Linear Algebra – Week 6

Elementary Matrices and Inverses

This week we will learn about:

- Elementary matrices,
- The inverse of a matrix, and
- How awesome inverses are.

Extra reading and watching:

- Section 2.2 in the textbook
- Lecture videos 22, 23, 24, and 25 on YouTube
- Elementary matrix at Wikipedia
- Invertible matrix at Wikipedia

Extra textbook problems:

$\star$ 2.2.1, 2.2.2

$\star\star$ 2.2.4–2.2.6, 2.2.8, 2.2.9, 2.2.13, 2.2.15, 2.2.20

$\star\star\star$ 2.2.7, 2.2.10, 2.2.11, 2.2.21, 2.2.22

☠ 2.2.23

Elementary Matrices

Last week, we learned how to solve systems of linear equations by repeatedly applying one of three row operations to the augmented matrix associated with that linear system. Remarkably, all three of those row operations can be carried out by matrix multiplication (on the left) by carefully-chosen matrices.

For example, if we wanted to swap the first and second rows of the matrix

$$\begin{bmatrix} 0 & 2 & 3 & -2 \\ 1 & 8 & 0 & 1 \\ 3 & -3 & 6 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 8 & 0 & 1 \\ 0 & 2 & 3 & -2 \\ 3 & -3 & 6 & -4 \end{bmatrix}$$

we could multiply it on the left by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}: \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 3 & -2 \\ 1 & 8 & 0 & 1 \\ 3 & -3 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 0 & 1 \\ 0 & 2 & 3 & -2 \\ 3 & -3 & 6 & -4 \end{bmatrix}$$

Similarly, to perform the row operations $R_3 - 3R_1$ and $\frac{1}{2}R_2$ we could multiply on the left by the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}: \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 8 & 0 & 1 \\ 0 & 2 & 3 & -2 \\ 3 & -3 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 0 & 1 \\ 0 & 2 & 3 & -2 \\ 0 & -27 & 6 & -7 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}: \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 8 & 0 & 1 \\ 0 & 2 & 3 & -2 \\ 0 & -27 & 6 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 0 & 1 \\ 0 & 1 & 3/2 & -1 \\ 0 & -27 & 6 & -7 \end{bmatrix}$$

Matrices that implement one of these three row operations in this way have a name:

Definition 6.1 — Elementary Matrices

A square matrix $A \in \mathcal{M}_n$ is called an **elementary matrix** if it can be obtained from the identity matrix via a single row operation.

For example, the elementary matrix corresponding to the "Swap" row operation $R_i \leftrightarrow R_j$ looks like

$$\begin{array}{l} \\ \\ \text{rows } i \text{ and} \rightarrow \\ j \text{ swapped} \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Similarly, the elementary matrices corresponding to the "Addition" row operation $R_i + cR_j$ and the "Multiplication" row operation cR_i look like

$$\begin{array}{r} \\ \\ R_j \\ \\ R_i + cR_j \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \vdots & \ddots & & \\ & & c & \cdots & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \quad \text{and} \quad \begin{array}{r} \\ \\ cR_j \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & c & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

Notice that if the elementary matrices $E_1, E_2, \ldots, E_k$ are used to row reduce a matrix A to its reduced row echelon form R, then

$$R = E_k \cdots E_2 E_1 A.$$

In particular, $E_1, E_2, \ldots, E_k$ act as a log that keeps track of which row operations should be performs to put A into RREF. Furthermore, if we define $E = E_k \cdots E_2 E_1$, then $R = EA$, so E acts as a condensed version of that log. Let's now do an example to see how to construct this matrix E.

Example. Let $A = \begin{bmatrix} 0 & 2 & 4 & 0 \\ 1 & 1 & 0 & -1 \\ 3 & 4 & 2 & 1 \end{bmatrix}$.

