

Solutions to Royden & Fitzpatrick's Real Analysis

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2026-09-01T01:00:00+10:00

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Solutions to every problem in Royden & Fitzpatrick’s *Real Analysis* (fourth edition, 2010) — 1062 problems across the book’s 22 chapters: Lebesgue measure and integration for functions of a single real variable, differentiation and integration, the L^p spaces, metric and topological spaces, Banach and Hilbert spaces, and measure and integration in full generality. The book is filed at Real Analysis (Royden & Fitzpatrick).

1 The Real Numbers: Sets, Sequences, and Functions

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1.1 Problems 1.1–1.7

Problem 1.1. For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$.

Solution. Throughout we use only the Field Axioms.

Step 0: $c \cdot 0 = 0$ for every real number c . Since 0 is the additive identity, $0 + 0 = 0$, and hence by the Distributive Property

$$c \cdot 0 = c(0 + 0) = c \cdot 0 + c \cdot 0.$$

Let d be an additive inverse of $c \cdot 0$. Adding d to both sides and using associativity of addition,

$$0 = c \cdot 0 + d = (c \cdot 0 + c \cdot 0) + d = c \cdot 0 + (c \cdot 0 + d) = c \cdot 0 + 0 = c \cdot 0.$$

Step 1: $ab \neq 0$, so that $(ab)^{-1}$ is defined. Suppose to the contrary that $ab = 0$. Since $a \neq 0$, the multiplicative inverse a^{-1} exists, and by associativity of multiplication together with Step 0,

$$b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0,$$

contradicting $b \neq 0$. Hence $ab \neq 0$, and the Multiplicative Inverse axiom provides $(ab)^{-1}$.

Step 2: Multiplicative inverses are unique. Suppose $x \neq 0$ and $xy = 1 = xz$. Then, using commutativity and associativity of multiplication,

$$y = 1 \cdot y = (zx)y = z(xy) = z \cdot 1 = z.$$

Thus a nonzero real number has exactly one real number whose product with it is 1; this is the uniqueness noted in the text just after the Field Axioms.

Step 3: The computation. Using commutativity and associativity of multiplication repeatedly,

$$(ab)(a^{-1}b^{-1}) = (ab)(b^{-1}a^{-1}) = a(b(b^{-1}a^{-1})) = a((bb^{-1})a^{-1}) = a(1 \cdot a^{-1}) = aa^{-1} = 1.$$

So $a^{-1}b^{-1}$ is a real number whose product with ab equals 1. Since $ab \neq 0$ by Step 1, the uniqueness established in Step 2 forces

$$(ab)^{-1} = a^{-1}b^{-1},$$

as claimed.

Problem 1.2. Verify the following:

- (i) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
- (ii) For each positive number a , its multiplicative inverse a^{-1} also is positive.
- (iii) If $a > b$, then

$$ac > bc \quad \text{if } c > 0 \quad \text{and} \quad ac < bc \quad \text{if } c < 0.$$

Solution. Recall the definitions in force: $\mathcal{P}$ denotes the set of positive numbers, subject to P1 (if $a, b \in \mathcal{P}$ then $ab \in \mathcal{P}$ and $a + b \in \mathcal{P}$) and P2 (for each real a exactly one of $a \in \mathcal{P}$, $-a \in \mathcal{P}$, $a = 0$ holds); and $a > b$ means $a - b \in \mathcal{P}$. In particular $a > 0$ means $a - 0 = a \in \mathcal{P}$, i.e. a is positive.

We first record two consequences of the Field Axioms. As in Problem 1, $c \cdot 0 = 0$ for every c . Next, for all a and b ,

$$ab + (-a)b = (a + (-a))b = 0 \cdot b = 0,$$

so $(-a)b$ is the additive inverse of ab ; since additive inverses are unique, $(-a)b = -(ab)$. Applying this twice,

$$(-a)(-b) = -(a(-b)) = -(-(ab)) = ab,$$

the last equality because ab is an additive inverse of $-(ab)$ and additive inverses are unique.

(i) Let $a \neq 0$. By P2, either $a \in \mathcal{P}$ or $-a \in \mathcal{P}$. In the first case $a^2 = a \cdot a \in \mathcal{P}$ by P1. In the second case, by the identity above and P1,

$$a^2 = a \cdot a = (-a)(-a) \in \mathcal{P}.$$

In either case $a^2 \in \mathcal{P}$, i.e. $a^2 > 0$. By the Nontriviality Assumption $1 \neq 0$, and $1 = 1 \cdot 1 = 1^2$ by the Multiplicative Identity axiom; hence $1 > 0$.

(ii) Let $a \in \mathcal{P}$. Then $a \neq 0$ by P2, so a^{-1} exists. Also $a^{-1} \neq 0$: otherwise $1 = aa^{-1} = a \cdot 0 = 0$, contradicting Nontriviality. By P2, either $a^{-1} \in \mathcal{P}$ or $-a^{-1} \in \mathcal{P}$. Suppose the latter. Then, by P1, $a \cdot (-a^{-1}) \in \mathcal{P}$. But

$$a \cdot (-a^{-1}) = -(aa^{-1}) = -1,$$

so $-1 \in \mathcal{P}$. This contradicts P2, since by part (i) we already have $1 \in \mathcal{P}$, and exactly one of $1 \in \mathcal{P}$, $-1 \in \mathcal{P}$, $1 = 0$ may hold. Therefore $a^{-1} \in \mathcal{P}$, i.e. $a^{-1} > 0$.

