Linear Algebra – Week 9

Determinants

This week we will learn about:

- Determinants of matrices, and
- That's it. Determinants, determinants, determinants.

Extra reading and watching:

- Section 3.2 in the textbook
- Lecture videos 34, 35, and 36 on YouTube
- Determinant at Wikipedia

Extra textbook problems:

$\star$ 3.2.1, 3.2.3, 3.2.4, 3.2.9

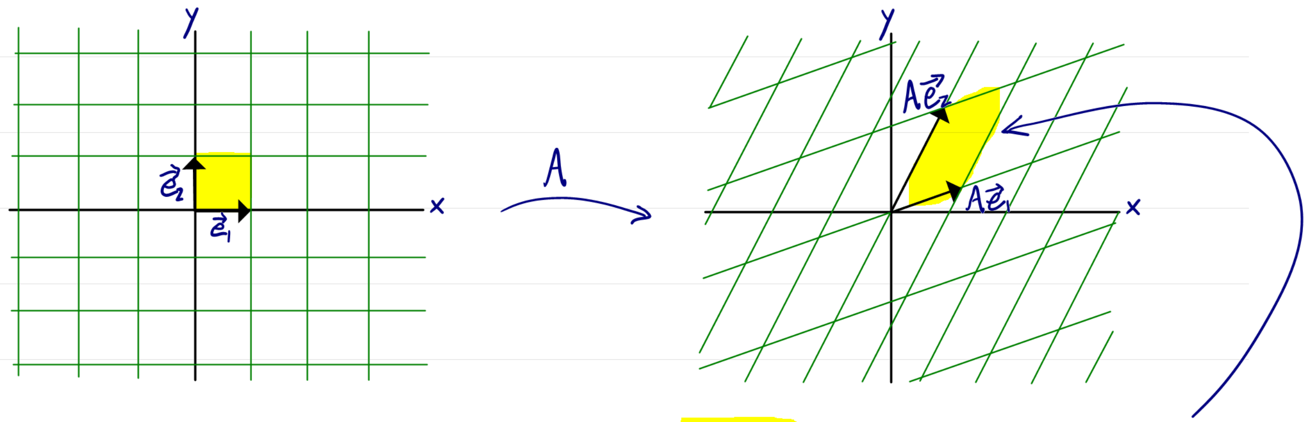
$\star\star$ 3.2.5–3.2.8, 3.2.10, 3.2.12, 3.2.17

$\star\star\star$ 3.2.14, 3.2.16, 3.2.18

☠ 3.2.19–3.2.21

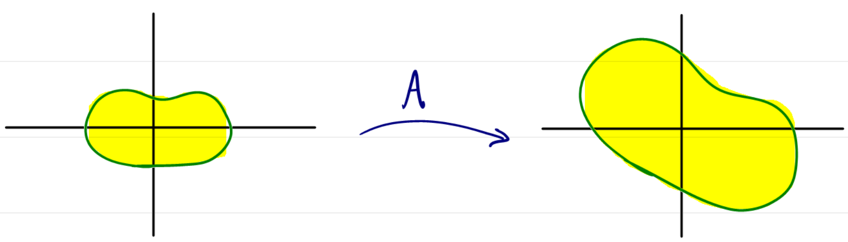
We now introduce one of the most important properties of a matrix: its **determinant**, which roughly is a measure of how "large" the matrix is. More specifically, recall that...

Matrices act as linear transformations:



The determinant of A, which we denote by $\det(A)$, is the area (or volume) of this image of the unit hypercube. In other words, it measures how much A expands space when acting as a linear transformation.

$$\det(A) = \frac{\text{area/volume of output } \cancel{\text{parallelogram/epiped}} \text{ region}}{\text{area/volume of input } \cancel{\text{square/cube}} \text{ region}}$$



Let's now start looking at some of the properties of determinants, so that we can (eventually!) learn how to compute it.

