# Eigenvalues and Eigenvectors

This week we will learn about:

- Complex numbers,
- Eigenvalues, eigenvectors, and eigenspaces,
- The characteristic polynomial of a matrix, and
- Algebraic and geometric multiplicity.

Extra reading:

- Section 3.3 in the textbook
- Lecture videos 37, 38, and 39 on YouTube
- Complex number at Wikipedia
- Eigenvalues and eigenvectors at Wikipedia

Extra textbook problems:

⋆ 3.3.1, 3.3.2

⋆⋆ 3.3.3, 3.3.5, 3.3.7, 3.3.9, 3.3.16, 3.3.20

⋆⋆⋆ 3.3.6, 3.3.11–3.3.14

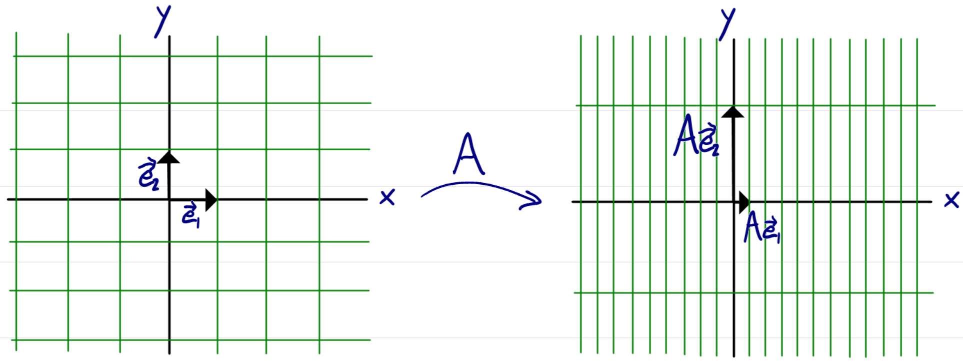
☠ 3.3.19, 3.3.23, 3.3.24

# Eigenvalues and Eigenvectors

Some linear transformations behave very well when they act on certain specific vectors. For example, diagonal matrices behave very well on the standard basis vectors:

$$A = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}$$

$$A\vec{e}_1 = \tfrac{1}{3}\vec{e}_1$$
$$A\vec{e}_2 = 2\vec{e}_2$$



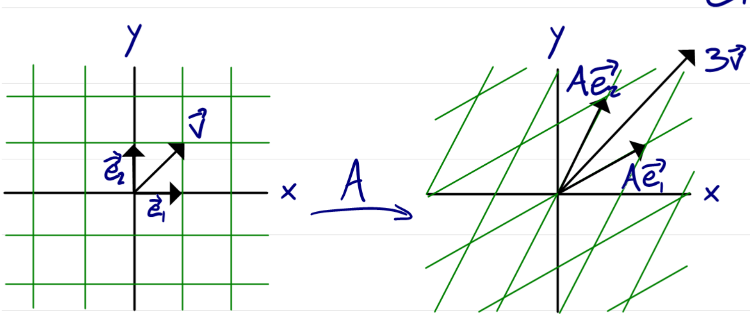

In the above example, we saw that there are vectors such that matrix multiplication behaved just like scalar multiplication: $A\mathbf{v} = \lambda\mathbf{v}$. This is extremely desirable in many situations: we often want matrix multiplication to behave like scalar multiplication, and we often want general matrices to behave like diagonal matrices. This leads to the following definition.

### Definition 10.1 — Eigenvalues and Eigenvectors

Let $A$ be a square matrix. A scalar $\lambda$ is called an **eigenvalue** of $A$ if there is a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Such a vector $\mathbf{v}$ is called an **eigenvector** of $A$ corresponding to $\lambda$.

**Example.** Show that $\mathbf{v} = (1, 1)$ is an eigenvector of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and find the corresponding eigenvalue.

$$A\vec{v} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{v},$$ so the corresponding eigenvalue is 3.




Most vectors (like $\vec{e}_1$ and $\vec{e}_2$) change dir. Eigenvectors don't.

