MATH3611 / MATH5705

Chapter 2: Metric spaces

Recall the definition of a limit: $\lim_{x \to a} f(x) = b$ means that "for any number $\epsilon > 0$, there is a number $\delta(\epsilon)$ such that $|f(x) - b| < \epsilon$ whenever $|x - a| < \delta$."

Definition

A *metric space* is a pair (X, d), where X is a (non-empty) set and $d : X \times X \to [0, \infty)$ is a function, such that the following conditions hold for all $x, y, z \in X$:

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) + d(y, z) \geq d(x, z)$ (*triangle inequality*)

Example: $X = \mathbb{R}$; $d(x, y) = |x - y|$.

Example: $X = \mathbb{R}^n$; $d_p((x_1, ..., x_n), (y_1, ..., y_n)) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}}$,
where $p \geq 1$ is a fixed number.

(**Question:** How easy is it to prove the triangle inequality for $p = 1$ or $p = 2$? What about in general?)

Example: $X = \mathbb{R}^n$;
$d_\infty((x_1, ..., x_n), (y_1, ..., y_n)) = \max\{|x_1 - y_1|, ..., |x_n - y_n|\}$.

Example: X is any (non-empty) set, and $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ (*discrete* metric).

Example: Let Γ be a finite connected graph, and let X be its set of vertices. Define $d(x, y)$ to be the length of the shortest path between x and y.

Example: Let Γ be a finite connected graph, and assign a positive number to each edge. Let X be the set of vertices, and define $d(x, y)$ to be the minimal *weighted* length of paths between x and y.

Example: Let X be the set of squares on an $n \times n$ chessboard, where n is a positive integer. Let $d(x, y)$ be the minimum number of "knight moves" to get from the square x to the square y. Does this give a well-defined metric?

Example: Let $X = c_{00}$, the set of infinite sequences with only finitely many non-zero coordinates. Define metrics d_p, $p \geq 1$ and d_∞ in a similar way as for $\mathbb{R}^n$.

Example: Let $X = C[0,1]$, the set of continuous real-valued functions on the interval $[0,1]$. For fixed $p \geq 1$, define

$$d_p(f,g) = \left(\int_0^1 |f(x) - g(x)|^p dx \right)^{\frac{1}{p}}.$$

Similarly, define

$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Example: Let X be the unit sphere in $\mathbb{R}^3$, and define $d(x, y)$ to be the arclength of the shortest path along the sphere between x and y. (Is the shortest path unique?)

Example: Let (X, d) be any metric space, and let Y be any (non-empty) subset of X. Then $d_{Y \times Y}$ is a metric on Y (*subset metric*).

(non)Example: $X = R[0,1]$, the set of Riemann integrable functions on $[0,1]$, with d_1 .

(non)Example: $X = \mathbb{R}^2$ with "$d_{\frac{1}{2}}$" .

(non)Example: The set of all international airports with "flying time metric" or "flight price metric" .

(non)Example: $X = \mathbb{R}^{[0,1]}$ (the set of real-valued functions on $[0,1]$), with "d_∞" .

Definition

A *sequence* in a set X is a function from $\mathbb{N}$ (or $\mathbb{Z}_+$) to X. (Notation: $\{x_n\}_{n=0}^{\infty}$).

Recall: a sequence $\{x_n\}_{n=0}^{\infty} \subset \mathbb{R}$ converges to a limit $x \in \mathbb{R}$ if for every $\epsilon > 0$, there is a $K(\epsilon) \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n > K(\epsilon)$.

Theorem

A sequence in a metric space can have at most one limit.

Recall: a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for every $x \in \mathbb{R}$ and every $\epsilon > 0$, there is a $\delta(x, \epsilon) > 0$ such that $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta(x, \epsilon)$.

Exercise

For metric spaces X and Y, a function $f : X \to Y$ is said to be *sequentially continuous* if for every convergent sequence $\{x_n\}_{n=0}^{\infty} \subseteq X$ with limit x, the sequences $\{f(x_n)\}_{n=0}^{\infty}$ converges to $f(x)$ (in Y). Show that sequential continuity is equivalent to the definition of continuity above.

Definition

For a point x in a metric space (X, d) and a number $\epsilon > 0$, define the (open) ϵ-ball

$$B(x, \epsilon) = \{y \in X : d(y, x) < \epsilon\}.$$

Let $X = \mathbb{R}^2$. Consider the two metrics d_1 and d_2.

