Chapter 7
Measures on Locally Compact Spaces

Let $\mathscr{K}(\mathbb{R})$ be the vector space consisting of those continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that have compact support—that is, for which the set $\{x \in \mathbb{R} : f(x) \neq 0\}$ has a compact closure. Then $f \mapsto \int_{\mathbb{R}} f\,d\lambda$ defines a positive[1] linear functional on $\mathscr{K}(\mathbb{R})$. More generally, if μ is a measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ that has finite values on the compact subsets of $\mathbb{R}$, then $f \mapsto \int_{\mathbb{R}} f\,d\mu$ defines a positive linear functional on $\mathscr{K}(\mathbb{R})$. According to a special case of the Riesz representation theorem (see Theorem 7.2.8), the converse also holds: for every positive linear functional $I\colon \mathscr{K}(\mathbb{R}) \to \mathbb{R}$ there is a Borel measure μ on $\mathbb{R}$ that is finite on compact sets and represents I, in the sense that $I(f) = \int_{\mathbb{R}} f\,d\mu$ holds for each f in $\mathscr{K}(\mathbb{R})$.

This chapter is devoted to the Riesz representation theorem and related results. The first section (Sect. 7.1) contains some basic facts about locally compact Hausdorff spaces, the spaces that provide the natural setting for the Riesz representation theorem, while the second section (Sect. 7.2) gives a proof of the Riesz representation theorem. The next two sections (Sects. 7.3 and 7.4) contain some useful and relatively basic related material. The results of Sects. 7.5 and 7.6 are needed for dealing with large locally compact Hausdorff spaces; for relatively small locally compact Hausdorff spaces (those that have a countable base), Proposition 7.6.2 is the only result from these sections that one really needs (see also Proposition 7.2.5 and Theorems 4.5.1 and 5.2.2).

The Daniell–Stone integral gives another way to deal with integration on locally compact Hausdorff spaces. A measure-theoretic version of the basic result of the Daniell–Stone theory is given by Theorem 7.7.4; the general Daniell–Stone setup is outlined in the exercises at the end of Sect. 7.7.

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[1]Recall that a linear functional I on a vector space of functions is *positive* if $I(f) \geq 0$ holds for each nonnegative function f in the domain of I.

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7.1 Locally Compact Spaces

In this chapter we will be dealing with measures and integrals on locally compact Hausdorff spaces. This first section contains a summary of some of the necessary topological facts and constructions; the main development begins in Sect. 7.2.

Recall that a topological space is *locally compact* if each of its points has an open neighborhood whose closure is compact.

Examples 7.1.1. Examples of locally compact Hausdorff spaces include the Euclidean spaces $\mathbb{R}^d$, spaces with the discrete[2] topology (for example, the set $\mathbb{Z}$ of all integers), and compact Hausdorff spaces. The space ℓ^2 of sequences $\{x_n\}$ such that $\sum_n x_n^2 < +\infty$, with the topology given by the norm $\{x_n\} \mapsto (\sum_n x_n^2)^{1/2}$, is not locally compact (inside each open ball there is an infinite sequence that has no convergent subsequence; see item D.39 in Appendix D). □

The following elementary proposition will be a basic tool for what follows.

Proposition 7.1.2. *Let X be a Hausdorff space, and let K and L be disjoint compact subsets of X. Then there are disjoint open subsets U and V of X such that $K \subseteq U$ and $L \subseteq V$.*

Proof. We can assume that K and L are both nonempty (otherwise we could use $\varnothing$ as one of our open sets and X as the other). Let us begin with the case where K contains exactly one point, say x. For each y in L there is a pair U_y, V_y of disjoint open sets such that $x \in U_y$ and $y \in V_y$ (recall that X is Hausdorff). Since L is compact, there is a finite family $y_1, \ldots, y_n$ such that the sets $V_{y_1}, \ldots, V_{y_n}$ cover L. The sets U and V defined by $U = \cap_{i=1}^n U_{y_i}$ and $V = \cup_{i=1}^n V_{y_i}$ are then the required open sets.

Next consider the case where K has more than one element. We have just shown that for each x in K there are disjoint open sets, say U_x and V_x, such that $x \in U_x$ and $L \subseteq V_x$. Since K is compact, there is a finite family $x_1, \ldots, x_k$ such that $U_{x_1}, \ldots, U_{x_k}$ cover K. The proof is complete if we define U and V by $U = \cup_{i=1}^k U_{x_i}$ and $V = \cap_{i=1}^k V_{x_i}$. □

The sets U and V just constructed are said to *separate* the sets K and L.

Let us note several useful results (Propositions 7.1.3–7.1.6), each of which makes at least indirect use of Proposition 7.1.2.

Proposition 7.1.3. *Let X be a locally compact Hausdorff space, let x be a point in X, and let U be an open neighborhood of x. Then x has an open neighborhood whose closure is compact and included in U.*

Proof. Since X is locally compact, there is an open neighborhood of x, say W, whose closure is compact. By replacing W with $W \cap U$, we can assume that W is included in U. The difficulty is that $\overline{W}$ may extend outside U. Use Proposition 7.1.2

[2] A topological space is *discrete* (or has the *discrete topology*) if each of its subsets is open.

to choose disjoint open sets V_1 and V_2 that separate the compact sets $\{x\}$ and $\overline{W} - W$. The closure of $V_1 \cap W$ is then compact and included in W and hence in U; thus $V_1 \cap W$ is the required open neighborhood of x. □

Proposition 7.1.4. *Let X be a locally compact Hausdorff space, let K be a compact subset of X, and let U be an open subset of X that includes K. Then there is an open subset V of X that has a compact closure and satisfies $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. Proposition 7.1.3 implies that each point in K has an open neighborhood whose closure is compact and included in U. Since K is compact, some finite collection of these neighborhoods covers K. Let V be the union of the sets in such a finite collection; then V is the required set. □

A subset of a topological space X is a G_δ if it is the intersection of a sequence of open subsets of X and is an F_σ if it is the union of a sequence of closed subsets of X.

Proposition 7.1.5. *Let X be a locally compact Hausdorff space that has a countable base for its topology. Then each open subset of X is an F_σ and is in fact the union of a sequence of compact sets. Likewise, each closed subset of X is a G_δ.*

Proof. Suppose that $\mathscr{U}$ is a countable base for the topology of X. Let U be an open subset of X, and let $\mathscr{U}_U$ be the collection of those sets V in $\mathscr{U}$ for which $\overline{V}$ is compact and included in U. Proposition 7.1.3 implies that U is the union of the closures of the sets in $\mathscr{U}_U$ (the open neighborhoods provided by Proposition 7.1.3 can be replaced with smaller sets that belong to $\mathscr{U}$). Thus U is the union of a countable collection of sets that are closed and, in fact, compact.

Now suppose that C is a closed subset of X. Then C^c is open and so is the union of a sequence $\{F_n\}$ of closed sets. Consequently C is the intersection of the open sets F_n^c, $n = 1, 2, \ldots$. □

Recall that a topological space (or a subset thereof) is *σ-compact* if it is the union of a countable collection of compact sets.

Proposition 7.1.6. *Every locally compact Hausdorff space that has a countable base for its topology is σ-compact.*

Proof. Since a topological space is an open subset of itself, this proposition is an immediate corollary of Proposition 7.1.5. □

We turn to the continuous functions on a locally compact Hausdorff space.

Recall that a topological space is *normal* if it is Hausdorff and each pair of disjoint closed subsets of it can be separated by a pair of disjoint open sets.

Proposition 7.1.7. *Every compact Hausdorff space is normal.*

Proof. Note that every closed subset of a compact space is compact, and use Proposition 7.1.2 □

We will need the following standard result. A proof of it is sketched in Exercise 5.

Theorem 7.1.8 (Urysohn's Lemma). *Let X be a normal topological space, and let E and F be disjoint closed subsets of X. Then there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ holds at each x in E and $f(x) = 1$ holds at each x in F.*

Let f be a continuous real- or complex-valued function on a topological space X. The *support* of f, written $\operatorname{supp}(f)$, is the closure of $\{x \in X : f(x) \neq 0\}$. In case X is a locally compact Hausdorff space, we will denote by $\mathscr{K}(X)$ the set of those continuous functions $f\colon X \to \mathbb{R}$ for which $\operatorname{supp}(f)$ is compact. Likewise, we will denote by $\mathscr{K}^{\mathbb{C}}(X)$ the set of those continuous functions $f\colon X \to \mathbb{C}$ for which $\operatorname{supp}(f)$ is compact.

It is clear that $\mathscr{K}(X)$ and $\mathscr{K}^{\mathbb{C}}(X)$ are vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively, and that each function in $\mathscr{K}(X)$ or in $\mathscr{K}^{\mathbb{C}}(X)$ is bounded (recall that continuous functions are bounded on compact sets).

The following fact about $\mathscr{K}(X)$ is essential for the development of measure theory on locally compact Hausdorff spaces.

Proposition 7.1.9. *Let X be a locally compact Hausdorff space, let K be a compact subset of X, and let U be an open subset of X that includes K. Then there is a function f that belongs to $\mathscr{K}(X)$, satisfies $\chi_K \leq f \leq \chi_U$, and is such that* $\operatorname{supp}(f) \subseteq U$.

Proof. Use Proposition 7.1.4 to choose an open set V that has a compact closure and satisfies $K \subseteq V \subseteq \overline{V} \subseteq U$. According to Urysohn's lemma (applied to the compact Hausdorff space $\overline{V}$), there is a continuous function $g\colon \overline{V} \to [0,1]$ such that $g(x) = 1$ holds at each x in K and $g(x) = 0$ holds at each x in $\overline{V} - V$. Now define the function $f\colon X \to [0,1]$ by requiring that f agree with g on $\overline{V}$ and vanish outside $\overline{V}$. The continuity of f follows from D.6 (note that f is continuous on $\overline{V}$ and is constant, and hence continuous, on $X - V$). The support of f is included in $\overline{V}$ and so is compact and included in U. □

Next we derive two consequences of Proposition 7.1.9 (Propositions 7.1.11 and 7.1.12); they will be needed in Sects. 7.2 and 7.3, respectively. Let us begin with the following lemma.

Lemma 7.1.10. *Let X be a Hausdorff space, let K be a compact subset of X, and let U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then there are compact sets K_1 and K_2 such that $K = K_1 \cup K_2$, $K_1 \subseteq U_1$, and $K_2 \subseteq U_2$.*

Proof. Let $L_1 = K - U_1$ and $L_2 = K - U_2$. Then L_1 and L_2 are disjoint and compact, and so according to Proposition 7.1.2 they can be separated by disjoint open sets, say by V_1 and V_2. If we define K_1 and K_2 by $K_1 = K - V_1$ and $K_2 = K - V_2$, then K_1 and K_2 are compact, are included in U_1 and U_2, respectively, and have K as their union. □

Proposition 7.1.11. *Let X be a locally compact Hausdorff space, let f belong to $\mathscr{K}(X)$, and let $U_1, \ldots, U_n$ be open subsets of X such that* $\operatorname{supp}(f) \subseteq \bigcup_{i=1}^n U_i$. *Then there are functions $f_1, \ldots, f_n$ in $\mathscr{K}(X)$ such that $f = f_1 + f_2 + \cdots + f_n$ and such* *that for each i the support of f_i is included in U_i. Furthermore, if the function f is nonnegative, then the functions f_i, ..., f_n can be chosen so that all are nonnegative.*

Proof. In case $n = 1$ we need only let f_1 be f. So we can begin by supposing that $n = 2$. Use Lemma 7.1.10 to construct compact sets K_1 and K_2 such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, and $\operatorname{supp}(f) = K_1 \cup K_2$, and then use Proposition 7.1.9 to construct functions h_1 and h_2 that belong to $\mathscr{K}(X)$ and satisfy $\chi_{K_i} \le h_i \le \chi_{U_i}$ and $\operatorname{supp}(h_i) \subseteq U_i$ for $i = 1, 2$. Define functions g_1 and g_2 by $g_1 = h_1$ and $g_2 = h_2 - (h_1 \wedge h_2)$. Then g_1 and g_2 are non-negative, their supports are included in U_1 and U_2, respectively, and they satisfy

$$g_1(x) + g_2(x) = (h_1 \vee h_2)(x) = 1$$

at each x in $\operatorname{supp}(f)$. We can complete the proof in the case where $n = 2$ by defining f_1 and f_2 to be fg_1 and fg_2.

The general case can be dealt with by induction: use what we have just proved to write f as the sum of two functions, having supports included in $\cup_{i=1}^{n-1} U_i$ and U_n, respectively, and then use the induction hypothesis to decompose the first of these functions into the sum of $n-1$ suitable functions. □

Proposition 7.1.12. *Let X be a locally compact Hausdorff space, let K_1, ..., K_n be disjoint compact subsets of X, and let α_1, ..., α_n be real (or complex) numbers. Then there is a function f that belongs to $\mathscr{K}(X)$ (or to $\mathscr{K}^{\mathbb{C}}(X)$) and satisfies*

(a) $f(x) = \alpha_i$ *if $x \in K_i$, $i = 1$, ..., n, and*
(b) $\|f\|_\infty = \max\{|a_1|, \ldots, |a_n|\}$.

Proof. We begin by constructing disjoint open sets U_1, ..., U_n such that $K_i \subseteq U_i$ holds for each i. If $n = 2$, such sets are provided by Proposition 7.1.2. The general case follows by induction: use Proposition 7.1.2 to choose disjoint open sets V_1 and V_2 that separate $\cup_{i=1}^{n-1} K_i$ from K_n, use the induction hypothesis to choose disjoint open sets W_1, ..., W_{n-1} that separate K_1, ..., K_{n-1} from one another, and then define $U_1, \ldots, U_n$ to be $V_1 \cap W_1, \ldots, V_1 \cap W_{n-1}, V_2$.

Next we use Proposition 7.1.9 to choose functions f_1, ..., f_n that belong to $\mathscr{K}(X)$ and satisfy $\chi_{K_i} \le f_i \le \chi_{U_i}$ for $i = 1$, ..., n. The required function f is now given by $f = \sum_{i=1}^n \alpha_i f_i$. □

We will have use for the *one-point compactification* of a locally compact Hausdorff space X; it is constructed as follows. The underlying set X^* consists of the points in X, together with one additional point, called the *point at infinity*. The open subsets of X^* are, by definition, the open subsets of X, together with the complements (with respect to X^*) of the compact subsets of X. It is not hard to check that this does define a topology on X^* and that the topology induced by it on the subspace X is the original topology on X; the details are left to the reader. We need to verify that X^* is a compact Hausdorff space. Let us begin by checking that X^* is Hausdorff. Suppose that x and y are distinct points in X^*. If both points belong to X, then they can be separated with sets that are open in X and hence in X^*. If one of these points, say y, is the point at infinity and if we choose an open neighborhood U of x whose closure (in X) is compact, then U and $X^* - \overline{U}$ are disjoint open neighborhoods in X^* of x and y. Hence X^* is Hausdorff. We turn to the compactness of X^*. Let $\mathscr{U}$ be an open cover of X^*. The point at infinity must belong to some set in $\mathscr{U}$, say U_0. Then $X^* - U_0$ is a compact subset of X and so is covered by some finite subfamily of $\mathscr{U}$. The sets in this subfamily, together with U_0, form a finite cover of X^*. Thus X^* is compact.

The remaining results in this section will be used in a few exercises, but otherwise they will not be used until Chap. 8. They do, however, provide some perspective on the spaces considered here.

Proposition 7.1.13. *A compact Hausdorff space is metrizable if and only if there is a countable base for its topology.*

Proof. First suppose that X is compact and metrizable. Then X is separable (Corollary D.40) and so has a countable base (see D.32).

Now suppose that X is a compact Hausdorff space that has a countable base, and let $\mathscr{U}$ be such a base. The space X is normal (Proposition 7.1.7), and so for each pair of sets U, V that belong to $\mathscr{U}$ and satisfy $\overline{U} \cap \overline{V} = \varnothing$ there is, by Urysohn's lemma, a continuous function $f \colon X \to [0,1]$ that vanishes on $\overline{U}$ and has value 1 everywhere on $\overline{V}$. Form a sequence $\{f_n\}$ by choosing one such function for each such pair of sets. Our next step is to check that this sequence of functions separates the points of X, and for this it is enough to show that for each pair x, y of distinct points in X there are sets U and V that belong to $\mathscr{U}$, have disjoint closures, and contain x and y, respectively. To construct such sets, choose disjoint open neighborhoods W_1 and W_2 of x and y, and use Proposition 7.1.3 to choose open sets U and V such that $x \in U \subseteq \overline{U} \subseteq W_1$ and $y \in V \subseteq \overline{V} \subseteq W_2$. By making U and V a bit smaller, if necessary, we can assume that they belong to $\mathscr{U}$. Thus the required sets U and V exist, and the sequence $\{f_n\}$ separates the points of X.

Define a function $d \colon X \times X \to \mathbb{R}$ by setting

$$d(x,y) = \sum_n \frac{1}{2^n} |f_n(x) - f_n(y)|.$$

It is easy to use the fact that the functions $f_1, f_2, \dots$ separate the points of X to check that d is a metric on the set X and to use the fact that the functions f_1, f_2, $\dots$ are continuous (with respect to the original topology on X) to check that the topology induced by d is weaker than the original topology. Since the original topology makes X a compact space, while the topology induced by d is weaker and Hausdorff, the two topologies must be the same (see D.17). Thus the original topology on X is metrizable and in fact is metrized by d. □

Our next task is to prove that each locally compact Hausdorff space that has a countable base is metrizable. For this we need the following lemma.

Lemma 7.1.14. *Let X be a locally compact Hausdorff space. If there is a countable base for the topology of X, then there is a countable base for the topology of the one-point compactification of X.*

Proof. Let $\mathscr{U}$ be a countable base for the topology of X, and let $\mathscr{U}_0$ be the collection of those sets V in $\mathscr{U}$ for which $\overline{V}$ is compact. Arrange the sets in $\mathscr{U}_0$ in a sequence, say $\{V_k\}$. Then $X = \cup_k V_k$, and so for each compact subset K of X there is a positive integer n such that $K \subseteq \cup_{i=1}^n V_k$. Thus if U is an open neighborhood in X^* of the point at infinity and if $K = X^* - U$, then there is a positive integer n such that $K \subseteq \cup_{k=1}^n V_k$ and hence such that $X^* - \overline{(\cup_{k=1}^n V_k)} \subseteq U$. It follows that the sets in $\mathscr{U}$, together with the sets $X^* - \overline{(\cup_{k=1}^n V_k)}$, $n = 1, 2, \ldots,$ form a countable base for the topology of X^*. □

Proposition 7.1.15. *Each locally compact Hausdorff space that has a countable base for its topology is metrizable.*

Proof. Let X be a locally compact Hausdorff space whose topology has a countable base. Proposition 7.1.13 and Lemma 7.1.14 imply that the one-point compactification X^* of X is metrizable. Then X, as a subspace of the metrizable space X^*, is metrizable. □

A locally compact Hausdorff space can be metrizable without having a countable base; see Exercise 1.

Exercises

1. (a) Show that each discrete topological space is metrizable and locally compact.
   (b) Conclude that there are metrizable locally compact Hausdorff spaces that are not second countable.
2. Let X be a locally compact Hausdorff space, and let Y be a subspace of X. Show that if Y is open or closed (as a subset of X), then Y is locally compact.
3. Let X be a locally compact Hausdorff space, and let Y be a subspace of X. Show that Y is locally compact if and only if $Y = U \cap F$ for some open subset U and some closed subset F of X. (Hint: See Exercise 2. Also show that if Y is locally compact, then Y is an open subset of $\overline{Y}$ (of course $\overline{Y}$ is the closure of Y in X and is to be given the topology it inherits from X).)
4. Find all continuous functions $f: \mathbb{Q} \to \mathbb{R}$ such that $\text{supp}(f)$ is compact.
5. Prove Urysohn's lemma, Theorem 7.1.8. (Hint: Let D be the set of all dyadic rationals in the interval $(0,1)$ (that is, let D be the set of all numbers of the form $m/2^n$, where m and n are positive integers and $m < 2^n$). Use the normality of X first to choose an open set $U_{1/2}$ such that $E \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq F^c$ and then to choose open sets $U_{1/4}$ and $U_{3/4}$ such that $E \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2}$ and $\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq F^c$. Continue inductively, producing an indexed family $\{U_r\}_{r \in D}$ of open subsets of X such that
$$E \subseteq U_r \subseteq \overline{U_r} \subseteq U_s \subseteq \overline{U_s} \subseteq F^c$$

holds whenever r and s belong to D and satisfy $r < s$. Define a function $f: X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \notin \bigcup_r U_r, \\ \inf\{r : x \in U_r\} & \text{otherwise,} \end{cases}$$

and check that it has the required properties.)

6. Prove the Tietze extension theorem: if X is a normal topological space, if E is a closed subspace of X, and if $f: E \to \mathbb{R}$ is bounded and continuous, then there is a bounded continuous function $g: X \to \mathbb{R}$ whose restriction to E is f. (Hint: Check that we can assume that $f(E) \subseteq [-1,1]$. Use Urysohn's lemma to choose a continuous function $g_1 : X \to [-1/3, 1/3]$ such that $g_1(x) = -1/3$ if $x \in \{x \in E : f(x) \le -1/3\}$ and $g_1(x) = 1/3$ if $x \in \{x \in E : f(x) \ge 1/3\}$. Show that $|f(x) - g_1(x)| \le 2/3$ holds at each x in E. Continue inductively, choosing continuous functions $g_2, g_3, \ldots$ such that $|g_n(x)| \le 2^{n-1}/3^n$ holds at each x in X and $|f(x) - (g_1 + \cdots + g_n)(x)| \le (2/3)^n$ holds at each x in E. Then define g by $g = \sum_{n=1}^{\infty} g_n$.)
7. Let X be a compact Hausdorff space that contains at least two points, and let I be an uncountable set. Show that the product space X^I (which is of course compact[3] and Hausdorff) does not have a countable base. (Hint: Use D.11 to show that if X^I has a countable base and if $\mathscr{U}$ is the base for X^I constructed in D.19, then some countable subset of $\mathscr{U}$ is a base for X^I. Then show that no countable subfamily of $\mathscr{U}$ can be a base for X^I.)
8. Let X be the set consisting of those step functions $f: [0,1] \to [0,1]$ such that

   (i) each value of f is rational, and
   (ii) each jump in the graph of $y = f(x)$ occurs at a rational value of x.