(iii) Assume $a > b$, that is, $a - b \in \mathcal{P}$. By the Distributive Property and the identity $(-b)c = -(bc)$,

$$(a - b)c = (a + (-b))c = ac + (-b)c = ac + (-(bc)) = ac - bc.$$

Suppose $c > 0$, so $c \in \mathcal{P}$. Then P1 gives $(a - b)c \in \mathcal{P}$, i.e. $ac - bc \in \mathcal{P}$, which by definition says $ac > bc$. Suppose instead $c < 0$, i.e. $0 > c$, which means $0 - c = -c \in \mathcal{P}$. Then P1 gives $(a - b)(-c) \in \mathcal{P}$. Now

$$(a - b)(-c) = -((a - b)c) = -(ac - bc) = bc - ac,$$

where the last equality holds because $bc + (-(ac))$ added to $ac + (-(bc))$ gives 0, and additive inverses are unique. Hence $bc - ac \in \mathcal{P}$, i.e. $bc > ac$, which is exactly $ac < bc$.

Problem 1.3. For a nonempty set of real numbers E , show that $\inf E = \sup E$ if and only if E consists of a single point.

Solution. Throughout, $\sup E$ and $\inf E$ are understood in the extended real numbers, as in the text: $\sup E = \infty$ if E is not bounded above and $\inf E = -\infty$ if E is not bounded below, so that every nonempty set of real numbers has both a supremum and an infimum.

First a remark on finiteness. Since $E \neq \emptyset$, choose $x_0 \in E$. Then $\inf E \leq x_0 \leq \sup E$, because an infimum is a lower bound and a supremum is an upper bound. Hence $\inf E < \infty$ and $\sup E > -\infty$. Consequently, if $\inf E = \sup E$, this common value c satisfies $c < \infty$ and $c > -\infty$, so c is a real number; in particular E is then bounded above and below, and the Completeness Axiom (together with Problem 6 for the infimum) guarantees that $\sup E$ and $\inf E$ are the least upper bound and the greatest lower bound of E .

Suppose $\inf E = \sup E$, and call the common (real) value c . Let $x \in E$ be arbitrary. Since $\sup E$ is an upper bound for E ,

$$x \leq \sup E = c,$$

and since $\inf E$ is a lower bound for E ,

$$x \geq \inf E = c.$$

Thus $x \leq c$ and $c \leq x$. If $x \neq c$, then by P2 exactly one of $x - c$ and $-(x - c) = c - x$ is positive; but $x \leq c$ gives $c - x \in \mathcal{P}$ or $x = c$, and $c \leq x$ gives $x - c \in \mathcal{P}$ or $x = c$, so $x \neq c$ would force both $c - x$ and $x - c$ to be positive, contradicting P2. Hence $x = c$. As $x \in E$ was arbitrary and $E \neq \emptyset$, we conclude $E = \{c\}$, a single point.

Conversely, suppose $E = \{x_0\}$ for some real number x_0 . Then x_0 is an upper bound for E (indeed $x_0 \leq x_0$), and if b is any upper bound for E then $b \geq x_0$ since $x_0 \in E$. So x_0 is the least upper bound: $\sup E = x_0$. Symmetrically, x_0 is a lower bound for E and every lower bound a satisfies $a \leq x_0$, so x_0 is the greatest lower bound: $\inf E = x_0$. Therefore

$$\inf E = x_0 = \sup E.$$

Problem 1.4. Let a and b be real numbers.

- (i) Show that if $ab = 0$, then $a = 0$ or $b = 0$.
- (ii) Verify that $a^2 - b^2 = (a - b)(a + b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

- (iii) Let c be a positive real number. Define $E = \{x \in \mathbb{R} \mid x^2 < c\}$. Verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$.

Solution. (i) Suppose $ab = 0$ and $a \neq 0$. Then a^{-1} exists, and since $a^{-1} \cdot 0 = 0$ (Step 0 of Problem 1),

$$b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0.$$

So either $a = 0$ or $b = 0$.

(ii) By the Distributive Property and commutativity of multiplication, and using $(-b)x = -(bx)$ from Problem 2,

$$(a - b)(a + b) = a(a + b) + (-b)(a + b) = (a^2 + ab) + (-(ba) - b^2) = a^2 + (ab - ab) - b^2 = a^2 - b^2.$$

Now assume $a^2 = b^2$. Then $a^2 - b^2 = 0$, so $(a - b)(a + b) = 0$, and part (i) gives $a - b = 0$ or $a + b = 0$. In the first case $a = b$; in the second, a is the additive inverse of b , i.e. $a = -b$.

(iii) Let $c > 0$ and $E = \{x \in \mathbb{R} \mid x^2 < c\}$.

E is nonempty: $0^2 = 0 < c$, so $0 \in E$.

E is bounded above by $1 + c$: note $1 > 0$ by Problem 2(i), and $(1 + c) - 1 = c$ is positive, so $1 + c > 1 > 0$. Let $x \in E$ and suppose $x > 1 + c$. Then $x > 1 > 0$, so x is positive. Multiplying the inequality $x > 1 + c$ by the positive number x gives $x^2 > (1 + c)x$, and multiplying the inequality $x > 1$ by the positive number $1 + c$ gives $(1 + c)x > (1 + c) \cdot 1$; both multiplications are legitimate by Problem 2(iii). Combining,

$$x^2 > (1 + c)x > (1 + c) \cdot 1 = 1 + c > c,$$

contradicting $x^2 < c$. So no $x \in E$ satisfies $x > 1 + c$, and hence, by P2 applied to $x - (1 + c)$, every $x \in E$ satisfies $x \leq 1 + c$.

