Advanced Linear Algebra – Week 4

Isomorphisms and Properties of Linear Transformations

This week we will learn about:

- Invertibility of linear transformations,
- Isomorphisms,
- Properties of linear transformations, and
- Non-integer powers of linear transformations.

Extra reading and watching:

- Sections 1.2.4 and 1.3.1 in the textbook
- Lecture videos 13, 14, 15, and 16 on YouTube
- Definition and Examples of Isomorphisms at WikiBooks
- Isomorphism at Wikipedia (be slightly careful – this page talks about isomorphisms on a broader context than just linear algebra)

Extra textbook problems:

- ⋆ 1.2.4(i,j), 1.3.1, 1.3.4(a–c), 1.3.5
- ⋆⋆ 1.2.10, 1.2.13–1.2.15, 1.2.17, 1.2.24, 1.2.25, 1.3.6
- ⋆⋆⋆ 1.2.19, 1.2.21, 1.2.33
- ☠ none this week

This week, we look at several important properties of linear transformations that you already saw for matrices back in introductory linear algebra. Thanks to standard matrices, all of these properties can be computed or determined using methods that we are already familiar with.

Invertibility of Linear Transformations

A linear transformation $T : \mathcal{V} \to \mathcal{W}$ is called **invertible** if there exists a linear transformation $T^{-1} : \mathcal{W} \to \mathcal{V}$ such that

$T^{-1}(T(\vec{v})) = \vec{v}$ and $T(T^{-1}(\vec{w})) = \vec{w}$ for all $\vec{v} \in V$, $\vec{w} \in W$.

That is, $T^{-1} \circ T = \underbrace{I_V}_{\text{identity on } V}$ and $T \circ T^{-1} = \underbrace{I_W}_{\text{identity on } W}$.

The following theorem shows us that we can find the inverse of a linear transformation (if it exists) simply by inverting its standard matrix.

Theorem 4.1 — Invertibility of Linear Transformations

Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation between n-dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$, which have bases B and D, respectively. Then T is invertible if and only if the matrix $[T]_{D\leftarrow B}$ is invertible. Furthermore,

$$([T]_{D\leftarrow B})^{-1} = \left[T^{-1}\right]_{B\leftarrow D}.$$

Proof. For the "only if" direction, note that if T is invertible then we have

$$[T^{-1}]_{B\leftarrow D}[T]_{D\leftarrow B} \underset{\text{Theorem from Week 3}}{=} [T^{-1}\circ T]_B = [I_V]_B = I.$$

If two matrices multiply to I, they are inverses.

$\therefore$ $[T]_{D\leftarrow B}$ is invertible and $([T]_{D\leftarrow B})^{-1} = [T^{-1}]_{B\leftarrow D}$.

The "if" direction is similar – try on your own! ∎

Example. *Compute* $\int x^2 e^{3x}\,dx$. *(wait, what course is this?)*

Could do integration by parts twice, or...

Let $B = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ and $\mathcal{V} = \mathrm{span}(B)$.

this is linearly independent (check!)

Let $D : \mathcal{V} \to \mathcal{V}$ be the differentiation map.

$D(e^{3x}) = 3e^{3x}$, $D(xe^{3x}) = e^{3x} + 3xe^{3x}$, $D(x^2e^{3x}) = 2xe^{3x} + 3x^2e^{3x}$

$[D(e^{3x})]_B = (3,0,0)$, $[D(xe^{3x})]_B = (1,3,0)$, $[D(x^2e^{3x})]_B = (0,2,3)$.

$$\therefore\ [D]_B = \Big[\, [D(e^{3x})]_B \;\Big|\; [D(xe^{3x})]_B \;\Big|\; [D(x^2e^{3x})]_B \,\Big] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then $$[D^{-1}]_B = [D]_B^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \frac{1}{27}\begin{bmatrix} 9 & -3 & 2 \\ 0 & 9 & -6 \\ 0 & 0 & 9 \end{bmatrix},$$ so

$$\left[\int x^2e^{3x}\,dx\right]_B = \left[D^{-1}(x^2e^{3x})\right]_B = \frac{1}{27}\begin{bmatrix} 9 & -3 & 2 \\ 0 & 9 & -6 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 2 \\ -6 \\ 9 \end{bmatrix}.$$

$$\therefore \int x^2e^{3x}\,dx = \frac{1}{27}\left(2e^{3x} - 6xe^{3x} + 9x^2e^{3x}\right) + C$$

Be careful: Differentiation is usually not an invertible transformation (why not?). The only reason it was invertible in the previous example was because we were able to choose the vector space $\mathcal{V}$ to not have any constant functions in it.

