MATH3611 / MATH5705

Chapter 4: Topological spaces

Convergence and topology

Recall that if (X, d) is a metric space, then the collection $\mathcal{O}(X)$ has the following properties.

1. $\emptyset, X \in \mathcal{O}(X)$
2. If $\{V_i\}_{i \in I} \subseteq \mathcal{O}(X)$, then $\bigcup_{i \in I} V_i \in \mathcal{O}(X)$.
3. If $V_1, V_2 \in \mathcal{O}(X)$, then $V_1 \cap V_2 \in \mathcal{O}(X)$.

Definition

A topological space is a set X together with a set of subsets $\tau = \mathcal{O}(X) \subseteq \mathcal{P}(X)$ satisfying:

1. $\emptyset, X \in \mathcal{O}(X)$
2. If $\{V_i\}_{i \in I} \subseteq \mathcal{O}(X)$, then $\bigcup_{i \in I} V_i \in \mathcal{O}(X)$.
3. If $V_1, V_2 \in \mathcal{O}(X)$, then $V_1 \cap V_2 \in \mathcal{O}(X)$.

Example: Let (X, d) be a metric space. Then we have already seen the *metric topology* τ_d.

Example: Let X be any set. The *coarse topology* is $\tau = \{\emptyset, X\}$.

Example: Let X be any set. The *discrete topology* is $\tau = \mathcal{P}(X)$.

Exercise: Let X be a set. Show that the discrete topology on X is the metric topology of the discrete metric on X.

Example: Let X be any set. The *cofinite topology* is

$$\tau = \{Y \subseteq X : Y^C \text{ is finite }\} \cup \{\emptyset\}.$$

Example: Let X be any set. The *cocountable topology* is

$$\tau = \{Y \subseteq X : Y^C \text{ is countable } \} \cup \{\emptyset\}.$$

Definition

Let (X, τ) be a topological space, and let $Y \subseteq X$. The *subspace topology* (also called *relative topology*) is $\tau|_Y = \{V \cap Y : V \in \tau\}$.

Exercise: If (X, d) is a metric space and $Y \subseteq X$, then the subspace topology on Y coming from the metric topology on X is the same as the topology of the subset metric on Y. (Formally: $\tau_d|_Y = \tau_{d|_{Y \times Y}}$.)

Definition

Let (X, τ) be a topological space. A subset $Y \subseteq X$ is closed if Y^C is open.

As with metric spaces, denote the closed sets by $\mathcal{C}(X))$. Then we have:

1. $\emptyset, X \in \mathcal{C}(X)$
2. If $\{V_i\}_{i \in I} \subseteq \mathcal{C}(X)$, then $\bigcap_{i \in I} V_i \in \mathcal{C}(X)$.
3. If $V_1, V_2 \in \mathcal{C}(X)$, then $V_1 \cup V_2 \in \mathcal{C}(X)$.

Definition

Let (X, τ) be a topological space.

1. Let $x \in X$ be a point. An *open neighborhood* of x is a set $V \in \tau$ such that $x \in V$. A *neighborhood* of x is any set containing an open neighborhood of x. We will denote the collection of neighborhoods of x by $\mathrm{Nbhd}(x)$.
2. Let $Y \subseteq X$ be a subset. The *interior* of Y is

$$\mathrm{Int}(Y) = \{y \in Y : \exists V_y \in \mathrm{Nbhd}(y) \text{ such that } V_y \subseteq Y\}.$$

Exercise: Show that

$$\mathrm{Int}(Y) = \bigcup_{\{V \in \tau : V \subseteq Y\}} V.$$

Corollary

Let (X, τ) be a topological space. For any subset $Y \subseteq X$, the interior $Int(Y)$ is an open set.

Definition

Let (X, τ) be a topological space, and let $Y \subseteq X$.

1. The *boundary* is

$$\mathrm{Bd}(Y) = X \backslash (\mathrm{Int}(Y) \sqcup \mathrm{Int}(Y^C)).$$

2. The *closure* is

$$\mathrm{cl}(Y) = \mathrm{Int}(Y) \cup \mathrm{Bd}(Y).$$

Exercise: We have $\mathrm{cl}(Y) = \bigcap_{\{V \in \mathcal{C}(X) : V \supseteq Y\}} V$.

