(A) = \int_A f\,d\mu$. Then*

$$|\nu|(A) = \int_A |f|\,d\mu$$

holds for each A in $\mathscr{A}$.

Proof. Let A belong to $\mathscr{A}$ and let $\{A_j\}_{j=1}^k$ be a finite sequence of disjoint $\mathscr{A}$-measurable sets whose union is A. Then

$$\sum_j |\nu(A_j)| = \sum_j \left| \int_{A_j} f\,d\mu \right| \le \sum_j \int_{A_j} |f|\,d\mu = \int_A |f|\,d\mu.$$

Since $|\nu|(A)$ is the supremum of the sums that can appear on the left side of this inequality, it follows that $|\nu|(A) \le \int_A |f|\,d\mu$.

Next construct a sequence $\{g_n\}$ of $\mathscr{A}$-measurable simple functions for which the relations $|g_n(x)| = 1$ and $\lim_n g_n(x)f(x) = |f(x)|$ hold at each x in X (the details of the construction are left to the reader). Suppose that $a_{n,j}$, $j = 1, \ldots, k_n$, are the values of g_n and that these values are attained on the sets $A_{n,j}$, $j = 1, \ldots, k_n$. Then for an arbitrary set A in $\mathscr{A}$ we have

$$\begin{aligned} \left| \int_A g_n f\,d\mu \right| &= \left| \sum_j a_{n,j} \int_{A\cap A_{n,j}} f\,d\mu \right| \\ &= \left| \sum_j a_{n,j} \nu(A\cap A_{n,j}) \right| \le \sum_j |\nu(A\cap A_{n,j})| \le |\nu|(A). \end{aligned}$$

Since the dominated convergence theorem implies that $\lim_n \int_A g_n f\,d\mu = \int_A |f|\,d\mu$, it follows that $\int_A |f|\,d\mu \le |\nu|(A)$. Thus $|\nu|(A) = \int_A |f|\,d\mu$, and the proof is complete. □

Corollary 4.2.6. *Let ν be a finite signed or complex measure on the measurable space $(X,\mathscr{A})$. Then the Radon–Nikodym derivative of ν with respect to $|\nu|$ has absolute value 1 at $|\nu|$-almost every point in X.*

Proof. Proposition 4.2.5, applied in the case where $f = \frac{d\nu}{d|\nu|}$ and $\mu = |\nu|$, implies that

$$|\nu|(A) = \int_A \left|\frac{d\nu}{d|\nu|}\right| d|\nu|$$

holds for each A in $\mathscr{A}$. Thus $|\frac{d\nu}{d|\nu|}|$ is a Radon–Nikodym derivative of $|\nu|$ with respect to $|\nu|$. Since the constant 1 is another such Radon–Nikodym derivative, it follows that $|\frac{d\nu}{d|\nu|}| = 1$ almost everywhere. □

Recall that in Sect. 4.1 we used the formulas

$$\int f\,d\nu = \int f\,d\nu^+ - \int f\,d\nu^-$$

and

$$\int f\,d\nu = \int f\,d\nu_1 - \int f\,d\nu_2 + i\int f\,d\nu_3 - i\int f\,d\nu_4$$

to define the integral of a bounded $\mathscr{A}$-measurable function f with respect to a finite signed or complex measure ν. Let $\frac{d\nu}{d|\nu|}$ be a Radon–Nikodym derivative of ν with respect to $|\nu|$. Then the relation

$$\int f\,d\nu = \int f\frac{d\nu}{d|\nu|}\,d|\nu| \tag{4}$$

holds for each bounded $\mathscr{A}$-measurable function f on X; this is clear in case f is the characteristic function of an $\mathscr{A}$-measurable set and then follows in the general case from the linearity of the integral and the dominated convergence theorem.

Exercises

1. Define a measure ν on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ by $\nu(A) = \int_A |x|\,\lambda(dx)$. Show that $\nu \ll \lambda$, but that for no positive ε does there exist a positive δ such that $\nu(A) < \varepsilon$ holds whenever A is a Borel set for which $\lambda(A) < \delta$. Thus the assumption that ν is finite is essential in Lemma 4.2.1.
2. Let $\{r_n\}$ be an enumeration of the rational numbers, and for each positive integer n let $f_n\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative Borel function that satisfies $\int f_n\,d\lambda = 1$ and vanishes outside the closed interval of length $1/2^n$ centered at r_n. Define μ on $\mathscr{B}(\mathbb{R})$ by $\mu(A) = \int_A \sum_n f_n\,d\lambda$.
   (a) Show that $\sum_n f_n(x) < +\infty$ holds at λ-almost every x in $\mathbb{R}$. (Hint: See Exercise 1.2.9.)
   (b) Show that μ is σ-finite, that $\mu \ll \lambda$, and that each nonempty open subset of $\mathbb{R}$ has infinite measure under μ.

