Appendix E
The Bochner Integral

Let $(X, \mathscr{A})$ be a measurable space, let E be a real or complex Banach space (that is, a Banach space over $\mathbb{R}$ or $\mathbb{C}$), and let $\mathscr{B}(E)$ be the σ-algebra of Borel subsets of E (that is, let $\mathscr{B}(E)$ be the σ-algebra on E generated by the open subsets of E). We will sometimes denote the norm on E by $|\cdot|$, rather than by the more customary $\|\cdot\|$. This will allow us to use $\|\cdot\|$ for the norm of elements of certain spaces of E-valued functions; see, for example, formula (7) below. A function $f: X \to E$ is *Borel measurable* if it is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E)$, and is *strongly measurable* if it is Borel measurable and has a separable range (here by the range of f we mean the subset $f(X)$ of E). The function f is *simple* if it has only finitely many values. Of course, a simple function is strongly measurable if and only if it is Borel measurable.

It is easy to see that if f is Borel measurable, then $x \mapsto |f(x)|$ is $\mathscr{A}$-measurable (use Lemma 7.2.1 and Proposition 2.6.1).

Note that if E is separable, then every E-valued Borel measurable function is strongly measurable. On the other hand, if E is not separable and if $(X, \mathscr{A}) = (E, \mathscr{B}(E))$, then the identity map from X to E is Borel measurable, but is not strongly measurable.

E.1. (Proposition) *Let $(X, \mathscr{A})$ be a measurable space, and let E be a real or complex Banach space. Then*

(a) *the collection of Borel measurable functions from X to E is closed under the formation of pointwise limits, and*
(b) *the collection of strongly measurable functions from X to E is closed under the formation of pointwise limits.*

Proof. Part (a) is a special case of Proposition 8.1.10, and so we can turn to part (b).

Let $\{f_n\}$ be a sequence of strongly measurable functions from X to E, and suppose that $\{f_n\}$ converges pointwise to f. It follows from the separability of the sets $f_n(X)$, $n = 1, 2, \ldots$, that $\cup_n f_n(X)$ is separable, that the closure of $\cup_n f_n(X)$ is separable, and finally that $f(X)$ is separable (see D.33). Since f is Borel measurable (part (a)), the proof is complete. □

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E.2. (Proposition) *Let $(X, \mathscr{A})$ be a measurable space, let E be a real or complex Banach space, and let $f : X \to E$ be strongly measurable. Then there is a sequence $\{f_n\}$ of strongly measurable simple functions such that*

$$f(x) = \lim_n f_n(x)$$

and

$$|f_n(x)| \le |f(x)|, \textit{ for } n = 1, 2, \ldots,$$

hold at each x in X.

Proof. We can certainly assume that $f(X)$ contains at least one nonzero element of E. Let C be a countable dense subset of $f(X)$, let $C^\sim$ be the set of rational multiples of elements of C, and let $\{y_n\}$ be an enumeration of $C^\sim$. We can assume that $y_1 = 0$. It is easy to check (do so) that

$$\text{for each } y \text{ in } f(X) \text{ and each positive number } \varepsilon \text{ there is a term } y_m \text{ of } \{y_n\} \text{ that satisfies } |y_m| \le |y| \text{ and } |y_m - y| < \varepsilon. \tag{1}$$

For each x in X and each positive integer n define a subset $A_n(x)$ of E by

$$A_n(x) = \{y_j : j \le n \text{ and } |y_j| \le |f(x)|\}.$$

Since $y_1 = 0$, each $A_n(x)$ is nonempty.

