Lecture 1:

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Measure: $(\Omega, \mathcal{F})$ ← Measurable Space

a measure $\mu : \mathcal{F} \to \mathbb{R}^+$

1. $\mu(\emptyset) = 0$
2. μ is countably additive

Note: There are also signed measures that return $\mathbb{R}$.

for a pairwise disjoint collection $\{A_i\}_{i=1}^{\infty}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Lecture 1: What is a Measure?

09/03/26

How is $\mathbb{R}^n$ related to its norm?

Ω sample space.

↓

$\mathbb{R}$ or $\mathbb{R}^2$

$A \subset \Omega A := \{x \in \Omega : x \in A\}$

$\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$, $\emptyset$

$A^c \subset \Omega$ $A^c := \{x \in \Omega : x \notin A\}$

$A^c = \Omega / A$

set difference

$A^c = \Omega \setminus A$

union:

symmetric difference is the same thing as XOR.

For a collection of sets: $\{A_i\}_{i=1}^{\infty}$ they are pairwise disjoint if $A_i \cap A_j = \emptyset$, $i \neq j$.

Special Cases:

$(\Omega, \mathcal{F}, \mu)$

Measure space (not be confused with measurable space)

- if $\mu(\Omega) = 1$ then we say that $(\Omega, \mathcal{F}, \mu)$ is a probability space and μ is a probability measure
- if $\mu(\Omega) < \infty$ then μ is a finite measure

• if $\Omega = \bigcup_{i=1}^{\infty} A_i$ s.t. $\mu(A_i) < \infty \quad \forall i$

then μ is a σ-finite measure

Example: $\Omega = \mathbb{R}$, $\mu([a,b]) = b-a$

Then $\mathbb{R} = \bigcup_{i=1}^{\infty} [i-1, i] \cup [-i, -i+1]$

Notation: $\mathcal{P}(\Omega)$ = power set.

$\Rightarrow$ set of all subsets of Ω

Note: $\mathcal{F} \subset \mathcal{P}(\Omega)$ and power set is often too big to be useful.

Example: (counting measure) $\Omega = \{1, 2, \dots n\}$ Then $\mathcal{P}(\Omega) = 2^{\Omega}$ is the set of possible subsets of Ω (hence the notation 2^{Ω}, as from 2^n subsets)

• $\mu(\{1, 3, 7\}) = 3$, $\mu(\Omega) = n$

• define $\nu(A) = \frac{1}{n}\mu(A)$

In contrast, consider the binomial (n,p) distribution, each point gets a measure of $\binom{n}{i} p^i (1-p)^{n-i}$ for $p \in (0,1)$

σ-field (σ-algebra)

→ what sets (subsets of Ω) am I allowed to measure.

Definition 1: For some set Ω, a σ-field $\mathcal{F}$ is a collection of sets $A \subseteq \Omega$ s.t.

countable dense

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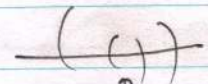


0 ↑ 1

sub-intervals

(1) $\phi, \Omega \in \mathcal{F}$

(2) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

(3) for a countable collection of sets $\{A_i\}_{i=1}^{\infty}$ s.t. $A_i \in \mathcal{F}$ for ~~$i=1 \ldots \infty$~~ all $i \in \mathbb{N}$

Then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

consequence: combining (2) and (3) gives you countable intersections.

(3*) $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

why? $\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$

Lecture 2: Existence Caratheodory Extension Theorem 10/03/26

Idea: if the measure of $(a,b]$ is $b-a$ for any $b>a$

Then, what else can we measure?

Start: The set of all such intervals $(a,b]$ is not a σ-field.

(they are a semi-ring)

Definition: (semi-ring)

$\mathcal{A}$ is a collection of subsets of Ω, is called a semi-ring when

- $\emptyset \in \mathcal{A}$
- for any $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$ and

$$B \setminus A = \bigcup_{i=1}^{n} C_i \quad \text{for } C_i \in \mathcal{A}$$

"the set difference might not be in $\mathcal{A}$, but could be created with a finite union of sets from $\mathcal{A}$" (C_i)

Example: $\mathcal{A}$ all intervals $(a,b]$ is a semi-ring

Defn: (Ring) $\mathcal{A}$, a collection of subsets of Ω, is called a Ring when

- $\emptyset \in \mathcal{A}$
- for any $A, B \in \mathcal{A}$
    - $B \setminus A \in \mathcal{A}$
    - $A \cup B \in \mathcal{A}$ (finite union)

Example: all finite unions of half open intervals e.g. $(a,b] \cup (c,d] \in \mathcal{A}$

NOT countable union ↳ which would take us to sigma fields!

