Real Analysis

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1. Foundations

Definition 1.1 **Analysis Concepts**

1. **Metric:** An abstract notion of distance in a space (not necessarily $\mathbb{R}^n$).
2. **Topology:** An abstract notion of convergence (even in spaces with no underlying notion of distance).

2. Russell's Paradox

Let

$$S = \{T : T \text{ is a set and } T \notin T\}. \tag{1}$$

Is $S \in S$?

3. Constructing Sets

1. **Unions:** If $S = \{T_i\}_{i \in I}$, then

   $$\bigcup_{i \in I} T_i = \{x : \exists i \in I \text{ such that } x \in T_i\} \tag{2}$$

   is a set.

2. **Subsets with Conditions:** If S is a set and $\varphi(x)$ is a condition on elements, then

   $$\{x \in S : \varphi(x)\} \tag{3}$$

   is a set.

3. **Power Set:** If S is a set, then

   $$\mathscr{P}(S) = \{T : T \subseteq S\} \tag{4}$$

   is a set.

4. Cartesian Product

If A and B are sets, then

$$A \times B = \{(a, b) : a \in A, b \in B\}. \tag{5}$$

More generally, if $\{S_i\}_{i \in I}$ is a collection of sets, we can form the product

$$\prod_{i \in I} S_i. \tag{6}$$

An element is a tuple $(s_i)_{i \in I}$ such that $s_i \in S_i$. Formally,

$$\prod_{i \in I} S_i = \left\{ f : I \to \bigcup_{i \in I} S_i : f(i) \in S_i \text{ for all } i \in I \right\}. \tag{7}$$

5. Axiom of Choice (AC)

Proposition 5.1 — **Axiom of Choice**

A Cartesian product of non-empty sets is non-empty.

6. Functions

A function $f : A \to B$ assigns each element of A exactly one element of B. Formally,

$$f \subseteq A \times B \text{ is a function} \iff \forall x \in A, \exists! y \in B \text{ such that } (x, y) \in f. \quad (8)$$

6.1. Types of Functions

1. **Injective:** $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$.
2. **Surjective:** $\forall y \in B, \exists x \in A$ such that $f(x) = y$.
3. **Bijective:** f is both injective and surjective.

Definition 6.1 — **Cardinality Equivalence**

Two sets A and B have the same cardinality if there exists a bijection $f : A \to B$. We write $A \sim B$.

Theorem 6.2 — **Cantor's Theorem**

For any set S, the power set $\mathscr{P}(S)$ has strictly greater cardinality than S: $S \neg \sim \mathscr{P}(S)$.

7. Cardinality

7.1. Properties

1. $A \sim A$ (reflexive)
2. $A \sim B \implies B \sim A$ (symmetric)
3. $A \sim B$ and $B \sim C \implies A \sim C$ (transitive)

7.2. Notations

1. $A \leq B$: there exists an injective map $f : A \to B$
2. $A = B$: $A \sim B$
3. $A < B$: $A \leq B$ and $A \neg \sim B$

8. Schröder-Bernstein Theorem

Theorem 8.1 **Schröder-Bernstein Theorem**

If there are injective maps $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$.

9. Finite and Infinite Sets

Definition 9.1 **Finite Sets**

A set S is finite if $| S | = \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. Otherwise it is infinite.

Definition 9.2 **Dedekind-Infinite Sets**

A set S is Dedekind-infinite if there exists a bijection from S to a proper subset of itself. Otherwise, it is Dedekind-finite.

10. Countability

Definition 10.1 **Countable Sets**

A set S is **countable** if $S \leq \mathbb{N}$. If countable and infinite, we say it is **countably infinite**. Otherwise, it is **uncountable**.

Theorem 10.2 **Countable Union of Countable Sets**

Let I be a countable set, and let $\{S_i\}_{i \in I}$ be a countable collection of countable sets. Then

$$\bigcup_{i \in I} S_i \tag{9}$$

is countable.

11. Metric Spaces

11.1. Basic Definitions and Properties

Definition 11.1 **Metric Space**

A **metric space** is a pair (X, d), where X is a non-empty set and $d : X \times X \to [0, \infty)$ is a function such that for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Definition 11.2 **Sequence in Metric Space**

A **sequence** in a set X is a function from $\mathbb{N}$ (or $\mathbb{Z}^+$) to X.

Theorem 11.3 **Uniqueness of Limits**

A sequence in a metric space can have at most one limit.

Definition 11.4 **Open Ball**

For a point x in a metric space (X, d) and $\varepsilon > 0$, the **open ε-ball** is

$$B(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}. \quad (10)$$

11.2. Topology in Metric Spaces

Definition 11.5 **Interior and Boundary**

Let $Y \subseteq X$ in a metric space (X, d). Define:

1. $\text{Int}(Y) = \{y \in Y : \exists \varepsilon > 0 \text{ such that } B(y, \varepsilon) \subseteq Y\}$
2. $\text{Bd}(Y) = X \setminus (\text{Int}(Y) \cup \text{Int}(X \setminus Y))$

Definition 11.6 **Open Sets**

Y is **open** if $Y = \text{Int}(Y)$.

Definition 11.7 **Closed Sets**

Y is **closed** if $X \setminus Y$ is open.

Lemma 11.8 **Interior is Idempotent**

Let (X, d) be a metric space and $Y \subseteq X$. Then $\text{Int}(\text{Int}(Y)) = \text{Int}(Y)$.

