Chapter 2
Functions and Integrals

This chapter is devoted to the definition and basic properties of the Lebesgue integral. We first introduce measurable functions—the functions that are simple enough that the integral can be defined for them if their values are not too large (Sect. 2.1). After a brief look in Sect. 2.2 at properties that hold almost everywhere (that is, that may fail on some set of measure zero, as long as they hold everywhere else), we turn to the definition of the Lebesgue integral and to its basic properties (Sects. 2.3 and 2.4). The chapter ends with a sketch of how the Lebesgue integral relates to the Riemann integral (Sect. 2.5) and then with a few more details about measurable functions (Sect. 2.6).

2.1 Measurable Functions

In this section we introduce measurable functions and study some of their basic properties. We begin with the following elementary result.

Proposition 2.1.1. *Let $(X, \mathscr{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathscr{A}$. For a function $f: A \to [-\infty, +\infty]$ the conditions*

(a) *for each real number t the set $\{x \in A : f(x) \le t\}$ belongs to $\mathscr{A}$,*
(b) *for each real number t the set $\{x \in A : f(x) < t\}$ belongs to $\mathscr{A}$,*
(c) *for each real number t the set $\{x \in A : f(x) \ge t\}$ belongs to $\mathscr{A}$, and*
(d) *for each real number t the set $\{x \in A : f(x) > t\}$ belongs to $\mathscr{A}$*

are equivalent.

Proof. The identity

$$\{x \in A : f(x) < t\} = \bigcup_n \{x \in A : f(x) \le t - 1/n\}$$

D.L. Cohn, *Measure Theory: Second Edition*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4614-6956-8_2,

implies that each of the sets appearing in condition (b) is the union of a sequence of sets appearing in condition (a); hence condition (a) implies condition (b). The sets appearing in condition (c) can be expressed in terms of those appearing in condition (b) by means of the identity

$$\{x \in A : f(x) \geq t\} = A - \{x \in A : f(x) < t\};$$

thus condition (b) implies condition (c). Similar arguments, the details of which are left to the reader, show that condition (c) implies condition (d) and that condition (d) implies condition (a). □

Let $(X, \mathscr{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathscr{A}$. A function $f \colon A \to [-\infty, +\infty]$ is *measurable with respect to* $\mathscr{A}$ if it satisfies one, and hence all, of the conditions of Proposition 2.1.1. A function that is measurable with respect to $\mathscr{A}$ is sometimes called $\mathscr{A}$*-measurable* or, if the σ-algebra $\mathscr{A}$ is clear from context, simply *measurable*. In case $X = \mathbb{R}^d$, a function that is measurable with respect to $\mathscr{B}(\mathbb{R}^d)$ is called *Borel measurable* or a *Borel function*, and a function that is measurable with respect to $\mathscr{M}_{\lambda^*}$ is called *Lebesgue measurable* (recall that $\mathscr{M}_{\lambda^*}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^d$). Of course every Borel measurable function on $\mathbb{R}^d$ is Lebesgue measurable.

We turn to a few examples and then to some of the basic facts about measurable functions.

Examples 2.1.2. (a) Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be continuous. Then for each real number t the set $\{x \in \mathbb{R}^d : f(x) < t\}$ is open and so is a Borel set. Thus f is Borel measurable.

(b) Let I be a subinterval of $\mathbb{R}$, and let $f \colon I \to \mathbb{R}$ be nondecreasing. Then for each real number t the set $\{x \in I : f(x) < t\}$ is a Borel set (it is either an interval, a set consisting of only one point, or the empty set). Thus f is Borel measurable.

(c) Let $(X, \mathscr{A})$ be a measurable space, and let B be a subset of X. Then χ_B, the characteristic function of B, is $\mathscr{A}$-measurable if and only if $B \in \mathscr{A}$.

(d) A function is called *simple* if it has only finitely many values. Let $(X, \mathscr{A})$ be a measurable space, let $f \colon X \to [-\infty, +\infty]$ be simple, and let $\alpha_1, \ldots, \alpha_n$ be the values of f. Then f is $\mathscr{A}$-measurable if and only if $\{x \in X : f(x) = \alpha_i\} \in \mathscr{A}$ for $i = 1, \ldots, n$. □

Proposition 2.1.3. *Let $(X, \mathscr{A})$ be a measurable space, let A be a subset of X that belongs to $\mathscr{A}$, and let f and g be $[-\infty, +\infty]$-valued measurable functions on A. Then the sets $\{x \in A : f(x) < g(x)\}$, $\{x \in A : f(x) \leq g(x)\}$, and $\{x \in A : f(x) = g(x)\}$ belong to $\mathscr{A}$.*

Proof. Note that the inequality $f(x) < g(x)$ holds if and only if there is a rational number r such that $f(x) < r < g(x)$. Thus

$$\{x \in A : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} (\{x \in A : f(x) < r\} \cap \{x \in A : r < g(x)\}),$$

and so $\{x \in A : f(x) < g(x)\}$, as the union of a countable collection of sets that belong to $\mathscr{A}$, itself belongs to $\mathscr{A}$. The set $\{x \in A : g(x) < f(x)\}$ likewise belongs to $\mathscr{A}$. This and the identity

$$\{x \in A : f(x) \leq g(x)\} = A - \{x \in A : g(x) < f(x)\}$$

imply that $\{x \in A : f(x) \leq g(x)\}$ belongs to $\mathscr{A}$. Finally $\{x \in A : f(x) = g(x)\}$ is the difference of $\{x \in A : f(x) \leq g(x)\}$ and $\{x \in A : f(x) < g(x)\}$ and so belongs to $\mathscr{A}$. □

Let f and g be $[-\infty,+\infty]$-valued functions having a common domain A. The *maximum* and *minimum* of f and g, written $f \vee g$ and $f \wedge g$, are the functions from A to $[-\infty,+\infty]$ defined by

$$(f \vee g)(x) = \max(f(x), g(x))$$

and

$$(f \wedge g)(x) = \min(f(x), g(x)).$$

Equivalently, we can define $f \vee g$ by

$$(f \vee g)(x) = \begin{cases} f(x) & \text{if } f(x) > g(x) \text{ and,} \\ g(x) & \text{otherwise,} \end{cases}$$

with $f \wedge g$ getting a corresponding definition.

If $\{f_n\}$ is a sequence of $[-\infty,+\infty]$-valued functions on A, then $\sup_n f_n \colon A \to [-\infty,+\infty]$ is defined by

$$(\sup_n f_n)(x) = \sup\{f_n(x) : n = 1, 2, \ldots\}$$

and $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$, and $\lim_n f_n$ are defined in analogous ways. The domain of $\lim_n f_n$ consists of those points in A at which $\limsup_n f_n$ and $\liminf_n f_n$ agree; the domain of each of the other four functions is A. Each of these functions can have infinite values, even if all the f_n's have only finite values; in particular, $\lim_n f_n(x)$ can be $+\infty$ or $-\infty$.

Proposition 2.1.4. *Let $(X, \mathscr{A})$ be a measurable space, let A be a subset of X that belongs to $\mathscr{A}$, and let f and g be $[-\infty,+\infty]$-valued measurable functions on A. Then $f \vee g$ and $f \wedge g$ are measurable.*

Proof. The measurability of $f \vee g$ follows from the identity

$$\{x \in A : (f \vee g)(x) \leq t\} = \{x \in A : f(x) \leq t\} \cap \{x \in A : g(x) \leq t\},$$

and the measurability of $f \wedge g$ follows from the identity

$$\{x \in A : (f \wedge g)(x) \le t\} = \{x \in A : f(x) \le t\} \cup \{x \in A : g(x) \le t\}. \qquad \square$$

Proposition 2.1.5. *Let $(X, \mathscr{A})$ be a measurable space, let A be a subset of X that belongs to $\mathscr{A}$, and let $\{f_n\}$ be a sequence of $[-\infty, +\infty]$-valued measurable functions on A. Then*

(a) *the functions* $\sup_n f_n$ *and* $\inf_n f_n$ *are measurable,*
(b) *the functions* $\limsup_n f_n$ *and* $\liminf_n f_n$ *are measurable, and*
(c) *the function* $\lim_n f_n$ *(whose domain is* $\{x \in A : \limsup_n f_n(x) = \liminf_n f_n(x)\}$*) is measurable.*

Proof. The measurability of $\sup_n f_n$ and $\inf_n f_n$ follows from the identities

$$\{x \in A : \sup_n f_n(x) \le t\} = \bigcap_n \{x \in A : f_n(x) \le t\}$$

and

$$\{x \in A : \inf_n f_n(x) < t\} = \bigcup_n \{x \in A : f_n(x) < t\}.$$

For each positive integer k define functions g_k and h_k by $g_k = \sup_{n \ge k} f_n$ and $h_k = \inf_{n \ge k} f_n$. Part (a) of the proposition implies first that each g_k is measurable and that each h_k is measurable and then that $\inf_k g_k$ and $\sup_k h_k$ are measurable. Since $\limsup_n f_n$ and $\liminf_n f_n$ are equal to $\inf_k g_k$ and $\sup_k h_k$, they too are measurable.

Let A_0 be the domain of $\lim_n f_n$. Then A_0 is equal to $\{x \in A : \limsup_n f_n(x) = \liminf_n f_n(x)\}$, which according to Proposition 2.1.3 belongs to $\mathscr{A}$. Since

$$\{x \in A_0 : \lim_n f_n(x) \le t\} = A_0 \cap \{x \in A : \limsup_n f_n(x) \le t\},$$

the measurability of $\lim_n f_n$ follows. $\square$

In the following two propositions we deal with arithmetic operations on $[0, +\infty]$-valued measurable functions (see B.4) and on $\mathbb{R}$-valued measurable functions. Arithmetic operations on $[-\infty, +\infty]$-valued functions are trickier and are seldom needed.

Proposition 2.1.6. *Let $(X, \mathscr{A})$ be a measurable space, let A be a subset of X that belongs to $\mathscr{A}$, let f and g be $[0, +\infty]$-valued measurable functions on A, and let α be a nonnegative real number. Then αf and $f + g$ are measurable.*[1]

Proof. For the measurability of αf, note that if $\alpha = 0$, then αf is identically 0 and so measurable, while if $\alpha > 0$, then for each t the set $\{x \in A : \alpha f(x) < t\}$ is equal to $\{x \in A : f(x) < t/\alpha\}$ and so belongs to $\mathscr{A}$.

[1]Recall that $0 \cdot (+\infty) = 0$ and that if $x \ne -\infty$, then $x + (+\infty) = (+\infty) + x = +\infty$. See Appendix B.

Exercises

1. Let $(X, \mathscr{A})$ be a measurable space. Use Proposition 2.6.1 and Example 2.1.2(a) to give another proof that if $f, g\colon X \to \mathbb{R}$ are measurable, then $f+g$ and fg are measurable. (Hint: Consider the function $H\colon X \to \mathbb{R}^2$ defined by $H(x) = (f(x), g(x))$.)
2. Show that if f is a measurable complex-valued function on $(X, \mathscr{A})$, then $|f|$ is also measurable.
3. Let $(X, \mathscr{A})$ be a measurable space, and let $f, g\colon X \to \mathbb{C}$ be measurable. Show that if g does not vanish, then f/g is measurable.
4. (a) Show that $\mathscr{B}(\overline{\mathbb{R}})$ is the σ-algebra on $\overline{\mathbb{R}}$ generated by the intervals of the form $[-\infty, t]$.
   (b) Use part (a) of this exercise, together with Proposition 2.6.2, to give another proof of Proposition 2.6.4.
5. Let X and Y be sets, and let f be a function from X to Y. Show that
   (a) if $\mathscr{A}$ is a σ-algebra on X, then $\{B \subseteq Y : f^{-1}(B) \in \mathscr{A}\}$ is a σ-algebra on Y,
   (b) if $\mathscr{B}$ is a σ-algebra on Y, then $\{f^{-1}(B) : B \in \mathscr{B}\}$ is a σ-algebra on X, and
   (c) if $\mathscr{C}$ is a collection of subsets of Y, then
$$\sigma(\{f^{-1}(C) : C \in \mathscr{C}\}) = \{f^{-1}(B) : B \in \sigma(\mathscr{C})\}.$$
6. Let μ be a nonzero finite Borel measure on $\mathbb{R}$, and let $F\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $F(x) = \mu((-\infty, x])$. Define a function g on the interval $(0, \lim_{x\to+\infty} F(x))$ by
$$g(x) = \inf\{t \in \mathbb{R} : F(t) \geq x\}.$$
   (a) Show that g is nondecreasing, finite valued, and Borel measurable.
   (b) Show that $\mu = \lambda g^{-1}$. (Hint: Start by showing that $\mu(B) = \lambda(g^{-1}(B))$ when B has the form $(-\infty, b]$.)
7. Show that a convex subset of $\mathbb{R}^2$ need not be a Borel set. (Hint: Consider an open ball, together with part of its boundary.)

Notes

See the notes for Chap. 1 for some alternative expositions of basic integration theory. At some point the reader should work through the constructions of the integral given in some of those references. The construction given by Halmos [54] is useful for the study of vector-valued functions (see also Appendix E).

There is an approach to integration theory, due to Daniell [32] and Stone [114], in which the integral is developed before measures are introduced. For an outline of this approach, see Sect. 7.7, and see the notes at the end of Chap. 7.

it follows that $\mu f^{-1}(\cup_n B_n) = \sum_n \mu f^{-1}(B_n)$ and hence that μf^{-1} is a measure on $(Y, \mathscr{B})$. The measure[4] μf^{-1} is sometimes called the *image of* μ *under* f.