Find a matrix E such that $EA = R$, where R is the RREF of A.

$$\left[\begin{array}{cccc|ccc} 0 & 2 & 4 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 3 & 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-3R_1}\left[\begin{array}{cccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4 & 0 & -3 & 1 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 4 & 0 & -3 & 1 \\ 0 & 2 & 4 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3-2R_2]{R_1-R_2}\left[\begin{array}{cccc|ccc} 1 & 0 & -2 & -5 & 0 & 4 & -1 \\ 0 & 1 & 2 & 4 & 0 & -3 & 1 \\ 0 & 0 & 0 & -8 & 1 & 6 & -2 \end{array}\right] \xrightarrow{-\frac{1}{8}R_3} \left[\begin{array}{cccc|ccc} 1 & 0 & -2 & -5 & 0 & 4 & -1 \\ 0 & 1 & 2 & 4 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & -\frac{1}{8} & -\frac{3}{4} & \frac{1}{4} \end{array}\right]$$

$$\xrightarrow[R_2-4R_3]{R_1+5R_3}\left[\begin{array}{cccc|ccc} 1 & 0 & -2 & 0 & -\frac{5}{8} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{8} & -\frac{3}{4} & \frac{1}{4} \end{array}\right].$$

$\therefore$ If we define $E = \begin{bmatrix} \frac{5}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$ then EA is the RREF of A.

Check: $$EA = \begin{bmatrix} \frac{5}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & 2 & 4 & 0 \\ 1 & 1 & 0 & -1 \\ 3 & 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \checkmark$$

The fact that the method of the previous example works in general can be seen by combining some block matrix multiplication trickery with the fact that multiplication on the left by an elementary matrix is equivalent to performing the corresponding row operation. In particular, if row reducing $[\,A \mid I\,]$ to some other matrix $[\,R \mid E\,]$ makes use of the row operations corresponding to elementary matrices $E_1, E_2, \ldots, E_k$, then

$$[\,R \mid E\,] = E_k \cdots E_2 E_1 [\,A \mid I\,] = [\,E_k \cdots E_2 E_1 A \mid E_k \cdots E_2 E_1\,].$$

This means (by looking at the right half of the above block matrix) that $E = E_k \cdots E_2 E_1$, which then implies (by looking at the left half of the block matrix) that $R = EA$. We state this observation as a theorem:

Theorem 6.1 — Row Reduction is Multiplication on the Left

If the block matrix $[\,A \mid I\,]$ can be row reduced to $[\,R \mid E\,]$ then... $R = EA$.

This theorem says that, not only is performing a single row operation equivalent to multiplication on the left by an elementary matrix, but performing a *sequence* of row operations is also equivalent to multiplication on the left (by some potentially non-elementary matrix).

The Inverse of a Matrix

When working with (non-zero) real numbers, we have an operation called "division," which acts as an inverse of multiplication. In particular, $a(1/a) = 1$ for all $a \neq 0$. It turns out that we can (usually) do something very similar for matrix multiplication:

Definition 6.2 — Inverse of a Matrix

If A is a square matrix, the **inverse** of A, denoted by A^{-1}, is a matrix (of the same size as A) with the property that

$$A^{-1}A = AA^{-1} = I.$$

If such a matrix A^{-1} exists, then A is called **invertible**.

Inverses (when they exist) are unique (i.e., every matrix has at most one inverse). To see this...

Suppose A has two inverses, B and C.

Then $BAC = B(AC) = BI = B$, and
$BAC = (BA)C = IC = C$, so
$B = C$. ✓

Example. Show that $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

Just multiply: $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Also: $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

So if we are given a particular pair of matrices, it is easy to check whether or not they are inverses of each other. But how could we *find* the inverse of a matrix in the first place? We'll see how soon!

As always, let's think about what properties our new mathematical operation (matrix inversion) has.

Theorem 6.2 — Properties of Matrix Inverses

Let A and B be invertible matrices of the same size, and let c be a non-zero real number. Then

a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

b) cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$

c) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

d) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ ← not a typo! not equal to $A^{-1}B^{-1}$!

Proof. Most parts of this theorem are intuitive enough, so we just prove part (d) (you can prove parts (a), (b) and (c) on your own: they're similar). To this end...