1. What do ϵ-balls look like in d_1 and d_2?
2. Does the sequence $(1/n, 1/n^2)$ converge to $(0, 0)$ in d_1? What about in d_2?
3. More generally, is there a difference between convergence in d_1 and d_2?

Interior and boundary

Definition

Let (X, d) be a metric space, and consider $Y \subseteq X$. Define the *interior*

$$\mathrm{Int}(Y) = \{y \in Y : \exists \epsilon > 0 \text{ such that } B(y, \epsilon) \subseteq Y\}.$$

Define the *boundary*

$$\mathrm{Bd}(Y) = X \backslash (\mathrm{Int}(Y) \cup \mathrm{Int}(Y^c)).$$

Exercise: Show that $x \in \mathrm{Bd}(Y)$ iff $\forall \epsilon > 0$, the sets $B(x, \epsilon) \cap Y$ and $B(x, \epsilon) \cap Y^c$ are both nonempty

Definition

A subset Y in (X, d) is open if $Y = \mathrm{Int}(Y)$.

Definition

A subset Y in (X, d) is closed if Y^c is open.

Lemma

Let (X, d) *be a metric space, and let* $Y \subseteq X$. *Then* $Int(Int(Y)) = Int(Y)$,

Corollary: For a subset Y of a metric space (X, d), the set $\mathrm{Int}(Y)$ is open.

Definition

The *closure* of Y is $\mathrm{Cl}(Y) = \mathrm{Int}(Y) \sqcup \mathrm{Bd}(Y)$.

Example: What are the closures in Y in X for:

1. $X = Y = \mathbb{R}$
2. $X = \mathbb{R}, \quad Y = [0, 1)$
3. $X = \mathbb{R}\backslash\{0\}, \quad Y = (-\infty, 0)$

(with the standard metric on $\mathbb{R}$ and its subsets)?

Definition

We say Y is *dense* if $\mathrm{Cl}(Y) = X$.

Exercise

What are the interior, boundary, and closure for:

1. $\mathbb{R} \subset \mathbb{R}^2$ (where R is the x-axis, with the standard Euclidean metric on $\mathbb{R}^2$)
2. $\mathbb{Q} \subset \mathbb{R}$ (with the standard metric on $\mathbb{R}$)
3. (Challenging:) $c_{00} \subset \ell^\infty$

Definition

Let (X, d) be a metric space. An *open neighborhood* of a point $x \in X$ is an open set $V \subseteq X$ such that $x \in V$. A *neighborhood* of x is a set $U \subseteq X$ such that there is an open neighborhood V of x with $V \subseteq U$.

Exercise: Show that a sequence $\{x_n\}_{n=0}^{\infty}$ converges to x if for every (open) neighborhood V of x, there is a $K(V) \in \mathbb{N}$ such that $x_n \in V$ whenever $n > K(V)$.

Definition

The set of open sets in a metric space X is called the *topology* of X. We will denote the topology by $\mathcal{O}(X)$.

Theorem

Let (X, d) be a metric space. The topology has the following properties.

1. $\emptyset, X \in \mathcal{O}(X)$
2. *If* $\{V_i\}_{i \in I} \subseteq \mathcal{O}(X)$, *then* $\bigcup_{i \in I} V_i \in \mathcal{O}(X)$.
   ("a union of open sets is open")
3. *If* $V_1, V_2 \in \mathcal{O}(X)$, *then* $V_1 \cap V_2 \in \mathcal{O}(X)$.
   ("a **finite** *intersection of open sets is open")*

Example: Give an example of an infinite collection of open sets in $\mathbb{R}$ whose intersection is not open

.

Example: What is the topology for the discrete metric on a set X?

Exercise

Let (X,d) be a metric space, and let $\mathcal{C}(X)$ be the set of closed sets in X. Show that:

1. $\emptyset, X \in \mathcal{C}(X)$
2. If $\{V_i\}_{i\in I} \in \mathcal{C}(X)$, then $\bigcap_{i\in I} V_i \in \mathcal{C}(X)$.
   (*"an intersection of closed sets is closed"*)
3. If $V_1, V_2 \in \mathcal{C}(X)$, then $V_1 \cup V_2 \in \mathcal{C}(X)$.
   (*"a* **finite** *union of closed sets is closed"*)

Warning: Is $[0,1]$ open? Is $[0,1]$ closed?

