Bases of Subspaces and the Rank of a Matrix

This week we will learn about:

- The dimension of a subspace,
- Bases of subspaces,
- The rank of a matrix, and
- The rank–nullity theorem.

Extra reading and watching:

- Sections 2.4 in the textbook
- Lecture videos 30, 31, 32, and 33 on YouTube
- Basis (linear algebra) at Wikipedia
- Rank (linear algebra) at Wikipedia

Extra textbook problems:

⋆ 2.4.1, 2.4.2, 2.4.8

⋆⋆ 2.4.5, 2.4.6, 2.4.9, 2.4.10

⋆⋆⋆ 2.4.11, 2.4.12, 2.4.13, 2.4.25, 2.4.30

☠ 2.4.27

Bases of Subspaces

A plane in $\mathbb{R}^3$ is spanned by any two vectors that are parallel to the plane, but not parallel to each other (i.e., are linearly independent). More than two vectors could be used to span the plane, but they would necessarily be linearly dependent. On the other hand, there is no way to use *fewer* than two vectors to span a plane (the span of just one vector is just a line). This leads to the idea of a *basis* of a subspace:

> **Definition 8.1 — Bases**
>
> A **basis** of a subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is a set of vectors in $\mathcal{S}$ that
>
> **a)** spans $\mathcal{S}$, and
>
> **b)** is linearly independent.

The idea of a basis is that it is a set that is "big enough" to span the subspace, but it is not "so big" that it contains redundancies. That is, it is "just" big enough to span the subspace.

It's not too big and it's not too small – it's just right!

Example. The standard basis of $\mathbb{R}^n$.

The set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

(a) Spans $\mathbb{R}^n$: $(v_1, v_2, \ldots, v_n) = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$

(b) Lin. indep.: if $c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n = \vec{0}$ then $c_1 = \cdots = c_n = 0$.

Example. Show that the set $\{(2,1),(1,3)\}$ is a basis of $\mathbb{R}^2$.

(a) Spans $\mathbb{R}^2$: solve $(x,y) = c_1(2,1) + c_2(1,3)$ for c_1, c_2.

$$\left[\begin{array}{cc|c} 2 & 1 & x \\ 1 & 3 & y \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & 1 & x \\ 0 & 5/2 & y - x/2 \end{array}\right]$$

This linear system has a solution for all (x,y), so $\{(2,1),(1,3)\}$ spans $\mathbb{R}^2$.
(b) Lin. indep.: $(2,1)$ and $(1,3)$ are not scalar multiples of each other.
$\therefore \{(2,1),(1,3)\}$ is a basis of $\mathbb{R}^2$.

The above example demonstrates that the same subspace can (and will!) have more than one basis:

$\{\vec{e}_1,\vec{e}_2\}$ and $\{(2,1),(1,3)\}$ are both bases of $\mathbb{R}^2$ (and there are many others).

However, the *number* of vectors in a basis of a given subspace is always the same, which we now state as a theorem.

Theorem 8.1 — Uniqueness of Size of Bases

Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$. Then every basis of $\mathcal{S}$ has the same number of vectors.

We don't prove the above theorem (it is a fairly long and ugly mess), but we can use it to pin down something we have been hand-wavey about up until now: we have never actually defined exactly what we mean by the "dimension" of a subspace of $\mathbb{R}^n$. We now fill in this gap:

Definition 8.2 — Dimension of a Subspace

Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$. The number of vectors in a basis of $\mathcal{S}$ is called the **dimension** of $\mathcal{S}$.

As one minor technicality, we notice that the set $\mathcal{S} = \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$. However, the only basis of this subspace is the empty set $\{\}$ (why?), so its dimension is 0.

Example. What is the dimension of $\mathbb{R}^n$?

The standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$. It has n vectors, so $\dim(\mathbb{R}^n) = n$.

Example. Find a basis for $\mathcal{S} = \operatorname{span}(\mathbf{v}, \mathbf{w}, \mathbf{x})$, where $\mathbf{v} = (1, 2, 3)$, $\mathbf{w} = (3, 2, 1)$, $\mathbf{x} = (1, 1, 1)$. What is the dimension of this subspace?

$\{\vec{v}, \vec{w}, \vec{x}\}$ spans S (by definition), so just need to check linear independence:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & -8 & -2 & 0 \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinitely many solutions, so linearly dependent.
$\therefore$ NOT a basis of S! $\{\vec{v}, \vec{w}\}$ is a basis of S (check this), so S is 2-dimensional.

We will now show how to find bases of the fundamental subspaces associated with a matrix: $\operatorname{range}(A)$ and $\operatorname{null}(A)$.

Example. Find bases for the range and null space of the following matrix and thus compute their dimensions:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 & 4 \\ 2 & 1 & 1 & -1 & -3 \end{bmatrix}$$

Null space: solve $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & 4 & 0 \\ 2 & 1 & 1 & -1 & -3 & 0 \end{array}\right] \xrightarrow[\substack{R_3+R_1 \\ R_4-2R_1}]{R_2-R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -1 & -3 & 0 \end{array}\right] \xrightarrow{R_4 + R_3} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x_1, x_2, x_4 are leading, x_3, x_5 are free

$x_1 = -x_3 + x_5$, $x_2 = x_3 - 2x_5$, $x_4 = -3x_5$

$$\therefore (x_1, x_2, x_3, x_4, x_5) = (-x_3 + x_5, x_3 - 2x_5, x_3, -3x_5, x_5)$$
$$= x_3(-1, 1, 1, 0, 0) + x_5(1, -2, 0, -3, 1),$$

so $\{(-1,1,1,0,0), (1,-2,0,-3,1)\}$ is a basis of null(A).

range(A) is the span of A's columns. But not all columns are needed. RREF:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so use columns 1, 2, and 4 of A to get the basis $\{(1,1,-1,2), (0,1,0,1), (0,0,1,-1)\}$.