By the Completeness Axiom, $x_0 = \sup E$ exists. We record two facts.

First, $x_0 > 0$. Put $t = c(1 + c)^{-1}$. Since $c > 0$ and $1 + c > 0$, Problem 2(ii) and P1 give $t > 0$. From $c < 1 + c$, multiplying by the positive number $(1 + c)^{-1}$ gives $t < 1$; from $1 \leq 1 + c$, multiplying by the positive number $c(1 + c)^{-1}$ gives $t \leq c$. Since $0 < t < 1$, multiplying $t < 1$ by $t > 0$ gives $t^2 < t \leq c$, so $t \in E$ and therefore $x_0 \geq t > 0$.

Second, a monotonicity lemma: if $0 < u < v$ then $u^2 < v^2$. Indeed, multiplying $u < v$ by $u > 0$ gives $u^2 < uv$, and multiplying $u < v$ by $v > 0$ gives $uv < v^2$; combine.

We now show $x_0^2 = c$ by excluding the other two alternatives of P2 applied to $x_0^2 - c$.

Suppose $x_0^2 < c$ and set $\epsilon = c - x_0^2 > 0$. Since $2x_0 + 1 > 0$, the number

$$h = \min \left\{ \frac{1}{2}, \epsilon(2(2x_0 + 1))^{-1} \right\}$$

is positive, and satisfies $h < 1$ and $h(2x_0 + 1) < \epsilon$. Because $0 < h < 1$ we have $h^2 < h$, hence

$$(x_0 + h)^2 = x_0^2 + 2x_0h + h^2 < x_0^2 + 2x_0h + h = x_0^2 + h(2x_0 + 1) < x_0^2 + \epsilon = c.$$

So $x_0 + h \in E$ while $x_0 + h > x_0 = \sup E$, contradicting that x_0 is an upper bound for E .

Suppose instead $x_0^2 > c$ and set $\epsilon = x_0^2 - c > 0$. Since $2x_0 > 0$, the number

$$k = \min \left\{ \frac{1}{2}x_0, \epsilon(2 \cdot 2x_0)^{-1} \right\}$$

is positive and satisfies $k < x_0$ and $2x_0k < \epsilon$. Then $x_0 - k > 0$ and, since $k^2 > 0$,

$$(x_0 - k)^2 = x_0^2 - 2x_0k + k^2 > x_0^2 - 2x_0k > x_0^2 - \epsilon = c.$$

We claim $x_0 - k$ is an upper bound for E . Let $x \in E$. If $x \leq 0$ then $x < x_0 - k$ since $x_0 - k > 0$. If $x > 0$ and $x \geq x_0 - k$, then either $x = x_0 - k$, whence $x^2 = (x_0 - k)^2 > c$, or $x > x_0 - k > 0$, whence the monotonicity lemma gives $x^2 > (x_0 - k)^2 > c$; both contradict $x^2 < c$. So $x < x_0 - k$ in every case,

and $x_0 - k$ is an upper bound for E strictly smaller than x_0 , contradicting that x_0 is the least upper bound.

By P2 the only remaining alternative is $x_0^2 - c = 0$, that is, $x_0^2 = c$, and $x_0 > 0$.

Finally, uniqueness. Suppose $x > 0$ and $x^2 = c = x_0^2$. By part (ii), $x = x_0$ or $x = -x_0$. Since $x_0 > 0$, P2 gives $-x_0 < 0$, so $x = -x_0$ is incompatible with $x > 0$. Hence $x = x_0$. Thus there is exactly one positive real number whose square is c ; it is denoted $\sqrt{c}$.

Problem 1.5. Let a , b , and c be real numbers such that $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, \quad x \in \mathbb{R}.$$

- (i) Suppose $b^2 - 4ac > 0$. Use the Field Axioms and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- (ii) Now suppose $b^2 - 4ac < 0$. Show that the quadratic equation fails to have any solution.

Solution. Write $2 = 1 + 1$ and $4 = 2 \cdot 2$. By Problem 2(i), $1 > 0$, so P1 gives $2 > 0$ and $4 > 0$; in particular $2 \neq 0$ and $4 \neq 0$ by P2. Since also $a \neq 0$, Problem 4(i) shows $2a \neq 0$ and $4a \neq 0$, so $(2a)^{-1}$ and $(4a)^{-1}$ exist.

Completing the square. For every real x , the Field Axioms give

$$(2ax + b)^2 = 4a^2x^2 + 4abx + b^2 = 4a(ax^2 + bx + c) + (b^2 - 4ac).$$

Hence

$$4a(ax^2 + bx + c) = (2ax + b)^2 - (b^2 - 4ac).$$

Since $4a \neq 0$, we have $4a(ax^2 + bx + c) = 0$ if and only if $ax^2 + bx + c = 0$: one direction is $4a \cdot 0 = 0$, and the other is Problem 4(i). Therefore, for every real x ,

$$ax^2 + bx + c = 0 \iff (2ax + b)^2 = b^2 - 4ac.$$

Call this equivalence (*).

