Appendix H
The Henstock–Kurzweil and McShane Integrals

In this appendix we look at the consequences of making what may seem to be a small change to the definition of the Riemann integral. The modified definition gives what is often called the Henstock–Kurzweil integral or the generalized Riemann integral. It will be easy to see that the Henstock–Kurzweil integral is an extension of the Riemann integral; we will see later that it is in fact also an extension of the Lebesgue integral.

Near the end of this appendix we look at another modification of the definition of the Riemann integral; this modification gives the McShane integral. We will see that the McShane integral turns out to be equivalent to the Lebesgue integral.

Most of the results in this appendix are presented as exercises, often with hints.

Let $[a,b]$ be a closed bounded interval. Recall (see Sect. 2.5) that a *partition* of $[a,b]$ is a finite sequence $\{a_i\}_{i=0}^k$ of real numbers such that

$$a = a_0 < a_1 < \cdots < a_k = b,$$

and that a *tagged partition* of $[a,b]$ is a partition of $[a,b]$, together with a sequence $\{x_i\}_{i=1}^k$ of real numbers (called *tags*) such that $a_{i-1} \le x_i \le a_i$ holds for each i (in other words, such that for each i the value x_i belongs to the interval $[a_{i-1}, a_i]$). We will often denote a partition or a tagged partition by a letter such as $\mathscr{P}$. Recall also that the *norm* or *mesh* of a partition or tagged partition $\mathscr{P}$, written $\|\mathscr{P}\|$, is defined by $\|\mathscr{P}\| = \max_i(a_i - a_{i-1})$.

Let f be a real-valued function on an interval $[a,b]$, and let $\mathscr{P}$ be a tagged partition of $[a,b]$. Recall that the *Riemann sum* $\mathscr{R}(f,\mathscr{P})$ corresponding to f and $\mathscr{P}$ is the weighted sum of values of f given by

$$\mathscr{R}(f,\mathscr{P}) = \sum_{i=1}^k f(x_i)(a_i - a_{i-1}).$$

We saw in Proposition 2.5.7 that the Riemann integral of f over the interval $[a,b]$ is the limit of Riemann sums $\mathscr{R}(f,\mathscr{P})$, where the limit is taken as the mesh of $\mathscr{P}$

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approaches 0. More precisely, f is *Riemann integrable*, with *integral* L, if and only if for every positive number ε there is a positive number δ such that

$$|\mathscr{R}(f,\mathscr{P})-L|<\varepsilon \text{ holds for every } \mathscr{P} \text{ that satisfies } \|\mathscr{P}\|<\delta.$$

It seems plausible that it might be worthwhile to require some of the subintervals in a tagged partition $\mathscr{P}$ to be rather narrow (perhaps in regions where the function f is varying rapidly), while allowing other subintervals to be wider. This is what the Henstock–Kurzweil integral does; we turn to the details.

A real-valued function δ whose domain includes the interval $[a,b]$ is said to be a *gauge* on $[a,b]$ if it satisfies $\delta(x)>0$ at each x in $[a,b]$. Given a gauge δ, a tagged partition $\mathscr{P}$ of $[a,b]$ is said to be δ*-fine*, or *subordinate to* δ, if

$$[a_{i-1},a_i]\subseteq(x_i-\delta(x_i),x_i+\delta(x_i))$$

holds for each i. So the subintervals in a δ-fine tagged partition $\mathscr{P}$ must be very short in the parts of $[a,b]$ where all the values of δ are close to 0, while the subintervals in other parts of $[a,b]$ can be longer.

Now consider a function $f\colon [a,b]\to\mathbb{R}$. Note that, in contrast to our discussion of the Riemann integral, we are *not* assuming that f is bounded, although we are for now still assuming that it is real-valued (and not $[-\infty,+\infty]$-valued). Then f is *Henstock–Kurzweil integrable* on $[a,b]$ if there is a number L such that for every positive number ε there is a gauge δ on $[a,b]$ such that

$$|\mathscr{R}(f,\mathscr{P})-L|<\varepsilon \text{ holds for every } \delta\text{-fine tagged partition } \mathscr{P} \text{ of } [a,b].$$

The number L is called the *Henstock–Kurzweil integral* of f over the interval $[a,b]$ and is denoted by $(H)\int_a^b f$ or by $(H)\int_a^b f(x)\,dx$. In cases where there does not seem to be a significant chance of confusion, we may simply write $\int_a^b f$ or $\int_a^b f(x)\,dx$.

See Exercises 11 and 12 for some nontrivial examples of Henstock–Kurzweil integrable functions.

The preceding definition would not make sense if for some function f there were two values of L, each satisfying the definition of the integral of f. The following exercise gives the tool needed to check (in Exercise 2) that such pathology does not occur.

Exercises

1. Cousin's lemma says that if δ is a gauge on an interval $[a,b]$, then there is a δ-fine partition of $[a,b]$. Prove Cousin's lemma

   (a) with a bisection argument (if $[a,b]$ fails to have a δ-fine partition, then so does either its left half or its right half, ...), and

(b) by analyzing

$$\sup\{t \in [a,b] : \text{there is a } \delta\text{-fine partition of } [a,t]\}.$$

2. Show that the value of the Henstock–Kurzweil integral is well defined. That is, show that if f is Henstock–Kurzweil integrable and if L_1 and L_2 are real numbers, each of which satisfies the definition of the Henstock–Kurzweil integral of f, then $L_1 = L_2$. (Hint: Use Exercise 1.)
3. Show that if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then f is Henstock–Kurzweil integrable and $(H)\int_a^b f = (R)\int_a^b f$. (The proof can be very short.)
4. Show that the set of Henstock–Kurzweil integrable functions on $[a,b]$ is a vector space and that the Henstock–Kurzweil integral is a positive linear functional on it.
5. (Cauchy criterion for Henstock–Kurzweil integrability) Show that a function $f\colon [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable if and only if for every positive number ε there is a gauge δ such that $|\mathscr{R}(f,\mathscr{P}_1) - \mathscr{R}(f,\mathscr{P}_2)| < \varepsilon$ holds whenever $\mathscr{P}_1$ and $\mathscr{P}_2$ are δ-fine tagged partitions of $[a,b]$.
6. Suppose that δ is a gauge on $[a,b]$ and that x is a point in $[a,b]$. Then there is a gauge δ' that satisfies $\delta' \le \delta$ and is such that each δ'-fine tagged partition contains x as one of its tags. In many situations this allows us to force specified points to be tags in the partitions under consideration. (Hint: Use
$$\delta'(t) = \begin{cases} \min(\delta(t), |t-x|/2) & \text{if } t \ne x, \text{ and} \\ \delta(x) & \text{if } t = x \end{cases}$$
to define δ'.)
7. Suppose that δ is a gauge on $[a,b]$ and that $\mathscr{P}$ is a δ-fine tagged partition of $[a,b]$ that contains x among its tags. If x belongs to the interior of one of the subintervals of $\mathscr{P}$, say, $a_{i-1} < x < a_i$, and if we define a partition $\mathscr{P}'$ to contain the same intervals and tags as $\mathscr{P}$, except that the interval $[a_{i-1}, a_i]$ is replaced with the two intervals $[a_{i-1}, x]$ and $[x, a_i]$, with x serving as tag in each of these new intervals, then $\mathscr{P}'$ is also a δ-fine partition of $[a,b]$ and $\mathscr{R}(f,\mathscr{P}') = \mathscr{R}(f,\mathscr{P})$ holds for each function f on $[a,b]$.
8. Show that if $f\colon [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable on $[a,b]$ and if $g\colon [a,b] \to \mathbb{R}$ agrees with f everywhere in $[a,b]$ except perhaps at a finite number of points, then g is Henstock–Kurzweil integrable on $[a,b]$ and $\int_a^b g = \int_a^b f$.
9. Show that if $a < c < b$ and if f is Henstock–Kurzweil integrable on $[a,c]$ and on $[c,b]$, then f is Henstock–Kurzweil integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$. (Hint: Use Exercises 6 and 7.)
10. Show that if f is Henstock–Kurzweil integrable on $[a,b]$ and if $[c,d]$ is a subinterval of $[a,b]$, then f is Henstock–Kurzweil integrable on $[c,d]$.

11. Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} n & \text{if } x \in [1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}),\ n = 1, 2, \ldots, \text{ and} \\ 0 & \text{if } x = 1. \end{cases}$$

Using only the definition and basic properties of the Henstock–Kurzweil integral (that is, without using deeper results, such as those given in Exercises 14 and 17), verify that f is Henstock–Kurzweil integrable on $[0,1]$ and that

$$\int_0^1 f\,dx = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

12. Let $f\colon [0,1] \to \mathbb{R}$ be the characteristic function of the set of rational numbers in $[0,1]$. Show that f is Henstock–Kurzweil integrable, with $\int_0^1 f$ equal to 0. (Hint: Let $\{r_n\}_1^\infty$ be an enumeration of the rationals in $[0,1]$. Given a positive value ε, define a gauge δ by letting $\delta(r_n) = \varepsilon/2^{n+1}$ for each n, while letting $\delta(x) = 1$ for all other values of x. Check that each δ-fine partition $\mathscr{P}$ satisfies $|\mathscr{R}(f,\mathscr{P})| < \varepsilon$.)

13. (a) Let $f\colon [a,b] \to \mathbb{R}$ be a function that vanishes almost everywhere. Show that f is Henstock–Kurzweil integrable, with $\int_a^b f$ equal to 0. (Hint: Suppose that $\varepsilon > 0$. For each positive integer n first define A_n by $A_n = \{x \in [a,b] : n-1 < |f(x)| \le n\}$ and then choose an open set U_n such that $A_n \subseteq U_n$ and $\lambda(U_n) < \varepsilon/n2^n$. Define a gauge δ by letting $\delta(x)$ be the distance from x to the complement of U_n if $x \in A_n$ and letting $\delta(x) = 1$ if $x \notin \cup_n A_n$. Find an upper bound for $|\mathscr{R}(f,\mathscr{P})|$ that is valid for all δ-fine partitions $\mathscr{P}$ of $[a,b]$.)
    (b) Suppose that the functions $f, g\colon [a,b] \to \mathbb{R}$ agree almost everywhere and that f is Henstock–Kurzweil integrable. Show that g is Henstock–Kurzweil integrable and that $\int_a^b g = \int_a^b f$.

We can now define the Henstock–Kurzweil integral for $[-\infty,+\infty]$-valued functions: one calls a function $f\colon [a,b] \to [-\infty,+\infty]$ *Henstock–Kurzweil integrable* if there is a function $g\colon [a,b] \to \mathbb{R}$ that is Henstock–Kurzweil integrable and agrees with f almost everywhere. The *Henstock–Kurzweil integral* of f is then defined to be that of g. Exercise 13(b) implies that the resulting concepts of integrability and integral are well defined. One can deal in a similar way with the Henstock–Kurzweil integral for functions that are defined only almost everywhere.

A *tagged subpartition* of an interval $[a,b]$ is a finite indexed collection $\{[c_i,d_i]\}_{i=1}^k$ of nonoverlapping[1] subintervals of $[a,b]$, together with tags $\{x_i\}_{i=1}^k$ such that $x_i \in [c_i,d_i]$ holds for each i. So a tagged subpartition is like a tagged partition, except that the intervals involved may not cover the entire interval $[a,b]$. Note that with subpartitions we cannot do as we did with partitions and use a

[1] Let $\{I_i\}$ be an indexed collection of intervals. These intervals are *nonoverlapping* if for all i and j, the intersection $I_i \cap I_j$ contains at most one point.

sequence of division points $\{a_i\}$ to specify the subintervals, since now there may be gaps between the subintervals.

Let δ be a gauge on $[a,b]$. A tagged subpartition is said to be *δ-fine*, or *subordinate to* δ, if $[c_i,d_i] \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))$ holds for each i. The *Riemann sum* associated to a function f and tagged subpartition $\mathscr{P}$ is, of course, defined by $\mathscr{R}(f,\mathscr{P}) = \sum_i f(x_i)(d_i - c_i)$.

The following result gives some useful estimates involving Riemann sums over subpartitions.