3. Suppose that μ and ν are σ-finite positive measures on $(X, \mathscr{A})$, that $\nu \ll \mu$, and that g is a Radon–Nikodym derivative of ν with respect to μ. Show that
   (a) an $\mathscr{A}$-measurable function $f\colon X \to \mathbb{R}$ is ν-integrable if and only if fg is μ-integrable, and
   (b) if those functions are integrable, then $\int f\,d\nu = \int fg\,d\mu$.
4. Suppose that ν_1, ν_2, and ν_3 are σ-finite positive measures on $(X, \mathscr{A})$, that $\nu_1 \ll \nu_2$, and that $\nu_2 \ll \nu_3$.
   (a) Show that $\nu_1 \ll \nu_3$.
   (b) Make precise and prove the assertion that
$$\frac{d\nu_1}{d\nu_3} = \frac{d\nu_1}{d\nu_2}\frac{d\nu_2}{d\nu_3}.$$
5. Let $(X, \mathscr{A})$ be a measurable space, let μ be a σ-finite positive measure on $(X, \mathscr{A})$, and let ν_1 and ν_2 be finite signed measures on $(X, \mathscr{A})$ that are absolutely continuous with respect to μ.
   (a) Show that $(\nu_1 \vee \nu_2) \ll \mu$ and $(\nu_1 \wedge \nu_2) \ll \mu$ (see Exercise 4.1.3).
   (b) Express the Radon–Nikodym derivatives (with respect to μ) of $\nu_1 \vee \nu_2$ and $\nu_1 \wedge \nu_2$ in terms of those of ν_1 and ν_2.
6. Show that the assumption that ν is σ-finite can be removed from Theorem 4.2.2 if g is allowed to have values in $[0, +\infty]$. (Hint: Reduce the general case to the case where μ is finite. For each positive integer n choose a Hahn decomposition (P_n, N_n) for $\nu - n\mu$; then consider the measures $A \mapsto \nu(A \cap (\cap_n P_n))$ and $A \mapsto \nu(A \cap (\cap_n P_n)^c)$.)
7. Let μ be a σ-finite positive measure on $(X, \mathscr{A})$.
   (a) Show that
$$\{\nu \in M(X, \mathscr{A}, \mathbb{R}) : \nu \ll \mu\}$$
   is a closed linear subspace of the normed linear space $M(X, \mathscr{A}, \mathbb{R})$.
   (b) Find an isometric isomorphism of $L^1(X, \mathscr{A}, \mu, \mathbb{R})$ onto the subspace of $M(X, \mathscr{A}, \mathbb{R})$ considered in part (a).
8. Let $(X, \mathscr{A})$ be a measurable space, let μ be a finite signed or complex measure on $(X, \mathscr{A})$, and let f be a bounded real- or complex-valued $\mathscr{A}$-measurable function on X. Show that $|\int f\,d\mu| \le \int |f|\,d|\mu|$.
9. Let μ and ν be σ-finite positive measures on $(X, \mathscr{A})$. Show that the conditions
   (i) $\nu \ll \mu$ and $\mu \ll \nu$,
   (ii) μ and ν have exactly the same sets of measure zero, and
   (iii) there is an $\mathscr{A}$-measurable function g that satisfies $0 < g(x) < +\infty$ at each x in X and is such that $\nu(A) = \int_A g\,d\mu$ holds for each A in $\mathscr{A}$

   are equivalent.

10. Show that if μ is a σ-finite measure on $(X,\mathscr{A})$, then there is a finite measure ν on $(X,\mathscr{A})$ such that $\nu \ll \mu$ and $\mu \ll \nu$. (Hint: See Exercise 9.)
11. Supply the missing details in the following proof of the Radon–Nikodym theorem for finite positive measures. Let $(X,\mathscr{A})$ be a measurable space, and let μ and ν be finite positive measures on $(X,\mathscr{A})$.

    (a) Show that the formula $F(\langle f\rangle) = \int f\,d\nu$ defines a bounded linear functional on $L^2(X,\mathscr{A},\mu+\nu,\mathbb{R})$.
    (b) Use Exercises 3.3.3 and 3.5.7 to obtain a function g in $\mathscr{L}^2(X,\mathscr{A},\mu+\nu,\mathbb{R})$ such that $F(\langle f\rangle) = \int fg\,d(\mu+\nu)$ holds for each f in $\mathscr{L}^2(X,\mathscr{A},\mu+\nu,\mathbb{R})$.
    (c) Show that if $\nu \ll \mu$, then the function g satisfies $0 \le g(x) < 1$ at $(\mu+\nu)$-almost every x in X and hence can be redefined so that $0 \le g(x) < 1$ holds at every x in X.
    (d) Show that if $\nu \ll \mu$ and if g has been redefined as in part (c), then $\nu(A) = \int_A g/(1-g)\,d\mu$ holds for each A in $\mathscr{A}$.

12. Let $(X,\mathscr{A},\mu)$ be a finite measure space, and let $\mathscr{F}$ be a subset of $\mathscr{L}^1(X,\mathscr{A},\mu)$. Then $\mathscr{F}$ is called *L^1-bounded* if the set $\{\|f\|_1 : f \in \mathscr{F}\}$ is bounded above, is called *uniformly absolutely continuous* if for each positive ε there is a positive δ such that $\int_A |f|\,d\mu < \varepsilon$ holds whenever $f \in \mathscr{F}$, $A \in \mathscr{A}$, and $\mu(A) < \delta$, and is called *uniformly integrable* if it is L^1-bounded and uniformly absolutely continuous. Show that $\mathscr{F}$ is uniformly integrable if and only if it satisfies
$$\lim_{a\to+\infty} \sup\left\{\int_{\{|f|>a\}} |f|\,d\mu : f \in \mathscr{F}\right\} = 0.$$
(Hint: Recall Proposition 2.3.10.)
13. Show that if $(X,\mathscr{A},\mu)$ is a finite measure space, then every finite subset of $\mathscr{L}^1(X,\mathscr{A},\mu)$ is uniformly integrable.
14. Let $(X,\mathscr{A},\mu)$ be a finite measure space, and let g be a nonnegative function that belongs to $\mathscr{L}^1(X,\mathscr{A},\mu)$. Show that if $\mathscr{F}$ is a collection of measurable functions such that $|f(x)| \le g(x)$ holds for each f in $\mathscr{F}$ and each x in X, then $\mathscr{F}$ is uniformly integrable.
15. Construct a finite measure space $(X,\mathscr{A},\mu)$ and a sequence $\{f_n\}$ of $\mathscr{A}$-measurable functions on X such that $\{f_n : n = 1, 2, \dots\}$ is uniformly integrable, but $\sup_n |f_n|$ is not integrable. (Compare this with Exercise 14.)
16. Let $(X,\mathscr{A},\mu)$ be a finite measure space, let $\{f_n\}$ be a sequence of functions in $\mathscr{L}^1(X,\mathscr{A},\mu)$, and let f be an $\mathscr{A}$-measurable real- or complex-valued function on X.

    (a) Show that if $\{f_n\}$ is uniformly integrable and if $\{f_n\}$ converges to f in measure, then f is integrable and $\int f\,d\mu = \lim_n \int f_n\,d\mu$. (Hint: Use Proposition 3.1.3, Theorem 2.4.4, and the inequality
$$\int |f_n - f|\,d\mu \le \int_A |f_n - f|\,d\mu + \int_{A^c} |f_n|\,d\mu + \int_{A^c} |f|\,d\mu.)$$

(b) Now suppose that f belongs to $\mathscr{L}^1(X,\mathscr{A},\mu)$. Show that $\{f_n\}$ converges to f in mean if and only if $\{f_n\}$ is uniformly integrable and converges to f in measure.

(c) Use part (a) to give another proof of the dominated convergence theorem in the case where μ is finite. (See Exercise 14.)