All of our methods of checking invertibility of matrices carry over straightforwardly to linear transformations on finite-dimensional vector spaces. For example...

if $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional with $\dim(\mathcal{V}) = \dim(\mathcal{W})$ then $T : \mathcal{V} \to \mathcal{W}$ is invertible if and only if $T(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$.

Isomorphisms

Recall that every finite-dimensional vector space $\mathcal{V}$ has a basis B, and we can use that basis to represent a vector $\mathbf{v} \in \mathcal{V}$ as a coordinate vector $[\mathbf{v}]_B \in \mathbb{F}^n$, where $\mathbb{F}$ is the ground field. We used this correspondence between $\mathcal{V}$ and $\mathbb{F}^n$ to motivate the idea that...

$\mathcal{V}$ and $\mathbb{F}^n$ are "the same": we can solve linear algebra problems in $\mathcal{V}$ by working in $\mathbb{F}^n$.

We now make this idea of vector spaces being "the same" a bit more precise and clarify under exactly which conditions this "sameness" happens.

Definition 4.1 — Isomorphisms

Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces over the same field. We say that $\mathcal{V}$ and $\mathcal{W}$ are **isomorphic**, denoted by $\mathcal{V} \cong \mathcal{W}$, if there exists an invertible linear transformation $T : \mathcal{V} \to \mathcal{W}$ (called an **isomorphism** from $\mathcal{V}$ to $\mathcal{W}$).

The idea behind this definition is that if $\mathcal{V}$ and $\mathcal{W}$ are isomorphic then they have the same structure as each other—the only difference is the label given to their members ($\mathbf{v}$ for the members of $\mathcal{V}$ and $T(\mathbf{v})$ for the members of $\mathcal{W}$).

Example. *Show that $\mathcal{M}_{1,n}$ and $\mathcal{M}_{n,1}$ are isomorphic.*

Let $T : \mathcal{M}_{1,n} \to \mathcal{M}_{n,1}$ be the transpose map. T is clearly invertible (its inverse is... the transpose map), so $\mathcal{M}_{1,n} \cong \mathcal{M}_{n,1}$.

Similarly, $\mathcal{M}_{1,n}$ and $\mathcal{M}_{n,1}$ are both isomorphic to... $\mathbb{F}^n$.

$[v_1 \ v_2 \ \cdots \ v_n] \in \mathcal{M}_{1,n}$ $\qquad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathcal{M}_{n,1}$ $\qquad (v_1, v_2, \ldots, v_n) \in \mathbb{F}^n$.

This is why we often treat row vectors ($\mathcal{M}_{1,n}$), column vectors ($\mathcal{M}_{n,1}$), and vectors ($\mathbb{F}^n$) the same.

Example. *Show that $\mathcal{P}^3$ and $\mathbb{R}^4$ are isomorphic.*

Define $T : \mathcal{P}^3 \to \mathbb{R}^4$ by $T(a+bx+cx^2+dx^3) = (a,b,c,d)$.
Clearly invertible, with $T^{-1}(a,b,c,d) = a+bx+cx^2+dx^3$.

That's it! Since we have found an invertible linear transformation, we are done.

More generally, we have the following theorem that pins down the idea that every finite-dimensional vector space "behaves like" $\mathbb{F}^n$:

Theorem 4.2 — Isomorphisms of Finite-Dimensional Vector Spaces

Suppose $\mathcal{V}$ is an n-dimensional vector space over a field $\mathbb{F}$. Then $\mathcal{V} \cong \mathbb{F}^n$.