   Show that X is a countable dense subset of the product space $[0,1]^{[0,1]}$. Conclude that a compact Hausdorff space can be separable without being second countable. (See Exercise 7.)
9. Let X be a second countable compact Hausdorff space (in other words, a compact metrizable space), and let $C(X)$ be the vector space of all real-valued continuous functions on X. Give $C(X)$ the norm $\|\cdot\|_\infty$ defined by $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Show that $C(X)$ is separable. (Hint: We saw in the proof of Proposition 7.1.13 that one can choose a countable collection S of continuous functions on X such that S separates the points of X. The Stone–Weierstrass theorem (Theorem D.22) implies that the polynomials in the functions belonging to S form a dense subset of $C(X)$. Those polynomials that have rational coefficients form a countable dense subset of $C(X)$.)
10. Let Ω be the smallest uncountable ordinal, let X be the set of all ordinal numbers α such that $\alpha \le \Omega$, and let Y be the set of all ordinal numbers α such that $\alpha < \Omega$

[3] See Theorem D.20.

(thus Y consists of the countable ordinals). Give X and Y the order topology (see D.24). Show that
(a) X is a compact Hausdorff space, and
(b) Y is a locally compact Hausdorff space.
(Hint: Use transfinite induction to show that for each α in X the set $\{\beta \in X : \beta \leq \alpha\}$ is compact.)

7.2 The Riesz Representation Theorem

Let X be a Hausdorff topological space. Then $\mathscr{B}(X)$, the *Borel σ-algebra* on X, is the σ-algebra generated by the open subsets of X; the *Borel subsets* of X are those that belong to $\mathscr{B}(X)$. Note that $\mathscr{B}(X)$ is also the σ-algebra generated by the closed subsets of X.

We will need the following two elementary facts about the Borel subsets of Hausdorff spaces.

Lemma 7.2.1. *Let X and Y be Hausdorff topological spaces, and let $f: X \to Y$ be continuous. Then f is Borel measurable (that is, measurable with respect to $\mathscr{B}(X)$ and $\mathscr{B}(Y)$).*

Proof. The continuity of f implies that if U is an open subset of Y, then $f^{-1}(U)$ is an open and hence a Borel subset of X. Since the collection of open subsets of Y generates $\mathscr{B}(Y)$, the measurability of f follows from Proposition 2.6.2. □

Lemma 7.2.2. *Let X be a Hausdorff topological space, and let Y be a subspace of X. Then*

$$\mathscr{B}(Y) = \{A : \textit{there is a set } B \textit{ in } \mathscr{B}(X) \textit{ such that } A = B \cap Y\}.$$

Proof. Let $\mathscr{B}(X)_Y$ denote the collection of subsets of Y that have the form $B \cap Y$ for some B in $\mathscr{B}(X)$. We need to show that $\mathscr{B}(Y) = \mathscr{B}(X)_Y$. Let f be the standard injection of Y into X (in other words, let $f(y) = y$ hold at each y in Y). Then f is continuous and hence measurable with respect to $\mathscr{B}(Y)$ and $\mathscr{B}(X)$. Since $f^{-1}(B) = B \cap Y$ holds for each subset B of X, the measurability of f implies that $\mathscr{B}(X)_Y \subseteq \mathscr{B}(Y)$. On the other hand, it is easy to check that $\mathscr{B}(X)_Y$ is a σ-algebra on Y that contains all the open subsets of Y and hence includes $\mathscr{B}(Y)$. With this we have shown that $B(Y) = \mathscr{B}(X)_Y$. □

We turn to terminology for measures. Let X be a Hausdorff topological space. A *Borel measure* on X is a measure whose domain is $\mathscr{B}(X)$. Suppose that $\mathscr{A}$ is a σ-algebra on X such that $\mathscr{B}(X) \subseteq \mathscr{A}$. A positive measure μ on $\mathscr{A}$ is *regular* if

(a) each compact subset K of X satisfies $\mu(K) < +\infty$,
(b) each set A in $\mathscr{A}$ satisfies

$$\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ and } U \text{ is open}\}, \text{ and}$$

(c) each open subset U of X satisfies

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ is compact}\}.$$

A *regular Borel measure* on X is a regular measure whose domain is $\mathscr{B}(X)$. A measure that satisfies condition (b) is often called *outer regular*, and a measure that satisfies condition (c), *inner regular*.

We have already seen that Lebesgue measure on $\mathbb{R}^d$ is regular (Proposition 1.4.1) and that every finite Borel measure on $\mathbb{R}^d$ is regular (Proposition 1.5.6).

The regularity of a measure allows many approximations and calculations that would be impossible without it. In particular, various linear functionals can be represented in a useful way with regular measures; see Theorems 7.2.8 and 7.3.6.

On certain rather complicated locally compact Hausdorff spaces there exist finite Borel measures that are not regular; see Exercise 7. However, for a locally compact Hausdorff space that has a countable base, we have the following result.

Proposition 7.2.3. *Let X be a locally compact Hausdorff space that has a countable base, and let μ be a Borel measure on X that is finite on compact sets. Then μ is regular.*

Proof. First consider the inner regularity of μ. Let U be an open subset of X. Proposition 7.1.5 implies that U is the union of a sequence $\{K_j\}$ of compact sets, and Proposition 1.2.5 then implies that $\mu(U) = \lim_n \mu(\cup_{j=1}^n K_j)$. The inner regularity of μ follows.

We will use the following reformulation of Lemma 1.5.7 in our proof of the outer regularity of μ.

Lemma 7.2.4. *Let X be a Hausdorff space in which each open set is an F_σ, and let μ be a finite Borel measure on X. Then each Borel subset A of X satisfies*

$$\mu(A) = \inf\{\mu(U) : A \subseteq U \textit{ and } U \textit{ is open}\} \tag{1}$$

and

$$\mu(A) = \sup\{\mu(F) : F \subseteq A \textit{ and } F \textit{ is closed}\}. \tag{2}$$

Proof. The arguments used to prove Lemma 1.5.7 also prove this lemma; the details will not be repeated. □

Let us continue with the proof of Proposition 7.2.3. We still need to check the outer regularity of μ. In order to apply Lemma 7.2.4, we will consider certain finite measures that are closely related to μ. Let $\{U_n\}$ be a sequence of open sets such that $X = \cup_n U_n$ and such that $\mu(U_n) < +\infty$ holds for each n (for instance, take a countable base $\mathscr{U}$ for X and arrange in a sequence those sets U in $\mathscr{U}$ for which $\overline{U}$ is compact). For each n define a Borel measure μ_n on X by $\mu_n(A) = \mu(A \cap U_n)$. The measures μ_n are finite, and so Proposition 7.1.5 and Lemma 7.2.4 imply that they are outer regular. Hence if A belongs to $\mathscr{B}(X)$ and if ε is a positive number, then for each n there is an open set V_n that includes A and satisfies $\mu_n(V_n) < \mu_n(A) + \varepsilon/2^n$.

Consequently $\mu((U_n \cap V_n) - A) < \varepsilon/2^n$. The set V defined by $V = \cup_n (U_n \cap V_n)$ is open, includes A, and satisfies

$$\mu(V - A) \leq \sum_n \mu((U_n \cap V_n) - A) < \varepsilon.$$

Hence $\mu(V) \leq \mu(A) + \varepsilon$, and the outer regularity of μ follows. □

Proposition 7.2.5. *Let X be a locally compact Hausdorff space that has a countable base. Then every regular measure on X is σ-finite.*

Proof. The space X is, according to Proposition 7.1.6, the union of a sequence of compact sets. Since the measure of a compact set is finite under a regular measure, the proposition follows. □

The following proposition enables one to approximate many sets from within by compact sets.

Proposition 7.2.6. *Let X be a Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $\mathscr{A}$. If A belongs to $\mathscr{A}$ and is σ-finite under μ, then*

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \textit{ and } K \textit{ is compact}\}. \tag{3}$$

Proof. First suppose that $\mu(A) < +\infty$. Let ε be a positive number, and use the regularity of μ first to choose an open set V such that $A \subseteq V$ and $\mu(V) < \mu(A) + \varepsilon$ and then to choose a compact subset L of V such that $\mu(L) > \mu(V) - \varepsilon$. Since $\mu(V - A) < \varepsilon$, we can choose an open set W that includes $V - A$ and satisfies $\mu(W) < \varepsilon$. The set $L - W$ is then a compact subset of A, and it satisfies

$$\mu(L - W) = \mu(L) - \mu(L \cap W) > \mu(V) - 2\varepsilon \geq \mu(A) - 2\varepsilon.$$

Since ε is arbitrary, relation (3) follows in the case where $\mu(A)$ is finite.

In the case where $\mu(A) = +\infty$, we can suppose that $A = \cup_n A_n$, where for each n we have $A_n \in \mathscr{A}$ and $\mu(A_n) < +\infty$. For each positive number α, we need to construct a compact subset K of A such that $\mu(K) > \alpha$. We can construct such a set by first choosing N large enough that $\mu(\cup_{n=1}^N A_n) > \alpha$ and then using the construction in the first part of the proof to produce an appropriate compact subset of $\cup_{n=1}^N A_n$. □

Let X be a locally compact Hausdorff space. Recall that $\mathscr{K}(X)$ is the vector space consisting of all real-valued functions on X that are continuous and have compact support. We will study the relationship between regular measures on X and linear functionals on $\mathscr{K}(X)$. The first thing to note is that each function in $\mathscr{K}(X)$ is integrable with respect to each regular measure on X (each such function is measurable (Lemma 7.2.1) and so, since it is bounded and vanishes outside a set that is compact and hence of finite measure, is integrable). It follows that if μ is a regular Borel measure on X, then $f \mapsto \int f\,d\mu$ defines a linear functional on $\mathscr{K}(X)$. Two questions arise immediately. Can several regular Borel measures induce the same functional? Which functionals arise in this way? Both of these questions will be answered in Theorem 7.2.8.

For dealing with such questions the concept of positivity for linear functionals is essential. A linear functional I on $\mathscr{K}(X)$ is *positive* if for each nonnegative f in $\mathscr{K}(X)$ we have $I(f) \geq 0$. Note that if μ is a regular Borel measure on X, then the functional $f \mapsto \int f\,d\mu$ is positive. Note also that a positive linear functional I on $\mathscr{K}(X)$ is order preserving, in the sense that if f and g belong to $\mathscr{K}(X)$ and satisfy $f \leq g$, then $I(f) \leq I(g)$ (if $f \leq g$, then $g - f$ is nonnegative, and we have $I(g) - I(f) = I(g-f) \geq 0$).

Let U be an open subset of the locally compact Hausdorff space X. We will often deal with functions f that belong to $\mathscr{K}(X)$ and satisfy

$$0 \leq f \leq \chi_U. \tag{4}$$

Among the functions f in $\mathscr{K}(X)$ that satisfy (4), those that also satisfy $\operatorname{supp}(f) \subseteq U$ are especially nice to deal with; accordingly we will write $f \prec U$ to indicate that f satisfies both (4) and the relation $\operatorname{supp}(f) \subseteq U$.

Lemma 7.2.7. *Let X be a locally compact Hausdorff space, and let μ be a regular Borel measure on X. If U is an open subset of X, then*

$$\begin{aligned}\mu(U) &= \sup\left\{\int f\,d\mu : f \in \mathscr{K}(X) \textit{ and } 0 \leq f \leq \chi_U\right\}\\ &= \sup\left\{\int f\,d\mu : f \in \mathscr{K}(X) \textit{ and } f \prec U\right\}.\end{aligned}$$

Proof. It is clear that $\mu(U)$ is at least as large as the first supremum and that the first supremum is at least as large as the second. So it is enough to prove that

$$\mu(U) \leq \sup\left\{\int f\,d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\}.$$

Let α be a number that satisfies $\alpha < \mu(U)$, and use the regularity of μ to choose a compact subset K of U such that $\alpha < \mu(K)$. Proposition 7.1.9 provides a function f in $\mathscr{K}(X)$ that satisfies $\chi_K \leq f$ and $f \prec U$. Then $\alpha < \int f\,d\mu$, and so

$$\alpha < \sup\left\{\int f\,d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\}.$$

Since α was an arbitrary number less than $\mu(U)$, the proof is complete. □

We are now in a position to prove the main result of this section.

Theorem 7.2.8 (Riesz Representation Theorem). *Let X be a locally compact Hausdorff space, and let I be a positive linear functional on $\mathscr{K}(X)$. Then there is a unique regular Borel measure μ on X such that*

$$I(f) = \int f\,d\mu$$

holds for each f in $\mathscr{K}(X)$.

Proof. We first prove the uniqueness of μ. Suppose that μ and ν are regular Borel measures on X such that

$$\int f \, d\mu = \int f \, d\nu = I(f)$$

holds for each f in $\mathscr{K}(X)$. It follows from Lemma 7.2.7 that $\mu(U) = \nu(U)$ holds for each open subset U of X and then from the outer regularity of μ and ν that $\mu(A) = \nu(A)$ holds for each Borel subset A of X. Thus μ and ν are equal, and the uniqueness is proved.

We turn to the construction of a measure representing the functional I. Lemma 7.2.7 and condition (b) in the definition of regularity suggest how to proceed. Define a function μ^* on the open subsets of X by

$$\mu^*(U) = \sup\{I(f) : f \in \mathscr{K}(X) \text{ and } f \prec U\}, \tag{5}$$

and then extend it to all subsets of X by

$$\mu^*(A) = \inf\{\mu^*(U) : U \text{ is open and } A \subseteq U\} \tag{6}$$

(it is easy to check that Eq. (6) is consistent with Eq. (5), in the sense that an open set is assigned the same value by both). We will presently see that the required measure μ can be obtained by restricting μ^* to $\mathscr{B}(X)$.

The rest of the proof of Theorem 7.2.8 will be given by Proposition 7.2.9, Lemma 7.2.10, and Proposition 7.2.11.

Proposition 7.2.9. *Let X and I be as in the statement of Theorem 7.2.8, and let μ^* be defined by (5) and (6). Then μ^* is an outer measure on X, and every Borel subset of X is μ^*-measurable.*

Proof. The relation $\mu^*(\varnothing) = 0$ and the monotonicity of μ^* are clear. We need to check the countable subadditivity of μ^*. First suppose that $\{U_n\}$ is a sequence of open subsets of X; we will verify that

$$\mu^*\Big(\bigcup_n U_n\Big) \le \sum_n \mu^*(U_n). \tag{7}$$

Let f be a function that belongs to $\mathscr{K}(X)$ and satisfies $f \prec \cup_n U_n$. Then $\operatorname{supp}(f)$ is a compact subset of $\cup_n U_n$, and so there is a positive integer N such that $\operatorname{supp}(f) \subseteq \cup_{n=1}^N U_n$. Proposition 7.1.11 implies that f is the sum of functions $f_1, \ldots, f_N$ that belong to $\mathscr{K}(X)$ and satisfy $f_n \prec U_n$ for $n = 1, \ldots, N$. It follows that

$$I(f) = \sum_{n=1}^N I(f_n) \le \sum_{n=1}^N \mu^*(U_n) \le \sum_{n=1}^\infty \mu^*(U_n).$$

This and Eq. (5) yield inequality (7).

Now suppose that $\{A_n\}$ is an arbitrary sequence of subsets of X. The inequality $\mu^*(\cup_n A_n) \le \sum_n \mu^*(A_n)$ is clear if $\sum_n \mu^*(A_n) = +\infty$. So suppose that

$\sum_n \mu^*(A_n) < +\infty$, let ε be a positive number, and for each n use (6) to choose an open set U_n that includes A_n and satisfies $\mu^*(U_n) \le \mu^*(A_n) + \varepsilon/2^n$. Then (see inequality (7))

$$\mu^*(\cup_n A_n) \le \mu^*(\cup_n U_n) \le \sum_{n=1}^{\infty} \mu^*(U_n) \le \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Since ε is arbitrary, the relation $\mu^*(\cup_n A_n) \le \sum_n \mu^*(A_n)$ follows. Thus μ^* is countably subadditive and so is an outer measure.

Since the family of μ^*-measurable sets is a σ-algebra, we can show that every Borel subset of X is μ^*-measurable by checking that each open subset of X is μ^*-measurable. So let U be an open subset of X. According to the discussion preceding Proposition 1.3.5, we can prove that U is μ^*-measurable by showing that

$$\mu^*(A) \ge \mu^*(A \cap U) + \mu^*(A \cap U^c) \tag{8}$$

holds for each subset A of X that satisfies $\mu^*(A) < +\infty$. Let A be such a set, let ε be a positive number, and use (6) to choose an open set V that includes A and satisfies $\mu^*(V) < \mu^*(A) + \varepsilon$. If we show that

$$\mu^*(V) > \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon, \tag{9}$$

it will follow that

$$\mu^*(A) + \varepsilon > \mu^*(A \cap U) + \mu^*(A \cap U^c) - 2\varepsilon,$$

and, since ε is arbitrary, that (8) holds. So let us verify (9). Choose a function f_1 in $\mathscr{K}(X)$ that satisfies $f_1 \prec V \cap U$ and $I(f_1) > \mu^*(V \cap U) - \varepsilon$, let $K = \operatorname{supp}(f_1)$, and then choose a function f_2 in $\mathscr{K}(X)$ that satisfies $f_2 \prec V \cap K^c$ and $I(f_2) > \mu^*(V \cap K^c) - \varepsilon$. (This would be a good time to draw a sketch of the sets involved here.) Since $f_1 + f_2 \prec V$ and $V \cap U^c \subseteq V \cap K^c$, we have

$$\mu^*(V) \ge I(f_1 + f_2) > \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon.$$

Thus (9) holds and proof of Proposition 7.2.9 is complete. □

Lemma 7.2.10. *Let X and I be as in the statement of Theorem 7.2.8, and let μ^* be defined by (5) and (6). Suppose that A is a subset of X and that f belongs to $\mathscr{K}(X)$. If $\chi_A \le f$, then $\mu^*(A) \le I(f)$, while if $0 \le f \le \chi_A$ and if A is compact,[4] then $I(f) \le \mu^*(A)$.*

Proof. First assume that $\chi_A \le f$. Let ε satisfy $0 < \varepsilon < 1$, and define U_ε by $U_\varepsilon = \{x \in X : f(x) > 1 - \varepsilon\}$. Then U_ε is open, and each g in $\mathscr{K}(X)$ that satisfies $g \le \chi_{U_\varepsilon}$ also satisfies $g \le \frac{1}{1-\varepsilon} f$; hence (5) implies that $\mu^*(U_\varepsilon) \le \frac{1}{1-\varepsilon} I(f)$. Since $A \subseteq U_\varepsilon$ and since ε can be made arbitrarily close to 0, it follows that $\mu^*(A) \le I(f)$.

[4] The assumption that A is compact simplifies the proof, but is not actually necessary.

Now suppose that $0 \le f \le \chi_A$ and that A is compact. Let U be an open set that includes A. Then $f \prec U$ and so (5) implies that $I(f) \le \mu^*(U)$. Since U was an arbitrary open set that includes A, (6) implies that $I(f) \le \mu^*(A)$. □

Proposition 7.2.11. *Let X and I be as in the statement of Theorem 7.2.8, let μ^* be defined by (5) and (6), let μ be the restriction of μ^* to $\mathscr{B}(X)$, and let μ_1 be the restriction of μ^* to the σ-algebra $\mathscr{M}_{\mu^*}$ of μ^*-measurable sets. Then μ and μ_1 are regular measures, and*

$$\int f\,d\mu = \int f\,d\mu_1 = I(f)$$

holds for each f in $\mathscr{K}(X)$.

Proof. Theorem 1.3.6 implies that μ_1 is a measure on $\mathscr{M}_{\mu^*}$ and, since $\mathscr{B}(X) \subseteq \mathscr{M}_{\mu^*}$ (Proposition 7.2.9), that μ is a measure on $\mathscr{B}(X)$. Since for each compact subset K of X there is a function f that belongs to $\mathscr{K}(X)$ and satisfies $\chi_K \le f$ (Proposition 7.1.9), the first part of Lemma 7.2.10 implies that μ and μ_1 are finite on compact sets. The outer regularity of μ and μ_1 follows from (6), and the inner regularity of these measures follows from (5) and the second part of Lemma 7.2.10 (where we let A be the support of f).

We turn to the identity $I(f) = \int f\,d\mu = \int f\,d\mu_1$. Since each function in $\mathscr{K}(X)$ is the difference of two nonnegative functions in $\mathscr{K}(X)$, we can restrict our attention to the nonnegative functions in $\mathscr{K}(X)$. Let f be such a function. Let ε be a positive number, and for each positive integer n define a function $f_n\colon X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } f(x) \le (n-1)\varepsilon, \\ f(x) - (n-1)\varepsilon & \text{if } (n-1)\varepsilon < f(x) \le n\varepsilon, \\ \varepsilon & \text{if } n\varepsilon < f(x). \end{cases} \tag{10}$$

(See Fig. 7.1 below.) Then each f_n belongs to $\mathscr{K}(X)$, $f = \sum_n f_n$, and there is a positive integer N such that $f_n = 0$ if $n > N$. Let $K_0 = \operatorname{supp}(f)$ and for each positive integer n let $K_n = \{x \in X : f(x) \ge n\varepsilon\}$. Then $\varepsilon\chi_{K_n} \le f_n \le \varepsilon\chi_{K_{n-1}}$ holds for each n, and so Lemma 7.2.10 and the basic properties of the integral imply that $\varepsilon\mu(K_n) \le I(f_n) \le \varepsilon\mu(K_{n-1})$ and $\varepsilon\mu(K_n) \le \int f_n\,d\mu \le \varepsilon\mu(K_{n-1})$ hold for each n. Since $f = \sum_{n=1}^N f_n$, the relations

$$\sum_{n=1}^{N} \varepsilon\mu(K_n) \le I(f) \le \sum_{n=0}^{N-1} \varepsilon\mu(K_n)$$

and

$$\sum_{n=1}^{N} \varepsilon\mu(K_n) \le \int f\,d\mu \le \sum_{n=0}^{N-1} \varepsilon\mu(K_n)$$

follow. Thus $I(f)$ and $\int f\,d\mu$ both lie in the interval $[\sum_{n=1}^N \varepsilon\mu(K_n), \sum_{n=0}^{N-1} \varepsilon\mu(K_n)]$, which has length $\varepsilon\mu(K_0) - \varepsilon\mu(K_N)$. Since ε is arbitrary and this length is at most

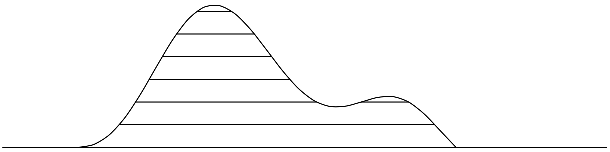

Fig. 7.1 Decomposing f as $\sum f_n$ (see Eq. (10))

$\varepsilon\mu(K_0)$, $I(f)$ and $\int f\,d\mu$ must be equal. It is clear that $\int f\,d\mu_1 = \int f\,d\mu$, and so the proof of Proposition 7.2.11, and hence that of Theorem 7.2.8, is complete. □

Exercises

1. Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $(X,\mathscr{A})$. Show that if A belongs to $\mathscr{A}$ and is σ-finite under μ, then for each positive ε there is an open set U that includes A and satisfies $\mu(U-A)<\varepsilon$. (Be sure to consider the case where $\mu(A)$ is infinite.)
2. Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $(X,\mathscr{A})$. Show that the completion of μ is regular.
3. Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $(X,\mathscr{A})$. Show that if A belongs to $\mathscr{A}$ and is σ-finite under μ, then there are sets E and F in $\mathscr{B}(X)$ such that $E\subseteq A\subseteq F$ and $\mu(F-E)=0$. (In particular, if μ is σ-finite, then $\mathscr{A}$ is included in the completion of $\mathscr{B}(X)$ under the restriction of μ to $\mathscr{B}(X)$.)
4. Let us construct a topological space X by letting the underlying set be $\mathbb{R}^2$ and declaring that the open subsets U of X are those for which each section of the form U_x is an open subset of $\mathbb{R}$.