Definition: (Field)

$\mathcal{A}$ is a field if it is a Ring and $\Omega \in \mathcal{A}$

note: field + countable unions = σ field.

Defn: (Set function)

$\mu : \mathcal{A} \to \mathbb{R}^+$ (not nec. a measure)

for $A, B \in \mathcal{A}$, we say that

- μ is increasing if $A \subset B \Rightarrow \mu(A) \leq \mu(B)$
- μ is additive if for A, B disjoint then $\mu(A \cup B) = \mu(A) + \mu(B)$
- μ is countably additive if for $\{A_i\}_{i=1}^{\infty}$ s.t. $A_i \cap A_j = \emptyset \quad \forall i \neq j$ (i.e pairwise disjoint) and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

  Then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

- μ is countably sub-additive if $\{A_i\}_{i=1}^{\infty}$ are in $\mathcal{A}$, and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

  (not necessarily pairwise disjoint)

  Then $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Often, a set function on μ on a ring $\mathcal{A}$ s.t. $\mu(\emptyset) = 0$ and μ countably additive, then we say that μ is a pre-measure.

For a pre-measure μ on some ~~$\mathcal{A}$~~ Ring $\mathcal{A}$
we have an outer measure.

$$\mu^*(E) = \inf \sum_i \mu(A_i), \text{ for any } E \subseteq \Omega$$

where the inf is taken over all collections (finite or countable) of A_i s.t. $E \subset \bigcup_{i=1} A_i$

Question: Can we "measure" (outer measure) any $E \subset \Omega$?

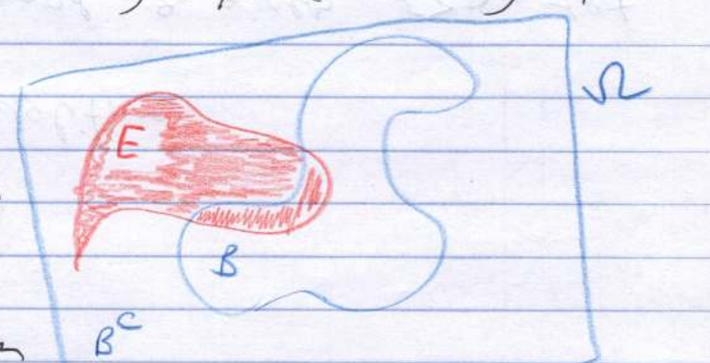
Not necessarily.

Denote (script m) $\mathcal{M}$ to be the collection of all μ^* measurable sets.

where we say that $B \subseteq \Omega$ is μ^* measurable when

$$\mu^*(E \cap B) + \mu^*(E \cap B^c) = \mu^*(E)$$

for all $E \subseteq \Omega$



Theorem: (Carathéodory Extension Theorem)

Let $\mathcal{A}$ be a Ring on Ω and μ be a pre-measure, then μ extends to a measure on $\sigma(\mathcal{A})$.

Note: $\sigma(\mathcal{A})$ is the σ-field that comes from extending $\mathcal{A}$ by including countable unions and Ω itself.

Note: The "correct" extension is the outer measure.

λ is for Lebesgue measure

Proof: Assume $B \subseteq \Omega$ and $\mu^*(B) < \infty$

(Step 1) Prove stuff about μ^*

- $\mu^*(\emptyset) = 0$ as μ is a pre-measure
- μ^* is non-negative $\forall B \subseteq \Omega$ as μ is non-neg
- μ^* is monotone (increasing)

  Let $B_1, B_2 \subseteq \Omega$ ~~$\in \mathcal{A}$~~ and $B_1 \subset B_2$ then for any $\{A_i\}$ s.t. $B_2 \subseteq \bigcup_i A_i$

  Then $B_1 \subseteq \bigcup_i A_i$

$\therefore \mu^*(B_1) \leq \mu^*(B_2)$

- μ^* is countably sub additive

  For $\{B_i\}_{i=1}^{\infty}$ and a given $\varepsilon > 0$, let

  $B_i \subseteq \bigcup_j A_{ij}$ for $A_{ij} \in \mathcal{A}$ s.t.

$$\sum_j \mu(A_{ij}) \leq \mu^*(B_i) + \varepsilon 2^{-i}$$

As $\bigcup_{i=1}^{\infty} B_i \subseteq \bigcup_{i,j} A_{ij}$ and as μ^* is monotone and μ is sub-additive, ~~then~~ then

$$\mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \mu\left(\bigcup_{i,j} A_{ij}\right)$$

$$\leq \sum_{ij} \mu(A_{ij})$$

$$\leq \sum_i \mu^*(B_i) + \varepsilon$$

Take $\varepsilon \to 0$ to get that μ^* is countably subadditive

μ is nicer to work than μ^*

outer measure is defined in terms of inf.