We now construct the required sequence $\{f_n\}$ by letting $f_n(x)$ be the element of $A_n(x)$ that lies closest to $f(x)$ (in case

$$|f(x) - y_j| = \inf\{|f(x) - y_i| : y_i \in A_n(x)\} \tag{2}$$

holds for several elements y_j of $A_n(x)$, let $f_n(x)$ be y_{j_0}, where j_0 is the smallest value of j for which y_j belongs to $A_n(x)$ and satisfies (2)). It is clear that each f_n is a simple function and that $|f_n(x)| \le |f(x)|$ holds for each n and x. Since the sets $\{x \in X : f_n(x) = y_j\}$ can be described by means of inequalities involving $|f(x)|$, $|y_i|$, $i = 1, \ldots, n$, and $|f(x) - y_i|$, $i = 1, \ldots, n$, each f_n is strongly measurable. Finally, observation (1) implies that $\{f_n\}$ converges pointwise to f (if y_m satisfies the inequalities $|y_m| \le |f(x)|$ and $|y_m - f(x)| < \varepsilon$, then $|f_n(x) - f(x)| < \varepsilon$ holds whenever $n \ge m$). □

Let us note two consequences of Propositions E.1 and E.2. The first is immediate: a function from X to E is strongly measurable if and only if it is the pointwise limit of a sequence of Borel (or strongly) measurable simple functions. The second is given by the following corollary (see, however, Exercise 2).

E.3. (Corollary) *Let $(X, \mathscr{A})$ be a measurable space, and let E be a real or complex Banach space. Then the set of all strongly measurable functions from X to E is a vector space.*

Proof. Suppose that f and g are strongly measurable and that a and b are real (or complex) numbers. Choose sequences $\{f_n\}$ and $\{g_n\}$ of strongly measurable simple functions that converge pointwise to f and g respectively (Proposition E.2). Since $\{af_n + bg_n\}$ converges pointwise to $af + bg$, and since each $af_n + bg_n$ is strongly measurable (it is simple and each of its values is attained on a measurable set), Proposition E.1 implies that $af + bg$ is strongly measurable. □

We turn to the integration of functions with values in a Banach space. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let E be a real or complex Banach space. A function $f\colon X \to E$ is *integrable* (or *strongly integrable*, or *Bochner integrable*) if it is strongly measurable and the function $x \to |f(x)|$ is integrable.[1]

The integral of such functions is defined as follows. First suppose that $f\colon X \to E$ is simple and integrable. Let $a_1, \ldots, a_n$ be the nonzero values of f, and suppose that these values are attained on the sets $A_1, \ldots, A_n$. Then Proposition 2.3.10, applied to the real-valued function $x \mapsto |f(x)|$, implies that each A_i has finite measure under μ. Thus the expression $\sum_{i=1}^n a_i \mu(A_i)$ makes sense; we define the *integral* of f, written $\int f\,d\mu$, to be this sum. It is easy to see that

$$\left| \int f\,d\mu \right| \le \int |f|\,d\mu. \tag{3}$$

It is also easy to see that if f and g are simple integrable functions and a and b are real (or complex) numbers, then $af + bg$ is a simple integrable function, and

$$\int (af + bg)\,d\mu = a \int f\,d\mu + b \int g\,d\mu. \tag{4}$$

Now suppose that f is an arbitrary integrable function. Choose a sequence $\{f_n\}$ of simple integrable functions such that $f(x) = \lim_n f_n(x)$ holds at each x in X and such that the function $x \mapsto \sup_n |f_n(x)|$ is integrable (see Proposition E.2). The dominated convergence theorem for real-valued functions (Theorem 2.4.5) implies that $\lim_n \int |f_n - f|\,d\mu = 0$, and hence that $\lim_{m,n} \int |f_m - f_n|\,d\mu = 0$. Thus (see (3) and (4)) $\{\int f_n\,d\mu\}$ is a Cauchy sequence in E, and so is convergent. The *integral* (or *Bochner integral*) of f, written $\int f\,d\mu$, is defined to be the limit of the sequence $\{\int f_n\,d\mu\}$. (It is easy to check that the value of $\int f\,d\mu$ does not depend on the choice of the sequence $\{f_n\}$: if $\{g_n\}$ is another sequence having the properties required of $\{f_n\}$, then $\lim_n \int |f_n - g_n|\,d\mu = 0$, from which it follows that $\lim_n \int (f_n - g_n)\,d\mu = 0$ and hence that $\lim_n \int f_n\,d\mu = \lim_n \int g_n\,d\mu$.)