4.3 Singularity

Let $(X,\mathscr{A})$ be a measurable space. A positive measure μ on $(X,\mathscr{A})$ is *concentrated* on the $\mathscr{A}$-measurable set E if $\mu(E^c) = 0$. A signed or complex measure μ on $(X,\mathscr{A})$ is *concentrated* on the $\mathscr{A}$-measurable set E if the variation $|\mu|$ of μ is concentrated on E, or equivalently, if each $\mathscr{A}$-measurable subset A of E^c satisfies $\mu(A) = 0$ (see Exercise 4.1.1). Now suppose that μ and ν are positive, signed, or complex measures on $(X,\mathscr{A})$. Then μ and ν are *mutually singular* if there is an $\mathscr{A}$-measurable set E such that μ is concentrated on E and ν is concentrated on E^c. One sometimes writes $\mu \perp \nu$ to indicate that μ and ν are mutually singular. Instead of saying that μ and ν are mutually singular, one sometimes says that μ and ν are singular, that ν is singular with respect to μ, or that μ is singular with respect to ν. A positive, signed, or complex measure on $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d))$ is simply called *singular* if it is singular with respect to d-dimensional Lebesgue measure.

Examples 4.3.1.

(a) Let μ be a signed measure on the measurable set $(X,\mathscr{A})$. Then the positive and negative parts μ^+ and μ^- of μ are mutually singular; they are concentrated on the pair of disjoint sets appearing in a Hahn decomposition of μ.

(b) Next let us consider some measures on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ that are singular with respect to Lebesgue measure. If μ is a finite discrete measure on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$, then there is a countable subset C of $\mathbb{R}$ on which μ is concentrated; since Lebesgue measure is concentrated on the complement of C, μ is singular with respect to Lebesgue measure. However not every finite measure on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ that is singular with respect to Lebesgue measure is discrete; for example, the measure induced by the Cantor function (defined in Sect. 2.1) is singular with respect to Lebesgue measure but assigns measure zero to each point in $\mathbb{R}$ (see Exercise 2.1.7). □

Theorem 4.3.2 (Lebesgue Decomposition Theorem). *Let $(X,\mathscr{A})$ be a measurable space, let μ be a positive measure on $(X,\mathscr{A})$, and let ν be a finite signed, complex, or σ-finite positive measure on $(X,\mathscr{A})$. Then there are unique finite signed, complex, or positive measures ν_a and ν_s on $(X,\mathscr{A})$ such that*

(a) *ν_a is absolutely continuous with respect to μ,*
(b) *ν_s is singular with respect to μ, and*
(c) *$\nu = \nu_a + \nu_s$.*

The decomposition $\nu = \nu_a + \nu_s$ is called the *Lebesgue decomposition* of ν, while ν_a and ν_s are called the *absolutely continuous* and *singular parts* of ν.

Proof. We begin with the case in which ν is a finite positive measure. Define $\mathscr{N}_\mu$ by

$$\mathscr{N}_\mu = \{B \in \mathscr{A} : \mu(B) = 0\},$$

and choose a sequence $\{B_j\}$ of sets in $\mathscr{N}_\mu$ such that

$$\lim_j \nu(B_j) = \sup\{\nu(B) : B \in \mathscr{N}_\mu\}.$$

Let $N = \cup_j B_j$, and define measures ν_a and ν_s on $(X, \mathscr{A})$ by $\nu_a(A) = \nu(A \cap N^c)$ and $\nu_s(A) = \nu(A \cap N)$. Of course $\nu = \nu_a + \nu_s$. The countable subadditivity of μ implies that $\mu(N) = 0$ and hence that ν_s is singular with respect to μ. Since

$$\nu(N) = \sup\{\nu(B) : B \in \mathscr{N}_\mu\},$$

each $\mathscr{A}$-measurable subset B of N^c that satisfies $\mu(B) = 0$ also satisfies $\nu(B) = 0$ (otherwise $N \cup B$ would belong to $\mathscr{N}_\mu$ and satisfy $\nu(N \cup B) > \nu(N)$). The absolute continuity of ν_a follows.

In case ν is a finite signed or complex measure, we can apply the preceding construction to the finite positive measure $|\nu|$, obtaining a μ-null set N such that the Lebesgue decomposition of $|\nu|$ is given by $|\nu|_a(A) = |\nu|(A \cap N^c)$ and $|\nu|_s(A) = |\nu|(A \cap N)$. It is easy to check that the signed or complex measures ν_a and ν_s defined by $\nu_a(A) = \nu(A \cap N^c)$ and $\nu_s(A) = \nu(A \cap N)$ form a Lebesgue decomposition of ν.

Now suppose that ν is a σ-finite positive measure, and let $\{D_k\}$ be a partition of X into $\mathscr{A}$-measurable sets that have finite measure under ν. For each k let $\mathscr{A}_k$ be the σ-algebra on D_k that consists of the $\mathscr{A}$-measurable subsets of D_k, and apply the construction above to the restrictions of the measures μ and ν to the spaces $(D_k, \mathscr{A}_k)$. Let N_1, N_2, ... be the μ-null subsets of D_1, D_2, ... thus constructed, and let $N = \cup_k N_k$. Then the measures ν_a and ν_s defined by $\nu_a(A) = \nu(A \cap N^c)$ and $\nu_s(A) = \nu(A \cap N)$ form a Lebesgue decomposition of ν.

We turn to the uniqueness of the Lebesgue decomposition. Let $\nu = \nu_a + \nu_s$ and $\nu = \nu'_a + \nu'_s$ be Lebesgue decompositions of ν. First suppose that ν is a finite signed, complex, or finite positive measure. Then

$$\nu_a - \nu'_a = \nu'_s - \nu_s,$$

and since $(\nu_a - \nu'_a) \ll \mu$ and $(\nu'_s - \nu_s) \perp \mu$, it follows that

$$\nu_a - \nu'_a = \nu'_s - \nu_s = 0$$

(see Exercise 1). Thus $\nu_a = \nu'_a$ and $\nu_s = \nu'_s$. The case where ν is a σ-finite positive measure can be dealt with by choosing a partition $\{D_k\}$ of X into $\mathscr{A}$-measurable subsets that have finite measure under ν, and applying the preceding argument to the restrictions of ν_a, ν_s, ν'_a, and ν'_s to the $\mathscr{A}$-measurable subsets of the sets D_k. $\square$

See Exercise 6 for another proof of the uniqueness of the Lebesgue decomposition.