(Step 2) check that μ and μ^* coincide for all $A \in \mathcal{A}$

For any $A \in \mathcal{A}$ we have $\mu^*(A) \leq \mu(A)$ because $A \subseteq A$

For the reverse, if $A \subset \bigcup_i A_i$, then by countable sub-additivity and monotonicity

$$\mu(A) \leq \sum_i \mu(A \cap A_i) \leq \sum_i \mu(A_i)$$

$$\therefore \mu(A) \leq \mu^*(A)$$

$$\therefore \mu(A) = \mu^*(A) \quad \forall A \in \mathcal{A}$$

(Step 3) check that $\mathcal{A} \subset \mathcal{M}$

i.e. for any $A \in \mathcal{A}$ we want to show that A is μ^*-measurable.

$$\text{i.e. } \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subseteq \Omega$$

Note $\mu^*(E \cap A) + \mu^*(E \cap A^c) \geq \mu^*(E)$

as μ^* is sub-additive

Next, for some $\varepsilon > 0$, choose $\{A_i\}$ s.t.

$$E \subseteq \bigcup_i A_i \text{ and } \sum_i \mu(A_i) \leq \mu^*(E) + \varepsilon$$

Furthermore, $E \cap A \subseteq \bigcup_i (A \cap A_i)$

$E \cap A^c \subseteq \bigcup_i (A^c \cap A_i)$

Thus, $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq$

$$\leq \sum_i \mu(A \cap A_i) + \sum_i \mu(A \cap A_i)$$

$$= \sum_i \mu(A_i) \leq \mu^*(E) + \varepsilon$$

and take $\varepsilon \to 0$.

collection of all μ^* measurable sets

(Step 4) Show that $\mathcal{M}$ is a σ-field

- $\emptyset \in \mathcal{M}$ since $\emptyset \in \mathcal{A}$
- $\Omega \in \mathcal{M}$ since for any $E \subseteq \Omega$ (a sigma-field requires Ω to be in it).

$\mu^*(E \cap \Omega) + \mu^*(E \cap \emptyset) = \mu^*(E)$

Next since, $A \cap B = (A^c \cup B^c)^c$, we will show $\mathcal{M}$ is closed under intersections

For $B_1, B_2 \in \mathcal{M}$ and any $E \subseteq \Omega$ $\mu^*(E) = \mu^*(B_1 \cap E) + \mu^*(B_1^c \cap E)$

$= \mu^*(B_2 \cap B_1 \cap E) + \mu^*(B_2 \cap B_1^c \cap E) + \mu^*(B_2^c \cap B_1 \cap E) + \mu^*(B_2^c \cap B_1^c \cap E)$

(via subadditivity)

$\geq \mu^*(B_2 \cap B_1 \cap E) + \mu^*(\{B_2 \cap B_1^c \cap E\} \cup \{B_2^c \cap B_1 \cap E\} \cup \{B_2^c \cap B_1^c \cap E\})$

$= \underbrace{\mu^*(\{B_2 \cap B_1\} \cap E) + \mu^*(\{B_2 \cap B_1\}^c \cap E)}_{\to\ \mu^*(E)}$

$\geq \mu^*(E)$

$\therefore B_1 \cap B_2 \in \mathcal{M}$

Lastly, note that $B \setminus A = B \cap A^c$

$\therefore$ we need to show that $\mathcal{M}$ is closed under complementation because $\mu^*(E \cap B^c) + \mu^*(E \cap (B^c)^c) = \mu^*(E)$

$\therefore \mathcal{M}$ is a field.