Proposition 2.6.8. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let $(Y, \mathscr{B})$ be a measurable space, and let $f\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ be measurable. Let g be an extended real-valued $\mathscr{B}$-measurable function on Y. Then g is μf^{-1}-integrable if and only if $g \circ f$ is μ-integrable. If these functions are integrable, then*

$$\int_Y g\, d(\mu f^{-1}) = \int_X (g \circ f)\, d\mu.$$

Proof. The measurability of $g \circ f$ follows from Propositions 2.6.1 and 2.6.4. We turn to the integrability of g and $g \circ f$. First suppose that g is the characteristic function of a set B in $\mathscr{B}$. Then $g \circ f$ is the characteristic function of $f^{-1}(B)$, and $\int_Y g\, d(\mu f^{-1})$ and $\int_X (g \circ f)\, d\mu$ are both equal to $\mu(f^{-1}(B))$. Thus the identity

$$\int_Y g\, d(\mu f^{-1}) = \int_X (g \circ f)\, d\mu$$

holds for characteristic functions. The additivity and homogeneity of the integral (Proposition 2.3.4) imply that this identity holds for nonnegative simple $\mathscr{B}$-measurable functions, and an approximation argument (use Proposition 2.1.8 and Theorem 2.4.1) shows that it holds for all $[0, +\infty]$-valued $\mathscr{B}$-measurable functions. Since an arbitrary $\mathscr{B}$-measurable function can be separated into its positive and negative parts, the proposition follows. □

We derive two elementary consequences of Proposition 2.6.8. First suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = -x$. Then $\lambda f^{-1} = \lambda$, and so a Borel function g on $\mathbb{R}$ is Lebesgue integrable if and only if the function $x \mapsto g(-x)$ is Lebesgue integrable. If these functions are integrable, then

$$\int g(x)\, \lambda(dx) = \int g(-x)\, \lambda(dx).$$

A similar argument shows that if $y \in \mathbb{R}$, then a Borel function g is Lebesgue integrable if and only if the function $x \mapsto g(x+y)$ is Lebesgue integrable. If these functions are integrable, then

$$\int g(x)\, \lambda(dx) = \int g(x+y)\, \lambda(dx).$$

[4] Another notation for μf^{-1} is $\mu \circ f^{-1}$.

We turn to $f+g$. It is easy to check that $(f+g)(x)<t$ holds if and only if there is a rational number r such that $f(x)<r$ and $g(x)<t-r$. Thus

$$\begin{aligned}&\{x\in A:(f+g)(x)<t\}\\&\qquad=\bigcup_{r\in\mathbb{Q}}(\{x\in A:f(x)<r\}\bigcap\{x\in A:g(x)<t-r\}),\end{aligned}$$

and so $\{x\in A:(f+g)(x)<t\}$, as the union of a countable collection of sets that belong to $\mathscr{A}$, itself belongs to $\mathscr{A}$. The measurability of $f+g$ follows. □

Proposition 2.1.7. *Let $(X,\mathscr{A})$ be a measurable space, let A be a subset of X that belongs to $\mathscr{A}$, let f and g be measurable real-valued functions on A, and let α be a real number. Then αf, $f+g$, $f-g$, fg, and f/g (where the domain of f/g is $\{x\in A:g(x)\neq 0\}$) are measurable.*

Proof. The measurability of αf and $f+g$ can be verified by modifying the proof of Proposition 2.1.6, and so the details are omitted (note that if $\alpha<0$, then $\{x\in A:\alpha f(x)<t\}=\{x\in A:f(x)>t/\alpha\}$). The measurability of $f-g$ follows from the identity $f-g=f+(-1)g$.

We turn to the product of measurable functions and begin by showing that if $h\colon A\to\mathbb{R}$ is measurable, then h^2 is measurable. For this note that if $t\leq 0$, then

$$\{x\in A:h^2(x)<t\}=\varnothing,$$

while if $t>0$, then

$$\{x\in A:h^2(x)<t\}=\{x\in A:-\sqrt{t}<h(x)<\sqrt{t}\};$$

the measurability of h^2 follows. Hence if f and g are measurable, then f^2, g^2, and $(f+g)^2$ are measurable, and the measurability of fg follows from the identity

$$fg=\frac{1}{2}((f+g)^2-f^2-g^2).$$

Let $A_0=\{x\in A:g(x)\neq 0\}$, so that A_0 is the domain of f/g. It is easy to check (do so) that A_0 belongs to $\mathscr{A}$. Since for each t the set $\{x\in A_0:(f/g)(x)<t\}$ is the union of

$$\{x\in A:g(x)>0\}\cap\{x\in A:f(x)<tg(x)\}$$

and

$$\{x\in A:g(x)<0\}\cap\{x\in A:f(x)>tg(x)\},$$

the measurability of f/g follows (see Proposition 2.1.3). □

Let A be a set, and let f be an extended real-valued function[2] on A. The *positive part* f^+ and the *negative part* f^- of f are the extended real-valued functions defined by

$$f^+(x) = \max(f(x), 0)$$

and

$$f^-(x) = -\min(f(x), 0).$$

Thus $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. It is easy to check that if $(X, \mathscr{A})$ is a measurable space and if f is a $[-\infty, +\infty]$-valued function defined on a subset of X, then f is measurable if and only if f^+ and f^- are both measurable. It follows from this remark, together with Proposition 2.1.6, that the absolute value $|f|$ of a measurable function f is measurable (note that $|f| = f^+ + f^-$).

Let $(X, \mathscr{A})$ be a measurable space, let A be a subset of X that belongs to $\mathscr{A}$, and let f be a $[-\infty, +\infty]$-valued function on A. The following relationships between the measurability of f and the measurability of restrictions of f to subsets of A are sometimes useful:

(a) If f is $\mathscr{A}$-measurable and if B is a subset of A that belongs to $\mathscr{A}$, then the restriction f_B of f to B is $\mathscr{A}$-measurable; this follows from the identity

$$\{x \in B : f_B(x) < t\} = B \cap \{x \in A : f(x) < t\}.$$

(b) If $\{B_n\}$ is a sequence of sets that belong to $\mathscr{A}$, if $A = \cup_n B_n$, and if for each n the restriction f_{B_n} of f to B_n is $\mathscr{A}$-measurable, then f is $\mathscr{A}$-measurable; this follows from the identity

$$\{x \in A : f(x) < t\} = \bigcup_n \{x \in B_n : f_{B_n}(x) < t\}.$$

We will repeatedly have need for the following basic result.

Proposition 2.1.8. *Let $(X, \mathscr{A})$ be a measurable space, let A be a subset of X that belongs to $\mathscr{A}$, and let f be a $[0, +\infty]$-valued measurable function on A. Then there is a sequence $\{f_n\}$ of simple $[0, +\infty)$-valued measurable functions on A that satisfy*

$$f_1(x) \le f_2(x) \le \dots \tag{1}$$

and

$$f(x) = \lim_n f_n(x) \tag{2}$$

at each x in A.

[2] An *extended real-valued function* is, of course, a $[-\infty, +\infty]$-valued function.

arguments show that the product of two measurable complex-valued functions on X is measurable; in particular, the product of a complex number and a complex-valued measurable function is measurable. □

Let $(X,\mathscr{A},\mu)$ be a measure space. A complex-valued function f on X is *integrable* if its real and imaginary parts $\Re(f)$ and $\Im(f)$ are integrable; if f is integrable, then its *integral* is defined by

$$\int f\,d\mu = \int \Re(f)\,d\mu + i\int \Im(f)\,d\mu.$$

It is easy to check that if f and g are integrable complex-valued functions on X and if α is a complex number, then

(a) $f+g$ and αf are integrable,
(b) $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$, and
(c) $\int (\alpha f)\,d\mu = \alpha \int f\,d\mu$.

The dominated convergence theorem (Theorem 2.4.5) is valid if the functions f and f_1, f_2, ... appearing in it are complex-valued (consider the real and imaginary parts of these functions separately).

Proposition 2.6.7. *Let $(X,\mathscr{A},\mu)$ be a measure space, and let f be a complex-valued function on X that is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\mathbb{C})$. Then f is integrable if and only if $|f|$ is integrable. If these functions are integrable, then $|\int f\,d\mu| \le \int |f|\,d\mu$.*

Proof. The measurability of $|f|$ is easy to check (see Exercise 2). Let $\Re(f)$ and $\Im(f)$ be the real and imaginary parts of f. If f is integrable, then the integrability of $|f|$ follows from the inequality $|f| \le |\Re(f)| + |\Im(f)|$, while if $|f|$ is integrable, then the integrability of f follows from the inequalities $|\Re(f)| \le |f|$ and $|\Im(f)| \le |f|$ (see Proposition 2.3.8). Now suppose that f is integrable. Write the complex number $\int f\,d\mu$ in its polar form, letting w be a complex number of absolute value 1 such that

$$\int f\,d\mu = w\left|\int f\,d\mu\right|.$$

If we divide by w and use that fact that $|w^{-1}| = 1$, we find

$$\left|\int f\,d\mu\right| = w^{-1}\int f\,d\mu = \int (w^{-1}f)\,d\mu = \int \Re(w^{-1}f)\,d\mu \le \int |f|\,d\mu,$$

and the proof is complete. □

Let $(X,\mathscr{A},\mu)$ be a measure space, let $(Y,\mathscr{B})$ be a measurable space, and let $f\colon (X,\mathscr{A}) \to (Y,\mathscr{B})$ be measurable. Define a $[0,+\infty]$-valued function μf^{-1} on $\mathscr{B}$ by letting $\mu f^{-1}(B) = \mu(f^{-1}(B))$ for each B in $\mathscr{B}$. Clearly $\mu f^{-1}(\varnothing) = 0$. Note that if $\{B_n\}$ is a sequence of disjoint sets that belong to $\mathscr{B}$, then $\{f^{-1}(B_n)\}$ is a sequence of disjoint sets that belong to $\mathscr{A}$ and satisfy $f^{-1}(\cup_n B_n) = \cup_n f^{-1}(B_n)$;

$f^{-1}(\cup_n B_n) = \cup_n f^{-1}(B_n)$ imply that $\mathscr{F}$ is a σ-algebra on Y. Since $\mathscr{F}$ includes $\mathscr{B}_0$, it must include the σ-algebra generated by $\mathscr{B}_0$, namely $\mathscr{B}$. Thus f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$. □

Example 2.6.3. Suppose that $(X,\mathscr{A})$ is a measurable space and that f is a real-valued function on X. Proposition 2.1.9 implies that f is $\mathscr{A}$-measurable (in the sense of Sect. 2.1) if and only if it is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\mathbb{R})$. This conclusion can also be derived from Proposition 2.6.2 (let the collection $\mathscr{B}_0$ in Proposition 2.6.2 consist of all intervals of the form $(-\infty,t]$; see Proposition 1.1.4). □

Next we consider extended real-valued functions. Let $\mathscr{B}(\overline{\mathbb{R}})$ be the collection of all subsets of $\overline{\mathbb{R}}$ of the form $B \cup C$, where $B \in \mathscr{B}(\mathbb{R})$ and $C \subseteq \{-\infty,+\infty\}$. It is easy to check that $\mathscr{B}(\overline{\mathbb{R}})$ is a σ-algebra on $\overline{\mathbb{R}}$.

Proposition 2.6.4. *Let $(X,\mathscr{A})$ be a measurable space, and let f be an extended real-valued function on X. Then f is $\mathscr{A}$-measurable (in the sense of Sect. 2.1) if and only if it is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\overline{\mathbb{R}})$.*

Proof. If f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\overline{\mathbb{R}})$, then for each t in $\mathbb{R}$ the set $\{x \in X : f(x) \le t\}$, as the inverse image under f of the set $\{-\infty\} \cup (-\infty,t]$, belongs to $\mathscr{A}$; hence f must be $\mathscr{A}$-measurable.

Now assume that f is $\mathscr{A}$-measurable. Then $f^{-1}(\{+\infty\})$ and $f^{-1}(\{-\infty\})$ are equal to $\cap_{n=1}^{\infty}\{x \in X : f(x) > n\}$ and $\cap_{n=1}^{\infty}\{x \in X : f(x) < -n\}$, respectively, and so the inverse image under f of each subset of $\{-\infty,+\infty\}$ belongs to $\mathscr{A}$. In addition $\{x \in X : -\infty < f(x) < +\infty\}$ belongs to $\mathscr{A}$, and Proposition 2.1.9 (applied to the restriction of f to $\{x \in X : -\infty < f(x) < +\infty\}$) implies that $f^{-1}(B)$ belongs to $\mathscr{A}$ whenever B is a Borel subset of $\mathbb{R}$. Thus $f^{-1}(B \cup C) \in \mathscr{A}$ if $B \in \mathscr{B}(\mathbb{R})$ and $C \subseteq \{-\infty,+\infty\}$, and so f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\overline{\mathbb{R}})$. □

See Exercise 4 for another proof of Proposition 2.6.4.

Example 2.6.5. Let $(X,\mathscr{A})$ be a measurable space, and let f be an $\mathbb{R}^d$-valued function on X. Let $f_1, \ldots, f_d$ be the components of f, i.e., the real-valued functions on X that satisfy $f(x) = (f_1(x), f_2(x), \ldots, f_d(x))$ at each x in X. Then Proposition 2.6.2 and part (b) of Proposition 1.1.5 imply that f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\mathbb{R}^d)$ if and only if $f_1, \ldots, f_d$ are $\mathscr{A}$-measurable. It follows from this remark and Propositions 2.1.5 and 2.1.7 that the class of measurable functions from $(X,\mathscr{A})$ to $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is closed under the formation of sums, scalar multiples, and limits. □

Example 2.6.6. Now consider the space $\mathbb{R}^2$, and identify it with the set $\mathbb{C}$ of complex numbers. The remarks just above imply that a complex-valued function on $(X,\mathscr{A})$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}(\mathbb{C})$, that is, with respect to $\mathscr{A}$ and $\mathscr{B}(\mathbb{R}^2)$, if and only if its real and imaginary parts are $\mathscr{A}$-measurable, and that the collection of measurable functions from $(X,\mathscr{A})$ to $(\mathbb{C},\mathscr{B}(\mathbb{C}))$ is closed under the formation of sums and limits and under multiplication by real constants. Similar

Proof. For each positive integer n and for $k = 1, 2, \ldots, n2^n$ let $A_{n,k} = \{x \in A : (k-1)/2^n \le f(x) < k/2^n\}$. The measurability of f implies that each $A_{n,k}$ belongs to $\mathscr{A}$. Define a sequence $\{f_n\}$ of functions from A to $\mathbb{R}$ by requiring f_n to have value $(k-1)/2^n$ at each point in $A_{n,k}$ (for $k = 1, 2, \ldots, n2^n$) and to have value n at each point in $A - \cup_k A_{n,k}$. The functions so defined are simple and measurable, and it is easy to check that they satisfy (1) and (2) at each x in A. □

Suppose that $(X, \mathscr{A})$ is a measurable space and that f is a $[-\infty, +\infty]$-valued $\mathscr{A}$-measurable function defined on an $\mathscr{A}$-measurable subset A of X. Then by applying Proposition 2.1.8 to the positive and negative parts of f, we can construct a sequence $\{f_n\}$ of simple $\mathscr{A}$-measurable functions from A to $\mathbb{R}$ such that $f(x) = \lim_n f_n(x)$ holds at each x in A.