One sometimes goes a step further for a finite measure ν on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Let $C = \{x \in \mathbb{R} : \nu(\{x\}) \neq 0\}$, and note that C is countable (for each positive integer n, there are only finitely many points x such that $\nu(\{x\}) \geq 1/n$). Let ν_1 be the measure on $\mathscr{B}(\mathbb{R})$ defined by $\nu_1(A) = \nu(A \cap C)$, and let ν_2 and ν_3 be the singular and absolutely continuous (with respect to Lebesgue measure) parts of the measure $A \mapsto \nu(A \cap C^c)$. Then $\nu = \nu_1 + \nu_2 + \nu_3$ is a decomposition of ν into the sum of a discrete measure, a continuous but singular measure, and an absolutely continuous measure. It is easy to check that the measures appearing in this decomposition are unique.

Exercises

1. Let μ be a positive measure on $(X, \mathscr{A})$, and let ν be a positive, signed, or complex measure on $(X, \mathscr{A})$. Show that if $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$. (Hint: Use the definitions of absolute continuity and of singularity.)
2. Let μ be a positive measure on $(X, \mathscr{A})$. Show that
$$\{\nu \in M(X, \mathscr{A}, \mathbb{R}) : \nu \perp \mu\}$$
is a closed linear subspace of the normed linear space $M(X, \mathscr{A}, \mathbb{R})$.
3. Let μ be a positive measure on $(X, \mathscr{A})$, let ν be a finite signed or complex measure on $(X, \mathscr{A})$, and let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν. Show that $\|\nu\| = \|\nu_a\| + \|\nu_s\|$.
4. Let μ and ν be positive measures on $(X, \mathscr{A})$ such that for each positive ε there is a set A in $\mathscr{A}$ that satisfies $\mu(A) < \varepsilon$ and $\nu(A^c) < \varepsilon$. Show that $\mu \perp \nu$. (Hint: Choose sets $A_1, A_2, \ldots$ in such a way that the set A defined by $A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ satisfies $\mu(A) = 0$ and $\nu(A^c) = 0$.)
5. Show by example that in the Lebesgue decomposition theorem, we cannot allow ν to be an arbitrary positive measure. (Hint: Let $(X, \mathscr{A}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$, let μ be Lebesgue measure on $(X, \mathscr{A})$, and let ν be counting measure on $(X, \mathscr{A})$.)
6. (a) Let μ and ν be as in Theorem 4.3.2, let $\nu = \nu_a + \nu_s$ be a Lebesgue decomposition of ν, and suppose that ν_s is concentrated on the μ-null set N. Show that each A in $\mathscr{A}$ satisfies $\nu_s(A) = \nu(A \cap N)$ and $\nu_a(A) = \nu(A \cap N^c)$.
   (b) Use part (a) to give another proof of the uniqueness assertion in Theorem 4.3.2.
7. (Continuation of Exercise 4.1.3.) Let μ and ν be finite positive measures on $(X, \mathscr{A})$. Show that the conditions

   (i) $\mu \perp \nu$,
   (ii) $\mu \wedge \nu = 0$, and
   (iii) $\mu \vee \nu = \mu + \nu$

   are equivalent.

4.4 Functions of Finite Variation

In Sect. 1.3 we constructed a bijection between the set of all finite positive measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and the set of all bounded nondecreasing right-continuous functions $F : \mathbb{R} \to \mathbb{R}$ that vanish at $-\infty$.[4] In this section we will extend this correspondence to a bijection between the set of all finite signed measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and a certain set of real-valued functions on $\mathbb{R}$, and we will use this bijection to give a classical characterization of those finite signed measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ that are absolutely continuous with respect to Lebesgue measure.

Suppose that F is a real-valued function whose domain includes the interval $[a,b]$. Let $\mathscr{S}$ be the collection of finite sequences $\{t_i\}_{i=0}^n$ such that

$$a \le t_0 < t_1 < \cdots < t_n \le b.$$

Then $V_F[a,b]$, the *variation of F over $[a,b]$*, is defined by

$$V_F[a,b] = \sup\left\{ \sum_i |F(t_i) - F(t_{i-1})| : \{t_i\} \in \mathscr{S} \right\}.$$

The function F is *of finite variation* (or *of bounded variation*) *on* $[a,b]$ if $V_F[a,b]$ is finite.

The *variation of F over* the interval $(-\infty, b]$ and the *variation of F over* $\mathbb{R}$, written $V_F(-\infty, b]$ and $V_F(-\infty, +\infty)$, respectively, are defined in a similar way, now using finite sequences whose members belong to $(-\infty, b]$ or to $(-\infty, +\infty)$. Of course, F is said to be *of finite variation on* $(-\infty, b]$ if $V_F(-\infty, b]$ is finite, and to be *of finite variation* if $V_F(-\infty, +\infty)$ is finite. If $F : \mathbb{R} \to \mathbb{R}$ is of finite variation, then the *variation* of F is the function $V_F : \mathbb{R} \to \mathbb{R}$ defined by $V_F(x) = V_F(-\infty, x]$.

Suppose that μ is a finite signed measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Define a function $F_\mu : \mathbb{R} \to \mathbb{R}$ by letting

$$F_\mu(x) = \mu((-\infty, x]) \tag{1}$$

hold at each x in $\mathbb{R}$. If $\{t_i\}_{i=0}^n$ is an increasing sequence of real numbers, then

$$\sum_{i=1}^n |F_\mu(t_i) - F_\mu(t_{i-1})| = \sum_{i=1}^n |\mu((t_{i-1}, t_i])| \le |\mu|(\mathbb{R});$$

it follows that $V_{F_\mu}(-\infty, +\infty) \le |\mu|(\mathbb{R})$ and hence that F_μ is of finite variation. It is easy to check that F_μ vanishes at $-\infty$ and is right-continuous (use Proposition 1.3.9 and the Jordan decomposition of μ). We will soon see that every right-continuous function of finite variation that vanishes at $-\infty$ arises from a finite signed measure in this way.

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[4] Recall that a function $F : \mathbb{R} \to \mathbb{R}$ is said to *vanish at* $-\infty$ if $\lim_{x \to -\infty} F(x) = 0$.