To verify that matrices are inverses of each other, just multiply:

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$ ✓

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$ ✓

$\therefore$ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. ∎

The fact that $(AB)^{-1} = B^{-1}A^{-1}$ (as opposed to the incorrect $(AB)^{-1} = A^{-1}B^{-1}$) is actually intuitive enough: you put on your socks before your shoes, but when reversing that operation, you take off your shoes before your socks.

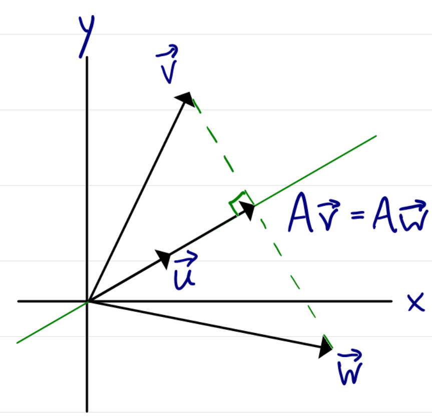
Not every matrix is invertible. For example,

the zero matrix: $OA = O$ for all A, so $OA \neq I$.
1×1 case: 0 (the number) is not invertible.

However, there are even more exotic examples of non-invertible matrices. For example, recall that if $\mathbf{u}$ is a unit vector then the matrix $A = \mathbf{u}\mathbf{u}^T$...

projects $\mathbb{R}^n$ onto the line in the direction of $\vec{u}$.

There exist $\vec{v} \neq \vec{w}$ such that $A\vec{v} = A\vec{w}$. If A were invertible, we would have $A^{-1}(A\vec{v}) = A^{-1}(A\vec{w})$, so $\vec{v} = \vec{w}$ (contradiction).



In order to come up with a general method for determining whether or not a matrix is invertible (and constructing its inverse if it exists), we first notice that if A has reduced row echelon form equal to I, then Theorem 6.1 tells us that

row-reducing $[A \mid I]$ to $[I \mid E]$ gives $EA = I$, which hints at $E = A^{-1}$.

It thus seems like A being invertible is closely related to whether or not it can be row reduced to the identity matrix. The following theorem shows that this is indeed the case (along with a whole lot more):

Theorem 6.3 — Characterization of Invertible Matrices

Let $A \in \mathcal{M}_n$. The following are equivalent:

a) A is invertible.

b) The reduced row echelon form of A is I (the identity matrix).

c) There exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $A = E_1 E_2 \cdots E_k$.

d) The linear system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$.

e) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.

f) The linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

Example. Determine whether or not the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is invertible.

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ This RREF is not I!

$\therefore$ A is not invertible.

$\begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} \xrightarrow{\frac{-1}{5}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓ $\therefore$ Invertible.

We won't rigorously prove the above theorem, but we'll try to give a rough idea for why some of its equivalences hold. First, notice that every elementary matrix is invertible:

Every row operation can be undone by a row operation of the same type:

- $R_i + cR_j$ is undone by $R_i - cR_j$
- cR_j is undone by $\frac{1}{c}R_j$
- $R_i \leftrightarrow R_j$ is undone by itself.

For example, $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$,

and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Since the product of invertible matrices is still invertible, it follows that any matrix of the form $A = E_1E_2\cdots E_k$ (where $E_1, E_2, \ldots, E_k$ are elementary) is invertible, which shows why (c) $\implies$ (a).

The connection between invertibility and linear systems can be clarified by noting that if A is invertible, then we can rearrange the linear system

$$A\vec{x} = \vec{b} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}. \qquad \text{(unique!)}$$

Thus (a) $\implies$ (e), which implies each of (d) and (f).

When we combine our previous two theorems, we get a method for not only determining whether or not a matrix is invertible, but also for computing its inverse if it exists:

Theorem 6.4 — How to Compute Inverses

A matrix $A \in \mathcal{M}_n$ is invertible if and only if the RREF of $[\,A \mid I\,]$ has the form $[\,I \mid B\,]$ for some $B \in \mathcal{M}_n$. If the RREF has this form then $A^{-1} = B$.