The following proposition gives some additional ways of viewing measurable functions; part (d) suggests a way to deal with more general situations (see Sect. 2.6).

Proposition 2.1.9. *Let $(X, \mathscr{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathscr{A}$. For a function $f : A \to \mathbb{R}$, the conditions*

(a) *f is measurable with respect to $\mathscr{A}$,*
(b) *for each open subset U of $\mathbb{R}$ the set $f^{-1}(U)$ belongs to $\mathscr{A}$,*
(c) *for each closed subset C of $\mathbb{R}$ the set $f^{-1}(C)$ belongs to $\mathscr{A}$, and*
(d) *for each Borel subset B of $\mathbb{R}$ the set $f^{-1}(B)$ belongs to $\mathscr{A}$*

are equivalent.

Proof. Let $\mathscr{F} = \{B \subseteq \mathbb{R} : f^{-1}(B) \in \mathscr{A}\}$. Then the fact that $f^{-1}(\mathbb{R}) = A$ and the identities

$$f^{-1}(B^c) = A - f^{-1}(B)$$

and

$$f^{-1}\left(\bigcup_n B_n\right) = \bigcup_n f^{-1}(B_n)$$

imply that $\mathscr{F}$ is a σ-algebra on $\mathbb{R}$. To require that f be measurable is to require that $\mathscr{F}$ contain all the intervals of the form $(-\infty, b]$ or equivalently (since $\mathscr{F}$ is a σ-algebra) to require that $\mathscr{F}$ include the σ-algebra on $\mathbb{R}$ generated by these intervals. Since the σ-algebra generated by these intervals is the σ-algebra of Borel subsets of $\mathbb{R}$ (Proposition 1.1.4), conditions (a) and (d) are equivalent. However the σ-algebra of Borel subsets of $\mathbb{R}$ is also generated by the collection of all open subsets of $\mathbb{R}$ and by the collection of all closed subsets of $\mathbb{R}$, and so conditions (b) and (c) are equivalent to the others. □

We close this section by returning to one of the promises made in Sect. 1.3 and proving that there are Lebesgue measurable subsets of $\mathbb{R}$ that are not Borel sets. For this we will use the following example.

Example 2.1.10. Recall the construction of the Cantor set given in Sect. 1.4. There we let K_0 be the interval $[0,1]$, and for each positive integer n we constructed a compact set K_n by removing from K_{n-1} the open middle third of each of the intervals making up K_{n-1}. The Cantor set K is given by $K = \cap_n K_n$.

The *Cantor function* (also known as the *Cantor singular function*) is the function $f\colon [0,1] \to [0,1]$ defined as follows (the concept of singularity will be defined and studied in Chap. 4). For each x in the interval $(1/3, 2/3)$ let $f(x) = 1/2$. Thus f is now defined at each point removed from $[0,1]$ in the construction of K_1. Next define f at each point removed from K_1 in the construction of K_2 by letting $f(x) = 1/4$ if $x \in (1/9, 2/9)$ and letting $f(x) = 3/4$ if $x \in (7/9, 8/9)$. Continue in this way, letting $f(x)$ be $1/2^n$, $3/2^n$, $5/2^n$, $\ldots$ on the various intervals removed from K_{n-1} in the construction of K_n. After all these steps, f is defined on the open set $[0,1] - K$, is nondecreasing, and has values in $[0,1]$. Extend it to all of $[0,1]$ by letting $f(0) = 0$ and letting

$$f(x) = \sup\{f(t) : t \in [0,1] - K \text{ and } t < x\}$$

if $x \in K$ and $x \neq 0$. This completes the definition of the Cantor function.

It is easy to check that f is nondecreasing and continuous, and it is clear that $f(0) = 0$ and $f(1) = 1$. The intermediate value theorem (Theorem C.13) thus implies that for each y in $[0,1]$ there is at least one x in $[0,1]$ such that $f(x) = y$, and so we can define a function $g\colon [0,1] \to [0,1]$ by

$$g(y) = \inf\{x \in [0,1] : f(x) = y\}. \tag{3}$$

The continuity of f implies that $f(g(y)) = y$ holds for each y in $[0,1]$; hence g is injective. It is easy to check that all the values of g lie in the Cantor set. The fact that f is nondecreasing implies that g is nondecreasing and hence that g is Borel measurable (see Example 2.1.2(b)). □

Proposition 2.1.11. *There is a Lebesgue measurable subset of* $\mathbb{R}$ *that is not a Borel set.*

Proof. Let g be the function constructed above, let A be a subset of $[0,1]$ that is not Lebesgue measurable (see Theorem 1.4.9), and let $B = g(A)$. Then B is a subset of the Cantor set and so is Lebesgue measurable (recall that $\lambda(K) = 0$ and that Lebesgue measure on the σ-algebra of Lebesgue measurable sets is complete). If B were a Borel set, then $g^{-1}(B)$ would also be a Borel set (recall that g is Borel measurable, and see Proposition 2.1.9). However the injectivity of g implies that $g^{-1}(B)$ is the set A, which is not Lebesgue measurable and hence is not a Borel set. Consequently the Lebesgue measurable set B is not a Borel set. □

Example 2.1.12. The proof of Proposition 2.1.11 gives a Borel set of Lebesgue measure 0 (the Cantor set) that has a subset that is not a Borel set. It follows that Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is not complete. □

(b) Prove the fundamental theorem of calculus, in the form given in the Introduction to this book. (Hint: Use part (a) to estimate $\frac{F(x)-F(x_0)}{x-x_0}$.)

7. (a) Suppose that f and g are bounded functions on the interval $[a,b]$ and that ε is positive. Show that if $|f(x)-g(x)| \le \varepsilon$ holds for all x in $[a,b]$, then $|\underline{\int}_a^b f - \underline{\int}_a^b g| \le \varepsilon(b-a)$ and $|\overline{\int}_a^b f - \overline{\int}_a^b g| \le \varepsilon(b-a)$.

(b) Suppose that $\{f_n\}$ is a sequence of Riemann integrable functions on the interval $[a,b]$ and that $\{f_n\}$ converges uniformly to a function f. Show that f is Riemann integrable and that $\int_a^b f(x)\,dx = \lim_n \int_a^b f_n(x)\,dx$. (Hint: Use part (a).)

8. Show that as n approaches infinity, the mean of the n values $n/(n+1)$, $n/(n+2)$, ..., $n/(n+n)$ approaches $\ln(2)$. (Hint: Write the mean of those values as a Riemann sum for the integral $\int_0^1 \frac{1}{1+x}\,dx$.)

2.6 Measurable Functions Again, Complex-Valued Functions, and Image Measures

In this section we give a general definition of measurable functions, and then we discuss some related concepts and some examples.

Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be measurable spaces. A function $f\colon X \to Y$ is *measurable with respect to* $\mathscr{A}$ and $\mathscr{B}$ if for each B in $\mathscr{B}$ the set $f^{-1}(B)$ belongs to $\mathscr{A}$. Instead of saying that f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$, we will sometimes say that f is a *measurable function* from $(X,\mathscr{A})$ to $(Y,\mathscr{B})$ or simply that $f\colon (X,\mathscr{A}) \to (Y,\mathscr{B})$ is *measurable*. Likewise, if A belongs to $\mathscr{A}$, a function $f\colon A \to Y$ is *measurable* if $f^{-1}(B) \in \mathscr{A}$ holds whenever B belongs to $\mathscr{B}$.

Proposition 2.6.1. *Let $(X,\mathscr{A})$, $(Y,\mathscr{B})$, and $(Z,\mathscr{C})$ be measurable spaces, and let $f\colon (Y,\mathscr{B}) \to (Z,\mathscr{C})$ and $g\colon (X,\mathscr{A}) \to (Y,\mathscr{B})$ be measurable. Then $f\circ g\colon (X,\mathscr{A}) \to (Z,\mathscr{C})$ is measurable.*

Proof. Suppose that $C \in \mathscr{C}$. Then $f^{-1}(C) \in \mathscr{B}$, and so $g^{-1}(f^{-1}(C)) \in \mathscr{A}$. Since $(f\circ g)^{-1}(C) = g^{-1}(f^{-1}(C))$, the measurability of $f\circ g$ follows. □

See Exercises 1 and 2 for some applications of the preceding proposition.

The following result is often useful for verifying the measurability of a function.

Proposition 2.6.2. *Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be measurable spaces, and let $\mathscr{B}_0$ be a collection of subsets of Y such that $\sigma(\mathscr{B}_0) = \mathscr{B}$. Then a function $f\colon X \to Y$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$ if and only if $f^{-1}(B) \in \mathscr{A}$ holds for each B in $\mathscr{B}_0$.*

Proof. Of course, every function f that is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$ satisfies $f^{-1}(B) \in \mathscr{A}$ for each B in $\mathscr{B}_0$. We turn to the converse, and assume that $f^{-1}(B) \in \mathscr{A}$ holds for each B in $\mathscr{B}_0$. Let $\mathscr{F}$ be the collection of all subsets B of Y such that $f^{-1}(B) \in \mathscr{A}$. The identities $f^{-1}(Y) = X$, $f^{-1}(B^c) = (f^{-1}(B))^c$, and

$$\int_{\mathbb{R}} f\,d\lambda = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_{[a,b]} f\,d\lambda$$

holds, but as a consequence of the dominated convergence theorem (see Exercise 5), and not as a definition.

Exercises

1. Suppose that $a < b < c$ and that f is a real-valued function on $[a,c]$. Show directly (i.e., without using Theorem 2.5.4) that f is Riemann integrable on $[a,c]$ if and only if it is Riemann integrable on $[a,b]$ and $[b,c]$. Also show that

$$\int_a^c f = \int_a^b f + \int_b^c f$$

if f is Riemann integrable on these intervals.

2. Let $\mathscr{R}_{[a,b]}$ be the set of all Riemann integrable functions on the interval $[a,b]$. Show directly (i.e., without using Theorem 2.5.4) that
   (a) $\mathscr{R}_{[a,b]}$ is a vector space over $\mathbb{R}$, and
   (b) $f \mapsto \int_a^b f$ is a linear functional on $\mathscr{R}_{[a,b]}$.
3. Show that a Riemann integrable function is not necessarily Borel measurable. (Hint: Consider χ_B, where B is the set constructed in the proof of Proposition 2.1.11.)
4. Show that there is an increasing sequence $\{f_n\}$ of continuous functions on $[0,1]$ such that

   (i) $0 \le f_n(x) \le 1$ holds for each n and x, and
   (ii) $\lim_n f_n$ is not Riemann integrable.

   (Hint: Let C be one of the closed sets constructed in Exercise 1.4.4, let $U = [0,1] - C$, and choose $\{f_n\}$ so that $\lim_n f_n = \chi_U$.)
5. Show that if $f \in \mathscr{L}^1(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda, \mathbb{R})$, then

$$\int_{\mathbb{R}} f\,d\lambda = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_{[a,b]} f\,d\lambda.$$

   (Hint: Use the dominated convergence theorem and a modification of the hint given for Exercise 2.4.9.)
6. (a) Show that if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable and if $m \le f(t) \le M$ holds for all t in the subinterval $[c,d]$ of $[a,b]$, then

$$m(d-c) \le \int_c^d f(t)\,dt \le M(d-c).$$

Exercises

1. Let X be a set, let $\{A_k\}$ be a sequence of subsets of X, let $B = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$, and let $C = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$. Show that
   (a) $\liminf_k \chi_{A_k} = \chi_B$, and
   (b) $\limsup_k \chi_{A_k} = \chi_C$.
2. Show that the supremum of an uncountable family of $[-\infty, +\infty]$-valued Borel measurable functions on $\mathbb{R}$ can fail to be Borel measurable.
3. Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable everywhere on $\mathbb{R}$, then its derivative f' is Borel measurable.
4. Let $(X, \mathscr{A})$ be a measurable space, and let $\{f_n\}$ be a sequence of $[-\infty, +\infty]$-valued measurable functions on X. Show that
$$\{x \in X : \lim_n f_n(x) \text{ exists and is finite}\}$$
   belongs to $\mathscr{A}$.
5. Let $(X, \mathscr{A})$ be a measurable space.
   (a) Show directly (i.e., without using Proposition 2.1.6 or Proposition 2.1.7) that if $f, g\colon X \to \mathbb{R}$ are $\mathscr{A}$-measurable *simple* functions, then $f + g$ and fg are $\mathscr{A}$-measurable.
   (b) Now let $f, g\colon X \to \mathbb{R}$ be arbitrary $\mathscr{A}$-measurable functions. Use Propositions 2.1.4, 2.1.5, and 2.1.8, together with part (a) of this exercise, to show that $f + g$ and fg are $\mathscr{A}$-measurable.
6. Let $(X, \mathscr{A})$ be a measurable space, and let $f, g\colon X \to \mathbb{R}$ be measurable. Give still another proof of the measurability of $f + g$, this time by checking that for each real t the function $x \mapsto t - f(x)$ is measurable and then using Proposition 2.1.3. (Hint: Consider $\{x : g(x) < t - f(x)\}$.)
7. Let f be the Cantor function, and let μ be the Borel measure on $\mathbb{R}$ associated to f by Proposition 1.3.10 (actually, one should apply Proposition 1.3.10 to the function from $\mathbb{R}$ to $\mathbb{R}$ that agrees with f on $[0, 1]$, vanishes on $(-\infty, 0)$, and is identically 1 on $(1, +\infty)$). Show that
   (a) each of the 2^n intervals remaining after the nth step in the construction of the Cantor set has measure $1/2^n$ under μ,
   (b) the Cantor set has measure 1 under μ, and
   (c) each x in $\mathbb{R}$ satisfies $\mu(\{x\}) = 0$.
   Thus all the mass of μ is concentrated on a set of Lebesgue measure zero (the Cantor set), but μ is not a sum of multiples of point masses.
8. Let g be the inverse of the Cantor function (that is, let g be defined by formula (3)). Show that the points x that have the form $x = g(y)$ for some y in $[0, 1]$ are exactly those that belong to the Cantor set and are not right-hand endpoints of intervals removed from $[0, 1]$ during the construction of the Cantor set.
9. Let $(X, \mathscr{A})$ be a measurable space and let C be a subset of X that does not belong to $\mathscr{A}$. Show that a function $f\colon X \to \mathbb{R}$ is $\sigma(\mathscr{A} \cup \{C\})$-measurable if and only if there exist $\mathscr{A}$-measurable functions $f_1, f_2\colon X \to \mathbb{R}$ such that $f = f_1\chi_C + f_2\chi_{C^c}$. (See part (a) of Exercise 1.5.12.)