(i) Suppose $d := b^2 - 4ac > 0$. By Problem 4(iii) there is a unique real number $\sqrt{d} > 0$ with $(\sqrt{d})^2 = d$. By (*), x solves the quadratic equation if and only if $(2ax + b)^2 = (\sqrt{d})^2$, which by Problem 4(ii) happens if and only if

$$2ax + b = \sqrt{d} \quad \text{or} \quad 2ax + b = -\sqrt{d}.$$

Since $2a \neq 0$, the first alternative is equivalent to $2ax = -b + \sqrt{d}$, i.e.

$$x = (2a)^{-1}(-b + \sqrt{d}) = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

and the second is equivalent, in the same way, to

$$x = (2a)^{-1}(-b - \sqrt{d}) = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Every step above is an equivalence, so these two numbers are solutions and there are no others. They are distinct: if they were equal, multiplying by $2a$ would give $-b + \sqrt{d} = -b - \sqrt{d}$, hence $2\sqrt{d} = 0$, and since $2 \neq 0$ Problem 4(i) would force $\sqrt{d} = 0$, contradicting $\sqrt{d} > 0$. Thus the equation has exactly two solutions, as stated.

(ii) Suppose $b^2 - 4ac < 0$ and suppose, for contradiction, that some real x solves the equation. Put $y = 2ax + b$. By (*), $y^2 = b^2 - 4ac < 0$. But every square is nonnegative: if $y \neq 0$ then $y^2 > 0$ by Problem 2(i), and if $y = 0$ then $y^2 = 0$. In either case $y^2 \geq 0 > b^2 - 4ac$, and P2 forbids y^2 from being both equal to and greater than $b^2 - 4ac$. This contradiction shows the quadratic equation has no solution when $b^2 - 4ac < 0$.

Problem 1.6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x \mid x \in E\}.$$

Solution. We first record the reflection lemma: for real numbers u and v ,

$$u \leq v \iff -v \leq -u.$$

Indeed, since $-(-v) = v$ (additive inverses are unique), we have $(-u) - (-v) = (-u) + v = v - u$. So $v - u$ is positive or zero exactly when $(-u) - (-v)$ is positive or zero, which is the assertion.

Let E be a nonempty set of real numbers that is bounded below, and set

$$-E = \{-x \mid x \in E\}.$$

Since $E \neq \emptyset$, also $-E \neq \emptyset$. Let m be a lower bound for E , so $m \leq x$ for all $x \in E$. By the reflection lemma, $-x \leq -m$ for all $x \in E$, i.e. $-m$ is an upper bound for $-E$. Thus $-E$ is a nonempty set of real numbers that is bounded above, and the Completeness Axiom provides its least upper bound

$$s = \sup\{-x \mid x \in E\}.$$

We claim $-s$ is the greatest lower bound of E .

$-s$ is a lower bound for E : let $x \in E$. Then $-x \in -E$, so $-x \leq s$, and the reflection lemma gives $-s \leq -(-x) = x$.

$-s$ is the greatest lower bound: let ℓ be any lower bound for E . Then $\ell \leq x$ for all $x \in E$, so by the reflection lemma $-x \leq -\ell$ for all $x \in E$, i.e. $-\ell$ is an upper bound for $-E$. Since s is the least upper bound of $-E$, we get $s \leq -\ell$, and the reflection lemma once more gives $\ell = -(-\ell) \leq -s$.

Hence among the lower bounds of E there is a largest, namely $-s$; that is, E has an infimum and

$$\inf E = -s = -\sup\{-x \mid x \in E\}.$$

Problem 1.7. For real numbers a and b , verify the following:

- (i) $|ab| = |a| |b|$.
- (ii) $|a + b| \leq |a| + |b|$.
- (iii) For $\epsilon > 0$,

$$|x - a| < \epsilon \quad \text{if and only if} \quad a - \epsilon < x < a + \epsilon.$$

Solution. Recall the definition: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Two immediate consequences will be used repeatedly.

Fact A: $|x| \geq 0$, and $|x|$ equals x or $-x$. If $x \geq 0$ this is clear; if $x < 0$ then $-x > 0$ by P2 and $|x| = -x > 0$.

Fact B: $|-x| = |x|$. If $x > 0$ then $-x < 0$, so $|-x| = -(-x) = x = |x|$; if $x < 0$ then $-x > 0$, so $|-x| = -x = |x|$; and $|-0| = |0| = 0$.

Fact C: $-|x| \leq x \leq |x|$. If $x \geq 0$ then $|x| = x$, so $x \leq |x|$, while $-|x| = -x \leq 0 \leq x$. If $x < 0$ then $|x| = -x > 0 > x$, so $x < |x|$, while $-|x| = x$.

Fact D: inequalities add. If $u \leq v$ and $u' \leq v'$, then

$$(v + v') - (u + u') = (v - u) + (v' - u'),$$

and each of $v - u$, $v' - u'$ is positive or zero. If both are positive their sum is positive by P1; if exactly one is zero the sum equals the other, which is positive; if both are zero the sum is zero. In every case $(v + v') - (u + u')$ is positive or zero, i.e. $u + u' \leq v + v'$.

(i) We use $(-x)y = -(xy)$ and $(-x)(-y) = xy$, established in Problem 2.

If $a = 0$ or $b = 0$, then $ab = 0$ and both sides equal 0, since $|0| = 0$.

If $a > 0$ and $b > 0$, then $ab > 0$ by P1, so $|ab| = ab = |a| |b|$.

If $a > 0$ and $b < 0$, then $-b > 0$, so $a(-b) = -(ab) > 0$ by P1, hence $ab < 0$ by P2, and

$$|ab| = -(ab) = a(-b) = |a| |b|.$$

The case $a < 0$, $b > 0$ is identical with the roles of a and b exchanged.