Exercise: Let X be a metric space, and let $Y \subseteq X$. Then

$$\mathrm{Int}(Y) = \bigcup_{V \in \mathcal{O}(X) \& V \subseteq Y} V \quad \text{and} \quad \mathrm{cl}(Y) = \bigcap_{W \in \mathcal{C}(X) \& Y \subseteq W} W.$$

Exercise: Let X be a metric space, and let $Y \subseteq X$. Show that Y is closed iff the limit of every convergent sequence $\{y_n\}_{n=1}^{\infty} \subseteq Y$ is in Y.

Recall: $f : X \to Y$ is *continuous* if

$$\forall x \in X, \epsilon > 0, \exists \delta(x,\epsilon) \text{ s.t. } d_X(y,x) < \delta(x,\epsilon) \implies d_Y(f(y), f(x)) < \epsilon.$$

Let's rephrase this in terms of *pre-images*.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is *continuous* if

$$\text{For every } V \in \mathcal{O}(Y), \text{ we have } f^{-1}(V) \in \mathcal{O}(X).$$

(In words: *"the pre-image of every open set is open."*)

Exercise: Show that for a function f between metric are (X, d_X) and (Y, d_Y), the two definitions of continuity are equivalent:

1. For every $x \in X$ and $\epsilon > 0$, the pre-image of the ϵ-ball around $f(x)$ in Y contains a δ-ball around x in X.
2. The pre-image under f of every open set in Y is open in X.

Topological concepts

Exercise

Show that the metrics d_2 and d_∞ on $\mathbb{R}^2$ give the same topology (in other words, show that $Y \subset \mathbb{R}^2$ is open with the respect to the metric d_2 iff Y is open with respect to d_∞).

What about d_1? What about the discrete metric?

Theorem

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ *be metric spaces, and suppose* $f : X \to Y$ *and* $g : Y :\to Z$ *are continuous functions. Then the composition* $g \circ f : X \to Z$ *is continuous.*

Example: The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

is not continuous.

Lemma

Let (X, d) *be a metric space, and let* $\emptyset \neq Y \subseteq X$. *The following are equivalent:*

1. *For every* $x \in X$, *there is an* $R(x) > 0$ *such that* $Y \subseteq B(x, R)$
2. *There exists* $y \in Y$ *and* R *such that* $Y \subseteq B(y, R)$
3. *There is an* $R > 0$ *such that for any* $y_1, y_2 \in Y$, *we have* $d(y_1, y_2) < R$

Definition

A subset of a metric space $Y \subseteq X$ satisfying these equivalent conditions is called *bounded*. (If $Y = X$ we say the metric space is bounded).

Example: Let $X = C[0,1]$, and consider the sequence of functions $\{f_n\}_{n=1,2,\ldots}$.

$$f_n(x) = \begin{cases} n - n^2 x & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \le 1 \end{cases}.$$

Is this sequence bounded?

Completeness

Definition

A sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space (X, d) is a *Cauchy sequence* if for every $\epsilon > 0$, there is a $K(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ whenever $m, n > K(\epsilon)$.

Example: Consider the sequence in $\mathbb{R}$ defined by

$$x_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Is this a Cauchy sequence?

Exercise: Every Cauchy sequence is bounded.

Exercise: Every convergent sequence is a Cauchy sequence.

Example: Let $X = (0, 1)$ with the usual metric. Consider the sequence $x_n = \frac{1}{n}$. Is this sequence a Cauchy sequence? Does it converge?

Example: Let $X = C[0,1]$ with the metric d_1. Consider the sequence of piecewise linear functions

$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ \frac{1}{2} - \frac{1}{n} + \frac{n}{2}x & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}, \quad n \geq 2.$$

Is this a Cauchy sequence? Does it converge?

Definition

A metric space (X, d) is called complete if every Cauchy sequence in X converges to a point in X.

Example: $(0,1)$ and $\mathbb{Q}$, with the metrics inherited from $\mathbb{R}$, are not complete.

Example: $C[0, 1]$ with the metric d_1 is not complete.

Example: A discrete metric space is complete.

Example: $\mathbb{R}$ (with the usual metric) is complete.