14. (Saks–Henstock lemma) Suppose that $f\colon [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable, that ε is a positive number, and that δ is a gauge on $[a,b]$ such that every δ-fine tagged partition $\mathscr{P}$ of $[a,b]$ satisfies $|\mathscr{R}(f,\mathscr{P}) - (H)\int_a^b f| < \varepsilon$. Show that if $\mathscr{P}'$ is a δ-fine tagged subpartition of $[a,b]$, with subintervals $\{[c_i,d_i]\}$ and tags $\{x_i\}$, then
$$\left|\sum_i f(x_i)(d_i - c_i) - \sum_i (H)\int_{c_i}^{d_i} f\right| \leq \varepsilon \tag{1}$$
and
$$\sum_i \left| f(x_i)(d_i - c_i) - (H)\int_{c_i}^{d_i} f\right| \leq 2\varepsilon. \tag{2}$$
(Hint: Suppose that f, ε, δ, and $\mathscr{P}'$ are as specified above. Let $\{[g_j,h_j]\}$ be the closures of the maximal subintervals of $[a,b]$ that are disjoint from all the subintervals of $\mathscr{P}'$, and for each j choose a partition $\mathscr{P}_j$ of $[g_j,h_j]$ that is subordinate to δ and moreover is such that $\mathscr{R}(f,\mathscr{P}_j)$ is extremely close to $(H)\int_{g_j}^{h_j} f$. To prove (1), consider the partition of $[a,b]$ formed by combining $\mathscr{P}'$ and all the $\mathscr{P}_j$. What happens when the partitions $\mathscr{P}_j$ are made finer and finer? In order to derive (2) from (1), look at two subpartitions, one where the differences $f(x_i)(d_i - c_i) - (H)\int_{c_i}^{d_i} f$ are all positive, and one where they are all negative.)
15. Suppose that $f\colon [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable and that $F\colon [a,b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f$. Show that F is continuous. (Hint: Use the Saks–Henstock lemma (Exercise 14) to show that given a positive ε and an element x_0 of $[a,b]$, we have $|F(x) - F(x_0) - f(x_0)(x - x_0)| < \varepsilon$ for all x sufficiently close to x_0.)
16. (a) Suppose that $f\colon [a,b) \to \mathbb{R}$ is Henstock–Kurzweil integrable on $[a,c]$ for each c in (a,b). Show that for each positive ε there is a positive function δ on $[a,b)$ such that for each c in (a,b) and each δ-fine partition $\mathscr{P}$ of $[a,c]$ we have $|\mathscr{R}(f,\mathscr{P}) - \int_a^c f| < \varepsilon$. (Hint: Let $\{a_n\}_1^\infty$ be a strictly increasing sequence such that $a_1 = a$ and $\lim_n a_n = b$. For each n choose a gauge δ_n on $[a_n, a_{n+1}]$ such that each δ_n-fine partition $\mathscr{P}$ of $[a_n, a_{n+1}]$ satisfies $|\mathscr{R}(f,\mathscr{P}) - \int_{a_n}^{a_{n+1}} f| < \varepsilon/2^n$. Form δ by combining the gauges δ_n, $n = 1$, $2, \ldots$, suitably. See Exercises 6, 7, and 14.)
    (b) Show that if $f\colon [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable on $[a,c]$ for each c in (a,b) and if $\lim_{c\to b}\int_a^c f$ exists, then f is Henstock–Kurzweil integrable

on $[a,b]$ and $\int_a^b f = \lim_{c \to b} \int_a^c f$. Thus the improper Henstock–Kurzweil integral is no more general than the Henstock–Kurzweil integral. (Hint: By modifying f, if necessary, we can assume that $f(b) = 0$. Use the function δ from part (a) of this exercise in your proof.)

17. (The monotone convergence theorem) This exercise is devoted to a proof of the monotone convergence theorem for the Henstock–Kurzweil integral, which can be stated as follows: Suppose that f and f_1, f_2, ... are $[-\infty, +\infty]$-valued functions on $[a,b]$ that are finite almost everywhere and satisfy

$$f_1(x) \le f_2(x) \le \dots \tag{3}$$

and

$$f(x) = \lim_n f_n(x) \tag{4}$$

at almost every x in $[a,b]$. If each f_n is Henstock–Kurzweil integrable and if the sequence $\{(H)\int_a^b f_n\}$ is bounded above, then f is Henstock–Kurzweil integrable and $(H)\int_a^b f = \lim_n (H)\int_a^b f_n$.

(a) Check that for proving the monotone convergence theorem it is enough to consider the case where all the functions involved are $[0, +\infty)$-valued and relations (3) and (4) hold at *every* x in $[a,b]$.