If $a < 0$ and $b < 0$, then $-a > 0$ and $-b > 0$, so $ab = (-a)(-b) > 0$ by P1, and

$$|ab| = ab = (-a)(-b) = |a| |b|.$$

All cases are covered by P2, so $|ab| = |a| |b|$ always.

(ii) By Fact C, $a \leq |a|$ and $b \leq |b|$, so Fact D gives

$$a + b \leq |a| + |b|.$$

Again by Fact C, $-|a| \leq a$ and $-|b| \leq b$, so Fact D gives $(-|a|) + (-|b|) \leq a + b$. Now $(-|a|) + (-|b|) = -(|a| + |b|)$, since adding $|a| + |b|$ to $(-|a|) + (-|b|)$ yields 0 and additive inverses are unique. Hence $-(|a| + |b|) \leq a + b$, and applying the reflection lemma of Problem 6,

$$-(a + b) \leq |a| + |b|.$$

By Fact A, $|a + b|$ is either $a + b$ or $-(a + b)$, and both of these are at most $|a| + |b|$. Hence

$$|a + b| \leq |a| + |b|.$$

(iii) Let $\epsilon > 0$ and let x and a be real numbers.

Suppose $|x - a| < \epsilon$. By Fact C, $x - a \leq |x - a| < \epsilon$, so $x - a < \epsilon$ and therefore $x < a + \epsilon$. Also, by Fact C applied to $-(x - a)$ together with Fact B,

$$-(x - a) \leq |-(x - a)| = |x - a| < \epsilon,$$

so $a - x < \epsilon$, which gives $a - \epsilon < x$. Hence $a - \epsilon < x < a + \epsilon$.

Conversely, suppose $a - \epsilon < x < a + \epsilon$. From $x < a + \epsilon$ we get $x - a < \epsilon$, and from $a - \epsilon < x$ we get $-(x - a) = a - x < \epsilon$. By Fact A, $|x - a|$ equals either $x - a$ or $-(x - a)$, and each of these is less than ϵ ; hence $|x - a| < \epsilon$.

1.2 Problems 1.8–1.14

Problem 1.8. Use an induction argument to show that for each natural number n , the interval $(n, n + 1)$ fails to contain any natural number.

Solution. Recall that a set E of real numbers is inductive provided $1 \in E$ and $x \in E$ implies $x + 1 \in E$, and that $\mathbb{N}$ is defined to be the intersection of all inductive subsets of $\mathbb{R}$. Consequently, to prove that every natural number belongs to a set E it suffices to prove that E is inductive; this is exactly the content of the Principle of Mathematical Induction.

We first record two lemmas.

Lemma 1. Every natural number n satisfies $n \geq 1$.

Indeed, the set $\{x \in \mathbb{R} \mid x \geq 1\}$ is inductive: it contains 1, and if $x \geq 1$ then $x + 1 \geq 1 + 1 > 1$, since $1 > 0$. Hence $\mathbb{N}$ is contained in this set.

Lemma 2. If n is a natural number and $n \neq 1$, then $n - 1$ is a natural number.

Define

$$F = \{x \in \mathbb{R} \mid x = 1 \text{ or } x - 1 \in \mathbb{N}\}.$$

Then $1 \in F$. Suppose $x \in F$. If $x = 1$, then $(x + 1) - 1 = 1 \in \mathbb{N}$, so $x + 1 \in F$. If $x - 1 \in \mathbb{N}$, then, since $\mathbb{N}$ is inductive, $x = (x - 1) + 1 \in \mathbb{N}$, and therefore $(x + 1) - 1 = x \in \mathbb{N}$, so again $x + 1 \in F$. Thus F is inductive and hence $\mathbb{N} \subseteq F$, which is the assertion of the lemma. (This is the first assertion of Problem 9; its proof uses only the definition of $\mathbb{N}$ as an intersection of inductive sets, so there is no circularity.)

We now prove, by induction on n , the assertion

$$S(n) : (n, n+1) \cap \mathbb{N} = \emptyset.$$

The base case $S(1)$. Define

$$E = \{1\} \cup \{x \in \mathbb{R} \mid x \geq 2\}.$$

Then $1 \in E$. If $x \in E$ and $x = 1$, then $x+1 = 2 \geq 2$, so $x+1 \in E$; if $x \in E$ and $x \geq 2$, then $x+1 \geq 3 > 2$, so $x+1 \in E$. Hence E is inductive and therefore $\mathbb{N} \subseteq E$. Since no point of E lies strictly between 1 and 2, no natural number lies in $(1, 2)$. Thus $S(1)$ holds.

The induction step. Suppose $S(k)$ holds for the natural number k , that is, no natural number lies in $(k, k+1)$. We claim $S(k+1)$ holds. Suppose, to the contrary, that n is a natural number with

$$k+1 < n < k+2.$$

By Lemma 1, $k \geq 1$, so $n > k+1 \geq 2 > 1$ and in particular $n \neq 1$. By Lemma 2, $n-1$ is a natural number. Subtracting 1 from each term of the displayed inequality gives

$$k < n-1 < k+1,$$

so $n-1$ is a natural number lying in $(k, k+1)$. This contradicts $S(k)$. Hence no natural number lies in $(k+1, k+2)$, that is, $S(k+1)$ holds.

By the Principle of Mathematical Induction, $S(n)$ holds for every natural number n : for each natural number n the interval $(n, n+1)$ contains no natural number.