10. Let $\mathscr{V}_0$ be the collection of all Borel measurable functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $\mathscr{V}_0$ is the smallest of those collections $\mathscr{V}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ for which

    (i) $\mathscr{V}$ is a vector space over $\mathbb{R}$,
    (ii) $\mathscr{V}$ contains each continuous function, and
    (iii) if $\{f_n\}$ is an increasing sequence of nonnegative functions in $\mathscr{V}$ and if $\lim_n f_n(x)$ is finite for each x in $\mathbb{R}$, then $\lim_n f_n$ belongs to $\mathscr{V}$.

    (Hint: Suppose that $\mathscr{V}$ satisfies conditions (a), (b), and (c), and define $S(\mathscr{V})$ by $S(\mathscr{V}) = \{A \subseteq \mathbb{R} : \chi_A \in \mathscr{V}\}$. Show that $S(\mathscr{V})$ contains each interval of the form $(-\infty, a)$, and then use Theorem 1.6.2 to show that $S(\mathscr{V})$ contains each Borel set.)

2.2 Properties That Hold Almost Everywhere

Let $(X, \mathscr{A}, \mu)$ be a measure space. A property of points of X is said to hold *μ-almost everywhere* if the set of points in X at which it fails to hold is μ-negligible. In other words, a property holds μ-almost everywhere if there is a set N that belongs to $\mathscr{A}$, satisfies $\mu(N) = 0$, and contains every point at which the property fails to hold. More generally, if E is a subset of X, then a property is said to hold *μ-almost everywhere on E* if the set of points in E at which it fails to hold is μ-negligible. The expression μ-almost everywhere is often abbreviated to μ-a.e. or to a.e.$[\mu]$. In cases where the measure μ is clear from context, the expressions almost everywhere and a.e. are also used.

Consider a property that holds almost everywhere, and let F be the set of points in X at which it fails. Then it is not necessary that F belong to $\mathscr{A}$; it is only necessary that there be a set N that belongs to $\mathscr{A}$, includes F, and satisfies $\mu(N) = 0$. Of course, if μ is complete, then F will belong to $\mathscr{A}$.

Examples 2.2.1. Suppose that f and g are functions on X. Then $f = g$ almost everywhere if the set of points x at which $f(x) \neq g(x)$ is μ-negligible, and $f \geq g$ almost everywhere if the set of points x at which $f(x) < g(x)$ is μ-negligible. Note that the sets $\{x \in X : f(x) \neq g(x)\}$ and $\{x \in X : f(x) < g(x)\}$ belong to $\mathscr{A}$ if f and g are $\mathscr{A}$-measurable; otherwise these sets may fail to belong to $\mathscr{A}$. If $\{f_n\}$ is a sequence of functions on X and f is a function on X, then $\{f_n\}$ converges to f almost everywhere if the set of points x at which the relation $f(x) = \lim_n f_n(x)$ fails to hold is μ-negligible. In this case one also says that $f = \lim_n f_n$ almost everywhere. □

Proposition 2.2.2. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and g be extended real-valued functions on X that are equal almost everywhere. If μ is complete and if f is $\mathscr{A}$-measurable, then g is $\mathscr{A}$-measurable.*

Proof. Let t be a real number and let N be a set that belongs to $\mathscr{A}$, satisfies $\mu(N) = 0$, and is such that f and g agree everywhere outside N. Then

$$\{x \in X : g(x) \leq t\} = (\{x \in X : f(x) \leq t\} \cap N^c) \cup (\{x \in X : g(x) \leq t\} \cap N). \tag{1}$$

Now suppose that f is Riemann integrable. Let ε be a positive number, and choose a partition $\mathscr{P}_0$ such that $u(f,\mathscr{P}_0) - l(f,\mathscr{P}_0) < \varepsilon$ (see Lemma 2.5.1). Let N be the number of subintervals in $\mathscr{P}_0$. We will produce a positive number δ such that each tagged partition $\mathscr{P}$ with mesh less than δ satisfies $|\mathscr{R}(f,\mathscr{P}) - \int_a^b f| < 2\varepsilon$. We begin by assuming that δ is smaller than the mesh of $\mathscr{P}_0$; we will presently see how much smaller we should make it. So let $\mathscr{P}$ be a tagged partition with mesh less than δ. If it happens that $\mathscr{P}$ is a refinement of $\mathscr{P}_0$ (i.e., every subinterval of $\mathscr{P}$ is a subset of some subinterval of $\mathscr{P}_0$), then $\mathscr{R}(f,\mathscr{P})$ satisfies

$$l(f,\mathscr{P}_0) \le \mathscr{R}(f,\mathscr{P}) \le u(f,\mathscr{P}_0)$$

and so belongs to the interval $[l(f,\mathscr{P}_0), u(f,\mathscr{P}_0)]$. Since $\int_a^b f$ also belongs to this interval, it follows that

$$\left|\mathscr{R}(f,\mathscr{P}) - \int_a^b f\right| \le u(f,\mathscr{P}_0) - l(f,\mathscr{P}_0) \le \varepsilon.$$

We turn to the general case, where $\mathscr{P}$ might not be a refinement of $\mathscr{P}_0$. Some of the intervals $[a_{i-1}, a_i]$ in $\mathscr{P}$ might contain a division point of $\mathscr{P}_0$ as an interior point. Since there are only N subintervals in $\mathscr{P}_0$, at most $N-1$ subintervals of $\mathscr{P}$ can have a division point of $\mathscr{P}_0$ as an interior point. Build a new tagged partition $\mathscr{P}'$ of $[a,b]$ by taking the subintervals and tags from $\mathscr{P}$ but splitting each subinterval whose interior contains a division point into two subintervals (dividing it at the corresponding division point) and choosing arbitrary tags in the new intervals. The differences between $\mathscr{R}(f,\mathscr{P}')$ and $\mathscr{R}(f,\mathscr{P})$ arise only from the split intervals, and it is easy to check that $|\mathscr{R}(f,\mathscr{P}) - \mathscr{R}(f,\mathscr{P}')| \le 2M(N-1)\delta$, where M is an upper bound for the values of $|f|$. If we require that δ be so small that $2M(N-1)\delta < \varepsilon$ and note that $|\mathscr{R}(f,\mathscr{P}') - \int_a^b f| \le \varepsilon$ (since $\mathscr{P}'$ is a refinement of $\mathscr{P}_0$), then we have

$$\begin{aligned}\left|\mathscr{R}(f,\mathscr{P}) - \int_a^b f\right| &\le \left|\mathscr{R}(f,\mathscr{P}) - \mathscr{R}(f,\mathscr{P}')\right| + \left|\mathscr{R}(f,\mathscr{P}') - \int_a^b f\right| \\ &\le 2M(N-1)\delta + \varepsilon < 2\varepsilon,\end{aligned}$$

and the proof is complete. □

Note that although in the Riemann theory integrals over all of $\mathbb{R}$ are defined as improper integrals, in the Lebesgue theory they can be[3] defined directly. If f is a Lebesgue integrable function on $\mathbb{R}$, then the relation

[3]There are also cases of functions defined on $\mathbb{R}$ that are not Lebesgue integrable over $\mathbb{R}$ but for which the corresponding improper integral exists. For instance, define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$ and $f(x) = (-1)^n/n$ if $n \le x < n+1$, where $n = 1, 2, \ldots$.

assertion is still not as powerful as the dominated convergence theorem for the Lebesgue integral, since it can only be applied when we can prove the Riemann integrability of the limit function. □

It is sometimes useful to view Riemann integrals as the limits of what are called Riemann sums. For this we need a couple of definitions. A *tagged partition* of an interval $[a,b]$ is a partition $\{a_i\}_{i=0}^k$ of $[a,b]$, together with a sequence $\{x_i\}_{i=1}^k$ of numbers (called *tags*) such that $a_{i-1} \le x_i \le a_i$ holds for $i = 1, \dots, k$. (In other words, each tag x_i must belong to the corresponding interval $[a_{i-1}, a_i]$.) As with partitions, we will often denote a tagged partition by a symbol such as $\mathscr{P}$.

The *mesh* or *norm* $\|\mathscr{P}\|$ of a partition (or a tagged partition) $\mathscr{P}$ is defined by $\|\mathscr{P}\| = \max_i(a_i - a_{i-1})$, where $\{a_i\}$ is the sequence of division points for $\mathscr{P}$. In other words, the mesh of a partition is the length of the longest of its subintervals.

The Riemann sum $\mathscr{R}(f, \mathscr{P})$ corresponding to the function f and the tagged partition $\mathscr{P}$ is defined by

$$\mathscr{R}(f, \mathscr{P}) = \sum_{i=1}^{k} f(x_i)(a_i - a_{i-1}).$$

Since for each i the value $f(x_i)$ lies between the infimum m_i and the supremum M_i of the values of f on the interval $[a_{i-1}, a_i]$, we have

$$l(f, \mathscr{P}) \le \mathscr{R}(f, \mathscr{P}) \le u(f, \mathscr{P})$$

for each tagged partition $\mathscr{P}$.

Proposition 2.5.7. *A function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if there is a real number L such that*

$$\lim_{\mathscr{P}} \mathscr{R}(f, \mathscr{P}) = L, \tag{2}$$

where the limit is taken as the mesh of the tagged partition $\mathscr{P}$ approaches 0*. If this limit exists, then it is equal to the Riemann integral $\int_a^b f$.*

We can make this more precise if we note that saying $\lim_{\mathscr{P}} \mathscr{R}(f, \mathscr{P}) = L$ is the same as saying that for every positive ε there is a positive δ such that $|\mathscr{R}(f, \mathscr{P}) - L| < \varepsilon$ holds whenever $\mathscr{P}$ is a tagged partition whose mesh is less than δ.

Proof. Suppose there exists a number L such that $\lim_{\mathscr{P}} \mathscr{R}(f, \mathscr{P}) = L$. Let ε be a positive number, choose a corresponding δ, and then choose a partition $\mathscr{P}_0$ whose mesh is less than δ. Consider the collection of all tagged partitions $\mathscr{P}$ that have the same division points as $\mathscr{P}_0$. Each of these tagged partitions has mesh less than δ and so satisfies $|\mathscr{R}(f, \mathscr{P}) - L| < \varepsilon$. By choosing the tags appropriately, we can find tagged partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ in this collection that make $\mathscr{R}(f, \mathscr{P}_1)$ and $\mathscr{R}(f, \mathscr{P}_2)$ arbitrarily close to $l(f, \mathscr{P}_0)$ and $u(f, \mathscr{P}_0)$, which gives us $|l(f, \mathscr{P}_0) - L| \le \varepsilon$ and $|u(f, \mathscr{P}_0) - L| \le \varepsilon$. It then follows from Lemma 2.5.1 that f is Riemann integrable. It is easy to check that $L = \int_a^b f$ (note that $\int_a^b f$ lies between $l(f, \mathscr{P}_0)$ and $u(f, \mathscr{P}_0)$).

The measurability of f and N implies that $\{x \in X : f(x) \leq t\} \cap N^c$ belongs to $\mathscr{A}$, while the completeness of μ implies that $\{x \in X : g(x) \leq t\} \cap N$ belongs to $\mathscr{A}$. The measurability of g follows. □

Corollary 2.2.3. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of extended real-valued functions on X, and let f be an extended real-valued function on X such that $\{f_n\}$ converges to f almost everywhere. If μ is complete and if each f_n is $\mathscr{A}$-measurable, then f is $\mathscr{A}$-measurable.*

Proof. According to Proposition 2.1.5 the function $\liminf_n f_n$ is $\mathscr{A}$-measurable. Since f and $\liminf_n f_n$ agree almost everywhere, Proposition 2.2.2 implies that f is $\mathscr{A}$-measurable. □

Example 2.2.4. Suppose that $(X, \mathscr{A}, \mu)$ is a measure space that is not complete, and let N be a μ-negligible subset of X that does not belong to $\mathscr{A}$. Then the characteristic function χ_N and the constant function 0 agree almost everywhere, but 0 is $\mathscr{A}$-measurable while χ_N is not. Thus Proposition 2.2.2 would fail if the hypothesis of completeness were removed. Furthermore, the sequence each term of which is 0 converges almost everywhere to χ_N; consequently Corollary 2.2.3 would also fail if the hypothesis of completeness were removed. □

Proposition 2.2.5. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $\mathscr{A}_\mu$ be the completion of $\mathscr{A}$ under μ. Then a function $f : X \to [-\infty, +\infty]$ is $\mathscr{A}_\mu$-measurable if and only if there are $\mathscr{A}$-measurable functions $f_0, f_1 : X \to [-\infty, +\infty]$ such that*

$$f_0 \leq f \leq f_1 \text{ holds everywhere on } X \tag{2}$$

and

$$f_0 = f_1 \text{ holds } \mu\text{-almost everywhere on } X. \tag{3}$$

In the context of Proposition 2.2.5, it is natural to ask whether it is always possible, given an $\mathscr{A}_\mu$-measurable function f with values in $\mathbb{R}$, rather than in $[-\infty, +\infty]$, to find *real*-valued functions f_0 and f_1 that satisfy (2) and (3). It turns out that the answer is no; see Exercise 8.3.3.

