MATH3611 / MATH5705

Chapter 3: Sequences and Series of Functions

A sequence of numbers $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ converges to a number x if:

"for every $\epsilon > 0$, there is a $K(\epsilon) \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ when $n \geq K$."

Definition

A sequence of functions $f_n : X \to \mathbb{R}$ converges *pointwise* to f if for every $x \in X$ and $\epsilon > 0$, there is a $K(x, \epsilon) \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ when $n \geq K$.

Definition

A sequence of functions $f_n : X \to \mathbb{R}$ converges *uniformly* to f if for every $\epsilon > 0$, there is a $K(\epsilon) \in \mathbb{N}$ such that for *every* $x \in X$, $|f_n(x) - f(x)| < \epsilon$ when $n \geq K$.

Example: Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 1 & x = n \\ 0 & x \neq n \end{cases}.$$

Example: In $C[0,1]$, let each f_n be the piecewise linear function whose graph connects the points $(0,0)$, $(\frac{2^n-1}{2^n},0)$, $(\frac{2^{n+2}-3}{2^{n+2}},1)$, $(\frac{2^{n+1}-1}{2^{n+1}},0)$, and $(1,0)$ (Spikes - draw a couple!).

Uniform norm

.

Definition

Let X be a set, and let $B(X, \mathbb{R})$ denote the set of bounded real-valued functions on X. The uniform norm is

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Theorem

$(B(X, \mathbb{R}), \|\cdot\|_\infty)$ *is a Banach space.*

Banach space valued functions

Let X be a set and let E be a Banach space. Then $B(X, E)$ (bounded E-valued functions) is a Banach space with the uniform norm. If X is a metric space, then so is $C_b(X, E)$ (continuous bounded E-valued functions).

Uniform convergence and continuity

Example: Does the sequence of functions $f_n(x) = x^n$ converge pointwise and/or uniformly on the interval $[0, 1]$?

Definition

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on an interval $[a, b]$. We say that the sequence *converges in L^p*, for some $p \geq 1$, to an integrable function f if

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p dx = 0.$$

Example: Consider the sequence of functions $\{f_n\}_{n=1,2,\ldots}$, where

$$f_n(x) = \begin{cases} n & 0 \leq x \leq \frac{1}{n^2} \\ 0 & \frac{1}{n^2} < x \leq 1 \end{cases}.$$

Example: Consider the sequence of functions on $[0,1]$:

$$f_1(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} < x \le 1 \end{cases}, \quad f_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases},$$

$$f_3(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{3} \\ 0 & \frac{1}{3} < x \le 1 \end{cases}, \quad f_4(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{3} \\ 1 & \frac{1}{3} < x \le \frac{2}{3} \\ 0 & \frac{2}{3} < x \le 1 \end{cases},$$

$$f_5(x) = \begin{cases} 0 & 0 \le x \le \frac{2}{3} \\ 1 & \frac{2}{3} < x \le 1 \end{cases}, \quad f_6(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{4} \\ 0 & \frac{1}{4} < x \le 1 \end{cases}$$

etc (hopefully the pattern is clear).

Example: Consider the sequence of functions on $[0,1]$:

$$f_n(x) = \begin{cases} n & \frac{n-2}{n} \leq x \leq \frac{n-1}{n} \\ 0 & \text{otherwise} \end{cases} .$$

Exercise: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on an interval $[a, b]$. If f_n converges uniformly to f, then it also converges to f in L^p for all $p \geq 1$.

Example: Consider the sequence of functions on $(0, \infty)$

$$f_n(x) = \begin{cases} \frac{1}{n} & x \leq n \\ 0 & x > n \end{cases}.$$

Series of functions

If $\{x_n\}_{n=0}^{\infty}$ is a sequence of numbers, then we write $\sum_{n=0}^{\infty} x_n$ for the corresponding series (meaning sequence of partial sums and/or its limit!).

Similarly, if $\{f_n(x)\}_{n=0}^{\infty}$ is a sequence of functions, we can consider the corresponding series $\sum_{n=0}^{\infty} f_n(x)$.

A series $\sum_{n=0}^{\infty} x_n$ *converges absolutely* if $\sum_{n=0}^{\infty} |x_n|$ converges. Recall:

If a series converges absolutely, then it converges.