Example. Determine whether or not the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible, and find its inverse if it exists.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{=I\ \checkmark}} & \multicolumn{2}{c}{\overbrace{\begin{matrix} -2 & 1 \\ 3/2 & -1/2 \end{matrix}}^{=A^{-1}}} \end{array}\right].$$

$\therefore$ A is invertible an $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$.

Check: $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Could check $A^{-1}A = I$ too.

Example. Solve the linear system $x + 2y = 3$, $3x + 4y = 5$.

In matrix notation, $\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 5 \end{bmatrix}}_{\vec{b}}$.

We just showed that $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, so

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

is the unique solution.

Example. Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$ if it exists.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \\ 0 & -2 & 6 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 + 2R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & 0 & -2 \\ 0 & 1 & -2 & -1 & 0 & 1 \\ 0 & 0 & 2 & -4 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & 0 & -2 \\ 0 & 1 & -2 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[R_2+2R_3]{R_1-3R_3} \left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{=I} & \underbrace{\begin{matrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{matrix}}_{\text{inverse}} \end{array}\right]$$

$$\therefore \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$$

Example. Find the inverse of $\begin{bmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}$ if it exists.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ -4 & -1 & 6 & 0 & 1 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+2R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & -5 & -3 & 1 \end{array}\right]$$

zero row, so not invertible

Using our characterization of invertible matrices, we can prove all sorts of nice properties of them. For example, even though the definition of invertibility required that both $AA^{-1} = I$ and $A^{-1}A = I$, the following theorem shows that it is enough to just multiply on the left *or* the right: you don't need to check both.

Theorem 6.5 — One-Sided Matrix Inverses

Let $A \in \mathcal{M}_n$ be a square matrix. If $B \in \mathcal{M}_n$ is a matrix such that either $AB = I$ or $BA = I$, then A is invertible and $A^{-1} = B$.

Proof. Suppose $BA = I$, and consider the equation $A\mathbf{x} = \mathbf{0}$.

Then $BA\vec{x} = B(A\vec{x}) = B\vec{0} = \vec{0}$, but also
$BA\vec{x} = (BA)\vec{x} = I\vec{x} = \vec{x}$, so
$\vec{x} = \vec{0}$ is the unique solution of $A\vec{x} = \vec{0}$.

By Theorem 6.3, A is invertible. Multiplying $BA = I$ on the right shows that $B = A^{-1}$.

This completes the proof of the $BA = I$ case. Try to prove the case when $AB = I$ on your own. ■

Similarly, we can even come up with an explicit formula for the inverse of matrices in certain small cases. For example, for 2×2 matrices, we have the following formula:

Theorem 6.6 — Inverse of a 2×2 Matrix

Suppose A is the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then A is invertible if and only if $ad - bc \neq 0$, and if it is invertible then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. If $ad - bc \neq 0$ then we can show that the inverse of A is as claimed just by multiplying it by A:

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

On the other hand, if $ad - bc = 0$ then $ad = bc$.

Case 1: $a = 0$ or $b = 0$. Then A has a $\vec{0}$ row or column.

Case 2: $a \neq 0$ and $b \neq 0$. Then $d/b = c/a$, so the bottom row is a multiple of the top row.

In either case, the RREF of A has a $\vec{0}$ row, so A is not invertible. ■

Example. Compute the inverse (or show that none exists) of the following matrices:

$A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$ $ad - bc = 12 + 2 = 14$, so $A^{-1} = \frac{1}{14}\begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$.

$B = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$ $ad - bc = 12 - 12 = 0$, so B not invertible.

$C = \begin{bmatrix} \pi & 3.5 \\ -2 & 6 \end{bmatrix}$ $ad - bc = 6\pi + 7$, so $C^{-1} = \frac{1}{6\pi + 7}\begin{bmatrix} 6 & -3.5 \\ 2 & \pi \end{bmatrix}$.

Keep in mind that you can always use the general method of computing inverses (row reduce $[\, A \mid I \,]$ to $[\, I \mid A^{-1} \,]$) if you forget this formula for the 2×2 case.