Appendix D
Topological Spaces and Metric Spaces

A number of the results in this appendix are stated without proof. For additional details, the reader should consult a text on point-set topology (for example, Kelley [69], Munkres [91], or Simmons [109]).

D.1. Let X be a set. A *topology* on X is a family $\mathscr{O}$ of subsets of X such that

(a) $X \in \mathscr{O}$,
(b) $\varnothing \in \mathscr{O}$,
(c) if $\mathscr{S}$ is an arbitrary collection of sets that belong to $\mathscr{O}$, then $\cup\mathscr{S} \in \mathscr{O}$, and
(d) if $\mathscr{S}$ is a finite collection of sets that belong to $\mathscr{O}$, then $\cap\mathscr{S} \in \mathscr{O}$.

A *topological space* is a pair $(X, \mathscr{O})$, where X is a set and $\mathscr{O}$ is a topology on X (we will generally abbreviate the notation and simply call X a topological space). The *open* subsets of X are those that belong to $\mathscr{O}$. An *open neighborhood* of a point x in X is an open set that contains x.

The collection of all open subsets of $\mathbb{R}^d$ (as defined in Appendix C) is a topology on $\mathbb{R}^d$; it is sometimes called the *usual* topology on $\mathbb{R}^d$.

D.2. Let $(X, \mathscr{O})$ be a topological space. A subset F of X is *closed* if F^c is open. The union of a finite collection of closed sets is closed, as is the intersection of an arbitrary collection of closed sets (use De Morgan's laws and parts (c) and (d) of the definition of a topology). It follows that if $A \subseteq X$, then there is a smallest closed set that includes A, namely the intersection of all the closed subsets of X that include A; this set is called the *closure* of A and is denoted by $\overline{A}$ or by A^-. A point x in X is a *limit point* of A if each open neighborhood of x contains at least one point of A other than x (the point x itself may or may not belong to A). A set is closed if and only if it contains each of its limit points. The closure of the set A consists of the points in A, together with the limit points of A.

D.3. Let $(X, \mathscr{O})$ be a topological space, and let A be a subset of X. The *interior* of A, written A^o, is the union of the open subsets of X that are included in A; thus A^o is the largest open subset of A. It is easy to check that $A^o = ((A^c)^-)^c$.

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D.4. Let $(X, \mathscr{O})$ be a topological space, let Y be a subset of X, and let $\mathscr{O}_Y$ be the collection of all subsets of Y that have the form $Y \cap U$ for some U in $\mathscr{O}$. Then $\mathscr{O}_Y$ is a topology on Y; it is said to be *inherited from* X, or to be *induced* by $\mathscr{O}$. The space $(Y, \mathscr{O}_Y)$ (or simply Y) is called a *subspace* of $(X, \mathscr{O})$ (or of X).

Note that if Y is an open subset of X, then the members of $\mathscr{O}_Y$ are exactly the subsets of Y that are open as subsets of X. Likewise, if Y is a closed subset of X, then the closed subsets of the topological space $(Y, \mathscr{O}_Y)$ are exactly the subsets of Y that are closed as subsets of $(X, \mathscr{O}_X)$.

D.5. Let X and Y be topological spaces. A function $f\colon X \to Y$ is *continuous* if $f^{-1}(U)$ is an open subset of X whenever U is an open subset of Y. It is easy to check that f is continuous if and only if $f^{-1}(C)$ is closed whenever C is a closed subset of Y. A function $f\colon X \to Y$ is a *homeomorphism* if it is a bijection such that f and f^{-1} are both continuous. Equivalently, f is a homeomorphism if it is a bijection such that $f^{-1}(U)$ is open exactly when U is open. The spaces X and Y are *homeomorphic* if there is a homeomorphism of X onto Y.

