MATH3611 — Final Solutions

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Contents

1. (15 points) Set Theory

   (a) (5 points) i. $|A| \leq |B|$

   Solution: means there exists an **injective** map $f : A \to B$.

   ii. $|A| = |B|$

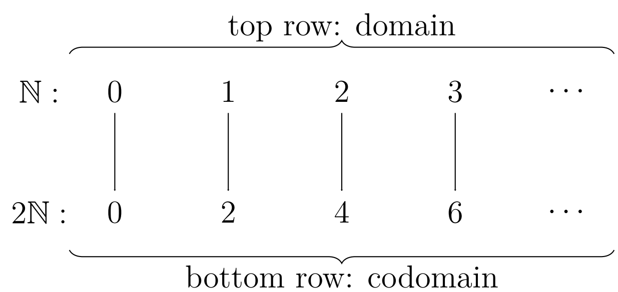
   Solution: means there exists a **bijective** map $f : A \to B$.

   iii. $|A| < |B|$

   Solution: means $\exists$ injective but not **surjective** map $A \to B$.

   iv. Prove that $|\mathbb{N}| = |2\mathbb{N}|$.

   Solution: Consider $f : \mathbb{N} \to 2\mathbb{N}$ given by $f(n) = 2n$. It is clearly injective and surjective. The correspondence is illustrated below.



   Thus f is a bijection and $|\mathbb{N}| = |2\mathbb{N}|$.

   (b) (10 points) i. State the Schroder–Bernstein Theorem. This is also known as the Cantor Bernstein Theorem.

   Solution: If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

   ii. If A is infinite, show $|\mathbb{N}| \leq |A|$.

   Solution: Pick distinct $a_0, a_1, \dots \in A$ recursively; $n \mapsto a_n$ is injective $\mathbb{N} \hookrightarrow A$.

   iii. Deduce $|A \cup \mathbb{N}| = |A|$ for infinite A.

   Solution: Trivially $|A| \leq |A \cup \mathbb{N}|$. From (b) get injection $\mathbb{N} \hookrightarrow A$; combine with inclusion $A \hookrightarrow A$ to build an injection $A \cup \mathbb{N} \hookrightarrow A$ by sending $n \mapsto a_{2n+1}$ and $a_k \mapsto a_{2k}$. Apply Schröder–Bernstein.

   iv. If A is countably infinite prove that $|\mathbb{N}| \leq |A|$.

Solution: Since A is countably infinite, there exists a bijection $h : \mathbb{N} \to A$, hence a fortiori an injection $\mathbb{N} \hookrightarrow A$. Therefore $|\mathbb{N}| \leq |A|$.

2. (13 points) Metric Spaces

(a) Define a Metric Space (X, d).

Solution: A metric space is a set X with $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

$$d(x, y) = 0 \iff x = y, \quad d(x, y) = d(y, x), \quad d(x, z) \leq d(x, y) + d(y, z).$$

(b) Define an open set $Y \subseteq X$.

Solution: $U \subseteq X$ is open if for each $x \in U$ there exists $r > 0$ with the open ball $B(x, r) = \{y \in X : d(x, y) < r\} \subseteq U$.

(c) Define a boundary point.

Solution: A point $x \in X$ is a *boundary point* of $A \subseteq X$ if every open ball $B(x, r)$ meets both A and $X \setminus A$. The boundary is $\partial A = \mathrm{cl}(A) \setminus \mathrm{Int}(A)$.

(d) (4 points) Prove that the interior of Y is open.

Solution: By definition,

$$\mathrm{Int}(Y) = \bigcup \{ B(x, r) : x \in Y,\ r > 0,\ B(x, r) \subseteq Y \},$$

a union of open balls. Unions of open sets are open, so $\mathrm{Int}(Y)$ is open.

3. (5 points) Suppose $\limsup x_n = a$ and $\limsup x_n = b$. Prove $a = b$.

Solution: Assume $a < b$ and set $\varepsilon = \frac{b-a}{3}$. By the lim sup characterization, eventually $x_n < a + \varepsilon$, but for infinitely many n, $x_n > b - \varepsilon$. Thus for some n,

$$b - \varepsilon < x_n < a + \varepsilon \ \Rightarrow\ b - a < 2\varepsilon = \tfrac{2}{3}(b - a),$$

a contradiction. Symmetrically $b \leq a$. Hence $a = b$.

4. (11 points) Norm Topology

(a) Define a Normed Space.

Solution: A normed space is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ with $\|\cdot\| : V \to [0,\infty)$ such that for all $x, y \in V$, α scalar:

$$\|x\| = 0 \iff x = 0, \quad \|\alpha x\| = |\alpha|\,\|x\|, \quad \|x+y\| \le \|x\| + \|y\|.$$

(b) Define a Banach Space.