To get to a σ-field, let $\{B_i\}$ in $\mathcal{M}$ be countable and pairwise disjoint and $\cup_i B_i \in \mathcal{M}$. Let $B = \bigcup_{i=1}^{\infty} B_i$

Then, $\mu^*(E) = \mu^*(E \cap B_1) + \mu^*(E \cap B_1^c)$

$$= \mu^*(E \cap B_1) + \mu^*(E \cap B_2) + \mu^*(E \cap B_1^c \cap B_2^c)$$

$$\mu^*(E) = \sum_{i=1}^{n} \mu^*(E \cap B_i) + \mu^*(E \cap \{\bigcap_{i=1}^{n} B_i^c\})$$

Then, by monotonicity, subadditivity and $n \to \infty$ we get

$$\mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*(E \cap B^c)$$

$$\geq \mu^*(E \cap B) + \mu^*(E \cap B^c)$$

$$\geq \mu^*(E)$$

$\therefore$ $\mathcal{M}$ is closed under countable unions

choose $E = B$ and

$$\therefore \mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i)$$

$\therefore$ μ^* countably additive.

Conclusion: μ^* is a set function $P(\Omega) \to \mathbb{R}^+$, but it's also a measure on $\mathcal{M}$ and since $\mathcal{A} \subset \mathcal{M}$ then $\sigma(\mathcal{A}) \subseteq \mathcal{M}$

Lastly, μ^* is a measure on $\mathcal{M}$ it is also a measure on $\sigma(\mathcal{A})$

Note: $\sigma(\mathcal{A}) \subset \mathcal{M}$ (sometimes)

Borel?

11/03/26

Lecture 3: Uniqueness and Dykin's Pi-~~s~~ lambda system.

Question: μ_1, μ_2 on $\sigma(\mathcal{A})$

if $\mu_1(A) = \mu_2(A)$ $\forall A \in \mathcal{A}$

Does $\mu_1(B) = \mu_2(B)$ $\forall B \in \sigma(\mathcal{A})$

Def (π-system)

a collection of sets $\mathcal{A}$ is a π-system if for any $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$

Def (λ-system)

a collection of set $\mathcal{L}$ is a λ-system

- $\Omega \in \mathcal{L}$
- $A, B \in \mathcal{L}$ s.t. $A \subset B$ then $B \setminus A \in \mathcal{L}$
- $\{A_i\}_{i=1}^{\infty}$ pairwise disjoint, $A_i \in \mathcal{L}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$

Similar to σ-field.

Note: a field is a π-system

Theorem (uniqueness of extension)

Let μ_1, μ_2 be σ-finite measures on $\sigma(\mathcal{A})$ where $\mathcal{A}$ is a π-system. Then, if $\mu_1(A) = \mu_2(A)$ $\forall A \in \mathcal{A}$ then μ_1 and μ_2 agree on $\sigma(\mathcal{A})$

Theorem (Dynkin π-λ theorem)

Let $\mathcal{A}$ be a π system, $\mathcal{L}$ be a lambda system and $\mathcal{A} \subset \mathcal{L}$. Then $\sigma(\mathcal{A}) \subset \mathcal{L}$

Proof: Let $\mathcal{L}_0$ be the smallest λ-system such that $\mathcal{A} \subset \mathcal{L}_0$, then $\mathcal{L}_0 \subseteq \mathcal{L}$

Goal is to show that $\mathcal{L}_0$ is also a π-system and a collection of sets that is both a π-system and a λ-system is a σ-field. Then we necessarily have that $\sigma(\mathcal{A}) \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$

Show that $\mathcal{L}_0$ is closed under intersections

Let $\mathcal{L}' = \{B \in \mathcal{L}_0 : B \cap A \in \mathcal{L}_0, \forall A \in \mathcal{A}\}$

Then $\mathcal{A} \subset \mathcal{L}'$ as $\mathcal{A}$ is a π-system

Let's show that $\mathcal{L}'$ is also a λ-system.

- $\Omega \in \mathcal{L}'$ because $\mathcal{A} \in \mathcal{L}_0$

- if $B_1, B_2 \in \mathcal{L}'$ s.t. $B_1 \subset B_2$ then for any $A \in \mathcal{A}$ we have that $B_1 \cap A, B_2 \cap A \in \mathcal{L}_0$

Thus $(B_2 \cap A) \setminus (B_1 \cap A) = (B_2 \setminus B_1) \cap A \in \mathcal{L}_0$

$\therefore B_2 \setminus B_1 \in \mathcal{L}'$

- if $\{B_i\}_{i=1}^{\infty} \in \mathcal{L}'$ are pairwise disjoint

then for any $A \in \mathcal{A}$, $A \cap B_i \in \mathcal{L}_0$

$\therefore \bigcup_{i=1}^{\infty} (A \cap B_i) = A \cap \left(\bigcup_{i=1}^{\infty} B_i\right) \in \mathcal{L}_0$

Hence, $\bigcup_{i=1}^{\infty} B_i \in \mathcal{L}'$

(that contains $\mathcal{A}$)