It is easy to check that the function F_μ defined by (1) is continuous if and only if $\mu(\{x\}) = 0$ holds for each x in $\mathbb{R}$. In this case

$$\mu((a,b)) = \mu([a,b]) = \mu([a,b)) = \mu((a,b]) = F_\mu(b) - F_\mu(a)$$

holds whenever $a < b$.

Let us turn to some general properties of functions of finite variation.

Suppose that $F\colon \mathbb{R} \to \mathbb{R}$ is of finite variation. It is easy to check that F is bounded and that if $-\infty < a < b < +\infty$, then

$$V_F(-\infty,b] = V_F(-\infty,a] + V_F[a,b]. \tag{2}$$

Furthermore, if $b \in \mathbb{R}$, then

$$V_F(-\infty,b] = \lim_{a\to-\infty} V_F[a,b]; \tag{3}$$

to prove this, let ε be a positive number, choose an increasing sequence $\{t_i\}_{i=0}^n$ of numbers that belong to $(-\infty,b]$ and satisfy

$$\sum_{i=1}^n |F(t_i) - F(t_{i-1})| > V_F(-\infty,b] - \varepsilon,$$

and note that for each a that satisfies $a \le t_0$ we have

$$V_F(-\infty,b] - \varepsilon < V_F[a,b] \le V_F(-\infty,b].$$

A similar argument shows that if $a < c$ and if F is right-continuous at a, then

$$V_F[a,c] = \lim_{b\to a^+} V_F[b,c]. \tag{4}$$

Lemma 4.4.1. *Let F be a function of finite variation on $\mathbb{R}$. Then*

(a) *V_F is bounded and nondecreasing,*
(b) *V_F vanishes at $-\infty$, and*
(c) *if F is right-continuous, then V_F is right-continuous.*

Proof. Part (a) is clear. Equations (2) and (3) justify the calculation

$$\begin{aligned}\lim_{x\to-\infty} V_F(x) &= \lim_{x\to-\infty} V_F(-\infty,x] \\ &= \lim_{x\to-\infty} (V_F(-\infty,b] - V_F[x,b]) \\ &= V_F(-\infty,b] - V_F(-\infty,b] = 0,\end{aligned}$$

and so part (b) is proved. A similar argument, using Eqs. (2) and (4), yields part (c). □

Proposition 4.4.2. *Let F be a function of finite variation on $\mathbb{R}$. Then there are bounded nondecreasing functions F_1 and F_2 such that $F = F_1 - F_2$.*

Proof. It is easy to check that the functions defined by $F_1 = (V_F + F)/2$ and $F_2 = (V_F - F)/2$ have the required properties. □

Let $F: \mathbb{R} \to \mathbb{R}$ be of finite variation, and let F_1 and F_2 be the functions constructed in the proof of Proposition 4.4.2. Lemma 4.4.1 implies that if F is right-continuous, then F_1 and F_2 are right-continuous, and that if F vanishes at $-\infty$, then F_1 and F_2 vanish at $-\infty$.

Proposition 4.4.3. *Equation* (1) *defines a bijection $\mu \mapsto F_\mu$ between the set of all finite signed measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and the set of all right-continuous functions of finite variation that vanish at $-\infty$.*

Proof. We have already checked that F_μ is a right-continuous function of finite variation that vanishes at $-\infty$. If μ and ν are finite signed measures such that $F_\mu = F_\nu$ and if $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ are their Jordan decompositions, then $F_{\mu^+} - F_{\mu^-} = F_{\nu^+} - F_{\nu^-}$; since this implies that $F_{\mu^+} + F_{\nu^-} = F_{\nu^+} + F_{\mu^-}$, it follows from Proposition 1.3.10 that $\mu^+ + \nu^- = \nu^+ + \mu^-$ and hence that $\mu = \nu$. The injectivity of the map $\mu \mapsto F_\mu$ follows. The surjectivity follows from Proposition 1.3.10, Proposition 4.4.2, and the remarks following the proof of Proposition 4.4.2. □

A function $F: \mathbb{R} \to \mathbb{R}$ is *absolutely continuous* if for each positive number ε there is a positive number δ such that $\sum_i |F(t_i) - F(s_i)| < \varepsilon$ holds whenever $\{(s_i, t_i)\}$ is a finite sequence of disjoint open intervals for which $\sum_i (t_i - s_i) < \delta$.

It is clear that every absolutely continuous function is continuous and, in fact, uniformly continuous. There are, however, functions that are uniformly continuous and of finite variation, but are not absolutely continuous (see Exercise 3). It is easy to check that an absolutely continuous function is of finite variation on each closed bounded interval (see Exercise 5), but is not necessarily of finite variation on $\mathbb{R}$ (consider the function F defined by $F(x) = x$).

We turn to the relationship between absolute continuity for signed measures and absolute continuity for functions of a real variable.

Lemma 4.4.4. *If $F: \mathbb{R} \to \mathbb{R}$ is absolutely continuous and of finite variation, then V_F is absolutely continuous.*

Proof. Let ε be a positive number, and use the absolute continuity of F to choose a corresponding δ. If $\{(s_i, t_i)\}$ is a finite sequence of disjoint open intervals such that $\sum_i (t_i - s_i) < \delta$, then each finite sequence $\{(u_j, v_j)\}$ of disjoint open subintervals of $\cup_i (s_i, t_i)$ satisfies $\sum (v_j - u_j) < \delta$ and so satisfies $\sum_j |F(v_j) - F(u_j)| < \varepsilon$. Since the sequence $\{(u_j, v_j)\}$ can be chosen so as to make $\sum_j |F(v_j) - F(u_j)|$ arbitrarily close to $\sum_i V_F[s_i, t_i]$, we have

$$\sum_i |V_F(t_i) - V_F(s_i)| = \sum_i V_F[s_i, t_i] \le \varepsilon.$$

The absolute continuity of V_F follows. □

Proposition 4.4.5. *Let μ be a finite signed measure on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$, and let $F_\mu : \mathbb{R} \to \mathbb{R}$ be defined by (1). Then F_μ is absolutely continuous if and only if μ is absolutely continuous with respect to Lebesgue measure.*

Proof. First suppose that μ is absolutely continuous with respect to Lebesgue measure. Let ε be a positive number, and use Lemma 4.2.1 to choose a positive number δ such that $|\mu|(A) < \varepsilon$ holds whenever A is a Borel set that satisfies $\lambda(A) < \delta$. If $\{(s_i,t_i)\}$ is a finite sequence of disjoint open intervals such that $\sum_i(t_i - s_i) < \delta$, then $\lambda(\cup_i(s_i,t_i]) < \delta$, and so

$$\sum_i |F_\mu(t_i) - F_\mu(s_i)| = \sum_i |\mu((s_i,t_i])| \le |\mu|\Big(\bigcup_i (s_i,t_i]\Big) < \varepsilon.$$

Hence F_μ is absolutely continuous.