The quantities dim(range(A)) and dim(null(A)) that we computed in the previous example highlight a lot of the structure of the matrix A, so let's have a closer look at them now.

The Rank of a Matrix

With many of the technical details of this course out of the way, we are now in a position to introduce one of the most important properties of a matrix: its rank.

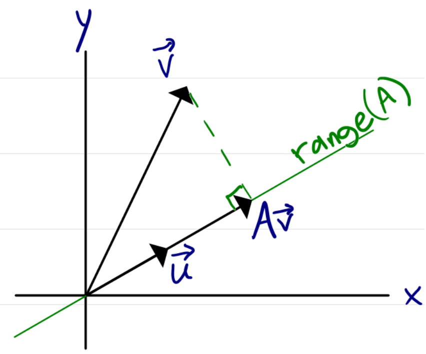
Definition 8.3 — Rank of a Matrix

Let $A \in \mathcal{M}_{m,n}$ be a matrix. Then its **rank**, denoted by rank(A), is the dimension of its range.

Rank can be thought of as a measure of how degenerate a matrix is, as it describes how much of the output space can actually be reached by A.

Example. Suppose $A \in \mathcal{M}_n$ is the standard matrix of a projection onto a line. What is $\text{rank}(A)$?

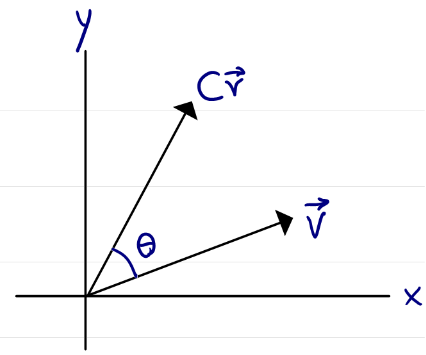
$A = \vec{u}\vec{u}^T$, where $\vec{u}$ is a unit vector on the line.
$\text{range}(A)$ is that (1-dim.) line,
so $\text{rank}(A) = \dim(\text{range}(A)) = 1$.



Example. Suppose $C \in \mathcal{M}_2$ is the standard matrix of a rotation. What is $\text{rank}(C)$?

$$C = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$\text{range}(C) = \mathbb{R}^2$, so
$\text{rank}(C) = \dim(\text{range}(C)) = 2$.



One of the reasons why the rank of a matrix is so useful is that it can be interpreted in so many different ways. While it equals the dimension of the range, it also equals some other quantities that we have already seen as well:

Theorem 8.2 — Characterization of Rank

Let $A \in \mathcal{M}_{m,n}$ be a matrix. Then the following quantities are all equal to each other:

a) $\text{rank}(A)$

b) $\text{rank}(A^T)$.

c) The number of non-zero rows in any row echelon form of A.

d) The number of leading columns in any row echelon form of A.

Proof. To see the equivalence of (c) and (d)...

recall that every non-zero row of a row echelon form has exactly one leading entry, all in different columns.

To see the equivalence of (a) and (d)...

recall our earlier example: one basis of range(A) consists of the columns of A corresponding to the leading columns in REF.
$\therefore$ rank(A) = dim(range(A)) = # of leading columns

The equivalence of (b) and (c) is similar:

range(A) is span of A's columns, so range(A^T) is span of A's rows. Basis: rows of REF. ■

Example. Find the rank of the matrix $A = \begin{bmatrix} 0 & 0 & -2 & 2 & -2 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -1 & 1 & -1 & 0 & -3 \\ 2 & -2 & -1 & 3 & 3 \\ 0 & 0 & -2 & 2 & -2 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} -1 & 1 & -1 & 0 & -3 \\ 0 & 0 & -3 & 3 & -3 \\ 0 & 0 & -2 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{2}{3}R_2} \begin{bmatrix} -1 & 1 & -1 & 0 & -3 \\ 0 & 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is in REF.

$\therefore$ The rank of this matrix is 2 (the number of non-zero rows in REF).

Similarly, the **nullity** of a matrix, denoted by nullity(A), is the dimension of its null space (i.e., the dimension of the solution set of the linear system $A\mathbf{x} = \mathbf{0}$). The following theorem demonstrates the close connection between the rank and nullity of a matrix:

Theorem 8.3 — Rank–Nullity

Let $A \in \mathcal{M}_{m,n}$ be a matrix. Then $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

Proof. We use the equivalence of the quantities (a) and (d) from the previous theorem:

rank(A) = # of leading columns in an REF of A.

Recall that each column of A corresponds to a variable in the linear system $A\vec{x} = \vec{0}$.

rank(A) = # of leading variables
nullity(A) = # of free variables
n = total # of variables ∎

Example. Find the nullity of the matrix $A = \begin{bmatrix} 0 & 0 & -2 & 2 & -2 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}$

REF (from earlier): $\begin{bmatrix} -1 & 1 & -1 & 0 & -3 \\ 0 & 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

rank(A) + nullity(A) = n, so 2 + nullity(A) = 5, so nullity(A) = 3. (also, 3 free variables)

The previous theorem makes some geometric sense—there are n dimensions that go into A. $\text{rank}(A)$ of them are sent to the output space, and the other $\text{nullity}(A)$ of them are "squashed away" by A. This observation leads immediately to *yet another* characterization of invertibility:

Theorem 8.4 — Rank and Invertible Matrices

Let $A \in \mathcal{M}_n$. The following are equivalent:

a) A is invertible.

b) $\text{rank}(A) = n$

c) $\text{nullity}(A) = 0$