Definition

Let (X, τ) be a topological space. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to $x \in X$ if for every $V \in \text{Nbhd}(x)$, there is a $K(V) \in \mathbb{N}$ such that $x_n \in V$ when $n \geq K$.

Definition

Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is continuous if for every $V \in \tau_Y$, we have $f^{-1}(V) \in \tau_X$.

Theorem

Let (X, τ_X), (Y, τ_Y) , and (Z, τ_Z) be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $(g \circ f) : X \to Z$ is also continuous.

Example: Let X be a set with the coarse topology. Which sequences converge to which points?

Example: Let $X = \mathbb{N}$ with the cofinite topology. Which sequences converge to which points?

Example: Let $X = \mathbb{R}$ with the cocountable topology. Which sequences converge to which points?

Definition

A topological space X has the *Hausdorff property* if for every pair of distinct points $x, y \in X$, there are neighborhoods $V(x, y) \in \mathrm{Nbhd}(x)$ and $U(x, y) \in \mathrm{Nbhd}(y)$ such that $V(x, y) \cap U(x, y) = \emptyset$.

Exercise: A sequence in a Hausdorff space has at most one limit.

Example: Consider $\mathbb{R}$ with the cocountable topology. Is this a Hausdorff topology, and do sequences have unique limits?

Example: Let $X = \{a, b\}$ and let $\tau = \{\emptyset, \{a\}, X\}$. Is this a Hausdorff space, and do sequences have unique limits?

Exercise: "Line with two origins": $\mathbb{R}$ with two copies of 0 (call them 0_a and 0_b). We want an "open ϵ-interval" around 0_a to look like $(-\epsilon, 0) \cup 0_a \cup (0, \epsilon)$, and similarly for 0_b. How can we define this topology precisely?

Question: Given a set X, how can we describe a topology on X?

Example: In $\mathbb{R}$, with the standard topology, a set is open iff it is a countable disjoint union of open intervals.

Exercise: Show that not every open set in $\mathbb{R}^2$ can be expressed as a disjoint union of open disks.

Let (X, d) be a metric space. Notice:

1. $Y \subseteq X$ is open iff Y is a union of open ϵ-balls (we'll allow the empty union).
2. $Y \subseteq X$ is a neighborhood of $x \in X$ iff Y contains an open ϵ-ball around x.

Definition

Let (X, τ) be a topological space.

1. A *base* for τ is a subset $\mathcal{B} \subset \tau$ such that every $V \in \tau$ can be expressed as as a union of elements of $\mathcal{B}$:

$$V = \bigcup_{i \in I} V_i, \text{ where } V_i \in \mathcal{B}, \ \forall i \in I.$$

2. A *local base* for τ at a point $x \in X$ is a collection $\mathcal{LB}_x \subseteq \tau$ of open neighborhoods of x such that if U is any neighborhood of x, there is a $V \in \mathcal{LB}_x$ such that $V \subseteq U$.

Example: If (X, τ) is a topological space, then τ is a base for itself. For any $x \in X$, the collection of all open neighborhoods of x is a local base for τ at x.

Example: Let (X, d) be a metric space. Then

$$\mathcal{B} = \{B(x, \epsilon)\}_{x \in X, \epsilon > 0}$$

is a base for the metric topology. For any $x \in X$, the set

$$\mathcal{LB}_x = \{B(x, \epsilon)\}_{\epsilon > 0}$$

is a local base at x.

Example: In a metric space, we don't actually need all values of ϵ for a base: the set

$$\mathcal{B} = \{B(x, \frac{1}{n})\}_{x \in X, n \in \mathbb{Z}_+}$$

is a base for the metric topology. For any $x \in X$, the set

$$\mathcal{LB}_x = \{B(x, \frac{1}{n})\}_{n \in \mathbb{Z}_+}$$

is a local base at x

Example: Let $X = \mathbb{R}^2$. As we have seen, all of the metrics $d_p, p \geq 1$ give the same topology. So this topology has a base consisting of open disks, a base of open diamonds, etc. But you can describe many others - let's try a couple.

Example: Let X be a set with the discrete topology. Then the set of singleton sets $\mathcal{B} = \{\{x\}\}_{x \in X}$ is a base. For each $x \in X$, the set $\mathcal{LB}_x = \{\{x\}\}$ is a local base at x.