Let us note a few basic properties of the Bochner integral.

E.4. (Proposition) *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let E be a real or complex Banach space. Suppose that $f, g\colon X \to E$ are integrable and that a and b are real (or complex) numbers. Then $af + bg$ is integrable, and*

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[1] See Exercise 4 for an indication of another standard definition of Bochner integrability.

$$\int (af+bg)\,d\mu = a\int f\,d\mu + b\int g\,d\mu. \tag{5}$$

Proof. The integrability of $af+bg$ follows from Corollary E.3 and the inequality $|(af+bg)(x)| \le |a|\,|f(x)|+|b|\,|g(x)|$. Let $\{f_n\}$ and $\{g_n\}$ be sequences of simple integrable functions that converge pointwise to f and g respectively and are such that $x \mapsto \sup_n |f_n(x)|$ and $x \mapsto \sup_n |g_n(x)|$ are integrable. Then the functions af_n+bg_n are simple and integrable, and they satisfy

$$\int (af_n+bg_n)\,d\mu = a\int f_n\,d\mu + b\int g_n\,d\mu \tag{6}$$

(see (4)). Furthermore $x \mapsto \sup_n |(af_n+bg_n)(x)|$ is integrable, and so according to the definition of the integral, we can take limits in (6), obtaining (5). □

E.5. (Proposition) *Let $(X,\mathscr{A},\mu)$ be a measure space, and let E be a real or complex Banach space. If $f\colon X \to E$ is integrable, then $|\int f\,d\mu| \le \int |f|\,d\mu$.*

Proof. Let f be an integrable function, and let $\{f_n\}$ be a sequence of simple integrable functions such that $\sup_n |f_n(x)| \le |f(x)|$ and $f(x) = \lim_n f_n(x)$ hold at each x in X (Proposition E.2). Then

$$\left|\int f_n\,d\mu\right| \le \int |f_n|\,d\mu \le \int |f|\,d\mu$$

(see (3)); since $\int f\,d\mu = \lim_n \int f_n\,d\mu$, the proposition follows. □

The dominated convergence theorem can be formulated as follows for E-valued functions.

E.6. (Theorem) *Let $(X,\mathscr{A},\mu)$ be a measure space, let E be a real or complex Banach space, and let g be a $[0,+\infty]$-valued integrable function on X. Suppose that f and f_1, f_2, ... are strongly measurable E-valued functions on X such that the relations*

$$f(x) = \lim_n f_n(x)$$

and

$$|f_n(x)| \le g(x), \textit{ for } n = 1, 2, \ldots,$$

hold at almost every x in X. Then f and f_1, f_2, ... are integrable, and $\int f\,d\mu = \lim_n \int f_n\,d\mu$.

Proof. The integrability of f and $f_1, f_2, \ldots$ is immediate. Since $|f_n - f| \le 2g$ holds almost everywhere, the dominated convergence theorem for real-valued functions (Theorem 2.4.5) implies that $\lim_n \int |f_n - f|\,d\mu = 0$. In view of Propositions E.4 and E.5, this implies that $\int f\,d\mu = \lim_n \int f_n\,d\mu$. □

Let $\mathscr{L}^1(X,\mathscr{A},\mu,E)$ be the set of all E-valued integrable functions on X. Then $\mathscr{L}^1(X,\mathscr{A},\mu,E)$ is a vector space (see Proposition E.4). It is easy to check that the collection $L^1(X, \mathscr{A}, \mu, E)$ of equivalence classes (under almost everywhere equality) of elements of $\mathscr{L}^1(X, \mathscr{A}, \mu, E)$ can be made into a vector space in the natural way, and that the formula

$$\|f\|_1 = \int |f|\, d\mu \tag{7}$$

induces a norm on $L^1(X, \mathscr{A}, \mu, E)$ (and, of course, a seminorm on $\mathscr{L}^1(X, \mathscr{A}, \mu, E)$). The proof of Theorem 3.4.1 can be modified so as to show that $L^1(X, \mathscr{A}, \mu, E)$ is complete under $\|\cdot\|_1$.

