Linear Algebra – Week 11

Diagonalization

This week we will learn about:

- Diagonalization of matrices,
- Matrix functions, and
- Why diagonalization is amazing.

Extra reading:

- Section 3.4 in the textbook
- Lecture videos 40, 41, 42, 43, and 44 on YouTube
- Diagonalizable matrix at Wikipedia
- Matrix exponential at Wikipedia

Extra textbook problems:

$\star$ 3.4.1

$\star\star$ 3.4.2, 3.4.4, 3.4.6, 3.4.7, 3.4.22

$\star\star\star$ 3.4.8–3.4.12, 3.4.21

☠ 3.4.23

Diagonalization

One of the primary uses of eigenvalues and eigenvectors is that they let us put (most) matrices into a form that makes them almost as easy to work with as diagonal matrices.

Definition 11.1 — Diagonalizable Matrices

A square matrix A is called **diagonalizable** if there is a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

To get an idea of why diagonalizability is useful, consider the problem of computing a large power of a matrix, like A^{500}.

- If A is a general matrix...

$$A^{500} = \underbrace{AA\cdots A}_{\text{499 matrix multiplications}}$$

No thanks!

- If A is a diagonal matrix...

If $A = \begin{bmatrix} a_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n,n} \end{bmatrix}$ then $A^{500} = \begin{bmatrix} a_{1,1}^{500} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n,n}^{500} \end{bmatrix}$. Easier!

- If A is diagonalizable...

$A = PDP^{-1}$,

$$A^{500} = (PDP^{-1})^{500} = (PD\underbrace{\cancel{P^{-1})(P}}_{=I}D\underbrace{\cancel{P^{-1})(P}}_{=I}DP^{-1})\cdots(PDP^{-1})$$

$$= PD^{500}P^{-1}$$

entrywise!

only 2 matrix multiplications!

But how could we ever hope to determine whether or not a matrix is diagonalizable? It turns out that eigenvalues and eigenvectors give us the answer:

Theorem 11.1 — Diagonalizability

Let A be an $n \times n$ matrix. Then the following are equivalent:

a) A is diagonalizable.

b) A has a set of n linearly independent eigenvectors.

Furthermore, if A is diagonalizable then $A = PDP^{-1}$, where P is the matrix whose columns are the n linearly independent eigenvectors, and D is the diagonal matrix whose diagonal entries are the eigenvalues corresponding to the eigenvectors in P in the same order.

Proof. Start by noticing that the equation $A = PDP^{-1}$ is equivalent to...

$AP = PD$ (as long a P is invertible).
Let's write P in terms of its columns and use block matrix multiplication:
If $P = [\vec{p}_1 \mid \vec{p}_2 \mid \cdots \mid \vec{p}_n]$ then
$AP = [A\vec{p}_1 \mid A\vec{p}_2 \mid \cdots \mid A\vec{p}_n]$ and $PD = [d_{1,1}\vec{p}_1 \mid d_{2,2}\vec{p}_2 \mid \cdots \mid d_{n,n}\vec{p}_n]$.

These are equal if and only if
$A\vec{p}_j = d_{j,j}\vec{p}_j$ for all $1 \leq j \leq n$ (i.e., if and only if each $\vec{p}_j$ is an eigenvector with corresponding eigenvalue $d_{j,j}$).

From Week 7, P is invertible (so $A = PDP^{-1}$) if and only if its columns are linearly independent. ∎

Example. Diagonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

We need eigenvalues and eigenvectors. From last week: $\lambda_1 = -1$, $\vec{v}_1 = (-1,1)$, $\lambda_2 = 6$, $\vec{v}_2 = (2,5)$.
Since $\{(-1,1),(2,5)\}$ is linearly independent,
$A = PDP^{-1}$ with $D = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}$, $P^{-1} = \frac{1}{7}\begin{bmatrix} -5 & 2 \\ 1 & 1 \end{bmatrix}$.

OK, now that we've diagonalized this matrix... so what? How does it make our lives easier? Well...