Exercise: Let (X, τ) be a topological space. If $\mathcal{B}$ is a base for τ, then for any $x \in X$, the set $\mathcal{LB}_x = \{V \in \mathcal{B} : x \in V\}$ is a local base at x. Conversely, if we have a local base $\mathcal{LB}_x$ for each point $x \in X$, then $\mathcal{B} = \bigcup_{x \in X} \mathcal{LB}_x$ is a base.

Theorem

Let X *be a set, and let* $\mathcal{B} \subseteq \mathcal{P}(X)$ *be a collection of subsets. Then*

$$\tau = \{V \subseteq X : V \text{ is a union of sets in } \mathcal{B}\}$$

is a topology iff the following conditions hold:

1. $\bigcup_{V \in \mathcal{B}} V = X$ (*"$\mathcal{B}$ covers X"*)
2. *for every* V_1 *and* V_2 *in* $\mathcal{B}$ *and every* $x \in V_1 \cap V_2$, *there is* $V \in \mathcal{B}$ *such that* $x \in V \subseteq V_1 \cap V_2$.

Exercise: Let (X, τ_X) and (Y, τ_Y) be topological spaces, and suppose $\mathcal{B}$ is a base for τ_Y. Then a function $f : X \to Y$ is continuous iff for every $V \in \mathcal{B}$, we have $f^{-1}(V) \in \tau_X$.

Exercise: Let (X, τ) be a topological space, and suppose that $\mathcal{LB}_x$ is a local base for τ at $x \in X$. Then a sequence $\{x_n\}_{n=1}^{\infty}$ converges to x iff for every $V \in \mathcal{LB}_x$, there is a $K(V)$ such that $x_n \in V$ for all $n \geq K$.

Definition

Let X be a set, and let $S \subseteq \mathcal{P}(X)$ be any collection of subsets. Define $\mathcal{B}$ to be the set of all **finite** intersections of sets in S:

$$\mathcal{B} = \{V_1 \cap ... \cap V_n : V_k \in S,\ k = 1, ..., n\}.$$

(We allow the empty intersection X). Then $\mathcal{B}$ satisfies the conditions for a base in the previous theorem (why?), so

$$\tau(S) = \{V \subseteq X : V \text{ is a union of sets in } \mathcal{B}\}$$

is a topology. We call S a *subbase* for $\tau(S)$, and say that τ is generated by S.

Exercise:

1. Let X be a set, and let $\{\tau_i\}_{i\in I}$ be a set of topologies on X. Then

$$\tau = \bigcap_{i\in I} \tau_i$$

is a topology.

2. Let X be a set, and let $S \subseteq X$ be a subset. Show that $\tau(S)$, as defined above, is the intersection of all topologies on X which contain S.

Example: The set of infinite open intervals $S = \{(a, \infty)\}_{a \in \mathbb{R}} \cup \{(-\infty, a)\}_{a \in \mathbb{R}}$ is a subbase for the standard topology on $\mathbb{R}$.

Example: The set of open half-planes is a subbase for the standard topology on $\mathbb{R}^2$.

Example: Let X be a set, and let $S = \{X \backslash \{a\}\}_{a \in X}$. What is the topology $\tau(S)$ generated by S?

Example: Let $S = \{[a,b]\}_{a<b\in\mathbb{R}}$. What is the topology $\tau(S)$ on $\mathbb{R}$?

Exercise: Let S be a subbase for a topology τ. Then a sequence $\{x_n\}_{n=1}^{\infty}$ converges to x iff for every $V \in S$ such that $x \in V$, there is a $K(V)$ such that $x_n \in V$ for $n \geq K$.

Topologies defined by modes of convergence

General procedure for defining a topology for a type of convergence:

1. Start by writing down precisely what we mean by convergence
2. Use the description of convergence to say what "open neighborhoods" should look like
3. Define the topology generated by such neighborhoods

Pointwise convergence

Let X be any set, and let $Y = F(X, \mathbb{R})$ be the set of real-valued functions on X. A sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise if

$$\forall x \in X, \epsilon > 0,\ \exists K(x, \epsilon) \text{ such that } \forall n \geq K,\ \text{ we have}$$

$$|f_n(x) - f(x)| < \epsilon.$$

Question: How can we rephrase that last part as f_n eventually belonging to some "neighborhood" of f?

