Linear Algebra – Week 7

Subspaces, Spans, and Linear Independence

This week we will learn about:

- Subspaces,
- The span of a set of vectors, and
- Linear (in)dependence.

Extra reading and watching:

- Section 2.3 in the textbook
- Lecture videos 26, 27, 28, and 29 on YouTube
- Linear subspace at Wikipedia
- Linear independence at Wikipedia

Extra textbook problems:

$\star$ 2.3.1, 2.3.2, 2.3.4

$\star\star$ 2.3.3, 2.3.5, 2.3.6, 2.3.9–2.3.11, 2.3.18, 2.3.19

$\star\star\star$ 2.3.12, 2.3.14, 2.3.16, 2.3.22

☠ 2.3.27

Subspaces

Recall that linear systems can be interpreted geometrically as asking for the point(s) of intersection of a collection of lines or planes (depending on the number of variables involved). The following definition introduces "subspaces", which can be thought of as any-dimensional analogues of lines and planes.

Definition 7.1 — Subspaces

A **subspace** of $\mathbb{R}^n$ is a non-empty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ such that:

a) If $\mathbf{v}$ and $\mathbf{w}$ are in $\mathcal{S}$ then $\mathbf{v}+\mathbf{w}$ is in $\mathcal{S}$.

b) If $\mathbf{v}$ is in $\mathcal{S}$ and c is a scalar, then $c\mathbf{v}$ is in $\mathcal{S}$.

Properties (a) and (b) above together are equivalent to requiring that $\mathcal{S}$ is closed under linear combinations:

if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in S$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ then
$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \in S$.

Example. Is the set of vectors (x, y) satisfying $y = x^2$ a subspace of $\mathbb{R}^2$?

Need both properties (a) and (b) to hold:

(a): $(2,4) \in S$ $(2^2=4)$

$(3,9) \in S$ $(3^2=9)$

But $(2,4)+(3,9)=(5,13) \notin S$ $(5^2 \neq 13)$

$\therefore$ S is NOT a subspace.

Example. Is the set of vectors (x, y, z) satisfying $x = 3y$ and $z = -2y$ a subspace of $\mathbb{R}^3$?

Equivalently, set S of vectors of the form
$(x,y,z) = (3y, y, -2y) = y(3,1,-2)$.

(a): If $\vec{v}_1 = y_1(3,1,-2)$ and $\vec{v}_2 = y_2(3,1,-2)$ then
$\vec{v}_1 + \vec{v}_2 = y_1(3,1,-2) + y_2(3,1,-2) = (y_1+y_2)(3,1,-2)$,
which is in S. ✓

(b): If $\vec{v} = y(3,1,-2)$ and $c \in \mathbb{R}$ then
$c\vec{v} = cy(3,1,-2)$, which is in S. ✓

$\therefore$ Yes! S is a subspace.

Example. Is the set of vectors (x, y, z) satisfying $x = 3y + 1$ and $z = -2y$ a subspace of $\mathbb{R}^3$?

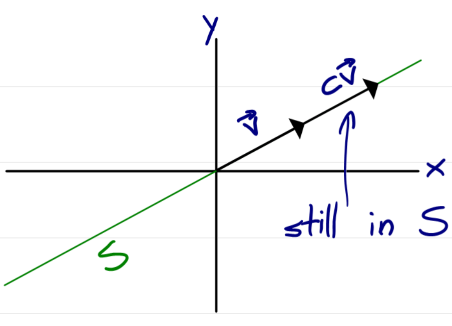
Equivalently, set S of vectors of the form
$(x,y,z) = (3y+1,\ y,\ -2y) = y(3,1,-2) + (1,0,0)$.

(b): $(1,0,0) \in S$, but $0(1,0,0) = (0,0,0) \notin S$. ☆

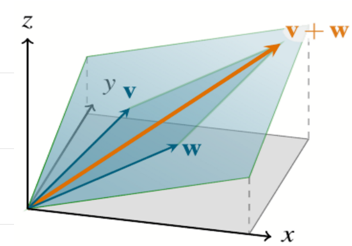
$\therefore$ S is NOT a subspace.