OK, how do we go about actually *finding* eigenvalues and eigenvectors? It's easy enough when the eigenvector is given to us, but the real world isn't that nice.

Well, we find them via a two-step process: first, we find the eigenvalues, then we find the eigenvectors.

**Step 1: Find the eigenvalues.** Recall that $\lambda$ is an eigenvalue of $A$ if and only if there is a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. This is equivalent to...

$$A\vec{v} - \lambda\vec{v} = \vec{0} \iff (A - \lambda I)\vec{v} = \vec{0} \iff \vec{v} \in \text{null}(A - \lambda I).$$

Wrong: $(A - \lambda)\vec{v} = \vec{0}$

matrix − scalar = ??

In other words, $\lambda$ is an eigenvalue of $A$ if and only if the matrix $A - \lambda I$ has non-zero null space. How can we find when a matrix has a non-zero null space? Well...

- $\underbrace{\dim(\text{null}}_{\text{nullity!}}(A - \lambda I)) > 0$ if and only if...

  $A - \lambda I$ is not invertible

- ...if and only if...

  $\det(A - \lambda I) = 0.$

A-ha! This is the type of equation we can actually solve! So to find the eigenvalues of $A$, we find all numbers $\lambda$ such that $\det(A - \lambda I) = 0$.

**Example.** Find all eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

$$\begin{aligned} 0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix}\right) &= (1-\lambda)(4-\lambda) - 10 \\ &= \lambda^2 - 5\lambda - 6 \\ &= (\lambda - 6)(\lambda + 1) \end{aligned}$$

$\therefore$ The eigenvalues are $\lambda = 6$ and $\lambda = -1$.

**Step 2: Find the eigenvectors.** Once you know the eigenvalues (from step 1), the associated eigenvectors are the vectors $\mathbf{v}$ satisfying $A\mathbf{v} = \lambda\mathbf{v}$. But this equation holds if and only if...

$(A - \lambda I)\vec{v} = \vec{0}$ (i.e., $\vec{v} \in \text{null}(A - \lambda I)$).

In other words, to find all eigenvectors of $A$ associated with the eigenvalue $\lambda$, we compute $\text{null}(A - \lambda I)$.

**Example.** Find all eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

$\underline{\lambda = -1:}$ $(A - \lambda I)\vec{v} = \vec{0} \iff (A + I)\vec{v} = \vec{0}$:

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 5 & 5 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 5 & 5 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

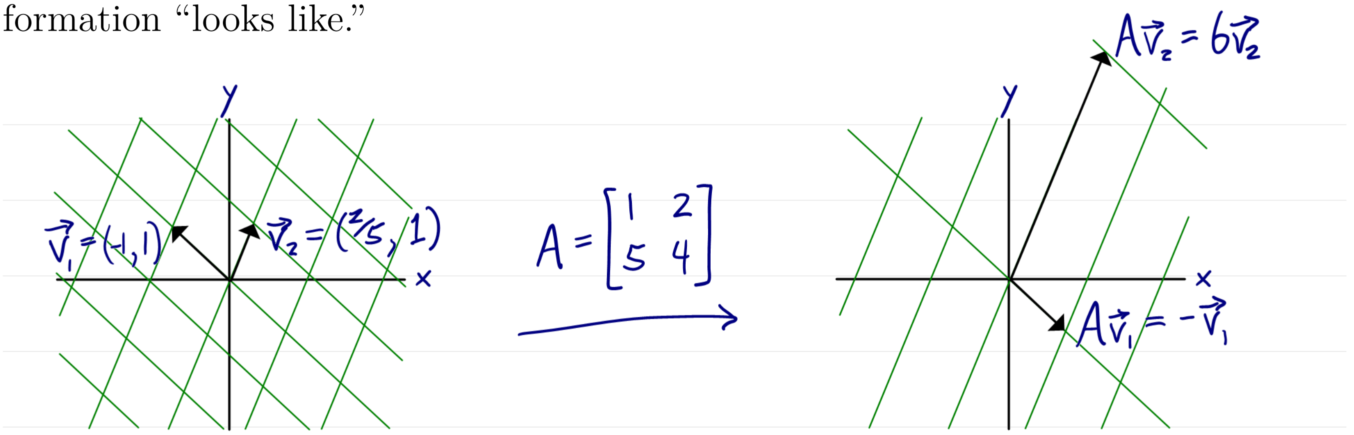
$\therefore$ $v_2$ free, $v_1 = -v_2$, so $\vec{v} = (-v_2, v_2) = v_2(-1, 1)$.