By definition $\mathcal{L}'$, $\mathcal{L}' \subseteq \mathcal{L}_0$, but $\mathcal{L}_0$ is minimal

$\therefore \mathcal{L}_0 = \mathcal{L}'$ $\therefore \mathcal{L}_0$ contains all intersections with elements of $\mathcal{A}$

Lastly, let $\mathcal{L}'' = \{B \in \mathcal{L}_0 : B \cap C \in \mathcal{L}_0 \ \forall C \in \mathcal{L}_0\}$

since $\mathcal{L}_0 = \mathcal{L}'$, $\mathcal{A} \subset \mathcal{L}''$

Then do the same thing we did to $\mathcal{L}'$ to $\mathcal{L}''$ to show that $\mathcal{L}''$ is a λ-system and thus $\mathcal{L}'' = \mathcal{L}_0$

$\therefore \mathcal{L}_0$ is closed under intersections

$\therefore \mathcal{L}_0$ is a σ-field.

$\therefore \sigma(\mathcal{A}) \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$ $\square$

Proof of Uniqueness. (finite measures)

- $\mu_1(\Omega) = \mu_2(\Omega) < \infty$
- Let $\mathcal{L} = \{B \subset \Omega : \mu_1(B) = \mu_2(B)\}$

if $\mathcal{L}$ is a λ-system, we're done!
$\mathcal{A} \subset \mathcal{L} \therefore \sigma(\mathcal{A}) \subset \mathcal{L}$

→ show that $\mathcal{L}$ is a λ-system

- $\Omega \in \mathcal{L}$ (by assumption)
- Next, if $A, B \in \mathcal{L}$ with $A \subset B$ then

$$\mu_1(B \setminus A) + \mu_1(A) = \mu_1(B)$$
$$= \mu_2(B)$$
$$= \mu_2(B \setminus A) + \mu_2(A) \quad < \infty$$

only valid subtraction, because we are restricting to finite measure.

Hence $B \setminus A \in \mathcal{L}$

- $\{A_i\}_{i=1}^{\infty}$ are pairwise disjoint $A_i \in \mathcal{L}$

$$\mu_1\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_1(A_i) = \sum_{i=1}^{\infty} \mu_2(A_i)$$

$$= \mu_2\left(\bigcup_{i=1}^{\infty} A_i\right) < \infty$$

$\therefore \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$

$\Rightarrow \mathcal{L}$ is a λ-system! and $\mathcal{A} \subset \mathcal{L}$

$\therefore$ contains $\sigma(\mathcal{A}) \subset \mathcal{L}$

$\therefore \mu_1 = \mu_2$ for all sets in $\sigma(\mathcal{A})$. $\square$

Proof of uniqueness (σ-finite measure)

- For any $A \in \mathcal{A}$ s.t. $\mu_1(A) = \mu_2(A) < \infty$ we define $\mathcal{L}_A$ to be all $B \subseteq \Omega$ s.t. $\mu_1(A \cap B) = \mu_2(A \cap B)$

Proceeding as in the proof for finite measures we can show that $\mathcal{L}_A$ is a λ-system and $\therefore \sigma(\mathcal{A}) \subset \mathcal{L}_A$ (by Dynkin π-λ)

- By σ-finiteness we decompose $\Omega = \bigcup_{i=1}^{\infty}$, $A_i \in \mathcal{A}$ and $\mu_1(A_i) = \mu_2(A) < \infty$

For any $B \in \sigma(\mathcal{A})$ and any $n \in \mathbb{N}$,

$$\mu_1\left(\bigcup_{i=1}^{n} (B \cap A_i)\right) = \sum_{i=1}^{n} \mu_1(B \cap A_i) - \sum_{i,j} \mu_1(B \cap A_i \cap A_j) + \ldots$$ (by inclusion-exclusion formula)

this also works for μ_2

Since $\mathcal{A}$ is a π-system, $A_i \cap A_j \in \mathcal{A}$ as well as further intersections

$$\therefore \mu_1\left(\bigcup_{i=1}^{n}(B\cap A_i)\right) = \mu_2\left(\bigcup_{i=1}^{n}(B\cap A_i)\right)$$

any $n \in \mathbb{N}$ σ-finite

Let $n \to \infty$ and conclude

$\square$

(write faster!)

<Remark> a π-system is very natural in probability theory as $\cap \equiv$ 'and'

<Remark> without σ-finiteness, uniqueness may fail!