Example. Find a formula for A^n when $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

Since $A = PDP^{-1}$,

$$A^n = PD^nP^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}^n \left(\frac{1}{7}\begin{bmatrix} -5 & 2 \\ 1 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{7}\begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 6^n \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 5(-1)^{n+1} & 2(-1)^n \\ 6^n & 6^n \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 5(-1)^n + 2\cdot 6^n & 2(-1)^{n+1} + 2\cdot 6^n \\ 5(-1)^{n+1} + 5\cdot 6^n & 2(-1)^n + 5\cdot 6^n \end{bmatrix}$$

For example, $A^1 = \frac{1}{7}\begin{bmatrix} -5+12 & 2+12 \\ 5+30 & -2+30 \end{bmatrix}$

$$= \frac{1}{7}\begin{bmatrix} 7 & 14 \\ 35 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} = A. \checkmark$$

Example. Find a formula for A^n when $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Method 1: Diagonalize! From last week:

$\lambda_1 = i, \ \vec{v}_1 = (i, 1)$ and $\lambda_2 = -i, \ \vec{v}_2 = (-i, 1)$.

linearly independent

$\therefore A = PDP^{-1}$, where $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$,

so
$$A^n = PD^nP^{-1} = \frac{1}{2}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i^n & 0 \\ 0 & (-i)^n \end{bmatrix}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -i^{n+1} & i^n \\ (-1)^n i^{n+1} & (-i)^n \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i^{n+2} + (-1)^{n+1} i^{n+2} & i^{n+1} + (-i)^{n+1} \\ -i^{n+1} + (-1)^n i^{n+1} & i^n + (-i)^n \end{bmatrix}.$$

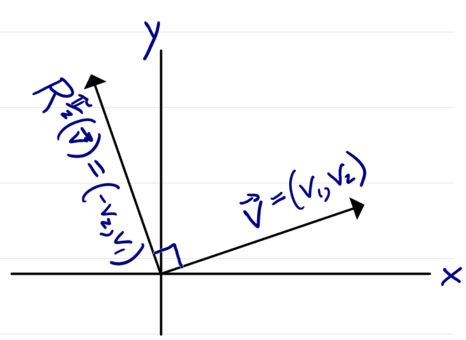
This matrix is always real, even though the formula involves complex numbers.

E.g.,
$$A^3 = \frac{1}{2}\begin{bmatrix} -i^5 + (-1)^4 i^5 & i^4 + (-i)^4 \\ -i^4 + (-1)^3 i^4 & i^3 + (-i)^3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i+i & 1+1 \\ -1-1 & -i+i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Method 2: Geometry!

The matrix A is a rotation counter-clockwise by $\pi/2$ (from Week 4).

$$\therefore A^n = \left[R^{\frac{n\pi}{2}}\right] = \begin{bmatrix} \cos\left(\frac{n\pi}{2}\right) & -\sin\left(\frac{n\pi}{2}\right) \\ \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \end{bmatrix}.$$



The previous theorem is nice since it completely characterizes when a matrix is diagonalizable. However, there is one special case that is worth pointing out where it is actually much easier to prove that a matrix is diagonalizable.

Theorem 11.2 — Matrices with Distinct Eigenvalues

Let A be an $n \times n$ matrix with **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then A is diagonalizable.

Proof. We just need to prove that eigenvectors corresponding to different eigenvalues are linearly independent. To show this...

let $\vec{v}_j$ be an eigenvector of A corresponding to λ_j (for each $1 \le j \le n$).
If A is not diagonalizable then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ must be linearly dependent.
If k is the largest integer for which $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent, then

(1) $\vec{v}_{k+1} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ for some $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Multiplying (1) by A gives

(2) $\lambda_{k+1}\vec{v}_{k+1} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_k\lambda_k\vec{v}_k$.

Multiplying (1) by λ_{k+1} gives

(3) $\lambda_{k+1}\vec{v}_{k+1} = c_1\lambda_{k+1}\vec{v}_1 + c_2\lambda_{k+1}\vec{v}_2 + \cdots + c_k\lambda_{k+1}\vec{v}_k$.

Subtracting (3) from (2) gives

$$\vec{0} = c_1(\lambda_1 - \lambda_{k+1})\vec{v}_1 + c_2(\lambda_2 - \lambda_{k+1})\vec{v}_2 + \cdots + c_k(\lambda_k - \lambda_{k+1})\vec{v}_k.$$

By linear independence of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, this implies $\lambda_j = \lambda_{k+1}$ for some $1 \le j \le k$. ∎

Example. Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable.

Eigenvalues: $0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right)$

$$= (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$\therefore\ \lambda = \frac{1 \pm \sqrt{5}}{2}$ (Quadratic formula: $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$)

Since these two eigenvalues are distinct, A is diagonalizable.
(To find a diagonalization, though, we need eigenvectors.)

However, to actually perform the diagonalization itself, we still need to know the eigenvectors.

The Fibonacci Sequence

As an example to demonstrate the usefulness of diagonalization, let's investigate the Fibonacci sequence, which is the sequence of integers that starts as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

A bit more properly, it is defined by

$$F_1 = 1,\ F_2 = 1,\ F_{n+1} = F_n + F_{n-1} \quad \text{for all} \quad n \geq 2.$$

My question to you is: can we find a simple closed-form formula for the n-th term of this sequence, without computing all of the previous terms first?