   (a) Show that X is locally compact and Hausdorff.
   (b) Characterize the functions $f\colon X\to\mathbb{R}$ that belong to $\mathscr{K}(X)$ in terms of their sections f_x.
   (c) Show that the formula

   $$I(f)=\sum_x\int f_x\,d\lambda$$

   (where λ is Lebesgue measure on $\mathbb{R}$) defines a positive linear functional on $\mathscr{K}(X)$ and that the regular Borel measure associated to I by the Riesz representation theorem is the restriction to $\mathscr{B}(X)$ of the measure defined in Exercise 3.3.6. (We will see in Exercise 9.4.12 that the σ-algebra $\mathscr{A}$ in Exercise 3.3.6 is strictly larger than $\mathscr{B}(X)$.)
   (d) Show that if μ is the regular Borel measure on X that corresponds to I, then

   $$\mu(A)=\sup\{\mu(K):K\subseteq A\text{ and }K\text{ is compact}\}$$

   fails for some Borel subset A of X.

5. Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $(X,\mathscr{A})$. Define $\mu^\bullet$ as in Exercise 1.2.8.
   (a) Show that
   $$\mu^\bullet(A) = \sup\{\mu^\bullet(K) : K \subseteq A \text{ and } K \text{ is compact}\}$$
   holds for each A in $\mathscr{A}$. (In particular, $\mu^\bullet$ is inner regular.)
   (b) Show that the conditions
   (i) $\mu^\bullet$ is regular,
   (ii) $\mu^\bullet = \mu$, and
   (iii) every locally μ-null set in $\mathscr{A}$ is μ-null

   are equivalent.
6. Let X be a locally compact Hausdorff space, and let μ be a regular Borel measure on X. Suppose that $\mu(\{x\}) = 0$ holds for each x in X. Show that if B is a Borel subset of X such that $\mu(B) < +\infty$ and if a is a real number such that $0 < a < \mu(B)$, then there is a Borel subset A of B that satisfies $\mu(A) = a$. Can the Borel set A be replaced with a compact set?
7. Let Y be the collection of all countable ordinals, with the order topology (see Exercise 7.1.10).
   (a) Show that a subset A of Y is uncountable if and only if for each countable ordinal α there is an ordinal β that belongs to A and satisfies $\beta > \alpha$.
   (b) Show that if $\{C_n\}$ is a sequence of uncountable closed subsets of Y, then $\cap_n C_n$ is an uncountable closed set. (Hint: Use part (a); show that if $\{\alpha_k\}$ is an increasing sequence of countable ordinals such that each C_n contains infinitely many terms of $\{\alpha_k\}$, then $\lim_k \alpha_k$ exists and belongs to $\cap_n C_n$.)
   (c) Show that if $A \in \mathscr{B}(Y)$, then exactly one of A and A^c includes an uncountable closed subset of Y.
   (d) Suppose that we define a function μ on $\mathscr{B}(Y)$ by letting $\mu(A)$ be 1 if A includes an uncountable closed set and letting $\mu(A)$ be 0 otherwise. Show that μ is a Borel measure that is not regular. Find the regular Borel measure μ' on Y that satisfies $\int f\,d\mu' = \int f\,d\mu$ for each f in $\mathscr{K}(Y)$.
   (e) Let X be the collection of all ordinal numbers that are less than or equal to the first uncountable ordinal, and give X the order topology (again see Exercise 7.1.10). Show that the formula $\nu(A) = \mu(A \cap Y)$ defines a non-regular Borel measure ν on X. Find the regular Borel measure ν' on X that satisfies $\int f\,d\nu' = \int f\,d\nu$ for each f in $\mathscr{K}(X)$.
8. Let X be a compact Hausdorff space, and let $C(X)$ be the set of all real-valued continuous functions on X. Then $\mathscr{B}_0(X)$, the *Baire σ-algebra* on X, is the smallest σ-algebra on X that makes each function in $C(X)$ measurable; the sets that belong to $\mathscr{B}_0(X)$ are called the *Baire subsets* of X. A *Baire measure* on X is a *finite* measure on $(X,\mathscr{B}_0(X))$.
   (a) Show that $\mathscr{B}_0(X)$ is the σ-algebra generated by the closed G_δ's in X. (Hint: Check that if $f \in C(X)$ and if $a \in \mathbb{R}$, then $\{x : f(x) \le a\}$ is a closed G_δ, and use Proposition 7.1.9 to check that every closed G_δ in X arises in this way.)

(b) Show that if the compact Hausdorff space X is second countable, then $\mathscr{B}_0(X) = \mathscr{B}(X)$.

9. Show that if μ is a Baire measure on a compact Hausdorff space X, then μ is regular, in the sense that

$$\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ is open, and } U \in \mathscr{B}_0(X)\}$$

holds for each set A in $\mathscr{B}_0(X)$ and

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ is compact, and } K \in \mathscr{B}_0(X)\}$$

holds for each open set U in $\mathscr{B}_0(X)$. (Hint: Modify the proof of Lemma 1.5.7; show that the collection of Baire sets that can be approximated from above by open Baire sets and from below by compact Baire sets is a σ-algebra and that this σ-algebra contains all the closed G_δ's in X. See Exercise 8.)
10. Let X be a compact Hausdorff space, and let I be a positive linear functional on $C(X)$ (note that since X is compact, $C(X) = \mathscr{K}(X)$). Show that there is a unique Baire measure μ on X such that $I(f) = \int f\, d\mu$ holds for each f in $C(X)$. (Hint: First check that the restriction to $\mathscr{B}_0(X)$ of the measure given by Theorem 7.2.8 works. Then modify the part of the proof of Theorem 7.2.8 that deals with uniqueness; see Exercise 9.)
11. Let X be a compact Hausdorff space. Show that if K is a closed Baire subset of X, then K is a G_δ. (Hint: Use Exercise 1.1.7 to choose a sequence $\{f_n\}$ of functions in $C(X)$ such that K belongs to the smallest σ-algebra making f_1, f_2, … measurable; then define $F: X \to \mathbb{R}^\mathbb{N}$ by letting F take x to the sequence $\{f_n(x)\}$. Show that $F(K)$ is a compact subset of the second countable space $\mathbb{R}^\mathbb{N}$ and so is a G_δ; then check that $K = F^{-1}(F(K))$ (see Exercise 2.6.5), and conclude that K is a G_δ in X.)
12. Let I be the interval $[0,1]$, and let X be the product space I^I, with the product topology (here the interval $[0,1]$, when considered as a factor in the product space, is to have its usual topology). Thus each element x of X is an indexed family $\{x_i\}_{i\in I}$ of elements of I.

    (a) Show that if f belongs to $C(X)$, then $f(x)$ depends on only countably many of the components x_i of x (in other words, for each f in $C(X)$ there is a countable subset C of I such that if $x_i = y_i$ holds for each i in C, then $f(x) = f(y)$). (Hint: First consider the case where f is a polynomial in the components of x, and then use the Stone–Weierstrass theorem.)
    (b) Show that if $A \in \mathscr{B}_0(X)$, then $\chi_A(x)$ depends on only countably many of the components of x. (Hint: Check that the collection of sets A such that $\chi_A(x)$ depends on only countably many of the components of x is a σ-algebra.)
    (c) Show that if $f: X \to \mathbb{R}$ is $\mathscr{B}_0(X)$-measurable, then $f(x)$ depends on only countably many of the components of x.
    (d) Show that the one-element subsets of X belong to $\mathscr{B}(X)$ but not to $\mathscr{B}_0(X)$. Conclude that $\mathscr{B}_0(X) \neq \mathscr{B}(X)$.

13. Let X be the space of all ordinals less than or equal to the first uncountable ordinal, and let X have the order topology (see Exercise 7.1.10). Find $\mathscr{B}_0(X)$, the Baire σ-algebra on X. Is $\mathscr{B}_0(X)$ equal to $\mathscr{B}(X)$?

7.3 Signed and Complex Measures; Duality

This section is devoted to regularity for finite signed and complex Borel measures. The main result is a measure-theoretic representation for the duals of certain Banach spaces of continuous functions.

Let X be a locally compact Hausdorff space, and let f be a real- or complex-valued continuous function on X. Then f is said to *vanish at infinity* if for every positive number ε there is a compact subset K of X such that $|f(x)| < \varepsilon$ holds at each x outside K. We will denote by $C_0(X)$ the set of all real-valued continuous functions on X that vanish at infinity and by $C_0^{\mathbb{C}}(X)$ the set of all complex-valued continuous functions on X that vanish at infinity.

Examples 7.3.1. Note that a continuous function f on $\mathbb{R}$ vanishes at infinity if and only if $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = 0$. Note also that every continuous function on a compact Hausdorff space vanishes at infinity. See Exercises 1, 2, and 9 for some more examples, and see Exercise 3 for another characterization of the continuous functions that vanish at infinity. □

Of course, $C_0(X)$ and $C_0^{\mathbb{C}}(X)$ are vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively. These spaces are normed spaces: each continuous function that vanishes at infinity is bounded (since a continuous function is bounded on a compact set), and so the formula

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

defines norms on $C_0(X)$ and $C_0^{\mathbb{C}}(X)$.

Proposition 7.3.2. *Let X be a locally compact Hausdorff space. Then $\mathscr{K}(X)$ and $\mathscr{K}^{\mathbb{C}}(X)$ are dense subspaces of $C_0(X)$ and $C_0^{\mathbb{C}}(X)$, respectively.*

Proof. It is clear that $\mathscr{K}(X)$ and $\mathscr{K}^{\mathbb{C}}(X)$ are linear subspaces of $C_0(X)$ and $C_0^{\mathbb{C}}(X)$. We need only show that they are dense. Suppose that f belongs to $C_0(X)$ or to $C_0^{\mathbb{C}}(X)$ and that ε is a positive number. Choose a compact set K such that $|f(x)| \le \varepsilon$ holds at each x outside K, and use Proposition 7.1.9 to choose a function $g\colon X \to [0,1]$ that belongs to $\mathscr{K}(X)$ and satisfies $g(x) = 1$ at each x in K. Let $h = fg$. Then h belongs to $\mathscr{K}(X)$ or to $\mathscr{K}^{\mathbb{C}}(X)$ and satisfies $\|f-h\|_\infty \le \varepsilon$. Since ε is arbitrary, the proof is complete. □

Proposition 7.3.3. *Let X be a locally compact Hausdorff space. Then $C_0(X)$ and $C_0^{\mathbb{C}}(X)$ are Banach spaces.*

Proof. The only issue is the completeness of these spaces. So let $\{f_n\}$ be a Cauchy sequence in one of them. A standard argument (see the proof given in Sect. 3.2 of the completeness of $C[a,b]$) shows that there is a continuous function f such that $\{f_n\}$ converges uniformly to f. We need only check that f vanishes at infinity. Let ε be a positive number, choose a positive integer n such that $|f(x) - f_n(x)| < \varepsilon$ holds at each x in X, and use the fact that f_n vanishes at infinity to choose a compact set K such that $|f_n(x)| < \varepsilon$ holds at each x outside K. Then

$$|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| < 2\varepsilon$$

holds at each x outside K, and since ε is arbitrary, the proof is complete. □

Let X be a locally compact Hausdorff space. A finite signed or complex measure μ on $(X, \mathscr{B}(X))$ is *regular* if its variation $|\mu|$ is regular (in the sense of Sect. 7.2). It is convenient to note the following equivalent formulations of regularity.

Proposition 7.3.4. *Let X be a locally compact Hausdorff space, and let μ be a finite signed or complex measure on $(X, \mathscr{B}(X))$. Then the conditions*

(a) *μ is regular,*
(b) *each of the positive measures appearing in the Jordan decomposition of μ is regular, and*
(c) *μ is a linear combination of finite positive regular Borel measures*

are equivalent.

Proof. Suppose that condition (a) holds, and let μ' be one of the measures appearing in the Jordan decomposition of μ. Then μ' satisfies $\mu' \le |\mu|$ (consider how μ' arises from a Hahn decomposition). Thus if $A \in \mathscr{B}(X)$, if ε is a positive number, and if U is an open set that includes A and satisfies $|\mu|(U) < |\mu|(A) + \varepsilon$, then $\mu'(U - A) \le |\mu|(U - A) < \varepsilon$, and so

$$\mu'(U) = \mu'(A) + \mu'(U - A) < \mu'(A) + \varepsilon.$$

The outer regularity of μ' follows. The inner regularity of μ' can be proved in a similar manner. Hence condition (a) implies condition (b).

Condition (b) certainly implies condition (c).

The proof that (c) implies (a) is similar to the proof that (a) implies (b) and makes use of the fact that if $\mu = \alpha_1\mu_1 + \cdots + \alpha_n\mu_n$, where each α_i is a real or complex number and each μ_i is positive, then $|\mu| \le |\alpha_1|\mu_1 + \cdots + |\alpha_n|\mu_n$. □

Regularity makes possible the following approximation (see also Exercise 5).

Lemma 7.3.5. *Let X be a locally compact Hausdorff space, and let μ be a finite signed or complex regular Borel measure on X. Then for each A in $\mathscr{B}(X)$ and each positive number ε there is a compact subset K of A such that $|\mu(A) - \mu(B)| < \varepsilon$ holds whenever B is a Borel set that satisfies $K \subseteq B \subseteq A$.*

Proof. Let A and ε be as in the statement of the lemma. Use the regularity of $|\mu|$ and Proposition 7.2.6 to choose a compact subset K of A such that $|\mu|(A - K) < \varepsilon$. Then each Borel set B that satisfies $K \subseteq B \subseteq A$ also satisfies

$$|\mu(A) - \mu(B)| = |\mu(A - B)| \leq |\mu|(A - B) \leq |\mu|(A - K) < \varepsilon,$$

which completes the proof of the lemma. □

Let X be a locally compact Hausdorff space. We will denote by $M_r(X,\mathbb{R})$ the set of all finite signed regular Borel measures on X and by $M_r(X,\mathbb{C})$ the set of all complex regular Borel measures on X. It is easy to check that $M_r(X,\mathbb{R})$ and $M_r(X,\mathbb{C})$ are linear subspaces of the vector spaces $M(X,\mathscr{B}(X),\mathbb{R})$ and $M(X,\mathscr{B}(X),\mathbb{C})$ of all finite signed or complex measures on $(X,\mathscr{B}(X))$. These larger spaces are Banach spaces under the total variation norm (Proposition 4.1.8). Moreover $M_r(X,\mathbb{R})$ and $M_r(X,\mathbb{C})$ are closed subspaces of $M(X,\mathscr{B}(X),\mathbb{R})$ and $M(X,\mathscr{B}(X),\mathbb{C})$ (to check this, note that if μ is regular, if $\|\mu - \nu\| < \varepsilon$, and if A is a Borel set and U is an open set chosen so that $A \subseteq U$ and $|\mu|(U - A) < \varepsilon$, then

$$|\nu|(U - A) \leq \|\nu - \mu\| + |\mu|(U - A) < 2\varepsilon).$$

It follows that $M_r(X,\mathbb{R})$ and $M_r(X,\mathbb{C})$ are themselves Banach spaces under the total variation norm.

Recall (see Sect. 4.1) that if $(X,\mathscr{A})$ is a measurable space, if f is a bounded $\mathscr{A}$-measurable function on X, and if μ is a finite signed measure on $(X,\mathscr{A})$ with Jordan decomposition $\mu = \mu^+ - \mu^-$, then the integral of f with respect to μ is defined by

$$\int f\,d\mu = \int f\,d\mu^+ - \int f\,d\mu^-.$$

Likewise, if μ is a complex measure with Jordan decomposition $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, then

$$\int f\,d\mu = \int f\,d\mu_1 - \int f\,d\mu_2 + i\int f\,d\mu_3 - i\int f\,d\mu_4.$$

Theorem 7.3.6. *Let X be a locally compact Hausdorff space. Then the map that takes the finite signed (or complex) regular Borel measure μ to the functional $f \mapsto \int f\,d\mu$ is an isometric isomorphism of the Banach space $M_r(X,\mathbb{R})$ (or $M_r(X,\mathbb{C})$) onto the dual of the Banach space $C_0(X)$ (or $C_0^{\mathbb{C}}(X)$).*

Proof. For each finite signed regular Borel measure μ on X define a functional Φ_μ on $C_0(X)$ by $\Phi_\mu(f) = \int f\,d\mu$. It is easy to see that Φ_μ is a linear functional on $C_0(X)$ and that

$$|\Phi_\mu(f)| \leq \|f\|_\infty \|\mu\|$$

holds for each f and μ (see the discussion at the end of Sect. 4.1). Thus Φ_μ is continuous and its norm satisfies

$$\|\Phi_\mu\| \le \|\mu\|. \tag{1}$$

Moreover, $\mu \mapsto \Phi_\mu$ defines a linear map Φ from $M_r(X,\mathbb{R})$ to the dual of $C_0(X)$. Analogous results hold for complex measures and complex-valued functions.

We need to show that Φ is norm preserving and surjective. Let us begin with the first of these tasks. In view of (1), it is enough to show that

$$\|\Phi_\mu\| \ge \|\mu\| \tag{2}$$

holds for each μ. So let μ belong to $M_r(X,\mathbb{R})$ or to $M_r(X,\mathbb{C})$, and let ε be a positive number. We can assume that $\|\mu\| \ne 0$. According to the definition of $\|\mu\|$, we can choose a finite partition of X into Borel sets A_j, $j = 1, \dots, n$, such that $\sum_{j=1}^n |\mu(A_j)| > \|\mu\| - \varepsilon$. Now choose compact subsets $K_1, \dots, K_n$ of $A_1, \dots, A_n$ such that

$$\|\mu\| - \varepsilon < \sum_j |\mu(K_j)| \le \sum_j |\mu|(K_j)$$

(see Lemma 7.3.5). We can assume that $\mu(K_j) \ne 0$ holds for each j. Choose a continuous function f that has compact support (and hence vanishes at infinity), satisfies $\|f\|_\infty \le 1$, and is such that $f(x) = \overline{\mu(K_j)}/|\mu(K_j)|$ holds for each j and each x in K_j (see Proposition 7.1.12). Let $K = \cup_j K_j$. Then $\int_K f\,d\mu = \sum_j |\mu(K_j| > \|\mu\| - \varepsilon$, while $|\int_{K^c} f\,d\mu| \le |\mu|(K^c) < \varepsilon$. It follows that $|\int f\,d\mu| > \|\mu\| - 2\varepsilon$. Since f satisfies $\|f\|_\infty \le 1$ and ε is arbitrary, relation (2) follows. Thus Φ is norm preserving.

We turn to the surjectivity of Φ. First consider the case of real-valued functions and measures. Suppose that L is a continuous linear functional on $C_0(X)$ that is positive, in the sense that $L(f) \ge 0$ holds for each nonnegative f in $C_0(X)$. The restriction of L to $\mathscr{K}(X)$ is also positive, and so the Riesz representation theorem (Theorem 7.2.8) provides a regular Borel measure μ on X such that $L(f) = \int f\,d\mu$ holds for each f in $\mathscr{K}(X)$. Lemma 7.2.7 implies that

$$\mu(X) = \sup\{L(f) : f \in \mathscr{K}(X) \text{ and } 0 \le f \le 1\},$$

and hence that $\mu(X) \le \|L\|$; in particular, μ is finite. Note that so far we have only proved that $L(f) = \Phi_\mu(f)$ holds when f belongs to the subspace $\mathscr{K}(X)$ of $C_0(X)$. However, since $\mathscr{K}(X)$ is dense in $C_0(X)$ (Proposition 7.3.2), while L and Φ_μ are continuous, the equality of L and Φ_μ on $C_0(X)$ follows. With this we have proved that each positive continuous linear functional on $C_0(X)$ is of the form Φ_μ.

We need the following lemma to complete the proof of Theorem 7.3.6.

Lemma 7.3.7. *Let X be a locally compact Hausdorff space. Then for each continuous linear functional L on $C_0(X)$ there are positive continuous linear functionals L_+ and L_- on $C_0(X)$ such that $L = L_+ - L_-$.*

Proof. For each nonnegative f in $C_0(X)$ define $L_+(f)$ by

$$L_+(f) = \sup\{L(g) : g \in C_0(X) \text{ and } 0 \leq g \leq f\}. \tag{3}$$

The relation

$$|L(g)| \leq \|L\|\|g\|_\infty \leq \|L\|\|f\|_\infty,$$

which is valid if $0 \leq g \leq f$, implies that the supremum involved in the definition of $L_+(f)$ is finite and in fact that

$$L_+(f) \leq \|L\|\|f\|_\infty. \tag{4}$$

We need to check that if $t \geq 0$ and if f, f_1, and f_2 are nonnegative functions in $C_0(X)$, then

$$0 \leq L_+(f),$$
$$L_+(tf) = tL_+(f), \text{ and}$$
$$L_+(f_1 + f_2) = L_+(f_1) + L_+(f_2).$$

The first two of these properties are easy to check, and so we turn to the third. If g_1 and g_2 belong to $C_0(X)$ and satisfy $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2$, then $0 \leq g_1 + g_2 \leq f_1 + f_2$, and so

$$L(g_1) + L(g_2) = L(g_1 + g_2) \leq L_+(f_1 + f_2).$$

Since g_1 and g_2 can be chosen so as to make $L(g_1) + L(g_2)$ arbitrarily close to $L_+(f_1) + L_+(f_2)$, the inequality

$$L_+(f_1) + L_+(f_2) \leq L_+(f_1 + f_2)$$

follows. Now consider the reverse inequality. Suppose that g belongs to $C_0(X)$ and satisfies $0 \leq g \leq f_1 + f_2$, and define functions g_1 and g_2 by $g_1 = g \wedge f_1$ and $g_2 = g - g_1$. Then g_1 and g_2 belong to $C_0(X)$ and satisfy $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2$, and so

$$L(g) = L(g_1) + L(g_2) \leq L_+(f_1) + L_+(f_2).$$

Since g can be chosen so as to make $L(g)$ arbitrarily close to $L_+(f_1 + f_2)$, the inequality

$$L_+(f_1 + f_2) \leq L_+(f_1) + L_+(f_2)$$

follows. With this the third of our properties is proved.