Example: $(\mathbb{R}^2, d_2)$ is complete.

Similarly, $(\mathbb{R}^n, d_p)$ is complete for any n and any $p \in [1, \infty]$.

Theorem

Let (X, d) *be a complete metric space, and let* $Y \subseteq X$. *Then* Y *is complete (with the subset metric) iff* Y *is closed.*

Theorem

The metric space $(C[0,1], d_\infty)$ *is complete.*

Outline of proof: Given a Cauchy sequence $\{f_n\}_{n=0}^{\infty}$ in $(C[0,1], d_\infty)$, we want to show that this sequence converges to some function f.

There are three steps:

1. Show that for each specific $x \in [0,1]$, the sequence $\{f_n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ (and therefore converges to a number by completeness of $\mathbb{R}$.)
2. Define $f(x)$ to be the limit of $\{f_n(x)\}_{n=0}^{\infty}$ for each x. Show that $f(x)$ is continuous (and therefore belongs to $C[0,1]$).
3. Show that $\{f_n\}_{n=0}^{\infty}$ converges to $f(x)$ in the d_∞ metric.

Step 1: Let $\{f_n\}_{n=0}^{\infty}$ be a Cauchy sequence in $(C[0,1], d_\infty)$. We want to show that for each specific $x \in [0,1]$, the sequence $\{f_n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.

Step 2: We now want to show that the pointwise limit $f(x)$ is a continuous function.

Need to show: given $x \in [0, 1]$, $\epsilon > 0$, there is a $\delta(x, \epsilon)$ such that

$$|f(y) - f(x)| < \epsilon \text{ whenever } |y - x| < \delta.$$

Step 3: We've now seen that the pointwise limit f is continuous, and the last step is to show that the sequence $\{f_n\}_{n=0}^{\infty}$ converges to f in $(C[0,1], d_\infty)$.

Exercise: Show that if I and J are closed and bounded intervals, then $C(I, J)$ (the set of continuous functions from I to J) is complete with the d_∞ metric.

Completion

Question: The metric space $(0, 1)$ is not complete, but $[0, 1]$ is. How can we describe the fact that $(0, 1)$ is "missing points" **without** mentioning any external points like 0 or 1?

Definition

Two Cauchy sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in a metric space (X, d) are said to be *equivalent* if the sequence $\{d(a_n, b_n)\}_{n=1}^{\infty}$ converges to 0 (in $\mathbb{R}$).

Exercise: Two Cauchy sequences in a complete metric space are equivalent iff they have the same limit.

Exercise: Describe the equivalence classes of Cauchy sequences in $(0, 1)$.

Definition

Let (X, d) be a metric space. Let $\bar{X}$ be the set of *equivalence classes of Cauchy sequences* in X. We write $[\{a_n\}]$ for the equivalence class of the sequence $\{a_n\}$.

Define $\bar{d} : \bar{X} \times \bar{X} \to [0, \infty)$ as follows:

$$\bar{d}([\{a_n\}], [\{b_n\}]) = \lim_{n\to\infty} d(a_n, b_n)$$

(Note that this definition assumes that the limit exists, and does not depend on which representatives of the equivalence classes are taken!)

Theorem

Let (X, d) be a metric space.

1. *The completion $(\bar{X}, \bar{d})$ (as defined above) is a complete metric space.*
2. *Consider the function $i : X \to \bar{X}$ which sends $x \in X$ to the equivalence class of the constant sequence $\{x\}$. Then i is an isometry (i.e. $\bar{d}(i(x), i(y)) = d(x, y),\ \forall x, y \in X$), and $i(X)$ is dense in $\bar{X}$.*
3. *The completion is unique in the following sense. Suppose Y is another complete metric space and $j : X \to Y$ is an isometry such that $j(X)$ is dense in Y. Then there is a bijective isometry $f : Y \to \bar{X}$ such that $f \circ j = i$.*

Completeness of $\mathbb{R}$

We have seen that $C[0,1]$ **is** complete with the metric d_∞, but **is not** complete with the metric d_1.

Can we describe the completion of $(C[0,1], d_1)$ concretely?

