Proof. Let $\mathcal{O}$ denote the collection of all open intervals. Since every open set in $\mathbb{R}$ is an at most countable union of (disjoint) open intervals, we have $\sigma(\mathcal{O}) = \mathcal{B}(\mathbb{R})$.

Let $\mathcal{D}$ be the collection of all intervals of the form $(-\infty, a]$, $a \in \mathbb{Q}$. For any $a < b$, $a, b \in \mathbb{Q}$,

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - \tfrac{1}{n}] = \bigcup_{n=1}^{\infty} (-\infty, b - \tfrac{1}{n}] \cap (-\infty, a]^c,$$

which implies that $(a, b) \in \sigma(\mathcal{D})$, i.e. $\mathcal{O} \subseteq \sigma(\mathcal{D}) \Rightarrow \sigma(\mathcal{O}) \subseteq \sigma(\mathcal{D})$. However, every element of $\mathcal{D}$ is a closed set (complement of an open set) so $\mathcal{D} \subseteq \mathcal{B}(\mathbb{R}) \Rightarrow \sigma(\mathcal{D}) \subseteq \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O})$, completing the proof. ■