Now use the formula

$$L_+(f) = L_+(f^+) - L_+(f^-), \tag{5}$$

where f^+ and f^- are the positive and negative parts of f, to extend the definition of L_+ to all of $C_0(X)$. By imitating some arguments used in the construction of the integral (see Lemma 2.3.5 and Proposition 2.3.6), the reader can show that L_+ is a linear functional on $C_0(X)$. The positivity of L_+ is clear. Relations (4) and (5), together with the positivity of L_+, imply that $|L_+(f)| \le \|L\| \|f\|_\infty$ and hence that L_+ is continuous.

Define a functional L_- on $C_0(X)$ by $L_- = L_+ - L$. The linearity and continuity of L_- are immediate. Its positivity follows from its definition and the fact that $L_+(f) \ge L(f)$ holds for each nonnegative f (let $g = f$ in relation (3)). Since $L = L_+ - L_-$, the proof of the lemma is complete. □

Let us return to the proof of Theorem 7.3.6. Since we have already checked that each positive continuous linear functional on $C_0(X)$ is of the form Φ_μ, the surjectivity of $\Phi\colon M_r(X,\mathbb{R}) \to C_0(X)^*$ follows from Lemma 7.3.7. The extension to the case of complex-valued functions and measures is easy: if $L \in C_0^{\mathbb{C}}(X)^*$, then there are functionals L_1 and L_2 in $C_0(X)^*$ such that $L(f) = L_1(f) + iL_2(f)$ for each f in $C_0(X)$ (that is, for each *real-valued* f in $C_0^{\mathbb{C}}(X)$), and so if μ_1 and μ_2 are the finite signed regular Borel measures that represent L_1 and L_2, then $\mu_1 + i\mu_2$ is a complex regular Borel measure that represents L. □

We close this section by turning to those finite signed or complex measures ν on $(X, \mathscr{B}(X))$ that have the form $\nu(A) = \int_A f\,d\mu$ for some μ-integrable f (here μ is a positive regular Borel measure on X). Two questions arise: Does the regularity of ν follow from the regularity of μ? Can such measures ν be characterized by a version of the Radon–Nikodym theorem, even if μ is not σ-finite? The next two propositions answer these questions.

These results will be used only in Sect. 9.4.

Proposition 7.3.8. *Let X be a locally compact Hausdorff space, let μ be a regular Borel measure on X, let f belong to $\mathscr{L}^1(X, \mathscr{B}(X), \mu)$, and let ν be the finite signed or complex measure on $(X, \mathscr{B}(X))$ defined by $\nu(A) = \int_A f\,d\mu$. Then ν is regular.*

Proof. For each f in $\mathscr{L}^1(X, \mathscr{B}(X), \mu)$ define a finite signed or complex measure ν_f on $(X, \mathscr{B}(X))$ by $\nu_f(A) = \int_A f\,d\mu$. Let us deal first with the case where f is the characteristic function of a Borel set B for which $\mu(B) < +\infty$. In this case ν_f is the positive measure given by $\nu_f(A) = \mu(A \cap B)$, and for each A in $\mathscr{B}(X)$ Proposition 7.2.6, applied to the measure μ and the set $A \cap B$, implies that

$$\nu_f(A) = \sup\{\nu_f(K) : K \subseteq A \text{ and } K \text{ is compact}\}. \tag{6}$$

Thus ν_f is inner regular. The outer regularity of ν_f follows if for each A in $\mathscr{B}(X)$ we use (6) (with A replaced by A^c) to approximate A^c from below by compact sets and hence to approximate A from above by open sets.

We can use the regularity of ν_f for such characteristic functions to conclude first that ν_f is regular if f is simple and integrable and then that ν_f is regular if f is an arbitrary integrable function (see Propositions 3.4.2 and 4.2.5). □

Proposition 7.3.9. *Let X be a locally compact Hausdorff space, let μ be a regular Borel measure on X, and let ν be a finite signed or complex regular Borel measure on X. Then the conditions*

(a) there is a function f in $\mathscr{L}^1(X,\mathscr{B}(X),\mu)$ such that $\nu(A) = \int_A f\,d\mu$ holds for each A in $\mathscr{B}(X)$,
(b) ν is absolutely continuous with respect to μ (each Borel subset A of X that satisfies $\mu(A) = 0$ also satisfies $\nu(A) = 0$), and
(c) each compact subset K of X that satisfies $\mu(K) = 0$ also satisfies $\nu(K) = 0$

are equivalent.

Proof. It is clear that condition (a) implies condition (b) and that condition (b) implies condition (c).

If $A \in \mathscr{B}(X)$, then Lemma 7.3.5 implies that for every positive ε there is a compact subset K of A such that $|\nu(A) - \nu(K)| < \varepsilon$. Consequently if condition (c) holds and if A satisfies $\mu(A) = 0$, then A must also satisfy $\nu(A) = 0$. Thus condition (c) implies condition (b).

Next suppose that condition (b) holds. The difficulty in using the Radon–Nikodym theorem (Theorem 4.2.4) to derive condition (a) is that we are not assuming that μ is σ-finite. We take care of this as follows. Use the regularity of ν to choose an increasing sequence $\{K_n\}$ of compact sets such that $\lim_n |\nu|(K_n) = |\nu|(X)$. Then, because of the regularity of μ, $\mu(K_n)$ is finite for each n, and so the measure μ_0 defined by $\mu_0(A) = \mu(A \cap (\cup_n K_n))$ is σ-finite. Since ν is absolutely continuous with respect to μ and since $|\nu|(X - (\cup_n K_n)) = 0$, ν is also absolutely continuous with respect to μ_0. Thus the Radon–Nikodym theorem provides a function f in $\mathscr{L}^1(X,\mathscr{B}(X),\mu_0)$ such that $\nu(A) = \int_A f\,d\mu_0$ holds for each A in $\mathscr{B}(X)$. If we modify f so that it vanishes outside $\cup_n K_n$, then $\nu(A) = \int_A f\,d\mu$ holds for each A in $\mathscr{B}(X)$. With this we have shown that condition (b) implies condition (a). □

Proposition 7.3.10. *Let X be a locally compact Hausdorff space, and let μ be a regular Borel measure on X. For each f in $\mathscr{L}^1(X,\mathscr{B}(X),\mu)$ define a finite signed or complex measure ν_f on $(X,\mathscr{B}(X))$ by means of the formula $\nu_f(A) = \int_A f\,d\mu$. Then the map $f \mapsto \nu_f$ induces a linear isometry of $L^1(X,\mathscr{B}(X),\mu)$ onto the subspace of $M_r(X,\mathbb{R})$ (or of $M_r(X,\mathbb{C})$) that consists of those ν that are absolutely continuous with respect to μ.*

Proof. The proposition is an immediate consequence of Propositions 7.3.8 and 7.3.9 and the fact that $\|\nu_f\| = \int |f|\,d\mu$ (see Proposition 4.2.5). □

Exercises

1. Describe $\mathscr{K}(X)$ and $C_0(X)$ rather explicitly in the case where X is the space $\{(x_1,x_2) \in \mathbb{R}^2 : (x_1,x_2) \neq (0,0)\}$.
2. Give an example of a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that does not belong to $C_0(\mathbb{R}^2)$ but satisfies $\lim_{t\to\infty} f(tx_1,tx_2) = 0$ for each nonzero (x_1,x_2) in $\mathbb{R}^2$.

3. Let X be a locally compact Hausdorff space, let X^* be its one-point compactification, and let x_∞ be the point at infinity. Show that a function $f: X \to \mathbb{R}$ belongs to $C_0(X)$ if and only if the function $f^*: X^* \to \mathbb{R}$ defined by
$$f^*(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x = x_\infty, \end{cases}$$
is continuous.
4. Show that the decomposition $L = L_+ - L_-$ given in the proof of Lemma 7.3.7 is minimal, in the sense that if $L = L_1 - L_2$ is another decomposition of L into a difference of positive linear functionals, then $L_1(f) \geq L_+(f)$ and $L_2(f) \geq L_-(f)$ hold for each nonnegative f in $C_0(X)$.
5. Prove the converse of Lemma 7.3.5: if X is a locally compact Hausdorff space, if μ is a finite signed or complex measure on $(X, \mathscr{B}(X))$, and if μ satisfies the conclusion of Lemma 7.3.5, then μ is regular.
6. Let X be a locally compact Hausdorff space, and let μ be a regular Borel measure on X such that $\mu(X) = +\infty$. Show that there is a nonnegative function f in $C_0(X)$ such that $\int f\,d\mu = +\infty$.
7. Let X be a locally compact Hausdorff space. Show that each positive linear functional on $C_0(X)$ is continuous.
8. Show that if X is a second countable locally compact Hausdorff space, then $C_0(X)$ is separable. (Hint: Use Exercises 7.1.9 and 7.3.3.)
9. Let Y be the space of all countable ordinals, with the order topology (see Exercise 7.1.10). Show that Y is not compact, but $C_0(Y) = \mathscr{K}(Y)$.
10. Let X be a compact Hausdorff space, let $\mathscr{B}_0(X)$ be the Baire σ-algebra on X (see Exercise 7.2.8), and let $C(X)$ be the space of all continuous real- (or complex-) valued functions on X. Give $C(X)$ the norm $\|\cdot\|_\infty$ defined by $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Show that the map that assigns to a finite signed (or complex) measure μ on $(X, \mathscr{B}_0(X))$ the functional $f \mapsto \int f\,d\mu$ is an isometric isomorphism of $M(X, \mathscr{B}_0(X), \mathbb{R})$ (or of $M(X, \mathscr{B}_0(X), \mathbb{C})$) onto $C(X)^*$. (Hint: Modify the proof of Theorem 7.3.6; see Exercises 7.2.9 and 7.2.10.)

7.4 Additional Properties of Regular Measures

This section is devoted to several useful facts about regular measures.

Proposition 7.4.1. *Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure*[5] *on $(X, \mathscr{A})$. Then the union of all the open subsets of X that have measure zero under μ is itself an open set that has measure zero under μ.*

[5] Note that μ is a positive measure, since its specification has no modifier such as "signed" or "complex."

Proof. Let $\mathscr{U}$ be the collection of all open subsets of X that have measure zero under μ, and let U be the union of the sets in $\mathscr{U}$. Then U is open and so belongs to $\mathscr{A}$. If K is a compact subset of U, then K can be covered by a finite collection U_1, $U_2, \ldots, U_n$ of sets that belong to $\mathscr{U}$, and so we have

$$\mu(K) \leq \sum_{i=1}^{n} \mu(U_i) = 0.$$

This and the inner regularity of μ imply that $\mu(U) = 0$. □

Let us continue for a moment with X and μ having the same meaning as in the statement of Proposition 7.4.1. Then X has a largest open subset of μ-measure zero, namely the union of all its open subsets of μ-measure zero. The complement of this open set is called the *support* of μ and is denoted by $\operatorname{supp}(\mu)$. Of course $\operatorname{supp}(\mu)$ is the smallest closed set whose complement has measure zero under μ. Furthermore, a point x belongs to $\operatorname{supp}(\mu)$ if and only if every open neighborhood of x has positive measure under μ.

If μ is a finite signed or complex regular Borel measure on a locally compact Hausdorff space, then its *support* is defined to be the support of its variation $|\mu|$.

Examples 7.4.2. It is easy to check that if, as usual, λ is Lebesgue measure on $\mathbb{R}$, then $\operatorname{supp}(\lambda) = \mathbb{R}$. At the other extreme, if δ_x is the point mass on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ concentrated at x, then $\operatorname{supp}(\delta_x) = \{x\}$. See Exercises 1 through 5 for more information about supports. □

We turn to two theorems that deal with the approximation of measurable functions by continuous functions. These results are often useful, since continuous functions are in many ways easier to handle than are measurable functions.

Proposition 7.4.3. *Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $(X, \mathscr{A})$. Suppose that $1 \leq p < +\infty$. Then $\mathscr{K}(X)$ is a dense subspace of $\mathscr{L}^p(X, \mathscr{A}, \mu, \mathbb{R})$ and so determines a dense subspace of $L^p(X, \mathscr{A}, \mu, \mathbb{R})$.*

Note that Proposition 7.4.3 is a generalization of Proposition 3.4.4.

Proof. It is clear that $\mathscr{K}(X) \subseteq \mathscr{L}^p(X, \mathscr{A}, \mu, \mathbb{R})$. Since the simple functions in $\mathscr{L}^p(X, \mathscr{A}, \mu, \mathbb{R})$ are dense in $\mathscr{L}^p(X, \mathscr{A}, \mu, \mathbb{R})$ (Proposition 3.4.2), it suffices to show that if A belongs to $\mathscr{A}$ and has finite measure under μ, then there are functions f in $\mathscr{K}(X)$ that make $\|\chi_A - f\|_p$ arbitrarily small.

So let A be as specified above, and let ε be a positive number. Use the outer regularity of μ to choose an open set U that includes A and satisfies $\mu(U) < \mu(A) + \varepsilon$, and use Proposition 7.2.6 to choose a compact set K that is included in A and satisfies $\mu(K) > \mu(A) - \varepsilon$. Let f belong to $\mathscr{K}(X)$ and satisfy $\chi_K \leq f \leq \chi_U$ (see Proposition 7.1.9). Then $|\chi_A - f| \leq \chi_U - \chi_K$, and so

$$\|\chi_A - f\|_p \leq \|\chi_U - \chi_K\|_p = (\mu(U - K))^{1/p} < (2\varepsilon)^{1/p};$$

since $(2\varepsilon)^{1/p}$ can be made arbitrarily small by a suitable choice of ε, the proof is complete. □

Theorem 7.4.4 (Lusin's Theorem). *Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, let μ be a regular measure on $(X,\mathscr{A})$, and let $f: X \to \mathbb{R}$ be $\mathscr{A}$-measurable. If A belongs to $\mathscr{A}$ and satisfies $\mu(A) < +\infty$ and if ε is a positive number, then there is a compact subset K of A such that $\mu(A-K) < \varepsilon$ and such that the restriction of f to K is continuous. Moreover, there is a function g in $\mathscr{K}(X)$ that agrees with f at each point in K; if $A \neq \varnothing$ and f is bounded on A, then the function g can be chosen so that*

$$\sup\{|g(x)| : x \in X\} \leq \sup\{|f(x)| : x \in A\}. \tag{1}$$

Proof. First suppose that f has only countably many values, say $a_1, a_2, \ldots,$ and that these values are attained on the sets $A_1, A_2, \ldots$. Use Proposition 1.2.5 to choose a positive integer n such that $\mu(A - (\cup_{i=1}^n A_i)) < \varepsilon/2$, and then use Proposition 7.2.6 to choose compact subsets $K_1, \ldots, K_n$ of $A \cap A_1, \ldots, A \cap A_n$ that satisfy $\sum_{i=1}^n \mu((A \cap A_i) - K_i) < \varepsilon/2$. Let $K = \cup_{i=1}^n K_i$. Then K is a compact subset of A, and

$$\mu(A-K) = \mu(A - (\cup_{i=1}^n A_i)) + \sum_{i=1}^n \mu((A \cap A_i) - K_i) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Furthermore, since f is constant on each K_i, its restriction to K is continuous (see D.6). Thus K is the required set.

Now let f be an arbitrary $\mathscr{A}$-measurable function. Then f is the uniform limit of a sequence $\{f_n\}$ of functions, each of which is $\mathscr{A}$-measurable and has only countably many values (for example, f_n might be defined by letting $f_n(x)$ be k/n, where k is the integer that satisfies $k/n \leq f(x) < (k+1)/n$). According to what we have just proved, for each n there is a compact subset K_n of A such that $\mu(A - K_n) < \varepsilon/2^n$ and such that the restriction of f_n to K_n is continuous. Let $K = \cap_n K_n$. Then K is a compact subset of A,

$$\mu(A-K) \leq \sum_n \mu(A - K_n) < \sum_n \varepsilon/2^n = \varepsilon,$$

and f, as the uniform limit of the functions f_n, each of which is continuous on K, is itself continuous on K. With this the first part of the theorem is proved.

We turn to the construction of a function g in $\mathscr{K}(X)$ that agrees with f on K. The one-point compactification X^* of X is normal (Proposition 7.1.7), and so the Tietze extension theorem (Exercise 7.1.6) provides a continuous function $h^*: X^* \to \mathbb{R}$ that agrees with f on K. Let $g: X \to \mathbb{R}$ be the product hp, where h is the restriction of h^* to X and p is a function that belongs to $\mathscr{K}(X)$ and satisfies $p(x) = 1$ at each x in K (Proposition 7.1.9). Then g belongs to $\mathscr{K}(X)$ and agrees with f on K. In order to make sure that g satisfies inequality (1), let $B = \sup\{|f(x)| : x \in A\}$, define $\varphi: \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} -B & \text{if } t < -B, \\ t & \text{if } -B \leq t \leq B, \\ B & \text{if } B < t, \end{cases}$$

and replace g with $\varphi \circ g$. □

Note that Proposition 7.4.3 and Theorem 7.4.4 can be extended to apply to complex-valued functions. Everything except for inequality (1) can be proved by dealing with real and imaginary parts separately. For the proof of (1), let $B = \sup\{|f(x)| : x \in A\}$, and define $\varphi\colon \mathbb{C} \to \mathbb{C}$ by

$$\varphi(t) = \begin{cases} t & \text{if } |t| \leq B, \\ \frac{t}{|t|}B & \text{if } |t| > B. \end{cases}$$

Then φ is continuous, and, as before, the function g can be replaced with $\varphi \circ g$.

The reader should note that in certain cases Lusin's theorem can be used to characterize measurable functions (see Exercise 7.5.2). In fact, Bourbaki defines a function to be measurable if it satisfies the conclusion of Lusin's theorem.

For our next result we need to recall two definitions. Let X be a topological space. A function $f\colon X \to (-\infty, +\infty]$ is *lower semicontinuous* if for each x in X and each real number A that satisfies $A < f(x)$ there is an open neighborhood V of x such that $A < f(t)$ holds at each t in V. It is easy to see that f is lower semicontinuous if and only if for each real number A the set $\{x \in X : A < f(x)\}$ is open. It follows that the supremum of a collection of continuous (or lower semicontinuous) functions is lower semicontinuous and that each lower semicontinuous function on a Hausdorff space is Borel measurable.

Now suppose that X is an arbitrary set and that $\mathscr{H}$ is a family of $[-\infty, +\infty]$-valued functions on X. Then $\mathscr{H}$ is *directed upward* if for each pair h_1, h_2 of functions in $\mathscr{H}$ there is a function h in $\mathscr{H}$ that satisfies $h_1 \leq h$ and $h_2 \leq h$. Note that if $\mathscr{H}$ is directed upward and if $h_1, \ldots, h_n$ belong to $\mathscr{H}$, then there is a function h in $\mathscr{H}$ that satisfies $h_i \leq h$ for $i = 1, \ldots, n$.

Proposition 7.4.5. *Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $(X, \mathscr{A})$. Suppose that $f\colon X \to [0, +\infty]$ is lower semicontinuous and that $\mathscr{H}$ is a family of nonnegative lower semicontinuous functions that is directed upward and satisfies*

$$f(x) = \sup\{h(x) : h \in \mathscr{H}\} \tag{2}$$

at each x in X. Then

$$\int f\,d\mu = \sup\{\int h\,d\mu : h \in \mathscr{H}\}.$$

Proof. Certainly $\int h\,d\mu \le \int f\,d\mu$ holds whenever h belongs to $\mathscr{H}$. Thus we need only show that for each real number A that satisfies $A < \int f\,d\mu$ there is a function h that belongs to $\mathscr{H}$ and satisfies $A < \int h\,d\mu$. So let A be a real number (which we will hold fixed) that satisfies $A < \int f\,d\mu$.

We begin by approximating f with simple functions in the following way. For each positive integer n define open sets $U_{n,i}$, $i = 1, \dots, n2^n$, by

$$U_{n,i} = \{x \in X : f(x) > i/2^n\},$$

and then define a function $f_n \colon X \to \mathbb{R}$ by

$$f_n = \frac{1}{2^n} \sum_{i=1}^{n2^n} \chi_{U_{n,i}}.$$

Each f_n is Borel measurable and hence $\mathscr{A}$-measurable. It is easy to check that $f_n(x) = 0$ if $f(x) = 0$, that $f_n(x) = i/2^n$ if $0 < f(x) \le n$ and i is the integer that satisfies $i/2^n < f(x) \le (i+1)/2^n$, and that $f_n(x) = n$ if $n < f(x)$. Consequently $\{f_n\}$ is a nondecreasing sequence of nonnegative functions for which $f(x) = \lim_n f_n(x)$ holds at each x in X, and so the monotone convergence theorem (Theorem 2.4.1) implies that $\int f\,d\mu = \lim_n \int f_n\,d\mu$. Hence we can choose a positive integer N such that $A < \int f_N\,d\mu$. The plan now is to choose a function g that satisfies $A < \int g\,d\mu$ but is a bit more convenient than f_N and then to choose a function h in $\mathscr{H}$ that is at least as large as g.

Since $\int f_N\,d\mu = (1/2^N)\sum_i \mu(U_{N,i})$, we can use the regularity of μ to get compact subsets K_i of $U_{N,i}$, $i = 1, \dots, N2^N$, such that $A < (1/2^N)\sum_i \mu(K_i)$. Let $g = (1/2^N)\sum_i \chi_{K_i}$.