The set

$$V = \{g \in Y : |g(x) - f(x)| < \epsilon\} \subseteq Y$$

depends on three things:

1. a point $x \in X$
2. a number $f(x) \in \mathbb{R}$
3. a number $\epsilon > 0$

Definition

Let X be a set and let $Y = F(X, \mathbb{R})$. For each $x \in X$, $y \in \mathbb{R}$, and $\epsilon > 0$, define

$$V_{x,y,\epsilon} = \{g \in Y : |g(x) - y| < \epsilon\}.$$

Then let

$$S = \{V_{x,y,\epsilon}\}_{x \in X, y \in \mathbb{R}, \epsilon > 0}.$$

Finally, define the topology of pointwise convergence to be $\tau_{pt} = \tau(S)$, the topology generated by the subbase S.

Exercise: The topology τ_{pt} is Hausdorff.

Theorem

A sequence of functions $f_n : X \to \mathbb{R}$ *converges pointwise to* f *iff* $f_n \to f$ *in the topology* τ_{pt}.

A base for τ_{pt} is given by finite intersections of elements of S. We can visualise these for $X = [0, 1]$ as *gate sets*.

Weak convergence

Definition

Let $\mathbf{H}$ be a Hilbert space, such as $\mathbb{R}^n$ or ℓ^2. We say that a sequence of vectors $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges *weakly* to a vector $\mathbf{x} \in \mathbf{H}$ if for every vector $\mathbf{y} \in \mathbf{H}$, we have

$$\langle \mathbf{x}_n, \mathbf{y} \rangle \to \langle \mathbf{x}, \mathbf{y} \rangle.$$

Exercise: Show that if a sequence of vectors in a Hilbert space converges in the norm topology, then it converges weakly.

Example: Consider the sequence of standard basis vector $\{\mathbf{e}_n\}_{n=1}^{\infty}$ in ℓ_2 (where $(\mathbf{e}_n)_k = \delta_{n,k}$). Then $\mathbf{e}_n$ converges to $\mathbf{0}$ weakly but not in norm.

A sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges weakly to $\mathbf{x}$ if for every $\mathbf{y} \in \mathbf{H}$, we have $\langle \mathbf{x}_n - \mathbf{x}, \mathbf{y} \rangle \to 0$.

This means that for every $\mathbf{y} \in \mathbf{H}$ and every $\epsilon > 0$, there is a $K(\mathbf{y}, \epsilon)$ such that $\forall n \geq K$, we have

$$|\langle \mathbf{x}_n - \mathbf{x}, \mathbf{y} \rangle| < \epsilon.$$

How can we rephrase this last part as $\mathbf{x}_n$ eventually belonging to some "neighborhood" of $\mathbf{x}$?

The set

$$V_{\mathbf{x},\mathbf{y},\epsilon} = \{\mathbf{z} : |\langle \mathbf{z} - \mathbf{x}, \mathbf{y} \rangle| < \epsilon\}$$

depends on three things: vectors $\mathbf{x}$ and $\mathbf{y}$, and a number $\epsilon > 0$.

Definition

Let $\mathbf{H}$ be a Hilbert space, and for each $\mathbf{x}, \mathbf{y} \in \mathbf{H}$ and $\epsilon > 0$, let

$$V_{\mathbf{x},\mathbf{y},\epsilon} = \{\mathbf{z} \in \mathbf{H} : |\langle \mathbf{z} - \mathbf{x}, \mathbf{y} \rangle| < \epsilon\}.$$

Let

$$S = \{V_{\mathbf{x},\mathbf{y},\epsilon}\}_{\mathbf{x},\mathbf{y} \in \mathbf{H}, \epsilon > 0}.$$

The *weak* topology on $\mathbf{H}$ is $\tau_{weak} = \tau(S)$.

Exercise:

1. The weak topology is Hausdorff.
2. A sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges weakly to $\mathbf{x}$ in $\mathbf{H}$ if it converges in the topology τ_{weak}.

Slice sets

The weak topology agrees with the standard topology on $\mathbb{R}^n$.

Definition

Let (a, b) be an open interval in $\mathbb{R}$. A sequence of functions is said to converge compactly if it converges uniformly on every closed subinterval $[c, d] \subseteq (a, b)$.

Exercise: Define a topology τ_{cpt} on $C((0,1),\mathbb{R})$ (the set of continuous functions from the open interval $(0,1)$ to $\mathbb{R}$) such that a sequence of functions converges compactly iff it converges in the topology τ_{cpt} .

Definition

A topological space (X, τ) is said to be:

1. *First countable* if every point in X has a countable local base for τ.
2. *Second countable* if X has a countable base for τ.