What we will do is represent the Fibonacci sequence via matrix multiplication. Notice that

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \quad \text{for all } n \geq 2.$$

Well, if we iterate this line of thinking, we get

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Matrix multiplication is kinda nasty, but fortunately we already saw that this matrix is diagonalizable, so we can compute large powers of it easily! So let's find its eigenvectors (we already computed its eigenvalues):

Eigenvalues: $\lambda = \frac{1 \pm \sqrt{5}}{2}$. Let $\varphi = \frac{1+\sqrt{5}}{2}$, so $1-\varphi = \frac{1-\sqrt{5}}{2}$.

Eigenvectors:

$\lambda = \varphi$: $\left[\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 1 & -\varphi & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -\varphi & 0 \\ 1-\varphi & 1 & 0 \end{array}\right] \xrightarrow{R_2 - (1-\varphi)R_1} \left[\begin{array}{cc|c} 1 & -\varphi & 0 \\ 0 & 0 & 0 \end{array}\right]$

$= 1 - (1-\varphi)(-\varphi)$
$= -\varphi^2 + \varphi + 1$
$= 0.$

v_1 leading, v_2 free

$v_1 = \varphi v_2$, so $\vec{v} = (\varphi v_2, v_2) = v_2(\varphi, 1)$.

Basis of eigenspace: $\{(\varphi, 1)\}$.

try on your own

$\lambda = 1-\varphi$: Basis of eigenspace: $\{(1-\varphi, 1)\}$.

Thus its diagonalization is:

$A = PDP^{-1}$, where $D = \begin{bmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{bmatrix}$, $P = \begin{bmatrix} \varphi & 1-\varphi \\ 1 & 1 \end{bmatrix}$, and

$$P^{-1} = \frac{1}{\varphi-(1-\varphi)}\begin{bmatrix} 1 & \varphi-1 \\ -1 & \varphi \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \varphi-1 \\ -1 & \varphi \end{bmatrix}.$$

Finally, we can use this diagonalization to compute arbitrary powers of the matrix:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = PD^{n-1}P^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \varphi & 1-\varphi \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \varphi^{n-1} & 0 \\ 0 & (1-\varphi)^{n-1} \end{bmatrix}\begin{bmatrix} 1 & \varphi-1 \\ -1 & \varphi \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \varphi & 1-\varphi \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \varphi^{n-1} & 0 \\ 0 & (1-\varphi)^{n-1} \end{bmatrix}\begin{bmatrix} \varphi \\ \varphi-1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \varphi & 1-\varphi \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \varphi^{n} \\ -(1-\varphi)^{n} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \varphi^{n+1}-(1-\varphi)^{n+1} \\ \varphi^{n}-(1-\varphi)^{n} \end{bmatrix}.$$

Thus we obtain the following simple formula for the n-th Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (1-\varphi)^n\right).$$

The idea used throughout this example applies in a lot of generality: if we can represent something by matrix multiplication, then there's a good chance that diagonalization (via eigenvalues/eigenvectors) can shed light on the problem.

Arbitrary Matrix Powers

Once we have diagonalized a matrix, performing an operation on it is almost as easy as performing that operation on a number. We already saw this with computing large powers of a matrix: our procedure was...

a) First, diagonalize $A = PDP^{-1}$.

b) Next, compute D^n.

c) Finally, "undiagonalize": compute PD^nP^{-1}.

This same basic idea works in lots of generality, and helps us talk about things we wouldn't even know how to define otherwise. For example...

What is a square root B of the matrix A?

It is a matrix B with the property that $B^2 = A$.

OK, how could we find a square root of A?

If $A = PDP^{-1}$, set $B = P\sqrt{D}P^{-1}$, where $\sqrt{D}$ is computed entrywise. $B^2 = (P\sqrt{D}\underbrace{\cancel{P^{-1})(P}}_{= I}\sqrt{D}P^{-1}) = PDP^{-1} = A$.

Example. Find a square root of the matrix $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$.

Eigenvalues: $0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 4 \\ -1 & 5-\lambda \end{bmatrix}\right)$

$= -\lambda(5-\lambda) + 4$

$= \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$.

$\therefore\ \lambda = 1$ or $\lambda = 4$.

Eigenvectors:

$\lambda = 1$: $\left[\begin{array}{cc|c} -1 & 4 & 0 \\ -1 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

v_1 leading, v_2 free, $-v_1 + 4v_2 = 0$, so $v_1 = 4v_2$, so $\vec{v} = (v_1, v_2) = (4v_2, v_2) = v_2(4, 1)$.

