1 Lecture 5 — Simple and Measurable Functions

Having built measures (Lectures 1–4), we now turn to the objects we actually integrate: *functions* between measurable spaces. We start with *simple functions* — finite linear combinations of indicators — which are concrete enough to define an integral by hand, and then promote the construction to arbitrary *measurable functions* via the right preservation lemmas. In probability language, simple functions are finite-valued random variables, and measurable functions are random variables.

Notation. Throughout, $(\Omega, \mathcal{F}, \mu)$ is a measure space; for the probabilistic statements we write $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(\Omega) = 1$. For a set A, the indicator $\mathbf{1}[\omega \in A]$ equals 1 when $\omega \in A$ and 0 otherwise. For $f : \mathbb{X} \to \mathbb{Y}$ and $B \subseteq \mathbb{Y}$, the preimage is $f^{-1}(B) = \{x \in \mathbb{X} : f(x) \in B\}$. When the codomain is $\mathbb{R}$ we always equip it with its Borel σ-field $\mathcal{B}(\mathbb{R})$ unless stated otherwise.

1.1 Simple functions and simple random variables

A simple function is, by design, the simplest object on which an integral can be defined: it takes only finitely many values, each on a measurable piece of Ω.

Definition 1.1: Simple random variable

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *simple random variable* is a real-valued function $X : \Omega \to \mathbb{R}$ such that

1. X takes only finitely many values $x_1, \ldots, x_p \in \mathbb{R}$;
2. for every i, the level set $\{\omega \in \Omega : X(\omega) = x_i\} \in \mathcal{F}$.

Equivalently, there is a finite partition $\{A_i\}_{i=1}^{p} \subseteq \mathcal{F}$ of Ω (so $\bigsqcup_{i=1}^{p} A_i = \Omega$ with $A_i \cap A_j = \emptyset$ for $i \neq j$) and constants $x_1, \ldots, x_p \in \mathbb{R}$ with

$$X(\omega) = \sum_{i=1}^{p} x_i \, \mathbf{1}[\omega \in A_i].$$

Remark 1.1. For a simple random variable as above, $\mathbb{P}(X = x_i) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x_i\}) = \mathbb{P}(A_i)$, and the *expectation* of X is the natural finite sum

$$\mathbb{E}X = \sum_{i=1}^{p} x_i \, \mathbb{P}(X = x_i).$$

■ **Example 1.1 (Binary steps on $(0, 1]$).** Take $\Omega = (0, 1]$ with Lebesgue measure λ and partition

$$A_1 = (0, 0.25], \; A_2 = (0.25, 0.5], \; A_3 = (0.5, 0.75], \; A_4 = (0.75, 1],$$

each of measure $\lambda(A_i) = 0.25$. Set $x_i = (i - 1)/4$. The simple random variable $X^{(4)}(\omega) = \sum_{i=1}^{4} x_i \, \mathbf{1}[\omega \in A_i]$ takes the values $0, 0.25, 0.5, 0.75$ each with probability $1/4$, so $\mathbb{E}X^{(4)} = (0 + 0.25 + 0.5 + 0.75)/4 = 0.375$. Refining the partition into 2^m pieces and letting $m \to \infty$ recovers the uniform distribution on $(0, 1]$ in the limit.

The same definition makes sense on a general (not necessarily probability) measure space, and is the starting point for the abstract integral.

Definition 1.2: Simple function and its integral

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A *simple function* is a function $f : \Omega \to \mathbb{R}$ of the form

$$f(\omega) = \sum_{i=1}^{p} x_i \, \mathbf{1}[\omega \in B_i], \qquad x_i \in \mathbb{R},\ B_i \in \mathcal{F}.$$

Its integral with respect to μ is defined by

$$\int f \, d\mu := \sum_{i=1}^{p} x_i \, \mu(B_i).$$

The sets B_i need not be disjoint, but any simple function admits such a representation with $\{B_i\}$ disjoint, and the value of the integral is independent of the chosen representation.

Proposition 1.3: Algebra of simple functions

If $f, g : \Omega \to \mathbb{R}$ are simple functions, then so are $f + g$, fg, $\max\{f, g\}$ and $\min\{f, g\}$. For non-negative simple functions f, g and a scalar $c \geq 0$, the integral is linear:

$$\int (f+g)\, d\mu = \int f \, d\mu + \int g \, d\mu, \qquad \int cf \, d\mu = c \int f \, d\mu.$$

1.2 Measurable functions

To extend the simple-function picture beyond finitely many values, we replace "X takes the value x_i on a measurable set" with "X pulls back *every* Borel set to a measurable set". Working between two abstract measurable spaces costs nothing extra.

Definition 1.4: Measurable function

Let $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ be measurable spaces. A function $f : \mathbb{X} \to \mathbb{Y}$ is *measurable* (with respect to $\mathcal{X}/\mathcal{Y}$) if

$$f^{-1}(B) \in \mathcal{X} \qquad \text{for every } B \in \mathcal{Y}.$$

When $(\mathbb{Y}, \mathcal{Y}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say f is *Borel measurable*; if $\mathcal{B}(\mathbb{R})$ is replaced by the Lebesgue σ-field $\mathcal{M}_\lambda(\mathbb{R})$, f is *Lebesgue measurable*. A measurable function on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a *random variable*.