Proof. First suppose that there exist $\mathscr{A}$-measurable functions f_0 and f_1 that satisfy (2) and (3). Then f_0 is $\mathscr{A}_\mu$-measurable and $f = f_0$ holds $\overline{\mu}$-almost everywhere, and so Proposition 2.2.2, applied to the space $(X, \mathscr{A}_\mu, \overline{\mu})$, implies that f is $\mathscr{A}_\mu$-measurable.

Now suppose that $f : X \to [-\infty, +\infty]$ is $\mathscr{A}_\mu$-measurable. If f is simple and $[0, +\infty)$-valued, say attaining values $a_1, \ldots, a_k$ on the sets $A_1, \ldots, A_k$, then there are sets $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ that belong to $\mathscr{A}$ and satisfy $C_i \subseteq A_i \subseteq B_i$ and $\mu(B_i - C_i) = 0$ for each i. The functions f_0 and f_1 defined by $f_0 = \sum_i a_i \chi_{C_i}$ and $f_1 = \sum_i a_i \chi_{B_i}$ then satisfy (2) and (3).

We can deal with the case where f is simple and real-valued by applying the preceding argument to the positive and negative parts of f.

Finally, let $f\colon X \to [-\infty,+\infty]$ be an arbitrary $\mathscr{A}_\mu$-measurable function, and choose a sequence $\{g_n\}$ of simple $\mathscr{A}_\mu$-measurable functions from X to $\mathbb{R}$ such that $f(x) = \lim_n g_n(x)$ holds at each x in X (see the remark following the proof of Proposition 2.1.8). If for each n we choose $\mathscr{A}$-measurable functions $g_{0,n}$ and $g_{1,n}$ such that

$$g_{0,n} \le g_n \le g_{1,n} \text{ holds everywhere on } X$$

and

$$g_{0,n} = g_{1,n} \text{ holds } \mu\text{-almost everywhere on } X,$$

then the required functions f_0 and f_1 can be constructed by letting f_0 be $\overline{\lim}_n g_{0,n}$ and f_1 be $\underline{\lim}_n g_{1,n}$. □

Exercises

1. Give Borel functions $f, g\colon \mathbb{R} \to \mathbb{R}$ that agree on some dense subset of $\mathbb{R}$ but are such that $f(x) \ne g(x)$ holds at λ-almost every x in $\mathbb{R}$.
2. Let $\{x_n\}$ be a sequence of real numbers, and define μ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ by $\mu = \sum_n \delta_{x_n}$ (see Exercise 1.2.6). Show that functions $f, g\colon \mathbb{R} \to \mathbb{R}$ agree μ-almost everywhere if and only if $f(x_n) = g(x_n)$ holds for each n.
3. Let f and g be continuous real-valued functions on $\mathbb{R}$. Show that if $f = g$ λ-almost everywhere, then $f = g$ (i.e., $f(x) = g(x)$ for every x in $\mathbb{R}$).
4. Let μ be the finite Borel measure on $\mathbb{R}$ that is associated to the Cantor function by Proposition 1.3.10 (see Exercise 2.1.7). Show that continuous real-valued functions on $\mathbb{R}$ agree μ-almost everywhere if and only if they agree at every point in the Cantor set.
5. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and f_1, f_2, ... be $[-\infty,+\infty]$-valued $\mathscr{A}$-measurable functions on X. Show that if $\{f_n\}$ converges to f almost everywhere, then there are $\mathscr{A}$-measurable functions g_1, g_2, ... that are equal to f_1, f_2, ... almost everywhere and satisfy $f = \lim_n g_n$ everywhere.
6. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is nowhere continuous and that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x \text{ is irrational,} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are relatively prime and } q > 0 \end{cases}$$

is continuous λ-almost everywhere.

there having the values $\inf\{f(x) : a_{i-1} \le x \le a_i\}$ and $\sup\{f(x) : a_{i-1} \le x \le a_i\}$, respectively. Then $\{g_n\}$ is an increasing sequence of simple Borel functions that satisfy $g_n \le f$ and $\int_{[a,b]} g_n \, d\lambda = l(f, \mathscr{P}_n)$ for each n, and $\{h_n\}$ is a decreasing sequence of simple Borel functions that satisfy $h_n \ge f$ and $\int_{[a,b]} h_n \, d\lambda = u(f, \mathscr{P}_n)$ for each n. Since f is bounded, the sequences $\{g_n\}$ and $\{h_n\}$ are bounded. Define functions g and h by $g = \lim_n g_n$ and $h = \lim_n h_n$. Then g and h are Borel measurable, and the dominated convergence theorem (Theorem 2.4.5) implies that g and h are Lebesgue integrable, with $\int_{[a,b]} g \, d\lambda$ and $\int_{[a,b]} h \, d\lambda$ equal to $\lim_n l(f, \mathscr{P}_n)$ and $\lim_n u(f, \mathscr{P}_n)$, respectively, and so to the Riemann integral of f. Thus $\int_{[a,b]} (h - g) \, d\lambda = 0$. Since in addition $h - g \ge 0$, Corollary 2.3.12 implies that

$$g(x) = h(x) \text{ for almost every } x \text{ in } [a,b]. \tag{1}$$

This relation has two consequences. For the first, note that if $g(x) = h(x)$ and if x is a point in $[a,b]$ that is not a division point in any of the partitions $\mathscr{P}_n$, then f is continuous at x. Thus (1) implies that f is continuous almost everywhere, and so half of part (a) is proved. Note also that $g \le f \le h$, and so (1) implies that f is equal to g almost everywhere. It follows that f is Lebesgue measurable and Lebesgue integrable (Propositions 2.2.2 and 2.3.9) and that the Riemann and Lebesgue integrals of f are the same; thus part (b) is proved.

We turn to the remaining half of part (a). For this suppose that f is continuous almost everywhere. For each n let $\mathscr{P}_n$ be the partition of $[a,b]$ that divides $[a,b]$ into 2^n subintervals of equal length. Use these partitions $\mathscr{P}_n$ to construct functions g_n and h_n as in the first part of this proof. The relations $f(x) = \lim_n g_n(x)$ and $f(x) = \lim_n h_n(x)$ clearly hold at each x at which f is continuous and so at almost every x in $[a,b]$. Thus $\lim_n (h_n - g_n) = 0$ holds almost everywhere, and so, since $\int_{[a,b]} g_n \, d\lambda = l(f, \mathscr{P}_n)$ and $\int_{[a,b]} h_n \, d\lambda = u(f, \mathscr{P}_n)$, the dominated convergence theorem implies that

$$\lim_n (u(f, \mathscr{P}_n) - l(f, \mathscr{P}_n)) = 0.$$

Thus for each ε there is a partition $\mathscr{P}$ of $[a,b]$ such that $u(f, \mathscr{P}) - l(f, \mathscr{P}) < \varepsilon$, and the Riemann integrability of f follows. □

Example 2.5.5. Since the characteristic function of the set of rational numbers in $[0,1]$ is not continuous anywhere in $[0,1]$, part (a) of Theorem 2.5.4 gives another proof that this characteristic function is not Riemann integrable. □

Example 2.5.6. We saw in the Introduction that the pointwise limit of a bounded sequence of Riemann integrable functions may fail to be Riemann integrable. Thus a simple rewriting of the dominated convergence theorem so as to apply to the Riemann integral will fail. However, in view of Theorem 2.5.4 and the dominated convergence theorem for the Lebesgue integral, we can repair this difficulty by adding the hypothesis that the limit function be Riemann integrable. The repaired on $[a,b]$, and the common value of $\underline{\int}_a^b f$ and $\overline{\int}_a^b f$ is called the *Riemann integral* of f over $[a,b]$ and is denoted by $\int_a^b f$ or $\int_a^b f(x)\,dx$ (we'll occasionally write $(R)\int_a^b f$ or $(R)\int_a^b f(x)\,dx$ when we need to make clear which integral we mean).

The following reformulation of the definition of Riemann integrability is often useful.

Lemma 2.5.1. *A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every positive ε there is a partition $\mathscr{P}$ of $[a,b]$ such that $u(f,\mathscr{P}) - l(f,\mathscr{P}) < \varepsilon$.*

Proof. This is an immediate consequence of the fact that f is Riemann integrable if and only if

$$\sup_{\mathscr{P}} l(f,\mathscr{P}) = \inf_{\mathscr{P}} u(f,\mathscr{P}),$$

together with the fact that if $\mathscr{P}_1$ and $\mathscr{P}_2$ are partitions such that

$$u(f,\mathscr{P}_1) - l(f,\mathscr{P}_2) < \varepsilon,$$

then taking a common refinement $\mathscr{P}$ of $\mathscr{P}_1$ and $\mathscr{P}_2$ gives a partition $\mathscr{P}$ such that $u(f,\mathscr{P}) - l(f,\mathscr{P}) < \varepsilon$. □

Example 2.5.2. Suppose that f is a continuous, and hence bounded, function on $[a,b]$. Then f is uniformly continuous (Theorem C.12), and so for each positive number ε there is a positive number δ such that if x and y are elements of $[a,b]$ that satisfy $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. If ε and δ are related in this way and if $\mathscr{P}$ is a partition of $[a,b]$ into intervals each of which has length less than δ, then $u(f,\mathscr{P}) - l(f,\mathscr{P}) \le \varepsilon(b-a)$. It follows that every continuous function on $[a,b]$ is Riemann integrable. □

Example 2.5.3. Let $f\colon [0,1] \to \mathbb{R}$ be the characteristic function of the set of rational numbers in $[0,1]$. Then f is Lebesgue integrable, and $\int_{[0,1]} f\,d\lambda = 0$. However, as we noted in the Introduction, every lower sum of f is equal to 0 and every upper sum of f is equal to 1; thus f is not Riemann integrable. □

Theorem 2.5.4. *Let $[a,b]$ be a closed bounded interval, and let f be a bounded real-valued function on $[a,b]$. Then*

(a) *f is Riemann integrable if and only if it is continuous at almost every point of $[a,b]$, and*
(b) *if f is Riemann integrable, then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.*

Proof. Suppose that f is Riemann integrable. Then for each positive integer n we can choose a partition $\mathscr{P}_n$ of $[a,b]$ such that $u(f,\mathscr{P}_n) - l(f,\mathscr{P}_n) < 1/n$. By replacing the $\mathscr{P}_n$'s with finer partitions if necessary, we can assume that for each n the partition $\mathscr{P}_{n+1}$ is a refinement of the partition $\mathscr{P}_n$. Define sequences $\{g_n\}$ and $\{h_n\}$ of functions on $[a,b]$ by letting g_n and h_n agree with f at the point a and letting them be constant on each interval of the form $(a_{i-1},a_i]$ determined by $\mathscr{P}_n$,

2.3 The Integral

In this section we construct the integral and study some of its basic properties. The construction will take place in three stages.

We begin with the simple functions. Let $(X, \mathscr{A})$ be a measurable space. We will denote by $\mathscr{S}$ the collection of all simple real-valued $\mathscr{A}$-measurable functions on X and by $\mathscr{S}_+$ the collection of nonnegative functions in $\mathscr{S}$.

Let μ be a measure on $(X, \mathscr{A})$. If f belongs to $\mathscr{S}_+$ and is given by $f = \sum_{i=1}^m a_i \chi_{A_i}$, where $a_1, \ldots, a_m$ are nonnegative real numbers and $A_1, \ldots, A_m$ are disjoint subsets of X that belong to $\mathscr{A}$, then $\int f\,d\mu$, the *integral* of f with respect to μ, is defined to be $\sum_{i=1}^m a_i \mu(A_i)$ (note that this sum is either a nonnegative real number or $+\infty$). We need to check that $\int f\,d\mu$ depends only on f and not on $a_1, \ldots, a_m$ and $A_1, \ldots, A_m$. So suppose that f is also given by $\sum_{j=1}^n b_j \chi_{B_j}$, where $b_1, \ldots, b_n$ are nonnegative real numbers and $B_1, \ldots, B_n$ are disjoint subsets of X that belong to $\mathscr{A}$. We can assume that $\cup_{i=1}^m A_i = \cup_{j=1}^n B_j$ (if necessary eliminate those sets A_i for which $a_i = 0$ and those sets B_j for which $b_j = 0$). Then the additivity of μ and the fact that $a_i = b_j$ if $A_i \cap B_j \neq 0$ imply that

$$\begin{aligned}
\sum_{i=1}^m a_i \mu(A_i) &= \sum_{i=1}^m \sum_{j=1}^n a_i \mu(A_i \cap B_j) \\
&= \sum_{i=1}^m \sum_{j=1}^n b_j \mu(A_i \cap B_j) = \sum_{j=1}^n b_j \mu(B_j);
\end{aligned}$$

hence $\int f\,d\mu$ does not depend on the representation of f used in its definition.

Before proceeding to the next stage of our construction, we verify a few properties of the integral of a nonnegative simple function.

Proposition 2.3.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let f and g belong to $\mathscr{S}_+$, and let α be a nonnegative real number. Then*

(a) $\int \alpha f\,d\mu = \alpha \int f\,d\mu$,
(b) $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$, *and*
(c) *if $f(x) \leq g(x)$ holds at each x in X, then $\int f\,d\mu \leq \int g\,d\mu$.*

Proof. Suppose that $f = \sum_{i=1}^m a_i \chi_{A_i}$, where $a_1, \ldots, a_m$ are nonnegative real numbers and $A_1, \ldots, A_m$ are disjoint subsets of X that belong to $\mathscr{A}$, and that $g = \sum_{j=1}^n b_j \chi_{B_j}$, where $b_1, \ldots, b_n$ are nonnegative real numbers and $B_1, \ldots, B_n$ are disjoint subsets of X that belong to $\mathscr{A}$. We can again assume that $\cup_{i=1}^m A_i = \cup_{j=1}^n B_j$. Then parts (a) and (b) follow from the calculations

$$\int \alpha f\,d\mu = \sum_{i=1}^m \alpha a_i \mu(A_i) = \alpha \sum_{i=1}^m a_i \mu(A_i) = \alpha \int f\,d\mu$$

and

$$\begin{aligned}\int (f+g)\,d\mu &= \sum_{i=1}^{m}\sum_{j=1}^{n}(a_i+b_j)\mu(A_i\cap B_j)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}a_i\mu(A_i\cap B_j)+\sum_{i=1}^{m}\sum_{j=1}^{n}b_j\mu(A_i\cap B_j)\\ &= \sum_{i=1}^{m}a_i\mu(A_i)+\sum_{j=1}^{n}b_j\mu(B_j)=\int f\,d\mu+\int g\,d\mu.\end{aligned}$$

Next suppose that $f(x) \le g(x)$ holds at each x in X. Then $g - f$ belongs to $\mathscr{S}_+$, and so part (c) follows from the calculation

$$\int g\,d\mu = \int (f+(g-f))\,d\mu = \int f\,d\mu + \int (g-f)\,d\mu \ge \int f\,d\mu. \qquad \square$$

Proposition 2.3.2. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let f belong to $\mathscr{S}_+$, and let $\{f_n\}$ be a nondecreasing sequence of functions in $\mathscr{S}_+$ such that $f(x) = \lim_n f_n(x)$ holds at each x in X. Then $\int f\,d\mu = \lim_n \int f_n\,d\mu$.*

This proposition is a weak version of one of the fundamental properties of the Lebesgue integral, the monotone convergence theorem (Theorem 2.4.1). We need this weakened version now for use as a tool in completing the definition of the integral.