$\Omega = (0,1]$, $\mathcal{A}$ be all finite unions of half open intervals $(a,b]$

- μ is a set function that assigns 0 to $\emptyset$ and ∞ to any non-empty set

$\therefore \mu^*$ assigns ∞ to any subset of Ω that is not $\emptyset$

"require" σ-finiteness

- Counting measure: "counts the # of elements" also assign 0 to $\emptyset$ and ∞ $(a,b]$

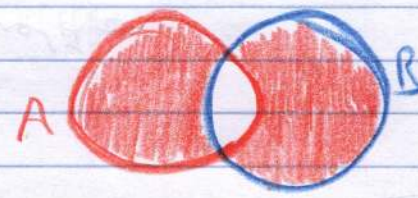
However $\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\} \to 3$

Does not coincide with μ^*

Completeness: We want to "complete" a measure.
If E and A only differ on a set of measure 0, then we want both E and A to be measurable and have the same measure.

Def: Symmetric Difference

For sets A, B, $A \triangle B = (A \setminus B) \cup (B \setminus A)$



For a measure space $(\underbrace{X}_{\text{space}}, \underbrace{\mathcal{F}}_{\substack{\text{sigma field} \\ \subset \mathcal{P}(X)}}, \underbrace{\mu}_{\text{measure}})$

(outer measure)

$$\mu^*(B) = \inf\{\mu(A) : B \subset A, A \in \mathcal{F}\}$$

Then, the μ-null sets are $\mathcal{N}_\mu \subset \mathcal{P}(X)$

s.t. $\mu^*(N) = 0 \ \forall N \in \mathcal{N}_\mu$

We say that a measure space is complete

if $\mathcal{N}_\mu \subset \mathcal{F}$ (Exercise: Show that $\mathcal{N}_\mu$ is a ring)

If $(X, \mathcal{F}, \mu)$ is not complete, we can complete it by replacing $\mathcal{F}$ with

$$\mathcal{F} \vee \mathcal{N}_\mu = \{A \cup N : A \in \mathcal{F}, N \in \mathcal{N}_\mu\}$$

Prop 3.3.2 in Dudley

This completion is equal to $\{B \subseteq X : \exists A \in \mathcal{F} \text{ s.t. } A \triangle B \in \mathcal{N}_\mu\}$

and this is the smallest σ-field that contains both $\mathcal{F}$ and $\mathcal{N}_\mu$

$(X, \mathcal{F} \vee \mathcal{N}_\mu, \bar{\mu})$ where $\bar{\mu}(A \cup N) = \mu(A)$.

Lecture 4: Lebesgue measure 12/03/26

$\Omega = \cancel{[0,1)}$ $(0,1]$ or $\Omega = \mathbb{R}$

we want $\lambda((a,b]) = b - a$

For now, λ is a pre-measure on $\mathcal{A}$ where $\mathcal{A}$ is any finite union of half-open intervals.

$$(a,b] \cup (c,d], \quad a \le b < c < d$$

Claim: $\mathcal{A}$ is a π-system

"pi-system basically means you can do intersections"

$$\Rightarrow (a,b] \cap (c,d] = \begin{cases} \emptyset & \text{if } b<c,\ d<a \\ (c,b] & \text{if } a<c<b<d \end{cases}$$

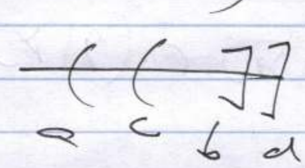
[$\forall$ half-open interval if you intersect, you get another half-open interval]

Claim: $\mathcal{A}$ is a ring

Recall, a ring has $\emptyset$ and is closed under set subtraction and finite unions.

"there are different ways to come up with Lebesgue measure"

- $a=b$ $(a,a] = \emptyset$
- $(a,b] \setminus (c,d]$
- by definition (finite unions condition)



$\mathcal{B}$ = Borel σ-field

$\mathcal{B}(\mathbb{R}) = \sigma(\text{open sets})$

We can generate $\mathcal{B}$ from $\mathcal{A}$. That is, $\mathcal{B} = \sigma(\mathcal{A})$

We also have $\mathcal{M}_\lambda$ = set of all Lebesgue measurable sets.

Question: $\mathcal{B} = \mathcal{M}_\lambda$? NO! $\mathcal{B} \subset \mathcal{M}_\lambda$ "strictly subset"

in fact, $\mathcal{M}_\lambda$ is the completion of $\mathcal{B}$ with $\mathcal{N}_\lambda$ ← Lebesgue Null sets.