Example:

1. Every metric space is first countable.
2. $\mathbb{R}$ is second countable.
3. The cofinite topology on $\mathbb{N}$ is second countable.

Definition

A topological space is *separable* if it contains a countable dense subset.

Exercise:

1. Every second countable topological space is separable.
2. Every separable *metric* space is second countable.

Exercise:

1. For each $1 \leq p < \infty$, the space ℓ^p is separable.
2. The space ℓ^∞ is not separable.

Exercise: Show that $\mathbb{R}$ with the cocountable topology is **not** first countable.

Recall that if (X, d) is a metric space, then $Y \subseteq X$ is closed iff Y contains the limits of all of its sequences. Is this true in a topological space?

Example: Consider $\mathbb{R}$ with the cocountable topology.

Definition

Let (X, τ) be a topological space. A local base $\{V_n\}_{n\in\mathbb{N}}$ at a point x is called *nested* if $V_n \subseteq V_m, \ \forall n \geq m$.

Exercise: Show that a point in a first countable space always has a nested local base.

Exercise: Show that if (X, τ) is a first countable topological space and $Y \subseteq X$, then every point in $\mathrm{cl}(Y)$ is the limit of a sequence in Y.

Theorem

Let (X, τ) be a first countable topological space. Then a subset $Y \subseteq X$ is closed iff for every sequence in Y which converges (in X), the limit is in Y.

Nets

A sequence is a function from $\mathbb{N}$ to some set or space. When we talk about the limit of a sequence as $n \to \infty$, we are using the fact that the index set/domain $\mathbb{N}$ has a *direction*: there is a clear meaning to moving "further along" in the sequence.

Definition

A *directed set* is a set Λ, together with a binary relation $\leq$ satisfying, for all $i, j, k \in \Lambda$:

1. $i \leq i$
2. $i \leq j \,\&\, j \leq k \implies i \leq k$
3. $\exists m \in \Lambda$ such that $i, j \leq m$.

Example: Any totally ordered set is directed (under either $\leq$ or $\geq$!).

Example:

There are many ways to make $\mathbb{N} \times \mathbb{N}$ a directed set. Here are a couple:

- $(m1, n1) \leq (m_2, n_2)$ iff $m_1 \leq m_2$ and $n_1 \leq n_2$
- $(m1, n1) \leq (m_2, n_2)$ iff either $m_1 > m_2$ or $m_1 = m_2$ and $n_1 \geq n_2$

.

Example: Let (X, τ) be a topological space, and fix $x \in X$. Then Nbhd(x) is a directed set, with $U \leq V$ iff $V \subseteq U$. (Note the direction!)

Example: Let S be the set whose *tagged partitions* of $[0, 1]$, meaning partitions of $[0, 1]$ into finitely many subintervals, together with choices of points from each of these subintervals.

Define the *mesh* of a tagged partition to be the maximum length of its subintervals. Then for two tagged partitions s and t, we say that $s \geq t$ if s has a *smaller* mesh than t.

Definition

A *net* is a function from a directed set $(\Lambda, \leq)$ to a set X. As for sequences, we'll use the notation $\{x_\lambda\}_{\lambda\in\Lambda}$.

Definition

Let (X, τ) be a topological space. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ converges to a point $x \in X$ if for every neighborhood V of x, there is an $\alpha(V) \in \Lambda$ such that $x_\lambda \in V$ when $\lambda \geq \alpha$.

Example: Let $X = \mathbb{R}$ with the standard topology, and let $\Lambda = \mathbb{R}$ with the standard direction $\leq$. Then a Λ-net $\{f(x)\}_{x \in \mathbb{R}}$ is simply a function $f : \mathbb{R} \to \mathbb{R}$. What does it mean for this net to converge? What if you use the opposite direction $\geq$?

Example: Let $X = \mathbb{R}$, and let $\Lambda = \mathbb{N} \times \mathbb{N}$, with one of the two directions described above. Consider the net $\{\frac{1}{n+1}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$. Does this net converge?

Example: Let $X = \mathbb{R}$, and let Λ be the directed set of tagged partitions of $[0, 1]$. Fix a function $f : [0, 1] \to \mathbb{R}$. Then we can define the Riemann net $\{f_\lambda\}_{\lambda \in \lambda}$, where for a tagged partition

$$\lambda = [0, x_1], x_1^*; [x_1, x_2], x_2^*; \ldots; [x_{n-1}, 1], x_n^*,$$

we define the Riemann sum

$$f_\lambda = \sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}).$$

What does it mean for this net to converge?