D.6. We will on occasion need the following techniques for verifying the continuity of a function. Let X and Y be topological spaces, and let f be a function from X to Y. If $\mathscr{S}$ is a collection of open subsets of X such that $X = \cup\mathscr{S}$, and if for each U in $\mathscr{S}$ the restriction f_U of f to U is continuous (as a function from U to Y), then f is continuous (to prove this, note that if V is an open subset of Y, then $f^{-1}(V)$ is the union of the sets $f_U^{-1}(V)$, and so is open). Likewise, if $\mathscr{S}$ is a *finite* collection of closed sets such that $X = \cup\mathscr{S}$, and if for each C in $\mathscr{S}$ the restriction of f to C is continuous, then f is continuous.

D.7. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are topologies on the set X, and if $\mathscr{O}_1 \subseteq \mathscr{O}_2$, then $\mathscr{O}_1$ is said to be *weaker* than $\mathscr{O}_2$.

Now suppose that $\mathscr{A}$ is an arbitrary collection of subsets of the set X. There exist topologies on X that include $\mathscr{A}$ (for instance, the collection of all subsets of X). The intersection of all such topologies on X is a topology; it is the weakest topology on X that includes $\mathscr{A}$ and is said to be *generated* by $\mathscr{A}$.

We also need to consider topologies generated by sets of functions. Suppose that X is a set and that $\{f_i\}$ is a collection of functions, where for each i the function f_i maps X to some topological space Y_i. A topology on X makes all these functions continuous if and only if $f_i^{-1}(U)$ is open (in X) for each index i and each open subset U of Y_i. The topology *generated* by the family $\{f_i\}$ is the weakest topology on X that makes each f_i continuous, or equivalently, the topology generated by the sets $f_i^{-1}(U)$.

D.8. A subset A of a topological space X is *dense* in X if $\overline{A} = X$. The space X is *separable* if it has a countable dense subset.

D.9. Let $(X, \mathscr{O})$ be a topological space. A collection $\mathscr{U}$ of open subsets of X is a *base* for $(X, \mathscr{O})$ if for each V in $\mathscr{O}$ and each x in V there is a set U that belongs to $\mathscr{U}$ and satisfies $x \in U \subseteq V$. Equivalently, $\mathscr{U}$ is a base for X if the open subsets of

X are exactly the unions of (possibly empty) collections of sets in $\mathscr{U}$. A topological space is said to be *second countable*, or to *have a countable base*, if it has a base that contains only countably many sets.

D.10. It is easy to see that if X is second countable, then X is separable (if $\mathscr{U}$ is a countable base for X, then we can form a countable dense subset of X by choosing one point from each nonempty set in $\mathscr{U}$). The converse is not true. (Construct a topological space $(X, \mathscr{O})$ by letting $X = \mathbb{R}$ and letting $\mathscr{O}$ consist of those subsets A of X such that either $A = \varnothing$ or $0 \in A$. Then $\{0\}$ is dense in X, and so X is separable; however, X is not second countable. Exercise 7.1.8 contains a more interesting example.)

D.11. If X is a second countable topological space, and if $\mathscr{V}$ is a collection of open subsets of X, then there is a countable subset $\mathscr{V}_0$ of $\mathscr{V}$ such that $\cup\mathscr{V}_0 = \cup\mathscr{V}$. (Let $\mathscr{U}$ be a countable base for X, and let $\mathscr{U}_0$ be the collection of those elements U of $\mathscr{U}$ for which there is a set in $\mathscr{V}$ that includes U. For each U in $\mathscr{U}_0$ choose an element of $\mathscr{V}$ that includes U. The collection of sets chosen is the required subset of $\mathscr{V}$.)

D.12. A topological space X is *Hausdorff* if for each pair x, y of distinct points in X there are open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

D.13. Let A be a subset of the topological space X. An *open cover* of A is a collection $\mathscr{S}$ of open subsets of X such that $A \subseteq \cup\mathscr{S}$. A *subcover* of the open cover $\mathscr{S}$ is a subfamily of $\mathscr{S}$ that is itself an open cover of A. The set A is *compact* if each open cover of A has a finite subcover. A topological space X is *compact* if X, when viewed as a subset of the space X, is compact.