In $\mathbb{R}^3$, lines and planes through the origin are subspaces (this is hopefully not difficult to see for lines, and it can be seen for planes by using the parallelogram law):

In $\mathbb{R}^2$:



$\mathbb{R}^3$:



Subspaces: $\{\vec{0}\}$, lines through origin, $\mathbb{R}^2$

Also planes through the origin

Even though we can't visualize subspaces in higher dimensions, you should keep the line/plane intuition in mind: a subspace of $\mathbb{R}^n$ looks like a copy of $\mathbb{R}^m$ (for some $m < n$) going through the origin.

Subspaces Associated with Matrices

Let's now look at some other natural examples of subspaces that appear frequently when working with matrices.

> **Definition 7.2 — Matrix Subspaces**
>
> Let $A \in \mathcal{M}_{m,n}$ be an $m \times n$ matrix.
>
> **a)** The **range** of A is the subspace of $\mathbb{R}^m$, denoted by $\mathrm{range}(A)$, that consists of all vectors of the form $A\mathbf{x}$.
>
> **b)** The **null space** of A is the subspace of $\mathbb{R}^n$, denoted by $\mathrm{null}(A)$, that consists of all solutions $\mathbf{x}$ of the linear system $A\mathbf{x} = \mathbf{0}$.

Some remarks about these matrix subspaces are in order:

- $\mathrm{null}(A)$ is a subspace. Why?

(a): If $\vec{v},\vec{w} \in \mathrm{null}(A)$ then $A\vec{v}=\vec{0}$ and $A\vec{w}=\vec{0}$,
so $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = \vec{0}+\vec{0}=\vec{0}$ too.
That is, $\vec{v}+\vec{w} \in \mathrm{null}(A)$.
(b): If $\vec{v} \in \mathrm{null}(A)$ and $c \in \mathbb{R}$ then $A\vec{v}=\vec{0}$,
so $A(c\vec{v}) = cA\vec{v} = c\vec{0} = \vec{0}$. That is, $c\vec{v} \in \mathrm{null}(A)$.
$\therefore$ $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$.

- $\mathrm{range}(A)$ is a subspace. Why?

(a): If $\vec{v},\vec{w} \in \mathrm{range}(A)$ then there exist
$\vec{x},\vec{y}$ such that $\vec{v}=A\vec{x}$ and $\vec{w}=A\vec{y}$.

Then $\vec{v}+\vec{w} = A\vec{x}+A\vec{y} = A(\vec{x}+\vec{y})$, so $\vec{v}+\vec{w} \in \text{range}(A)$. ✓

(b): If $\vec{v} \in \text{range}(A)$ and $c \in \mathbb{R}$ then there exists $\vec{x}$ such that $\vec{v} = A\vec{x}$. Then $c\vec{v} = c(A\vec{x}) = A(c\vec{x})$, so $c\vec{v} \in \text{range}(A)$. ✓

- The term "range" is being used here in the exact same sense as in previous courses.

$$\left.\begin{aligned} \text{range}(f) &= \{f(x) : x \in \mathbb{R}\} \\ \text{range}(A) &= \{A\vec{x} : \vec{x} \in \mathbb{R}^n\} \end{aligned}\right\} \leftarrow \text{set of possible outputs of the function.}$$

Example. Describe the range and null space of the 2×3 matrix $A = \begin{bmatrix} 3 & 1 & -4 \\ 6 & 2 & -8 \end{bmatrix}$.

<u>Range:</u> Find all $\vec{b} \in \mathbb{R}^2$ such that $A\vec{x} = \vec{b}$ has a solution.

$$\left[\begin{array}{ccc|c} 3 & 1 & -4 & b_1 \\ 6 & 2 & -8 & b_2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 3 & 1 & -4 & b_1 \\ 0 & 0 & 0 & \underbrace{b_2 - 2b_1}_{\text{must equal } 0} \end{array}\right]$$

$$\therefore \text{range}(A) = \{\vec{b} \in \mathbb{R}^2 : b_2 = 2b_1\}$$
$$= \{(b_1, 2b_1) : b_1 \in \mathbb{R}\} = \{b_1(1,2) : b_1 \in \mathbb{R}\}.$$

<u>Null space:</u> Find all $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 3 & 1 & -4 & 0 \\ 6 & 2 & -8 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 3 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x_1 leading, x_2 and x_3 free

$$\therefore \text{null}(A) = \{\vec{x} \in \mathbb{R}^3 : x_1 = \tfrac{-1}{3}x_2 + \tfrac{4}{3}x_3\}$$
$$= \{(\tfrac{-1}{3}x_2 + \tfrac{4}{3}x_3, x_2, x_3) : x_2, x_3 \in \mathbb{R}\}.$$

The Span of a Set of Vectors

One way to turn a set that is *not* a subspace into a subspace is to add linear combinations to it. For example, the set containing only the vector $(2, 1)$ is not a subspace of $\mathbb{R}^2$ because

$2(2,1) = (4,2)$ is not in the set.

