1 Lecture 2 — Constructing σ-Fields and Measures (Existence; Carathéodory's Extension)

Motivating question. Suppose we declare the measure of a half-open interval to be its length, $\mu((a,b]) = b - a$ for $b > a$. What *else* can we then measure? The collection of all such intervals is not a σ-field — e.g. $(a,b] \cup (c,d]$ is not in general a half-open interval — so we need a procedure that grows the class of "measurable" sets and extends μ to it. The extension is delivered by *Carathéodory's extension theorem*, the existence half of the construction of a measure space. Uniqueness is taken up next lecture.

1.1 Semirings, rings, and fields

We climb a small ladder of set systems sitting below a σ-field. At each rung the candidate measure has more room to manoeuvre.

Definition 1.1: Semiring

A collection $\mathcal{A}$ of subsets of Ω is a *semiring* if

- $\emptyset \in \mathcal{A}$,
- $A \cap B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$, and
- for all $A, B \in \mathcal{A}$ the set difference splits as a finite disjoint union

$$B \setminus A = \bigsqcup_{i=1}^{n} C_i, \qquad C_i \in \mathcal{A}.$$

The set difference itself need not lie in $\mathcal{A}$, but it must be *expressible* as a finite union of members of $\mathcal{A}$.

■ Example 1.1 (Half-open intervals form a semiring). The collection of all half-open intervals $(a,b] \subseteq \mathbb{R}$ (together with $\emptyset$) is a semiring: intersections of half-open intervals are half-open intervals, and a difference $(a,b] \setminus (c,d]$ is the union of at most two half-open intervals.

Definition 1.2: Ring

A collection $\mathcal{A}$ of subsets of Ω is a *ring* if

- $\emptyset \in \mathcal{A}$, and
- for all $A, B \in \mathcal{A}$, both $B \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

A ring is closed under finite (set-theoretic) unions and differences.

■ Example 1.2 (Finite unions of half-open intervals). The collection of all *finite* unions of half-open intervals $(a,b] \subseteq \mathbb{R}$ is a ring. It is not yet a σ-field — it fails to absorb countable unions.

Definition 1.3: Field

A ring $\mathcal{A}$ is a *field* (or *algebra*) on Ω if additionally $\Omega \in \mathcal{A}$.

Remark 1.1. A field that is closed under *countable* unions is a σ-field. The progression

$$\text{semiring} \subset \text{ring} \subset \text{field} \subset \sigma\text{-field}$$

mirrors a progression in stability under set operations: pairwise intersection only, then finite unions/differences, then Ω, finally countable unions.

1.2 Set functions and pre-measures

Before defining a measure we collect the regularity properties a set function may enjoy.

Definition 1.4: Set function and its properties

Let $\mathcal{A}$ be a collection of subsets of Ω. A *set function* is any map $\mu \colon \mathcal{A} \to [0, \infty]$ (not necessarily a measure). For $A, B \in \mathcal{A}$ we say:

- μ is *increasing* (monotone) if $A \subseteq B \implies \mu(A) \le \mu(B)$;
- μ is *(finitely) additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ are disjoint and $A \cup B \in \mathcal{A}$;
- μ is *countably additive* if for every pairwise disjoint sequence $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ with $\bigcup_i A_i \in \mathcal{A}$,
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i);$$
- μ is *countably subadditive* if for every (not necessarily disjoint) sequence $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ with $\bigcup_i A_i \in \mathcal{A}$,
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 1.5: Pre-measure

A set function $\mu \colon \mathcal{A} \to [0, \infty]$ on a ring $\mathcal{A}$ is a *pre-measure* if $\mu(\emptyset) = 0$ and μ is countably additive on $\mathcal{A}$.

A pre-measure is exactly what a measure looks like *before* the underlying set system has been closed up to a σ-field. Carathéodory's theorem will perform that closing-up.

1.3 Outer measure and μ^*-measurability

A pre-measure on a ring can be extended in a canonical way to *every* subset of Ω by approximating from above.

Definition 1.6: Outer measure

Let μ be a pre-measure on a ring $\mathcal{A}$ on Ω. The *outer measure* induced by μ is

$$\mu^*(E) = \inf\left\{\sum_i \mu(A_i) \;:\; A_i \in \mathcal{A},\ E \subseteq \bigcup_i A_i\right\}, \qquad E \subseteq \Omega,$$

where the infimum runs over finite or countable covers of E by elements of $\mathcal{A}$.

The outer measure μ^* is defined on *all* of $\mathcal{P}(\Omega)$, but in general it is only countably *sub*additive there — not additive. To recover countable additivity we restrict attention to sets that "split" every test set cleanly.

Definition 1.7: μ^*-measurable set

A set $B \subseteq \Omega$ is μ^**-measurable* (in the sense of Carathéodory) if for every $E \subseteq \Omega$,

$$\mu^*(E \cap B) + \mu^*(E \cap B^c) = \mu^*(E).$$

Write $\mathcal{M}$ for the collection of all μ^*-measurable subsets of Ω.

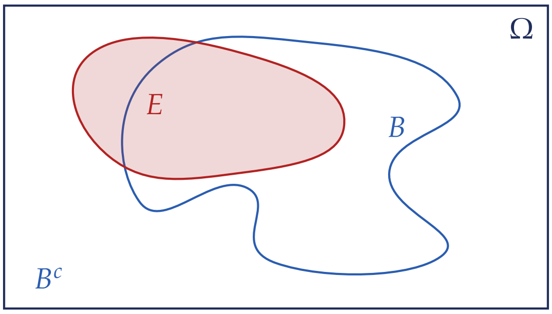

Figure 1. The Carathéodory criterion. A set $B \subseteq \Omega$ is μ^*-measurable when every test set E (red) is split additively by B and its complement: $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$.

Remark 1.2. Countable subadditivity of μ^* already gives "$\geq$" in the defining identity, so the substantive condition is the reverse inequality: B does not waste mass at its boundary. The class $\mathcal{M}$ is the largest natural domain on which μ^* is genuinely additive.

1.4 Carathéodory's extension theorem

We now assemble the pieces. Starting from a pre-measure on a ring, the outer measure machinery delivers a measure on a σ-field containing the original ring.

Theorem 1.8: Carathéodory Extension

Let $\mathcal{A}$ be a ring on Ω and let μ be a pre-measure on $\mathcal{A}$. Let μ^* be the outer measure induced by μ and $\mathcal{M}$ the collection of μ^*-measurable sets. Then:

1. $\mu^*(\emptyset) = 0$, and μ^* is monotone and countably subadditive on $\mathcal{P}(\Omega)$;
2. μ^* and μ agree on $\mathcal{A}$, i.e. $\mu^*(A) = \mu(A)$ for every $A \in \mathcal{A}$;

3. $\mathcal{A} \subseteq \mathcal{M}$;
4. $\mathcal{M}$ is a σ-field on Ω and μ^* restricted to $\mathcal{M}$ is a measure;
5. consequently

$$\mathcal{A} \subseteq \sigma(\mathcal{A}) \subseteq \mathcal{M} \subseteq \mathcal{P}(\Omega),$$

and $\mu^*\big|_{\sigma(\mathcal{A})}$ is a measure on $\sigma(\mathcal{A})$ extending μ.

Remark 1.3. The chain in (5) records the precise sense in which Carathéodory "extends" μ: the original ring $\mathcal{A}$ sits inside the generated σ-field $\sigma(\mathcal{A})$, which sits inside the larger σ-field $\mathcal{M}$ of μ^*-measurable sets, which sits inside the full power set. Both $\sigma(\mathcal{A})$ and $\mathcal{M}$ carry the measure μ^*; the "correct" extension is the outer measure restricted to $\sigma(\mathcal{A})$. It can happen that $\sigma(\mathcal{A})$ is a strict subset of $\mathcal{M}$ (this is the gap filled by completion).

Remark 1.4. We have shown *existence*: at least one measure on $\sigma(\mathcal{A})$ agreeing with μ on $\mathcal{A}$ exists. The companion question — *is the extension unique?* — is

$$\mu_1(A) = \mu_2(A)\ \forall A \in \mathcal{A} \overset{?}{\Longrightarrow} \mu_1(B) = \mu_2(B)\ \forall B \in \sigma(\mathcal{A}),$$

and is the subject of the next lecture, via Dynkin's π-λ theorem.