D.14. A collection $\mathscr{C}$ of subsets of a set X satisfies the *finite intersection property* if each finite subcollection of $\mathscr{C}$ has a nonempty intersection. It follows from De Morgan's laws that a topological space X is compact if and only if each collection of closed subsets of X that satisfies the finite intersection property has a nonempty intersection.

D.15. If X and Y are topological spaces, if $f \colon X \to Y$ is continuous, and if K is a compact subset of X, then $f(K)$ is a compact subset of Y.

D.16. Every closed subset of a compact set is compact. Conversely, every compact subset of a Hausdorff space is closed (this is a consequence of Proposition 7.1.2; in fact, the first half of the proof of that proposition is all that is needed in the current situation).

D.17. It follows from D.15 and D.16 that if X is a compact space, if Y is a Hausdorff space, and if $f \colon X \to Y$ is a continuous bijection, then f is a homeomorphism.

D.18. If X is a nonempty compact space, and if $f \colon X \to \mathbb{R}$ is continuous, then f is bounded and attains its supremum and infimum: there are points x_0 and x_1 in X such that $f(x_0) \le f(x) \le f(x_1)$ holds at each x in X.

D.19. Let $\{(X_\alpha, \mathscr{O}_\alpha)\}$ be an indexed family of topological spaces, and let $\prod_\alpha X_\alpha$ be the product of the corresponding indexed family of sets $\{X_\alpha\}$ (see A.5). The *product topology* on $\prod_\alpha X_\alpha$ is the weakest topology on $\prod_\alpha X_\alpha$ that makes each of the coordinate projections $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ continuous (the projection π_β is defined by $\pi_\beta(x) = x_\beta$); see D.7. If $\mathscr{U}$ is the collection of sets that have the form $\prod_\alpha U_\alpha$ for some family $\{U_\alpha\}$ for which

(a) $U_\alpha \in \mathscr{O}_\alpha$ holds for each α and
(b) $U_\alpha = X_\alpha$ holds for all but finitely many values of α,

then $\mathscr{U}$ is a base for the product topology on $\prod_\alpha X_\alpha$.

D.20. (Tychonoff's Theorem) *Let $\{(X_\alpha, \mathscr{O}_\alpha)\}$ be an indexed collection of topological spaces. If each $(X_\alpha, \mathscr{O}_\alpha)$ is compact, then $\prod_\alpha X_\alpha$, with the product topology, is compact.*

D.21. Let X be a set. A collection $\mathscr{F}$ of functions on X *separates the points of X* if for each pair x, y of distinct points in X there is a function f in $\mathscr{F}$ such that $f(x) \neq f(y)$. A vector space $\mathscr{F}$ of real-valued functions on X is an *algebra* if fg belongs to $\mathscr{F}$ whenever f and g belong to $\mathscr{F}$ (here fg is the product of f and g, defined by $(fg)(x) = f(x)g(x)$). Now suppose that $\mathscr{F}$ is a vector space of bounded real-valued functions on X. A subset of $\mathscr{F}$ is *uniformly dense* in $\mathscr{F}$ if it is dense in $\mathscr{F}$ when $\mathscr{F}$ is given the topology induced by the uniform norm (see Example 3.2.1(f) in Sect. 3.2).

D.22. (Stone–Weierstrass Theorem) *Let X be a compact Hausdorff space. If A is an algebra of continuous real-valued functions on X that contains the constant functions and separates the points of X, then A is uniformly dense in the space $C(X)$ of continuous real-valued functions on X.*

D.23. (Stone–Weierstrass Theorem) *Let X be a locally compact*[1] *Hausdorff space, and let A be a subalgebra of $C_0(X)$ such that*

(a) *A separates the points of X, and*
(b) *for each x in X there is a function in A that does not vanish at x.*

Then A is uniformly dense in $C_0(X)$.

Theorem D.23 can be proved by applying Theorem D.22 to the one-point compactification of X.

