# 1 Lecture 1 — Measures and $\sigma$-Fields

The opening question of the course is disarmingly simple: *what is a measure?* Intuitively, a measure assigns a size — length, area, volume, probability — to a set. To pin this down, we have to settle two things up front: which subsets of an ambient space $\Omega$ we are even allowed to talk about, and what rules the size-assignment must obey. The first question is answered by a $\sigma$-field; the second by the definition of a measure. The relationship of $\mathbb{R}^n$ to its norm sits in the background as the running prototype.

## 1.1 Notation and set-theoretic preliminaries

Throughout the course $\Omega$ denotes a fixed ambient set, the *sample space*. For $A \subseteq \Omega$ we write

$$A^c = \{ x \in \Omega : x \notin A \} = \Omega \setminus A$$

for the complement, and use $\Omega \setminus A$ and $A^c$ interchangeably. A countable collection $\{A_i\}_{i=1}^{\infty}$ of subsets of $\Omega$ is *pairwise disjoint* if

$$A_i \cap A_j = \emptyset \qquad \text{whenever } i \neq j.$$

The *power set* of $\Omega$, denoted $\mathcal{P}(\Omega)$, is the collection of all subsets of $\Omega$. When $\Omega$ has $n$ elements this collection has $2^n$ elements, which is the source of the alternative notation $2^{\Omega}$.

**Remark 1.1.** Symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the set-theoretic counterpart of the logical XOR. It will not feature heavily in this lecture, but is worth keeping in the toolkit.

## 1.2 $\sigma$-fields

The first object we need is a class of subsets that is closed under the operations we plan to perform on it: complement, countable union, and (as a consequence) countable intersection.

**Definition 1.1: $\sigma$-field**

For a set $\Omega$, a $\sigma$-*field* (or $\sigma$-*algebra*) is a collection $\mathcal{F}$ of subsets $A \subseteq \Omega$ such that

1. $\emptyset, \ \Omega \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$;
3. for any countable collection $\{A_i\}_{i=1}^{\infty}$ with $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$,

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

The pair $(\Omega, \mathcal{F})$ is then called a *measurable space*.

**Remark 1.2.** Combining (2) and (3) via De Morgan's laws delivers closure under countable intersection: for $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$,

$$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F},$$

and applying (2) once more yields $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

**Remark 1.3.** Any $\sigma$-field on $\Omega$ is a subcollection of the power set, $\mathcal{F} \subseteq \mathcal{P}(\Omega)$. The power set itself is a $\sigma$-field, but it is typically too large to be useful: on $\mathbb{R}$, for example, it is too generous a collection to admit a translation-invariant length-like measure (a fact made precise in Lecture 4).

## 1.3 Measures

With a class of admissible sets in hand, a measure is a rule for assigning a non-negative size to each.

**Definition 1.2: Measure**

Let $(\Omega, \mathcal{F})$ be a measurable space. A *measure* is a function $\mu : \mathcal{F} \to \mathbb{R}^+$ satisfying

1. $\mu(\emptyset) = 0$;
2. $\mu$ is *countably additive*: for any pairwise disjoint countable collection $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

The triple $(\Omega, \mathcal{F}, \mu)$ is then called a *measure space*.

**Remark 1.4.** Take care to distinguish a *measurable space* $(\Omega, \mathcal{F})$ from a *measure space* $(\Omega, \mathcal{F}, \mu)$: the former specifies only *which* sets can be sized, the latter also specifies *how*.

**Remark 1.5.** There also exist *signed measures*, which are allowed to take values in $\mathbb{R}$ rather than $\mathbb{R}^+$. These will not concern us in this lecture.

## 1.4 Special classes of measures

Three size-on-the-whole-space conditions get repeated names.

**Definition 1.3: Probability, finite, and $\sigma$-finite measures**

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

- $\mu$ is a *probability measure* if $\mu(\Omega) = 1$; we then call $(\Omega, \mathcal{F}, \mu)$ a *probability space* and usually write $\mu = \mathbb{P}$.
- $\mu$ is a *finite measure* if $\mu(\Omega) < \infty$.
- $\mu$ is a *$\sigma$-finite measure* if there exists a countable cover $\Omega = \bigcup_{i=1}^{\infty} A_i$ with $A_i \in \mathcal{F}$ and $\mu(A_i) < \infty$ for every $i$.

**Example 1.1 (Length on $\mathbb{R}$).** Take $\Omega = \mathbb{R}$ and (anticipating Lecture 2) define $\mu([a,b]) = b-a$. Then $\mu$ is not finite, but it is $\sigma$-finite: writing

$$\mathbb{R} = \bigcup_{i=1}^{\infty}\big([\,i-1,i\,] \cup [-i,-i+1]\big),$$

each piece has finite length. This is the prototype of *Lebesgue measure*.

## 1.5 Examples on a finite sample space

**Example 1.2 (Counting measure).** Let $\Omega = \{1,2,\ldots,n\}$ and take $\mathcal{F} = \mathcal{P}(\Omega)$, which has $2^n$ elements (hence the notation $2^\Omega$). The *counting measure* is

$$\mu(A) = \#A, \qquad A \in \mathcal{F}.$$

Thus $\mu(\{1,3,7\}) = 3$ and $\mu(\Omega) = n$. The normalised version

$$\nu(A) = \tfrac{1}{n}\,\mu(A)$$

is the uniform probability measure on $\{1,\ldots,n\}$.

**Example 1.3 (Binomial probability measure).** By contrast, on $\Omega = \{0,1,\ldots,n\}$ the binomial $(n,p)$ distribution assigns to each point $i$ the weight

$$\mu(\{i\}) = \binom{n}{i} p^i\,(1-p)^{n-i}, \qquad p \in (0,1),$$

extended additively to $\mathcal{P}(\Omega)$. This gives a probability measure on $\Omega$ which is *not* uniform.