Now suppose that F_μ is absolutely continuous. Then V_{F_μ} is absolutely continuous (Lemma 4.4.4), and so the functions F_1 and F_2 defined by $F_1 = (V_{F_\mu} + F_\mu)/2$ and $F_2 = (V_{F_\mu} - F_\mu)/2$ are absolutely continuous. Let μ_1 and μ_2 be the finite positive measures on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ that correspond to F_1 and F_2. Since $F_\mu = F_1 - F_2$, it follows (Proposition 4.4.3) that $\mu = \mu_1 - \mu_2$; thus we need only show that $\mu_1 \ll \lambda$ and $\mu_2 \ll \lambda$. Let ε be a positive number, and let δ be a positive number such that

$$\begin{aligned}&\sum_i |F_1(t_i) - F_1(s_i)| < \varepsilon \text{ holds whenever } \{(s_i,t_i)\} \text{ is a finite}\\ &\text{sequence of disjoint open intervals such that } \sum_i (t_i - s_i) < \delta.\end{aligned} \tag{5}$$

Suppose that A is a Borel subset of $\mathbb{R}$ such that $\lambda(A) < \delta$, and use the regularity of Lebesgue measure to choose an open set U that includes A and satisfies $\lambda(U) < \delta$. Then U is the union of a sequence $\{(s_i,t_i)\}$ of disjoint open intervals (see Proposition C.4), and it follows from (5) that

$$\mu_1\Big(\bigcup_{i=1}^n (s_i,t_i)\Big) = \sum_{i=1}^n (F_1(t_i) - F_1(s_i)) < \varepsilon$$

holds for each n. Hence $\mu_1(U) = \mu_1(\cup_{i=1}^\infty (s_i,t_i)) \le \varepsilon$ (see Proposition 1.2.5), and so $\mu_1(A) \le \varepsilon$. The absolute continuity of μ_1 now follows from Lemma 4.2.1. The case of μ_2 is similar, and so the proof is complete. □

Proposition 4.4.6. *The functions $F : \mathbb{R} \to \mathbb{R}$ that can be written in the form*

$$F(x) = \int_{-\infty}^x f(t)\,dt \tag{6}$$

for some f in $\mathscr{L}^1(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda,\mathbb{R})$ are exactly the absolutely continuous functions of finite variation that vanish at $-\infty$.

Proof. First suppose that f belongs to $\mathscr{L}^1(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda,\mathbb{R})$ and that F arises from f through (6). The signed measure μ defined by $\mu(A) = \int_A f\,d\lambda$ is absolutely continuous with respect to λ, and $F = F_\mu$; hence it follows from Propositions 4.4.3 and 4.4.5 that F is of finite variation, is absolutely continuous, and vanishes at $-\infty$.

Now suppose that $F: \mathbb{R} \to \mathbb{R}$ is of finite variation, is absolutely continuous, and vanishes at $-\infty$. Proposition 4.4.3 implies that there is a finite signed measure μ such that $F = F_\mu$, and Proposition 4.4.5 implies that $\mu \ll \lambda$. If $f = \frac{d\mu}{d\lambda}$, then (6) holds at each x in $\mathbb{R}$. □

The study of absolute continuity for functions of a real variable, and in particular of Eq. (6), will be continued in Sect. 6.3.

Exercises

1. Suppose that $F: \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x \sin \frac{1}{x} & \text{if } x > 0. \end{cases}$$

   Find the closed bounded intervals $[a,b]$ for which $V_F[a,b]$ is finite.
2. Show that if $F: \mathbb{R} \to \mathbb{R}$ is of finite variation, then the limits $\lim_{x \to -\infty} F(x)$ and $\lim_{x \to +\infty} F(x)$ exist.
3. Let F be the Cantor function, extended so as to vanish on the interval $(-\infty, 0)$ and to have value 1 on the interval $(1, +\infty)$. Show directly (i.e., without using Proposition 4.4.5) that F is uniformly continuous but not absolutely continuous.
4. Show that if $F: \mathbb{R} \to \mathbb{R}$ is continuous and of finite variation, then $V_F: \mathbb{R} \to \mathbb{R}$ is continuous.
5. Show that if $F: \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then F is of finite variation on each closed bounded interval. (Hint: Let δ be a positive number such that $\sum_i |F(t_i) - F(s_i)| < 1$ holds whenever $\{(s_i, t_i)\}$ is a finite sequence of disjoint open intervals such that $\sum_i (t_i - s_i) < \delta$, and let $[a,b]$ be a closed bounded interval. Show that if $\{u_i\}_{i=0}^n$ is a finite sequence such that

$$a \leq u_0 < u_1 < \ldots < u_n \leq b,$$

   then $\sum_{i=1}^n |F(u_i) - F(u_{i-1})| \leq (b-a)/\delta + 1$.)
6. Let μ be a finite signed measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Show that $V_{F_\mu}(-\infty, x] = |\mu|((-\infty, x])$ holds at each x in $\mathbb{R}$.

4.5 The Duals of the L^p Spaces

We return to the study, which we began in Sect. 3.5, of the duals of the L^p spaces. Let $(X, \mathscr{A}, \mu)$ be a measure space, let p satisfy $1 \leq p < +\infty$, and let q be defined by $1/p + 1/q = 1$. Recall that if f belongs to $\mathscr{L}^p(X, \mathscr{A}, \mu)$ (or to $\mathscr{L}^q(X, \mathscr{A}, \mu)$), then $\langle f \rangle$ is the coset in $L^p(X, \mathscr{A}, \mu)$ (or in $L^q(X, \mathscr{A}, \mu)$) to which f belongs. We have seen that each $\langle g \rangle$ in $L^q(X, \mathscr{A}, \mu)$ induces a bounded linear functional $T_{\langle g \rangle}$ on $L^p(X, \mathscr{A}, \mu)$ by means of the formula $T_{\langle g \rangle}(\langle f \rangle) = \int fg\,d\mu$ and that the operator T that takes $\langle g \rangle$ to $T_{\langle g \rangle}$ is an isometry of $L^q(X, \mathscr{A}, \mu)$ into $(L^p(X, \mathscr{A}, \mu))^*$ (Proposition 3.5.5). We now use the Radon–Nikodym theorem to show that in many situations the operator T is surjective and hence is an isometric isomorphism.

