# 1 Lecture 4 — Lebesgue Measure; Non-Measurable Sets; Product Measures and Independence

The Carathéodory and Dynkin machinery of Lecture 3 is now put to work. We build Lebesgue measure on $(0,1]$ (and on $\mathbb{R}$), exhibit a Vitali set that no translation-invariant measure can size, then sketch how the same toolkit yields product measures and independence of $\sigma$-fields.

## 1.1 Lebesgue measure on $(0,1]$ and $\mathbb{R}$

Take $\Omega = (0,1]$ (or $\Omega = \mathbb{R}$) and let $\mathcal{A}$ be the collection of all finite unions of half-open intervals $(a,b]$, together with $\emptyset$. The target is a set function with

$$\lambda\big((a,b]\big) \;=\; b-a,$$

extended by additivity to $\mathcal{A}$.

**Proposition 1.1: $\mathcal{A}$ is a $\pi$-system and a ring**

The collection $\mathcal{A}$ of finite (disjoint) unions of half-open intervals $(a,b] \subseteq (0,1]$, together with $\emptyset$, is a $\pi$-system: intersections of half-open intervals are again half-open intervals,

$$(a,b]\cap(c,d] \;=\; \begin{cases} \emptyset & \text{if } b \le c, \\ (c,b] & \text{if } a \le c < b \le d, \end{cases}$$

and unions of finitely many such intersections remain in $\mathcal{A}$. The class $\mathcal{A}$ is also a ring: it contains $\emptyset = (a,a]$, is closed under set difference $(a,b]\setminus(c,d]$, and closed under finite unions by definition.

**Proposition 1.2: $\lambda$ is a pre-measure on $\mathcal{A}$**

The set function $\lambda\colon \mathcal{A} \longrightarrow [0,\infty]$ defined by $\lambda((a,b]) = b-a$ and extended additively to finite disjoint unions is a pre-measure on the ring $\mathcal{A}$.

Combining the two propositions with the Carathéodory and $\pi$-$\lambda$ theorems of Lecture 3 produces the construction.

**Theorem 1.3: Lebesgue measure**

Let $\mathcal{B} = \sigma(\mathcal{A})$ be the Borel $\sigma$-field of $(0,1]$ (equivalently, the $\sigma$-field generated by the open sets). There exists a unique measure $\lambda$ on $\mathcal{B}$ with $\lambda((a,b]) = b-a$. It is obtained by Carathéodory extension (Step 2) of the pre-measure on $\mathcal{A}$ (Step 1); uniqueness on $\sigma(\mathcal{A})$ follows because $\mathcal{A}$ is a $\pi$-system (Step 3). Writing $\mathcal{M}_\lambda$ for the $\sigma$-field of *Lebesgue measurable* sets,

$$\mathcal{A} \;\subset\; \mathcal{B} \;=\; \sigma(\mathcal{A}) \;\subsetneq\; \mathcal{M}_\lambda \;\subsetneq\; \mathcal{P}\big((0,1]\big),$$

where $\mathcal{M}_\lambda$ is the completion of $\mathcal{B}$ with respect to the $\lambda$-null sets $\mathcal{N}_\lambda$.

**Remark 1.1.** Lebesgue measure is the *only* translation-invariant measure on $\mathbb{R}$ (up to a multiplicative constant) assigning length $b-a$ to $(a,b]$; the same statement holds on $\mathbb{R}^n$. The strict inclusion $\mathcal{M}_\lambda \subsetneq \mathcal{P}((0,1])$ is the content of the next subsection.

## 1.2 Non-measurable sets: the Vitali construction

We exhibit a set $H \subseteq (0,1]$ outside $\mathcal{M}_\lambda$. The construction uses the Axiom of Choice and addition modulo $1$, which wraps $(0,1]$ into a circle.

**Definition 1.4: Addition modulo $1$**

For $x, y \in (0,1]$ define

$$x \oplus y \;=\; \begin{cases} x+y & \text{if } x+y \le 1, \\ x+y-1 & \text{if } x+y > 1. \end{cases}$$

For $A \subseteq (0,1]$ and $x \in (0,1]$, set $A + x = \{y \in (0,1] : y - x \in A\}$, with subtraction taken mod $1$.

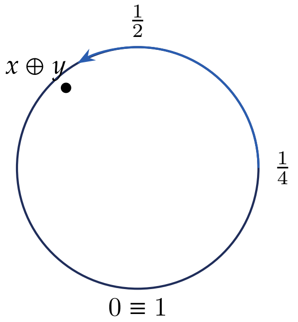

**Figure 1.** Addition mod $1$ wraps $(0,1]$ onto a circle: shifts $A \mapsto A + x$ become rotations.

**Lemma 1.5: Translation invariance on Borel sets**

Let $\mathcal{L} = \{A \in \mathcal{M}_\lambda : \lambda(A+x) = \lambda(A) \text{ for all } x \in (0,1]\}$. Then $\mathcal{L}$ is a $\lambda$-system, and $\mathcal{A} \subseteq \mathcal{L}$ since $\lambda((a,b]+x) = \lambda((a+x, b+x]) = b-a$. By Dynkin's $\pi$-$\lambda$ theorem, $\mathcal{B} = \sigma(\mathcal{A}) \subseteq \mathcal{L}$; equivalently, $\lambda$ is translation invariant on every Borel set.