(b) Prove the monotone convergence theorem. (Hint: Let L be the limit of the sequence $\{(H)\int_a^b f_n\}$. Here is a strategy for showing that f is integrable, with integral L: Let ε be a positive number, and for each n let δ_n be a gauge such that each δ_n-fine partition $\mathscr{P}$ satisfies $|\mathscr{R}(f_n, \mathscr{P}) - \int_a^b f_n| < \varepsilon/2^n$. For each x in $[a,b]$ let $n(x)$ be the smallest of those positive integers n that satisfy $\int f_n > L - \varepsilon$ and $f_n(x) > f(x) - \varepsilon$. Use the δ_n's to create a gauge δ by letting $\delta(x) = \delta_{n(x)}(x)$ for each x. Let $\mathscr{P}$ be a δ-fine partition, with division points $\{a_i\}$ and tags $\{x_i\}$. To bound $|\mathscr{R}(f, \mathscr{P}) - L|$, let m and M be the smallest and largest values of $n(x_i)$ as x_i ranges over the set of tags of $\mathscr{P}$, note that

$$\begin{aligned} &\left| \sum f(x_i)(a_i - a_{i-1}) - \sum \int_{a_{i-1}}^{a_i} f_{n(x_i)} \right| \\ &\qquad \le \left| \sum f(x_i)(a_i - a_{i-1}) - \sum f_{n(x_i)}(x_i)(a_i - a_{i-1}) \right| \\ &\qquad\qquad + \left| \sum f_{n(x_i)}(x_i)(a_i - a_{i-1}) - \sum \int_{a_{i-1}}^{a_i} f_{n(x_i)} \right|, \end{aligned}$$

use the definition of δ and the Saks–Henstock lemma to verify that the right side of the formula displayed above is at most $(b-a)\varepsilon + \varepsilon$, and then note that $\sum \int_{a_{i-1}}^{a_i} f_{n(x_i)}$ lies between $\int_a^b f_m$ and $\int_a^b f_M$, both of which are close to L.)

18. The goal of this exercise is to prove that the Henstock–Kurzweil integral is an extension of the Lebesgue integral—that is, that

$$f \text{ is Henstock–Kurzweil integrable and } (H)\int f = (L)\int f \tag{5}$$

holds for each Lebesgue integrable function $f\colon [a,b] \to \mathbb{R}$.

(a) Show that (5) holds if f is the characteristic function of a Borel subset of $[a,b]$. (Hint: Use Theorem 1.6.2.)
(b) Show that (5) also holds if f is the characteristic function of a Lebesgue measurable subset of $[a,b]$.
(c) Show that (5) holds if f is a nonnegative Lebesgue integrable function on $[a,b]$. (Hint: Use the monotone convergence theorems for the Lebesgue and Henstock–Kurzweil integrals.)
(d) Finally, show that (5) holds if f is an arbitrary Lebesgue integrable function.

19. Suppose that $f\colon [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable, and let $F\colon [a,b] \to \mathbb{R}$ be its indefinite integral—that is, the function defined by $F(x) = \int_a^x f$ for each x in $[a,b]$. Then F is differentiable, with derivative given by $F'(x) = f(x)$, at almost every x in $[a,b]$. (Hint: Define D^+ by

$$D^+(x) = \limsup_{t \to x^+} \frac{F(t) - F(x)}{t - x}.$$

Let α and ε be positive numbers, and use the Vitali covering theorem and the Saks–Henstock lemma to show that if the set $\{x : D^+(x) > f(x) + \alpha\}$ is nonempty, then we can choose a sequence $\{[a_i, b_i]\}$ of disjoint intervals that cover it up to a Lebesgue null set and satisfy

$$\varepsilon > \sum_i (F(b_i) - F(a_i) - f(a_i)(b_i - a_i)) > \alpha\lambda^*(\{x : D^+(x) > f(x) + \alpha\}).$$

Conclude that $D^+ \le f$ almost everywhere. Prove analogous results for lower limits and for limits from the left.)
20. Show that each Henstock–Kurzweil integrable function is Lebesgue measurable. (Hint: Use Exercises 15 and 19.)
21. Is every Henstock–Kurzweil integrable function Borel measurable?
22. (a) Show that the converse to part (c) of Exercise 18 also holds. Thus a nonnegative function is Henstock–Kurzweil integrable if and only if it is Lebesgue integrable. (Hint: Why was this not included as a part of Exercise 18, but delayed to this point?)
(b) Show that part (a) fails if the non-negativity condition is omitted. (Hint: Take a function on $[a,b]$ that has an improper Riemann integral but is not Lebesgue integrable.)
23. (A version of Theorem 6.3.11 for the Henstock–Kurzweil integral) Suppose that the function $F\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and is differentiable at all but a countable collection of points in $[a,b]$. Then its derivative F' is Henstock–Kurzweil integrable on $[a,b]$, and