Definition and Basic Properties

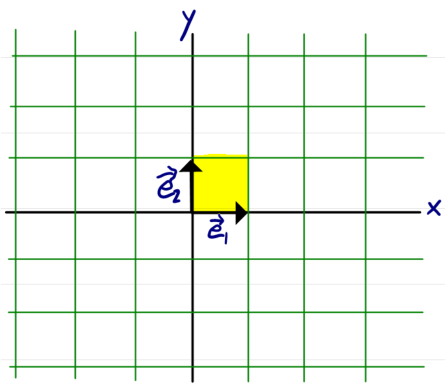
Before we even properly define the determinant, let's think about some properties that it should have. The first important property is that, since the identity matrix does not stretch or shrink $\mathbb{R}^n$ at all...

$\det(I) = 1.$

Next, since every $A \in \mathcal{M}_n$ expands space by a factor of $\det(A)$, and similarly each $B \in \mathcal{M}_n$ expands space by a factor of $\det(B)$...

AB stretches space by a factor of $\det(A)\det(B)$, so $\det(AB) = \det(A)\det(B)$.
"Multiplicativity of the determinant."

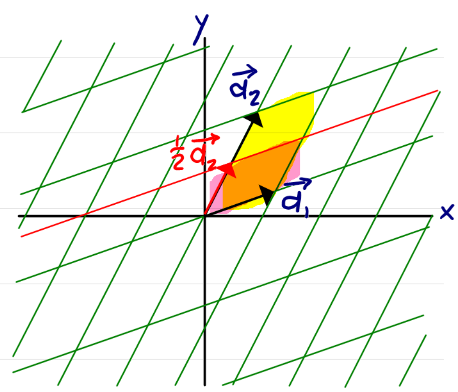
We will also need one more property of determinants, which is a bit more difficult to see. What happens to $\det(A)$ if we multiply one of the columns of A by a scalar $c \in \mathbb{R}$?



$A = [\vec{a}_1 \mid \vec{a}_2]$

$B = [\vec{a}_1 \mid \tfrac{1}{2}\vec{a}_2]$

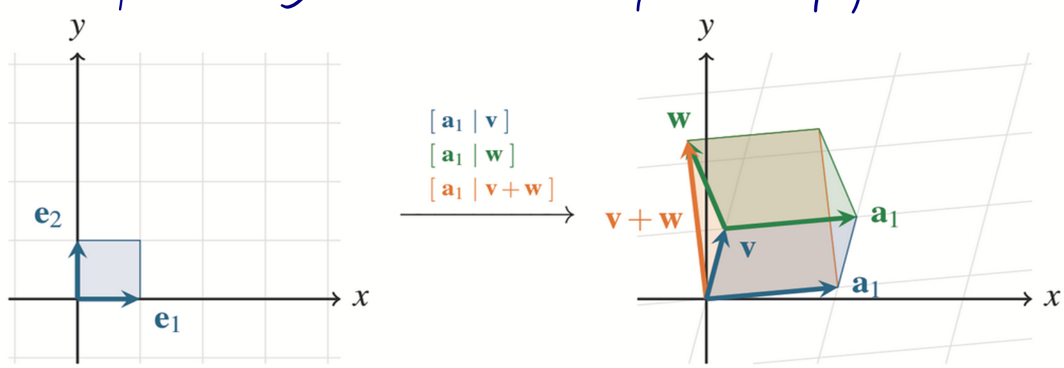
$\det(B) = \tfrac{1}{2}\det(A)$



In general, $\det([\vec{a}_1 | \cdots | c\vec{a}_j | \cdots | \vec{a}_n]) = c\det([\vec{a}_1 | \cdots | \vec{a}_j | \cdots | \vec{a}_n])$.

Similarly, if we add a vector to one of the columns of a matrix, then...

That vector is added to one side of the output parallelogram or parallelepiped:



Thus $\det([\vec{a}_1 | \vec{v}]) + \det([\vec{a}_1 | \vec{w}]) = \det([\vec{a}_1 | \vec{v} + \vec{w}])$.