Note that $g(x) \le f_N(x) < f(x)$ holds at each x for which $f(x) > 0$ and hence at each x in $\cup_1^{N2^N} K_i$. Thus (see also (2)) for each x in $\cup_1^{N2^N} K_i$ there is a function h_x in $\mathscr{H}$ such that $g(x) < h_x(x)$. Since h_x is lower semicontinuous and g is a positive multiple of a finite sum of characteristic functions of compact (and hence closed) sets, we can choose an open neighborhood U_x of x such that $g(t) < h_x(t)$ holds at each t in U_x. Carrying this out for each x in $\cup_1^{N2^N} K_i$ gives an open cover of $\cup_1^{N2^N} K_i$; since $\cup_1^{N2^N} K_i$ is compact, we can get first a finite subcover $U_{x_1}, \dots, U_{x_m}$ of $\cup_1^{N2^N} K_i$ and then a function h in $\mathscr{H}$ such that $h_{x_j} \le h$ holds for $j = 1, \dots, m$ (recall that $\mathscr{H}$ is directed upward). The function h satisfies $g \le h$ and so satisfies

$$A < \frac{1}{2^N} \sum_i \mu(K_i) = \int g\,d\mu \le \int h\,d\mu.$$

Thus we have produced the required function h, and the proof is complete. □

Exercises

1. Let $\{a_n\}$ be a sequence of positive real numbers such that $\sum_n a_n < +\infty$, let $\{x_n\}$ be an arbitrary sequence of real numbers, and let μ be the measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ defined by $\mu = \sum_n a_n \delta_{x_n}$. Find $\text{supp}(\mu)$.
2. Construct a finite signed regular Borel measure μ on $\mathbb{R}$ such that $\text{supp}(\mu^+)$ and $\text{supp}(\mu^-)$ are both equal to $\mathbb{R}$.
3. Let X be a locally compact Hausdorff space, and let μ be a regular Borel measure on X. Show that a point x in X belongs to $\text{supp}(\mu)$ if and only if every nonnegative function f in $\mathscr{K}(X)$ that satisfies $f(x) > 0$ also satisfies $\int f \, d\mu > 0$.
4. Let X be an uncountable space that has the discrete topology (and so is locally compact), and let X^* be the one-point compactification of X. Show that there is no regular Borel measure μ on X^* such that $\text{supp}(\mu) = X^*$.
5. Let X and Y be as in Exercise 7.1.10.
   (a) Show that there is no regular Borel measure μ on X such that $\text{supp}(\mu) = X$.
   (b) Is there a regular Borel measure μ on Y such that $\text{supp}(\mu) = Y$?
6. Give a proof of Lusin's theorem that does not depend on the Tietze extension theorem. (Hint: Construct real-valued $\mathscr{A}$-measurable functions f_1, f_2, ... such that each f_n has only countably many values and such that $|f_n(x) - f(x)| < 1/2^n$ holds for each n and x. Show that by applying part of the argument in the first paragraph of the proof of Theorem 7.4.4 to the functions f_1, $f_2 - f_1$, $f_3 - f_2$, ... and then using Proposition 7.1.12, we can construct functions g_1, g_2, ... in $\mathscr{K}(X)$ such that $\sum_n g_n$ belongs to $C_0(X)$ and agrees with f on a suitably large compact subset of A. Then modify $\sum_n g_n$ so that it belongs to $\mathscr{K}(X)$ and satisfies inequality (1).)
7. Let X be a topological space and let A be a subset of X. Show that χ_A is lower semicontinuous if and only if A is open.
8. Let X be a topological space and let $f\colon X \to (-\infty, +\infty]$ be lower semicontinuous. Show that if K is a nonempty compact subset of X, then
   (a) f is bounded below on K, and
   (b) there is a point x_0 in K such that $f(x_0) = \inf\{f(x) : x \in K\}$.
9. Let X be a locally compact Hausdorff space, and let f be a nonnegative lower semicontinuous function on X. Show that
   $$f(x) = \sup\{g(x) : g \in \mathscr{K}(X) \text{ and } 0 \le g \le f\}$$
   holds at each x in X.
10. Show by example that in Proposition 7.4.5 we can not replace the assumption that the functions in $\mathscr{H}$ are lower semicontinuous with the assumption that they are Borel measurable. (Hint: Let $X = \mathbb{R}$, let μ be Lebesgue measure, let f be the constant function 1, and choose $\mathscr{H}$ in such a way that $\int h \, d\mu = 0$ holds for each h in $\mathscr{H}$.)

7.5 The μ^*-Measurable Sets and the Dual of L^1

Let X be a locally compact Hausdorff space, and let I be a positive linear functional on $\mathscr{K}(X)$. In Sect. 7.2 we constructed an outer measure μ^* on X by using the equation

$$\mu^*(U) = \sup\{I(f) : f \in \mathscr{K}(X) \text{ and } f \prec U\}, \tag{1}$$

to define the outer measure of the open subsets of X, and then using the equation

$$\mu^*(A) = \inf\{\mu^*(U) : U \text{ is open and } A \subseteq U\} \tag{2}$$

to extend μ^* to all the subsets of X. Let $\mathscr{M}_{\mu^*}$ be the σ-algebra of μ^*-measurable sets. We showed that $\mathscr{B}(X) \subseteq \mathscr{M}_{\mu^*}$ and that the restrictions μ and μ_1 of μ^* to $\mathscr{B}(X)$ and to $\mathscr{M}_{\mu^*}$ are regular measures such that

$$\int f\,d\mu = \int f\,d\mu_1 = I(f)$$

holds for each f in $\mathscr{K}(X)$.

Although the Borel measure μ is appropriate for most purposes, its extension μ_1 is occasionally useful (see Theorem 7.5.4 and Exercise 2). In this section we will study a few of the properties of μ_1 and of $\mathscr{M}_{\mu^*}$.

Proposition 7.5.1. *Let X be a locally compact Hausdorff space, and let μ^* and $\mathscr{M}_{\mu^*}$ be as in the introduction to this section. If B is a subset of X, then the conditions*

(a) *$B \in \mathscr{M}_{\mu^*}$,*
(b) *$B \cap U \in \mathscr{M}_{\mu^*}$ whenever U is an open subset of X for which $\mu^*(U)$ is finite, and*
(c) *$B \cap K \in \mathscr{M}_{\mu^*}$ whenever K is a compact subset of X*

are equivalent.

Proof. Since the open subsets of X and the compact subsets of X belong to $\mathscr{M}_{\mu^*}$, condition (a) implies conditions (b) and (c).

Next assume that condition (b) holds. According to the discussion preceding Proposition 1.3.5, we can prove that B is μ^*-measurable by showing that

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c) \tag{3}$$

holds for each subset A of X that satisfies $\mu^*(A) < +\infty$. So let A be such a set, and let U be an open set that includes A and satisfies $\mu^*(U) < +\infty$. Then condition (b) says that $U \cap B$ is μ^*-measurable, and so

$$\mu^*(U) = \mu^*(U \cap B) + \mu^*(U \cap B^c) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

Since U can be chosen so as to make $\mu^*(U)$ arbitrarily close to $\mu^*(A)$, inequality (3) follows. With this the proof that (b) implies (a) is complete.

Finally, suppose that condition (c) holds. We will show that condition (b) follows. Let U be an open set such that $\mu^*(U) < +\infty$, and choose a sequence $\{K_n\}$ of compact subsets of U such that $\mu^*(U) = \sup_n \mu^*(K_n)$. Then on the one hand, condition (c) says that each $B \cap K_n$ belongs to $\mathscr{M}_{\mu^*}$, while on the other hand, $B \cap (U - \cup_n K_n)$, as a subset of $U - \cup_n K_n$, has μ^*-measure 0 and so belongs to $\mathscr{M}_{\mu^*}$. Since $B \cap U$ is the union of these sets, it also belongs to $\mathscr{M}_{\mu^*}$ and condition (b) follows. □

The following lemma is needed for our proof of Proposition 7.5.3, which is an important technical fact about the σ-algebra $\mathscr{M}_{\mu^*}$ of μ^*-measurable sets.

Lemma 7.5.2. *Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let μ be a regular measure on $(X, \mathscr{A})$. If K is a compact subset of X such that $\mu(K) > 0$, then there is a compact subset K_0 of K such that $\mu(K_0) = \mu(K)$ and such that each open subset U of X that meets K_0 satisfies $\mu(U \cap K_0) > 0$.*

Proof. The proof here is very similar to that of Proposition 7.4.1: here we let U be the union of the open sets V such that $\mu(V \cap K) = 0$, and we check that every compact subset of $U \cap K$ has measure zero. It then follows from Proposition 7.2.6 that $\mu(U \cap K) = 0$, and so we can let K_0 be $K \cap U^c$. □

Proposition 7.5.3. *Let X be a locally compact Hausdorff space, and let μ^*, $\mathscr{M}_{\mu^*}$, and μ_1 be as in the introduction to this section. Then there is a disjoint family $\mathscr{C}_0$ of compact subsets of X such that*

(a) *if $K \in \mathscr{C}_0$, then $\mu_1(K) > 0$,*
(b) *if U is open, if $K \in \mathscr{C}_0$, and if $U \cap K \neq \varnothing$, then $\mu_1(U \cap K) > 0$,*
(c) *if $A \in \mathscr{M}_{\mu^*}$ and if $\mu_1(A) < +\infty$, then $A \cap K \neq \varnothing$ for only countably many sets K in $\mathscr{C}_0$, and*
$$\mu_1(A) = \sum_K \mu_1(A \cap K),$$
(d) *a subset A of X belongs to $\mathscr{M}_{\mu^*}$ if and only if for each K in $\mathscr{C}_0$ the set $A \cap K$ belongs to $\mathscr{M}_{\mu^*}$, and*
(e) *a function $f\colon X \to \mathbb{R}$ is $\mathscr{M}_{\mu^*}$-measurable if and only if for each K in $\mathscr{C}_0$ the function $f\chi_K$ is $\mathscr{M}_{\mu^*}$-measurable.*

Proof. Let Ξ be the collection of all families $\mathscr{C}$ of compact subsets of X such that

(i) the sets in $\mathscr{C}$ are disjoint from one another,
(ii) if $K \in \mathscr{C}$, then $\mu_1(K) > 0$, and
(iii) if U is open, if $K \in \mathscr{C}$, and if $U \cap K \neq \varnothing$, then $\mu_1(U \cap K) > 0$.

Note that Ξ contains $\varnothing$ and so is nonempty, and that Ξ is partially ordered by inclusion. Furthermore, if Ξ_0 is a linearly ordered subcollection of Ξ, then $\bigcup \Xi_0$ belongs to Ξ and so is an upper bound for Ξ_0. Hence Zorn's lemma (see Theorem A.13) implies that Ξ has a maximal element.

Let $\mathscr{C}_0$ be a maximal element of Ξ. We will check that $\mathscr{C}_0$ satisfies properties (a) through (e). Properties (a) and (b) are immediate.

We turn to property (c). Suppose that A belongs to $\mathscr{M}_{\mu^*}$ and satisfies $\mu_1(A) < +\infty$, and use (2) to choose an open set U such that $A \subseteq U$ and $\mu_1(U) < +\infty$. Then each set K in $\mathscr{C}_0$ that meets A also meets U and so, by property (b), satisfies $\mu_1(U \cap K) > 0$. Since $\mu_1(U) < +\infty$ and the sets in $\mathscr{C}_0$ are disjoint from one another, there can for each n be only finitely many sets K in $\mathscr{C}_0$ such that $\mu_1(U \cap K) > 1/n$ and hence only countably many sets K in $\mathscr{C}_0$ such that $\mu_1(U \cap K) > 0$. Since $A \subseteq U$, it follows that only countably many of the sets in $\mathscr{C}_0$ meet A.

Now consider the second half of (c). To prove that $\mu_1(A) = \sum_K \mu_1(A \cap K)$, where K ranges over those sets in $\mathscr{C}_0$ that meet A, we need only show that $A - (\cup_K (A \cap K))$ has μ_1-measure zero. But if that set had positive measure, then according to Proposition 7.2.6 and Lemma 7.5.2, it would include a compact subset K that would satisfy $\mu_1(K) > 0$ and $\mu_1(U \cap K) > 0$ for each open set U such that $U \cap K \neq \varnothing$. Such a set K would be disjoint from all the sets in $\mathscr{C}_0$. This cannot happen, however, since it would contradict the maximality of the family $\mathscr{C}_0$. With this the proof of property (c) is complete.

To begin the proof of property (d), suppose that A is a set such that $A \cap K \in \mathscr{M}_{\mu^*}$ holds for each K in $\mathscr{C}_0$. According to Proposition 7.5.1, it is enough to show that $A \cap L \in \mathscr{M}_{\mu^*}$ for an arbitrary compact subset L of X. So let L be such a set. Part (c) of the current proposition says that L meets only countably many of the sets in $\mathscr{C}_0$ and that $\mu_1(L - \cup_n K_n) = 0$, where $\{K_n\}$ is the collection of sets in $\mathscr{C}_0$ that meet L. Thus $A \cap L$ is the union of the countable collection of sets of the form $A \cap K_n \cap L$, together with a subset of the μ_1-null set $L - \cup_n K_n$. Since all these sets are μ^*-measurable, the measurability of A follows and half of property (d) is proved. The converse half is immediate.

Property (e) follows from property (d), since for each Borel subset B of $\mathbb{R}$ and each K in $\mathscr{C}_0$ we have $f^{-1}(B) \cap K = (f\chi_K)^{-1}(B) \cap K$. □

Let us turn to an application of the preceding result. Suppose that $(X, \mathscr{A}, \mu)$ is an arbitrary measure space and that T is the map from $L^\infty(X, \mathscr{A}, \mu)$ to $(L^1(X, \mathscr{A}, \mu))^*$ that associates to each $\langle g \rangle$ in $L^\infty(X, \mathscr{A}, \mu)$ the functional $T_{\langle g \rangle}$ defined by

$$T_{\langle g \rangle}(\langle f \rangle) = \int fg\, d\mu \tag{4}$$

(see Sect. 3.5). Recall that T is an isometric isomorphism of $L^\infty(X, \mathscr{A}, \mu)$ onto a subspace of $(L^1(X, \mathscr{A}, \mu))^*$ (Proposition 3.5.5). Recall also that T is surjective if $(X, \mathscr{A}, \mu)$ is σ-finite but fails to be surjective in some other situations (see Theorem 4.5.1 and the example at the end of Sect. 4.5). We now use Proposition 7.5.3 to show that the map T is surjective for a large class of not necessarily σ-finite spaces.

Theorem 7.5.4. *Let X be a locally compact Hausdorff space, and let μ^*, $\mathscr{M}_{\mu^*}$, and μ_1 be as in the introduction to this section. Then the map T given by $T_{\langle g \rangle}(\langle f \rangle) = \int fg\, d\mu_1$ is an isometric isomorphism of $L^\infty(X, \mathscr{M}_{\mu^*}, \mu_1)$ onto $(L^1(X, \mathscr{M}_{\mu^*}, \mu_1))^*$.*

Proof. In view of the preceding discussion, only the surjectivity of T needs to be checked. Let F belong to $(L^1(X, \mathscr{M}_{\mu^*}, \mu_1))^*$, and let $\mathscr{C}_0$ be a disjoint family of compact subsets of X for which properties (a) through (e) of Proposition 7.5.3 hold. For each K in $\mathscr{C}_0$ consider the measure space $(K, \mathscr{M}_K, \mu_K)$, where $\mathscr{M}_K$ is the σ-algebra consisting of those subsets of K that belong to $\mathscr{M}_{\mu^*}$ and μ_K is the restriction of μ_1 to $\mathscr{M}_K$. Let F_K be the functional on $L^1(K, \mathscr{M}_K, \mu_K)$ defined by $F_K(\langle f \rangle) = F(\langle f' \rangle)$, where f' is the function on X that agrees on K with f and that vanishes outside K. Since μ_K is a finite measure, there is (Theorem 4.5.1) an $\mathscr{M}_K$-measurable function g_k on K such that

$$\sup\{|g_K(x)| : x \in K\} = \|F_K\| \leq \|F\| \tag{5}$$

and such that $F_K(\langle f \rangle) = \int_K f g_K \, d\mu_K$ holds for each $\langle f \rangle$ in $L^1(K, \mathscr{M}_K, \mu_K)$. For each K in $\mathscr{C}_0$ choose such a function g_K. Let g be the function on X that vanishes outside $\cup \mathscr{C}_0$ and that for each K in $\mathscr{C}_0$ agrees with g_K on K. It follows from part (e) of Proposition 7.5.3 and inequality (5) that $g \in \mathscr{L}^\infty(X, \mathscr{M}_{\mu^*}, \mu_1)$.

Let us check that $F = T_{\langle g \rangle}$. It is clear that if f is a member of $\mathscr{L}^1(X, \mathscr{M}_{\mu^*}, \mu_1)$ that vanishes outside some K in $\mathscr{C}_0$, then $F(\langle f \rangle) = T_{\langle g \rangle}(\langle f \rangle)$. If f is an arbitrary function in $\mathscr{L}^1(X, \mu_{\mu^*}, \mu_1)$, then f vanishes outside the union of a sequence of sets of finite measure (Corollary 2.3.11); thus according to part (c) of Proposition 7.5.3, there is a sequence $\{K_n\}$ of sets in $\mathscr{C}_0$ such that f vanishes almost everywhere outside $\cup_n K_n$. Since the functionals F and $T_{\langle g \rangle}$ agree on each $\langle f \chi_{K_n} \rangle$ and since $\lim_N \|f - \sum_{n=1}^N f \chi_{K_n}\|_1 = 0$, it follows that $F(\langle f \rangle) = T_{\langle g \rangle}(\langle f \rangle)$. Thus $F = T_{\langle g \rangle}$, and the proof is complete. □

It is natural to ask whether in Theorem 7.5.4 the measure space $(X, \mathscr{M}_{\mu^*}, \mu_1)$ can be replaced with $(X, \mathscr{B}(X), \mu)$. This change can of course be made if μ_1 and μ are σ-finite and can also be made in certain other situations (see Theorem 9.4.8); it cannot be made in general (see Fremlin [47]).

We are now in a position to sketch the relationship of the treatment of integration on locally compact Hausdorff spaces given here to that given by Bourbaki (see [18]).

Let X be a locally compact Hausdorff space and let $\mathscr{I}_+(X)$ be the set of all $[0, +\infty]$-valued lower semicontinuous functions on X. Suppose that I is a positive linear functional on $\mathscr{K}(X)$ (in Bourbaki's terminology, I is a *positive Radon measure* on X). Bourbaki defines a function $I^* : \mathscr{I}_+(X) \to [0, +\infty]$ by

$$I^*(f) = \sup\{I(g) : g \in \mathscr{K}(X) \text{ and } 0 \leq g \leq f\}$$

and then uses the formula

$$I^*(f) = \inf\{I^*(h) : h \in \mathscr{I}_+(X) \text{ and } f \leq h\}$$

to extend I^* to the set of all $[0, +\infty]$-valued functions on X. He checks that I^* satisfies

$$I^*(f + g) \leq I^*(f) + I^*(g) \tag{6}$$

and

$$I^*(af) = aI^*(f) \tag{7}$$

for all $f, g\colon X \to [0, +\infty]$ and all a in $[0, +\infty)$. Of course, $I^*(0) = 0$. It follows that the set $\mathscr{F}^1$ of functions $f\colon X \to \mathbb{R}$ for which $I^*(|f|) < +\infty$ is a vector space over $\mathbb{R}$ and that the function $N_1\colon \mathscr{F}^1 \to \mathbb{R}$ defined by $N_1(f) = I^*(|f|)$ is a seminorm on $\mathscr{F}^1$. Bourbaki then defines $\mathscr{L}^1(X, I)$ to be the closure of $\mathscr{K}(X)$ in[6] $\mathscr{F}^1$ (of course $\mathscr{F}^1$ is given the topology determined by N_1), and extends I from $\mathscr{K}(X)$ to $\mathscr{L}^1(X, I)$ by letting

$$I(f) = \lim_n I(f_n) \tag{8}$$

hold whenever $\{f_n\}$ is a sequence of functions in $\mathscr{K}(X)$ for which $\lim_n N_1(f_n - f) = 0$ (check that the limit in (8) exists and depends only on f). He calls the functions that belong to $\mathscr{L}^1(X, I)$ *I-integrable*, and he calls the extension of I to $\mathscr{L}^1(X, I)$ the *integral*; he often writes[7] $\int f\,dI$ in place of $I(f)$. He calls a function $f\colon X \to \mathbb{R}$ *I-measurable* if for each compact subset K of X and each positive number ε there is a compact subset L of K that satisfies $I^*(\chi_{K-L}) < \varepsilon$ and is such that the restriction of f to L is continuous. Furthermore, he calls a subset A of X *I-integrable* if χ_A is I-integrable and *I-measurable* if χ_A is I-measurable.

The following theorem shows how these concepts are related to those treated earlier in this chapter.

Theorem 7.5.5. *Let X be a locally compact Hausdorff space, let I be a positive linear functional on $\mathscr{K}(X)$, let $\mathscr{L}^1(X, I)$ be as defined in the preceding paragraphs, and let μ^*, $\mathscr{M}_{\mu^*}$, and μ_1 be as defined at the beginning of this section. Then*

(a) $\mathscr{L}^1(X, I) = \mathscr{L}^1(X, \mathscr{M}_{\mu^*}, \mu_1, \mathbb{R})$,
(b) $\int f\,dI = \int f\,d\mu_1$ *holds for each f in $\mathscr{L}^1(X, I)$,*
(c) *a subset A of X is I-measurable if and only if it belongs to $\mathscr{M}_{\mu^*}$, and*
(d) *a function $f\colon X \to \mathbb{R}$ is I-measurable if and only if it is $\mathscr{M}_{\mu^*}$-measurable.*

Proof (A Sketch). It follows from Proposition 7.4.5 and Exercise 7.4.9 that

$$I^*(f) = \int f\,d\mu_1 \tag{9}$$

holds for each f in $\mathscr{I}_+(X)$ and then from (9), together with the additivity and homogeneity of the integral, that (6) and (7) hold for all $f, g\colon X \to [0, +\infty]$ and

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[6] Note that $I^*(|f|) = I(|f|) < +\infty$ holds for each f in $\mathscr{K}(X)$ and hence that $\mathscr{K}(X)$ is included in $\mathscr{F}^1$.

[7] Actually, he usually calls his positive linear functional μ, and he writes $\mu(f)$ and $\int f\,d\mu$, rather than $I(f)$ and $\int f\,dI$; such notation will not be used in this book, since we have been using μ to denote a measure.

all a in $[0,+\infty)$.[8] Consequently $\mathscr{F}^1$ is a vector space and N_1 is a seminorm on it. The reader should check that

$$I^*(|f|) = \int |f|\,d\mu_1 \tag{10}$$

holds for each f in $\mathscr{L}^1(X,\mathscr{M}_{\mu^*},\mu_1,\mathbb{R})$ (use (9) and an appropriate extension of Lemma 6.3.12 to the case of functions on locally compact Hausdorff spaces).