Definition

Let V be a vector space (over $k = \mathbb{R}$ or $k = \mathbb{C}$). A *norm* on V is a function

$$\|\cdot\| : V \to [0, \infty), \quad \mathbf{x} \mapsto \|\mathbf{x}\|,$$

satisfying the following conditions for all $\mathbf{x}, \mathbf{y} \in V$ and $\lambda \in k$

1. $\|\mathbf{x}\| = 0 \implies \mathbf{x} = \mathbf{0}$
2. $\|\lambda \mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Theorem

Let $(V, \|\cdot\|)$ *be a normed vector space. Then* $(V, d_{\|\cdot\|})$ *is a metric space, where* $d_{\|\cdot\|}$ *is defined by*

$$d_{\|\cdot\|}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|, \ \forall \mathbf{x}, \mathbf{y} \in V.$$

Definition

A complete normed space is called a *Banach* space.

Example: Consider c_{00}, the vector space of real sequences $\{x_n\}_{n=1}^{\infty}$, such that $x_n = 0$ for all but *finitely* many n. This is a vector space with *pointwise* operations. Fix $p \in [1, \infty)$, and for $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \in c_{00}$ define

$$\|\mathbf{x}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

Then this is a norm (by *Minkowski's inequality*), which gives the metric

$$d_p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}.$$

Definition

For $p \in [1, \infty)$, let

$$\ell^p = \{\{x_n\}_{n=1}^{\infty} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

which is a vector space with pointwise operations. Define the norm

$$\|\{x_n\}_{n=1}^{\infty}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}.$$

Theorem

The normed vector space $(\ell^p, \|\cdot\|_p)$ *is a Banach space.*

Outline of proof: Given a Cauchy sequence (of sequences!) in ℓ^p

1. Show that for each coordinate $k \in \mathbb{N}$, we get a Cauchy sequence of numbers.
2. Show that the pointwise limit is a sequence in ℓ^p
3. Show that the sequence (of sequences) converges to the pointwise limit in d_p.

Exercise: Show that $(c_{00}, \|\cdot\|_p)$ is dense in $(\ell^p, \|\cdot\|_p)$.

Similarly, we can define the Banach space

$$\ell^\infty = \{\{x_n\}_{n=1}^\infty \subset \mathbb{R} : \sup_n |x_n| < \infty\}$$

with

$$\|\{x_n\}_{n=1}^\infty\|_\infty = \sup_n |x_n|.$$

Exercise: Describe the completion of $(c_{00}, \|\cdot\|_\infty)$.

Example: For $C[0,1]$, we can define the norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|,$$

which gives the metric d_∞. So $(C[0,1], \|\cdot\|_\infty)$ is a Banach space.

Example: Again for $C[0,1]$, and $p \in [1,\infty)$ define the metric

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}},$$

which gives the metric d_p. Is $(C[0,1], \|\cdot\|_p)$ a Banach space?

Definition

An inner product space is a vector space V (over $k = \mathbb{R}$ or $k = \mathbb{C}$), together with a function

$$\langle \cdot, \cdot \rangle : V \times V \to k, \quad (\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, \mathbf{y} \rangle,$$

such that for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\lambda \in k$, we have

1. $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for $\mathbf{x} \neq \mathbf{0}$
2. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
3. $\langle \mathbf{x} + \lambda \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \lambda \langle \mathbf{y}, \mathbf{z} \rangle$

Exercise: Show that if $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, then the function

$$\|\mathbf{x}\|_{\langle \cdot, \cdot \rangle} = \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$$

gives a norm (and hence a metric) on V. (**Hint:** Use the Cauchy-Schwartz inequality.)

Definition

A complete inner product space is called a *Hilbert space*.

Example:

1. $\mathbb{R}^n$ is a real Hilbert space with the dot product.
2. $\mathbb{C}^n$ is a complex Hilbert space with the inner product

$$\langle (x_1, ..., x_n), (y_1, ..., y_n) \rangle = \sum_{k=1}^{n} x_k \overline{y_k}.$$

Example: ℓ^2 (over $\mathbb{R}$) is a real Hilbert space with the inner product

$$\langle \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

Exercise: What about ℓ^1 with a similar inner product? What about ℓ^3?

Definition

A *contraction* on a metric space (X, d) is a function $f : X \to X$ such that there is a number $c < 1$ for which

$$d(f(x), f(y)) \leq c \cdot d(x, y), \ \forall x, y \in X.$$

Exercise: Let $X = \mathbb{C}$ (standard metric), and consider

$$f(z) = \frac{z^2 + 3}{5}.$$

For which $R > 0$ is f a contraction on the closed disk around the origin of radius R?