In other words, the determinant is linear in the columns of a matrix (sometimes called **multilinearity**). We now *define* the determinant to be the function that satisfies this multilinearity property, as well as the other two properties that we demonstrated earlier:

Definition 9.1 — Determinant

The **determinant** is the (unique!) function $\det : \mathcal{M}_n \to \mathbb{R}$ that satisfies the following three properties:

a) $\det(I) = 1$,

b) $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathcal{M}_n$, and

c) for all $c \in \mathbb{R}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^n$, it is the case that

$$\begin{aligned}&\det\big([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{v} + c\mathbf{w} \mid \cdots \mid \mathbf{a}_n\ ]\big)\\ &\quad = \det\big([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{v} \mid \cdots \mid \mathbf{a}_n\ ]\big) + c \cdot \det\big([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{w} \mid \cdots \mid \mathbf{a}_n\ ]\big).\end{aligned}$$

Example. Compute the determinant of the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Just use the properties above:

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) \underset{\text{property (c)}}{=} 2\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\right) \underset{\text{(c)}}{=} 6\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \underset{\text{(a)}}{=} 6.$$

Let's start looking at some of the basic properties of the determinant. First, if $A \in \mathcal{M}_n$ is invertible then properties (a) and (b) tell us that

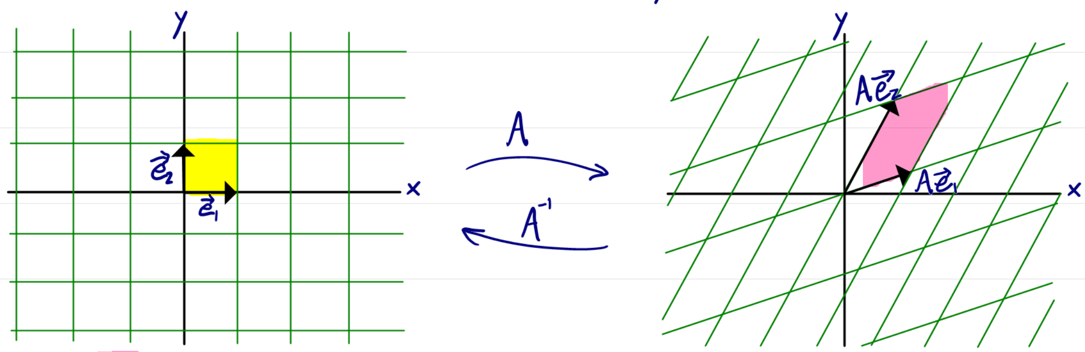
$\det(A^{-1}A) = \det(A^{-1})\det(A)$ (property (b))

$\det(A^{-1}A) = \det(I) = 1$ (property (a))

$\therefore \det(A^{-1})\det(A) = 1$, so $\det(A) \neq 0$ and $\det(A^{-1}) = 1/\det(A)$.

This makes sense geometrically, since if A expands space by a factor of $\det(A)$ then

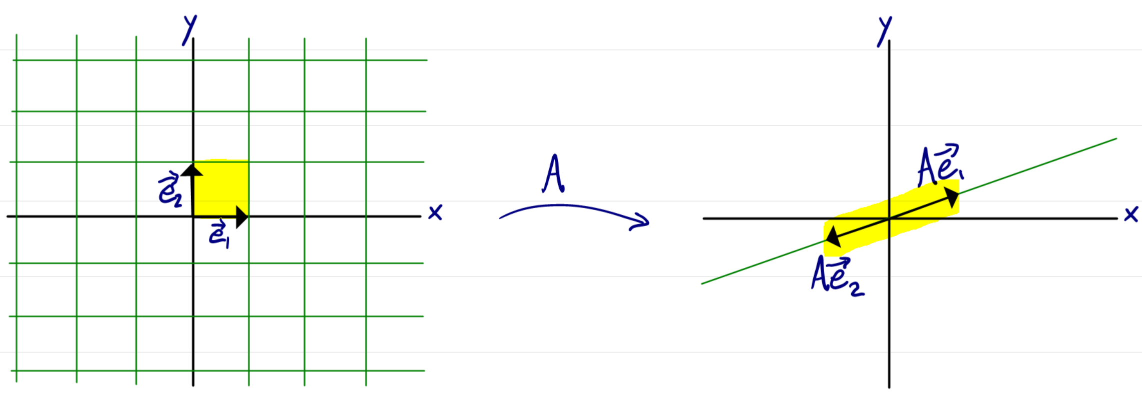
A^{-1} must shrink space by that same factor.