Theorem 4.5.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let p satisfy $1 \le p < +\infty$, and let q be defined by $1/p + 1/q = 1$. If $p = 1$ and μ is σ-finite, or if $1 < p < +\infty$ and μ is arbitrary, then the operator T defined above is an isometric isomorphism of $L^q(X, \mathscr{A}, \mu)$ onto $(L^p(X, \mathscr{A}, \mu))^*$.*

Proof. Since we know that T is an isometry (Proposition 3.5.5), we need only show that it is surjective.

Let F be an arbitrary element of $(L^p(X, \mathscr{A}, \mu))^*$. First suppose that $\mu(X) < +\infty$ and that p satisfies $1 \le p < +\infty$. We define a function ν on the σ-algebra $\mathscr{A}$ by means of the formula $\nu(A) = F(\langle \chi_A \rangle)$. If $\{A_k\}$ is a sequence of disjoint sets in $\mathscr{A}$ and if $A = \cup_k A_k$, then the dominated convergence theorem implies that $\lim_n \|\chi_A - \sum_{k=1}^n \chi_{A_k}\|_p = 0$; since F is continuous and linear, this implies that $F(\langle \chi_A \rangle) = \sum_k F(\langle \chi_{A_k} \rangle)$ and hence that $\nu(A) = \sum_k \nu(A_k)$. Thus ν is countably additive and so is a finite signed or complex measure. It is clear that ν is absolutely continuous with respect to μ. Hence the Radon–Nikodym theorem (Theorem 4.2.4) provides a function g in $\mathscr{L}^1(X, \mathscr{A}, \mu)$ that satisfies $\nu(A) = \int_A g\,d\mu$ for each A in $\mathscr{A}$. We will show that g belongs to $\mathscr{L}^q(X, \mathscr{A}, \mu)$ and that $F(\langle f \rangle) = \int fg\,d\mu$ holds for each f in $\mathscr{L}^p(X, \mathscr{A}, \mu)$.

For each positive integer n let $E_n = \{x \in X : |g(x)| \le n\}$. Then $g\chi_{E_n}$ is bounded and so belongs to $\mathscr{L}^q(X, \mathscr{A}, \mu)$ (recall that μ is finite). Define a functional F_{E_n} on $L^p(X, \mathscr{A}, \mu)$ by $F_{E_n}(\langle f \rangle) = F(\langle f\chi_{E_n} \rangle)$. Consider the relation

$$F_{E_n}(\langle f \rangle) = \int fg\chi_{E_n}\,d\mu. \tag{1}$$

If f is the characteristic function of an $\mathscr{A}$-measurable set A, then both sides of (1) are equal to $\nu(A \cap E_n)$); thus (1) holds if f is the characteristic function of an $\mathscr{A}$-measurable set and hence if f is an $\mathscr{A}$-measurable simple function. Since the $\mathscr{A}$-measurable simple functions determine a dense subspace of $L^p(X, \mathscr{A}, \mu)$ (Proposition 3.4.2), Eq. (1) holds for all $\langle f \rangle$ in $L^p(X, \mathscr{A}, \mu)$. It follows from Proposition 3.5.5 that

$$\|g\chi_{E_n}\|_q = \|F_{E_n}\| \le \|F\|.$$

If $q < +\infty$, then the monotone convergence theorem implies that $g \in \mathscr{L}^q(X, \mathscr{A}, \mu)$ and $\|g\|_q \le \|F\|$. If $q = +\infty$, then (since $E = \cup_n E_n$) we have

$$\mu(\{x \in X : |g(x)| > \|F\|\}) = \lim_n \mu(\{x \in E_n : |g(x)| > \|F\|\}) = 0,$$

and we can redefine g so that it will be bounded, in fact satisfying $|g(x)| \le \|F\|$ at every x in X. Thus $\|g\|_q \le \|F\|$, whether q is finite or infinite. Furthermore, in both cases we can take limits in (1) as n approaches infinity and conclude that $F(\langle f \rangle) = \int fg\,d\mu$. With this the theorem is proved in the case of finite measures.

We need some notation in order to deal with the case where μ is not finite. Suppose that B belongs to $\mathscr{A}$. Let $\mathscr{A}_B$ be the σ-algebra on B consisting of those subsets of B that belong to $\mathscr{A}$, and let μ_B be the restriction of μ to $\mathscr{A}_B$. If f is a real- or complex-valued function on B, then we will denote by f' the function on X that agrees with f on B and vanishes outside B. The formula $F_B(\langle f \rangle) = F(\langle f' \rangle)$ defines a linear functional F_B on $L^p(B, \mathscr{A}_B, \mu_B)$; this functional satisfies $\|F_B\| \le \|F\|$.

Now suppose that μ is σ-finite and that p satisfies $1 \le p < +\infty$. Let $\{B_k\}$ be a sequence of disjoint sets that belong to $\mathscr{A}$, have finite measure under μ, and satisfy $X = \cup_k B_k$. According to the first part of this proof there is for each k a function g_k in $\mathscr{L}^q(B_k, \mathscr{A}_{B_k}, \mu_{B_k})$ that represents F_{B_k} on $L^p(B_k, \mathscr{A}_{B_k}, \mu_{B_k})$ and satisfies $\|g_k\|_q \le \|F_{B_k}\|$. Define g on X so that it agrees on each B_k with g_k. It is not difficult to check (do so) that $g \in \mathscr{L}^q(X, \mathscr{A}, \mu)$ and that

$$F(\langle f \rangle) = \int fg\,d\mu$$

holds for each $\langle f \rangle$ in $L^p(X, \mathscr{A}, \mu)$.