D.24. Suppose that X is a set and that $\leq$ is a linear order on X. For each x in X define intervals $(-\infty, x)$ and $(x, +\infty)$ by

$$(-\infty, x) = \{z \in X : z < x\}$$

and

$$(x, +\infty) = \{z \in X : x < z\}.$$

[1]Locally compact spaces are defined in Sect. 7.1, and $C_0(X)$ is defined in Sect. 7.3.

The *order topology* on X is the weakest topology on X that contains all of these intervals. The set that consists of these intervals, the intervals of the form $\{z \in X : x < z < y\}$, and the set X, is a base for the order topology on X.

D.25. Let X be a set. A *metric* on X is a function $d \colon X \times X \to \mathbb{R}$ that satisfies

(a) $d(x,y) \geq 0$,
(b) $d(x,y) = 0$ if and only if $x = y$,
(c) $d(x,y) = d(y,x)$, and
(d) $d(x,z) \leq d(x,y) + d(y,z)$

for all x, y, and z in X. A *metric space* is a pair (X,d), where X is a set and d is a metric on X (of course, X itself is often called a metric space).

Let (X,d) be a metric space. If $x \in X$ and if r is a positive number, then the set $B(x,r)$ defined by

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

is called the *open ball* with center x and radius r; the *closed ball* with center x and radius r is the set

$$\{y \in X : d(x,y) \leq r\}.$$

A subset U of X is *open* if for each x in U there is a positive number r such that $B(x,r) \subseteq U$. The collection of all open subsets of X is a topology on X; it is called the topology *induced* or *generated* by d.[2] The open balls form a base for this topology.

D.26. A topological space $(X,\mathscr{O})$ (or a topology $\mathscr{O}$) is *metrizable* if there is a metric d on X that generates the topology $\mathscr{O}$; the metric d is then said to *metrize* X (or $(X,\mathscr{O})$).

D.27. Let X be a metric space. The *diameter* of the subset A of X, written $\operatorname{diam}(A)$, is defined by

$$\operatorname{diam}(A) = \sup\{d(x,y) : x,y \in A\}.$$

The set A is *bounded* if $\operatorname{diam}(A)$ is not equal to $+\infty$. The *distance* between the point x and the nonempty subset A of X is defined by

$$d(x,A) = \inf\{d(x,y) : y \in A\}.$$

Note that if x_1 and x_2 are points in X, then

$$d(x_1,A) \leq d(x_1,x_2) + d(x_2,A).$$

Since we can interchange the points x_1 and x_2 in the formula above, we find that

$$|d(x_1,A) - d(x_2,A)| \leq d(x_1,x_2),$$

[2] When dealing with a metric space (X,d), we will often implicitly assume that X has been given the topology induced by d.

from which it follows that $x \mapsto d(x,A)$ is continuous (and, in fact, uniformly continuous).

D.28. Each closed subset of a metric space is a G_δ, and each open subset is an F_σ. To check the first of these claims, note that if C is a nonempty closed subset of the metric space X, then

$$C = \bigcap_n \left\{ x \in X : d(x,C) < \frac{1}{n} \right\},$$

and so C is the intersection of a sequence of open sets. Now use De Morgan's laws (see Sect. A.1) to check that each open set is an F_σ.

D.29. Let x and $x_1, x_2, \ldots$ belong to the metric space X. The sequence $\{x_n\}$ *converges* to x if $\lim_n d(x_n,x) = 0$; if $\{x_n\}$ converges to x, we say that x is the *limit* of $\{x_n\}$, and we write $x = \lim_n x_n$.

D.30. Let X be a metric space. It is easy to check that a point x in X belongs to the closure of the subset A of X if and only if there is a sequence in A that converges to x.

D.31. Let (X,d) and (Y,d') be metric spaces, and give X and Y the topologies induced by d and d' respectively. Then a function $f\colon X \to Y$ is continuous (in the sense of D.5) if and only if for each x_0 in X and each positive number ε there is a positive number δ such that $d'(f(x), f(x_0)) < \varepsilon$ holds whenever x belongs to X and satisfies $d(x,x_0) < \delta$. The observation at the end of C.7 generalizes to metric spaces, and a small modification of the argument given there yields the following characterization of continuity in terms of sequences: the function f is continuous if and only if $f(x) = \lim_n f(x_n)$ holds whenever x and $x_1, x_2, \ldots$ are points in X such that $x = \lim_n x_n$.