To fix this problem, we could

put $(4,2)$ into the set, getting $\{(2,1),(4,2)\}$.

But it's still not a subspace since $3(2,1) = (6,3)$ is not in it.

OK, let's put <u>all</u> scalar multiples of $(2,1)$ in the set, getting $\{c(2,1) : c \in \mathbb{R}\}$, which IS a subspace of $\mathbb{R}^2$!

In general, if our starting set contains more than just one vector, we might also have to add general linear combinations of those vectors (not just their scalar multiples) in order to create a subspace. This idea of enlarging a set so as to create a subspace is an important one that we now give a name and explore.

Definition 7.3 — Span

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of those vectors is called their **span**, and is denoted by $\operatorname{span}(B)$ or $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.

For example, $\operatorname{span}\big((2, 1)\big)$ is the line through the origin and the point $(2, 1)$, as we discussed earlier.

Example. Show that $\text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

If $\vec{x} \in \mathbb{R}^3$ then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$, so $\vec{x} \in \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. ✓

The natural generalization of this fact holds in all dimensions:

$$\text{span}(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n) = \mathbb{R}^n.$$

Example. What is $\text{span}\big((1, 0, 3), (-1, 1, -3)\big)$ – a line, a plane, or something else?

Equivalent: which vectors (x,y,z) can be written in the form $(x,y,z) = c_1(1,0,3) + c_2(-1,1,-3)$?

$$\left.\begin{aligned} x &= c_1 - c_2 \\ y &= c_2 \\ z &= 3c_1 - 3c_2 \end{aligned}\right\} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 3 & -3 & z \end{array}\right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & \underbrace{z-3x}_{\text{must equal } 0} \end{array}\right]$$

$\therefore$ $\text{span}((1,0,3),(-1,1,-3))$ is the <u>plane</u> $z - 3x = 0$.

We motivated the span of a set of vectors as a way of turning that set into a subspace. We now state (but for the sake of time, do not prove) a theorem that says the span of a set of vectors is indeed always a subspace, as we would hope.

Theorem 7.1 — Spans are Subspaces

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbb{R}^n$.

In fact, you can think of the span of a set of vectors as the *smallest* subspace containing those vectors.

The range of a matrix can be expressed very conveniently as the span of a set of vectors in a way that requires no calculation whatsoever:

Theorem 7.2 — Range Equals the Span of Columns

If $A \in \mathcal{M}_{m,n}$ has columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ then $\text{range}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$.

This theorem follows immediately from

Theorem 3.6: $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$ is a linear combination of the columns of A.

For example, if we return to the 2×3 matrix from earlier, we see that its range is...

$$\text{range}(A) = \text{span}\{(3,6),(1,2),(-4,-8)\} = \text{span}\{(1,2)\}.$$

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 6 & 2 & -8 \end{bmatrix}$$

We close this section by introducing a connection between the range of a matrix and invertible matrices.

Theorem 7.3 — Spanning Sets and Invertible Matrices

Let $A \in \mathcal{M}_n$. The following are equivalent:

a) A is invertible.

b) $\text{range}(A) = \mathbb{R}^n$.

c) The columns of A span $\mathbb{R}^n$.

d) The rows of A span $\mathbb{R}^n$.

Proof. The fact that properties (a) and (c) are equivalent follows from combining...

Theorem 6.3: A is invertible $\Leftrightarrow$ $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$, and

Theorem 3.6: $A\vec{x}$ is a lin. comb. of A's columns.

The equivalence of properties (c) and (d) follows from the fact that

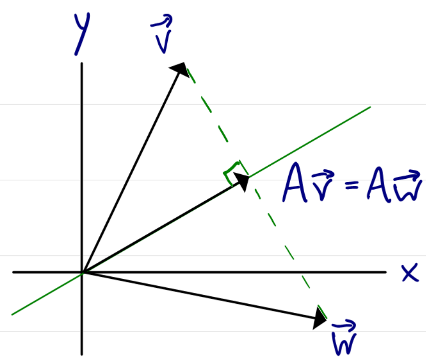
A is invertible if and only if A^T is invertible (Theorem 6.2).