$\underline{\lambda = 6:}$ $(A - \lambda I)\vec{v} = \vec{0} \iff (A - 6I)\vec{v} = \vec{0}$:

$$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ 5 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $v_2$ free, $v_1 = \frac{2}{5}v_2$, so $\vec{v} = (\frac{2}{5}v_2, v_2) = v_2(\frac{2}{5}, 1)$.

The eigenvalues and eigenvectors can help us understand what a linear transformation "looks like."



Since the set of all eigenvectors of $A$ corresponding to $\lambda$ is the nullspace of $A-\lambda I$, the set of eigenvectors forms a subspace (the nullspace is always a subspace). We give this subspace a name:

### Definition 10.2 — Eigenspace

Let $A$ be a square matrix and let $\lambda$ be an eigenvalue of $A$. The set of all eigenvectors of $A$ corresponding to $\lambda$, together with the zero vector, is called the **eigenspace** of $\lambda$.

**Example.** Find all eigenvalues, and bases of their corresponding eigenspaces, for the matrix $A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -1 & 3 \\ 0 & -2 & 4 \end{bmatrix}$.

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -3 & 3 \\ 0 & -1-\lambda & 3 \\ 0 & -2 & 4-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)(-1-\lambda)(4-\lambda) + 0 + 0 - (2-\lambda)(3)(-2) - 0 - 0$$

$$= (2-\lambda)(\lambda^2 - 3\lambda - 4 + 6)$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2)$$

$$= (2-\lambda)(\lambda-1)(\lambda-2)$$

$\therefore$ $A$ has eigenvalues $\lambda = 1$ and $\lambda = 2$.

$\underline{\lambda = 2:}$ $(A - \lambda I)\vec{v} = \vec{0}$ $\Leftrightarrow$ $(A - 2I)\vec{v} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow[R_3 - \frac{2}{3}R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $v_2$ is leading, $v_1$ and $v_3$ are free

$-3v_2 + 3v_3 = 0$, so $v_2 = v_3$, so

$(v_1, v_2, v_3) = (v_1, v_3, v_3) = v_1(1,0,0) + v_3(0,1,1)$.

Basis of eigenspace: $\{(1,0,0), (0,1,1)\}$.

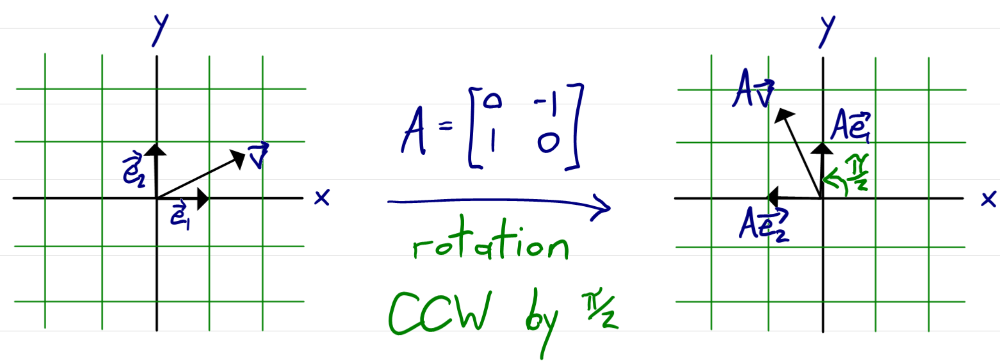
$\underline{\lambda = 1:}$ $\{(3,3,2)\}$ (try on your own)

There are some matrices that do not have any (real) eigenvalues or eigenvectors. For example...