D.32. We noted in D.10 that every second countable topological space is separable. The converse holds for metrizable spaces: if d metrizes the topology of X, and if D is a countable dense subset of X, then the collection consisting of those open balls $B(x,r)$ for which $x \in D$ and r is rational is a countable base for X.

D.33. If X is a second countable topological space, and if Y is a subspace of X, then Y is second countable (if $\mathscr{U}$ is a countable base for X, then $\{U \cap Y : U \in \mathscr{U}\}$ is a countable base for Y). It follows from this, together with D.10 and D.32, that every subspace of a separable metrizable space is separable.

D.34. Let (X,d) be a metric space. A sequence $\{x_n\}$ of elements of X is a *Cauchy sequence* if for each positive number ε there is a positive integer N such that $d(x_m,x_n) < \varepsilon$ holds whenever $m \geq N$ and $n \geq N$. The metric space X is *complete* if every Cauchy sequence in X converges to an element of X.

D.35. (Cantor's Nested Set Theorem) *Let X be a complete metric space. If $\{A_n\}$ is a decreasing sequence of nonempty closed sets of X such that* $\lim_n \operatorname{diam}(A_n) = 0$, *then $\cap_{n=1}^\infty A_n$ contains exactly one point.*

Proof. For each positive integer n choose an element x_n of A_n. Then $\{x_n\}$ is a Cauchy sequence whose limit belongs to $\cap_{n=1}^{\infty} A_n$. Thus $\cap_{n=1}^{\infty} A_n$ is not empty. Since $\lim_n \operatorname{diam}(A_n) = 0$, the set $\cap_{n=1}^{\infty} A_n$ cannot contain more than one point. □

D.36. A subset A of a topological space X is *nowhere dense* if the interior of $\overline{A}$ is empty.

D.37. (Baire Category Theorem) *Let X be a nonempty complete metric space* (*or a nonempty topological space that can be metrized with a complete metric*). *Then X cannot be written as the union of a sequence of nowhere dense sets. Moreover, if $\{A_n\}$ is a sequence of nowhere dense subsets of X, then $(\cup_n A_n)^c$ is dense in X.*

D.38. The metric space (X,d) is *totally bounded* if for each positive ε there is a finite subset S of X such that

$$X = \bigcup \{B(x,\varepsilon) : x \in S\}.$$

D.39. (Theorem) *Let X be a metric space. Then the conditions*

(a) *the space X is compact,*
(b) *the space X is complete and totally bounded, and*
(c) *each sequence of elements of X has a subsequence that converges to an element of X*

are equivalent.

D.40. (Corollary) *Each compact metric space is separable.*

Proof. Let X be a compact metric space. Theorem D.39 implies that X is totally bounded, and so for each positive integer n we can choose a finite set S_n such that $X = \cup\{B(x,1/n) : x \in S_n\}$. The set $\cup_n S_n$ is then a countable dense subset of X. □

D.41. Note, however, that a compact Hausdorff space can fail to be second countable and can even fail to be separable (see Exercises 7.1.7, 7.1.8, and 7.1.10).

D.42. Let $\{X_n\}$ be a sequence of nonempty metrizable spaces, and for each n let d_n be a metric that metrizes X_n. Let $\mathbf{x}$ and $\mathbf{y}$ denote the points $\{x_n\}$ and $\{y_n\}$ of the product space $\prod_n X_n$. Then the formula

$$d(\mathbf{x},\mathbf{y}) = \sum_n \frac{1}{2^n} \min(1, d_n(x_n,y_n))$$

defines a metric on $\prod_n X_n$ that metrizes the product topology. This fact, together with Theorem D.39, can be used to give a fairly easy proof of Tychonoff's theorem for *countable* families of compact *metrizable* spaces.