1 Lecture 6 — Integration and Convergence Theorems (MCT, Fatou, DCT)

Lectures **??** (and 5) built up Lebesgue measure and the supply of measurable functions; here we use that machinery to define the *Lebesgue integral* $\int f\,d\mu$ for a measurable function $f : (\Omega, \mathcal{F}, \mu) \to [-\infty, \infty]$, and state the three convergence theorems on which essentially all of subsequent measure theory rests: *Monotone Convergence*, *Fatou's Lemma*, and *Dominated Convergence*.

Notation. For a sequence of functions f_i we write $f_i \uparrow f$ to mean $f_i(\omega) \le f_{i+1}(\omega)$ for every ω and $f_i(\omega) \to f(\omega)$ pointwise; analogously for $f_i \downarrow f$. We work in the extended real line $[-\infty, \infty]$, with the conventions

$$0 \cdot \infty \;=\; 0, \qquad c \cdot \infty \;=\; \infty \;(c > 0), \qquad \infty - \infty \text{ undefined.}$$

Every measurable f is split into its positive and negative parts

$$f^+(\omega) \;=\; \max\{f(\omega), 0\}, \qquad f^-(\omega) \;=\; \max\{-f(\omega), 0\},$$

so that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

1.1 Building the integral via simple functions

Recall from Lecture 5 that a *simple function* $f = \sum_{i=1}^{p} x_i \mathbf{1}_{B_i}$ (with $B_i \in \mathcal{F}$) has the unambiguous integral

$$\int f\,d\mu \;=\; \sum_{i=1}^{p} x_i\,\mu(B_i),$$

and that this integral is linear and monotone on non-negative simple functions. The next theorem says that simple functions are dense, in the strong sense that every measurable function is the increasing limit of simple ones, so *any* property linear in f and stable under monotone limits which holds for indicators must hold for all measurable functions.

Theorem 1.1: Approximation by simple functions on a π-system

Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathcal{A}$ be a π-system that generates $\mathcal{F}$. Let $\mathcal{V}$ be a linear space of functions such that

1. $\mathbf{1}_\Omega \in \mathcal{V}$ and $\mathbf{1}_A \in \mathcal{V}$ for every $A \in \mathcal{A}$;
2. whenever $f_i \in \mathcal{V}$ and $f_i \uparrow f$, the limit $f \in \mathcal{V}$.

Then $\mathcal{V}$ contains every $\mathcal{F}$-measurable function. Concretely, for any non-negative measurable f the simple functions $f_i = 2^{-i}\lfloor 2^i f \rfloor$ satisfy $f_i \uparrow f$.

Definition 1.2: Lebesgue integral of a measurable function

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \to [-\infty, \infty]$ be measurable. For $f \ge 0$ define

$$\int f\,d\mu \;=\; \sup\left[\sum_i \left\{\inf_{\omega \in A_i} f(\omega)\right\}\mu(A_i)\right],$$

where the supremum is taken over all finite measurable partitions $\{A_i\}$ of Ω; equivalently, $\int f\,d\mu = \sup\left\{\int g\,d\mu : g \text{ simple},\ 0 \le g \le f\right\}$. For general measurable f, write $f = f^+ - f^-$

and set

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu,$$

provided not both terms are infinite. We say f is *integrable* if both $\int f^+\,d\mu$ and $\int f^-\,d\mu$ are finite, equivalently if $\int |f|\,d\mu < \infty$.

Remark 1.1. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the integral $\mathbb{E}X = \int X\,d\mathbb{P}$ is the *expectation* of the random variable X; the integrability condition $\int |X|\,d\mathbb{P} < \infty$ is written $X \in L^1(\Omega, \mathbb{P})$.

Remark 1.2. Following **??** the integral is linear and monotone:

$$\int (af + bg)\,d\mu = a\int f\,d\mu + b\int g\,d\mu, \qquad f \le g \implies \int f\,d\mu \le \int g\,d\mu,$$

whenever the right-hand sides are defined. It also respects μ-null modifications, as the next theorem records.

Theorem 1.3: Integrals ignore null sets

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f, g : \Omega \to [-\infty, \infty]$ be measurable with $f = g$ almost everywhere. Then f is integrable iff g is integrable, and in that case $\int f\,d\mu = \int g\,d\mu$.

