MATH3611/5705: Compactness

A continuous real valued function on a **closed** and **bounded** interval $[a, b]$ achieves a maximum. Both "closed" and "bounded" are necessary!

Definition

A topological space (X, τ) is *compact* if for every $\{V_i\}_{i \in I} \subseteq \tau$ such that

$$X = \bigcup_{i \in I} V_i,$$

there is a finite subset $\{i_1, ..., i_n\} \subseteq I$ such that

$$X = \bigcup_{k=1}^{n} V_{i_k}.$$

A subset $Y \subseteq X$ is compact iff it is compact in the subspace topology.

In words: "Every open cover has a **finite** subcover".

Exercise:

1. A coarse topological space is compact.
2. A discrete topological space is compact iff it is finite.

Compactness in terms of closed sets: A topological space (X, τ) is compact if every collection $\{V_i\}_{i \in I}$ of closed subsets of X such that *all finite intersections* $V_{i_1} \cap \ldots \cap V_{i_n}$ *are nonempty* has nonempty intersection.

Question: What does is it mean for a subset $Y \subseteq X$ to be compact?

Theorem

The interval $[0,1]$ *is compact.*

Theorem

Let (X, τ_X) *and* (Y, τ_Y) *be compact topological spaces. Then* $X \times Y$ *is compact (in the product topology).*

Corollary: Sets of the form $[a, b]^n \subseteq \mathbb{R}^n$ are compact (with the standard topology).

Exercise: Let (X, τ) be a compact topological space. If Y is a closed subset of X, then Y is compact.

Exercise: Let (X, τ) be a Hausdorff space. If Y is a compact subset of X, then Y is closed.

Example: In a coarse topological space, every subset is compact but only the empty set and the whole space are closed.

Theorem (Heine-Borel)

A subset $X \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

Theorem (Bolzano-Weierstrass)

Every bounded sequence of real numbers has a convergent subsequence.

To prove the B-W Theorem, we first show:

Lemma

Every sequence of reals numbers has a monotone subsequence.

Corollary

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Definition

A topological space (X, τ) is called *sequentially compact* if every sequence in X has a convergent subsequence. Similarly, a subset $Y \subseteq X$ is sequentially compact if it is sequentially compact in the subspace topology.

Theorem

Let X be a subset of $\mathbb{R}^n$. The following are equivalent:

1. *X is compact*
2. *X is sequentially compact*
3. *X is closed and bounded*

Theorem

Let (X, τ_X) *and* (X, τ_Y) *be topological spaces. If* $f : X \to Y$ *is continuous and* X *is compact, then* $f(X) \subseteq Y$ *is compact.*

Corollary (Min-max Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then* f *attains maximum and minimum values.*

For a function $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces:

f is continuous if $\forall x \in X,\ \forall \epsilon > 0,\ \exists \delta(x, \epsilon)$ such that $d_Y(f(x'), f(x)) < \epsilon$ whenever $d_X(x, x') < \delta$.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be *uniformly continuous* if $\forall \epsilon > 0,\ \exists \delta(\epsilon)$ such that $d_Y(f(x'), f(x)) < \epsilon$ whenever $d_X(x, x') < \delta$.

Exercise: If a function $f : X \to Y$ is Lipschitz continuous, then it is uniformly continuous.

Example: The function $f(x) = \sqrt{x}$ on $[0, 1]$ is not Lipschitz continuous but is uniformly continuous.

Theorem

Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.

Exercise: Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable, and let $g : [a, b] \times [a, b] \to \mathbb{R}$ be continuous. Show that

$$h(x) = \int_a^b f(t)g(x, t)\ dt$$

is a continuous function of x.

Question: Does the Heine-Borel Theorem apply in an arbitrary metric space?

Example: An infinite discrete metric space is complete and bounded, but is not compact.

Example: Consider the *closed unit ball* in $(C[0,1], \|\cdot\|_\infty)$:

$$B_1 = \{f \in C[0,1] : \|f\|_\infty \leq 1\}.$$

This is a closed and bounded set. Yet the sequence of functions $\{x^n\}_{n=1}^\infty$ does not have any convergent subsequence in $(C[0,1], \|\cdot\|_\infty)$ (why not?) Therefore B_1 is not sequentially compact.