$$(H)\int_a^b F'(x)\,dx = F(b) - F(a).$$

(Hint: It is enough to deal with the function f that agrees with F' where F is differentiable and that vanishes elsewhere. Let $\{t_i\}$ be a sequence consisting of the points at which F is not differentiable. Suppose that $\varepsilon > 0$, and define δ on the points t_i by choosing positive values $\delta(t_i)$ that are so small that $\sum_i |(F(b_i) - F(a_i)| < \varepsilon$ whenever $\{[a_i, b_i]\}$ is a finite sequence of intervals such that $t_i \in [a_i, b_i]$ and $[a_i, b_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ hold for each i. Check that δ can be extended to a gauge (also called δ) on $[a,b]$ such that each δ-fine partition $\mathscr{P}$ of $[a,b]$ satisfies $|\mathscr{R}(f,\mathscr{P}) - (F(b) - F(a))| < 2\varepsilon$.)

The McShane integral is another generalization of the Riemann integral; its definition is given by a slight modification of the definition of the Henstock–Kurzweil integral.

Let us consider a generalization of the concept of a tagged partition in which the tags x_i are no longer required to belong to the corresponding intervals $[a_{i-1}, a_i]$. More precisely, a *freely tagged*[2] partition of $[a,b]$ is a partition $\{a_i\}_{i=0}^k$ of $[a,b]$, together with a sequence $\{x_i\}_{i=1}^k$ of real numbers (tags) such that $x_i \in [a,b]$ for each i; it is not required that $x_i \in [a_{i-1}, a_i]$. If δ is a gauge on $[a,b]$, then a *δ-fine* freely tagged partition is a freely tagged partition such that

$$[a_{i-1}, a_i] \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))$$

holds for each i. Thus the subintervals in a δ-fine freely tagged partition are required to lie close to the corresponding tags, but are not required to contain the tags.

Note that every δ-fine tagged partition of $[a,b]$ is a δ-fine freely tagged partition of $[a,b]$, but that the converse does not hold. Note also that the δ-fine tagged partitions of $[a,b]$ are exactly the δ-fine freely tagged partitions of $[a,b]$ that are in fact tagged partitions.

Riemann sums are defined for freely tagged partitions just as they are for tagged partitions: if the freely tagged partition $\mathscr{P}$ has division points $\{a_i\}_{i=0}^k$ and tags $\{x_i\}_{i=1}^k$, then for a function $f\colon [a,b] \to \mathbb{R}$ we have $\mathscr{R}(f,\mathscr{P}) = \sum_{i=1}^k f(x_i)(a_i - a_{i-1})$.

A function $f\colon [a,b] \to \mathbb{R}$ is *McShane integrable* on $[a,b]$ if there is a number L such that for every positive number ε there is a gauge δ on $[a,b]$ such that

$$|\mathscr{R}(f,\mathscr{P}) - L| < \varepsilon \text{ holds for every } \delta\text{-fine freely tagged partition } \mathscr{P} \text{ of } [a,b];$$

the number L is then called the *McShane integral* of f over $[a,b]$. We will denote the McShane integral of f over the interval $[a,b]$ by $(M)\int_a^b f$ or $(M)\int_a^b f(x)\,dx$; in cases where there does not seem to be a significant chance of confusion, we may write simply $\int_a^b f$ or $\int_a^b f(x)\,dx$.

Arguments that show that the Henstock–Kurzweil integral is well defined (see Exercise 2) can also be used to show that the McShane integral is well defined.

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[2] Another term for a freely tagged partition is a *free tagged* partition.

Furthermore, it is easy to see that the Henstock–Kurzweil integral is an extension of the McShane integral: since every δ-fine tagged partition is a δ-fine freely tagged partition, it follows that if L is a value such that $|\mathscr{R}(f,\mathscr{P})-L|<\varepsilon$ holds for every δ-fine freely tagged partition, then this same inequality holds for every δ-fine tagged partition. We will soon see that the McShane integral is equivalent to the Lebesgue integral.