**■ Example 1.1 (Vitali set).** Define an equivalence relation on $(0,1]$ by $x \sim y \iff x - y \in \mathbb{Q}$; for instance $\frac{1}{\sqrt{2}} \sim \frac{1}{\sqrt{2}} + \frac{1}{100}$. By the Axiom of Choice pick a set $H \subseteq (0,1]$ containing exactly one representative from each equivalence class. Then for distinct $r_1, r_2 \in \mathbb{Q} \cap (0,1]$ we have $(H + r_1) \cap (H + r_2) = \emptyset$, and

$$(0,1] \;=\; \bigsqcup_{r \in \mathbb{Q} \cap (0,1]} (H + r).$$

If $H$ were Lebesgue measurable, countable additivity together with ?? would give

$$1 \;=\; \lambda\big((0,1]\big) \;=\; \sum_{r \in \mathbb{Q} \cap (0,1]} \lambda(H + r) \;=\; \sum_{r \in \mathbb{Q} \cap (0,1]} \lambda(H),$$

which is $0$ if $\lambda(H) = 0$ and $\infty$ if $\lambda(H) > 0$ — either way a contradiction. Hence $H \notin \mathcal{M}_\lambda$.

**Remark 1.2.** This shows the strict inclusion $\mathcal{M}_\lambda \subsetneq \mathcal{P}((0,1])$. A related fun fact: there is no infinite-dimensional analogue of Lebesgue measure: the only locally finite, translation-invariant Borel measure on an infinite-dimensional separable Banach space is the trivial one.

## 1.3 Product measures, briefly

The half-open construction generalises directly to $\mathbb{R}^p$ using half-open rectangles.

**Definition 1.6: Lebesgue measure on $\mathbb{R}^p$**

On $\mathbb{R}^p$, define

$$\lambda^{(p)}\big((a_1,b_1]\times\cdots\times(a_p,b_p]\big) \;=\; \prod_{i=1}^{p}\lambda\big((a_i,b_i]\big) \;=\; \prod_{i=1}^{p}(b_i-a_i).$$

The collection of half-open rectangles is a $\pi$-system, and the extension theorem gives a unique measure on $\mathcal{B}(\mathbb{R}^p)$. For $p = 2$,

$$\lambda^{(2)}\big((a,b]\times(c,d]\big) \;=\; (b-a)(d-c) \;=\; \lambda\big((a,b]\big)\,\lambda\big((c,d]\big).$$

**Definition 1.7: Product measure**

Given two $\sigma$-finite measure spaces $(\mathcal{X},\mathscr{X},\mu)$ and $(\mathcal{Y},\mathscr{Y},\nu)$, the *product measure space* is $(\mathcal{X}\times\mathcal{Y},\ \mathscr{X}\times\mathscr{Y},\ \pi)$, where the product $\sigma$-field is

$$\mathscr{X}\times\mathscr{Y} \;=\; \sigma\big(\{A\times B : A\in\mathscr{X},\ B\in\mathscr{Y}\}\big),$$

and $\pi$ is uniquely determined by

$$\pi(A\times B) \;=\; \mu(A)\,\nu(B), \qquad A\in\mathscr{X},\ B\in\mathscr{Y}.$$

**Remark 1.3.** The product of two Borel $\sigma$-fields satisfies $\mathcal{B}(\mathcal{X})\times\mathcal{B}(\mathcal{Y}) \subseteq \mathcal{B}(\mathcal{X}\times\mathcal{Y})$, and equality holds in "nice" (e.g. second-countable) settings, including $\mathcal{X} = \mathcal{Y} = \mathbb{R}$.

## 1.4 Independence

Switch perspective from measure theory to probability: let $(\Omega,\mathcal{F},\mu)$ be a probability space.

**Definition 1.8: Independence for sets**

A countable collection $\{A_i\}_{i\in I} \subseteq \mathcal{F}$ is *independent* if, for every finite $J \subseteq I$,

$$\mu\left(\bigcap_{j\in J} A_j\right) \;=\; \prod_{j\in J}\mu(A_j).$$

**■ Example 1.2 (Standard 52-card deck).** Draw one card uniformly at random and let $A_1 = \{\text{red}\}$, $A_2 = \{\text{heart or club}\}$, $A_3 = \{\text{queen}\}$. Then

$$\mu(A_1) = \tfrac{1}{2}, \quad \mu(A_2) = \tfrac{1}{2}, \quad \mu(A_3) = \tfrac{1}{13},$$

and one checks $\mu(A_1 \cap A_2) = \frac{1}{4}$, $\mu(A_1 \cap A_3) = \frac{1}{26}$, $\mu(A_2 \cap A_3) = \frac{1}{26}$, $\mu(A_1 \cap A_2 \cap A_3) = \frac{1}{52}$, so the three events are independent.

### Definition 1.9: Independence for $\sigma$-fields

A countable collection $\{\mathcal{F}_i\}_{i \in I}$ of sub-$\sigma$-fields of $\mathcal{F}$ is *independent* if every selection $\{A_i \in \mathcal{F}_i : i \in I\}$ is an independent collection of sets in the sense of **??**.

The next theorem is the workhorse result: independence on a generating $\pi$-system already forces independence of the generated $\sigma$-fields.

### Theorem 1.10: Independence from $\pi$-systems

Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{F}$ be $\pi$-systems. If

$$\mu(A_1 \cap A_2) \;=\; \mu(A_1)\,\mu(A_2) \quad \text{for all } A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2,$$

then $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent.