Linear Transformations

This week we will learn about:

- Understanding linear transformations geometrically,
- The standard matrix of a linear transformation, and
- Composition of linear transformations.

Extra reading and watching:

- Section 1.4 in the textbook
- Lecture videos 13, 14, 15, and 16 on YouTube
- Linear map at Wikipedia

Extra textbook problems:

$\star$ 1.4.1, 1.4.4, 1.4.5(a,b,e,f)

$\star\star$ 1.4.2, 1.4.3, 1.4.6, 1.4.7(a,b), 1.4.8, 1.4.14–1.4.16

$\star\star\star$ 1.4.18, 1.4.22, 1.4.23

☠ 1.4.19, 1.4.20

Linear Transformations

The final main ingredient of linear algebra, after vectors and matrices, are linear transformations: functions that act on vectors and that do not "mess up" vector addition and scalar multiplication:

Definition 4.1 — Linear Transformations

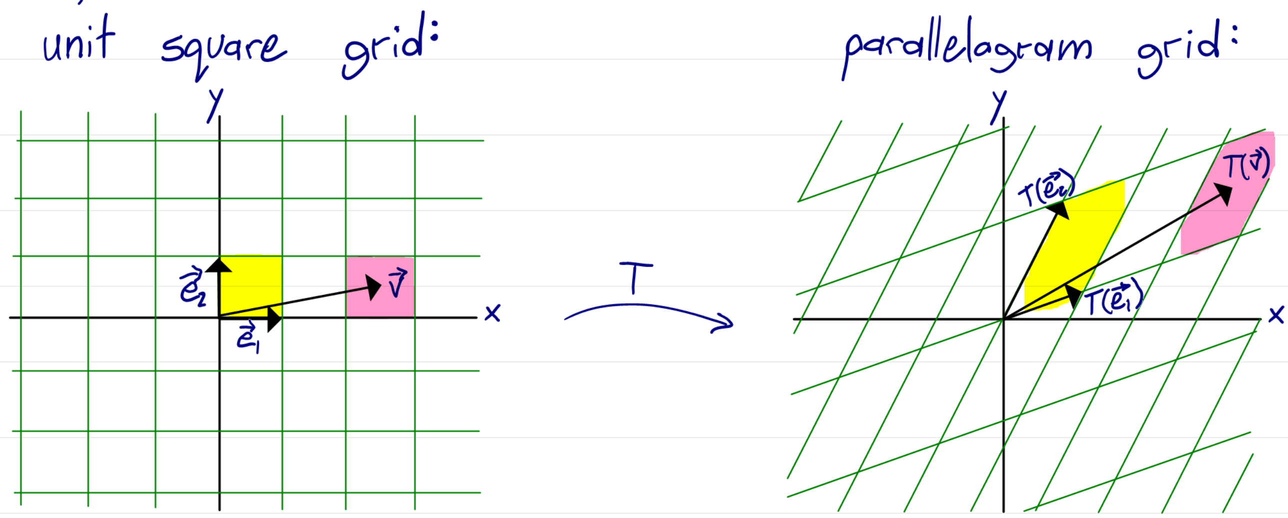
A **linear transformation** is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies the following two properties:

a) $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and

b) $T(c\mathbf{v}) = cT(\mathbf{v})$ for all vectors $\mathbf{v} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$.

Before looking at specific examples of linear transformations, let's think geometrically about what they do to $\mathbb{R}^n$:

Well, in $\mathbb{R}^2$ at least...



Another way of thinking about this: linear transformations are exactly the functions that preserve linear combinations:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_kT(\vec{v}_k)$$

for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$.

Example. Which of the following functions are linear transformations?

① Given $A \in M_{m,n}$, the function $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{v}) = A\vec{v}$. (as a column vector)

theorem from last week

(a) $T(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w})$ ✓

(b) $T(c\vec{v}) = A(c\vec{v}) = cA\vec{v} = cT(\vec{v})$ ✓

$\therefore$ T is a linear transformation.

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② $T: \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(x,y) = (x-y, x+y)$.

(a) $T(\vec{v}+\vec{w}) = T((v_1,v_2)+(w_1,w_2)) = T(v_1+w_1, v_2+w_2)$
$= ((v_1+w_1)-(v_2+w_2), (v_1+w_1)+(v_2+w_2))$, and

$T(\vec{v})+T(\vec{w}) = (v_1-v_2, v_1+v_2) + (w_1-w_2, w_1+w_2)$
$= ((v_1-v_2)+(w_1-w_2), (v_1+v_2)+(w_1+w_2))$ the same! ✓

(b) $T(c\vec{v}) = T(cv_1, cv_2) = (cv_1 - cv_2, cv_1 + cv_2)$
$= c(v_1-v_2, v_1+v_2) = cT(v_1,v_2)$. ✓

$\therefore$ T is a linear transformation.

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③ $T: \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(x,y) = (3x^2+y, 2x-4)$.

Let $\vec{v} = (1,1)$ and $\vec{w} = (2,3)$. Then

$T(\vec{v}+\vec{w}) = T(3,4) = (3(3^2)+4, 2(3)-4) = (31, 2)$, but

$T(\vec{v})+T(\vec{w}) = T(1,1) + T(2,3) = (4,-2) + (15,0) = (19,-2)$.

not the same!

$\therefore$ T is NOT a linear transformation.

Recall that every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ can be written in the form

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n. \qquad \Big(\vec{e}_j = (0,\ldots,0,\underset{\text{j-th entry}}{1},0,\ldots,0)\Big)$$

By using the fact that linear transformations preserve linear combinations, we see that

$$T(\vec{v}) = T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n) = v_1T(\vec{e}_1) + v_2T(\vec{e}_2) + \cdots + v_nT(\vec{e}_n).$$

But this is exactly what we said before: if $\mathbf{v} \in \mathbb{R}^2$ extends a distance of v_1 in the direction of $\mathbf{e}_1$ and a distance of v_2 in the direction of $\mathbf{e}_2$, then $T(\mathbf{v})$ extends the same amounts in the directions of $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, respectively.

This also tells us one of the most important facts to know about linear transformations:

Every linear transformation T is completely determined by the vectors $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$.
(This is very different from general functions! If you know $f(3)=4$ and $f(4)=7$, and any other number of points, we still don't know f.)

Example. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation for which $T(\mathbf{e}_1) = (1, 1)$ and $T(\mathbf{e}_2) = (-1, 1)$. Compute $T(2, 3)$ and then find a general formula for $T(v_1, v_2)$

$(2,3) = 2\vec{e}_1 + 3\vec{e}_2$, so

$$T(2,3) = T(2\vec{e}_1 + 3\vec{e}_2) = 2T(\vec{e}_1) + 3T(\vec{e}_2) = 2(1,1) + 3(-1,1) = (-1,5).$$

In general,

$$T(v_1,v_2) = T(v_1\vec{e}_1 + v_2\vec{e}_2) = v_1T(\vec{e}_1) + v_2T(\vec{e}_2)$$
$$= v_1(1,1) + v_2(-1,1) = (v_1 - v_2, v_1 + v_2).$$

One of the earlier examples showed that if $A \in \mathcal{M}_{m,n}$ is a matrix, then the function $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation. Amazingly, the converse is also true: *every* linear transformation can be written as matrix multiplication.

Theorem 4.1 — Standard Matrix of a Linear Transformation

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists a matrix $[T] \in \mathcal{M}_{m,n}$ such that

$$T(\mathbf{v}) = [T]\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathbb{R}^n.$$

Furthermore, the unique matrix $[T]$ with this property is called the **standard matrix** of T, and it is

$$[T] = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n) \right].$$

Proof. We already proved the "if" direction, so we just need to prove the "only if" direction. That is, we want to prove that if T is a linear transformation, then $T(\mathbf{v}) = [T]\mathbf{v}$, where the matrix $[T]$ is as defined in the theorem.

Well, $$[T]\vec{v} = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n) \right] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2) + \cdots + v_n T(\vec{e}_n) \qquad \text{(block matrix mult.)}$$

$$= T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n) \qquad \text{(T is a lin. trans.)}$$

$$= T(\vec{v}). \qquad \checkmark$$

We also need to show that $[T]$ is unique. To this end, suppose $A \in \mathcal{M}_{m,n}$ is such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$. Let $1 \leq j \leq n$ and notice that $A\vec{e}_j$ is the j-th column of A, which must equal $T(\vec{e}_j)$. That is, $A = [T]$. ■

Example. Find the standard matrix of the following linear transformations:

$T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(v_1, v_2) = (v_1 - v_2, v_1 + v_2)$.

$T(\vec{e}_1) = T(1,0) = (1,1)$ and $T(\vec{e}_2) = T(0,1) = (-1,1)$, so

$$[T] = [T(\vec{e}_1) \mid T(\vec{e}_2)] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

same!

Let's check: $[T]\vec{v} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 - v_2 \\ v_1 + v_2 \end{bmatrix}$.

$T : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $T(x,y) = (x + 2y, 3x - \frac{7}{4}y, \pi x)$.

$T(\vec{e}_1) = T(1,0) = (1, 3, \pi)$ and $T(\vec{e}_2) = (2, \frac{-7}{4}, 0)$.

$$[T] = [T(\vec{e}_1) \mid T(\vec{e}_2)] = \begin{bmatrix} 1 & 2 \\ 3 & -7/4 \\ \pi & 0 \end{bmatrix}.$$

the same!

Let's check: $[T]\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & -7/4 \\ \pi & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x - \frac{7}{4}y \\ \pi x \end{bmatrix}$.

A Catalog of Linear Transformations

To get more comfortable with the relationship between linear transformations and matrices, let's find the standard matrices of a few linear transformations that come up fairly frequently.

Example. The zero and identity transformations.

Zero transformation: $O(\vec{v}) = \vec{0}$ for all $\vec{v} \in \mathbb{R}^n$

$[O] = O$ (zero matrix)

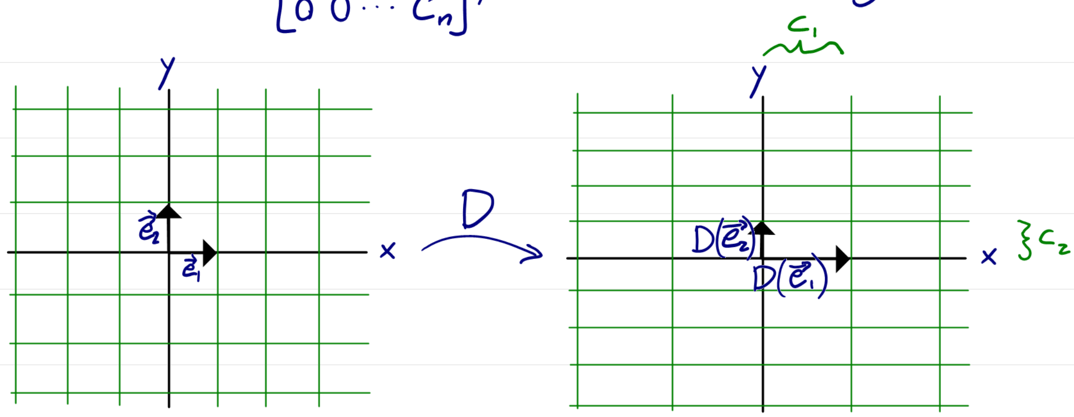
Identity transformation: $I(\vec{v}) = \vec{v}$ for all $\vec{v} \in \mathbb{R}^n$

$$[I] = I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$ (identity matrix)

Example. Diagonal transformations/matrices.

$D: \mathbb{R}^n \to \mathbb{R}^n$, defined by $D(v_1, v_2, \ldots, v_n) = (c_1 v_1, c_2 v_2, \ldots, c_n v_n)$ for some fixed $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

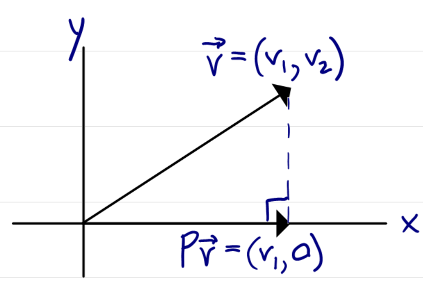
Then $$[D] = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix},$$ which is diagonal.



Example. Projection onto the x-axis.

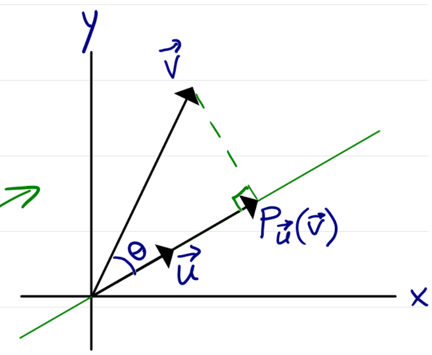
We want to find a matrix P so that $P\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$.

$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ works!



Example. Projection onto a line, $P_{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^n$.

$\vec{u}$ is a (given) unit vector specifying the line that $P_{\vec{u}}$ projects onto.



$P_{\vec{u}}(\vec{v})$ points in the same direction as $\vec{u}$, so...

$\|P_{\vec{u}}(\vec{v})\| = \|\vec{v}\| \cos(\theta)$ (From this triangle)

$$= \cancel{\|\vec{v}\|} \left(\frac{\vec{v} \cdot \vec{u}}{\cancel{\|\vec{v}\|} \underbrace{\cancel{\|\vec{u}\|}}_{=1}} \right) = \vec{v} \cdot \vec{u}, \quad \text{so}$$

$$P_{\vec{u}}(\vec{v}) = \|P_{\vec{u}}(\vec{v})\| \vec{u} = (\vec{v} \cdot \vec{u}) \vec{u} = \vec{u} (\vec{u}^T \vec{v}) = (\vec{u} \vec{u}^T) \vec{v}.$$

$\therefore \quad [P_{\vec{u}}] = \vec{u} \vec{u}^T$ ($n \times 1$ times $1 \times n$ = $n \times n$ matrix)

Example. Find the standard matrix of the linear transformation that projects $\mathbb{R}^3$ onto the line in the direction of the vector... $\vec{w} = (2, 1, -2)$.

First, normalize $\vec{w}$: $\|\vec{w}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$, so

$\vec{u} = \vec{w} / \|\vec{w}\| = (2, 1, -2)/3$.

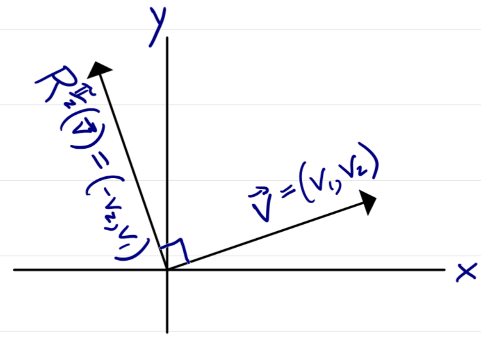
Then $$[P_{\vec{u}}] = \vec{u} \vec{u}^T = \left(\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right) \left(\frac{1}{3} \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \right) = \frac{1}{9} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}.$$

E.g., $$P_{\vec{u}}(1,2,3) = \frac{1}{9} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}.$$

Example. Rotation counter-clockwise around the origin by 90° ($\pi/2$ radians).

We know that $R^{\frac{\pi}{2}}(v_1, v_2)$ is orthagonal to (v_1, v_2), so it is a multiple of $(-v_2, v_1)$.

Since $\|R^{\frac{\pi}{2}}(\vec{v})\| = \|\vec{v}\|$, we see that $R^{\frac{\pi}{2}}(v_1, v_2) = (-v_2, v_1)$.

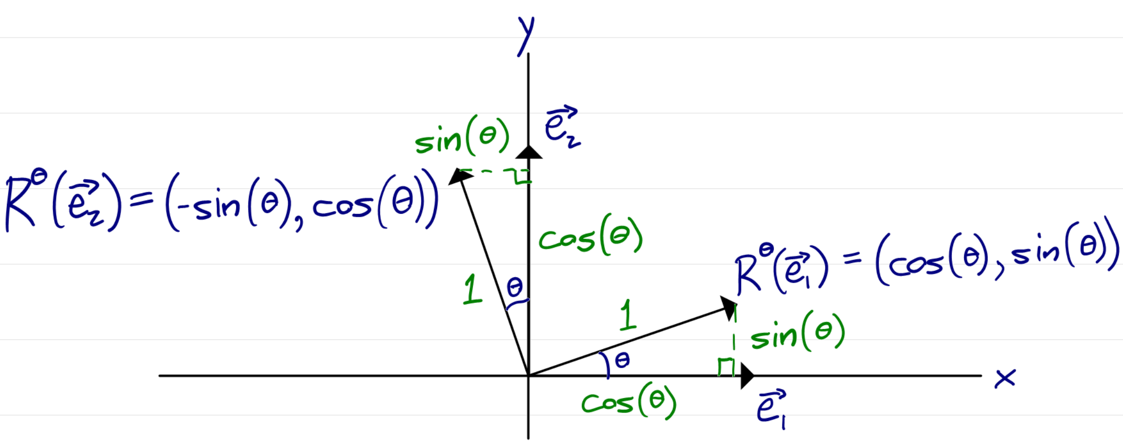


We want to find $[R^{\frac{\pi}{2}}]$ so that $[R^{\frac{\pi}{2}}]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$.

$[R^{\pi/2}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ works!

Example. Rotation $R^\theta : \mathbb{R}^2 \to \mathbb{R}^2$ counter-clockwise around the origin by an angle of θ.

We skip the proof of linearity and just compute $[R^\theta] = [R^\theta(\vec{e}_1) \mid R^\theta(\vec{e}_2)]$.



$$\therefore\ [R^\theta] = [R^\theta(\vec{e}_1) \mid R^\theta(\vec{e}_2)] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Example. What vector is obtained if we rotate $\mathbf{v} = (1, 3)$ by $\pi/4$ radians counter-clockwise?

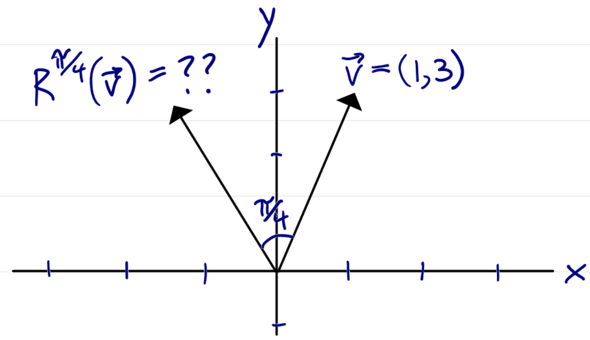
$$R^{\pi/4}(1,3) = \left[R^{\pi/4}\right]\begin{bmatrix}1\\3\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\pi/4) & -\sin(\pi/4)\\ \sin(\pi/4) & \cos(\pi/4)\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$$

$$= \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$$

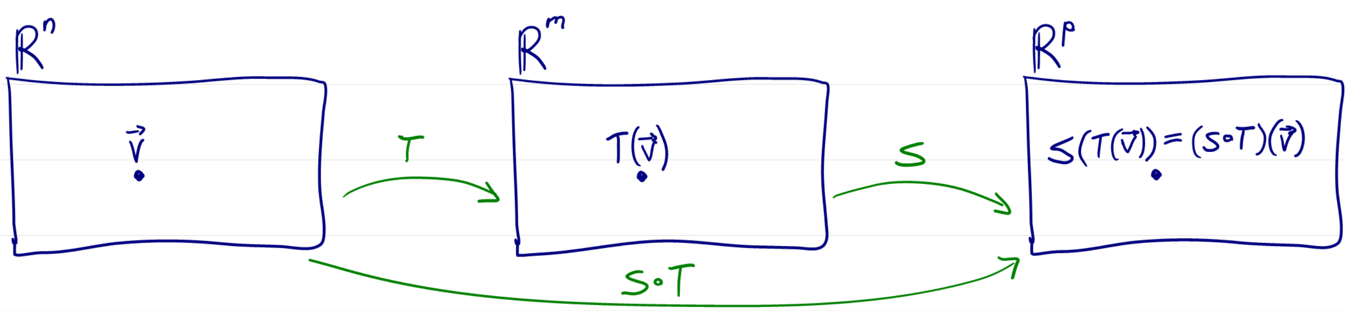
$$= \begin{bmatrix}-2/\sqrt{2}\\ 4/\sqrt{2}\end{bmatrix} = \begin{bmatrix}-\sqrt{2}\\ 2\sqrt{2}\end{bmatrix}.$$

$\cos(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$



Composing Linear Transformations

If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are linear transformations, then we can consider the function defined by first applying T to a vector, and then applying S. This function is called the **composition** of T and S, and is denoted by $S \circ T$.



Formally, the composition $S \circ T$ is defined by $(S \circ T)(\mathbf{v}) = S(T(\mathbf{v}))$ for all vectors $\mathbf{v} \in \mathbb{R}^n$. It turns out that $S \circ T$ is a linear transformation whenever S and T are linear transformations themselves, as shown by the next theorem.

Theorem 4.2 — Composition of Linear Transformations

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are linear transformations with standard matrices $[T] \in \mathcal{M}_{m,n}$ and $[S] \in \mathcal{M}_{p,m}$, respectively. Then $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation, and its standard matrix is $[S \circ T] = [S][T]$.

Proof. Let $\mathbf{v} \in \mathbb{R}^n$ and compute $(S \circ T)(\mathbf{v})$:

$$(S\circ T)(\vec{v}) = S(T(\vec{v})) = S([T]\vec{v}) = [S][T]\vec{v} = \underbrace{([S][T])}\vec{v}.$$

$\therefore$ $S\circ T$ is a linear transformation with standard matrix $[S\circ T] = [S][T]$. ■

The previous theorem shows us that matrix multiplication tells us how the composition of linear transformations behaves. In fact, this is exactly why matrix multiplication is defined the way it is.

Example. What vector is obtained if we rotate $\mathbf{v} = (4, 2)$ 45° counter-clockwise around the origin and then project it onto the line $y = 2x$?

Need $[R^{\pi/4}]$ and $[P_{\vec{u}}]$, where $\vec{u}$ is a unit vector on the line $y=2x$.

$$[R^{\pi/4}] = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

To find $\vec{u}$, set $\vec{w} = (1,2)$ and normalize:

$\|\vec{w}\| = \sqrt{1^2+2^2} = \sqrt{5}$, so $\vec{u} = \frac{1}{\sqrt{5}}(1,2)$, so

$$[P_{\vec{u}}] = \vec{u}\,\vec{u}^T = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Then $[P_{\vec{u}}\circ R^{\pi/4}] = [P_{\vec{u}}][R^{\pi/4}] = \frac{1}{5\sqrt{2}}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}.$

$$\therefore\ (P_{\vec{u}}\circ R^{\pi/4})(4,2) = \frac{1}{5\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix} 14 \\ 28 \end{bmatrix} = \frac{7\sqrt{2}}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Example. Find the standard matrix of the linear transformation T that projects $\mathbb{R}^2$ onto the line $y = (4/3)x$ and then stretches it in the x-direction by a factor of 2 and in the y-direction by a factor of 3.

Need $[P_{\vec{u}}]$, where $\vec{u}$ is a unit vector on the line $y = \frac{4}{3}x$. Set $\vec{w} = (3,4)$ and normalize: $\|\vec{w}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$, so $\vec{u} = \frac{1}{5}(3,4)$.

$$\therefore\ [P_{\vec{u}}] = \vec{u}\,\vec{u}^T = \left(\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right)\left(\frac{1}{5}[3,4]\right) = \frac{1}{25}\begin{bmatrix}9 & 12\\12 & 16\end{bmatrix}.$$

Also, $[D] = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$, so

$$[T] = [D\circ P_{\vec{u}}] = [D][P_{\vec{u}}] = \frac{1}{25}\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}9 & 12\\12 & 16\end{bmatrix} = \frac{1}{25}\begin{bmatrix}18 & 24\\36 & 48\end{bmatrix}.$$

Example. Derive the angle-sum formulas for sin and cos.

Rotating by an angle of θ and then φ is the same as rotating by an angle of $\theta+\varphi$, so $R^{\theta+\varphi} = R^{\theta}\circ R^{\varphi}$.

$$\therefore\ [R^{\theta+\varphi}] = [R^{\theta}\circ R^{\varphi}] = [R^{\theta}][R^{\varphi}]$$

compare entries!

Well, $[R^{\theta+\varphi}] = \begin{bmatrix}\cos(\theta+\varphi) & -\sin(\theta+\varphi)\\ \sin(\theta+\varphi) & \cos(\theta+\varphi)\end{bmatrix}$ and

$$[R^{\theta}][R^{\varphi}] = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\cos(\varphi) & -\sin(\varphi)\\ \sin(\varphi) & \cos(\varphi)\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) & -\cos(\theta)\sin(\varphi) - \sin(\theta)\cos(\varphi)\\ \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi) & -\sin(\theta)\sin(\varphi) + \cos(\theta)\cos(\varphi)\end{bmatrix}.$$