24. Show that the McShane integral is an extension of the Riemann integral: if $f\colon [a,b]\to\mathbb{R}$ is Riemann integrable, then f is McShane integrable and the McShane and Riemann integrals of f are equal. (Hint: Modify the proof of Proposition 2.5.7.)
25. Show that the set of McShane integrable functions on $[a,b]$ is a vector space and that the McShane integral is a positive linear functional on it (see Exercise 4).
26. Formulate and prove a Cauchy criterion for McShane integrability (see Exercise 5).
27. Show that Exercises 9 and 10, which relate integrals on an interval to integrals on its subintervals, also hold for the McShane integral.
28. Prove a version of Exercise 13 for the McShane integral. That is, prove that sets of Lebesgue measure zero behave as might be expected.
29. Formulate and prove the Saks–Henstock lemma (see Exercise 14) for the McShane integral (your new version should involve freely tagged partitions and subpartitions, and not just tagged ones).
30. Formulate and prove the monotone convergence theorem (see Exercise 17) for the McShane integral.
31. Show that a nonnegative function $f\colon [a,b]\to\mathbb{R}$ is McShane integrable if and only if it is Lebesgue integrable, and that in that case $(M)\int_a^b f=(L)\int_a^b f$. (Hint: Use ideas from Exercises 18 and 22.)
32. In this exercise, we prove that the McShane and Lebesgue integrals (for functions on $[a,b]$) are equivalent.

    (a) Show that if $f\colon [a,b]\to\mathbb{R}$ is McShane integrable, then $|f|$ is also McShane integrable. (Hint: Use the Cauchy criterion for McShane integrability. Suppose that $\mathscr{P}_1$ and $\mathscr{P}_2$ are δ-fine freely tagged partitions of $[a,b]$, where $\mathscr{P}_1$ has subintervals[3] $\{I_i\}$ and tags $\{x_i\}$ and $\mathscr{P}_2$ has subintervals$\{J_j\}$ and tags $\{y_j\}$. We will consider freely tagged partitions $\mathscr{P}_3$ and $\mathscr{P}_4$ of $[a,b]$ whose subintervals are the nondegenerate intervals of the form $I_i\cap J_j$ and whose tags (where $\mathscr{P}_3$ has tags $\{u_{i,j}\}$ and $\mathscr{P}_4$ has tags $\{v_{i,j}\}$) are such that both $u_{i,j}$ and $v_{i,j}$ belong to the set $\{x_i,y_j\}$. Check that in such cases $\mathscr{P}_3$ and $\mathscr{P}_4$ are both δ-fine. Check also that for each i and j we can choose $u_{i,j}$ and $v_{i,j}$ such that
$$\big||f(x_i)|-|f(y_j)|\big|\le f(u_{i,j})-f(v_{i,j}),$$

[3]Here we name the subintervals, rather than the division points, since we will also be considering partitions consisting of subintervals of the form $I_i\cap J_j$; we will need to relate $I_i\cap J_j$ to I_i and J_j, and this is awkward to do in terms of division points.

and that with this choice of $\mathscr{P}_3$ and $\mathscr{P}_4$ we have

$$|\mathscr{R}(|f|,\mathscr{P}_1)-\mathscr{R}(|f|,\mathscr{P}_2)| \le \mathscr{R}(f,\mathscr{P}_3)-\mathscr{R}(f,\mathscr{P}_4).$$

Use this inequality to derive the Cauchy condition for $|f|$ from the Cauchy condition for f.)

(b) Show that if $f\colon [a,b]\to\mathbb{R}$ is McShane integrable, then f^+ and f^-, the positive and negative parts of f, are McShane integrable. (Hint: Express f^+ and f^- as simple algebraic expressions involving $|f|$ and f.)

(c) Conclude that the McShane integral is equivalent to the Lebesgue integral. In other words, an arbitrary function $f\colon [a,b]\to\mathbb{R}$ is McShane integrable if and only if it is Lebesgue integrable, and in that case $(M)\int_a^b f = (L)\int_a^b f$. (See Exercise 31.)

(d) Show that part (a) of this exercise cannot be extended to the Henstock–Kurzweil integral. That is, show by example that the Henstock–Kurzweil integrability of a function $f\colon [a,b]\to\mathbb{R}$ does not imply the Henstock–Kurzweil integrability of $|f|$. (Hint: Once again, consider a function on $[a,b]$ that has an improper Riemann integral but is not Lebesgue integrable.)

Notes

There are many books and papers on the Henstock–Kurzweil integral. Two standard and thorough ones are by Bartle [5] and Gordon [52]. See also the paper by Bongiorno [16] in the handbook edited by Pap [95].