One often finds it useful to be able to deal with vector-valued functions in terms of real- (or complex-) valued functions. For this we need to recall the Hahn–Banach theorem.

E.7. (Hahn–Banach Theorem) *Let E be a real or complex normed linear space, let F be a linear subspace of E, and let φ_0 be a continuous linear functional on F. Then there is a continuous linear functional φ on E such that $\|\varphi\| = \|\varphi_0\|$ and such that φ_0 is the restriction of φ to F. In other words, φ_0 can be extended to a continuous linear functional on all of E without increasing its norm.*

A proof of the Hahn–Banach theorem can be found in almost any basic text on functional analysis (see, for example, Conway [31], Kolmogorov and Fomin [73], Royden [102], or Simmons [109]).

We also need the following consequence of the Hahn–Banach theorem.

E.8. (Corollary) *Let E be a real or complex normed linear space that does not consist of 0 alone. Then for each y in E there is a continuous linear functional φ on E such that $\|\varphi\| = 1$ and $\varphi(y) = \|y\|$.*

Proof. Let y be a nonzero element of E, let F be the subspace of E consisting of all scalar multiples of y, and let φ_0 be the linear functional on F defined by $\varphi_0(ty) = t\|y\|$. Then φ_0 satisfies $\|\varphi_0\| = 1$ and $\varphi_0(y) = \|y\|$, and we can produce the required functional φ by applying Theorem E.7 to φ_0. (In case $y = 0$, let φ be an arbitrary linear functional on E that satisfies $\|\varphi\| = 1$.) □

Let us now apply Theorem E.7 and Corollary E.8 to the study of vector-valued functions.

E.9. (Theorem) *Let $(X, \mathscr{A})$ be a measurable space, and let E be a real or complex Banach space. A function $f : X \to E$ is strongly measurable if and only if*

(a) *the image $f(X)$ of X under f is separable, and*
(b) *for each φ in E^* the function $\varphi \circ f$ is $\mathscr{A}$-measurable.*

We will use the following lemma in our proof of Theorem E.9.

E.10. (Lemma) *Let E be a separable normed linear space over $\mathbb{R}$ or $\mathbb{C}$. Then there is a sequence $\{\varphi_n\}$ of elements of E^* such that*

$$\|y\| = \sup\{|\varphi_n(y)| : n = 1, 2, \dots\} \tag{8}$$

holds for each y in E.

Proof. We can assume that E does not consist of 0 alone. Choose a sequence $\{y_n\}$ whose terms form a dense subset of E. According to Corollary E.8, we can choose, for each n, an element φ_n of E^* that satisfies $\|\varphi_n\| = 1$ and $\varphi_n(y_n) = \|y_n\|$. Let us check that the sequence $\{\varphi_n\}$ meets the requirements of the lemma. Since each φ_n satisfies $\|\varphi_n\| = 1$, it follows that

$$\sup\{|\varphi_n(y)| : n = 1, 2, \dots\} \le \|y\|$$

holds for each y in E. For an arbitrary y in E we can find terms in the sequence $\{y_n\}$ that lie arbitrarily close to y, and so the calculations

$$\varphi_n(y) = \varphi_n(y - y_n) + \varphi_n(y_n) = \varphi_n(y - y_n) + \|y_n\|$$

and $|\varphi_n(y - y_n)| \le \|\varphi_n\|\|y - y_n\| = \|y - y_n\|$ imply that

$$\|y\| = \sup\{|\varphi_n(y)| : n = 1, 2, \dots\}.$$

Relation (8) follows. □

Proof of Theorem E.9. Let us assume that we are dealing with Banach spaces over $\mathbb{R}$; the case of Banach spaces over $\mathbb{C}$ is similar.