**Example.** Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1$$

$\therefore$ $\lambda^2 = -1$, which does not have any (real) solutions.




# Complex Numbers

There are a few operations in your mathematical career that you have been told you cannot do:

- You cannot divide by 0.
- Cannot square root a negative number.

We now introduce something called **complex numbers** that let us "fix" one of these "problems": they let us work with square roots of negative numbers algebraically.

We define $i$ to be a number with the property that $i^2 = -1$. $(i \notin \mathbb{R})$

Remarkably, you can do arithmetic with $i$ just like you're used to with real numbers, and things have a way of just working out. But first, let's get some terminology out of the way:

- An **imaginary number** is a number of the form

  $bi$, where $b \in \mathbb{R}$

  do not take this name literally

- A **complex number** is a number of the form

  $a + bi$, where $a, b \in \mathbb{R}$

  real part imaginary part

Arithmetic with complex numbers works just like it does with real numbers, so nothing surprising happens when you add or multiply them.

**Example.** Add and multiply some complex numbers.

$$(3+4i) + (2-7i) = 5 - 3i$$

$$(3+4i)(2-7i) = 6 - 21i + 8i - \underbrace{28i^2}_{=28,\ \text{since } i^2=-1}$$

$$= 34 - 13i$$

Slightly more generally,

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

However, division of complex numbers requires one minor "trick" to get our hands on.

**Example.** Divide some complex numbers.

$$\frac{1+2i}{3+4i} = \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)} = \frac{11+2i}{25} = \frac{11}{25} + \left(\frac{2}{25}\right)i$$

imaginary cross terms cancel!

The number that we multiplied the top and bottom by in the above example was called the **complex conjugate** of the bottom (denominator). That is,

the complex conjugate of $a+bi$ is $\overline{a+bi} = a-bi$. Importantly,

$$(a+bi)(a-bi) = (a^2+b^2) + (ab-ab)i = a^2+b^2 \text{ is real.}$$

With just these basic tools under our belt, we can now find roots of quadratics that don't have real roots! We just use the quadratic formula like usual.

**Example.** Find the (potentially complex) solutions of the equation $x^2 - 2x + 2 = 0$.

$$x = \frac{1}{2a}\left(-b \pm \sqrt{b^2-4ac}\right) = \frac{1}{2}\left(2 \pm \sqrt{4-8}\right) = 1 \pm \frac{1}{2}\sqrt{-4}$$
$$= 1 \pm i$$

Let's double-check:

$$(1+i)^2 - 2(1+i) + 2 = 1 + 2i - 1 - 2 - 2i + 2 = 0. \checkmark$$

Also,

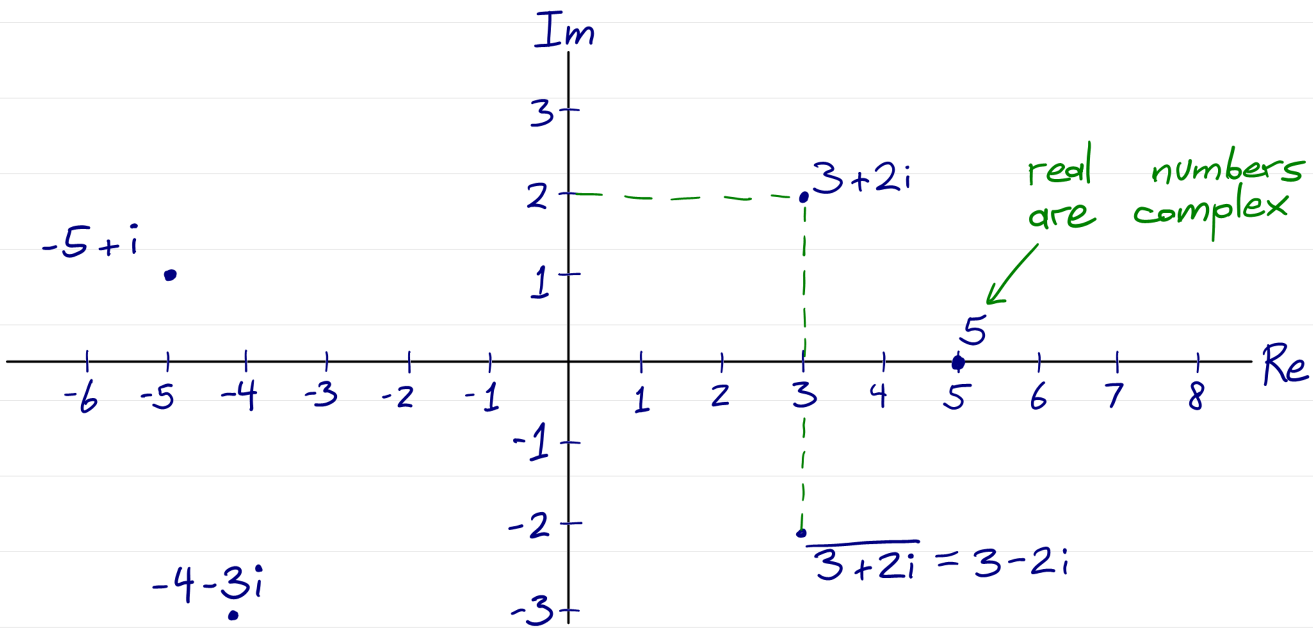
$$(x-(1+i))(x-(1-i)) = x^2 - (1+i+1-i)x + (1+i)(1-i) = x^2 - 2x + 2 \checkmark$$

The previous example hints at the following observation, which is indeed true:

If a polynomial has real coefficients, its roots come in complex conjugate pairs.

Proof sketch: suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$.
If $f(x_*) = 0$ then $f(\overline{x_*}) = a_n \overline{x_*}^n + \cdots + a_1 \overline{x_*} + a_0$
$= \overline{a_n x_*^n + \cdots + a_1 x_* + a_0} = \overline{f(x_*)} = \overline{0} = 0$.

Just like we think of $\mathbb{R}$ as a line, we can think of $\mathbb{C}$ as a plane, and the number $a + bi$ has coordinates $(a, b)$ on that plane.




**Example.** Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (again).

Eigenvalues: $\lambda^2 = -1 \iff \lambda = \pm\sqrt{-1} = \pm i$

$\underline{\lambda = i}$: $(A - \lambda I)\vec{v} = \vec{0} \iff (A - iI)\vec{v} = \vec{0}$:

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{R_2 - iR_1} \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $v_2$ free, $-iv_1 - v_2 = 0$, so $v_1 = -iv_2$, so
$\vec{v} = (-iv_2, v_2) = v_2(-i, 1)$. Basis: $\{(-i, 1)\}$.

$\underline{\lambda = -i}$: Basis is $\{(i, 1)\}$. (complex conjugate)

# Back to Eigenvalues and Eigenvectors

Recall that the eigenvalues of a matrix $A$ are the solutions $\lambda$ to the equation $\det(A - \lambda I) = 0$. This is a polynomial in $\lambda$, and we give it a special name:

> **Definition 10.3 — Characteristic Polynomial**
>
> Let $A$ be a square matrix. Then $\det(A - \lambda I)$ is called the **characteristic polynomial** of $A$, and $\det(A - \lambda I) = 0$ is called the **characteristic equation** of $A$.

The characteristic polynomial of an $n \times n$ matrix is always of degree $n$. Since every degree-$n$ polynomial has at most $n$ distinct roots, this immediately tells us that

every $n \times n$ (real) matrix has at most $n$ (real) eigenvalues.

**Example.** Find the characteristic polynomial, eigenvalues, and bases of the corresponding eigenspaces, of $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -1 & -1 \\ 2 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(2-\lambda)(2-\lambda) + 1 - 2 - (1-\lambda) + \cancel{2}(2-\lambda) - \cancel{(2-\lambda)}$$

$$= (1-\lambda)(2-\lambda)^2 - \cancel{1} - \cancel{(1-\lambda)} + \cancel{(2-\lambda)}$$

$$= (1-\lambda)(2-\lambda)^2 \leftarrow \text{characteristic polynomial}$$

$\therefore$ The eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$.