Point of Interest:

$\lambda((a,b]) = b-a$ ← lambda is the only measure that does this.

Step 1: λ is a pre-measure on a ring $\mathcal{A}$

Step 2: Carathéodory says that λ is a measure on $\sigma(\mathcal{A}) = \mathcal{B}$

Step 3: $\mathcal{A}$ is a π-system

$\therefore$ For any measure $\mu((a,b]) = b-a$

μ coincides with λ on $\mathcal{B}$

Are there any $A \subset \mathcal{P}((0,1])$ s.t. (powerset)

A is not λ-measurable "Lebesgue"

YES!

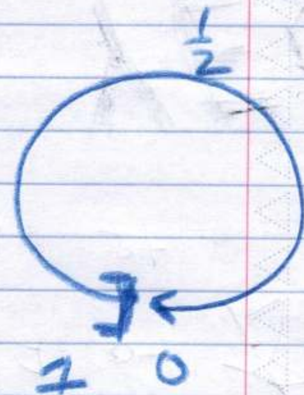
Example: Vitali sets

Example: Vitali set

$\Omega = (0,1]$, for $x, y \in (0,1]$

define addition mod 1

$$x + y = \begin{cases} x+y & \text{if } x+y \le 1 \\ x+y-1 & \text{if } x+y > 1 \end{cases}$$



Define $\mathcal{L}$ to contain all λ-measurable sets $A \subseteq (0,1]$ s.t. $\lambda(A) = \lambda(A+x)$

For any $x \in (0,1]$

where $A+x = \{y \in (0,1] : y-x \in A\}$

$A+x$ means shift A by x

Claim $\mathcal{L}$ is a λ-system

Since $\mathcal{A}$ from above is s.t. $\mathcal{A} \subset \mathcal{L}$

$$\lambda((a,b]) = b-a$$

$$\lambda\left((a,b]+x\right) = \lambda\left((a+x, b+x]\right) = b-a$$

$\therefore \sigma(\mathcal{A}) = \mathcal{B} \subset \mathcal{L}$ (Dynkin π-λ from Lecture 3)

"every Borel set is shift invariant with respect to Lebesgue measure"

Next, we say that $x \sim y$ if $x-y \in \mathbb{Q}$

e.g. $\underbrace{\frac{1}{\sqrt{2}} \text{ and } \frac{1}{\sqrt{2}} + \frac{1}{100}}_{\text{same!}}$

$\therefore$ we can decompose $(0,1]$ into disjoint equivalence classes.

Define $H \subset (0,1]$ s.t. H contains one element from each ~~equi~~ equivalence class.

(we can do this *if* we assume the axiom of choice)

No two points in H are equivalent

i.e. $r_1, r_2 \in \mathbb{Q}$ then $H+r_1 \neq H+r_2$ unless $r_1=r_2$

actually, $(H+r_1) \cap (H+r_2) = \phi$ for $r_1 \neq r_2$

$\therefore (0,1] = \bigcup_{r \in \mathbb{Q}} (H+r)$

$\therefore$ by countable additivity $1 = \lambda((0,1]) = \sum_{r \in \mathbb{Q}} \lambda(H+r)$

but, because λ-measure is translation invariant $\lambda(H+r_1) = \lambda(H+r_2)$

$$1 = \sum \lambda(H)$$

if $\lambda(H) = 0$ then $1=0$ contradiction!

if $\lambda(H) = \infty$ then $1 = \infty$

Conclusion: $\mathcal{M}_\lambda \subset \mathcal{P}((0,1])$

~~Fun~~ Fun Fact: Lebesgue Measure on $\mathbb{R}$ is the only translation invariant measure.

— same for $\mathbb{R}^n$.

— there is no ∞-dimensional Lebesgue measure.

↳ i.e. no translation Invariant Measure (functional analysis)

likelihoods don't exist.

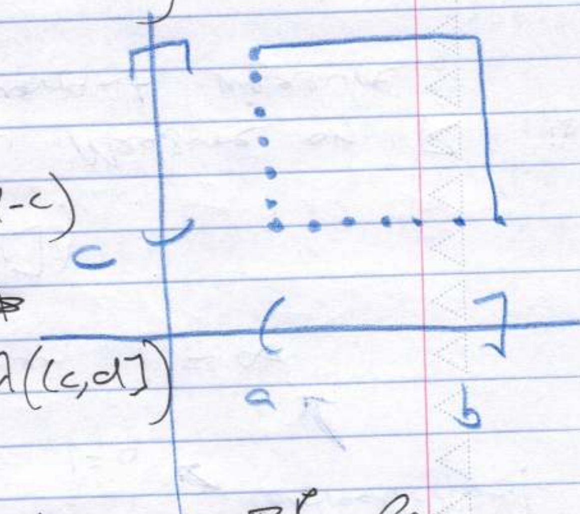
Product Measures.