Remark 1.2. Preimages preserve set operations: for $f : \mathbb{X} \to \mathbb{Y}$ and $A, A_i \subseteq \mathbb{Y}$,

$$f^{-1}\Big(\bigcup_i A_i\Big) = \bigcup_i f^{-1}(A_i), \qquad f^{-1}(\mathbb{Y} \setminus A) = \mathbb{X} \setminus f^{-1}(A).$$

Consequently $\{f^{-1}(B) : B \in \mathcal{Y}\}$ is itself a σ-field on $\mathbb{X}$; measurability of f is the statement that this σ-field is contained in $\mathcal{X}$.

The next proposition is the workhorse: to check measurability one need only inspect a generating family.

Proposition 1.5: Measurability via a generator

Let $\mathcal{A} \subseteq \mathcal{Y}$ be a collection of sets with $\sigma(\mathcal{A}) = \mathcal{Y}$. A function $f : \mathbb{X} \to \mathbb{Y}$ is measurable if and only if $f^{-1}(A) \in \mathcal{X}$ for every $A \in \mathcal{A}$. In particular, since the half-lines $\{(-\infty, t] : t \in \mathbb{R}\}$ generate $\mathcal{B}(\mathbb{R})$, $f : \mathbb{X} \to \mathbb{R}$ is Borel measurable iff

$$\{x \in \mathbb{X} : f(x) \le t\} \in \mathcal{X} \qquad \text{for every } t \in \mathbb{R}.$$

The class of measurable functions is closed under essentially every operation one performs in analysis.

Proposition 1.6: Stability properties of measurable functions

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space.

1. *Indicators.* For every $A \in \mathcal{X}$ the indicator $\mathbf{1}[x \in A]$ is measurable, and the σ-field generated by it is $\{\emptyset, A, A^c, \mathbb{X}\} \subseteq \mathcal{X}$.
2. *Algebraic operations.* If $f, g : \mathbb{X} \to \mathbb{R}$ are measurable, then so are $f + g$, fg, $\max\{f, g\}$ and $\min\{f, g\}$.
3. *Sequential operations.* If $\{f_i\}_{i=1}^{\infty}$ are measurable functions $\mathbb{X} \to \mathbb{R}$, then $\sup_i f_i$, $\inf_i f_i$, $\limsup_i f_i$, $\liminf_i f_i$, and $\lim_i f_i$ (when it exists pointwise) are all measurable.
4. *Continuity $\Rightarrow$ measurability.* Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.
5. *Generating measurable structure.* Given any family $\{f_i : \mathbb{X} \to \mathbb{Y}\}_{i \in I}$, the smallest σ-field on $\mathbb{X}$ making each f_i measurable is $\sigma(\{f_i\}_{i \in I}) = \sigma\big(\{f_i^{-1}(B) : i \in I,\ B \in \mathcal{Y}\}\big)$.

Remark 1.3. The sequential closure in (3) is the reason measure-theoretic integration interacts so well with limits: it lets us realise any non-negative measurable f as the increasing pointwise limit $f_i \uparrow f$ of simple functions, e.g. $f_i = 2^{-i}\lfloor 2^i f \rfloor$ (truncated above by i). This is what powers the construction of the Lebesgue integral in the next lecture.

1.3 Almost-everywhere equality

The final notion in this lecture lets us ignore behaviour on μ-negligible sets, which is essential once integration enters.

Definition 1.7: Almost everywhere / almost surely

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : \Omega \to \mathbb{R}$. We say $f = g$ *almost everywhere* (written $f = g$ a.e.) if the exceptional set

$$N = \{\omega \in \Omega : f(\omega) \neq g(\omega)\}$$

satisfies $\mu(N) = 0$. On a probability space this is also called *almost sure* equality, abbreviated a.s., and one says it holds *with probability one* (wp1).

■ **Example 1.2 (Equal almost everywhere on $(0,1]$).** Let $((0,1],\mathcal{B},\lambda)$ be the standard Borel measure space with Lebesgue measure, and define $f(t)=0$ for all $t\in(0,1]$ and

$$g(t) = \begin{cases} 0 & t\in(0,1]\setminus\mathbb{Q}, \\ 1 & t\in(0,1]\cap\mathbb{Q}. \end{cases}$$

Then $f = g$ a.e., because the exceptional set $(0,1]\cap\mathbb{Q}$ is countable and hence λ-null: enumerating $\mathbb{Q}\cap(0,1]=\{q_m\}_{m=1}^{\infty}$ and covering q_m by $(q_m-\varepsilon 2^{-m}, q_m+\varepsilon 2^{-m+1})$ gives $\lambda((0,1]\cap\mathbb{Q})\le 3\varepsilon$ for every $\varepsilon>0$, so $\lambda((0,1]\cap\mathbb{Q})=0$.