Definition

A metric space (X, d) is said to be *totally bounded* if for **every** $\epsilon > 0$, there is a **finite** set $\{x_1, ..., x_n\} \subseteq X$ such that

$$X = \bigcup_{k=1}^{n} B(x_k, \epsilon).$$

Similarly, a subset of a metric space is totally bounded if it is totally bounded with respect to the subset metric.

Exercise: Let (X, d) be a metric space, and let $Y \subseteq X$ be a totally bounded subset.

1. Every subset of Y is totally bounded.
2. The closure of Y is totally bounded.

Exercise: Show that a bounded subset of $\mathbb{R}^n$ is totally bounded.

Exercise: Show that, for any $p \geq 1$, the closed unit ball in ℓ^p

$$\{\mathbf{x} \in \ell^p : \|\mathbf{x}\|_p \leq 1\}$$

is not totally bounded.

Example: More generally, it can be shown that any ball of positive radius in an infinite-dimensional Banach space is not totally bounded.

Theorem

Let (X, d) be a metric space. The following are equivalent:

1. X *is compact*
2. X *is sequentially compact*
3. X *is complete and totally bounded*

Compact $\implies$ sequentially compact

Sequentially compact $\implies$ complete and totally bounded

Complete and totally bounded $\implies$ compact

Question: A metric space is compact iff it is complete and totally bounded. But what does "totally bounded" actually mean for a set of functions in $(C[0,1], \|\cdot\|_\infty)$?

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. A subset $S \subseteq C(X, Y)$ is said to be:

1. *Pointwise equicontinuous* if

$$\forall x \in X, \epsilon > 0,\ \exists \delta(x, \epsilon),\ \text{such that } \forall f \in S,$$

$$d_Y(f(x'), f(x)) < \epsilon \text{ whenever } d_X(x', x) < \delta.$$

2. *Uniformly equicontinuous* if

$$\forall \epsilon > 0,\ \exists \delta(\epsilon), \text{such that } \forall f \in S$$

$$d_Y(f(x'), f(x)) < \epsilon \text{ whenever } d_X(x', x) < \delta.$$

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, with X compact. Then $S \subseteq C(X, Y)$ is pointwise equicontinuous iff it is uniformly equicontinuous.

Example: Consider a sequence of "spike functions" $\{f_n\}_{n=2}^{\infty}$ in $C[0,1]$, where each f_n is 0 outside the interval $(\frac{n-2}{n}, \frac{n-1}{n})$ and spikes linearly to 1 and then back down to 0 within that interval. This sequence is uniformly bounded, but is not equicontinuous.

Note: On the open interval $(0,1)$, the above sequence is pointwise equicontinuous but not uniformly equicontinuous.

Exercise: Let $S \subseteq C[0,1]$ be a set of functions which are all differentiable on $(0,1)$. Suppose there exists a constant K such that $|f'(x)| \leq K, \ \forall f \in S, x \in (0,1)$. Then S is uniformly equicontinuous.

Theorem (Arzela-Ascoli)

A bounded subset of $(C[0,1], \|\cdot\|_\infty)$ *is totally bounded iff it is equicontinuous.*

(More generally, $[0,1]$ *can be replace by any compact metric space).*

Corollary

A subset of $(C[0,1], \|\cdot\|_\infty)$ *is compact iff it is closed, bounded and equicontinuous.*

Corollary

A uniformly bounded and equicontinuous sequence of functions on a closed interval $[a, b]$ *has a uniformly convergent subsequence.*

Example: Define a sequence of functions in $C[0,1]$ by

$$g_n(x) = \cos(n) + \int_0^x \sin(n\sqrt{t})\, dt.$$

Show that this sequence does not converge but has a uniformly convergent subsequence.

Theorem (Weierstrass Approximation Theorem)

Let f be a continuous function on a closed, bounded interval $[a, b]$. For any $\epsilon > 0$, there is a polynomial function $p(x)$ such that

$$\|f - p\|_\infty < \epsilon.$$

Definition

Consider a function $f \in C[0,1]$. The n^{th} Bernstein polynomial for f is defined as

$$B_{f,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \quad x \in [0,1].$$

Note that $B_{f,n}(x)$ is equal to the expected value $\mathbb{E}\left(f\left(\frac{X}{n}\right)\right)$, where X is a random variable with binomial distribution $B(n,x)$.