$\det(A) = \square / \square$ and $\det(A^{-1}) = \square / \square = 1/\det(A)$.

On the other hand, if A is not invertible, then

its range is less than n-dimensional;
at least one dimension is "squashed away":



We summarize our observations about the determinant of invertible and non-invertible matrices in the following theorem:

Theorem 9.1 — Determinants and Invertibility

Suppose $A \in \mathcal{M}_n$. Then A is invertible if and only if $\det(A) \neq 0$, and if it is invertible then $\det(A^{-1}) = 1/\det(A)$.

There are also a few other basic properties of determinants that are useful to know, so we state them here (but for time reasons we do not explicitly prove them):

Theorem 9.2 — Other Properties of the Determinant

Suppose $A \in \mathcal{M}_n$ and $c \in \mathbb{R}$. Then

a) $\det(cA) = c^n \det(A)$, and

b) $\det(A^T) = \det(A)$.

stretches the output by a factor of c in each of n directions

Example. Suppose $A, B \in \mathcal{M}_3$ are matrices with $\det(A) = 2$ and $\det(B) = 5$. Compute...

$$\begin{aligned} \det(A^2BA^{-1}) &= \det(A)\det(A)\det(B)\det(A^{-1}) && \text{(property (b))} \\ &= 2 \times 2 \times 5 \times (1/\det(A)) && \text{(Thm. 9.1)} \\ &= 20/2 = 10 \end{aligned}$$

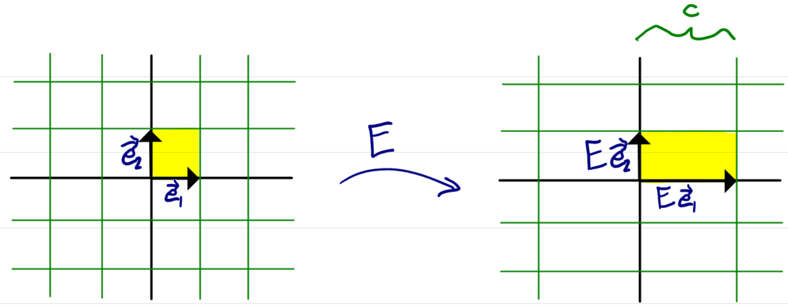
$$\begin{aligned} \det(2A^TB) &= 2^3 \det(A^TB) && \text{(Thm. 9.2(a))} \\ &= 8\det(A^T)\det(B) && \text{(property (b))} \\ &= 8 \times 2 \times 5 = 80. && \text{(Thm. 9.2(b))} \end{aligned}$$

Computation

In order to come up with a general method of computing the determinant, we start by computing it on elementary matrices.

The elementary matrix corresponding to the row operation cR_i has the form

$$\begin{array}{c} \\ \\ cR_j \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

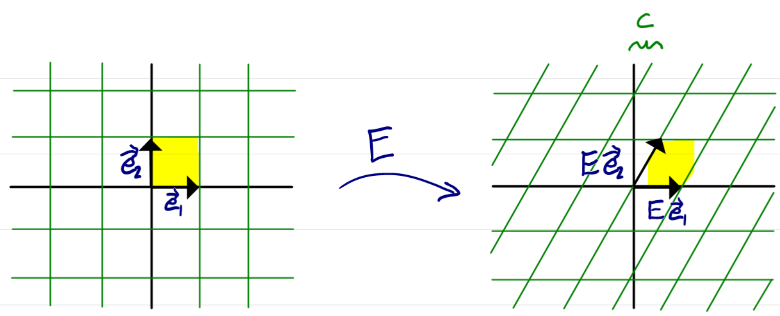


This matrix has determinant equal to...