In view of (10), Proposition 7.4.3 implies that $\mathscr{L}^1(X,\mathscr{M}_{\mu^*},\mu_1,\mathbb{R})$ is included in $\mathscr{L}^1(X,I)$ and that $\int f\,d\mu_1 = \int f\,dI$ holds for each f in $\mathscr{L}^1(X,\mathscr{M}_{\mu^*},\mu_1,\mathbb{R})$. The reverse inclusion is left to the reader (use (9) to show that if $f \in \mathscr{L}^1(X,I)$, then there is a sequence $\{f_n\}$ in $\mathscr{K}(X)$ that converges to f almost everywhere with respect to μ_1 and satisfies $\lim_n N_1(f-f_n) = 0$; then use the completeness of $L^1(X,\mathscr{M}_{\mu^*},\mu_1,\mathbb{R})$). With this parts (a) and (b) of the theorem are proved.

Part (d) follows from Lusin's theorem (Theorem 7.4.4) and Exercise 2. Finally, part (c) is a special case of part (d). □

Note that if I is a positive linear functional on $\mathscr{K}(X)$, then

$$\begin{aligned}&\text{for each compact subset } K \text{ of } X \text{ there is a number } c_K \text{ such}\\ &\text{that } |I(f)| \le c_K\|f\|_\infty \text{ holds whenever } f \text{ belongs to } \mathscr{K}(X)\\ &\text{and satisfies } \operatorname{supp}(f) \subseteq K\end{aligned} \tag{11}$$

(choose a function g that belongs to $\mathscr{K}(X)$ and satisfies $\chi_K \le g$, and let c_K be $I(g)$). Bourbaki calls a (not necessarily positive) linear functional I on $\mathscr{K}(X)$ a *Radon measure* on X if it satisfies (11). Since each difference of positive linear functionals on $\mathscr{K}(X)$ satisfies (11), each such difference is a Radon measure. The proof of Lemma 7.3.7 can be modified so as to show that every Radon measure on X is the difference of positive Radon measures on X (that is, of positive linear functionals on $\mathscr{K}(X)$). Thus the set of Radon measures on X is the vector space generated by the set of positive linear functionals on $\mathscr{K}(X)$.

Note that the formula

$$I(f) = \int_0^{+\infty} f(x)\,\lambda(dx) - \int_{-\infty}^0 f(x)\,\lambda(dx)$$

defines a Radon measure on $\mathbb{R}$; this Radon measure cannot be represented in terms of integration with respect to a signed measure on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ (recall that the positive and negative parts of a signed measure cannot both be infinite). See, however, Exercise 6.

[8] Bourbaki develops integration theory without first developing measure theory; his proofs, for example, of (6) and (7) are therefore quite different from those given here.

Exercises

1. Show that the assumption that $A \in \mathscr{M}_{\mu^*}$ can be omitted from part (c) of Proposition 7.5.3; that is, show that if $\mu^*(A) < +\infty$, then $A \cap K \neq \varnothing$ holds for only countably many of the sets K in $\mathscr{C}_0$, and
$$\mu^*(A) = \sum_K \mu^*(A \cap K).$$
2. Let X, μ^*, and μ_1 be as in the introduction to this section, and let f be a real-valued function on X. Suppose that for each compact subset K of X and each positive ε there is a compact subset L of K such that
   (i) $\mu_1(K - L) < \varepsilon$, and
   (ii) the restriction of f to L is continuous.

   Show that f is $\mathscr{M}_{\mu^*}$-measurable. (Note that this is a sort of converse to Lusin's theorem and that it explains one of the remarks following the proof of Theorem 7.4.4.)
3. Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, and let ν be a regular measure on $(X, \mathscr{A})$. Define a positive linear functional I on $\mathscr{K}(X)$ by $I(f) = \int f\, d\nu$. Show that if μ^*, $\mathscr{M}_{\mu^*}$, and μ_1 are associated to I as in this section, then $\mathscr{A} \subseteq \mathscr{M}_{\mu^*}$ and ν is the restriction of μ_1 to $\mathscr{A}$.
4. Show by example that the assumption of σ-finiteness can not be omitted in Exercise 7.2.3. (Hint: See Exercise 7.2.4.)
5. Let X, I, μ^*, $\mathscr{M}_{\mu^*}$, μ, and μ_1 be as in the introduction to this section. Suppose that $1 \leq p < +\infty$.
   (a) Show that if $f \in \mathscr{L}^p(X, \mathscr{M}_{\mu^*}, \mu_1)$, then there is a function that belongs to $\mathscr{L}^p(X, \mathscr{B}(X), \mu)$ and agrees with f μ-almost everywhere.
   (b) Conclude that $L^p(X, \mathscr{M}_{\mu^*}, \mu_1)$ and $L^p(X, \mathscr{B}(X), \mu)$ are isometrically isomorphic to one another.
6. Show that if I is a Radon measure on the locally compact Hausdorff space X, then there are regular Borel measures μ_1 and μ_2 on X such that $I(f) = \int f\, d\mu_1 - \int f\, d\mu_2$ holds for each f in $\mathscr{K}(X)$.

7.6 Products of Locally Compact Spaces

This section is devoted to the study of products of regular Borel measures on locally compact Hausdorff spaces. In Chap. 5 we proved that if μ and ν are σ-finite measures on measurable spaces $(X, \mathscr{A})$ and $(Y, \mathscr{B})$, then there is a unique measure $\mu \times \nu$ on $(X \times Y, \mathscr{A} \times \mathscr{B})$ such that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ holds for each A in $\mathscr{A}$ and each B in $\mathscr{B}$. Now assume that X and Y are locally compact Hausdorff spaces. Then $X \times Y$ is a locally compact Hausdorff space, and it would be convenient if for each pair of regular Borel measures on X and Y, the constructions in Chap. 5 gave a regular Borel measure on $X \times Y$. However, two problems arise. First, regular Borel measures can fail to be σ-finite, and so the earlier theory can fail to apply. Second, the product σ-algebra $\mathscr{B}(X) \times \mathscr{B}(Y)$ can fail to contain all the Borel subsets of $X \times Y$ (see Exercise 5.1.8), in which case no measure on $\mathscr{B}(X) \times \mathscr{B}(Y)$ can be regular.

We will begin by proving that these difficulties cannot arise if the spaces X and Y have countable bases for their topologies; then we will turn to a theory of product measures that is suitable for Borel measures on arbitrary locally compact Hausdorff spaces. Lemma 7.6.1 and Proposition 7.6.2 suffice for most applications. The remaining parts of this section will be used only in Chap. 9 and should be skipped by most readers.

Let us recall some notation. Suppose that X and Y are sets and that E is a subset of $X \times Y$. For each x in X and each y in Y the sections E_x and E^y are the subsets of Y and X given by

$$E_x = \{y \in Y : (x,y) \in E\}$$

and

$$E^y = \{x \in X : (x,y) \in E\}.$$

Likewise, if f is a function whose domain is $X \times Y$, then for each x in X and each y in Y the sections f_x and f^y are the functions on Y and X defined by

$$f_x(y) = f(x,y)$$

and

$$f^y(x) = f(x,y).$$

The following lemma summarizes some useful elementary facts.

Lemma 7.6.1. *Let X and Y be Hausdorff topological spaces, and let $X \times Y$ be their product. Then*

(a) *the product σ-algebra $\mathscr{B}(X) \times \mathscr{B}(Y)$ is included in $\mathscr{B}(X \times Y)$,*
(b) *if $E \in \mathscr{B}(X \times Y)$, then for each x in X the section E_x belongs to $\mathscr{B}(Y)$, and for each y in Y the section E^y belongs to $\mathscr{B}(X)$, and*
(c) *if $f\colon X \times Y \to \mathbb{R}$ is $\mathscr{B}(X \times Y)$-measurable, then for each x in X the section f_x is $\mathscr{B}(Y)$-measurable, and for each y in Y the section f^y is $\mathscr{B}(X)$-measurable.*

Proof. The projection π_1 of $X \times Y$ onto X is continuous and so is measurable with respect to $\mathscr{B}(X \times Y)$ and $\mathscr{B}(X)$ (Lemma 7.2.1). Likewise, the projection π_2 of $X \times Y$ onto Y is measurable with respect to $\mathscr{B}(X \times Y)$ and $\mathscr{B}(Y)$. Note that if $A \subseteq X$ and $B \subseteq Y$, then

$$A \times B = (A \times Y) \cap (X \times B) = \pi_1^{-1}(A) \cap \pi_2^{-1}(B).$$

Hence if $A \in \mathscr{B}(X)$ and $B \in \mathscr{B}(Y)$, then $A \times B \in \mathscr{B}(X \times Y)$. Since $\mathscr{B}(X) \times \mathscr{B}(Y)$ is the σ-algebra generated by the collection of all such rectangles $A \times B$, it follows that $\mathscr{B}(X) \times \mathscr{B}(Y) \subseteq \mathscr{B}(X \times Y)$. Thus part (a) is proved.

To check the first assertion in part (b), suppose that x belongs to X, and define $g\colon Y \to X \times Y$ by $g(y) = (x,y)$. Then g is continuous and so is measurable with respect to $\mathscr{B}(Y)$ and $\mathscr{B}(X \times Y)$. Each subset E of $X \times Y$ satisfies $E_x = g^{-1}(E)$; hence if $E \in \mathscr{B}(X \times Y)$, then $E_x \in \mathscr{B}(Y)$. The second assertion in part (b) is proved in the same way.

Part (c) follows from part (b) and the fact that if $B \subseteq \mathbb{R}$, then $(f_x)^{-1}(B) = (f^{-1}(B))_x$ and $(f^y)^{-1}(B) = (f^{-1}(B))^y$. □

Now we prove that the difficulties mentioned in the introduction to this section do not occur if each of the spaces X and Y has a countable base.

Proposition 7.6.2. *Let X and Y be locally compact Hausdorff spaces that have countable bases for their topologies. Then $\mathscr{B}(X \times Y) = \mathscr{B}(X) \times \mathscr{B}(Y)$. Furthermore, if μ and ν are regular Borel measures on X and Y, respectively, then μ and ν are σ-finite, and $\mu \times \nu$ is a regular Borel measure on $X \times Y$.*

Proof. Lemma 7.6.1 implies that $\mathscr{B}(X) \times \mathscr{B}(Y) \subseteq \mathscr{B}(X \times Y)$. We turn to the reverse inclusion. Let $\mathscr{U}$ and $\mathscr{V}$ be countable bases for X and Y, and let $\mathscr{W}$ be the collection of rectangles of the form $U \times V$, where $U \in \mathscr{U}$ and $V \in \mathscr{V}$. Then $\mathscr{W}$ is a countable base for $X \times Y$ and is included in $\mathscr{B}(X) \times \mathscr{B}(Y)$. Each open subset of $X \times Y$ is the union of a (necessarily countable) subfamily of the base $\mathscr{W}$ and so belongs to $\mathscr{B}(X) \times \mathscr{B}(Y)$. Since $\mathscr{B}(X \times Y)$ is generated by the open subsets of $X \times Y$, it follows that $\mathscr{B}(X \times Y) \subseteq \mathscr{B}(X) \times \mathscr{B}(Y)$. Thus $\mathscr{B}(X \times Y) = \mathscr{B}(X) \times \mathscr{B}(Y)$.

Now suppose that μ and ν are regular Borel measures on X and Y, respectively. Then μ and ν are σ-finite (Proposition 7.2.5), and so the constructions of Chap. 5 provide a unique product measure $\mu \times \nu$ on $\mathscr{B}(X) \times \mathscr{B}(Y)$. Since $\mathscr{B}(X) \times \mathscr{B}(Y) = \mathscr{B}(X \times Y)$, the measure $\mu \times \nu$ is a Borel measure. If K is a compact subset of $X \times Y$ and if K_1 and K_2 are the projections of K on X and Y, respectively, then K_1 and K_2 are compact, and so

$$(\mu \times \nu)(K) \le (\mu \times \nu)(K_1 \times K_2) = \mu(K_1)\nu(K_2) < +\infty.$$

Thus $\mu \times \nu$ is finite on the compact subsets of $X \times Y$. Since there is a countable base for $X \times Y$ (for example, the base $\mathscr{W}$ defined above), Proposition 7.2.3 implies that $\mu \times \nu$ is regular. □

Now let X and Y be arbitrary locally compact Hausdorff spaces, and let μ and ν be regular Borel measures on X and Y, respectively. As we noted in the introduction to this section, μ and ν can fail to be σ-finite, and the σ-algebra $\mathscr{B}(X) \times \mathscr{B}(Y)$ can fail to contain all the sets in $\mathscr{B}(X \times Y)$. Suppose, however, that we could prove that for each f in $\mathscr{K}(X \times Y)$ the iterated integrals $\int_X \int_Y f(x,y)\,\nu(dy)\,\mu(dx)$ and $\int_Y \int_X f(x,y)\,\mu(dx)\,\nu(dy)$ exist and are equal. We could then proceed in two steps, first defining a positive linear functional I on $\mathscr{K}(X \times Y)$ by letting $I(f)$ be the common value of these iterated integrals and then using the Riesz representation theorem to obtain the corresponding regular Borel measure on $X \times Y$. This is indeed the course that we will follow. The following propositions contain the necessary details.

Lemma 7.6.3. *Suppose that S and T are topological spaces, that T is compact, and that $f\colon S \times T \to \mathbb{R}$ is continuous. Then for each s_0 in S and each positive number ε there is an open neighborhood U of s_0 such that $|f(s,t) - f(s_0,t)| < \varepsilon$ holds for each s in U and each t in T.*

Proof. Suppose that s_0 belongs to S and that ε is a positive number. For each t in T choose open neighborhoods U_t of s_0 and V_t of t such that if $(s,t') \in U_t \times V_t$, then $|f(s,t') - f(s_0,t)| < \varepsilon/2$. It follows that if $s \in U_t$ and $t' \in V_t$, then

$$\begin{aligned} |f(s,t') - f(s_0,t')| &\le |f(s,t') - f(s_0,t)| + |f(s_0,t) - f(s_0,t')| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since T is compact, we can choose a finite collection $t_1, \ldots, t_n$ of points in T such that the neighborhoods $V_{t_1}, \ldots, V_{t_n}$ cover T. Then $\cap_{i=1}^n U_{t_i}$ is the required neighborhood of s_0. □

Proposition 7.6.4. *Let X and Y be locally compact Hausdorff spaces, let μ and ν be regular Borel measures on X and Y, respectively, and let f belong to $\mathscr{K}(X \times Y)$. Then*

(a) *for each x in X and each y in Y the sections f_x and f^y belong to $\mathscr{K}(Y)$ and $\mathscr{K}(X)$, respectively,*
(b) *the functions*

$$x \mapsto \int_Y f(x,y)\,\nu(dy)$$

and

$$y \mapsto \int_X f(x,y)\,\mu(dx)$$

belong to $\mathscr{K}(X)$ and $\mathscr{K}(Y)$, respectively, and
(c) $\int_X \int_Y f(x,y)\,\nu(dy)\,\mu(dx) = \int_Y \int_X f(x,y)\,\mu(dx)\,\nu(dy)$.

Proof. Let f belong to $\mathscr{K}(X \times Y)$, let K be the support of f, and let K_1 and K_2 be the projections of K on X and Y, respectively. Then K_1 and K_2 are compact.

If $x \in X$, then the section f_x is continuous, since it results from composing the continuous function $y \mapsto (x,y)$ with the continuous function f. The support of f_x is included in K_2 and so is compact. Thus $f_x \in \mathscr{K}(Y)$. A similar argument shows that $f^y \in \mathscr{K}(X)$.

It follows that the integrals in part (b) exist. We now check that the function $x \mapsto \int_Y f(x,y)\,\nu(dy)$ is continuous. Let x_0 belong to X and let ε be a positive number. According to Lemma 7.6.3, applied to the space $X \times K_2$, there is an open neighborhood U of x_0 such that if $x \in U$ and $y \in K_2$, then $|f(x,y) - f(x_0,y)| < \varepsilon$. Hence if $x \in U$, then

$$\left|\int_Y f(x,y)\,\nu(dy) - \int_Y f(x_0,y)\,\nu(dy)\right| \\ \leq \int_{K_2} |f(x,y) - f(x_0,y)|\,\nu(dy) \leq \varepsilon\nu(K_2).$$

Since ε was arbitrary, the continuity of $x \mapsto \int_Y f(x,y)\,\nu(dy)$ follows. In addition this function vanishes outside K_1, and so it belongs to $\mathscr{K}(X)$. A similar argument shows that the function $y \mapsto \int_X f(x,y)\,\mu(dx)$ belongs to $\mathscr{K}(Y)$.

We turn to part (c). Parts (a) and (b) imply that the integrals involved here exist. We prove that they are equal by approximating f with simpler functions. Let ε be an arbitrary positive number. For each x in K_1 choose a neighborhood U_x of x such that if $x' \in U_x$ and $y \in K_2$, then $|f(x',y) - f(x,y)| < \varepsilon$ (see Lemma 7.6.3). The set K_1 is compact, and so there exist points $x_1, \ldots, x_n$ in K_1 such that the sets $U_{x_1}, \ldots, U_{x_n}$ cover K_1. Now use these sets to construct disjoint Borel sets $A_1, \ldots, A_n$ such that $K_1 = \cup_i A_i$ and such that $A_i \subseteq U_{x_i}$ holds for $i = 1, \ldots, n$. Define $g\colon X \times Y \to \mathbb{R}$ by $g(x,y) = \sum_{i=1}^n \chi_{A_i}(x) f(x_i,y)$. The functions f and g vanish outside $K_1 \times K_2$ and satisfy $|f(x,y) - g(x,y)| < \varepsilon$ at each (x,y) in $K_1 \times K_2$; hence they satisfy

$$\left|\int_Y \int_X f(x,y)\,\mu(dx)\,\nu(dy) - \int_Y \int_X g(x,y)\,\mu(dx)\,\nu(dy)\right| \leq \varepsilon\mu(K_1)\nu(K_2)$$

and

$$\left|\int_X \int_Y f(x,y)\,\nu(dy)\,\mu(dx) - \int_X \int_Y g(x,y)\,\nu(dy)\,\mu(dx)\right| \leq \varepsilon\mu(K_1)\nu(K_2).$$

The two iterated integrals of g are both equal to $\sum \mu(A_i) \int f(x_i,y)\,\nu(dy)$; thus they are equal to each other, and so

$$\left|\int_Y \int_X f(x,y)\,\mu(dx)\,\nu(dy) - \int_X \int_Y f(x,y)\,\nu(dy)\,\mu(dx)\right| \leq 2\varepsilon\mu(K_1)\nu(K_2).$$

Since ε is arbitrary, the proof is complete. □

Let X and Y be locally compact Hausdorff spaces, and let μ and ν be regular Borel measures on X and Y, respectively. As promised earlier, we define $I\colon \mathscr{K}(X \times Y) \to \mathbb{R}$ by letting $I(f)$ be the common value of the iterated integrals $\int_X \int_Y f(x,y)\,\nu(dy)\,\mu(dx)$ and $\int_Y \int_X f(x,y)\,\mu(dx)\,\nu(dy)$. The *regular Borel product* of μ and ν is the regular Borel measure on $X \times Y$ induced by the functional I via the Riesz representation theorem. This measure will be denoted by $\mu \times \nu$.

Proposition 7.6.5. *Let X and Y be locally compact Hausdorff spaces, let μ and ν be regular Borel measures on X and Y, respectively, and let $\mu \times \nu$ be the regular Borel product of μ and ν. If U is an open subset of $X \times Y$, then*

(a) *the functions $x \mapsto \nu(U_x)$ and $y \mapsto \mu(U^y)$ are lower semicontinuous and hence Borel measurable, and*
(b) $(\mu \times \nu)(U) = \int_X \nu(U_x)\,\mu(dx) = \int_Y \mu(U^y)\,\nu(dy)$.

Proof. Of course U_x and U^y are open sets and therefore Borel sets. Let

$$\mathscr{F} = \{f \in \mathscr{K}(X \times Y) : 0 \le f \le \chi_U\},$$

and for each x in X and y in Y define sets $\mathscr{F}_x$ and $\mathscr{F}^y$ by

$$\mathscr{F}_x = \{f_x : f \in \mathscr{F}\} \text{ and}$$
$$\mathscr{F}^y = \{f^y : f \in \mathscr{F}\}.$$

Then $\mathscr{F}_x$ and $\mathscr{F}^y$ are included in $\mathscr{K}(Y)$ and $\mathscr{K}(X)$, respectively, are directed upward, and have χ_{U_x} and χ_{U^y} as their suprema. Since these characteristic functions are lower semicontinuous, Proposition 7.4.5 implies that

$$\nu(U_x) = \sup\left\{\int f_x\,d\nu : f_x \in \mathscr{F}_x\right\} \tag{1}$$

holds for each x in X and that

$$\mu(U^y) = \sup\left\{\int f^y\,d\mu : f^y \in \mathscr{F}^y\right\} \tag{2}$$

holds for each y in Y. Thus the functions $x \mapsto \nu(U_x)$ and $y \mapsto \mu(U^y)$ are suprema of collections of continuous functions (see part (b) of Proposition 7.6.4), and so are lower semicontinuous.

The first half of part (b) will follow, once we check the calculation

$$\begin{aligned}(\mu \times \nu)(U) &= \sup_{f \in \mathscr{F}} \int_X \int_Y f(x,y)\,\nu(dy)\,\mu(dx)\\ &= \int_X \left(\sup_{f \in \mathscr{F}} \int f_x\,d\nu\right)\mu(dx)\\ &= \int_X \nu(U_x)\,\mu(dx);\end{aligned}$$

here the first equality is a consequence of Lemma 7.2.7 and the definition of the functional I, the second a consequence of Proposition 7.4.5, and the third a consequence of Eq. (1). The other half of part (b) is proved in a similar way. □

Corollary 7.6.6. *Let X, Y, μ, ν, and $\mu \times \nu$ be as in Proposition 7.6.5. If E is a Borel subset of $X \times Y$ that is included in a rectangle whose sides are Borel sets that are σ-finite under μ and ν, respectively, then*

(a) *the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are Borel measurable, and*
(b) $(\mu \times \nu)(E) = \int_X \nu(E_x)\,\mu(dx) = \int_Y \mu(E^y)\,\nu(dy)$.