Finally we turn to the case where μ is arbitrary. Now we assume that $1 < p < +\infty$ and hence that $1 < q < +\infty$. Let $\mathscr{S}$ be the collection of sets in $\mathscr{A}$ that are σ-finite under μ. Note that if $B \in \mathscr{S}$, then $(B, \mathscr{A}_B, \mu_B)$ is σ-finite, and so by what we have just proved, there is a function g in $\mathscr{L}^q(B, \mathscr{A}_B, \mu_B)$ such that

$$F_B(\langle f \rangle) = \int fg\,d\mu_B$$

holds for each $\langle f \rangle$ in $L^p(B, \mathscr{A}_B, \mu_B)$. Furthermore if B_1 and B_2 are disjoint sets in $\mathscr{S}$, then

$$\|F_{B_1 \cup B_2}\|^q = \|F_{B_1}\|^q + \|F_{B_2}\|^q; \tag{2}$$

to prove this, choose a function g in $\mathscr{L}^q(B_1 \cup B_2, \mathscr{A}_{B_1 \cup B_2}, \mu_{B_1 \cup B_2})$ that represents $F_{B_1 \cup B_2}$, and note that

$$\begin{aligned}\|F_{B_1 \cup B_2}\|^q &= \int_{B_1 \cup B_2} |g|^q\,d\mu_{B_1 \cup B_2} \\ &= \int_{B_1} |g|^q\,d\mu_{B_1} + \int_{B_2} |g|^q\,d\mu_{B_2} = \|F_{B_1}\|^q + \|F_{B_2}\|^q.\end{aligned}$$

Now choose a sequence $\{C_n\}$ of sets in $\mathscr{S}$ such that

$$\lim_n \|F_{C_n}\| = \sup\{\|F_B\| : B \in \mathscr{S}\}.$$

Let $C = \cup_n C_n$. Then $C \in \mathscr{S}$,

$$\|F_C\| = \sup\{\|F_B\| : B \in \mathscr{S}\}, \tag{3}$$

and we can choose a function g_C in $\mathscr{L}^q(C,\mathscr{A}_C,\mu_C)$ such that

$$F_C(\langle f\rangle) = \int f g_C \, d\mu_C \tag{4}$$

holds for each $\langle f\rangle$ in $L^p(C,\mathscr{A}_C,\mu_C)$. Note that if f belongs to $\mathscr{L}^p(X,\mathscr{A},\mu)$ and vanishes on C, then $F(\langle f\rangle) = 0$ (otherwise, if $D = \{x \in X : f(x) \neq 0\}$, then D would belong to $\mathscr{S}$ (Corollary 2.3.11) and would satisfy $F_D \neq 0$, and so in view of (2), $F_{C\cup D}$ would satisfy

$$\|F_{C\cup D}\|^q = \|F_C\|^q + \|F_D\|^q > \|F_C\|^q,$$

contradicting (3)). It follows from this and (4) that if g is the function on X that agrees with g_C on C and vanishes off C, then $g \in \mathscr{L}^q(X,\mathscr{A},\mu)$ and

$$F(\langle f\rangle) = \int f g \, d\mu$$

holds for each $\langle f\rangle$ in $L^p(X,\mathscr{A},\mu)$ (decompose f into the sum of a function that vanishes on C and a function that vanishes on C^c). Hence $F = T_{\langle g\rangle}$ and the proof of the surjectivity of T is complete. □

Example 4.5.2. Let us consider an example that shows that the hypothesis of σ-finiteness cannot simply be omitted in Theorem 4.5.1 (see, however, Theorems 7.5.4 and 9.4.8). Let $X = \mathbb{R}$, let $\mathscr{A}$ be the σ-algebra consisting of those subsets A of $\mathbb{R}$ such that A or A^c is countable, and let μ be counting measure on $(X,\mathscr{A})$. Then $\mathscr{L}^1(X,\mathscr{A},\mu)$ consists of those functions f on $\mathbb{R}$ that vanish outside a countable set and satisfy $\sum_x |f(x)| < +\infty$, and for such functions we have $\|f\|_1 = \sum_x |f(x)|$. Define a functional F on $L^1(X,\mathscr{A},\mu)$ by $F(\langle f\rangle) = \sum_{x>0} f(x)$. Then F is continuous, and if g is a function that satisfies $F(\langle f\rangle) = \int fg\,d\mu$ for each f in $\mathscr{L}^1(X,\mathscr{A},\mu)$, then g must be the characteristic function of the interval $(0,+\infty)$. However this function is not $\mathscr{A}$-measurable, and so the functional F is induced by no function in $\mathscr{L}^\infty(X,\mathscr{A},\mu)$. □

Exercises

1. Let V be a normed linear space, and let v and v_1, v_2, ... belong to V. The sequence $\{v_n\}$ is said to *converge weakly* to v if $F(v) = \lim_n F(v_n)$ holds for each F in V^*.
   (a) Show that if $\{v_n\}$ converges to v in norm (that is, if $\lim_n \|v_n - v\| = 0$), then $\{v_n\}$ converges weakly to v.
   (b) Does the converse of part (a) hold if $V = L^2(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda)$?

2. Let $(X, \mathscr{A}, \mu)$ be a measure space. Show that the formula $T_{\langle g \rangle}(\langle f \rangle) = \int fg\,d\mu$ defines an isometry T of $L^1(X, \mathscr{A}, \mu)$ into $(L^\infty(X, \mathscr{A}, \mu))^*$. (Thus we could have allowed p to be $+\infty$ in Proposition 3.5.5. See, however, the following exercise.)
3. (This exercise depends on Exercise 3.5.8, and hence on the Hahn–Banach theorem.) Let $(X, \mathscr{A}, \mu)$ be a finite measure space. Show that the conditions

    (i) the map T in Exercise 2 is surjective,
    (ii) $L^1(X, \mathscr{A}, \mu)$ is finite dimensional,
    (iii) $L^\infty(X, \mathscr{A}, \mu)$ is finite dimensional, and
    (iv) there is a finite σ-algebra $\mathscr{A}_0$ on X such that $\mathscr{A}_0 \subseteq \mathscr{A}$ and such that each set in $\mathscr{A}$ differs from a set in $\mathscr{A}_0$ by a μ-null set

    are equivalent. (Hint: To show that (i) implies (iv), assume that (iv) fails and use ideas from Exercise 3.5.8 to show that (i) fails.)

Notes

The basic facts about absolute continuity and singularity of measures are contained in essentially all books on measure and integration, while the results given in the last part of Sect. 4.1 and in Sect. 4.4 are sometimes omitted. See Chap. 10, on probability, for applications of most of these results.

The proof of the Radon–Nikodym theorem outlined in Exercise 4.2.11 is due to von Neumann (see [120, pp. 124–131]).