$\underline{\lambda = 1:}$
$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $v_1, v_2$ leading, $v_3$ free

$v_1 = 0, \quad v_2 + v_3 = 0, \quad$ so $\quad v_2 = -v_3,\quad$ so
$(v_1, v_2, v_3) = (0, -v_3, v_3) = v_3(0,-1,1)$. Basis: $\{(0,-1,1)\}$

$\lambda = 2$: $\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore$ $v_1, v_2$ leading, $v_3$ free
$v_1 = -\frac{1}{2}v_3, \quad v_2 = -\frac{1}{2}v_3,\quad$ so
$(v_1, v_2, v_3) = (-\frac{1}{2}v_3, -\frac{1}{2}v_3, v_3) = v_3(-\frac{1}{2}, -\frac{1}{2}, 1)$. Basis: $\{(-\frac{1}{2}, -\frac{1}{2}, 1)\}$

In the previous example, we had a $3 \times 3$ matrix with only 2 distinct eigenvalues. However, the matrix has 3 eigenvalues if we count the multiplicities of the roots of the characteristic polynomial: the eigenvalue $\lambda = 1$ once and the eigenvalue $\lambda = 2$ twice.

There is actually another notion of multiplicity of an eigenvalue that is also important: the dimension of the corresponding eigenspace. These ideas lead to the following definition:

### Definition 10.4 — Multiplicity

Let $A$ be a square matrix with eigenvalue $\lambda$.

- The **algebraic multiplicity** of $\lambda$ is the multiplicity of $\lambda$ as a root of the characteristic polynomial of $A$.
- The **geometric multiplicity** of $\lambda$ is the dimension of its eigenspace.

In the previous example...

$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

The eigenvalues were...
$\lambda = 1$: algebraic multiplicity 1, geometric multiplicity 1
$\lambda = 2$: algebraic multiplicity 2, geometric multiplicity 1

The fact that the geometric multiplicity of each eigenvalue was $\leq$ the algebraic multiplicity was not a coincidence: it is our next theorem.

**Theorem 10.1 — Geo. Mult. $\leq$ Alg. Mult.**

Let $A$ be a square matrix. Then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

A remarkable fact called the Fundamental Theorem of Algebra says that every polynomial of degree $n$ has *exactly* $n$ roots, counted according to multiplicity. This immediately tells us that...

the sum of the algebraic multiplicities of the (potentially complex) eigenvalues of an $n \times n$ matrix always equals $n$.

However, the sum of the geometric multiplicities may be smaller.

**Example.** Compute the algebraic and geometric multiplicities of the eigenvalues of all matrices that we considered this week.

$A = \begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix}$: $\lambda = \frac{1}{3}, 2$ each have $AM = GM = 1$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$: $\lambda = 3, -1$ each have $AM = GM = 1$

$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$: $\lambda = 6, -1$ each have $AM = GM = 1$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $\lambda = i, -i$ each have $AM = GM = 1$

$A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -1 & 3 \\ 0 & -2 & 4 \end{bmatrix}$: $\lambda = 2$ has $AM = GM = 2$
$\lambda = 1$ has $AM = GM = 1$

Just like with determinants, our eigenvalue life becomes much easier when dealing with triangular matrices.

**Example.** Compute the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(4-\lambda)(6-\lambda) + 0 + 0 - 0 - 0 - 0$$

$$= (1-\lambda)(4-\lambda)(6-\lambda).$$

$\therefore$ The eigenvalues of $A$ are $\lambda = 1$, $\lambda = 4$, and $\lambda = 6$.

(eigen<u>vectors</u> are not so nice, though)

In general, because the determinant of a triangular matrix is just the product of its diagonal entries, the eigenvalues of a triangular matrix are exactly its diagonal entries:

**Theorem 10.2 — Eigenvalues of Triangular Matrices**

Let $A$ be a triangular matrix. Its eigenvalues are exactly the entries on its main diagonal (i.e., $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$).