$$\det\left(\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}\right) \underset{\substack{\uparrow\\ \text{defining property (c)}}}{=} c\det\left(\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}\right) \underset{\substack{\uparrow\\ \text{(a)}}}{=} c\cdot 1 = c$$

The elementary matrix corresponding to the row operation $R_i + cR_j$ has the form

$$\begin{array}{r} \\ R_j \\ R_i + cR_j \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & c & \cdots & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}.$$



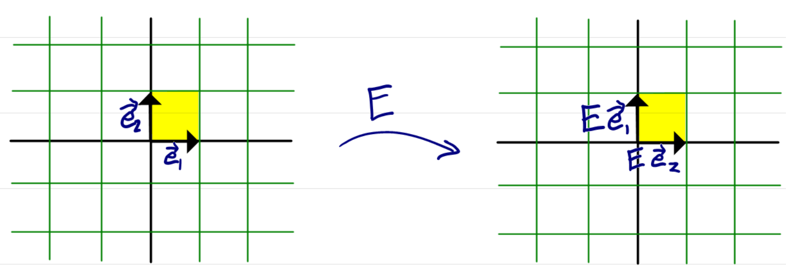
This matrix has determinant equal to...

$$\det\left(\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & c & \cdots & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}\right) \overset{\text{(c)}}{=} \det\left(\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & 0 & \cdots & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}\right) + c\det\left(\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & \vdots & \ddots & \\ & & 1 & \cdots & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}\right)$$

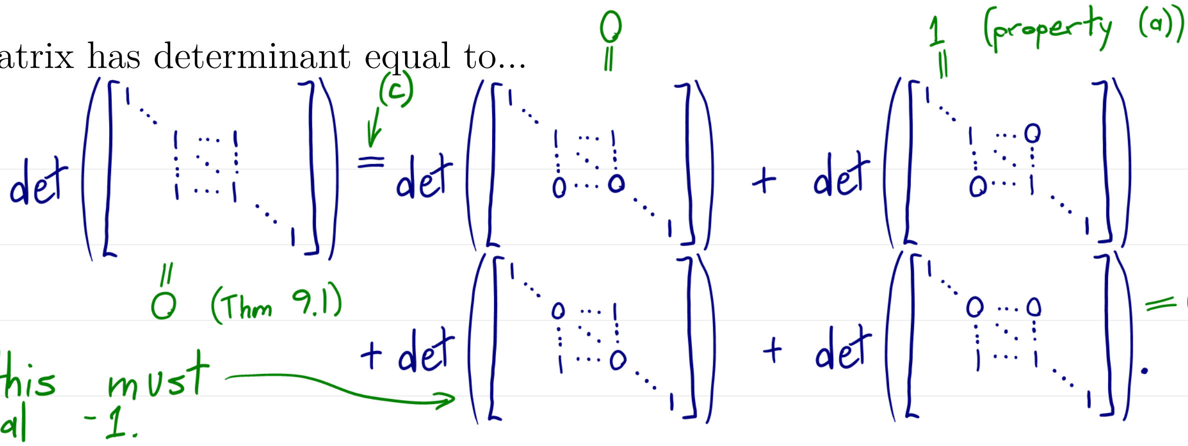
$$= \underset{\text{(a)}}{1} + c\cdot \underset{\text{Thm 9.1}}{0} = 1.$$

The elementary matrix corresponding to the row operation $R_i \leftrightarrow R_j$ has the form

$$\begin{array}{r} \\ R_i \\ R_j \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & \ddots & \vdots & \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$