Proof. It follows from Proposition 2.3.1 that

$$\int f_1\,d\mu \le \int f_2\,d\mu \le \cdots \le \int f\,d\mu;$$

hence $\lim_n \int f_n\,d\mu$ exists and satisfies $\lim_n \int f_n\,d\mu \le \int f\,d\mu$. We turn to the reverse inequality. Let ε be a number such that $0 < \varepsilon < 1$. We will construct a nondecreasing sequence $\{g_n\}$ of functions in $\mathscr{S}_+$ such that $g_n \le f_n$ holds for each n and such that $\lim_n \int g_n\,d\mu = (1-\varepsilon)\int f\,d\mu$. Since $\int g_n\,d\mu \le \int f_n\,d\mu$, this will imply that $(1-\varepsilon)\int f\,d\mu \le \lim_n \int f_n\,d\mu$ and, since ε is arbitrary, that $\int f\,d\mu \le \lim_n \int f_n\,d\mu$. Consequently $\int f\,d\mu = \lim_n \int f_n\,d\mu$.

We turn to the construction of the sequence $\{g_n\}$. Suppose that $a_1, \ldots, a_k$ are the nonzero values of f and that $A_1, \ldots, A_k$ are the sets on which these values occur. Thus $f = \sum_{i=1}^k a_i \chi_{A_i}$. For each n and i let

$$A(n,i) = \{x \in A_i : f_n(x) \ge (1-\varepsilon)a_i\}.$$

Then each $A(n,i)$ belongs to $\mathscr{A}$, and for each i the sequence $\{A(n,i)\}_{n=1}^\infty$ is nondecreasing and satisfies $A_i = \cup_n A(n,i)$. If we let $g_n = \sum_{i=1}^k (1-\varepsilon)a_i\chi_{A(n,i)}$, then g_n belongs to $\mathscr{S}_+$ and satisfies $g_n \le f_n$, and we can use Proposition 1.2.5 to conclude that we give a number of details that we omitted earlier. We also give the standard characterization of the Riemann integrable functions on a closed bounded interval as the bounded functions on that interval that are almost everywhere continuous.

Let $[a,b]$ be a closed bounded interval. A *partition* of $[a,b]$ is a finite sequence $\{a_i\}_{i=0}^k$ of real numbers such that

$$a = a_0 < a_1 < \cdots < a_k = b.$$

We will generally denote a partition by a symbol such as $\mathscr{P}$ or $\mathscr{P}_n$.

If $\{a_i\}_{i=0}^k$ and $\{b_i\}_{i=0}^j$ are partitions of $[a,b]$ and if each term of $\{a_i\}_{i=0}^k$ appears among the terms of $\{b_i\}_{i=0}^j$, then $\{b_i\}_{i=0}^j$ is a *refinement of* or is *finer than* $\{a_i\}_{i=0}^k$.

Let f be a bounded real-valued function on $[a,b]$. If $\mathscr{P}$ is the partition $\{a_i\}_{i=0}^k$ of $[a,b]$ and if $m_i = \inf\{f(x) : x \in [a_{i-1}, a_i]\}$ and $M_i = \sup\{f(x) : x \in [a_{i-1}, a_i]\}$ for $i = 1, \ldots, k$, then the *lower sum* $l(f,\mathscr{P})$ corresponding to f and $\mathscr{P}$ is defined to be $\sum_{i=1}^k m_i(a_i - a_{i-1})$, and the *upper sum* $u(f,\mathscr{P})$ corresponding to f and $\mathscr{P}$ is defined to be $\sum_{i=1}^k M_i(a_i - a_{i-1})$.

It is easy to check that if $\mathscr{P}$ is an arbitrary partition of $[a,b]$, then

$$l(f,\mathscr{P}) \le u(f,\mathscr{P})$$

and that if $\mathscr{P}_1$ and $\mathscr{P}_2$ are partitions of $[a,b]$ such that $\mathscr{P}_2$ is a refinement of $\mathscr{P}_1$, then

$$l(f,\mathscr{P}_1) \le l(f,\mathscr{P}_2)$$

and

$$u(f,\mathscr{P}_2) \le u(f,\mathscr{P}_1)$$

(first consider the case where $\mathscr{P}_2$ contains exactly one more point than $\mathscr{P}_1$, and then use induction on the difference between the number of points in $\mathscr{P}_2$ and the number of points in $\mathscr{P}_1$). It follows that if $\mathscr{P}_1$ and $\mathscr{P}_2$ are arbitrary partitions of $[a,b]$, then

$$l(f,\mathscr{P}_1) \le u(f,\mathscr{P}_2)$$

(let $\mathscr{P}_3$ be a partition of $[a,b]$ that is a refinement of both $\mathscr{P}_1$ and $\mathscr{P}_2$ and note that

$$l(f,\mathscr{P}_1) \le l(f,\mathscr{P}_3) \le u(f,\mathscr{P}_3) \le u(f,\mathscr{P}_2)).$$

Hence the set of all lower sums for f is bounded above by each of the upper sums for f. The supremum of this set of lower sums is the *lower integral* of f over $[a,b]$ and is denoted by $\underline{\int}_a^b f$. The lower integral satisfies $\underline{\int}_a^b f \le u(f,\mathscr{P})$ for each upper sum $u(f,\mathscr{P})$ and so is a lower bound for the set of all upper sums for f. The infimum of this set of upper sums is the *upper integral* of f over $[a,b]$ and is denoted by $\overline{\int}_a^b f$. It follows immediately that $\underline{\int}_a^b f \le \overline{\int}_a^b f$. If $\underline{\int}_a^b f = \overline{\int}_a^b f$, then f is *Riemann integrable*

Suppose that K is a kernel from $(X,\mathscr{A})$ to $(Y,\mathscr{B})$, that μ is a measure on $(X,\mathscr{A})$, and that f is a $[0,+\infty]$-valued $\mathscr{B}$-measurable function on Y. Show that
(a) $B \mapsto \int K(x,B)\,\mu(dx)$ is a measure on $(Y,\mathscr{B})$,
(b) $x \mapsto \int f(y)\,K(x,dy)$ is an $\mathscr{A}$-measurable function on X, and
(c) if ν is the measure on $(Y,\mathscr{B})$ defined in part (a), then $\int f(y)\,\nu(dy) = \int(\int f(y)\,K(x,dy))\,\mu(dx)$. (Hint: Begin with the case where f is a characteristic function.)

8. (Continuation.) Now suppose that μ is finite, that $\sup\{K(x,Y) : x \in X\}$ is finite, and that the measurable function f is bounded but not necessarily nonnegative. Show that
(a) $x \mapsto \int f(y)\,K(x,dy)$ is a bounded $\mathscr{A}$-measurable function on X, and
(b) $\int f(y)\,\nu(dy) = \int(\int f(y)\,K(x,dy))\,\mu(dx)$. (Here again ν is the measure defined in part (a) of Exercise 7.)
9. Let $(X,\mathscr{A},\mu)$ be a measure space, let g be a $[0,+\infty]$-valued integrable function on X, and let f and f_t (for t in $[0,+\infty)$) be real-valued $\mathscr{A}$-measurable functions on X such that
$$f(x) = \lim_{t\to+\infty} f_t(x)$$
and
$$|f_t(x)| \le g(x) \text{ for } t \text{ in } [0,+\infty)$$
hold at almost every x in X. Show that $\int f\,d\mu = \lim_{t\to+\infty}\int f_t\,d\mu$. (Hint: Give a simplified version of the argument in Example 2.4.6.)
10. Let I be an open subinterval of $\mathbb{R}$, and let $f\colon \mathbb{R}\to\mathbb{R}$ be a Borel measurable function such that $x \mapsto e^{tx}f(x)$ is Lebesgue integrable for each t in I. Define $h\colon I\to\mathbb{R}$ by $h(t) = \int_{\mathbb{R}} e^{tx}f(x)\,\lambda(dx)$. Show that h is differentiable, with derivative given by $h'(t) = \int_{\mathbb{R}} xe^{tx}f(x)\,\lambda(dx)$, at each t in I. Of course, it is part of your task to show that $x \mapsto xe^{tx}f(x)$ is integrable for each t in I. (Hint: Use the Maclaurin expansion of e^u to show that $|e^u - 1| \le |u|e^{|u|}$ holds for each u in $\mathbb{R}$, and use the argument from Example 2.4.6.)
11. Let $(X,\mathscr{A},\mu)$ be a measure space, and let f and f_1, f_2, … be nonnegative functions that belong to $\mathscr{L}^1(X,\mathscr{A},\mu,\mathbb{R})$ and satisfy

(i) $\{f_n\}$ converges to f almost everywhere, and
(ii) $\int f\,d\mu = \lim_n \int f_n\,d\mu$.

Show that $\lim_n \int |f_n - f|\,d\mu = 0$.

2.5 The Riemann Integral

This section contains the standard facts that relate the Lebesgue integral to the Riemann integral. We begin by recalling Darboux's definition of the Riemann integral, as given in the Introduction (we use it as our basic definition), and then

$$\lim_n \int g_n\,d\mu = \lim_n \sum_{i=1}^{k}(1-\varepsilon)a_i\mu(A(n,i))$$
$$= \sum_{i=1}^{k}(1-\varepsilon)a_i\mu(A_i) = (1-\varepsilon)\int f\,d\mu. \qquad \square$$

As our next step, we define the integral of an arbitrary $[0,+\infty]$-valued $\mathscr{A}$-measurable function on X. For such a function f, let

$$\int f\,d\mu = \sup\left\{\int g\,d\mu : g \in \mathscr{S}_+ \text{ and } g \le f\right\}.$$

It is easy to see that for functions f in $\mathscr{S}_+$, this agrees with the previous definition.

Let us check a few properties of the integral on the class of $[0,+\infty]$-valued measurable functions. The first of these properties is an extension of Proposition 2.3.2 and will itself be generalized in Theorem 2.4.1 (the monotone convergence theorem). It is included here so that it can be used in the proof of Proposition 2.3.4.

Proposition 2.3.3. *Let $(X,\mathscr{A},\mu)$ be a measure space, let f be a $[0,+\infty]$-valued $\mathscr{A}$-measurable function on X, and let $\{f_n\}$ be a nondecreasing sequence of functions in $\mathscr{S}_+$ such that $f(x) = \lim_n f_n(x)$ holds at each x in X. Then $\int f\,d\mu = \lim_n \int f_n\,d\mu$.*

Proof. It is clear that

$$\int f_1\,d\mu \le \int f_2\,d\mu \le \cdots \le \int f\,d\mu;$$

hence $\lim_n \int f_n\,d\mu$ exists and satisfies $\lim_n \int f_n\,d\mu \le \int f\,d\mu$. We turn to the reverse inequality. Recall that $\int f\,d\mu$ is the supremum of those elements of $[0,+\infty]$ of the form $\int g\,d\mu$, where g ranges over the set of functions that belong to $\mathscr{S}_+$ and satisfy $g \le f$. Thus to prove that $\int f\,d\mu \le \lim_n \int f_n\,d\mu$, it is enough to check that if g is a function in $\mathscr{S}_+$ that satisfies $g \le f$, then $\int g\,d\mu \le \lim_n \int f_n\,d\mu$. Let g be such a function. Then $\{g \wedge f_n\}$ is a nondecreasing sequence of functions in $\mathscr{S}_+$ for which $g = \lim_n (g \wedge f_n)$, and so Proposition 2.3.2 implies that $\int g\,d\mu = \lim_n \int (g \wedge f_n)\,d\mu$. Since $\int (g \wedge f_n)\,d\mu \le \int f_n\,d\mu$, it follows that $\int g\,d\mu \le \lim_n \int f_n\,d\mu$, and the proof is complete. $\square$

Proposition 2.3.4. *Let $(X,\mathscr{A},\mu)$ be a measure space, let f and g be $[0,+\infty]$-valued $\mathscr{A}$-measurable functions on X, and let α be a nonnegative real number. Then*

(a) $\int \alpha f\,d\mu = \alpha \int f\,d\mu$,
(b) $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$, *and*
(c) *if $f(x) \le g(x)$ holds at each x in X, then $\int f\,d\mu \le \int g\,d\mu$.*

Proof. Choose nondecreasing sequences $\{f_n\}$ and $\{g_n\}$ of functions in $\mathscr{S}_+$ such that $f = \lim_n f_n$ and $g = \lim_n g_n$ (see Proposition 2.1.8). Then $\{\alpha f_n\}$ and $\{f_n + g_n\}$ are nondecreasing sequences of functions in $\mathscr{S}_+$ that satisfy $\alpha f = \lim_n \alpha f_n$ and

$f+g=\lim_n(f_n+g_n)$, and so we can use Proposition 2.3.3, together with the homogeneity and additivity of the integral on $\mathscr{S}_+$, to conclude that

$$\int \alpha f\,d\mu = \lim_n \int \alpha f_n\,d\mu = \lim_n \alpha \int f_n\,d\mu = \alpha \int f\,d\mu$$

and

$$\begin{aligned}\int (f+g)\,d\mu &= \lim_n \int (f_n+g_n)\,d\mu \\ &= \lim_n \Big(\int f_n\,d\mu + \int g_n\,d\mu\Big) = \int f\,d\mu + \int g\,d\mu.\end{aligned}$$

Thus parts (a) and (b) are proved. For part (c), note that if $f \le g$, then the class of functions h in $\mathscr{S}_+$ that satisfy $h \le f$ is included in the class of functions h in $\mathscr{S}_+$ that satisfy $h \le g$; it follows that $\int f\,d\mu \le \int g\,d\mu$. □

Finally, let f be an arbitrary $[-\infty,+\infty]$-valued $\mathscr{A}$-measurable function on X. If $\int f^+\,d\mu$ and $\int f^-\,d\mu$ are both finite, then f is called *integrable* (or *μ-integrable* or *summable*), and its *integral* $\int f\,d\mu$ is defined by

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu.$$

The integral of f is said to *exist* if at least one of $\int f^+\,d\mu$ and $\int f^-\,d\mu$ is finite, and again in this case, $\int f\,d\mu$ is defined to be $\int f^+\,d\mu - \int f^-\,d\mu$. In either case one sometimes writes $\int f(x)\,\mu(dx)$ or $\int f(x)\,d\mu(x)$ in place of $\int f\,d\mu$.

