VECTORS

This week we will learn about:

- What vectors are,

- How to manipulate vectors, and

- Linear combinations.

Extra reading and watching:

- Section 1.1 in the textbook

- Lecture videos 1, 2, and 3 on YouTube

- Vector at Wikipedia

Extra textbook problems:

   ★ 1.1.1–1.1.3, 1.1.5–1.1.8

  ★★ 1.1.9–1.1.12

★★★ 1.1.13(a), 1.1.14, 1.1.15

  ☠ 1.1.13(b)

Linear algebra is one of the branches of mathematics where everything "just works". Most of the objects that we introduce in this course will seem rather simple and unremarkable at first, but we will be able to do a *lot* with them. Some of the things we will be able to do are motivated very geometrically...

Example. Lengths, angles, and deformations (oh my!)

- Distance between points ⎫
- Angle between lines   ⎬ in any dimension
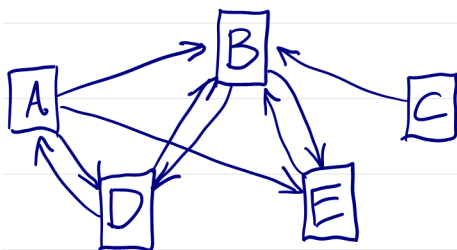- Transformations of geometric objects:

- Computer graphics
- Motion in 3D, project onto 2D screen

...while other applications will involve sifting through huge amounts of data:

Example. How does (well, *did*) Google work?

a "graph":

a "matrix":

$$\begin{array}{c c}
 & \begin{array}{c c c c c} A & B & C & D & E \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \left[\begin{array}{c c c c c}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
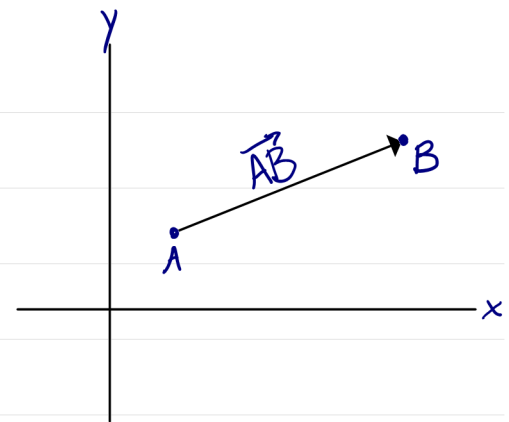0 & 1 & 0 & 0 & 0
\end{array}\right]
\end{array}$$

To find patterns in, or "important pieces" of, a matrix we compute "eigenvalues and eigenvectors".

Vectors

A **vector** is an ordered list of numbers like $(3, 1)$. These lists can be as long as we like, but we'll start by considering 2-dimensional vectors in order to establish some intuition for how they work, since we can interpret them geometrically in this case.

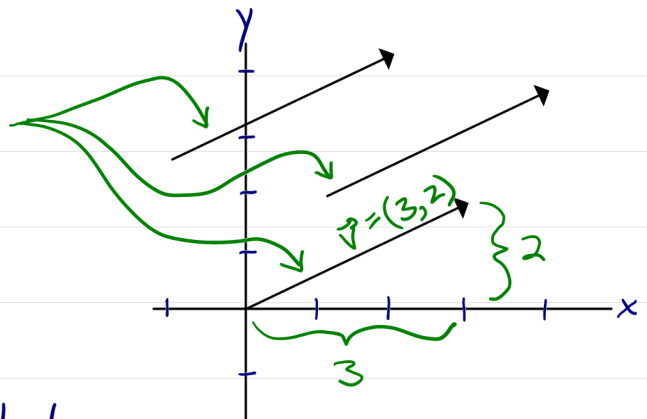Several different notations are used for vectors:

$\vec{v}$ (when I write)

v   (when I type)

$\vec{AB}$ (more common in physics)

↑ ↑
tail head

The **coordinates** or **entries** of a vector only tell us how far the vector stretches in the x- and y-directions; **not** where it is located in space.

Example. Coordinates of vectors.

These are all the SAME vector $(\vec{v} = (3, 2))$:

We usually position vectors so their "tail" (i.e., back end) is located at the origin (this is called "standard position").

That is, vectors represent...

motion/displacement, NOT location.

The order of the coordinates matters: for example, $(2,3) \neq (3,2)$. For this reason, 2D vectors are sometimes called "ordered pairs". In another math class, you might be introduced to objects called "sets", where order does *not* matter.

- Two vectors are equal if and only if...

  • same entries in same locations, or
  • same length and direction.

- The **zero vector** is...

  $(0,0)$, denoted by $\vec{0}$.

- Recall that the set of all real numbers is denoted by $\mathbb{R}$. Similarly,

  the set of 2D vectors is denoted by $\mathbb{R}^2$.

Sometimes we want to combine two (or more) vectors to get new ones. For example, we might want to think about what happens if we move along the path of multiple different vectors, one after another. Where do we end up after doing this? The answer is given by **vector addition**.
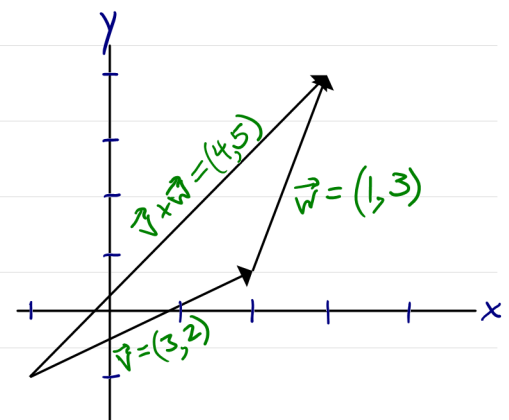
Example. Vector addition.

$\vec{V} = (3,2)$

$\vec{W} = (1,3)$

$\vec{V} + \vec{W} = (3,2) + (1,3)$
$\qquad = (4,5)$

Just add entrywise!

Specifically, if $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, then $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$ is the vector from the tail of $\mathbf{v}$ to the head of $\mathbf{w}$.

Another common way to manipulate vectors is to "scale" them (or "multiply by a scalar"). The idea here is that we want to move in the same direction as a given vector, but we want to change how *far* we move in that direction.
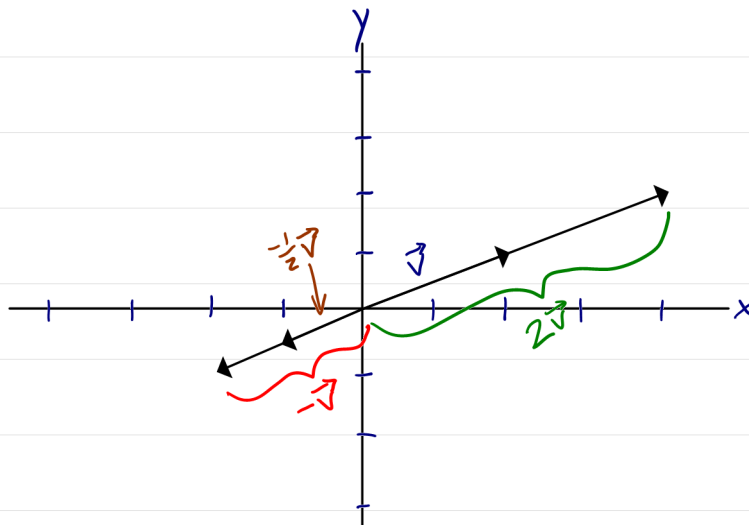
Example. Scalar multiplication.

$$\vec{v} = (2,1)$$

$$2\vec{v} = (4,2)$$

$$-\vec{v} = (-2,-1)$$

$$-\tfrac{1}{2}\vec{v} = \left(-1, \tfrac{-1}{2}\right)$$

Specifically, if $\mathbf{v} = (v_1, v_2)$ and c is a real number, then $c\mathbf{v} = (cv_1, cv_2)$ is the vector that points in the same direction as $\mathbf{v}$, but is c times as long (and if $c < 0$ then the direction of the vector is reversed).

Finally, how do you think vector subtraction might be defined? If $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, then

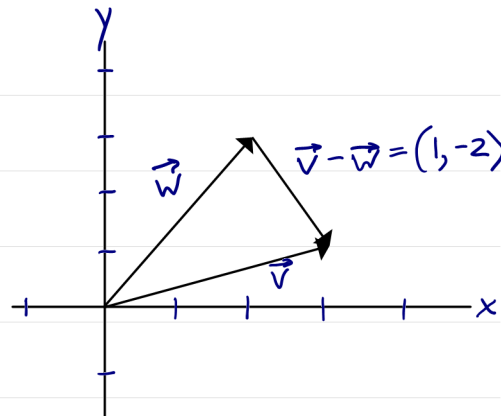$$\mathbf{v} - \mathbf{w} = \left(v_1 - w_1, \; v_2 - w_2\right)$$

is the vector from the   head of $\vec{w}$ to the   head of $\vec{v}$.

Example. Vector subtraction.
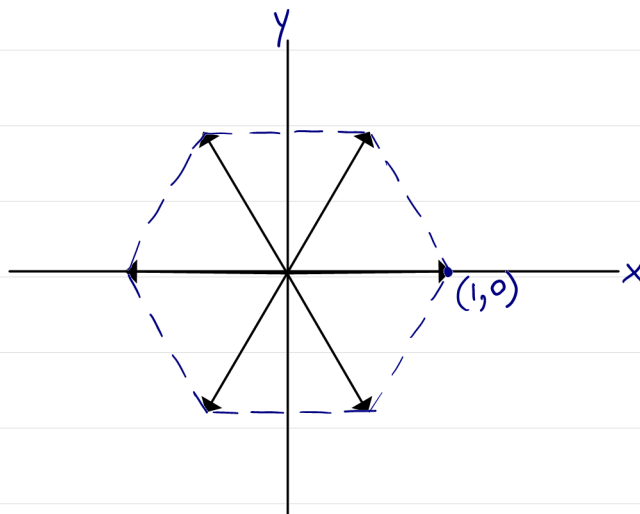
If $\vec{v} = (3,1)$ and $\vec{w} = (2,3)$
then

$$\vec{v} - \vec{w} = (3,1) - (2,3) = (1,-2).$$

Example. Suppose that a regular hexagon has its center at the point $(0,0)$ and one of its corners at the point $(1,0)$. Find the sum of the 6 vectors that point from its center to its corners.

Method 1: Compute entries
   of each vector using
   trig., add them up.
   (Ugly)
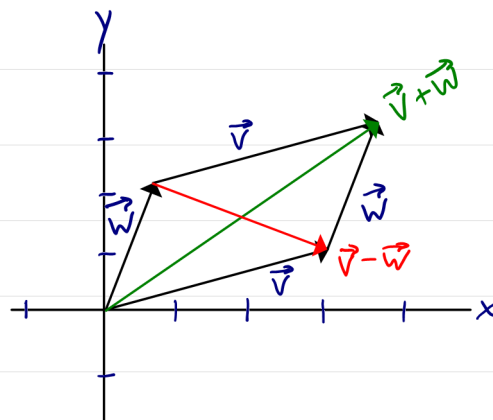
Method 2: Notice that the
   6 vectors come in
   3 $\pm$ pairs. Each pair
adds to $\vec{0}$, so the whole sum is $\vec{0}$.
(Much more elegant!)

If **v** and **w** are the (non-parallel) sides of a parallelogram, then $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ appear very naturally in that parallelogram...

Example. The parallelogram rule.

For a parallelogram with
side vectors $\vec{v}$ and $\vec{w}$,
the diagonals are

$\vec{v}+\vec{w}$ and $\vec{v}-\vec{w}$.

3-Dimensional Vectors

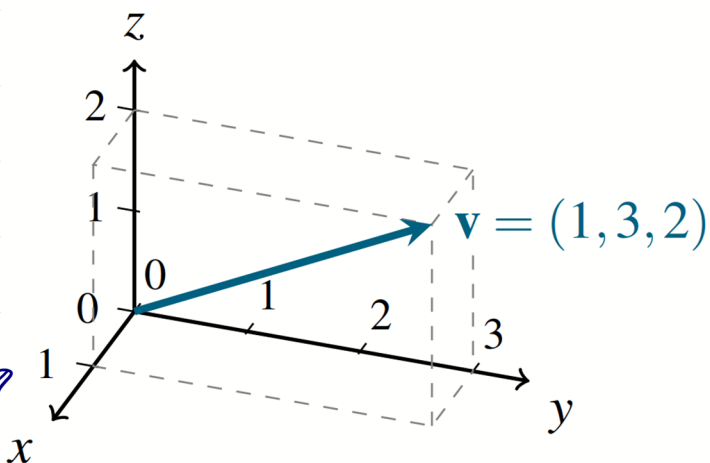Everything we have learned about vectors so far extends naturally to 3 dimensions.

There is nothing special about 2D!

A vector in 3 dimensions is an **ordered triple** like $(1, 3, 2)$, and the set of all ordered triples is denoted by $\mathbb{R}^3$. These are a bit harder to draw than their 2-dimensional counterparts, but it's still possible...

Example. Drawing 3D vectors.

We won't draw these by hand, but we need to at least be able to read computer-generated drawings.

$$z$$
$$\mathbf{v} = (1, 3, 2)$$
$$x \quad y$$

Everything we have seen in 2D carries over exactly how you would expect in 3D:

- Adding vectors still has the geometric interpretation of "following" both vectors, one after the other.

- Adding vectors has the same formula you might expect:

$$(v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

- Scalar multiplication still has the geometric interpretation of stretching the vector.

- Scalar multiplication has the same formula you might expect:

$$c(v_1, v_2, v_3) = (cv_1, cv_2, cv_3)$$

High-Dimensional Vectors

Everything we have learned about vectors so far extends naturally to 4 (and more!) dimensions.

There is nothing special about 3D!

We'll get a bit more general now and consider an *arbitrary* number of dimensions.

    A vector in n dimensions is an **ordered n-tuple** like $(1, 2, 3, \ldots, n)$, and the set of all ordered n-tuples is denoted by $\mathbb{R}^n$. These are a bit harder to draw than their 2- and 3-dimensional counterparts...

Example. Drawing 4D (and 5D, and 6D...) vectors.

NOPE!

We cannot quite picture these vectors like we could in 2D or 3D, but algebraically they work the same.

    However, everything *algebraic* that we have seen for 2D and 3D vectors carries over exactly how you would expect in higher dimensions:

- Adding vectors has the same formula you might expect:

$$\left(v_1, v_2, \ldots, v_n\right) + \left(w_1, w_2, \ldots, w_n\right) = \left(v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n\right)$$

- Scalar multiplication has the same formula you might expect:

$$c\left(v_1, v_2, \ldots, v_n\right) = \left(cv_1, cv_2, \ldots, cv_n\right)$$

Even though we can't draw vectors in $\mathbb{R}^n$, we still want to be able to manipulate them. We have seen that vector addition and scalar multiplication work the "naive" way. The following theorem shows some more properties that are similarly "obvious":

Theorem 1.1 — Properties of Vector Operations

Let $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^n$ be vectors and let $c, d \in \mathbb{R}$ be scalars. Then

a) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (commutativity)

b) $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$ (associativity)

c) $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$ (distributivity)

d) $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$ (distributivity)

e) $\mathbf{v} + \mathbf{0} = \mathbf{v}$

f) $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

g) $c(d\mathbf{v}) = (cd)\mathbf{v}$

Proof. We will prove property (a) in class; you can try to prove some of the others on your own (the method is quite similar).

Write $\vec{v} = (v_1, v_2, \ldots, v_n)$ and $\vec{w} = (w_1, w_2, \ldots, w_n)$.

Then

$$\vec{v} + \vec{w} = \left(v_1 + w_1, \ v_2 + w_2, \ldots, v_n + w_n \right)$$

$$= \left(w_1 + v_1, \ w_2 + v_2, \ldots, w_n + v_n \right) = \vec{w} + \vec{v}.$$

since addition on $\mathbb{R}$ is commutative

definition of vector addition

which completes the proof. ∎

Why did we even bother with the theorem on the previous page? They're all the type of thing you can just look at and "see" are true, right?

One reason is that we have to make sure that certain combinations of symbols even make sense when we are in new and unfamiliar settings. For example, associativity (property (b)) says that this expression makes sense:

$$\vec{v} + \vec{w} + \vec{x}.$$ We don't have to guess if it means $(\vec{v}+\vec{w})+\vec{x}$ or $\vec{v}+(\vec{w}+\vec{x})$.

We will soon introduce some operations that do not have these basic properties like commutativity, so we will have to start being very careful.

Example. Simplify $\mathbf{v} + 2(\mathbf{w} - \mathbf{v}) - 3(\mathbf{v} + 2\mathbf{w})$.

$$\vec{v} + 2(\vec{w}-\vec{v}) - 3(\vec{v}+2\vec{w}) = \vec{v} + 2\vec{w} - 2\vec{v} - 3\vec{v} - 6\vec{w}$$
$$= -4\vec{v} - 4\vec{w}$$
$$= -4(\vec{v}+\vec{w}) \longleftarrow \text{optional}$$

Only do vector addition and scalar multiplication!

Linear Combinations

One common task in linear algebra is to start out with some given collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and then use vector addition and scalar multiplication to construct new vectors out of them. The following definition gives a name to this concept.

Definition 1.1 — Linear Combinations

A **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is a vector of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Example. Show that $(1, 2, 3)$ is a linear combination of the vectors $(1, 1, 1)$ and $(-1, 0, 1)$.

We want to solve
$$(1,2,3) = c_1(1,1,1) + c_2(-1,0,1)$$
$$= (c_1 - c_2, c_1, c_1 + c_2).$$

$\therefore$ $c_1 = 2$ and $c_1 - c_2 = 1$, so $c_2 = 1$.

Example. Show that $(1, 2, 3)$ is *not* a linear combination of $(1, 1, 0)$ and $(2, 1, 0)$.

We want to solve
$$(1,2,3) = c_1(1,1,0) + c_2(2,1,0)$$
$$= (c_1 + 2c_2, c_1 + c_2, 0).$$

Third entry says $3 = 0$. No solution!

When working with linear combinations, some particularly important vectors are the ones with a single 1 in one of their entries, and all other entries equal to 0. These are called the **standard basis vectors**:

$$\vec{e}_1 = (1, 0, \ldots, 0), \quad \vec{e}_2 = (0, 1, 0, \ldots, 0), \quad \text{and} \quad \text{in} \quad \text{general}$$
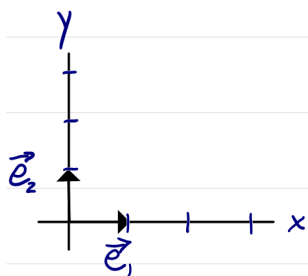$$\vec{e}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0). \qquad -j\text{-th entry}$$

Example. List and draw all of the standard basis vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

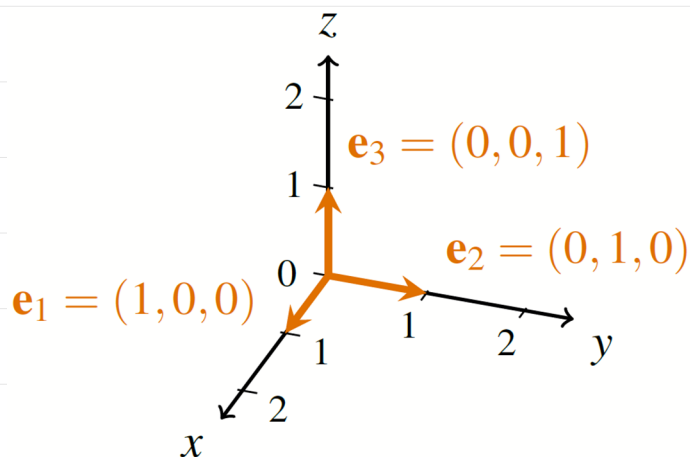<u>In $\mathbb{R}^2$:</u>

$\vec{e}_1 = (1, 0)$

$\vec{e}_2 = (0, 1)$

<u>In $\mathbb{R}^3$:</u>

$\vec{e}_1 = (1, 0, 0)$

$\vec{e}_2 = (0, 1, 0)$

$\vec{e}_3 = (0, 0, 1)$

$e_3 = (0, 0, 1)$

$e_2 = (0, 1, 0)$

$e_1 = (1, 0, 0)$

For now, the reason for our interest in these standard basis vectors is that every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ can be written as a linear combination of them. In particular,

$$\vec{v} = v_1 \vec{e_1} + v_2 \vec{e_2} + \cdots + v_n \vec{e_n} \, .$$

This idea of writing vectors in terms of the standard basis vectors is one of the most useful tricks that we make use of in linear algebra: in many situations, if we can prove that some property holds for the standard basis vectors, then we can use linear combinations to show that it must hold for *all* vectors.

Example. Compute $3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3 \in \mathbb{R}^3$.

$$3\vec{e_1} - 2\vec{e_2} + \vec{e_3} = 3(1,0,0) - 2(0,1,0) + (0,0,1)$$
$$= (3,0,0) + (0,-2,0) + (0,0,1) = (3,-2,1).$$

Example. Write $(3, 5, -2, -1)$ as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \in \mathbb{R}^4$.

$$(3,5,-2,-1) = 3\vec{e_1} + 5\vec{e_2} - 2\vec{e_3} - \vec{e_4}.$$

Just read the coefficients!