On the other hand $f(x) = f\left(\frac{nx}{n}\right) = f\left(\frac{\mathbb{E}(X)}{n}\right)$.

So

$$|f(x) - B_{f,n}(x)| = \left|f\left(\frac{\mathbb{E}(X)}{n}\right) - \mathbb{E}\left(f\left(\frac{X}{n}\right)\right)\right|.$$

- Constructive proof of W.A.T. - show that $\{B_{n,f}\}_{n=1}^{\infty}$ converges **uniformly** to f.
- As we will see, W.A.T. is a special case of a much more general result called the Stone-Weierstrass Theorem.

Exercise: Show that $(C[0,1], \|\cdot\|_\infty)$ is separable.

Definition

Let X and Y be sets. A set S of functions between X and Y is said to *separate points* if for every pair of distinct points $x, y \in X$, there is a function $f \in S$ such that $f(x) \neq f(y)$.

Theorem (Urysohn's Lemma)

Let X be a compact Hausdorff space. Then $C(X, \mathbb{R})$ separates points.

Let k be a vector space, and let X be a set. As we have seen, $F(X, k)$ (the set of functions from X to k) is a vector space with pointwise operations. Similarly, any (nonempty) subset of $F(X, k)$ which is closed under pointwise addition and scalar multiplication is a vector space.

Definition

An *algebra* of functions is a vector space of functions with pointwise operations which is also closed under pointwise multiplication of functions. An algebra of functions is called *unital* if it contains the constant function 1.

Example: Let (X, τ) be a topological space. Then $C(X, \mathbb{R})$ is a unital algebra.

Theorem (Stone-Weierstrass Theorem)

Let X be a compact Hausdorff space, and let $A \subseteq C(X, \mathbb{R})$ be a unital subalgebra. Then A is dense with respect to $\|\cdot\|_\infty$ iff A separates points.

Example: Consider the compact Hausdorff space $X = [0, 1]$. The function $f(x) = x$ by itself separates the points of $[0, 1]$. The smallest unital algebra containing x is the set of polynomials functions. Therefore the set of polynomials is uniformly dense in $C[0, 1]$ (i.e. W.A.T.)

Question: What about complex functions?

Example: Let $X = \mathbb{T}$ (the unit circle in the complex plane). The function $f(z) = z$ separates the points of $\mathbb{T}$. The smallest complex unital algebra containing z is the set of complex polynomials functions.

But this algebra is **not** uniformly dense in $C(\mathbb{T}, \mathbb{C})$. For example, $g(z) = \frac{1}{z}$ cannot be uniformly approximated by complex polynomials on $\mathbb{T}$ (why not?).

Definition

An algebra of complex-valued functions is called a $*$-*algebra* if it is closed under pointwise complex conjugation.

Theorem (Stone-Weierstrass Theorem - complex version)

Let X be a compact Hausdorff space, and let $A \subseteq C(X, \mathbb{C})$ be a unital $*$-subalgebra. Then A is dense with respect to $\|\cdot\|_\infty$ iff A separates points.

Example: Let $X = \mathbb{T}$. The function $f(z) = z$ separates the points of $\mathbb{T}$. The smallest complex unital $*$-algebra containing z is the set of Laurent polynomials functions (i.e. polynomials with negative powers of z allowed). By the Stone-Weierstrass Theorem, this algebra is uniformly dense in $C(\mathbb{T}, \mathbb{C})$.

Recall that a closed ball in an infinite-dimensional Banach is not compact in the norm topology.

For this reason, it is sometimes useful when studying Banach spaces to work with coarser topologies than the norm topology (for example the weak topology on a Hilbert space).

Theorem (Tychonoff's Theorem)

Let $\{(X_i, \tau_i)\}_{i \in I}$ *be a collection of compact topological spaces. Then* $\prod_{i \in I} X_i$ *is compact in the product topology.*

Theorem

Let H be a Hilbert space. The closed unit ball in H (i.e. $\{\mathbf{x} \in \mathbf{H} : \|\mathbf{x}\| \leq 1\}$) is compact in the weak topology.

This is a special case of an important result in functional analysis called the *Banach-Alaoglu Theorem*.