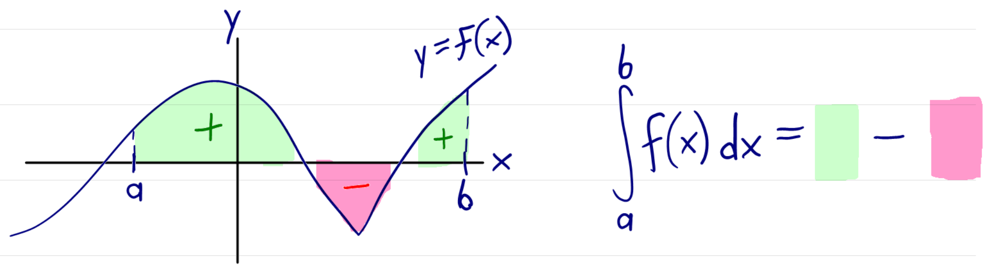
This matrix has determinant equal to...



Wait, so the determinant of a matrix can be ***negative***? But it measures area/volume!

No, it measures signed area/volume.

Analogous to definite integrals:



Since multiplication on the left by an elementary matrix corresponds to performing a row operation, we can rephrase our above calculations as the following theorem:

Theorem 9.3 — Computing Determinants via Row Operations

Suppose $A, B \in \mathcal{M}_n$. If B is obtained from A via a single row operation, then their determinants are related as follows:

$$cR_i\text{:}\ \det(B) = c \cdot \det(A),$$

$$R_i + cR_j\text{:}\ \det(B) = \det(A), \text{ and}$$

$$R_i \leftrightarrow R_j\text{:}\ \det(B) = -\det(A).$$

The above theorem gives us everything we need to know to be able to compute determinants in general – row reduce A to I, keeping track of the row operations that we performed along the way, and use the fact that $\det(I) = 1$. If we cannot row reduce to I, then A is not invertible, so $\det(A) = 0$.

Example. Compute the determinant of $A = \begin{bmatrix}1&1&1\\1&2&4\\1&3&9\end{bmatrix}$.

$$\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix}1&1&1\\0&1&3\\0&2&8\end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix}1&1&1\\0&1&3\\0&0&2\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{bmatrix}1&1&1\\0&1&3\\0&0&1\end{bmatrix} \xrightarrow[R_2-3R_3]{R_1-R_3} \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

The only row operation above that changed the determinant was $\frac{1}{2}R_3$. $\det(I) = 1$, so $\det(A) = \frac{1}{(\frac{1}{2})} = 2$.

In the previous example, the determinant of the row echelon form ended up being the product of its diagonal entries. We now state this observation as a theorem:

Theorem 9.4 — Determinant of a Triangular Matrix

Let $A \in \mathcal{M}_n$ be a triangular matrix. Then $\det(A)$ is the product of its diagonal entries:

$$\det(A) = a_{1,1}a_{2,2}\cdots a_{n,n}.$$

Proof. The idea is that a triangular matrix can be row-reduced to I just by operations of the form $R_i + cR_j$ (which do not affect the determinant) and $(1/a_{1,1})R_1$, …, $(1/a_{n,n})R_n$:

$$\underset{\det = a_{1,1}a_{2,2}\cdots a_{n,n}}{\begin{bmatrix}a_{1,1}&*&\cdots&*\\0&a_{2,2}&\cdots&*\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a_{n,n}\end{bmatrix}} \xrightarrow[R_n/a_{n,n}]{R_1/a_{1,1},\ R_2/a_{2,2},\ \ldots} \underset{\det = 1}{\begin{bmatrix}1&*&\cdots&*\\0&1&\cdots&*\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{bmatrix}} \xrightarrow{\text{addition row operations}} \underset{\det = 1}{I}.$$

(If $a_{j,j} = 0$ for any j then A not invertible.) ∎

By using this fact, we can compute determinants a bit more quickly, by just row-reducing to row echelon form (instead of *reduced* row echelon form). This method is best illustrated with another example.

