1 Lecture 7 — Lebesgue–Stieltjes Measure; Fubini–Tonelli

We are halfway through the integration arc. Today's two themes both push measures *between spaces*. First: a measurable map $\psi\colon \mathcal{X} \to \mathcal{Y}$ carries a measure μ on $\mathcal{X}$ to its *image measure* $\nu = \mu \circ \psi^{-1}$ on $\mathcal{Y}$; when μ is Lebesgue measure on $\mathbb{R}$ and ψ is built from a distribution function F, this produces the Lebesgue–Stieltjes measure dF. Second: given two σ-finite spaces, one constructs the product measure on $\mathcal{X}\times\mathcal{Y}$, and the Fubini–Tonelli theorem licences swapping the order of integration.

1.1 Image measures and Lebesgue–Stieltjes

Definition 1.1: Image measure

Let $(\mathcal{X}, \mathscr{X}, \mu)$ and $(\mathcal{Y}, \mathscr{Y})$ be measurable spaces and $\psi\colon \mathcal{X} \to \mathcal{Y}$ measurable. The *image* (or *push-forward*) measure of μ under ψ is the set function $\nu = \mu \circ \psi^{-1}$ on $\mathscr{Y}$,

$$\nu(B) \ = \ \mu(\psi^{-1}(B)), \qquad B \in \mathscr{Y}.$$

Remark 1.1. Measurability of ψ is what makes $\psi^{-1}(B) \in \mathscr{X}$, so $\nu(B)$ is defined. Inverse images preserve countable unions and complements, so ν is automatically a measure on $\mathscr{Y}$. This is the construction that turns Lebesgue measure on $\mathbb{R}$ into the Lebesgue–Stieltjes measure attached to a distribution function.

Theorem 1.2: Lebesgue–Stieltjes measure

Let $F\colon \mathbb{R} \to \mathbb{R}$ be non-constant, right-continuous, and non-decreasing. There exists a unique measure dF on $\mathcal{B}(\mathbb{R})$ such that for all $a < b$ in $\mathbb{R}$,

$$dF((a, b]) \ = \ F(b) - F(a).$$

Writing $F(\infty) = \lim_{x\to\infty} F(x)$, $F(-\infty) = \lim_{x\to-\infty} F(x)$, $I = (F(-\infty), F(\infty))$, and

$$g(y) \ = \ \inf\{x \in \mathbb{R} : y \le F(x)\}, \qquad y \in I,$$

the measure dF is realised as the image of Lebesgue measure under g:

$$dF \ = \ \lambda \circ g^{-1}.$$

Lemma 1.3: Properties of the left-inverse g

With F and g as in Result 1.2, the function g is left-continuous and non-decreasing on I, and

$$g(y) \le x \iff y \le F(x) \qquad \text{for } y \in I,\ x \in \mathbb{R}.$$

Concretely, for fixed $y \in I$ the set $J_y = \{x \in \mathbb{R} : y \le F(x)\}$ equals $[g(y), \infty)$, so g is Borel measurable.

Remark 1.2. Once g is Borel measurable and left-continuous, $\lambda \circ g^{-1}$ is a measure on $\mathcal{B}(\mathbb{R})$ and

$$dF((a, b]) = \lambda(\{y : g(y) > a,\ g(y) \le b\}) = \lambda((F(a), F(b)]) = F(b) - F(a).$$

Uniqueness follows from the π-λ argument used for Lebesgue measure: any other measure μ with $\mu((a,b]) = F(b) - F(a)$ must agree with dF on the π-system of half-open intervals, hence on every Borel set.

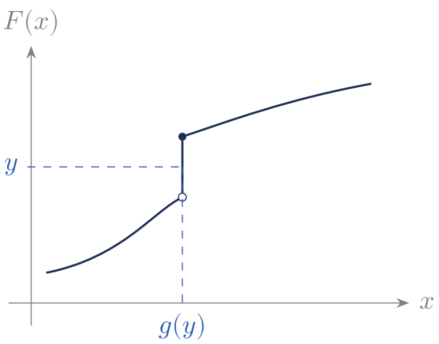

Figure 1. A right-continuous non-decreasing F with a jump. $g(y) = \inf\{x : y \le F(x)\}$ reads the picture sideways: a level y inside the jump still resolves to a single x-value.

Definition 1.4: Radon measure

A measure μ on $(\Omega, \mathcal{B})$, with $\mathcal{B}$ the Borel σ-field, is a *Radon measure* if $\mu(K) < \infty$ for every compact $K \in \mathcal{B}$.

Remark 1.3. dF is a Radon measure on $\mathbb{R}$, and conversely every non-zero Radon measure on $\mathcal{B}(\mathbb{R})$ can be written as $dF = \lambda \circ g^{-1}$ for some non-decreasing right-continuous F: take

$$F(x) \;=\; \begin{cases} \mu((0,x]) & \text{if } x \ge 0, \\ -\mu((x,0]) & \text{if } x < 0. \end{cases}$$

Then $F(b) - F(a) = \mu((a,b])$ for $a < b$, so $\mu = dF$ by uniqueness. Most measures one meets in practice are Radon.

1.2 Product σ-fields and the product measure

Switch settings: two measurable spaces, with the goal of building a measure on the Cartesian product.