Finally, the equivalence of properties (b) and (c) follows immediately from

Theorem 7.2: the range of a matrix is the span of its columns. ■

The geometric interpretation of the equivalence of properties (a) and (b) in the above theorem is

a matrix is invertible if and only if it does not "squish" space (that "squishing" cannot be undone!)



Linear Dependence and Independence

Recall from earlier that a row echelon form of a matrix can have entire rows of zeros at the end of it. For example, the reduced row echelon form of

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -1 & 1 & -2 \end{array}\right] \quad \text{is} \quad \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

This happens when there is some linear combination of the rows of the matrix that equals the zero row, and we interpret this roughly as saying that one row the rows of the matrix (i.e., one of the equations in the associated linear system) does not "contribute anything new." In the example above,

$$\begin{aligned} x - y &= 2 \\ -x + y &= -2 \end{aligned}$$

this equation is redundant — it follows from the first

The following definition captures this idea that a redundancy among vectors or linear equations can be identified by whether or not some linear combination of them equals zero.

Definition 7.4 — Linear Dependence and Independence

A set of vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$, at least one of which is not zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

If a set of vectors is not linearly dependent, it is called **linearly independent**.

For example, the set of vectors $\{(2,3),(1,0),(0,1)\}$ is linearly...

dependent, because $(2,3) - 2(1,0) - 3(0,1) = (0,0)$.

On the other hand, the set of vectors $\{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly...

independent, since $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$
$\Longrightarrow (c_1, c_2, c_3) = (0,0,0)$
$\Longrightarrow c_1 = c_2 = c_3 = 0.$

In general, to check whether or not a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, you should set

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$ unique solution $\Longrightarrow$ lin. indep.
∞ many solutions $\Longrightarrow$ lin. dep.

and then try to solve for the scalars $c_1, c_2, \ldots, c_k$. If they must all equal 0, then the set is linearly independent, and otherwise it is linearly dependent.

Example. Are these vectors linearly independent?

$\{(1,1,1),(1,2,3),(3,2,1)\}$

Solve: $c_1(1,1,1) + c_2(1,2,3) + c_3(3,2,1) = (0,0,0).$

$$c_1 + c_2 + 3c_3 = 0$$
$$c_1 + 2c_2 + 2c_3 = 0$$
$$c_1 + 3c_2 + c_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ There are infinitely many solutions, so $\{(1,1,1),(1,2,3),(3,2,1)\}$ is linearly dependent.

We saw in the previous example that we can check linear (in)dependence of a set of vectors by placing those vectors as columns in a matrix and augmenting with a $\mathbf{0}$ right-hand side. This is true in general:

Theorem 7.4 — Checking Linear Dependence

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ be vectors and let A be the $m \times n$ matrix with these vectors as its columns. The following are equivalent:

a) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly dependent set.

b) The linear system $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.

Some notes about linear (in)dependence are in order:

- A set of vectors is linearly dependent if and only if at least one of the vectors can be written as a linear combination of the others.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \quad \text{with some } c_j \text{ non-zero}$$
$$\Leftrightarrow \quad \vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \cdots - \frac{c_k}{c_j}\vec{v}_k$$

no $\vec{v}_j$ term

- Every set of vectors containing the zero vector is linearly...

dependent! For example, $\{\vec{0}, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is lin. dep. since $3\vec{0} + 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k = \vec{0}$.

non-zero

- Geometrically, linear dependence means that...

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ all live in a common subspace with dimension smaller than k.
For example, 2 vectors on a line,
3 vectors on a plane, etc.

- For a set of just 2 vectors, linear dependence means that...

they are scalar multiples of each other.

Example. Is this set linearly independent?

$\{(1,2,3,4), (-3,-6,-9,-12)\}$: linearly dependent

$\{(1,2,3,4), (-3,4,2,7)\}$: linearly independent

We close this section by introducing a connection between linear independence and invertible matrices, which we unfortunately have to state without proof due to time constraints.

Theorem 7.5 — Independence and Invertible Matrices

Let $A \in \mathcal{M}_n$. The following are equivalent:

a) A is invertible.

b) The columns of A form a linearly independent set.

c) The rows of A form a linearly independent set.