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 - R_2]{R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 4 & -3 \end{bmatrix} \xrightarrow{R_4 - 2R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

We only used addition row operations, which do not change the determinant.

Since the last matrix has determinant $1 \cdot (-1) \cdot 2 \cdot 3 = -6$, we know $\det(A) = -6$ too.

Explicit Formulas and Cofactor Expansions

Remarkably, the determinant can be computed via an explicit formula just in terms of multiplication and addition of the entries of the matrix. Before presenting the general formula for $n \times n$ matrices, let's start with what it looks like for 2×2 matrices.

Theorem 9.5 — Determinant of 2×2 Matrices

The determinant of a 2×2 matrix is given by

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

Proof. We prove this theorem by making use of multilinearity (i.e., defining property (c) of the determinant):

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}\right),$$

since these matrices differ only in left column.

Well,

$\det\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = ad$, since this matrix is upper triangular.

Similarly,

$$\det\left(\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}\right) \overset{\text{swap rows}}{=} -\det\left(\begin{bmatrix} c & d \\ 0 & b \end{bmatrix}\right) = -bc.$$

Adding these two quantities together gives the desired formula. ■

The above theorem is perhaps best remembered in terms of diagonals of the matrix – the determinant of a 2×2 matrix is the product of its forward diagonal minus the product of its backward diagonal.

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Method 1: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$, so $\det(A) = 1 \cdot (-2) = -2$.

Method 2: $\det(A) = ad - bc = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$.

The formula for the determinant of a 3×3 matrix is somewhat more complicated:

Theorem 9.6 — Determinant of 3×3 Matrices

The determinant of a 3×3 matrix is given by

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei + bfg + cdh - afh - bdi - ceg.$$

Proof. Again, we make use of multilinearity (i.e., defining property (c) of the determinant) to write

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & b & c \\ d & e & f \\ 0 & h & i \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & b & c \\ 0 & e & f \\ g & h & i \end{bmatrix}\right),$$

since $(a,d,g) = (a,0,0) + (0,d,0) + (0,0,g)$.

Let's compute the first of the three determinants on the right by using a similar trick on its second column:

$$\det\left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}\right) + \det\left(\begin{bmatrix} a & 0 & c \\ 0 & 0 & f \\ 0 & h & i \end{bmatrix}\right),$$ since

$(b,e,h) = (b,e,0) + (0,0,h)$.

Well, these two determinants are

$$\det\left(\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}\right) = aei,$$ since this matrix is upper triangular.

Similarly, $$\det\left(\begin{bmatrix} a & 0 & c \\ 0 & 0 & f \\ 0 & h & i \end{bmatrix}\right) = -\det\left(\begin{bmatrix} a & 0 & c \\ 0 & h & i \\ 0 & 0 & f \end{bmatrix}\right) = -afh.$$

The computation of the remaining terms in the determinant is similar. ■

We can also think of the formula for determinants of 3×3 matrices in terms of diagonals of the matrix – it is the sum of the products of its forward diagonals minus the sum of the products of its backward diagonals, with the understanding that the diagonals "loop around" the matrix:

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}$$

$$\det(A) = aei + bfg + cdh - afh - bdi - ceg.$$

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

$$\begin{aligned} \det(A) &= 1\cdot2\cdot9 + 1\cdot4\cdot1 + 1\cdot1\cdot3 \\ &\quad - 1\cdot4\cdot3 - 1\cdot1\cdot9 - 1\cdot2\cdot1 \\ &= 18 + 4 + 3 - 12 - 9 - 2 \\ &= 2 \qquad \text{(same as before).} \end{aligned}$$

The following theorem tells us how to come up with these formulas in general, and it is just a direct generalization of the 2×2 and 3×3 formulas that we already saw.