Suppose that $f\colon X \to [-\infty,+\infty]$ is $\mathscr{A}$-measurable and that $A \in \mathscr{A}$. Then f is *integrable over A* if the function $f\chi_A$ is integrable, and in this case $\int_A f\,d\mu$, the *integral of f over A*, is defined to be $\int f\chi_A\,d\mu$. Likewise, if $A \in \mathscr{A}$ and if f is a measurable function whose domain is A (rather than the entire space X), then the integral of f over A is defined to be the integral (if it exists) of the function on X that agrees with f on A and vanishes on A^c. In case $\mu(A^c)=0$, one often writes $\int f\,d\mu$ in place of $\int_A f\,d\mu$ and calls f integrable, rather than integrable over A.

In case $X=\mathbb{R}^d$ and $\mu=\lambda$, one often refers to *Lebesgue integrability* and the *Lebesgue integral*. The Lebesgue integral of a function f on $\mathbb{R}$ is often written $\int f(x)\,dx$. In case we are integrating over the interval $[a,b]$, we may write $\int_a^b f$ or $\int_a^b f(x)\,dx$ or, if we need to emphasize that we mean the Lebesgue integral, $(L)\int_a^b f$ or $(L)\int_a^b f(x)\,dx$.

We define $\mathscr{L}^1(X,\mathscr{A},\mu,\mathbb{R})$ (or sometimes simply $\mathscr{L}^1$) to be the set of all real-valued (rather than $[-\infty,+\infty]$-valued) integrable functions on X. According to Proposition 2.3.6 below, $\mathscr{L}^1(X,\mathscr{A},\mu,\mathbb{R})$ is a vector space and the integral is a linear functional on $\mathscr{L}^1(X,\mathscr{A},\mu,\mathbb{R})$.

Lemma 2.3.5. *Let $(X,\mathscr{A},\mu)$ be a measure space, and let f_1, f_2, g_1, and g_2 be nonnegative real-valued integrable functions on X such that $f_1-f_2=g_1-g_2$. Then $\int f_1\,d\mu - \int f_2\,d\mu = \int g_1\,d\mu - \int g_2\,d\mu$.*

$\lim_n t_n = t_0$. Then, in view of inequality (7), the dominated convergence theorem implies that $x \mapsto f_t(x,t_0)$ is integrable and that

$$\lim_n \frac{g(t_n) - g(t_0)}{t_n - t_0} = \int_X f_t(x,t_0)\,\mu(dx).$$

Combining this with item C.7 in Appendix C finishes the argument. □

Exercises

1. Give sequences $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ of functions in $\mathscr{L}^1(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda,\mathbb{R})$ that converge to zero almost everywhere, but satisfy
   (a) $\lim_n \int f_n\,d\lambda = +\infty$,
   (b) $\lim_n \int g_n\,d\lambda = 1$, and
   (c) $\limsup_n \int h_n\,d\lambda = 1$ and $\liminf_n \int h_n\,d\lambda = -1$.
2. Prove that the monotone convergence theorem still holds if the assumption that the functions f_1, f_2, ... are nonnegative is dropped, and the assumption that f_1 is integrable is added (note that in this case the integrals of the functions f and f_2, f_3, ... exist, but may be $+\infty$).
3. Let $(X,\mathscr{A},\mu)$ be a measure space. Use Exercise 2 to show that if $\{f_n\}$ is a decreasing sequence of measurable functions and if f_1 is integrable, then $\int \lim_n f_n\,d\mu = \lim_n \int f_n\,d\mu$ (as in Exercise 2 the integrals involved exist, but may be infinite).
4. Let f, g, and f_1, f_2, ... be as in the dominated convergence theorem, and define sequences $\{p_n\}$ and $\{q_n\}$ by $p_n = \inf_{k\geq n} f_k$ and $q_n = \sup_{k\geq n} f_k$. Use Exercises 2 and 3, together with the inequality $p_n \leq f_n \leq q_n$, to give another proof of the dominated convergence theorem.
5. Use Exercise 3, applied to the sequence $\{h_n\}$ defined by $h_n = \sup_{k\geq n}|f_k - f|$, to give still another proof of the dominated convergence theorem. (Of course the functions f and f_1, f_2, ... can be modified so that they are real valued and hence so that $f_k - f$ makes sense.)
6. Let $(X,\mathscr{A},\mu)$ be a measure space, and let $f\colon X \to [0,+\infty]$ be $\mathscr{A}$-measurable.
   (a) Show that if each value of f is a nonnegative integer or $+\infty$, then $\int f\,d\mu = \sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\})$.
   (b) Now suppose that the values of f are arbitrary elements of $[0,+\infty]$ and that μ is finite. Show that the integrability of f is equivalent to the convergence of the series $\sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\})$.
7. Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be measurable spaces. A function $K\colon X \times \mathscr{B} \to [0,+\infty]$ is called a *kernel* from $(X,\mathscr{A})$ to $(Y,\mathscr{B})$ if

   (i) for each x in X the function $B \mapsto K(x,B)$ is a measure on $(Y,\mathscr{B})$, and
   (ii) for each B in $\mathscr{B}$ the function $x \mapsto K(x,B)$ is $\mathscr{A}$-measurable.

Let us begin our proof that $\int f\,d\mu = \lim_n \int f_n\,d\mu$ by supposing that relations (4), (5), and

$$g(x) < +\infty \tag{6}$$

hold at *every* x in X. Then $\{g+f_n\}$ is a sequence of nonnegative $\mathscr{A}$-measurable functions such that $(g+f)(x) = \lim_n(g+f_n)(x)$ holds at each x in X, and so Fatou's lemma (Theorem 2.4.4) implies that

$$\int (g+f)\,d\mu \le \varliminf_n \int (g+f_n)\,d\mu$$

and hence that

$$\int f\,d\mu \le \varliminf_n \int f_n\,d\mu.$$

A similar argument, applied to the sequence $\{g-f_n\}$, shows that

$$\int (g-f)\,d\mu \le \varliminf_n \int (g-f_n)\,d\mu$$

and hence that

$$\varlimsup_n \int f_n\,d\mu \le \int f\,d\mu.$$

Consequently $\int f\,d\mu = \lim_n \int f_n\,d\mu$.

Next suppose that we only require that relations (4), (5), and (6) hold at almost every x in X (note that, according to Corollary 2.3.14, the hypothesis $\int g\,d\mu < +\infty$ implies that relation (6) holds at almost every x in X). We can reduce the present case to the one we have just dealt with by using a modified version of the final part of the proof of Theorem 2.4.1; the details are left to the reader. □

Example 2.4.6. Let us note how Theorem 2.4.5 can be used to justify "differentiation under the integral sign." Let $(X,\mathscr{A},\mu)$ be a measure space, let $g\colon X \to [0,+\infty]$ be an integrable function, let I be an open subinterval of $\mathbb{R}$, and let $f\colon X\times I \to \mathbb{R}$ be such that

(a) for each t in I the function $x \mapsto f(x,t)$ is integrable,
(b) for each x in X the function $t \mapsto f(x,t)$ is differentiable on I, and
(c) the inequality

$$\left|\frac{f(x,t)-f(x,t_0)}{t-t_0}\right| \le g(x) \tag{7}$$

holds for all t, t_0 in I and all x in X.

Define $g\colon I \to \mathbb{R}$ by $g(t) = \int_X f(x,t)\,\mu(dx)$. Let us use the dominated convergence theorem to show that g is differentiable on I, with g' given by $g'(t) = \int_X f_t(x,t)\,\mu(dx)$ at each t in I (here $f_t(x,t)$ denotes the partial derivative with respect to t). Suppose that $\{t_n\}$ is a sequence of elements of I, all different from t_0, such that

Proof. Since the functions f_1, f_2, g_1, and g_2 satisfy $f_1 - f_2 = g_1 - g_2$, they also satisfy $f_1 + g_2 = g_1 + f_2$ and so satisfy

$$\int f_1\,d\mu + \int g_2\,d\mu = \int g_1\,d\mu + \int f_2\,d\mu$$

(Proposition 2.3.4); since all the integrals involved are finite, this implies that

$$\int f_1\,d\mu - \int f_2\,d\mu = \int g_1\,d\mu - \int g_2\,d\mu. \qquad \square$$

Proposition 2.3.6. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let f and g be real-valued integrable functions on X, and let α be a real number. Then*

(a) *αf and $f+g$ are integrable,*
(b) *$\int \alpha f\,d\mu = \alpha \int f\,d\mu$,*
(c) *$\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$, and*
(d) *if $f(x) \le g(x)$ holds at each x in X, then $\int f\,d\mu \le \int g\,d\mu$.*

Proof. The integrability of αf and the relation $\int \alpha f\,d\mu = \alpha \int f\,d\mu$ are clear if $\alpha = 0$. If α is positive, then $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$; thus $(\alpha f)^+$ and $(\alpha f)^-$, and hence αf, are integrable, and

$$\begin{aligned}\int \alpha f\,d\mu &= \int (\alpha f)^+\,d\mu - \int (\alpha f)^-\,d\mu\\ &= \alpha \int f^+\,d\mu - \alpha \int f^-\,d\mu = \alpha \int f\,d\mu.\end{aligned}$$

If α is negative, then $(\alpha f)^+ = -\alpha f^-$ and $(\alpha f)^- = -\alpha f^+$, and we can modify the preceding argument so as to show that αf is integrable and that $\int \alpha f\,d\mu = \alpha \int f\,d\mu$.

Now consider the sum of f and g. Note that $(f+g)^+ \le f^+ + g^+$ and $(f+g)^- \le f^- + g^-$; thus (Proposition 2.3.4)

$$\int (f+g)^+\,d\mu \le \int f^+\,d\mu + \int g^+\,d\mu < +\infty$$

and

$$\int (f+g)^-\,d\mu \le \int f^-\,d\mu + \int g^-\,d\mu < +\infty,$$

and so $f+g$ is integrable. Since $f+g$ is equal to $(f+g)^+ - (f+g)^-$ and to $f^+ + g^+ - (f^- + g^-)$, it follows from Lemma 2.3.5 that

$$\int (f+g)\,d\mu = \int (f^+ + g^+)\,d\mu - \int (f^- + g^-)\,d\mu,$$

and hence that $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$.

If $f(x) \le g(x)$ holds at each x in X, then $g - f$ is a nonnegative integrable function; hence $\int (g-f)\,d\mu \ge 0$, and so $\int g\,d\mu - \int f\,d\mu = \int (g-f)\,d\mu \ge 0$. $\square$

Examples 2.3.7.

(a) If μ is a finite measure, then every bounded measurable function on $(X, \mathscr{A}, \mu)$ is integrable.
(b) In particular, every bounded Borel function, and hence every continuous function, on $[a,b]$ is Lebesgue integrable. (We'll see in Sect. 2.5 that the Lebesgue integral of a continuous function on $[a,b]$ can be found by calculating its Riemann integral.)
(c) Suppose that $\mathscr{A}$ is the σ-algebra on $\mathbb{N}$ containing all subsets of $\mathbb{N}$ and that μ is counting measure on $\mathscr{A}$. It follows from Proposition 2.3.3 that a nonnegative function f on $\mathbb{N}$ is μ-integrable if and only if the infinite series $\sum_n f(n)$ is convergent, and that in that case the integral and the sum of the series agree. Since a not necessarily nonnegative function f is integrable if and only if f^+ and f^- are integrable, it follows that f is integrable if and only if the infinite series $\sum_n f(n)$ is absolutely convergent. Once again, the integral and the sum of the series have the same value.
(d) Note that a simple measurable function that vanishes almost everywhere is integrable, with integral 0. We can reach the same conclusion for arbitrary measurable functions that vanish almost everywhere by first using Proposition 2.3.3 to deal with nonnegative functions and then using the decomposition $f = f^+ - f^-$. For a converse, see Corollary 2.3.12. □

We now consider a few elementary properties of the integral; the basic limit theorems for the integral will be presented in the next section.

Proposition 2.3.8. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a $[-\infty, +\infty]$-valued $\mathscr{A}$-measurable function on X. Then f is integrable if and only if $|f|$ is integrable. If these functions are integrable, then $|\int f\,d\mu| \le \int |f|\,d\mu$.*

Proof. Recall that by definition f is integrable if and only if f^+ and f^- are integrable. On the other hand, since $|f| = f^+ + f^-$, part (b) of Proposition 2.3.4 implies that $|f|$ is integrable if and only if f^+ and f^- are integrable. Thus the integrability of f is equivalent to the integrability of $|f|$. In case f and $|f|$ are integrable, the inequality $|\int f\,d\mu| \le \int |f|\,d\mu$ follows from the calculation

$$\left|\int f\,d\mu\right| = \left|\int f^+\,d\mu - \int f^-\,d\mu\right| \le \int f^+\,d\mu + \int f^-\,d\mu = \int |f|\,d\mu. \qquad \square$$

The reader should note that there are functions that are not measurable, and hence not integrable, but that have an integrable absolute value (see Exercise 3). Hence we needed to include the measurability of f among the hypotheses of Proposition 2.3.8.