Proof. We begin with Borel sets that are included in rectangles whose sides are Borel sets of finite measure. So let A and B be Borel subsets of X and Y that satisfy $\mu(A) < +\infty$ and $\nu(B) < +\infty$. Use the regularity of μ and ν to choose open sets U and V that include A and B and satisfy $\mu(U) < +\infty$ and $\nu(V) < +\infty$. Let $W = U \times V$ and let $\mathscr{S}$ consist of those Borel subsets D of $X \times Y$ for which the functions $x \mapsto \nu((D \cap W)_x)$ and $y \mapsto \mu((D \cap W)^y)$ are Borel measurable and for which the identity

$$\begin{aligned}(\mu \times \nu)(D \cap W) &= \int_X \nu((D \cap W)_x)\,\mu(dx) \\ &= \int_Y \mu((D \cap W)^y)\,\nu(dy)\end{aligned}$$

holds (according to Lemma 7.6.1, the sections $(D \cap W)_x$ and $(D \cap W)^y$ are Borel sets, and so these formulas make sense). According to Proposition 7.6.5, $\mathscr{S}$ contains all the open subsets of $X \times Y$. It is easy to check that

$$\text{if } D_1, D_2 \in \mathscr{S} \text{ and if } D_1 \subseteq D_2, \text{ then } D_2 - D_1 \in \mathscr{S}, \text{ and} \tag{3}$$

$$\text{if } D_1, D_2, \dots \in \mathscr{S} \text{ and if } D_1 \subseteq D_2 \subseteq \dots, \text{ then } \cup_n D_n \in \mathscr{S}. \tag{4}$$

Thus $\mathscr{S}$ is a d-system (see Sect. 1.6) that includes the π-system made up of the open subsets of $X \times Y$, and so Theorem 1.6.2 implies that $\mathscr{B}(X \times Y) \subseteq \mathscr{S}$. Thus if E is a Borel set that is included in $A \times B$, then E satisfies the conclusions of the corollary. Since a Borel set that is included in a rectangle with σ-finite sides is the union of an increasing sequence of Borel sets that are included in rectangles with sides of finite measure, the corollary follows. □

Theorem 7.6.7. *Let X and Y be locally compact Hausdorff spaces, let μ and ν be regular Borel measures on X and Y, respectively, and let $\mu \times \nu$ be the regular Borel product of μ and ν. If f belongs to $\mathscr{L}^1(X \times Y, \mathscr{B}(X \times Y), \mu \times \nu)$ and vanishes outside a rectangle whose sides are Borel sets that are σ-finite under μ and ν, respectively, then*

(a) $f_x \in \mathscr{L}^1(Y, \mathscr{B}(Y), \nu)$ *for μ-almost every x, and* $f^y \in \mathscr{L}^1(X, \mathscr{B}(X), \mu)$ *for ν-almost every y,*
(b) *the functions*

$$x \mapsto \begin{cases} \int f_x\,d\nu & \text{if } f_x \in \mathscr{L}^1(Y, \mathscr{B}(Y), \nu), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$y \mapsto \begin{cases} \int f^y \, d\mu & \textit{if } f^y \in \mathscr{L}^1(X, \mathscr{B}(X), \mu), \\ 0 & \textit{otherwise,} \end{cases}$$

belong to $\mathscr{L}^1(X, \mathscr{B}(X), \mu)$ *and* $\mathscr{L}^1(Y, \mathscr{B}(Y), \nu)$*, respectively, and*

(c) $\int f \, d(\mu \times \nu) = \int_X \int_Y f(x,y) \, \nu(dy) \mu(dx) = \int_Y \int_X f(x,y) \, \mu(dx) \nu(dy)$.

Proof. Let $\mathscr{F}$ be the collection of all functions in $\mathscr{L}^1(X \times Y, \mathscr{B}(X \times Y), \mu \times \nu)$ that vanish outside a rectangle with σ-finite sides. Corollary 7.6.6 implies that if f is a characteristic function that belongs to $\mathscr{F}$, then f satisfies the conclusions of the theorem (the finiteness of $\int f_x \, d\nu$ and $\int f^y \, d\mu$ for almost all x and y follows from Corollary 2.3.14). The linearity of the integral and the monotone convergence theorem imply that the same is true first for nonnegative simple functions in $\mathscr{F}$, then for nonnegative functions in $\mathscr{F}$, and finally for arbitrary functions in $\mathscr{F}$. □

See Exercises 3 and 4 for some techniques for computing $\int |f| \, d(\mu \times \nu)$ and hence for determining whether f is $(\mu \times \nu)$-integrable.

The reader should note several things about the hypotheses of Corollary 7.6.6 and Theorem 7.6.7:

(a) Corollary 7.6.6 would fail if E were only assumed to be a Borel subset of $X \times Y$; see Exercise 1.
(b) Corollary 7.6.6 would also fail if the Borel set E were only assumed to be σ-finite (or even of finite measure) under $\mu \times \nu$; see Exercise 2.
(c) We will see that if μ and ν are Haar measures on locally compact groups X and Y, then each Borel subset E of $X \times Y$ that satisfies $(\mu \times \nu)(E) < +\infty$ is included in a rectangle with σ-finite sides, and each integrable function on $X \times Y$ vanishes outside a rectangle with σ-finite sides (this follows from Lemma 9.4.2, applied to the group $X \times Y$).
(d) See Exercise 6 for an alternate version of Corollary 7.6.6 and Theorem 7.6.7.

Exercises

1. Show that the conclusions of Corollary 7.6.6 can fail if E is an arbitrary Borel (or even closed) subset of $X \times Y$. More precisely, show that part (a) can fail and that part (b) can fail even in cases where part (a) holds. (Hint: Let X be $\mathbb{R}$ with its usual topology, let Y be $\mathbb{R}$ with the discrete topology, let μ be Lebesgue measure on $(X, \mathscr{B}(X))$, let ν be counting measure on $(Y, \mathscr{B}(Y))$, and let E be a suitable subset of $\{(x,y) : x = y\}$.)
2. Let X be $\mathbb{R}$ with its usual topology, let Y be $\mathbb{R}$ with the discrete topology, let μ be a point mass on $(X, \mathscr{B}(X))$, and let ν be counting measure on $(Y, \mathscr{B}(Y))$. Suppose that A is a non-Borel subset of $\mathbb{R}$, and define E to be the set of all pairs (x,x) for which $x \in A$. Show that E is a Borel subset of $X \times Y$ that has finite measure under $\mu \times \nu$, but for which the conclusion of Corollary 7.6.6 fails.

3. Let X, Y, μ, ν, and $\mu \times \nu$ be as in Proposition 7.6.5. Show that if $f : X \times Y \to [0, +\infty]$ is lower semicontinuous, then
   (a) $x \mapsto \int f(x,y)\,\nu(dy)$ and $y \mapsto \int f(x,y)\,\mu(dx)$ are Borel measurable, and
   (b) $\int f\,d(\mu \times \nu) = \int\int f(x,y)\,\nu(dy)\mu(dx) = \int\int f(x,y)\,\mu(dx)\nu(dy)$.
4. Show that the conclusions of Exercise 3 also hold if f is a nonnegative Borel measurable function that vanishes outside a Borel rectangle with σ-finite sides.
5. Show that the Baire σ-algebras on compact Hausdorff spaces (see Exercise 7.2.8) behave "properly" under the formation of products, in the sense that $\mathscr{B}_0(X \times Y) = \mathscr{B}_0(X) \times \mathscr{B}_0(Y)$. (Hint: Use the Stone–Weierstrass theorem (Theorem D.22) to show that each function in $C(X \times Y)$ can be uniformly approximated by functions of the form $(x,y) \mapsto \sum_i f_i(x) g_i(y)$, where the sum is finite, each f_i belongs to $C(X)$, and each g_i belongs to $C(Y)$.)
6. Let X, Y, μ, ν, and $\mu \times \nu$ be as in Proposition 7.6.5, and consider the outer measures μ^*, ν^*, and $(\mu \times \nu)^*$ and measures μ_1, ν_1, and $(\mu \times \nu)_1$ that are associated to μ, ν, and $\mu \times \nu$ as in Sect. 7.5.
   (a) Show that if E belongs to $\mathscr{M}_{(\mu\times\nu)^*}$ and satisfies $(\mu \times \nu)_1(E) = 0$, then $\nu^*(E_x) = 0$ holds for μ_1-almost every x in X and $\mu^*(E^y) = 0$ holds for ν_1-almost every y in Y. (Note that E is not assumed to be included in a rectangle with σ-finite sides.)
   (b) Prove modifications of Corollary 7.6.6, Theorem 7.6.7, and Exercise 4 that apply to $\mathscr{M}_{(\mu\times\nu)^*}$-measurable, rather than Borel measurable, functions. Your modification of Theorem 7.6.7 should not contain the assumption that f vanishes outside a rectangle with σ-finite sides. (Hint: Replace $(X, \mathscr{B}(X), \mu)$ and $(Y, \mathscr{B}(Y), \nu)$ with $(X, \mathscr{M}_{\mu^*}, \mu_1)$ and $(Y, \mathscr{M}_{\nu^*}, \nu_1)$; see Exercises 7.2.3 and 5.2.6.)

7.7 The Daniell–Stone Integral

There is an alternate approach to integration theory, due to Daniell [32] and Stone [114], in which one does not begin with a measure but rather with a positive linear functional on a vector space of functions. One extends this functional to a larger collection of functions, proves analogues of the monotone and dominated convergence theorems for the extended functional, and finally shows that the extended functional can be viewed as integration with respect to a measure.

Exercises 3 through 36 at the end of this section contain an outline of these classical results. I hope that I have arranged these exercises in such a way that the student can supply the missing details without too much trouble. In the body of this section we simply give an argument due to Kindler [70] (see also Zaanen [130]) that shows that the functionals considered by Daniell and Stone in fact correspond to integration with respect to measures. This theorem does not, of course, give the entire Daniell–Stone theory, but it does provide what is needed for many applications.

We turn to some basic definitions. Let X be a nonempty set. Recall that for real-valued (or $[-\infty,+\infty]$-valued) functions f and g on X, the functions $f \vee g$ and $f \wedge g$ are defined by

$$(f \vee g)(x) = \max(f(x), g(x))$$

and

$$(f \wedge g)(x) = \min(f(x), g(x)).$$

A *vector lattice* on X is a vector space V of real-valued functions on X that is closed under the operations $\wedge$ and $\vee$. A vector lattice V satisfies *Stone's condition* if

$$f \wedge 1 \in V \text{ whenever } f \in V. \tag{1}$$

(Here 1 is the constant function whose value is 1 at every point in X. Note that the constant function 1 may or may not belong to V.)

A linear functional L on a vector lattice V is an *elementary integral* if it is positive (that is, $L(f) \geq 0$ holds for every nonnegative function f in V) and satisfies

$$\lim_n L(f_n) = 0 \text{ for every sequence } \{f_n\} \text{ in } V \text{ that decreases pointwise to } 0. \tag{2}$$

We have the following basic facts about elementary integrals.

Lemma 7.7.1. *Suppose that L is an elementary integral on the vector lattice V and that f and f_1, f_2, ..., are nonnegative functions in V.*

(a) *If the sequence $\{f_n\}$ increases to f, then $L(f) = \lim_n L(f_n)$.*
(b) *If $f = \sum_n f_n$, then $L(f) = \sum_n L(f_n)$.*
(c) *If $f \leq \sum_n f_n$, then $L(f) \leq \sum_n L(f_n)$.*

Proof. Part (a) follows from condition (2), applied to the sequence $\{f - f_n\}_{n=1}^{\infty}$. Then parts (b) and (c) are consequences of part (a), applied to the sequences $\{\sum_{i=1}^{n} f_i\}_{n=1}^{\infty}$ and $\{(\sum_{i=1}^{n} f_i) \wedge f\}_{n=1}^{\infty}$. □

The following lemma is often useful for verifying condition (2).

Lemma 7.7.2 (Dini's Theorem). *Suppose that X is a closed bounded subinterval of $\mathbb{R}$ (or, more generally, a compact Hausdorff space). Let $\{f_n\}$ be a sequence of nonnegative continuous functions on X that decreases to 0 (in the sense that $\{f_n(x)\}$ decreases to 0 for each x in X). Then the sequence $\{f_n\}$ converges uniformly to 0.*

Proof. We need to show that for each positive ε there is a positive integer N such that $\|f_n\|_\infty \leq \varepsilon$ holds whenever $n \geq N$.

So suppose that ε is a positive number. For each x in X choose a positive integer n_x such that $f_{n_x}(x) < \varepsilon$, and then use the continuity of f_{n_x} to choose an open neighborhood U_x of x such that $f_{n_x}(t) < \varepsilon$ holds for all t in U_x. The family $\{U_x\}_{x \in X}$ is an open cover of X, and so the compactness of X gives a finite subcover U_{x_i}, where $i = 1, \ldots, k$, of X. Let N be the maximum of n_{x_i}, for $i = 1, \ldots, k$. If $x \in X$, then $x \in U_{x_i}$ for some i, and so

$$0 \leq f_n(x) \leq f_{n_i}(x) < \varepsilon$$

holds for all n that satisfy $n \geq N$. Since this estimate is valid for every x in X, we have $\|f_n\|_\infty \leq \varepsilon$ and the proof is complete. □

Examples 7.7.3.

(a) Let $[a,b]$ be a closed bounded subinterval of $\mathbb{R}$ and let $C([a,b])$ be the set of all continuous real-valued functions on $[a,b]$. Then $C([a,b])$ is a vector lattice that satisfies Stone's condition. Suppose we define a functional $L\colon C([a,b]) \to \mathbb{R}$ by letting L be the Riemann integral: $L(f) = \int_a^b f$. Dini's theorem implies that L satisfies condition (2) and so is an elementary integral.
(b) Let X be a locally compact Hausdorff space, and (as in Sect. 7.1) let $\mathscr{K}(X)$ be the set of all continuous functions $f\colon X \to \mathbb{R}$ for which the support of f is compact. Then $\mathscr{K}(X)$ is a vector lattice that satisfies Stone's condition. (Note that the constant function 1 does not belong to $\mathscr{K}(X)$ if X is not compact.) If L is a positive linear functional on $\mathscr{K}(X)$, then L is an elementary integral (again use Dini's theorem to check that L satisfies condition (2)).
(c) The set of all differentiable real-valued functions on $\mathbb{R}$ is a vector space, but not a vector lattice.
(d) Let V be the set of all constant multiples of the function $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x$. Then V is a vector lattice, but it does not satisfy Stone's condition.
(e) Let V be the set of all continuous functions $f\colon [0,+\infty) \to \mathbb{R}$ such that $\lim_{x\to+\infty} f(x)$ exists, and define $L\colon V \to \mathbb{R}$ by $L(f) = \lim_{x\to+\infty} f(x)$. Then V is a vector lattice that satisfies Stone's condition, and L is a positive linear functional that does *not* satisfy condition (2)—consider, for example, the sequence $\{f_n\}$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x < n, \\ x - n & \text{if } n \leq x < n+1, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$ □

Before we look at the main theorem of this section, it will be convenient to look at a slight generalization of the concept of a σ-algebra. So let X be a set. A collection $\mathscr{R}$ of subsets of X is a *σ-ring* on X if

(a) $\varnothing$ belongs to $\mathscr{R}$,
(b) for all sets A, B that belong to $\mathscr{R}$, the set $A - B$ belongs to $\mathscr{R}$,
(c) for each infinite sequence $\{A_i\}$ of sets that belong to $\mathscr{R}$, the set $\cup_{i=1}^\infty A_i$ belongs to $\mathscr{R}$, and
(d) for each infinite sequence $\{A_i\}$ of sets that belong to $\mathscr{R}$, the set $\cap_{i=1}^\infty A_i$ belongs to $\mathscr{R}$.

Of course, every σ-algebra is a σ-ring. If X is an uncountable set, then the set of all countable subsets of X is a σ-ring but not a σ-algebra. It is sometimes useful to deal with σ-rings when one wants to deal only with sets that are in some sense not too large.

Here are a few properties of σ-rings; their proofs are left for the reader:

(a) If $\mathscr{F}$ is a collection of subsets of a set X, then there is a smallest σ-ring on X that includes $\mathscr{F}$.
(b) If $\mathscr{R}$ is a σ-ring on a set X, then the collection of subsets A of X such that either A or A^c belongs to $\mathscr{R}$ is a σ-algebra on X; it is in fact the σ-algebra $\sigma(\mathscr{R})$ generated by $\mathscr{R}$.
(c) Suppose that μ_0 is a measure on a σ-ring $\mathscr{R}$ (i.e., a countably additive $[0,+\infty]$-valued function on $\mathscr{R}$ such that $\mu_0(\varnothing) = 0$). Let $\mathscr{A}$ be the σ-algebra generated by $\mathscr{R}$. Then the function $\mu\colon \mathscr{A} \to [0,+\infty]$ defined by

$$\mu(A) = \begin{cases} \mu_0(A) & \text{if } A \in \mathscr{R}, \text{ and} \\ +\infty & \text{if } A \in \mathscr{A} - \mathscr{R} \end{cases}$$

is a measure on $\mathscr{A}$.

Now suppose that X is a set and that V is a vector lattice on X. Let $\mathscr{F}$ be the collection of sets of the form $\{x \in X : f(x) > B\}$, where f ranges over V and B ranges over the positive reals. Let $\mathscr{R}$ be the smallest σ-ring on X that includes $\mathscr{F}$, and let $\mathscr{A}$ be the smallest σ-algebra on X that includes $\mathscr{F}$. It is easy to check that $\mathscr{A}$ is the smallest σ-algebra on X that makes each function in V measurable.

The following theorem is the main result of this section.

Theorem 7.7.4. *Let X be a set, let V be a vector lattice on X that satisfies Stone's condition, let L be an elementary integral on V, and let $\mathscr{R}$ and $\mathscr{A}$ be as defined above. Then there is a measure μ on $(X,\mathscr{A})$ such that $L(f) = \int f\,d\mu$ holds for each f in V. The restriction of this measure to $\mathscr{R}$ is unique, in the sense that if μ_1 and μ_2 are measures on $(X,\mathscr{A})$ such that $\int f\,d\mu_1 = L(f) = \int f\,d\mu_2$ holds for all f in V, then $\mu_1(A) = \mu_2(A)$ holds for all A in $\mathscr{R}$.*

The uniqueness assertion in this theorem may seem rather weak, since it involves only sets in $\mathscr{R}$. Note, however, that if μ is a measure on $\mathscr{A}$ that represents L, if f is a nonnegative function in V, and if we let $A_{n,i} = \{x : i/2^n < f(x) \le (i+1)/2^n\}$ for each n and i, then the sequence $\{\frac{i}{2^n}\sum_{i=0}^{n2^n-1}\frac{i}{2^n}\chi_{A_{n,i}}\}_{n=1}^{\infty}$ increases pointwise to f, and so

$$L(f) = \int f\,d\mu = \lim_n \int \sum_{i=0}^{n2^n-1} \frac{i}{2^n}\chi_{A_{n,i}}\,d\mu = \lim_n \sum_{i=0}^{n2^n-1} \frac{i}{2^n}\mu(A_{n,i}).$$

Thus the sets in $\mathscr{R}$ are the only ones needed for computing $\int f\,d\mu$. Also see Exercise 2.

For functions f and g in V let $[f,g)$ be the subset of $X \times \mathbb{R}$ given by

$$[f,g) = \{(x,t) \in X \times \mathbb{R} : f(x) \le t < g(x)\}$$

(be careful: we are not assuming that $f \le g$). Note that if f is a nonnegative function in V, then $[0,f)$ can be interpreted as the region under the graph of f. Let $\mathscr{I}$ be the collection of all such sets $[f,g)$, and let $\mathscr{B}$ to be the smallest σ-algebra on $X \times \mathbb{R}$ that includes $\mathscr{I}$. We will begin the proof of Theorem 7.7.4 by constructing a measure

ν on $(X \times \mathbb{R}, \mathscr{B})$ such that $\nu([f,g)) = L(g-f)$ holds whenever f and g belong to V and satisfy $f \le g$. Next we will define a measure μ on $(X, \mathscr{A})$ that satisfies $\mu(A) = \nu(A \times [0,1))$ for each A in $\mathscr{R}$, and finally we will show that μ satisfies $L(f) = \int f\,d\mu$ for each f in V.

Here are a few basic facts about $\mathscr{I}$.

Lemma 7.7.5. *Suppose that V is a vector lattice of functions, that L is a positive linear functional on V, and that $\mathscr{I}$ is as defined above.*

(a) *If $I \in \mathscr{I}$, then there exist functions f and g in V such that $f \le g$ and $I = [f,g)$.*
(b) *If the member I of $\mathscr{I}$ can be written in the form $[f_1, g_1)$ and in the form $[f_2, g_2)$, where $f_1 \le g_1$ and $f_2 \le g_2$, then $g_1 - f_1 = g_2 - f_2$, and so $L(g_1 - f_1) = L(g_2 - f_2)$.*
(c) *If I_1 and I_2 belong to $\mathscr{I}$, then $I_1 \cap I_2$ also belongs to $\mathscr{I}$.*
(d) *If I_1 and I_2 belong to $\mathscr{I}$, then there are disjoint sets I' and I'' in $\mathscr{I}$ such that $I_1 \cap I_2^c = I' \cup I''$ and hence such that $I_1 = (I_1 \cap I_2) \cup I' \cup I''$.*

Proof. For part (a), note that if $f_0, g_0 \in V$ and if we let $f = f_0 \wedge g_0$ and $g = g_0$, then $f \le g$ and $[f,g) = [f_0, g_0)$. For part (b), note that if the section I_x is nonempty, then $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, while if the section I_x is empty, then $f_1(x) = g_1(x)$ and $f_2(x) = g_2(x)$. In either case we have $g_1(x) - f_1(x) = g_2(x) - f_2(x)$, and so $L(g_1 - f_1) = L(g_2 - f_2)$. Part (c) follows from the calculation $[f_1, g_1) \cap [f_2, g_2) = [f_1 \vee f_2, g_1 \wedge g_2)$. Finally, if $I_1 = [f_1, g_1)$ and $I_2 = [f_2, g_2)$, where $f_2 \le g_2$, then $I_1 \cap I_2^c = [f_1, g_1 \wedge f_2) \cup [f_1 \vee g_2, g_1)$, from which part (d) follows. □

In view of part (b) of Lemma 7.7.5, we can define a function $L_{\mathscr{I}} : \mathscr{I} \to \mathbb{R}$ by

$$L_{\mathscr{I}}(I) = L(g-f),$$

where f and g are elements of V such that $f \le g$ and $I = [f,g)$.