1.2 The three convergence theorems

The substance of Lebesgue integration — its main advantage over the Riemann integral — is the freedom with which we may exchange limits and integrals. The three theorems below give three different sufficient conditions; they form, in Koosis's words, "the most important results to learn" in measure theory.

Theorem 1.4: Monotone Convergence (Beppo Levi)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_i\}_{i=1}^{\infty}$ be measurable functions $\Omega \to \mathbb{R}$ with $f_i \uparrow f$ almost everywhere and $\int f_1\,d\mu > -\infty$. Then f is measurable and

$$\int f_i\,d\mu \uparrow \int f\,d\mu.$$

Remark 1.3. It is enough that $f_i \uparrow f$ hold *almost everywhere*: by **??** convergence may fail on a μ-null set without affecting the conclusion. A symmetric statement holds for $f_i \downarrow f$ provided $\int f_1\,d\mu < \infty$.

Theorem 1.5: Fatou's Lemma

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_i\}_{i=1}^{\infty}$ be *non-negative* measurable functions $\Omega \to \mathbb{R}$. Then

$$\int \liminf_{i\to\infty} f_i\,d\mu \le \liminf_{i\to\infty} \int f_i\,d\mu.$$

Remark 1.4. The inequality may be strict: take $\Omega = \mathbb{R}$ with Lebesgue measure and $f_i = \mathbf{1}_{[i,i+1]}$. Then $f_i \to 0$ pointwise, so the left-hand side is 0, while $\int f_i\,d\mu = 1$ for every i and the right-hand side is 1. Some hypothesis (monotonicity, domination, …) is needed to upgrade "$\le$" to equality.

Theorem 1.6: Dominated Convergence (Lebesgue)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $\{f_i\}_{i=1}^{\infty}$ be measurable, and let g be absolutely integrable (i.e. $\int |g| \, d\mu < \infty$). If

1. $|f_i(\omega)| \leq g(\omega)$ for all i and all $\omega \in \Omega$, and
2. $f_i(\omega) \to f(\omega)$ for each $\omega \in \Omega$ (pointwise convergence),

then f is absolutely integrable and

$$\int f_i \, d\mu \longrightarrow \int f \, d\mu.$$

Remark 1.5. On a probability space, **??** is the standard tool for passing limits inside expectations: if $|X_i| \leq Y$ with $\mathbb{E}Y < \infty$ and $X_i \to X$ almost surely, then $\mathbb{E}X_i \to \mathbb{E}X$. Bounded convergence ($g \equiv M$) on a *finite* measure space is the simplest and most-used special case.

Corollary 1.7: Bounded Convergence

Let $\mu(\Omega) < \infty$ and let $\{f_i\}$ be measurable with $|f_i| \leq M$ for some constant $M < \infty$. If $f_i \to f$ pointwise (or a.e.), then $\int f_i \, d\mu \to \int f \, d\mu$.

1.3 Image measures (Lebesgue–Stieltjes)

A measurable function $\psi : (\mathbb{X}, \mathcal{X}) \to (\mathbb{Y}, \mathcal{Y})$ turns a measure on the source into a measure on the target by "pushing forward."

Definition 1.8: Image (push-forward) measure

Let μ be a measure on $(\mathbb{X}, \mathcal{X})$ and $\psi : \mathbb{X} \to \mathbb{Y}$ be measurable. Define $\nu = \mu \circ \psi^{-1}$ on $\mathcal{Y}$ by

$$\nu(B) = \mu\big(\psi^{-1}(B)\big), \qquad B \in \mathcal{Y}.$$

Then ν is a measure on $(\mathbb{Y}, \mathcal{Y})$, called the *image* (or *push-forward*) of μ under ψ. Specialising $\mu = \lambda$ (Lebesgue) gives the family of *Lebesgue–Stieltjes measures*; the prototypical example is the distribution $\mathbb{P} \circ X^{-1}$ on $\mathbb{R}$ of a real random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, whose distribution function $F(t) = \mathbb{P}(X \leq t)$ is the corresponding Stieltjes measure of $(-\infty, t]$.