Proof. Pick some basis B of $\mathcal{V}$ and consider the function $T : \mathcal{V} \to \mathbb{F}^n$ defined by...

$T(\vec{v}) = [\vec{v}]_B$.

We noted earlier that $\underbrace{[\vec{v}+\vec{w}]_B}_{T(\vec{v}+\vec{w})} = \underbrace{[\vec{v}]_B + [\vec{w}]_B}_{T(\vec{v})+T(\vec{w})}$ and $[c\vec{v}]_B = c[\vec{v}]_B$ for all $\vec{v},\vec{w} \in V$ and $c \in \mathbb{F}$, so T is a linear transformation.

It's invertible because $\dim(V) = \dim(\mathbb{F}^n)$ and $T(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$ (since $[\vec{v}]_B = \vec{0}$ means $\vec{v} = 0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_n = \vec{0}$). ■

It is straightforward to check that if $\mathcal{V} \cong \mathcal{W}$ and $\mathcal{W} \cong \mathcal{X}$ then $\mathcal{V} \cong \mathcal{X}$. We thus get the following immediate corollary of the above theorem:

Two finite-dimensional vector spaces over the same field are isomorphic iff they have the same dimension.

Properties of Linear Transformations

Now that we know we can think of arbitrary linear transformations (on finite-dimensional vector spaces) as matrices, we can apply all of our machinery from the previous course to them. For example, we can talk about the eigenvalues, range, null space, and rank of a linear transformation, and the definitions are just "what you would expect":

$\mathrm{range}(T) = \{T(\vec{v}) : \vec{v} \in V\}$ ← subspace of W (if $T : V \to W$)

$\mathrm{rank}(T) = \dim(\mathrm{range}(T))$

$\mathrm{null}(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$ ← subspace of V

λ is an eigenvalue of T with corresponding eigenvector $\vec{v}$ if $T(\vec{v}) = \lambda\vec{v}$ (and $\vec{v} \neq \vec{0}$).

Furthermore, these properties can all be computed from the standard matrix.

Example. *Find the eigenvalues of the transposition map $T : \mathcal{M}_2 \to \mathcal{M}_2$, as well as a set of corresponding eigenvectors.*

Method 1: From definition.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $T(A) = \lambda A$ then $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$,

so $a = \lambda a$, $b = \lambda c$, $c = \lambda b$, $d = \lambda d$.

Two possibilities: $\lambda = 1$ and $b = c$, or

eigenvalues → $\lambda = -1$ and $a = d = 0$, $b = -c$. ← corresponding eigenvectors

Method 2: From standard matrix.

If $E = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$ then $[T]_E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Has eigenvalues $1, 1, 1, -1$.

Method 3: From another standard matrix.

If B is Pauli basis then $[T]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. ← eigenvalues

Example. *Find the range and rank of the differentiation map $D : \mathcal{P}^3 \to \mathcal{P}^3$.*

Method 1: From definition.

- $\text{range}(D) = \{D(a+bx+cx^2+dx^3) : a,b,c,d \in \mathbb{R}\}$
$= \{b+2cx+3dx^2 : b,c,d \in \mathbb{R}\} = \mathcal{P}^2$
- $\text{rank}(D) = \dim(\text{range}(D)) = \dim(\mathcal{P}^2) = 3$

Method 2: From standard matrix.

If $E = \{1, x, x^2, x^3\}$ then $[D]_E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

- $\text{range}([D]_E) = \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$
- $\text{rank}([D]_E) = 3$

Application: Diagonalization and Square Roots

Recall from introductory linear algebra that we can diagonalize many matrices. That is, for many $A \in \mathcal{M}_n$ we can write...

$A = PDP^{-1}$, where $D \in \mathcal{M}_n$ is diagonal and
$P \in \mathcal{M}_n$ is invertible. eigenvalues of A as diagonal entries
has eigenvectors of A as its columns

Doing so lets us easily take arbitrary (even non-integer) powers of matrices:

$$A^r = PD^rP^{-1}$$

where D^r can simply be computed entrywise.

Thanks to standard matrices, we can now do the same thing for most linear transformations. We illustrate what we mean via an example.

