MATH3611 — Final Solutions

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Contents

1. (15 points) Set Theory
   (a) (5 points) i. $|A| \leq |B|$
      ii. $|A| = |B|$
      iii. $|A| < |B|$
      iv. Prove that $|\mathbb{N}| = |2\mathbb{N}|$.
   (b) (10 points) i. State the Schroder–Bernstein Theorem. This is also known as the Cantor Bernstein Theorem.
      ii. If A is infinite, show $|\mathbb{N}| \leq |A|$.
      iii. Deduce $|A \cup \mathbb{N}| = |A|$ for infinite A.
      iv. If A is countably infinite prove that $|\mathbb{N}| \leq |A|$.
2. (13 points) Metric Spaces
   (a) Define a Metric Space (X, d).
   (b) Define an open set $Y \subseteq X$.
   (c) Define a boundary point.
   (d) (4 points) Prove that the interior of Y is open.
3. (5 points) Suppose $\limsup x_n = a$ and $\limsup x_n = b$. Prove $a = b$.
4. (11 points) Norm Topology
   (a) Define a Normed Space.
   (b) Define a Banach Space.
   (c) Consider a Cauchy sequence $(f_n)_{n \geq 1}$ in the $\| \cdot \|_\infty$ norm. Prove that (f_n) converges pointwise.
   (d) Hence or otherwise prove that the limit f is continuous (under the standard hypothesis).
   (e) Show c_{00} with the ℓ_1 metric is not complete.
5. (11 points) Topology, Compactness
   (a) Define a Hausdorff Space.
   (b) Define a compact space.
   (c) Consider
   $$\tau = \{\varnothing, \mathbb{R}\} \cup \{(-t, t) \subset \mathbb{R} : t > 0\}.$$
      i. Define a topology.
      ii. Prove τ is a topology on $\mathbb{R}$.
      iii. Find the limit(s) of the sequence $x_n = (-1)^n$ in $(\mathbb{R}, \tau)$.
   (d) Let X be Hausdorff and $Y \subseteq X$ compact. Prove Y is closed in X.