

## 1 Lecture 3 – Uniqueness; Dynkin's $\pi$ - $\lambda$ Theorem; Completeness

Lecture 2 produced an extension of a pre-measure to the generated  $\sigma$ -field via Carathéodory. The natural follow-up is the question that opens the handwritten page:

If  $\mu_1(A) = \mu_2(A)$  for every  $A \in \mathcal{A}$ , does it follow that  $\mu_1(B) = \mu_2(B)$  for every  $B \in \sigma(\mathcal{A})$ ?

The answer is *yes* provided  $\mathcal{A}$  is a  $\pi$ -system and the measures are  $\sigma$ -finite. The vehicle is Dynkin's  $\pi$ - $\lambda$  theorem, which we develop next, after which we take a brief detour through completeness.

### 1.1 $\pi$ - and $\lambda$ -systems

#### Definition 1.1: $\pi$ -system

A collection  $\mathcal{A}$  of subsets of  $\Omega$  is a  $\pi$ -system if it is closed under finite intersections: for every  $A, B \in \mathcal{A}$ ,  $A \cap B \in \mathcal{A}$ .

#### Definition 1.2: $\lambda$ -system

A collection  $\mathcal{L}$  of subsets of  $\Omega$  is a  $\lambda$ -system if

- $\Omega \in \mathcal{L}$ ;
- for any  $A, B \in \mathcal{L}$  with  $A \subset B$ ,  $B \setminus A \in \mathcal{L}$ ;
- for any pairwise-disjoint sequence  $\{A_i\}_{i=1}^{\infty} \subset \mathcal{L}$ ,  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$ .

**Remark 1.1.** A  $\lambda$ -system looks much like a  $\sigma$ -field, the difference being that countable unions are required only for *disjoint* sequences. Every field is automatically a  $\pi$ -system, and a collection that is *both* a  $\pi$ -system and a  $\lambda$ -system is a  $\sigma$ -field.

### 1.2 Dynkin's $\pi$ - $\lambda$ theorem

The lecture's central tool upgrades containment in a  $\lambda$ -system to containment of the full generated  $\sigma$ -field, provided one starts from a  $\pi$ -system.

#### Theorem 1.3: Dynkin $\pi$ - $\lambda$ theorem

Let  $\mathcal{A}$  be a  $\pi$ -system and  $\mathcal{L}$  a  $\lambda$ -system on  $\Omega$  with  $\mathcal{A} \subset \mathcal{L}$ . Then  $\sigma(\mathcal{A}) \subset \mathcal{L}$ .

**Remark 1.2.** The proof strategy from the handwriting: take  $\mathcal{L}_0$  to be the *smallest*  $\lambda$ -system containing  $\mathcal{A}$ . Show  $\mathcal{L}_0$  is also closed under intersections, by introducing

$$\mathcal{L}' = \{B \in \mathcal{L}_0 : B \cap A \in \mathcal{L}_0 \text{ for all } A \in \mathcal{A}\}$$

and verifying  $\mathcal{L}'$  is a  $\lambda$ -system containing  $\mathcal{A}$ , hence  $\mathcal{L}' = \mathcal{L}_0$ ; then repeat the argument with  $\mathcal{L}'' = \{B \in \mathcal{L}_0 : B \cap C \in \mathcal{L}_0 \text{ for all } C \in \mathcal{L}_0\}$ . A  $\lambda$ -system that is also a  $\pi$ -system is a  $\sigma$ -field, so  $\sigma(\mathcal{A}) \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$ .

### 1.3 Uniqueness of extension

The promised payoff:

#### Theorem 1.4: Uniqueness of extension

Let  $\mathcal{A}$  be a  $\pi$ -system on  $\Omega$  and let  $\mu_1, \mu_2$  be two  $\sigma$ -finite measures on  $\sigma(\mathcal{A})$ . If  $\mu_1(A) = \mu_2(A)$  for every  $A \in \mathcal{A}$ , then  $\mu_1(B) = \mu_2(B)$  for every  $B \in \sigma(\mathcal{A})$ .

**Remark 1.3.** The handwriting splits the proof in two:

- **Finite case.** Assume  $\mu_1(\Omega) = \mu_2(\Omega) < \infty$  and set  $\mathcal{L} = \{B \subset \Omega : \mu_1(B) = \mu_2(B)\}$ . One verifies  $\mathcal{L}$  is a  $\lambda$ -system:  $\Omega \in \mathcal{L}$  by hypothesis; if  $A \subset B$  lie in  $\mathcal{L}$  then  $\mu_i(B \setminus A) = \mu_i(B) - \mu_i(A)$  (here finiteness allows the subtraction), so  $B \setminus A \in \mathcal{L}$ ; countable additivity handles disjoint unions. Since  $\mathcal{A} \subset \mathcal{L}$ , Dynkin gives  $\sigma(\mathcal{A}) \subset \mathcal{L}$ .
- **$\sigma$ -finite case.** For each  $A \in \mathcal{A}$  with  $\mu_1(A) = \mu_2(A) < \infty$ , define  $\mathcal{L}_A = \{B \subseteq \Omega : \mu_1(A \cap B) = \mu_2(A \cap B)\}$ , a  $\lambda$ -system; by Dynkin  $\sigma(\mathcal{A}) \subset \mathcal{L}_A$ . Decompose  $\Omega = \bigcup_{i \geq 1} A_i$  with  $A_i \in \mathcal{A}$  and  $\mu_1(A_i) = \mu_2(A_i) < \infty$ . Inclusion-exclusion on the finitely many  $A_1, \dots, A_n$  (using that  $\mathcal{A}$  is a  $\pi$ -system, so  $A_i \cap A_j \in \mathcal{A}$  and so on) gives  $\mu_1(B \cap \bigcup_{i \leq n} A_i) = \mu_2(B \cap \bigcup_{i \leq n} A_i)$  for every  $B \in \sigma(\mathcal{A})$ ; let  $n \rightarrow \infty$ .