Theorem 9.7 — Cofactor Expansion

Let $A \in \mathcal{M}_n$. For each $1 \leq i, j \leq n$, define $c_{i,j} = (-1)^{i+j} \det(\overline{A_{i,j}})$, where $\overline{A_{i,j}}$ is the matrix obtained by removing the i-th row and j-th column of A. Then the determinant of A can be computed via

$$\begin{aligned} \det(A) &= a_{i,1}c_{i,1} + a_{i,2}c_{i,2} + \cdots + a_{i,n}c_{i,n} \quad \text{for all } 1 \leq i \leq n, \quad \text{and} \\ \det(A) &= a_{1,j}c_{1,j} + a_{2,j}c_{2,j} + \cdots + a_{n,j}c_{n,j} \quad \text{for all } 1 \leq j \leq n. \end{aligned}$$

That theorem is a mouthful! Several remarks are in order:

- If we use this theorem to compute the determinant of a 2×2 or 3×3 matrix,

  we get the same formulas as Theorems 9.5, 9.6.

- The above method of computing the determinant is called a "cofactor expansion," since the number $c_{i,j}$ is called the "(i, j)-cofactor of A."

- The theorem gives *multiple* formulas for $\det(A)$:

  We can "expand" along any row or column, and we will get the same answer.

Example. Compute the determinant of $A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ -1 & 3 & 0 & 2 \end{bmatrix}$.

Choose a row/column with lots of zeros to make life easier.

3rd row: $\det\left(\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ -1 & 3 & 0 & 2 \end{bmatrix}\right)$

$$= 0\det(\cdot) - 0\det(\cdot) + 1\det\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & 2 \end{bmatrix}\right) - 0\det(\cdot)$$

$$= -8 - 3 + 0 - 18 - 0 - 0 = -29.$$

1st col: $\det(A) = 2\det\left(\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}\right) - 0 + 0 - (-1)\det\left(\begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}\right)$

$$= 2(-4+0+0-0-0-9) + 1(0+0+0-3-0-0) = -26 - 3 = -29.$$

Example. Compute the determinant of $A = \begin{bmatrix} 0^{+} & -1^{-} & 2^{+} & 1^{-} & 3^{+} \\ 0^{-} & 0^{+} & 0^{-} & 2^{+} & 0^{-} \\ -2^{+} & 1^{-} & 1^{+} & -1^{-} & 0^{+} \\ 1^{-} & 0^{+} & -3^{-} & 1^{+} & 0^{-} \\ 2^{+} & 1^{-} & -1^{+} & 0^{-} & 0^{+} \end{bmatrix}$.

2nd row: $\det\left(\begin{bmatrix} 0 & -1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 \\ -2 & 1 & 1 & -1 & 0 \\ 1 & 0 & -3 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 \end{bmatrix}\right)$

$= -0 + 0 - 0 + 2\det\left(\begin{bmatrix} 0 & -1 & 2 & 3 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -3 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix}\right) - 0$

do another cofactor expansion (4th column)

$= 2\left(-3\det\left(\begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & -3 \\ 2 & 1 & -1 \end{bmatrix}\right) + 0 - 0 + 0\right)$

$= 2\cdot(-3)\cdot(0 - 6 + 1 - 6 + 1 - 0) = 2\cdot(-3)\cdot(-10) = 60.$

In general, computing determinants via cofactor expansions is extremely inefficient. It's not too bad for 2×2, 3×3, or maybe 4×4 matrices. But for an $n\times n$ matrix A, a cofactor expansion contains $n!$ terms being added up, and each of those terms is the product of n entries of A. For example,

$$\det\left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{bmatrix}\right) = \begin{aligned}[t] & afkp - af\ell o - agjp + ag\ell n + ahjo - ahkn \\ & - bekp + be\ell o + bgip - bg\ell m - bhio + bhkm \\ & + cejp - ce\ell n - cfip + cf\ell m + chin - chjm \\ & - dejo + dekn + dfio - dfkm - dgin + dgjm. \end{aligned}$$

Ugh! So for large matrices, use the Gaussian elminiation method instead. Nonetheless, cofactor expansions will be useful for us for theoretical reasons next week.