Proposition 2.3.9. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and g be $[-\infty, +\infty]$-valued $\mathscr{A}$-measurable functions on X that agree almost everywhere. If either $\int f\,d\mu$ or $\int g\,d\mu$ exists, then both exist, and $\int f\,d\mu = \int g\,d\mu$.*

Proof. First consider the case where f and g are nonnegative. Let $A = \{x \in X : f(x) \ne g(x)\}$, and let h be the function defined by

Example 2.4.3. Corollary 2.4.2 can be applied as follows to construct a large class of measures. Suppose that $(X,\mathscr{A},\mu)$ is a measure space and that $f\colon X \to [0,+\infty]$ is $\mathscr{A}$-measurable. Define a function $\nu\colon \mathscr{A} \to [0,+\infty]$ by $\nu(A) = \int_A f\,d\mu$. Then $\nu(\varnothing) = 0$, and Corollary 2.4.2, applied to the series $\sum_n f\chi_{A_n}$, implies that if $\{A_n\}$ is a sequence of disjoint sets in $\mathscr{A}$, then $\nu(\cup_n A_n) = \sum_n \nu(A_n)$. Thus ν is a measure on $(X,\mathscr{A})$. Moreover ν is a finite measure if and only if f is μ-integrable. □

The next result is often used to show that a function is integrable or to provide an upper bound for the value of an integral.

Theorem 2.4.4 (Fatou's Lemma). *Let $(X,\mathscr{A},\mu)$ be a measure space, and let $\{f_n\}$ be a sequence of $[0,+\infty]$-valued $\mathscr{A}$-measurable functions on X. Then*

$$\int \varliminf_n f_n\,d\mu \le \varliminf_n \int f_n\,d\mu.$$

Proof. For each positive integer n let $g_n = \inf_{k\ge n} f_k$. Each g_n is $\mathscr{A}$-measurable (Proposition 2.1.5), and the relations

$$g_1(x) \le g_2(x) \le \dots$$

and

$$\varliminf_n f_n(x) = \lim_n g_n(x)$$

hold at each x in X. It follows from the monotone convergence theorem (Theorem 2.4.1) and the inequality $g_n \le f_n$ that

$$\int \varliminf_n f_n\,d\mu = \int \lim_n g_n\,d\mu = \lim_n \int g_n\,d\mu \le \varliminf_n \int f_n\,d\mu.$$ □

Theorem 2.4.5 (Lebesgue's Dominated Convergence Theorem). *Let $(X,\mathscr{A},\mu)$ be a measure space, let g be a $[0,+\infty]$-valued integrable function on X, and let f and f_1, f_2, ... be $[-\infty,+\infty]$-valued $\mathscr{A}$-measurable functions on X such that*

$$f(x) = \lim_n f_n(x) \tag{4}$$

and

$$|f_n(x)| \le g(x),\ n = 1, 2, \dots \tag{5}$$

hold at μ-almost every x in X. Then f and f_1, f_2, ... are integrable, and $\int f\,d\mu = \lim_n \int f_n\,d\mu$.

Proof. The integrability of f and f_1, f_2, ... follows from that of g; see Proposition 2.3.8, Proposition 2.3.9, and part (c) of Proposition 2.3.4.

In this theorem the functions f and $f_1, f_2, \ldots$ are only assumed to be nonnegative and measurable; there are no assumptions about whether they are integrable.

Proof. First suppose that relations (1) and (2) hold at each x in X. The monotonicity of the integral (part (c) of Proposition 2.3.4) implies that

$$\int f_1 \, d\mu \leq \int f_2 \, d\mu \leq \cdots \leq \int f \, d\mu;$$

hence the sequence $\{\int f_n \, d\mu\}$ converges (perhaps to $+\infty$), and its limit satisfies $\lim_n \int f_n \, d\mu \leq \int f \, d\mu$. We turn to the reverse inequality. For each n choose a nondecreasing sequence $\{g_{n,k}\}_{k=1}^{\infty}$ of simple $[0,+\infty)$-valued measurable functions such that $f_n = \lim_k g_{n,k}$ (Proposition 2.1.8). For each n define a function h_n by

$$h_n = \max(g_{1,n}, g_{2,n}, \ldots, g_{n,n}).$$

Then $\{h_n\}$ is a nondecreasing sequence of simple $[0,+\infty)$-valued measurable functions that satisfy $h_n \leq f_n$ and $f = \lim_n h_n$. It follows from these remarks, Proposition 2.3.3, and the monotonicity of the integral that

$$\int f \, d\mu = \lim_n \int h_n \, d\mu \leq \lim_n \int f_n \, d\mu.$$

Hence $\int f \, d\mu = \lim_n \int f_n \, d\mu$.

Now suppose that we only require that relations (1) and (2) hold for almost every x in X. Let N be a set that belongs to $\mathscr{A}$, has measure zero under μ, and contains all points at which one or more of these relations fails. The function $f\chi_{N^c}$ and the sequence $\{f_n\chi_{N^c}\}$ satisfy the hypotheses made in the first part of the proof, and so

$$\int f\chi_{N^c} \, d\mu = \lim_n \int f_n\chi_{N^c} \, d\mu. \tag{3}$$

Since $f_n\chi_{N^c}$ agrees with f_n almost everywhere and $f\chi_{N^c}$ agrees with f almost everywhere, Eq. (3) and Proposition 2.3.9 imply that

$$\int f \, d\mu = \lim_n \int f_n \, d\mu. \qquad \square$$

Corollary 2.4.2 (Beppo Levi's Theorem). *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $\sum_{k=1}^{\infty} f_k$ be an infinite series whose terms are $[0,+\infty]$-valued $\mathscr{A}$-measurable functions on X. Then*

$$\int \sum_{k=1}^{\infty} f_k \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.$$

Proof. Use the linearity of the integral, and apply Theorem 2.4.1 to the sequence $\{\sum_{k=1}^{n} f_k\}_{n=1}^{\infty}$ of partial sums of the series $\sum_{k=1}^{\infty} f_k$. $\square$

$$h(x) = \begin{cases} +\infty & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then $\int h\,d\mu = 0$ (apply Proposition 2.3.3 to the sequence $\{h_n\}$ defined by $h_n = n\chi_A$). In view of Proposition 2.3.4 and the inequality $f \le g + h$, this implies that $\int f\,d\mu \le \int g\,d\mu + \int h\,d\mu = \int g\,d\mu$. A similar argument shows that $\int g\,d\mu \le \int f\,d\mu$. Thus $\int f\,d\mu = \int g\,d\mu$.

The case where f and g are not necessarily nonnegative can be reduced to the case just treated through the decompositions $f = f^+ - f^-$ and $g = g^+ - g^-$. □

Proposition 2.3.10. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a $[0, +\infty]$-valued $\mathscr{A}$-measurable function on X. If t is a positive real number and if A_t is defined by $A_t = \{x \in X : f(x) \ge t\}$, then*

$$\mu(A_t) \le \frac{1}{t}\int_{A_t} f\,d\mu \le \frac{1}{t}\int f\,d\mu.$$

Proof. The relation $0 \le t\chi_{A_t} \le f\chi_{A_t} \le f$ and part (c) of Proposition 2.3.4 imply that

$$\int t\chi_{A_t}\,d\mu \le \int_{A_t} f\,d\mu \le \int f\,d\mu.$$

Since $\int t\chi_{A_t}\,d\mu = t\mu(A_t)$, the proposition follows. □

Corollary 2.3.11. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a $[-\infty, +\infty]$-valued integrable function on X. Then $\{x \in X : f(x) \ne 0\}$ is σ-finite under μ.*

Proof. Proposition 2.3.10, applied to the function $|f|$, implies that the sets A_1, A_2, ... defined by

$$A_n = \left\{x \in X : |f(x)| \ge \frac{1}{n}\right\}$$

have finite measure under μ. Thus $\{x \in X : f(x) \ne 0\}$, since it is equal to $\cup_n A_n$, is σ-finite under μ. □

Corollary 2.3.12. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a $[-\infty, +\infty]$-valued $\mathscr{A}$-measurable function on X that satisfies $\int |f|\,d\mu = 0$. Then f vanishes μ-almost everywhere.*

Proof. Proposition 2.3.10, applied to the function $|f|$, implies that

$$\mu\left(\left\{x \in X : |f(x)| \ge \frac{1}{n}\right\}\right) \le n\int |f|\,d\mu = 0$$

holds for each positive integer n. Since

$$\{x \in X : f(x) \neq 0\} = \bigcup_n \left\{x \in X : |f(x)| \geq \frac{1}{n}\right\},$$

the countable subadditivity of μ implies that $\mu(\{x \in X : f(x) \neq 0\}) = 0$. Thus f vanishes almost everywhere. □

Corollary 2.3.13. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a $[-\infty, +\infty]$-valued integrable function on X such that $\int_A f\, d\mu \geq 0$ holds for all A in $\mathscr{A}$ (or even just for all A in the smallest σ-algebra on X that makes f measurable). Then $f \geq 0$ holds μ-almost everywhere.*

Proof. Let $A = \{x \in X : f(x) < 0\}$. Then $\int f\chi_A\, d\mu = \int_A f\, d\mu = 0$ (since $f < 0$ on A, yet we are assuming that $\int_A f\, d\mu \geq 0$). It follows from Corollary 2.3.12 that $f\chi_A$ vanishes almost everywhere and hence that $f \geq 0$ holds almost everywhere. □

Corollary 2.3.14. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a $[-\infty, +\infty]$-valued integrable function on X. Then $|f(x)| < +\infty$ holds at μ-almost every x in X.*

Proof. Proposition 2.3.10, applied to the function $|f|$, implies that

$$\mu(\{x \in X : |f(x)| \geq n\}) \leq \frac{1}{n} \int |f|\, d\mu$$

holds for each positive integer n. Thus

$$\mu(\{x \in X : |f(x)| = +\infty\}) \leq \mu(\{x \in X : |f(x)| \geq n\}) \leq \frac{1}{n} \int |f|\, d\mu$$

holds for each n, and so $\mu(\{x \in X : |f(x)| = +\infty\}) = 0$ □

Corollary 2.3.15. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a $[-\infty, +\infty]$-valued $\mathscr{A}$-measurable function on X. Then f is integrable if and only if there is a function in $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$ that is equal to f almost everywhere.*

In other words, a measurable $[-\infty, +\infty]$-valued function f is integrable if and only if there is an $\mathbb{R}$-valued function that is integrable and equal to f μ-almost everywhere.

Proof. If there is a function in $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$ that is equal to f almost everywhere, then the integrability of f follows from Proposition 2.3.9. Next suppose that f is integrable, and let $A = \{x \in X : |f(x)| = +\infty\}$. Then $A \in \mathscr{A}$, and Corollary 2.3.14 implies that $\mu(A) = 0$. It follows that the function f_0 defined by $f_0 = f\chi_{A^c}$ is $\mathscr{A}$-measurable and agrees with f almost everywhere. Proposition 2.3.9 now implies that f_0 is integrable and hence a member of $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$. □

Exercises

1. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and g belong to $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$. Show that $f \vee g$ and $f \wedge g$ belong to $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$.
2. Give Borel functions $f, g \colon \mathbb{R} \to \mathbb{R}$ that are Lebesgue integrable but are such that fg is not Lebesgue integrable.
3. Show that there is a function $f \colon \mathbb{R} \to \mathbb{R}$ that is not Lebesgue integrable, but is such that $|f|$ is Lebesgue integrable. (Hint: Let $f = \chi_A - \chi_B$, where A and B are suitable subsets of $\mathbb{R}$.)
4. Let $(X, \mathscr{A}, \mu)$ be a measure space, let $f, g \colon X \to [-\infty, +\infty]$ be integrable, and let $h \colon X \to [-\infty, +\infty]$ be an $\mathscr{A}$-measurable function that satisfies $h(x) = f(x) + g(x)$ at μ-almost every x in X. Show that h is integrable and that $\int h\, d\mu = \int f\, d\mu + \int g\, d\mu$.
5. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $f \colon X \to [-\infty, +\infty]$ be an $\mathscr{A}$-measurable function whose integral exists and is not equal to $-\infty$. Show that if $g \colon X \to [-\infty, +\infty]$ is an $\mathscr{A}$-measurable function that satisfies $f \le g$ μ-almost everywhere, then the integral of g exists and satisfies $\int f\, d\mu \le \int g\, d\mu$.
6. Let $(X, \mathscr{A}, \mu)$ be a measure space, let $\{f_n\}$ be a nondecreasing sequence of $[0, +\infty]$-valued $\mathscr{A}$-measurable functions on X, and let f be the function on X that satisfies $f(x) = \lim_n f_n(x)$ at each x in X.
   (a) Show that if g belongs to $\mathscr{S}_+$ and satisfies $g \le f$, then for each ε in the interval $(0,1)$, there is a sequence $\{g_n\}$ in $\mathscr{S}_+$ such that $g_n \le f_n$ holds for each n and such that $\lim_n \int g_n\, d\mu = (1-\varepsilon) \int g\, d\mu$. (Hint: See the proof of Proposition 2.3.2).
   (b) Use part (a) to prove that $\lim_n \int f_n\, d\mu = \int f\, d\mu$. Thus we have another proof of Proposition 2.3.3 and, at the same time, of Theorem 2.4.1 below (see, however, the last paragraph of the proof of Theorem 2.4.1).

2.4 Limit Theorems

In this section we prove the basic limit theorems of integration theory. These results are extremely important and account for much of the power of the Lebesgue integral. We will use them often in the rest of the book.

Theorem 2.4.1 (The Monotone Convergence Theorem). *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \ldots$ be $[0, +\infty]$-valued $\mathscr{A}$-measurable functions on X. Suppose that*

$$f_1(x) \le f_2(x) \le \ldots \tag{1}$$

and

$$f(x) = \lim_n f_n(x) \tag{2}$$

hold at μ-almost every x in X. Then $\int f\, d\mu = \lim_n \int f_n\, d\mu$.