Eigenvalues and Eigenvectors

This week we will learn about:

- Complex numbers,
- Eigenvalues, eigenvectors, and eigenspaces,
- The characteristic polynomial of a matrix, and
- Algebraic and geometric multiplicity.

Extra reading:

- Section 3.3 in the textbook
- Lecture videos 37, 38, and 39 on YouTube
- Complex number at Wikipedia
- Eigenvalues and eigenvectors at Wikipedia

Extra textbook problems:

⋆ 3.3.1, 3.3.2

⋆⋆ 3.3.3, 3.3.5, 3.3.7, 3.3.9, 3.3.16, 3.3.20

⋆⋆⋆ 3.3.6, 3.3.11–3.3.14

☠ 3.3.19, 3.3.23, 3.3.24

Eigenvalues and Eigenvectors

Some linear transformations behave very well when they act on certain specific vectors. For example, diagonal matrices behave very well on the standard basis vectors:

In the above example, we saw that there are vectors such that matrix multiplication behaved just like scalar multiplication: $A\mathbf{v} = \lambda\mathbf{v}$. This is extremely desirable in many situations: we often want matrix multiplication to behave like scalar multiplication, and we often want general matrices to behave like diagonal matrices. This leads to the following definition.

Definition 10.1 — Eigenvalues and Eigenvectors

Let A be a square matrix. A scalar λ is called an **eigenvalue** of A if there is a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Such a vector $\mathbf{v}$ is called an **eigenvector** of A corresponding to λ.

Example. Show that $\mathbf{v} = (1, 1)$ is an eigenvector of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and find the corresponding eigenvalue.

OK, how do we go about actually *finding* eigenvalues and eigenvectors? It's easy enough when the eigenvector is given to us, but the real world isn't that nice.

Well, we find them via a two-step process: first, we find the eigenvalues, then we find the eigenvectors.

Step 1: Find the eigenvalues. Recall that λ is an eigenvalue of A if and only if there is a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. This is equivalent to...

In other words, λ is an eigenvalue of A if and only if the matrix $A - \lambda I$ has non-zero null space. How can we find when a matrix has a non-zero null space? Well...

- $\dim(\text{null}(A - \lambda I)) > 0$ if and only if...
- ...if and only if...

A-ha! This is the type of equation we can actually solve! So to find the eigenvalues of A, we find all numbers λ such that $\det(A - \lambda I) = 0$.

Example. Find all eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

Step 2: Find the eigenvectors. Once you know the eigenvalues (from step 1), the associated eigenvectors are the vectors $\mathbf{v}$ satisfying $A\mathbf{v} = \lambda\mathbf{v}$. But this equation holds if and only if...

In other words, to find all eigenvectors of A associated with the eigenvalue λ, we compute $\mathrm{null}(A - \lambda I)$.

Example. Find all eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

The eigenvalues and eigenvectors can help us understand what a linear transformation "looks like."

Since the set of all eigenvectors of A corresponding to λ is the nullspace of $A-\lambda I$, the set of eigenvectors forms a subspace (the nullspace is always a subspace). We give this subspace a name:

Definition 10.2 — Eigenspace

Let A be a square matrix and let λ be an eigenvalue of A. The set of all eigenvectors of A corresponding to λ, together with the zero vector, is called the **eigenspace** of λ.

Example. Find all eigenvalues, and bases of their corresponding eigenspaces, for the matrix $A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -1 & 3 \\ 0 & -2 & 4 \end{bmatrix}$.

There are some matrices that do not have any (real) eigenvalues or eigenvectors. For example...

Example. Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Complex Numbers

There are a few operations in your mathematical career that you have been told you cannot do:

We now introduce something called **complex numbers** that let us "fix" one of these "problems": they let us work with square roots of negative numbers algebraically.

Remarkably, you can do arithmetic with i just like you're used to with real numbers, and things have a way of just working out. But first, let's get some terminology out of the way:

- An **imaginary number** is a number of the form

- A **complex number** is a number of the form

Arithmetic with complex numbers works just like it does with real numbers, so nothing surprising happens when you add or multiply them.

Example. Add and multiply some complex numbers.

Slightly more generally,

$(a + bi) + (c + di) =$

$(a + bi)(c + di) =$

However, division of complex numbers requires one minor "trick" to get our hands on.

Example. Divide some complex numbers.

The number that we multiplied the top and bottom by in the above example was called the **complex conjugate** of the bottom (denominator). That is,

With just these basic tools under our belt, we can now find roots of quadratics that don't have real roots! We just use the quadratic formula like usual.

Example. Find the (potentially complex) solutions of the equation $x^2 - 2x + 2 = 0$.

The previous example hints at the following observation, which is indeed true:

Just like we think of $\mathbb{R}$ as a line, we can think of $\mathbb{C}$ as a plane, and the number $a + bi$ has coordinates (a, b) on that plane.

Example. Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (again).

Back to Eigenvalues and Eigenvectors

Recall that the eigenvalues of a matrix A are the solutions λ to the equation $\det(A-\lambda I)=0$. This is a polynomial in λ, and we give it a special name:

Definition 10.3 — Characteristic Polynomial

Let A be a square matrix. Then $\det(A-\lambda I)$ is called the **characteristic polynomial** of A, and $\det(A-\lambda I)=0$ is called the **characteristic equation** of A.

The characteristic polynomial of an $n\times n$ matrix is always of degree n. Since every degree-n polynomial has at most n distinct roots, this immediately tells us that

Example. Find the characteristic polynomial, eigenvalues, and bases of the corresponding eigenspaces, of $A=\begin{bmatrix}1 & -1 & -1\\ 2 & 2 & 1\\ -1 & 1 & 2\end{bmatrix}$.

In the previous example, we had a 3×3 matrix with only 2 distinct eigenvalues. However, the matrix has 3 eigenvalues if we count the multiplicities of the roots of the characteristic polynomial: the eigenvalue $\lambda = 1$ once and the eigenvalue $\lambda = 2$ twice.

There is actually another notion of multiplicity of an eigenvalue that is also important: the dimension of the corresponding eigenspace. These ideas lead to the following definition:

Definition 10.4 — Multiplicity

Let A be a square matrix with eigenvalue λ.

- The **algebraic multiplicity** of λ is the multiplicity of λ as a root of the characteristic polynomial of A.
- The **geometric multiplicity** of λ is the dimension of its eigenspace.

In the previous example...

The fact that the geometric multiplicity of each eigenvalue was $\leq$ the algebraic multiplicity was not a coincidence: it is our next theorem.

Theorem 10.1 — Geo. Mult. $\leq$ Alg. Mult.

Let A be a square matrix. Then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

A remarkable fact called the Fundamental Theorem of Algebra says that every polynomial of degree n has *exactly* n roots, counted according to multiplicity. This immediately tells us that...

Example. Compute the algebraic and geometric multiplicities of the eigenvalues of all matrices that we considered this week.

Just like with determinants, our eigenvalue life becomes much easier when dealing with triangular matrices.

Example. Compute the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

In general, because the determinant of a triangular matrix is just the product of its diagonal entries, the eigenvalues of a triangular matrix are exactly its diagonal entries:

Theorem 10.2 — Eigenvalues of Triangular Matrices

Let A be a triangular matrix. Its eigenvalues are exactly the entries on its main diagonal (i.e., $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$).