Corollary 11.9 **Interior is Open**

$\text{Int}(Y)$ is open.

Definition 11.10 **Closure**

The **closure** of Y is $\text{Cl}(Y) = \text{Int}(Y) \cup \text{Bd}(Y)$.

Definition 11.11 **Dense Sets**

Y is **dense** if $\text{Cl}(Y) = X$.

Definition 11.12 **Neighborhood**

A **neighborhood** of x is a set $U \subseteq X$ such that there exists an open set V with $x \in V \subseteq U$.

Definition 11.13 **Topology**

The set of open subsets of X is called the **topology** $\mathcal{O}(X)$.

Theorem 11.14 **Properties of Topology**

The topology $\mathcal{O}(X)$ satisfies:

1. $\emptyset, X \in \mathcal{O}(X)$
2. Arbitrary unions of open sets are open
3. Finite intersections of open sets are open

11.3. Continuity and Boundedness

Definition 11.15 **Continuity in Metric Spaces**

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **continuous** if for every open $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Theorem 11.16 **Composition of Continuous Functions**

If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Definition 11.17 **Bounded Sets**

A subset $Y \subseteq X$ is **bounded** if there exists $R > 0$ and $x \in X$ such that $Y \subseteq B(x, R)$.

11.4. Completeness and Cauchy Sequences

Definition 11.18 **Cauchy Sequence**

A sequence $\{x_n\}$ in (X, d) is a **Cauchy sequence** if for all $\varepsilon > 0$, there exists N such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$.

Definition 11.19 **Complete Metric Space**

A metric space is **complete** if every Cauchy sequence converges to a point in the space.

Theorem 11.20 **Completeness and Closedness**

Let (X, d) be a complete metric space. A subset $Y \subseteq X$ is complete $\iff Y$ is closed.

Definition 11.21 **Equivalent Cauchy Sequences**

Two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ are equivalent if $\lim d(a_n, b_n) = 0$.

Definition 11.22 **Completion of Metric Space**

The **completion** of a metric space (X, d) is the space of equivalence classes of Cauchy sequences with distance

$$d([\{a_n\}], [\{b_n\}]) = \lim d(a_n, b_n). \quad (11)$$

Theorem 11.23 **Properties of Completion**

The completion $\overline{X}$ of X is a complete metric space. The map $x \mapsto [\{x\}]$ is an isometry, and its image is dense in $\overline{X}$. The completion is unique up to isometric bijection.

11.5. Normed and Inner Product Spaces

Definition 11.24 **Norm**

A **norm** on a vector space V is a function $\| \cdot \| : V \to [0, \infty)$ satisfying:

1. $\| x \| = 0 \iff x = 0$
2. $\| \lambda x \| = | \lambda | \cdot \| x \|$
3. $\| x + y \| \leq \| x \| + \| y \|$ (triangle inequality)

Theorem 11.25 **Norm Induces Metric**

Let $(V, \| \cdot \|)$ be a normed vector space. Then $d(x, y) = \| x - y \|$ defines a metric.

Definition 11.26 **Banach Space**

A **Banach space** is a complete normed vector space.

Definition 11.27 **ell^p Spaces**

For $p \in [1, \infty)$, define

$$\ell^p = \left\{ \{x_n\} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} | x_n |^p < \infty \right\}, \tag{12}$$

with norm $\| x \|_p = \left(\sum | x_n |^p \right)^{\frac{1}{p}}$.

Theorem 11.28 **ell^p is Banach**

$\left(\ell^p, \| \cdot \|_p \right)$ is a Banach space.

Definition 11.29 **Inner Product Space**

An **inner product space** is a vector space V with a function $\langle \cdot, \cdot \rangle$ such that:

1. $\langle x, x \rangle > 0$ if $x \neq 0$
2. $\langle x, y \rangle = \langle y, x \rangle$ (conjugate symmetry)
3. $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$ (linearity)

Definition 11.30 **Hilbert Space**

A **Hilbert space** is a complete inner product space.

11.6. Contraction and Lipschitz Mappings

Definition 11.31 **Contraction Mapping**

A **contraction** is a function $f : X \to X$ such that there exists $c < 1$ with $d(f(x), f(y)) \leq cd(x, y)$.

Lemma 11.32 **Contraction Generates Cauchy Sequence**

Let (X, d) be a metric space and f a contraction. Then the sequence $x_{n+1} = f(x_n)$ is Cauchy.

Theorem 11.33 **Contraction Mapping Theorem**

Let (X, d) be a complete metric space and $f : X \to X$ a contraction. Then f has a unique fixed point. Moreover, for any $x \in X$, the sequence $x_{n+1} = f(x_n)$ converges to that fixed point.

Definition 11.34 **Lipschitz Continuity**

A function $f : X \to \mathbb{R}$ is **Lipschitz continuous** if there exists $K > 0$ such that $| f(x) - f(y) | \leq K | x - y |$.

Definition 11.35 **Lipschitz in Second Variable**

A function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is **Lipschitz in the second variable** if

$$| f(x, y_1) - f(x, y_2) | \leq K | y_1 - y_2 |. \tag{13}$$

Theorem 11.36 **Picard–Lindelöf Theorem**

Let g be continuous near $(a, b) \in \mathbb{R}^2$ and Lipschitz in the second variable. Then the differential equation

$$y' = g(x, y), \quad y(a) = b \tag{14}$$

has a unique solution near a.