Example. *Find a square root of the transpose map acting on $\mathcal{M}_2$.*

We want $S:\mathcal{M}_2 \to \mathcal{M}_2$ such that $S^2 = T$. ← transpose

How about $S = T^{\frac{1}{2}}$?

Recall: If $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ then $[T]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$[T]_B$ is already diagonal, so $[T]_B^{\frac{1}{2}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$\therefore$ $S = T^{\frac{1}{2}}$ is the linear transformation for which

$S\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $S\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $S\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

and $S\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Equivalently, $S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & \frac{(1+i)b + (1-i)c}{2} \\ \frac{(1-i)b + (1+i)c}{2} & d \end{bmatrix}$.

As perhaps an even more striking example, recall from last week that we could take powers of the standard matrix of the derivative to compute (for example) the fourth derivative of a function. If we use this method based on diagonalization to take non-integer powers of the standard matrix, we can compute *fractional* derivatives!

Example. *Compute the half-derivative of $\sin(x)$ and $\cos(x)$. Then find a formula for the r-th derivative of these functions for arbitrary (not necessarily integer) $r \in \mathbb{R}$.*

Let $B = \{\sin(x), \cos(x)\}$ and $V = \mathrm{span}(B)$.

Let $D: V \to V$ be the differentiation map.

Then... $D(\sin(x)) = \cos(x)$, $D(\cos(x)) = -\sin(x)$, so

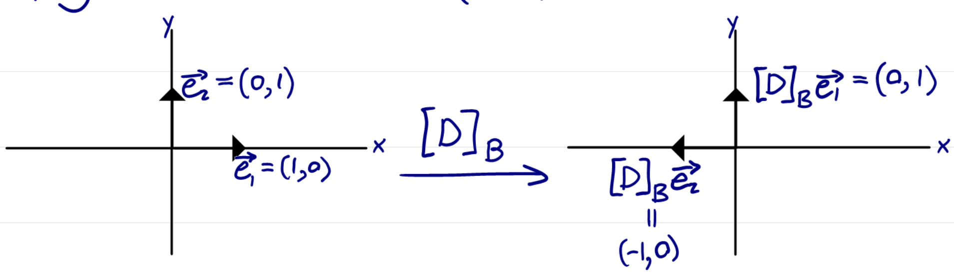
$[D(\sin(x))]_B = (0,1)$, $[D(\cos(x))]_B = (-1,0)$.

$\therefore [D]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which has eigenvalues $\pm i$ with corresponding eigenvectors $(1, \mp i)$.

Then $[D]_B = PSP^{-1}$, where $S = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$,
so $[D]_B^{\frac{1}{2}} = P \begin{bmatrix} i^{\frac{1}{2}} & 0 \\ 0 & (-i)^{\frac{1}{2}} \end{bmatrix} P^{-1} = P \begin{bmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{bmatrix} P^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

The half-derivative of $\sin(x)$ is thus $\frac{1}{\sqrt{2}}(\sin(x) + \cos(x))$, and the half-derivative of $\cos(x)$ is $\frac{1}{\sqrt{2}}(-\sin(x) + \cos(x))$.

<u>Another way:</u>
Notice that $[D]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix of the rotation counter-clockwise by an angle of $\frac{\pi}{2}$ (90°):



Therefore, $[D]_B^r$ is the rotation by angle $\frac{\pi r}{2}$:
$[D]_B^r = \begin{bmatrix} \cos(\frac{\pi r}{2}) & -\sin(\frac{\pi r}{2}) \\ \sin(\frac{\pi r}{2}) & \cos(\frac{\pi r}{2}) \end{bmatrix}$, so

$$D^r(\sin(x)) = \cos(\tfrac{\pi r}{2})\sin(x) + \sin(\tfrac{\pi r}{2})\cos(x) \quad \left(= \sin(x + \tfrac{\pi r}{2})\right)$$
$$D^r(\cos(x)) = -\sin(\tfrac{\pi r}{2})\sin(x) + \cos(\tfrac{\pi r}{2})\cos(x) \quad \left(= \cos(x + \tfrac{\pi r}{2})\right)$$