Example: Let $X = [1, \infty)$ (usual metric), and consider

$$f : X \to X, \quad f(x) = x + \frac{1}{x}.$$

Is f a contraction?

Lemma: Let (X, d) be a metric space, and let $f : X \to X$ be a contraction. Then for any $x \in X$, the recursively defined sequence

$$x_0 = x, \quad x_{n+1} = f(x_n)$$

is a Cauchy sequence.

Theorem (Contraction Mapping theorem)

Let (X, d) *be a* **complete** *metric space, and let* $f : X \to X$ *be a contraction. Then* f *has a unique fixed point* $x = f(x)$.

Moreover, for any $x_0 \in X$*, the recursively defined sequence* $x_{n+1} = f(x_n)$ *converges to the fixed point* x.

Exercise: Show that Newton's method for $f(x) = x^2 - 2$ starting at $x_0 = 1$ gives a sequence converging to $\sqrt{2}$.

Picard-Lindelöf Theorem

Question: Consider the differential equation

$$y' = \cos(xy).$$

Does this have a solution?

Basic idea: Rewrite the differential equation as a *fixed-point problem* for an *integral operator* .

General setup: We consider the initial value problem

$$y'(x) = g(x, y), \quad y(a) = b,$$

where g is a continuous function on a neighborhood of (a, b) of the form $I \times J$, where $I = [a - \delta, a + \delta]$ and $J = [b - \epsilon, b + \epsilon]$.

Fixed point version: Define the integral operator

$$T : C(I, J) \to C(I)$$

$$T(y)(x) = b + \int_a^x g(t, y(t))\, dt.$$

Question: Is T a contraction on $X = C(I, J)$?

Need to check:

1. $T(X) \subseteq X$, and
2. T satisfies the contraction property.

Condition 1: $T(X) \subseteq X$**.** This means that the range of $T(y)$ must be contained in J for any $y \in C(I, J)$.

Condition 2: T satisfies the contraction property. This means that there is a $c < 1$ such that

$$d_\infty(T(y_1), T(y_2)) \le c \cdot d_\infty(y_1, y_2), \ \forall y_1, y_2 \in C(I, J).$$

Summary of the 2 sufficient conditions:

1. $\delta \cdot \sup_{(x,y) \in I \times J} |g(x,y)| \leq \epsilon$
2. $\delta \cdot \sup_{x \in I} |g(x, y_1(x)) - g(x, y_2(x))| < c \cdot \sup_{x \in I} |y_1(x) - y_2(x)|$

Question: Do these conditions hold?

Definition

Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is called *Lipschitz continuous* if there is a $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \ \forall x, y \in X.$$

The number K is then called a *Lipschitz constant* for f.

Exercise: Suppose f is continuous on a closed interval $[a, b]$, differentiable on (a, b), and f' is bounded on (a, b). Then f is Lipschitz continuous on $[a, b]$.

Exercise: Give an example of a continuous function on a closed interval which is not Lipschitz continuous.

Definition

If $X \subseteq \mathbb{R}^2$, then $f : X \to \mathbb{R}$ is *Lipschitz continuous in the second variable* if there is a $K > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq K \cdot |y_1 - y_2|, \ \forall (x, y_1), (x, y_2) \in X.$$

We can now simplify the two sufficient conditions above as follows: For the function g on the rectangle $I \times J$, let M be an upper bound, and let K be a y-Lipschitz constant. Then

1. Condition (1) above holds if $\delta < \frac{\epsilon}{M}$
2. Condition (2) above holds if $\delta < \frac{\epsilon}{K}$

Theorem (Picard-Lindelöf Theorem)

Let g *be a continuous function on a neighborhood of* $(a, b) \in \mathbb{R}^2$ *which is Lipschitz continuous in the second variable. Then there is an interval around* a *on which the differential equation*

$$y' = g(x, y), \quad y(a) = b$$

has a unique solution.

Example: Consider the initial value problem

$$y' = \cos(xy), \quad y(0) = 0.$$

Does this have a unique solution on a neighborhood of 0?

Example: Consider the initial value problem

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0.$$

Does this have a unique solution on a neighborhood of 0?