We construct λ on $\mathbb{R}$ by considering the intervals $(a,b]$

For $\mathbb{R}^p$ we can do the same thing with p-dimensional rectangles

e.g. $(a,b] \times (c,d] \subset \mathbb{R}^2$

$$\lambda^{(2)}\Big((a,b] \times (c,d]\Big) = (b-a)\times(d-c)$$
$$= \cancel{\lambda((a,b]) \times}$$
$$\lambda((a,b])\,\lambda((c,d])$$



Note: the set of such rectangle in $\mathbb{R}^p$ form a π-system

Generally: For two measure spaces $(\underbrace{\mathbb{X}}_{\text{space}}, \underbrace{\mathcal{X}}_{\sigma\text{-field on } \mathbb{X}}, \underbrace{\mu}_{\text{measure}})$

$(\mathbb{Y}, \mathcal{Y}, \nu)$ we can define

$(\mathbb{X} \times \mathbb{Y}, \mathcal{X} \times \mathcal{Y}, \pi)$ where $\pi(A \times B) = \mu(A)\nu(B)$

for $A \in \mathcal{X}$ and $B \in \mathcal{Y}$

Question: How are these related?

↳ $\mathcal{B}(\mathbb{X}) \times \mathcal{B}(\mathbb{Y})$

↳ $\mathcal{B}(\mathbb{X} \times \mathbb{Y})$

From Dudley (4.1.7), $\mathcal{B}(\mathbb{X}) \times \mathcal{B}(\mathbb{Y}) \subset \mathcal{B}(\mathbb{X} \times \mathbb{Y})$

but these are "usually" equal to each other.

(↳ more precisely, equality occurs when the measurable spaces $(\mathbb{X}, \mathcal{X})$, $(\mathbb{Y}, \mathcal{Y})$ are "second countable") actually probability

Independence

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space

Def: (Independence for sets)

For a countable collection of sets $A_i, i \in I$, we say that this collection is independent if for all finite $J \subset I$, we have

$$\mu\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} \mu(A_j)$$

Example: $A_1 = \{$draw a red card$\}$

$A_2 = \{$draw a heart or club$\}$

$A_3 = \{$draw a queen$\}$

$\mu(A_1) = \frac{1}{2}$ $\quad \mu(A_1 \cap A_2) = \frac{1}{4} \quad (=\frac{1}{2} \times \frac{1}{2})$

$\mu(A_2) = \frac{1}{2}$ $\quad \mu(A_1 \cap A_3) = \frac{1}{26} \quad (=\frac{1}{2} \times \frac{1}{13})$

$\mu(A_3) = \frac{1}{13}$ $\quad \mu(A_2 \cap A_3) = \frac{1}{26}$

$$\mu(A_1 \cap A_2 \cap A_3) = \frac{1}{52}$$

Definition (independent sigma fields)

For a countable collection of σ-fields $\mathcal{F}_i \subset \mathcal{F}$, $i \in I$, we say the collection is independent if any set of sets $\{A_i \in \mathcal{F}_i : i \in I\}$ is independent

Theorem: Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{F}$ be π-systems.

If $\mu(A_1 \cap A_2) = \mu(A_1)\mu(A_2)$

for any $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, then $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent

Proof: Fix an $A_1 \in \mathcal{A}_1$ and define two measures for $B \in \mathcal{F}$

$$\nu_1(B) = \mu(A_1 \cap B)$$

$$\nu_2(B) = \mu(A_1)\,\mu(B)$$

By assumption, $\nu_1(A_2) = \nu_2(A_2)$

for any $A_2 \in \mathcal{A}_2$

Then by uniqueness of extension ν_1 and ν_2 have to coincide on $\sigma(\mathcal{A}_2)$.

$$\therefore \mu(A_1 \cap B_2) = \mu(A_1)\,\mu(B_2)$$

for any $B_2 \in \sigma(\mathcal{A}_2)$

Now do the same argument again, but by fixing some $B_2 \in \sigma(\mathcal{A}_2)$ to get that $\mu(B_1 \cap B_2) = \mu(B_1)\mu(B_2)$

for any $B_i \in \sigma(\mathcal{A}_i)$, $i = 1, 2$.

$\square$