MATH3611 | Assignment 3

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§ Question 1

Lemma 1.1. *For $x, y \in \mathbb{R}^3$ define*

$$d_1(x,y) = \sum_{i=1}^{3} |x_i - y_i|, \qquad d_\infty(x,y) = \max_{1 \le i \le 3} |x_i - y_i|.$$

Then

$$d_\infty(x,y) \ \le \ d_1(x,y) \ \le \ 3\, d_\infty(x,y).$$

Proof. Lower bound. The maximum of non-negative numbers is never larger than their sum, so $d_\infty(x,y) \le d_1(x,y)$.
Upper bound. Since each term $|x_i - y_i| \le d_\infty(x,y)$, summing over $i = 1, 2, 3$ yields $d_1(x,y) \le 3\, d_\infty(x,y)$. □

Theorem 1.2. *The metrics d_1 and d_∞ induce the same topology on $\mathbb{R}^3$.*

Proof. Fix $x \in \mathbb{R}^3$ and $\varepsilon > 0$.
d_1-open $\implies$ d_∞-open. If $y \in B_1(x, \varepsilon)$, then $d_\infty(x,y) \le d_1(x,y) < \varepsilon$; hence

$$B_1(x, \varepsilon) \subset B_\infty(x, \varepsilon).$$

Thus every d_1-open set is d_∞-open.
Converse ($\impliedby$). Let $y \in B_\infty(x, \varepsilon)$. By the lemma, $d_1(x,y) \le 3\, d_\infty(x,y) < 3\varepsilon$, so

$$B_\infty(x, \varepsilon) \subset B_1(x, 3\varepsilon).$$

Hence every d_∞-open set is d_1-open.
Because each topology is contained in the other, they coincide. □

§ Question 2

Definition 2.1.

$$\tau = \{ \varnothing \} \cup \{ U \subset \mathbb{R} : \mathbb{R} \setminus U \text{ is countable} \}$$

Theorem 2.1 (A sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ converges in $(\mathbb{R}, \tau)$ iff it is eventually constant)**.**

$$\exists K \in \mathbb{N} : \forall n \geq K : x_n = x \iff x_n \xrightarrow{\tau} x$$

Proof. ($\implies$) Assume $x_n \xrightarrow{\tau} x$. Set

$$C := \{ x_n : x_n \neq x \}$$

to be the countable collection of terms different from x, and set

$$U := \mathbb{R} \setminus C \cup \{ x \}$$

to be the neighbourhood.

Since $\mathbb{R} \setminus U = C$ is countable, $U \in \tau$ and $x \in U$.

By convergence, there exists K such that $x_n \in U$ for all $n \geq K$. But if any $n \geq K$ with $x_n \neq x$, then $x_n \in C = \mathbb{R} \setminus U$ which contradicts $x_n \in U$. Hence $x_n = x \; \forall n \geq K$.

($\impliedby$) Conversely, suppose $x_n = x$ for all $n \geq K$. Let $U \in \tau$ be any neighbourhood of x. Then $x_n \in U$ whenever $n \geq K$, so $x_n \xrightarrow{\tau} x$.

□

Corollary 2.1.1. *For the sequence*

$$x_n = \begin{cases} 1, & n \text{ odd}, \\ 1 - \dfrac{1}{n}, & n \text{ even} \end{cases}$$

no tail is constant, hence

$$x_n \not\xrightarrow{\tau} x \quad \text{for any } x \in \mathbb{R}$$

Proof. The set of odd indices is infinite so $x_{2k-1} = 1$ occurs infinitely often. Likewise, the even subsequence $(1 - \frac{1}{2k})_{k\geq 1}$ takes infinitely many distinct values. Thus the sequence $(x_n)_{n\in\mathbb{N}}$ cannot be eventually constant and by 2.1 does not converge in the co-countable topology. □

§ Question 3

Theorem 3.1 (Uniform convergence on closed sub-intervals)**.** *Let*

$$S(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad x \in \mathbb{R},$$

be a power series with (finite) radius of convergence $R > 0$. Then for every $\varepsilon > 0$ the series S converges uniformly *on the closed interval*

$$[-R + \varepsilon,\ R - \varepsilon].$$

Proof. Fix $\varepsilon > 0$ and set $r := R - \varepsilon > 0$. Let $I = [-r, r]$.

For $n \in \mathbb{N}$ define $f_n(x) := a_n x^n$ on I.

Because $|x| \le r$ for all $x \in I$,

$$|f_n(x)| \le |a_n|\, r^n =:\ M_n \quad (x \in I).$$

Since $|r| < R$, the power series converges *absolutely* at $x^* = r$; hence the series $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} |a_n| r^n$ converges.

With $|f_n(x)| \le M_n$ for every $x \in I$, the Weierstrass M-test guarantees that $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on I, i.e.