Problem 1.9. Use an induction argument to show that if $n > 1$ is a natural number, then $n-1$ also is a natural number. Then use another induction argument to show that if m and n are natural numbers with $n > m$, then $n-m$ is a natural number.

Solution. Throughout we use the definition of $\mathbb{N}$ as the intersection of all inductive subsets of $\mathbb{R}$; equivalently, we use the Principle of Mathematical Induction, which asserts that a set of real numbers which is inductive contains $\mathbb{N}$.

First assertion. Define

$$F = \{x \in \mathbb{R} \mid x = 1 \text{ or } x-1 \in \mathbb{N}\}.$$

We claim F is inductive. Certainly $1 \in F$. Let $x \in F$. There are two cases. If $x = 1$, then $(x+1)-1 = 1$, and $1 \in \mathbb{N}$, so $x+1 \in F$. If $x \neq 1$, then $x-1 \in \mathbb{N}$; since $\mathbb{N}$ is inductive, $x = (x-1) + 1$ belongs to $\mathbb{N}$, and therefore $(x+1)-1 = x \in \mathbb{N}$, so $x+1 \in F$. In either case $x+1 \in F$, so F is inductive.

Since F is inductive, $\mathbb{N} \subseteq F$. Now let n be a natural number with $n > 1$. Then $n \in F$ and $n \neq 1$, so by the definition of F we conclude that $n-1 \in \mathbb{N}$. This proves the first assertion.

Second assertion. For a natural number m , let $S(m)$ be the assertion

$$S(m) : \text{for every natural number } n \text{ with } n > m, \text{ the number } n-m \text{ is a natural number.}$$

The base case $S(1)$ is precisely the first assertion just proved.

The induction step. Suppose $S(m)$ holds for the natural number m . Let n be a natural number with $n > m+1$. Since $1 > 0$ we have $m+1 > m$, and hence $n > m$. By the induction hypothesis $S(m)$, the number $n-m$ is a natural number. Moreover, subtracting m from both sides of $n > m+1$ gives

$$n-m > 1.$$

Therefore $n-m$ is a natural number strictly greater than 1, so by the first assertion $(n-m)-1$ is a natural number. But

$$(n-m)-1 = n-(m+1),$$

so $n-(m+1)$ is a natural number. As n was an arbitrary natural number exceeding $m+1$, the assertion $S(m+1)$ holds.

By the Principle of Mathematical Induction, $S(m)$ holds for every natural number m . That is, if m and n are natural numbers with $n > m$, then $n-m$ is a natural number.

Problem 1.10. Show that for any integer n , there is exactly one integer in the interval $[n, n + 1)$.

Solution. Recall that the set $\mathbb{Z}$ of integers consists of the natural numbers, their negatives, and the number 0; thus

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N}), \quad -\mathbb{N} = \{-p \mid p \in \mathbb{N}\}.$$

As observed in the solution of Problem 8, every natural number is greater than or equal to 1, since $\{x \in \mathbb{R} \mid x \geq 1\}$ is inductive. Hence every member of $-\mathbb{N}$ is less than or equal to -1 , and in particular is negative. This yields the positivity observation: if k is an integer and $k > 0$, then k belongs to neither $\{0\}$ nor $-\mathbb{N}$, so k is a natural number and therefore $k \geq 1$.

We also need the following closure property.

Lemma A. $\mathbb{N}$ is closed under addition: if p and m are natural numbers, then $p + m$ is a natural number. Fix a natural number m and set $G = \{x \in \mathbb{R} \mid x + m \in \mathbb{N}\}$. Since $\mathbb{N}$ is inductive and $m \in \mathbb{N}$, we have $1 + m = m + 1 \in \mathbb{N}$, so $1 \in G$. If $x \in G$, then $x + m \in \mathbb{N}$, so $(x + 1) + m = (x + m) + 1 \in \mathbb{N}$ because $\mathbb{N}$ is inductive; hence $x + 1 \in G$. Thus G is inductive and therefore $\mathbb{N} \subseteq G$, which proves the lemma.

Lemma B (the difference lemma). If m and n are integers with $n > m$, then $n - m$ is a natural number. We consider cases according to the trichotomy for m .

Case $m \in \mathbb{N}$. If $n \in \mathbb{N}$, then $n - m \in \mathbb{N}$ by Problem 9. If $n = 0$ or $n \in -\mathbb{N}$, then $n \leq 0 < 1 \leq m$, contradicting $n > m$; so these subcases do not occur.

Case $m = 0$. Then $n > 0$, so by the positivity observation $n \in \mathbb{N}$, and $n - m = n \in \mathbb{N}$.

Case $m = -q$ with $q \in \mathbb{N}$. If $n \in \mathbb{N}$, then $n - m = n + q \in \mathbb{N}$ by Lemma A. If $n = 0$, then $n - m = q \in \mathbb{N}$. If $n = -p$ with $p \in \mathbb{N}$, then $-p > -q$ gives $q > p$, and $n - m = q - p \in \mathbb{N}$ by Problem 9.

The three cases are exhaustive, since $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$. In every case $n - m$ is a natural number, and Lemma B is proved. (It is used again in Problems 11 and 12.)

Now let n be an integer. Since $n \leq n$ and $n < n + 1$, the integer n itself lies in $[n, n + 1)$, so the interval contains at least one integer.

For uniqueness, suppose m is an integer with $m \in [n, n + 1)$ and $m \neq n$. Since $m \geq n$ and $m \neq n$, we have $m > n$. By the lemma, $m - n$ is a natural number, hence $m - n \geq 1$, that is, $m \geq n + 1$. This contradicts $m < n + 1$. Therefore n is the only integer in $[n, n + 1)$.