If f is strongly measurable, then (a) is immediate and (b) follows from Lemma 7.2.1 and Proposition 2.6.1.

Now suppose that f satisfies (a) and (b). In view of (a), it suffices to show that f is Borel measurable. Let E_0 be the smallest closed linear subspace of E that includes $f(X)$. Then E_0 is separable (if C is a countable dense subset of $f(X)$, then E_0 is the closure of the set of finite sums of rational multiples of elements of C). We can show that f is Borel measurable (that is, measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E)$) by showing that it is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E_0)$ (Lemma 7.2.2).

Let $\{\varphi_n\}$ be a sequence in $(E_0)^*$ such that

$$\|y\| = \sup\{|\varphi_n(y)| : n = 1, 2, \dots\} \tag{9}$$

holds for each y in E_0 (Lemma E.10). Since each continuous linear functional on E_0 is the restriction to E_0 of an element of E^* (Theorem E.7), condition (b) implies that for each n the function $\varphi_n \circ f$ is $\mathscr{A}$-measurable. If B is a closed ball in E_0, say with center y_0 and radius r, then $f^{-1}(B)$ is equal to

$$\bigcap_n \{x : |\varphi_n(f(x)) - \varphi_n(y_0)| \le r\},$$

and so belongs to $\mathscr{A}$. Since each open ball in E_0 is the union of a countable collection of closed balls, and since each open subset of E_0 is the union of a countable collection of open balls (recall that E_0 is separable), the collection of closed balls generates $\mathscr{B}(E_0)$. It now follows from Proposition 2.6.2 that f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(E_0)$ and the proof is complete. □

E.11. (Proposition) *Let $(X,\mathscr{A},\mu)$ be a measure space, let E be a real or complex Banach space, and let $f\colon X \to E$ be integrable. Then*

$$\int \varphi \circ f \, d\mu = \varphi\left(\int f \, d\mu\right) \tag{10}$$

holds for each φ in E^*.

The reader should see Exercise 3 for a strengthened form of Proposition E.11.

Proof. It is easy to check (do so) that the integrability of $\varphi \circ f$ follows from that of f. If f is a simple integrable function, attaining the nonzero values $a_1, \ldots, a_k$ on the sets $A_1, \ldots, A_k$, then each side of (10) is equal to $\sum_{i=1}^{k} \varphi(a_i)\mu(A_i)$; hence (10) holds for simple integrable functions. Next suppose that f is an arbitrary integrable function and that $\{f_n\}$ is a sequence of simple integrable functions such that $f(x) = \lim_n f_n(x)$ and $\sup_n |f_n(x)| \le |f(x)|$ hold at each x in X (Proposition E.2). Then Theorems E.6 and 2.4.5 enable us to take limits in the relation $\int \varphi \circ f_n \, d\mu = \varphi(\int f_n \, d\mu)$, and (10) follows for arbitrary integrable functions. □

The reader should note Exercises 5 and 7, which show some difficulties that arise in the extension of integration theory to vector-valued functions. The issues hinted at in these exercises have been the subject of much research over the years; see Diestel and Uhl [37] for a summary and for further references.

Exercises

1. Show that a simpler proof of Proposition E.2 could be given if the f_n's were not required to satisfy the inequality $|f_n(x)| \le |f(x)|$.
2. Suppose that $(X,\mathscr{A})$ is a measurable space and that E is a Banach space. Show by example that the set of Borel measurable functions from X to E can fail to be a vector space. (Hint: Let E be a Banach space with cardinality greater than that of the continuum, and let $(X,\mathscr{A})$ be $(E \times E, \mathscr{B}(E) \times \mathscr{B}(E))$. See Exercise 5.1.8.)
3. Let $(X,\mathscr{A},\mu)$ be a measure space, let E be a Banach space, and let $f\colon X \to E$ be Bochner integrable. Show that $\int f \, d\mu$ is the *only* element x_0 of E that satisfies $\varphi(x_0) = \int \varphi \circ f \, d\mu$ for each φ in E^*. (Hint: Use Corollary E.8.)
4. (This exercise hints at another, rather common, way to define strong measurability and Bochner measurability.) Suppose that $(X,\mathscr{A},\mu)$ is a measure space and that E is a Banach space. Let $f\colon X \to E$ be a function for which there is a sequence $\{f_n\}$ of strongly measurable simple functions such that $f(x) = \lim_n f_n(x)$ holds at μ-almost every x in X.