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges uniformly on } [-R + \varepsilon, R - \varepsilon].$$

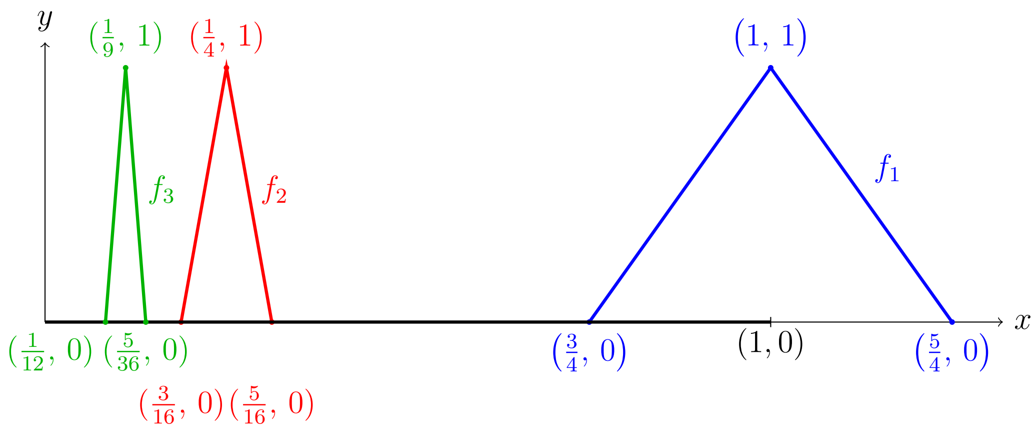
□

§ Question 4

For each integer $k \geq 1$ define

$$f_k : [0,1] \longrightarrow \mathbb{R}, \qquad f_k(x) = \max\{0, 1 - 4k^2|\,x - \tfrac{1}{k^2}|\}.$$

a) **Sketches of f_1, f_2, f_3.**



b) **Support of f_k.**

Solve $1 - 4k^2|x - \frac{1}{k^2}| > 0 \iff |x - \frac{1}{k^2}| < \frac{1}{4k^2}$. Hence

$$\boxed{\operatorname{supp}(f_k) = \left(\tfrac{3}{4k^2}, \tfrac{5}{4k^2}\right) \cap [0,1], k \geq 1}$$

whereby,

$$\operatorname{supp}(f_1) = (\tfrac{3}{4}, 1], \qquad \operatorname{supp}(f_k) = \left(\tfrac{3}{4k^2}, \tfrac{5}{4k^2}\right) \ (k \geq 2).$$

c) **Pointwise convergence and failure of uniform convergence of**

$$S(x) := \sum_{k=1}^{\infty} \frac{f_k(x)}{k}, \qquad x \in [0,1].$$

(i) *Pointwise convergence.*

Fix $x \in (0,1]$. The inequality $\frac{3}{4k^2} < x < \frac{5}{4k^2}$ is equivalent to

$$A(x) := \sqrt{\tfrac{3}{4x}} < k < B(x) := \sqrt{\tfrac{5}{4x}}.$$

Whose length is

$$L(x) := B(x) - A(x) = \frac{\sqrt{5} - \sqrt{3}}{2\sqrt{x}} < \frac{0.253}{\sqrt{x}} < \infty,$$

so the interval holds at most $\lceil L(x) \rceil$ integers. Hence only finitely many k satisfy $f_k(x) \neq 0$; the series $S(x)$ reduces to a finite sum and converges. For $x = 0$ every term is 0, so $S(0) = 0$. Thus S converges for every $x \in [0,1]$.

(ii) *Failure of uniform convergence.*
Let $S_N(x) := \sum_{k=1}^{N} \frac{f_k(x)}{k}$. For $N \geq 10$ choose

$$x_N := \frac{1}{N^2} \in [0,1].$$

Claim 1. *For every $k \in [N+1, N + \lfloor N/10 \rfloor]$ we have $f_k(x_N) \geq \frac{1}{2}$.*

Proof. For such k,

$$\left|x_N - \tfrac{1}{k^2}\right| = \frac{|k^2 - N^2|}{k^2 N^2} = \frac{(k-N)(k+N)}{k^2 N^2} \leq \frac{(N/10)(11N/10)}{k^2 N^2} < \frac{11}{100\,k^2} < \frac{1}{8\,k^2} < \frac{1}{4k^2},$$

so $x_N \in \operatorname{supp}(f_k)$ and $f_k(x_N) = 1 - 4k^2|x_N - \frac{1}{k^2}| \geq \frac{1}{2}$. □

Therefore the tail $T_N(x) := \sum_{k>N} \frac{f_k(x)}{k}$ satisfies

$$T_N(x_N) \geq \frac{1}{2} \sum_{k=N+1}^{N+\lfloor N/10 \rfloor} \frac{1}{k} \geq \frac{1}{2} \ln\left(1 + \tfrac{1}{10}\right) =: c > 0 \qquad (N \geq 10),$$

using $\sum_{k=m}^{n} \frac{1}{k} \geq \ln \dfrac{n}{m}, \forall n, m \in \mathbb{N}, n \geq m \geq 1$:

$$\|S - S_N\|_\infty = \sup_{x \in [0,1]} |S(x) - S_N(x)| \geq |T_N(x_N)| \geq c \quad \text{for all } N \geq 10,$$

so $\|S - S_N\|_\infty \not\to 0$. The convergence of the series is therefore *not* uniform on $[0,1]$.