Problem 1.11. Show that any nonempty set of integers that is bounded above has a largest member.

Solution. Let E be a nonempty set of integers that is bounded above. Since E is a nonempty set of real numbers that is bounded above, the Completeness Axiom provides a supremum; define

$$c = \sup E.$$

Since $c - 1 < c$ and c is the least upper bound for E , the number $c - 1$ is not an upper bound for E . Hence there is a member m of E with

$$m > c - 1.$$

We claim m is the largest member of E . Suppose otherwise. Then there is a member n of E with $n > m$. Both m and n are integers, so by the difference lemma (Lemma B in the solution of Problem 10), $n - m$ is a natural number, and therefore $n - m \geq 1$, since every natural number is at least 1. Thus

$$n \geq m + 1 > (c - 1) + 1 = c,$$

so $n > c$. This contradicts the fact that c is an upper bound for E and $n \in E$.

Therefore no member of E exceeds m ; that is, m is a member of E with $n \leq m$ for all $n \in E$. Hence E has a largest member, namely m . (In fact $m = c = \sup E$, so the supremum of such a set is attained.)

Problem 1.12. Show that the irrational numbers are dense in $\mathbb{R}$.

Solution. Recall the definition: a set E of real numbers is dense in $\mathbb{R}$ provided that between any two real numbers there lies a member of E . So we must show that if a and b are real numbers with $a < b$, then there is an irrational number x with $a < x < b$.

We first record that $\mathbb{Q}$ is closed under subtraction; for this we need that $\mathbb{Z}$ is closed under subtraction and under multiplication.

Recall from the solution of Problem 10 that $\mathbb{N}$ is closed under addition (Lemma A there). It is also closed under multiplication: for a fixed natural number m the set $\{x \in \mathbb{R} \mid xm \in \mathbb{N}\}$ contains 1, since $1 \cdot m = m \in \mathbb{N}$, and if $xm \in \mathbb{N}$ then $(x+1)m = xm + m \in \mathbb{N}$ by additive closure; hence this set is inductive and contains $\mathbb{N}$.

Now let m and n be integers, so that each lies in $\mathbb{N} \cup \{0\} \cup (-\mathbb{N})$. For the product: if either factor is 0, then $mn = 0 \in \mathbb{Z}$; otherwise write $m = \epsilon p$ and $n = \delta q$ with $p, q \in \mathbb{N}$ and $\epsilon, \delta \in \{1, -1\}$, so that $mn = (\epsilon\delta)(pq)$, and $pq \in \mathbb{N}$ by multiplicative closure, whence $mn \in \mathbb{N} \cup (-\mathbb{N}) \subseteq \mathbb{Z}$. For the difference: if $n > m$, then $n - m \in \mathbb{N} \subseteq \mathbb{Z}$ by the difference lemma (Lemma B in the solution of Problem 10); if $n = m$, then $n - m = 0 \in \mathbb{Z}$; and if $n < m$, then $m - n \in \mathbb{N}$ by that same lemma, so $n - m = -(m - n) \in -\mathbb{N} \subseteq \mathbb{Z}$. Thus $\mathbb{Z}$ is closed under multiplication and subtraction.

Consequently $\mathbb{Q}$ is closed under subtraction. Indeed, let s and t be rational, say $s = a_1/a_2$ and $t = b_1/b_2$ with a_1, a_2, b_1, b_2 integers and $a_2 \neq 0, b_2 \neq 0$. Then

$$s - t = \frac{a_1b_2 - b_1a_2}{a_2b_2}.$$

By the closure properties just established, the numerator and the denominator are integers, and $a_2b_2 \neq 0$ by Problem 4(i). Hence $s - t$ is a quotient of integers with nonzero denominator, that is, $s - t$ is rational.

By Problem 4(iii) there is a unique positive real number whose square is 2; it is denoted by $\sqrt{2}$. As shown in the text of Section 1.2, using the prime factorization theorem, $\sqrt{2}$ is irrational.

Now let a and b be real numbers with $a < b$. Then

$$a - \sqrt{2} < b - \sqrt{2},$$

so by Theorem 2, which asserts that the rational numbers are dense in $\mathbb{R}$, there is a rational number r with

$$a - \sqrt{2} < r < b - \sqrt{2}.$$

Set $x = r + \sqrt{2}$. Adding $\sqrt{2}$ to each term of the last display gives

$$a < x < b.$$

Finally, x is irrational. For if x were rational, then, since r is rational and the rationals are closed under subtraction, the number

$$\sqrt{2} = x - r$$

would be rational, contradicting the irrationality of $\sqrt{2}$.

Thus between any two real numbers there lies an irrational number, that is, the irrational numbers are dense in $\mathbb{R}$.

Problem 1.13. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

Solution. Let x be a real number.

The rational case. Define

$$E = \{r \in \mathbb{Q} \mid r < x\},$$

a set of rational numbers. We check that E is nonempty and bounded above, and that $\sup E = x$.

E is nonempty: since $x - 1 < x$, Theorem 2 (the rational numbers are dense in $\mathbb{R}$) provides a rational number r with $x - 1 < r < x$; this r belongs to E .