   (a) Show by example that f need not have a separable range.
   (b) Show that there is a strongly measurable function $g\colon X \to E$ that agrees with f μ-almost everywhere.

(c) Show that $x \mapsto |f(x)|$ is measurable with respect to the completion $\mathscr{A}_\mu$ of $\mathscr{A}$ under μ.
(d) How should $\int f\,d\mu$ be defined if $\int |f|\,d\overline{\mu}$ is finite? (Of course $\overline{\mu}$ is the completion of μ.)

5. Let $(X, \mathscr{A})$ be a measurable space, and let E be a Banach space. An *E-valued measure* on $(X, \mathscr{A})$ is a function $\nu\colon \mathscr{A} \to E$ such that $\nu(\varnothing) = 0$ and such that $\nu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \nu(A_i)$ holds for each infinite sequence $\{A_i\}$ of disjoint sets in $\mathscr{A}$. The *variation* $|\nu|\colon \mathscr{A} \to [0, +\infty]$ of the E-valued measure ν is defined by letting $|\nu|(A)$ be the supremum of the sums $\sum_{i=1}^n |\nu(A_i)|$, where $\{A_i\}_{i=1}^n$ ranges over all finite partitions of A into $\mathscr{A}$-measurable sets.

   (a) Show that the variation of an E-valued measure on $(X, \mathscr{A})$ is a positive measure on $(X, \mathscr{A})$.
   (b) Show by example that the variation of an E-valued measure may not be finite. (Hint: Let X be $\mathbb{N}$, let $\mathscr{A}$ be $\mathscr{P}(\mathbb{N})$, let E be ℓ^2, and define $\nu\colon \mathscr{A} \to E$ by letting $\nu(A)$ be the sequence

$$n \mapsto \begin{cases} \frac{1}{n} & \text{if } n \in A, \\ 0 & \text{if } n \notin A.) \end{cases}$$

6. Let $(X, \mathscr{A}, \mu)$ be a measure space, let E be a Banach space, and let $f\colon X \to E$ be Bochner integrable. Define $\nu\colon \mathscr{A} \to E$ by $\nu(A) = \int \chi_A f\,d\mu$.
   (a) Show that ν is an E-valued measure on $(X, \mathscr{A})$.
   (b) Show that the variation $|\nu|$ of ν is finite.
7. Let λ be Lebesgue measure on $([0,1], \mathscr{B}([0,1]))$, and let E be the Banach space $L^1([0,1], \mathscr{B}([0,1]), \lambda, \mathbb{R})$. Define $\nu\colon \mathscr{B}([0,1]) \to E$ by letting $\nu(A)$ be the element of E determined by the characteristic function χ_A of A.

   (a) Show that ν is an E-valued measure on $([0,1], \mathscr{B}([0,1]))$.
   (b) Show that $|\nu|$ is finite.
   (c) Show that ν is absolutely continuous with respect to λ (in other words, show that $\nu(A) = 0$ holds whenever A satisfies $\lambda(A) = 0$).
   (d) Show that there is no Bochner integrable function $f\colon [0,1] \to E$ that satisfies $\nu(A) = \int \chi_A f\,d\lambda$ for each A in $\mathscr{B}([0,1])$. Thus the Radon–Nikodym theorem fails for the Bochner integral. (Hint: Use Proposition E.11.)