Question: How do you prove this?

Theorem (Absolute convergence implies convergence)

Let E be a Banach space. Let $\{\mathbf{x}_n\}_{n=0}^{\infty}$ be a sequence of vectors in E whose series of norms $\sum_{n=0}^{\infty} \|x_n\|$ converges (in $\mathbb{R}$). Then the series of vectors $\sum_{n=0}^{\infty} x_n$ converges (in E).

Corollary (Weierstrass M-Test)

Let $\{f_n\}_{n=0}^{\infty}$ *be a sequence of real-valued functions on a set* X*. Suppose that there is a sequence of numbers* $M_n \geq 0$ *such that* M_n *is an upper bound for* f_n *for each* n*, and such that the series* $\sum_{n=0}^{\infty} M_n$ *converges. Then the the series of functions* $\sum_{n=0}^{\infty} f_n$ *converges uniformly.*

Example: Consider the series

$$\sum_{k=0}^{\infty} \frac{\cos(13^k \pi x)}{2^k}.$$

Theorem

Let $\{f_n\}_{n=1}^{\infty} \subseteq C[a,b]$ *be a sequence of functions which converges uniformly to* f. *Then* $\int_a^b f_n(x)dx$ *converges to* $\int_a^b f(x)dx$.

Uniform limits and differentiation

Example:

1. The sequence of functions $f_n(x) = \frac{1}{n}\sin(n^2 x)$ converges uniformly to the constant 0 function on $\mathbb{R}$ (why?), but the sequence of derivatives $f_n'(x)$ does not converge anywhere (why?).

2. The Weierstrass series
$$\sum_{k=0}^{\infty} \frac{\cos(13^k \pi x)}{2^k}$$
discussed earlier is a uniform limit of differentiable functions, but is not differentiable anywhere.

Theorem

Let $\{f_n\}_{n=1}^{\infty} \subseteq C[a, b]$ *be a uniformly convergent sequence of functions. Suppose the functions* f_n *are all differentiable on* (a, b)*, with continuous and bounded derivatives* f_n'*. Suppose further that the sequence of derivatives* $\{f_n'\}_{n=1}^{\infty}$ *converges uniformly on* (a, b)*.*

Then the limit function f *is also differentiable on* (a, b)*, and* f' *is the uniform limit of the sequence* $\{f_n'\}_{n=1}^{\infty}$*.*

Corollary

Suppose f_n *is a sequence of continuously differentiable functions on an interval* (a, b)*, and suppose that both the series*

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} f_n'(x)$$

converge uniformly. Then $f(x)$ *is differentiable, and we have* $f'(x) = g(x)$*.*

Consider the series $\sum_{n=0}^{\infty} x^n$ on $(-1, 1)$.

Question: Does this series converge uniformly?

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, and suppose the sequence $\{|a_n|^{\frac{1}{n}}\}_{n=0}^{\infty}$ is bounded. For each $n \geq 0$, let

$$b_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, |a_{n+2}|^{\frac{1}{n+2}}, \ldots\}.$$

and let

$$b = \lim_{n \to \infty} b_n$$

(b is the *limit superior* or $\limsup$ of the sequence $\{|a_n|^{\frac{1}{n}}\}_{n=0}^{\infty}$).

Theorem (Cauchy-Hadamard)

With notation as above, the power series converges absolutely if $|x| \cdot b < 1$ *and diverges if* $|x| \cdot b > 1$. *(True for complex numbers as well!)*

Definition

The number $R = \frac{1}{b}$, for $b \neq 0$, is called the *radius of convergence* of the power series. If $b = 0$ the radius of convergence is said to be ∞. If the sequence $\{|a_n|^{\frac{1}{n}}\}_{n=0}^{\infty}$ is unbounded, then the radius of convergence is said to be 0.

Corollary

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series, with radius of convergence* R. *Then the termwise derivative power series* $\sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$ *has the same radius of convergence* R.

Theorem

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series, with radius of convergence* $R > 0$. *Then the series is differentiable on the interval* $(-R, R)$, *with the derivative given by* $\sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$.

Compact convergence