Solution: A Banach space is a complete normed space, i.e. every Cauchy sequence converges in norm to a limit in the space.

(c) Consider a Cauchy sequence $(f_n)_{n\ge1}$ in the $\|\cdot\|_\infty$ norm. Prove that (f_n) converges pointwise.

Solution: Let X be any set and $f_n : X \to \mathbb{R}$ (or $\mathbb{C}$). Since (f_n) is Cauchy in $\|\cdot\|_\infty$, for all $\varepsilon > 0$ $\exists N$ s.t. $\|f_n - f_m\|_\infty < \varepsilon$ for $m, n \ge N$. Fix $x \in X$. Then

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \varepsilon \quad (m, n \ge N),$$

so $(f_n(x))$ is Cauchy in $\mathbb{R}$ (or $\mathbb{C}$) and hence convergent. Define $f(x) = \lim_{n\to\infty} f_n(x)$. Thus $f_n \to f$ pointwise.

(d) Hence or otherwise prove that the limit f is continuous (under the standard hypothesis).

Solution: If each f_n is *continuous* and $f_n \to f$ in $\|\cdot\|_\infty$ (i.e. uniformly), then f is continuous as a uniform limit of continuous functions. (No compactness assumption is needed for this implication.)

(e) Show c_{00} with the ℓ_1 metric is not complete.

Solution: Let $x^{(n)} = (1, 1/2, \ldots, 1/2^{n-1}, 0, 0, \ldots) \in c_{00}$. For $m > n$,

$$\|x^{(m)} - x^{(n)}\|_1 = \sum_{k=n}^{m-1} 2^{-k} \le 2^{-(n-1)} \xrightarrow{n\to\infty} 0,$$

so $(x^{(n)})$ is Cauchy. In ℓ^1, $x^{(n)} \to x = (1, 1/2, 1/4, \ldots)$, but $x \notin c_{00}$. Hence c_{00} is not complete.

5. (11 points) Topology, Compactness

(a) Define a Hausdorff Space.

Solution: (X, τ) is Hausdorff if for all $x \ne y$ there exist disjoint $U, V \in \tau$ with $x \in U$, $y \in V$.

(b) Define a compact space.

> **Solution:** (X, τ) is compact if every open cover admits a finite subcover.

(c) Consider

$$\tau = \{\varnothing, \mathbb{R}\} \;\cup\; \{(-t,t) \subset \mathbb{R} : \; t > 0\}.$$

i. Define a topology.

> **Solution:** A topology τ on X is a collection of subsets of X containing $\varnothing$ and X, closed under arbitrary unions and finite intersections. Members of τ are the open sets.

ii. Prove τ is a topology on $\mathbb{R}$.

> **Solution:** $\varnothing, \mathbb{R} \in \tau$ by definition. Arbitrary unions: a union of sets $(-t_i, t_i)$ is either $\mathbb{R}$ (if t_i unbounded) or $(-T, T)$ with $T = \sup_i t_i$; both in τ, and unions with $\mathbb{R}$ give $\mathbb{R}$. Finite intersections: $(-s, s) \cap (-t, t) = (-\min\{s,t\}, \min\{s,t\}) \in \tau$, and intersections with $\mathbb{R}$ return the other set. Hence τ is a topology.

iii. Find the limit(s) of the sequence $x_n = (-1)^n$ in $(\mathbb{R}, \tau)$.

> **Solution:** Nontrivial basic neighborhoods are $(-t,t)$ about 0. For $y \neq 0$, the only open set containing y is $\mathbb{R}$, so the neighborhood condition is vacuous and *every* sequence converges to y. For 0, neighborhoods are $(-t,t)$; since $(-1)^n \notin (-t,t)$ for $t < 1$, the sequence is not eventually in any neighborhood of 0. Therefore $(-1)^n$ converges to every $y \in \mathbb{R} \setminus \{0\}$ and to no other point.

(d) Let X be Hausdorff and $Y \subseteq X$ compact. Prove Y is closed in X.

> **Solution:** For $x \in X \setminus Y$ and each $y \in Y$, choose disjoint opens $U_y \ni x$, $V_y \ni y$. The $\{V_y\}_{y \in Y}$ cover Y, so compactness yields $y_1, \ldots, y_k$ with $Y \subset \bigcup_{i=1}^k V_{y_i}$. Then $U = \bigcap_{i=1}^k U_{y_i}$ is an open neighborhood of x disjoint from Y. Hence $X \setminus Y$ is open, so Y is closed.