Exercise: A net in a Hausdorff space has at most one limit.

Exercise: Let (X, τ) be a topological space, and let $Y \subseteq X$. Then any point in $\mathrm{cl}(Y)$ is the limit of a net in Y.

Theorem

Let (X, τ) be a topological space. The $Y \subseteq X$ is closed iff Y contains the limits of all of its convergent nets.

Comparison of topologies

Definition

Suppose τ and σ are two topologies on a set X such that $\tau \subseteq \sigma$. Then we say that τ is *coarser* and σ is *finer*.

If τ is a coarser topology than σ, and $\{x_\lambda\}_{\lambda\in\Lambda}$ is a net in X which converges to a point x with respect to σ, then $\{x_\lambda\}_{\lambda\in\Lambda}$ also converges to x with respect to τ.

Example:

1. For any set X, the coarse topology is coarser than any other topology, and the discrete topology is finer than any other topology.
2. On $C[0,1]$, the $\|\cdot\|_\infty$ topology is finer than the $\|\cdot\|_p$ topology for any p, and is also finer than the pointwise topology. (It is also true that the $\|\cdot\|_p$ topology is finer than the $\|\cdot\|_q$ topology for $p > q$).
3. On $C[0,1]$, the $\|\cdot\|_1$ topology and the pointwise topology are incomparable.
4. On a Hilbert space, the norm topology is finer than the weak topology.

Definition

Let (X, τ_X) and (Y, τ_Y) be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* if both f and f^{-1} are continuous. We say that X and Y are *homeomorphic* if there exists a homemorphism $f : X \to Y$.

Exercise: Show that homeomorphism satisfies the properties of an equivalence relation.

Exercise: Show that $\mathbb{R}$ is homeomorphic to $(0,1)$ (with the standard topologies).

Exercise: Give an example of a continuous bijection that is not a homeomorphism.

Two metric (X, d_X) and (Y, d_Y) are said to be *isometric* if there is a bijection $f : X \to Y$ such that

$$d_Y(f(x_1), f(x_2)) = d_Y(x_1, x_2), \ \forall x_1, x_2 \in X.$$

Definition

Two metric spaces (X, d_X) and (Y, d_Y) are said to be *equivalent* if there is a bijection $f : X \to Y$ and constants $k, K > 0$ such that

$$k \cdot d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K \cdot d_X(x_1, x_2), \ \forall x_1, x_2 \in X.$$

Two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are said to be *equivalent* if there is a bijection $f : X \to Y$ and constants $k, K > 0$ such that

$$k \cdot \|\mathbf{x}\|_X \leq \|f(\mathbf{x})\|_Y \leq K \cdot \|\mathbf{x}\|_X, \ \forall x_1, x_2 \in X.$$

Exercise:

1. Show that each of the two the relations defined above satisfies the properties of an equivalence relation.
2. Show that if two normed spaces are equivalent (as normed spaces), then the two metric spaces coming from the norms are equivalent (as metric spaces).
3. Show that if two metric spaces are equivalent (as metric spaces), then the two topological spaces coming from the metrics are homeomorphic.
4. Give an example of two metric spaces which are **not** equivalent as metric spaces, but whose metric topologies are homeomorphic.

Homeomorphism invariants

Definition

A topological space (X, τ) is *connected* if it is not the union of two disjoint nonempty open subsets. A subset $Y \subseteq X$ is connected if it is connected in the subspace topology.

Exercise: Unpack the definition of what it means for a subset of a topological space to be connected.

Example:

1. A discrete space (with at least two points) is not connected.
2. The set of invertible $n \times n$ matrices, $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$, is not connected in the norm topology inherited from $M_n(\mathbb{R})$.

Theorem

The interval $[0, 1]$ *is connected.*

Theorem

Let (X, τ_X) *and* (Y, τ_Y) *be topological spaces, and suppose* $f : X \to Y$ *is a continuous function. If* X *is connected, then so is* $f(X)$ *(as a subset of* Y*).*

Lemma

Let (X, τ) *be a topological space, and suppose that* $\{W_i\}_{i \in I}$ *are connected subsets such that* $\bigcap_{i \in I} W_i \neq \emptyset$*. Then* $\bigcup_{i \in I} W_i$ *is connected.*

Let (X, τ) be a topological space. For each $x \in X$, define

$$\text{Comp}(x) = \bigcup_{\{V \subseteq X : x \in V \ \&\ V \text{ is connected}\}} V.$$

Now, for points x and y in X, we'll say $x \sim^{ct} y$ if $y \in \text{Comp}(x)$.