**Remark 1.4.**  $\pi$ -systems are very natural in probability theory: the joint event  $A \cap B$  lives alongside  $A$  and  $B$ , so any sensible event class is closed under finite intersections.

■ **Example 1.1 ( $\sigma$ -finiteness is needed).** Take  $\Omega = (0, 1]$  and let  $\mathcal{A}$  be the collection of finite unions of half-open intervals  $(a, b]$ . Two measures on  $\sigma(\mathcal{A})$  agreeing on  $\mathcal{A}$  but disagreeing on  $\sigma(\mathcal{A})$ :

- $\mu$  sends  $\emptyset \mapsto 0$  and every non-empty element of  $\mathcal{A}$  to  $\infty$ ; the induced outer measure  $\mu^*$  then assigns  $\infty$  to every non-empty subset of  $\Omega$ .
- Counting measure  $\nu$  also sends  $\emptyset \mapsto 0$  and each non-empty  $(a, b]$  to  $\infty$ , but  $\nu(\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}) = 3$  while  $\mu^*(\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}) = \infty$ .

The pre-measure on  $\mathcal{A}$  is not  $\sigma$ -finite, and uniqueness breaks at the level of  $\sigma(\mathcal{A})$ .

### 1.4 Completeness

We now leave uniqueness behind and address a different deficiency: a measure space may contain sets of measure zero whose subsets are not themselves measurable. We patch this by enlarging the  $\sigma$ -field to absorb every such “negligible” set. The geometry behind the patch is the symmetric difference.

#### Definition 1.5: Symmetric difference

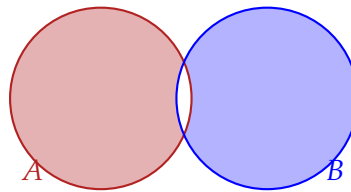
For sets  $A, B \subseteq \Omega$ ,

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

For a measure space  $(X, \mathcal{F}, \mu)$  the *outer measure* extends  $\mu$  to all of  $\mathcal{P}(X)$  by

$$\mu^*(B) = \inf\{\mu(A) : B \subset A, A \in \mathcal{F}\}.$$

A subset  $N \subseteq X$  is  $\mu$ -null if  $\mu^*(N) = 0$ ; write  $\mathcal{N}_\mu$  for the collection of all such sets.



**Figure 1.** The shaded region is  $A \Delta B$ ; the unshaded overlap is  $A \cap B$ .

#### Definition 1.6: Complete measure space

The measure space  $(X, \mathcal{F}, \mu)$  is *complete* if every  $\mu$ -null set already lies in  $\mathcal{F}$ , i.e.  $\mathcal{N}_\mu \subset \mathcal{F}$ .

**Remark 1.5** (Exercise from the lecture). Show that  $\mathcal{N}_\mu$  is a ring of sets.

When  $(X, \mathcal{F}, \mu)$  fails to be complete, we can replace  $\mathcal{F}$  by a slightly larger  $\sigma$ -field that contains all null sets.

#### Proposition 1.7: Completion (Dudley 3.3.2)

Let  $(X, \mathcal{F}, \mu)$  be a measure space with null-set collection  $\mathcal{N}_\mu$ . Define

$$\mathcal{F} \vee \mathcal{N}_\mu = \{A \cup N : A \in \mathcal{F}, N \in \mathcal{N}_\mu\}.$$

Then

$$\mathcal{F} \vee \mathcal{N}_\mu = \{B \subseteq X : \exists A \in \mathcal{F} \text{ with } A \Delta B \in \mathcal{N}_\mu\},$$

and this is the smallest  $\sigma$ -field containing both  $\mathcal{F}$  and  $\mathcal{N}_\mu$ . Setting  $\bar{\mu}(A \cup N) = \mu(A)$ , the triple  $(X, \mathcal{F} \vee \mathcal{N}_\mu, \bar{\mu})$  is a complete measure space, the *completion* of  $(X, \mathcal{F}, \mu)$ .

**Remark 1.6.** The two descriptions of  $\mathcal{F} \vee \mathcal{N}_\mu$  match the picture above:  $B$  is in the completion iff it differs from some  $A \in \mathcal{F}$  only on a set of outer measure zero, i.e.  $A \Delta B$  is null. Lebesgue measure on  $\mathbb{R}$  is the completion of its restriction to the Borel  $\sigma$ -field; this strict enlargement is the gap  $\mathcal{B} \subsetneq \mathcal{M}_\lambda$  used in the next lecture.