$\lambda = 4$: $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - \frac{1}{4}R_1} \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

v_1 leading, v_2 free, $-4v_1 + 4v_2 = 0$, so $v_1 = v_2$, so $\vec{v} = (v_1, v_2) = (v_2, v_2) = v_2(1, 1)$.

Diagonalize: $A = PDP^{-1}$, where

$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$, and $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$.

Square root: set $\sqrt{D} = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Undiagonalize:

Set $B = P\sqrt{D}P^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 8 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}$.

Sanity check:

$$B^2 = \left(\frac{1}{3}\begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}\right)\left(\frac{1}{3}\begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 4-4 & 8+28 \\ -2-7 & -4+49 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 36 \\ -9 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} = A. \checkmark$$

(Side note: there are 3 other square roots.)

Similarly, we can use this method to define the r-th power of a diagonalizable matrix for any real number r (i.e., r doesn't need to be an integer):

Definition 11.2 — Matrix Powers

Let r be a real number and let A be a diagonalizable matrix (i.e., $A = PDP^{-1}$ for some invertible P and diagonal D). Then $A^r = PD^rP^{-1}$, where D^r is obtained by raising each of its diagonal entries to the r-th power.

Example. Compute A^π when $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$.

From earlier: $A = PDP^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$,

$P = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$, and $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$. Then $A^\pi = PD^\pi P^{-1}$

$$= \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^\pi \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -4^\pi & 4^{\pi+1} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4-4^\pi & 4^{\pi+1}-4 \\ 1-4^\pi & 4^{\pi-1}-1 \end{bmatrix}$$

Question to ponder: What happens when $r = -1$ in the above definition?

You get A^{-1}!

Arbitrary Matrix Functions

The previous section just touched the tip of the iceberg: we can also extend any function with a Taylor series to matrices now. For example, let's consider the function e^x. Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

With that in mind, we define e^A, where A is a square matrix, as follows:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

That seems like it might be nasty to calculate in general. Fortunately, we can just do what we usually do: diagonalize, apply the function e^x to each diagonal entry, and then un-diagonalize.

Example. Compute e^A, where $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$.

From earlier: $A = PDP^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$, and $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$. Then $e^A = Pe^DP^{-1}$

$$= \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e & -e \\ -e^4 & 4e^4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4e-e^4 & 4e^4-4e \\ e-e^4 & 4e^4-e \end{bmatrix}.$$

Why does this work?

Because
$$\begin{aligned} e^A &= I + A + \tfrac{1}{2!}A^2 + \tfrac{1}{3!}A^3 + \cdots \\ &= I + PDP^{-1} + \tfrac{1}{2!}(PD^2P^{-1}) + \tfrac{1}{3!}(PD^3P^{-1}) + \cdots \\ &= P\left(I + D + \tfrac{1}{2!}D^2 + \tfrac{1}{3!}D^3 + \cdots\right)P^{-1} \\ &= Pe^DP^{-1}. \end{aligned}$$

(Ignoring some convergence concerns.)

The following properties of the matrix exponential are straightforward to check:

- $e^O = I$ (O is a diagonal matrix)
- $e^A e^{-A} = I$ (i.e., $(e^A)^{-1} = e^{-A}$)

If $A = PDP^{-1}$ then $-A = P(-D)P^{-1}$, so

$$e^A e^{-A} = (Pe^D \underbrace{\cancel{P^{-1})(P}}_{=I} e^{-D}P^{-1}) = Pe^D e^{-D}P^{-1} = PIP^{-1} = I.$$

There wasn't really anything too special about the function e^x here that let us define it for matrices: we can do the same thing for any function that equals its Taylor series, and the idea is exactly the same: just apply the function to the diagonal entries in the diagonalization of the matrix.

Example. Compute $\sin(A)$, where $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$.

From earlier: $A = PDP^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$, and $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$. Then

$$\sin(A) = P\sin(D)P^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \sin(1) & 0 \\ 0 & \sin(4) \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \sin(1) & -\sin(1) \\ -\sin(4) & 4\sin(4) \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4\sin(1)-\sin(4) & 4\sin(4)-4\sin(1) \\ \sin(1)-\sin(4) & 4\sin(4)-\sin(1) \end{bmatrix}.$$

So go forth, and compute the sin, arctan, and log of matrices to your heart's content!

(Unless the matrices aren't diagonalizable... in that case, take Advanced Linear Algebra (MATH 3221) to learn what to do.)