Exercise: Show that $\sim^{ct}$ is an equivalence relation on X.

Definition

The equivalence classes of X under $\sim^{ct}$ are called the *connected components* of X.

Example: Show that a plus sign is not homeomorphic to a minus sign.

Question: Are connected components open and/or closed?

Exercise: Let (X, τ) be a topological space, and let $Y \subseteq X$ be a connected subset. Show that $\mathrm{cl}(Y)$ is connected.

Exercise: Connected components are closed.

Example: What are the connected components of $\{0\} \cup \{\frac{1}{n}\}_{n \in \mathbb{Z}_+}$ (in the standard topology inherited from $\mathbb{R}$)?

Definition

A topological space (X, τ) is *path-connected* if for every pair of points $x, y \in X$, there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. A subset $Y \subseteq X$ is path-connected if it is path-connected in the subspace topology.

Theorem

If a topological space is path-connected, then it is connected.

Exercise: Consider the set $\{(x, \sin\frac{1}{x})\}_{x\in\mathbb{R}\setminus\{0\}} \cup \{(0,0)\} \subseteq \mathbb{R}^2$. Show that this space (with the subspace topology inherited from $\mathbb{R}^2$) is connected but not path-connected.

Example: The space $(C[0,1], \|\cdot\|_\infty)$ is connected.

Example: The spaces $(0, 1)$ and $(0, 1]$ are not homeomorphic.

Question: Let (X, τ_X) and (Y, τ_Y) be topological spaces. How can we define a topology on the Cartesian product $X \times Y$?

Definition

Let (X, τ_X) and (Y, τ_Y) be topological spaces. The product topology on $X \times Y$ is the topology generated by the base

$$\{U \times V\}_{U \in \tau_X, V \in \tau_Y}.$$

Exercise: Show that the product topology on $\mathbb{R} \times \mathbb{R}$ (where each copy of $\mathbb{R}$ has the standard topology) is the standard topology on $\mathbb{R}^2$.

Exercise: Let $\mathbb{T}$ be the unit circle in the complex plane, with the standard topology. Describe the product topology on $\mathbb{T} \times \mathbb{T}$.

Exercise: Let (X, τ_X) and (Y, τ_Y) be topological spaces, and suppose that $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for τ_X and τ_Y, respectively. Then $\{U \times V\}_{U \in \mathcal{B}_X, V \in \mathcal{B}_Y}$ is a base for the product topology on $X \times Y$.

Question: What about an arbitrary (possibly infinite) collection of topological spaces $\{(X_i, \tau_i)\}_{i \in I}$? Can we similarly put a topology on the Cartesian product $\prod_{i \in I} X_i$?

Definition

Let $\{(X_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces. The *box topology* $\prod_{i\in I} X_i$ is the topology with base

$$\mathcal{B} = \{\prod_{i\in I} U_i : U_i \in \tau_i,\ \forall i \in I\}$$

Example: Consider $\mathbb{R}^N$ (infinite Cartesian power of $\mathbb{R}$, i.e. the set of real sequences), with the box topology coming from the standard topology on each copy of $\mathbb{R}$. Show that the sequence

$$(1, 0, 0, \dots),\ (0, \frac{1}{2}, 0, \dots),\ (0, 0, \frac{1}{3}), \dots$$

does **not** converge to the zero seqeunce.

Definition

Let $\{(X_i, \tau_i)\}_{i \in I}$ be a set of topological spaces. The *product topology* on $\prod_{i \in I} X_i$ is the topology generated by the base

$$\{\prod_{i \in I} U_i : U_i \in \tau_i,\ \forall i \in I \ \&\ U_i = X_i \text{ for all but finitely many } i \in I\ \}.$$

Exercise: Show that a sequence in $\mathbb{R}^N$ with the product topology converges iff it converges pointwise.

Exercise: More generally, let X be a set. Show that the product topology on $\mathbb{R}^X = F(X, \mathbb{R})$ is the topology of pointwise convergence.