Lemma 7.7.6. *Suppose that I and I_1, I_2, ... are members of $\mathscr{I}$.*

(a) *If the sets I_n, $n = 1, 2, \ldots$, are disjoint and if $I = \cup_n I_n$, then $L_{\mathscr{I}}(I) = \sum_n L_{\mathscr{I}}(I_n)$.*
(b) *If $I \subseteq \cup_n I_n$, then $L_{\mathscr{I}}(I) \le \sum_n L_{\mathscr{I}}(I_n)$.*

Proof. Suppose that $I = \cup_n I_n$, and let $I = [f,g)$ and $I_n = [f_n, g_n)$, $n = 1, 2, \ldots$, where $f \le g$ and $f_n \le g_n$, $n = 1, 2, \ldots$. For each x in X the sections of these sets at x satisfy $I_x = \cup_n (I_n)_x$, and so the countable additivity of Lebesgue measure implies that

$$g(x) - f(x) = \lambda(I_x) = \sum_n \lambda((I_n)_x) = \sum_n (g_n(x) - f_n(x)).$$

It follows from Lemma 7.7.1 that $L_{\mathscr{I}}(I) = \sum_n L_{\mathscr{I}}(I_n)$, and so the proof of part (a) is complete. Part (b) can be proved with a similar argument. □

Proof of Theorem 7.7.4. We define a function ν^* on the subsets of $X \times \mathbb{R}$ by letting $\nu^*(A)$ be the infimum of the set of sums of the form $\sum_i L_{\mathscr{I}}(I_i)$, where $\{I_i\}$ is a sequence in $\mathscr{I}$ such that $A \subseteq \cup_i I_i$. (Of course, $\nu^*(A) = +\infty$ if there is no sequence $\{I_i\}$ such that $A \subseteq \cup_i I_i$.)

Lemma 7.7.7. *Let ν^* be as defined above. Then*

(a) *ν^* is an outer measure on X,*
(b) *every set in $\mathscr{I}$ is ν^*-measurable, and*
(c) *if $I \in \mathscr{I}$, then $\nu^*(I) = L_{\mathscr{I}}(I)$.*

Proof. It is immediate that ν^* is an outer measure. Now suppose that $I \in \mathscr{I}$. We can show that I is ν^*-measurable by checking that

$$\nu^*(A) \geq \nu^*(A \cap I) + \nu^*(A \cap I^c)$$

holds for each subset A of $X \times \mathbb{R}$ such that $\nu^*(A) < +\infty$ (see Sect. 1.3 and, in particular, the discussion of inequality (1) in that section). So suppose that A is such a set, that ε is a positive number, and that $\{I_n\}$ is a sequence of sets in $\mathscr{I}$ such that $A \subseteq \cup_n I_n$ and

$$\nu^*(A) + \varepsilon > \sum_n L_{\mathscr{I}}(I_n).$$

According to part (d) of Lemma 7.7.5, for each n there are sets I'_n and I''_n such that $I_n \cap I$, I'_n, and I''_n are disjoint and have I_n as their union. Thus

$$L_{\mathscr{I}}(I_n) = L_{\mathscr{I}}(I_n \cap I) + L_{\mathscr{I}}(I'_n) + L_{\mathscr{I}}(I''_n);$$

since $A \cap I \subseteq \cup_n (I_n \cap I)$ and $A \cap I^c \subseteq \cup_n (I'_n \cup I''_n)$, we have

$$\begin{aligned} \nu^*(A) + \varepsilon &> \sum_n L_{\mathscr{I}}(I_n) \\ &= \sum_n L_{\mathscr{I}}(I_n \cap I) + \sum_n (L_{\mathscr{I}}(I'_n) + L_{\mathscr{I}}(I''_n)) \\ &\geq \nu^*(A \cap I) + \nu^*(A \cap I^c). \end{aligned}$$

Thus I is measurable. Each I in $\mathscr{I}$ certainly satisfies $\nu^*(I) \leq L_{\mathscr{I}}(I)$. The reverse inequality follows from part (b) of Lemma 7.7.6, and with that the proof of Lemma 7.7.7 is complete. □

We return to the proof of Theorem 7.7.4. Let f be a nonnegative function in V, let B be a positive real number, and for each n define f_n by

$$f_n = 1 \wedge n(f - (f \wedge B)).$$

Since V is a vector lattice that satisfies Stone's condition, each f_n belongs to V. For each positive number C, the sequence $\{Cf_n\}$ is increasing and converges pointwise to $C\chi_{\{f>B\}}$, and the sets $[0, Cf_n)$ increase to the set $\{f > B\} \times [0, C)$. Let us consider three consequences of this.

First, for each f in V and each positive number B we have $\{f > B\} \times [0, 1) \in \mathscr{B}$ (recall that $\mathscr{B}$ is the σ-algebra generated by the family $\mathscr{I}$). It follows that each A in $\mathscr{R}$ satisfies $A \times [0, 1) \in \mathscr{B}$ and hence that $\mu(A) = \nu(A \times [0, 1))$ defines a measure on $\mathscr{R}$. As we noted earlier in this section, we can extend μ to a measure on the σ-algebra $\mathscr{A}$ by letting $\mu(A) = +\infty$ if A belongs to $\mathscr{A}$ but not to $\mathscr{R}$.

Second, the fact that the sets $[0, Cf_n)$ increase to $\{f > B\} \times [0, C)$ implies that

$$\begin{aligned}\nu(\{f > B\} \times [0,C)) &= \lim_n \nu([0, Cf_n)) = \lim_n L(Cf_n)\\ &= C \lim_n L(f_n) = C \lim_n \nu([0, f_n))\\ &= C\nu(\{f > B\} \times [0,1));\end{aligned}$$

that is,

$$\nu(\{f > B\} \times [0,C)) = C\mu(\{f > B\}). \tag{3}$$

Finally, for each n we have $f_n \le \chi_{\{f>B\}} \le f/B$ and so $L(f_n) \le L(f)/B$. Since $\nu(\{f > B\} \times [0,C)) = \lim_n CL(f_n) \le CL(f)/B$, it follows that the values in (3) are finite.

Now let n and i range over the positive integers. If we apply (3) twice, once with $B = i/2^n$ and $C = i/2^n$ and once with $B = (i+1)/2^n$ and $C = i/2^n$, we find that

$$\nu\left(\left\{\frac{i}{2^n} < f \le \frac{i+1}{2^n}\right\} \times [0, i/2^n)\right) = \frac{i}{2^n}\mu\left(\left\{\frac{i}{2^n} < f \le \frac{i+1}{2^n}\right\}\right)$$

and hence that

$$\nu\left(\bigcup_{i=1}^{n2^n}\left\{\frac{i}{2^n} < f \le \frac{i+1}{2^n}\right\} \times [0, i/2^n)\right) = \sum_{i=1}^{n2^n}\frac{i}{2^n}\mu\left(\left\{\frac{i}{2^n} < f \le \frac{i+1}{2^n}\right\}\right). \tag{4}$$

The countable additivity of ν implies that the left side of (4) approaches $\nu([0, f))$ as n approaches infinity, while the monotone convergence theorem implies that the right side approaches $\int f\,d\mu$. With this we have

$$L(f) = \nu([0, f)) = \int f\,d\mu.$$

Since each f in V can be separated into its positive and negative parts, we have $L(f) = \int f\,d\mu$ for each f in V, and the construction of μ is complete.

We turn to the uniqueness. Let μ_1 and μ_2 be measures on $\mathscr{A}$ such that $\int f\,d\mu_1 = L(f) = \int f\,d\mu_2$ holds for all f in V. Suppose that $f_1, \ldots, f_k$ belong to V, that $B_1, \ldots, B_k$ are positive numbers, and that $A = \cap_i \{f_i > B_i\}$. For each n let

$$g_n = \wedge_{i=1}^k (1 \wedge n(f_i - (f_i \wedge B_i))).$$

Then each g_n belongs to V and is nonnegative, and the sequence $\{g_n\}$ increases to χ_A. Hence

$$\mu_1(A) = \lim_n \int g_n\,d\mu_1 = \lim_n L(g_n) = \lim_n \int g_n\,d\mu_2 = \mu_2(A).$$

Now fix f_1 and B_1, and let $\mathscr{F}_{f_1,B_1}$ be the collection of all subsets of $\{f_1 > B_1\}$ that have the form $\cap_i\{f_i > B_i\}$. Then μ_1 and μ_2 agree on the π-system $\mathscr{F}_{f_1,B_1}$, and so it follows from Corollary 1.6.3 that μ_1 and μ_2 agree on the σ-algebra on $\{f_1 > B_1\}$ that $\mathscr{F}_{f_1,B_1}$ generates. However, it is easy to check that this σ-algebra is just the collection of all subsets of $\{f_1 > B_1\}$ that belong to $\mathscr{R}$. Finally, every set in $\mathscr{R}$ is included in a countable union of sets of the form $\{f > B\}$, and so μ_1 and μ_2 agree on $\mathscr{R}$. □

Exercises

1. (a) Let X be the interval $[-1,1]$. Find (i.e., describe precisely) the smallest vector lattice V on X that contains the function $x \mapsto x$.
   (b) Does V satisfy Stone's condition?
2. Let $X = \mathbb{R}$ and let V be the set of those continuous functions $f: X \to \mathbb{R}$ whose support is compact and included in $(0,+\infty)$. Define L on V by letting $L(f)$ be the Riemann integral of f, and let $\mathscr{A}$ and $\mathscr{R}$ be as in Theorem 7.7.4.
   (a) What sets do $\mathscr{A}$ and $\mathscr{R}$ contain? (Your answer should relate these families of sets to the Borel or the Lebesgue measurable subsets of $\mathbb{R}$.)
   (b) Give two measures on $\mathscr{A}$ that represent the functional L.

The following exercises contain an outline of the usual development of the Daniell–Stone integral. As noted at the beginning of this section, this development is not based on measure theory. Thus solutions to Exercises 3 through 32 should not contain any references to the earlier chapters of this book. Sigma-algebras, measurable functions, measures, and the Lebesgue integral appear first in Exercises 33–36

Suppose that V is a vector lattice of functions on a set X, that V satisfies Stone's condition, and that L is an elementary integral on V (i.e., it is a positive linear functional on V that satisfies relation (2)). Let $V^\bullet$ be the set of all $(-\infty,+\infty]$-valued functions on X that are pointwise limits of increasing sequences of functions in V, and define $L^\bullet: V^\bullet \to (-\infty,+\infty]$ by $L^\bullet(f) = \lim_n L(f_n)$, where $\{f_n\}$ is an increasing sequence of functions in V that converges pointwise to f (see Exercise 3). Likewise, let $V_\bullet$ be the set of all $[-\infty,+\infty)$-valued functions on X that are pointwise limits of decreasing sequences of functions in V, and define $L_\bullet: V_\bullet \to [-\infty,+\infty)$ by $L_\bullet(f) = \lim_n L(f_n)$, where $\{f_n\}$ is an decreasing sequence of functions in V that converges pointwise to f.

3. Show that $L^\bullet$ and $L_\bullet$ are well defined on $V^\bullet$ and $V_\bullet$. For example, to show that $L^\bullet$ is well defined, one needs to show that if $f \in V^\bullet$ and if $\{f_n\}$ and $\{g_n\}$ are sequences of functions in V that increase to f, then $\lim_n L(f_n) = \lim_n L(g_n)$. (Hint: Start by showing that if $g \in V$ and if $\{f_n\}$ is an increasing sequence in V such that $g(x) \le \lim_n f_n(x)$ holds for each x, then $L(g) \le \lim L(f_n)$.)
4. Show that $f \in V_\bullet$ if and only if there is a function g in $V^\bullet$ such that $f = -g$ and that in this case $L_\bullet(f) = -L^\bullet(g)$.

5. Suppose that $f_1, f_2 \in V^\bullet$ and that α is a nonnegative real number. Show that
   (a) $f_1 \wedge f_2, f_1 \vee f_2 \in V^\bullet$,
   (b) $f_1 + f_2 \in V^\bullet$ and $L^\bullet(f_1 + f_2) = L^\bullet(f_1) + L^\bullet(f_2)$, and
   (c) $\alpha f_1 \in V^\bullet$ and $L^\bullet(\alpha f_1) = \alpha L^\bullet(f_1)$.
6. Show that if $f_1, f_2 \in V^\bullet$ and $f_1 \le f_2$, then $L^\bullet(f_1) \le L^\bullet(f_2)$.
7. (a) Show that if $g \in V_\bullet$ and $h \in V^\bullet$, then $h - g \in V^\bullet$ and $L^\bullet(h-g) = L^\bullet(h) - L_\bullet(g)$. In particular, all the differences involved here are defined (that is, neither $+\infty$ or $-\infty$ is ever subtracted from itself here).
   (b) Conclude that if $g \in V_\bullet$, $h \in V^\bullet$, and $g \le h$, then $L_\bullet(g) \le L^\bullet(h)$.
8. Show that if $\{f_n\}$ is a sequence of functions in $V^\bullet$ and if $\{f_n\}$ increases to f, then $f \in V^\bullet$ and $L^\bullet(f) = \lim_n L^\bullet(f_n)$. (Hint: Use some ideas from the proof of Theorem 2.4.1.)

Suppose that f is an arbitrary $[-\infty, +\infty]$-valued function on X. Define $\overline{L}(f)$ and $\underline{L}(f)$ by

$$\overline{L}(f) = \inf\{L^\bullet(h) : h \in V^\bullet \text{ and } f \le h\}$$

and

$$\underline{L}(f) = \sup\{L_\bullet(g) : g \in V_\bullet \text{ and } g \le f\}$$

(of course, $\overline{L}(f) = +\infty$ if there is no h in $V^\bullet$ such that $f \le h$, and $\underline{L}(f) = -\infty$ if there is no g in $V_\bullet$ such that $g \le f$). For each f we have $\underline{L}(f) \le \overline{L}(f)$ (see Exercise 9). A function $f : X \to [-\infty, +\infty]$ is *L-summable*, or simply *summable*, if $\underline{L}(f)$ and $\overline{L}(f)$ are finite and equal. We define L_1 on the collection of summable functions by letting $L_1(f)$ be the common value of $\underline{L}(f)$ and $\overline{L}(f)$.

9. Show that each $f : X \to [-\infty, +\infty]$ satisfies $\underline{L}(f) \le \overline{L}(f)$.
10. Show that $f : X \to [-\infty, +\infty]$ is L-summable if and only if for every positive ε there exist g in $V_\bullet$ and h in $V^\bullet$ such that $g \le f \le h$ and $L^\bullet(h - g) < \varepsilon$. (Hint: See Exercise 7.)
11. Show that if $f \in V$, then f is summable and $L_1(f) = L(f)$. Thus L_1 is an extension of L.
12. Show that if f_1 and f_2 are $\mathbb{R}$-valued summable functions and $\alpha \in \mathbb{R}$, then
    (a) $f_1 + f_2$ is summable and $L_1(f_1 + f_2) = L_1(f_1) + L_1(f_2)$, and
    (b) αf_1 is summable and $L_1(\alpha f_1) = \alpha L_1(f_1)$.
13. Show that if f_1 and f_2 are $[0, +\infty]$-valued summable functions, then $f_1 + f_2$ is summable and $L_1(f_1 + f_2) = L_1(f_1) + L_1(f_2)$.
14. Show that if f_1 and f_2 are $[-\infty, +\infty]$-valued summable functions, then $f_1 \wedge f_2$ and $f_1 \vee f_2$ are summable.
15. (a) Generalize part (a) of Exercise 12 to the case where f_1 and f_2 are summable $[-\infty, +\infty]$-valued functions such that $f_1(x) + f_2(x)$ is defined for each x (i.e., such that for no x is this sum of the form $+\infty + (-\infty)$ or $-\infty + (+\infty)$).
    (b) Show that a function $f : X \to [-\infty, +\infty]$ is summable if and only if f^+ and f^- are summable and that in such cases $L_1(f) = L_1(f^+) - L_1(f^-)$.

16. Show that if $\{f_n\}$ is a sequence of $[0,+\infty]$-valued summable functions, if $\{f_n\}$ increases pointwise to f, and if $\sup_n L_1(f_n) < +\infty$, then f is summable and $L_1(f) = \lim_n L_1(f_n)$. (Hint: It might be useful to verify and use the fact that if f_1 and f_2 are nonnegative summable functions such that $f_1 \le f_2$, if ε_1 and ε_2 are positive numbers, and if g_1 and g_2 belong to $V^\bullet$ and satisfy $f_i \le g_i$ and $L^\bullet(g_i) < L_1(f_i) + \varepsilon_i$ for $i = 1, 2$, then $L^\bullet(g_1 \vee g_2) < L_1(f_2) + \varepsilon_1 + \varepsilon_2$.)
17. Suppose that $\{f_n\}$ is a sequence of $\mathbb{R}$-valued summable functions, that $f(x) = \lim_n f_n(x)$ holds for all x, and that h is a $[0,+\infty)$-valued summable function such that $|f_n(x)| \le h(x)$ holds for all n and all x. Show that f is summable and that $L_1(f) = \lim_n L_1(f_n)$.
18. Show that if f is a $[-\infty,+\infty]$-valued summable function, then $f \wedge 1$ is summable. (Hint: See Exercise 10.)
19. Show that if f is a $[0,+\infty]$-valued summable function, if α is a positive real number, and if $A = \{x \in X : f(x) > \alpha\}$, then χ_A is summable and $L_1(\alpha\chi_A) \le L_1(f)$. (Hint: Use Exercise 18. Reduce the question to one involving only $[0,+\infty)$-valued functions. Check that if f is $[0,+\infty)$-valued, then the sequence $\{f_n\}$, where $f_n = \alpha \wedge n(f - (f \wedge \alpha))$, increases to $\alpha\chi_A$.)

A subset A of X is *L-negligible* or *L-null* if χ_A is summable and $L_1(\chi_A) = 0$. A property of points x in X is said to hold *L-almost everywhere* if the set of points at which it fails is an L-negligible set.

20. Show that a subset A of X is L-negligible if and only if for every ε there is a function f in $V^\bullet$ such that $\chi_A \le f$ and $L^\bullet(f) < \varepsilon$.
21. Show that each subset of an L-negligible set is L-negligible.
22. Show that the union of a countable collection of L-negligible sets is L-negligible.
23. Suppose that f_1 and f_2 are $[-\infty,+\infty]$-valued functions that are equal L-almost everywhere. Show that if one of these functions is summable, then both are summable and $L_1(f_1) = L_1(f_2)$.
24. Show that if f is a $[-\infty,+\infty]$-valued summable function, then $\{x \in X : |f(x)| = +\infty\}$ is L-negligible.
25. Reformulate Exercises 16 and 17 by allowing the appropriate hypotheses to hold only L-almost everywhere and (in the case of Exercise 17) by allowing the functions involved to have infinite values on L-negligible sets. Prove your reformulated versions.

A function $f\colon X \to [-\infty,+\infty]$ is called *L-measurable*, or simply *measurable*, if $(g \vee f) \wedge h$ is summable for every choice of functions g and h, where g is a nonpositive summable function and h is a nonnegative summable function. A subset A of X is *L-measurable* if χ_A is a measurable function. Let $\mathscr{M}$ be the collection of all L-measurable subsets of X.

26. Show that every summable function is measurable.

27. Show that
    (a) a $[0,+\infty]$-valued function f is measurable if and only if for each nonnegative summable h the function $f \wedge h$ is also summable, and
    (b) a $[-\infty,+\infty]$-valued function f is measurable if and only if f^+ and f^- are measurable.
28. Show that the constant function 1 is measurable.
29. Let f_1 and f_2 be $[-\infty,+\infty]$-valued functions that are equal L-almost everywhere. Show that f_1 is measurable if and only if f_2 is measurable.
30. Show that if f_1 and f_2 are $[-\infty,+\infty]$-valued measurable functions, then $f_1 \wedge f_2$ and $f_1 \vee f_2$ are measurable.
31. Show that if $\{f_n\}$ is a sequence of $[-\infty,+\infty]$-valued measurable functions and if $f(x) = \lim_n f_n(x)$ holds for almost every x in X, then f is measurable.
32. Show that if f_1 and f_2 are $\mathbb{R}$-valued measurable functions and if $\alpha \in \mathbb{R}$, then $f_1 + f_2$ and αf_1 are measurable.
33. Show that the collection $\mathscr{M}$ of L-measurable sets is a σ-algebra.
34. Show that a function $f\colon X \to [-\infty,+\infty]$ is L-measurable if and only if it is measurable (in the sense of Chap. 2) with respect to the σ-algebra $\mathscr{M}$.

    Define a function $\mu\colon \mathscr{M} \to [0,+\infty]$ by

$$\mu(A) = \begin{cases} L_1(\chi_A) & \text{if } \chi_A \text{ is summable, and} \\ +\infty & \text{if } \chi_A \text{ is measurable but not summable.} \end{cases}$$

35. Show that μ is a measure on $\mathscr{M}$.
36. Show that a function $f\colon X \to [-\infty,+\infty]$ is L-summable if and only if it is $\mathscr{M}$-measurable and μ-integrable and that then $L_1(f) = \int f\,d\mu$.
37. Let $[a,b]$ be a closed bounded interval and let L be the Riemann integral on $C[a,b]$, as in Example 7.7.3(a). Show that the L-summable functions on $[a,b]$ are exactly the Lebesgue measurable functions on $[a,b]$ that are Lebesgue integrable, and that $L_1(f) = \int f\,d\lambda$ holds for each such function f.
38. Let V and L be as in Exercise 2. Characterize the set of L-summable functions in terms of concepts from the Lebesgue theory. Be very precise.

Notes

The historical notes in Chapter III of Hewitt and Ross [58] contain a nice summary of the history of integration theory on locally compact Hausdorff spaces.

The reader who wants to see another elementary treatment of integration on locally compact Hausdorff spaces might find Halmos [54], Hewitt and Stromberg [59], Rudin [105], or Hewitt and Ross [58] useful. He or she would also do well to look up the paper of Kakutani [67].

The definition given here for the collection of Borel subsets of X agrees with that given by Hewitt and Ross [58], Hewitt and Stromberg [59], and Rudin [105]; it agrees with that given by Halmos [54] only when X is σ-compact. The definition given in Exercise 7.2.8 for the σ-algebra $\mathscr{B}_0(X)$ of Baire subsets of a compact Hausdorff space X is a special case of that given by Halmos (Halmos considers σ-rings, in addition to σ-algebras, and so is able to give a definition of $\mathscr{B}_0(X)$ that can reasonably be applied to an arbitrary locally compact Hausdorff space X).

Bourbaki [18] and Hewitt and Ross [58] deal with the μ^*-, ν^*-, and $(\mu \times \nu)^*$-measurable sets, rather than with the Borel sets, when considering product measures. Proposition 7.6.5, Corollary 7.6.6, and Theorem 7.6.7 were suggested by de Leeuw [34]. (See also Godfrey and Sion [51] and Bledsoe and Wilks [13].) Point (b) in the discussion at the end of Sect. 7.6, and also Exercise 7.6.2, come from suggestions made by Roy Johnson.

See Loomis [84], Riesz and Nagy [99], Royden [102], or Taylor [116] for more details on the Daniell–Stone version of integration theory. Taylor's exposition is especially clear and detailed. The Daniell treatment of integration theory can also be used to prove a version of the Riesz representation theorem; this is done, for instance, by Loomis [84] and Royden [102].