Definition 1.5: Rectangles and product σ-field

Given measurable spaces $(\mathcal{X}, \mathscr{X})$ and $(\mathcal{Y}, \mathscr{Y})$, a *rectangle* is a set $A \times B$ with $A \in \mathscr{X}$, $B \in \mathscr{Y}$. Write $\mathcal{R}$ for the collection of all rectangles. The *product σ-field* is

$$\mathscr{X} \times \mathscr{Y} \;=\; \sigma(\mathcal{R}).$$

Theorem 1.6: Existence and uniqueness of the product measure

Let $(\mathcal{X}, \mathscr{X}, \mu)$ and $(\mathcal{Y}, \mathscr{Y}, \nu)$ be σ-finite measure spaces, and let π be the set function on rectangles defined by

$$\pi(A \times B) \;=\; \mu(A)\,\nu(B), \qquad A \in \mathscr{X},\; B \in \mathscr{Y}.$$

Then π extends uniquely to a measure on $(\mathcal{X} \times \mathcal{Y}, \mathscr{X} \times \mathscr{Y})$, and for every $E \in \mathscr{X} \times \mathscr{Y}$,

$$\pi(E) = \iint \mathbf{1}_E(x,y)\, d\mu(x)\, d\nu(y) = \iint \mathbf{1}_E(x,y)\, d\nu(y)\, d\mu(x).$$

Remark 1.4. σ-finiteness cannot be dropped: without it, the extension above need not be unique. The strategy of proof is (i) prove the result for finite measures, then (ii) stitch together σ-finite exhaustions $\{A_i\} \subseteq \mathscr{X}$, $\{B_j\} \subseteq \mathscr{Y}$ with $\mu(A_i), \nu(B_j) < \infty$. The technical engine for step (i) is not Dynkin's π-λ theorem but its sibling, the *monotone class theorem*.

1.3 The monotone class theorem

The monotone class theorem is the natural "π-λ" tool when one starts from a field rather than a π-system.

Definition 1.7: Monotone class

A collection $\mathcal{M}$ of subsets of Ω is a *monotone class* if it is closed under monotone limits:

1. for $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ with $A_i \uparrow A = \bigcup_{i=1}^{\infty} A_i$, one has $A \in \mathcal{M}$;
2. for $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ with $A_i \downarrow A = \bigcap_{i=1}^{\infty} A_i$, one has $A \in \mathcal{M}$.

Recall that a *field* (algebra) on Ω is a non-empty collection containing Ω, closed under complements and finite unions. A field that is also a monotone class is automatically a σ-field.

Theorem 1.8: Monotone class theorem

Let $\mathcal{A}$ be a field on Ω and let $\mathcal{M}$ be a monotone class with $\mathcal{A} \subseteq \mathcal{M}$. Then

$$\sigma(\mathcal{A}) \subseteq \mathcal{M}.$$

Remark 1.5. Equivalently, the smallest monotone class containing a field $\mathcal{A}$ coincides with $\sigma(\mathcal{A})$. One uses this in the product-measure proof as follows: the rectangles $\mathscr{R}$ are a semi-ring, finite disjoint unions of rectangles form a field $\mathcal{A}$, and π is countably additive on $\mathcal{A}$. The monotone class theorem is what lifts "equality on $\mathcal{A}$" to "equality on $\sigma(\mathcal{A}) = \mathscr{X} \times \mathscr{Y}$" for any two candidate extensions.

Lemma 1.9: Order of integration for indicators

Let $(\mathcal{X}, \mathscr{X}, \mu)$ and $(\mathcal{Y}, \mathscr{Y}, \nu)$ be finite measure spaces. Then for every $E \in \mathscr{X} \times \mathscr{Y}$,

$$\iint \mathbf{1}_E(x,y)\, d\mu(x)\, d\nu(y) = \iint \mathbf{1}_E(x,y)\, d\nu(y)\, d\mu(x).$$

The collection of E for which the displayed equality holds contains every rectangle $A \times B$ (since both sides equal $\mu(A)\nu(B)$), is closed under finite disjoint unions, and is a monotone class by Monotone Convergence. By Result 1.6 it equals $\mathscr{X} \times \mathscr{Y}$.

1.4 The Fubini–Tonelli theorem

The set-function identity in **??** extends to integrals of *measurable functions*, not just indicators.

Theorem 1.10: Fubini–Tonelli

Let $(\mathcal{X}, \mathscr{X}, \mu)$ and $(\mathcal{Y}, \mathscr{Y}, \nu)$ be σ-finite measure spaces, and let $f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be $\mathscr{X} \times \mathscr{Y}$-measurable. Suppose either

$$f \geq 0 \quad \text{(Tonelli)}, \qquad \text{or} \qquad \iint |f| \, d(\mu \times \nu) < \infty \quad \text{(Fubini)}.$$

Then

$$\int f \, d(\mu \times \nu) = \iint f(x, y) \, d\mu(x) \, d\nu(y) = \iint f(x, y) \, d\nu(y) \, d\mu(x).$$

Moreover $\int f(x, y) \, d\mu(x)$ is $\mathscr{Y}$-measurable in y, and $\int f(x, y) \, d\nu(y)$ is $\mathscr{X}$-measurable in x.

Remark 1.6. The proof slots together the three convergence theorems of Lecture 6 with the indicator case: the identity is immediate for indicators (**??**), extends by linearity to simple functions, extends by Monotone Convergence to non-negative measurable f (Tonelli), and finally extends to integrable $f = f^+ - f^-$ by splitting positive and negative parts (Fubini). Both halves are needed in practice: Tonelli to *check* integrability by computing one of the iterated integrals of $|f|$; Fubini to then *swap* the order in the actual integral.

Remark 1.7. By induction the theorem extends to any finite product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ of σ-finite spaces. The infinite product story is genuinely different: a countably infinite product of σ-finite spaces need not be σ-finite, but a countable product of *probability* spaces is again a probability space. We will return to this when constructing stochastic processes.