E is bounded above by x , directly from its definition. Hence by the Completeness Axiom $c = \sup E$

exists, and since x is an upper bound for E while c is the least upper bound,

$$c \leq x.$$

Suppose $c < x$. By Theorem 2 there is a rational number r with $c < r < x$. Since r is rational and $r < x$, we have $r \in E$; but $r > c = \sup E$, contradicting the fact that c is an upper bound for E . Therefore $c = x$, that is,

$$x = \sup\{r \in \mathbb{Q} \mid r < x\},$$

and x is the supremum of a set of rational numbers.

The irrational case. The argument is identical, with Problem 12 in place of Theorem 2. Define

$$E' = \{t \in \mathbb{R} \mid t \text{ is irrational and } t < x\},$$

a set of irrational numbers.

E' is nonempty: by Problem 12 the irrational numbers are dense in $\mathbb{R}$, so, as $x - 1 < x$, there is an irrational t with $x - 1 < t < x$, and this t belongs to E' .

E' is bounded above by x , so by the Completeness Axiom $c' = \sup E'$ exists and $c' \leq x$.

If $c' < x$, then by Problem 12 there is an irrational number t with $c' < t < x$; then $t \in E'$ while $t > c' = \sup E'$, a contradiction. Hence $c' = x$, that is,

$$x = \sup\{t \mid t \text{ irrational, } t < x\},$$

and x is also the supremum of a set of irrational numbers.

Problem 1.14. Show that if $r > 0$, then, for each natural number n , $(1 + r)^n \geq 1 + n \cdot r$.

Solution. Fix a real number $r > 0$. For a real number x the natural powers are defined recursively by $x^1 = x$ and $x^{k+1} = x^k \cdot x$. For a natural number n let $S(n)$ be the assertion

$$S(n): \quad (1 + r)^n \geq 1 + n \cdot r.$$

We prove $S(n)$ for every natural number n by the Principle of Mathematical Induction.

The base case $S(1)$. By definition $(1 + r)^1 = 1 + r = 1 + 1 \cdot r$, so $S(1)$ holds, with equality.

The induction step. Suppose $S(k)$ holds for the natural number k , that is,

$$(1 + r)^k \geq 1 + k \cdot r.$$

Since $r > 0$ and $1 > 0$ we have $1 + r > 0$. Multiplication by the positive number $1 + r$ preserves the inequality: if $(1 + r)^k > 1 + kr$, then $(1 + r)^k(1 + r) > (1 + kr)(1 + r)$ by Problem 2(iii), while if $(1 + r)^k = 1 + kr$ the two products are equal. In either case

$$(1 + r)^{k+1} = (1 + r)^k(1 + r) \geq (1 + kr)(1 + r).$$

Expanding the right-hand side by the distributive law,

$$\begin{aligned} (1 + kr)(1 + r) &= 1 + r + kr + kr^2 \\ &= 1 + (k + 1)r + kr^2. \end{aligned}$$

Now k is a natural number, so $k \geq 1 > 0$, and $r \neq 0$ gives $r^2 > 0$ (Problem 2(i)); hence $kr^2 > 0$ and therefore

$$1 + (k + 1)r + kr^2 \geq 1 + (k + 1)r.$$

Combining the two displays,

$$(1 + r)^{k+1} \geq 1 + (k + 1)r,$$

which is the assertion $S(k + 1)$.

By the Principle of Mathematical Induction, $S(n)$ holds for every natural number n ; that is, $(1 + r)^n \geq 1 + n \cdot r$ for every natural number n whenever $r > 0$.

We remark that the same argument works for every real r with $r > -1$, since the only properties of r used were $1 + r > 0$ and $kr^2 \geq 0$; this more general form is Bernoulli's inequality.

1.3 Problems 1.15–1.21

Problem 1.15. Use induction arguments to prove that for every natural number n ,

(i)

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}.$$

(ii)

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2.$$

(iii)

$$1 + r + \dots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad \text{if } r \neq 1.$$

Solution. Throughout we use the Principle of Mathematical Induction in the form stated in Section 1.2: if S is a set of natural numbers with $1 \in S$, and with the property that $n \in S$ implies $n + 1 \in S$, then S is all of $\mathbb{N}$. In each part let S be the set of natural numbers n for which the asserted identity holds; we verify that $1 \in S$ and that S is closed under successor.

(i) For $n = 1$ the left-hand side is $1^2 = 1$ and the right-hand side is $\frac{1 \cdot 2 \cdot 3}{6} = 1$, so $1 \in S$.

Suppose $n \in S$, that is, $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$. Then

$$\begin{aligned} \sum_{j=1}^{n+1} j^2 &= \left(\sum_{j=1}^n j^2 \right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} = \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}, \end{aligned}$$

where the factorization $2n^2 + 7n + 6 = (n+2)(2n+3)$ is checked by expanding the right-hand product. Hence $n+1 \in S$, and by induction $S = \mathbb{N}$.

(ii) We first record the companion identity

$$1 + 2 + \dots + n = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N},$$

which is itself proved by induction: it holds for $n = 1$ since $1 = \frac{1 \cdot 2}{2}$, and if it holds for n then

$$(1 + \dots + n) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2},$$

which is the same formula with n replaced by $n+1$.

Now let S be the set of n for which $1^3 + \dots + n^3 = (1 + \dots + n)^2$. For $n = 1$ both sides equal 1, so $1 \in S$. Suppose $n \in S$. Using the induction hypothesis and then the companion identity,

$$\begin{aligned} 1^3 + \dots + n^3 + (n+1)^3 &= (1 + \dots + n)^2 + (n+1)^3 = \left(\frac{n(n+1)}{2} \right)^2 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2[n^2 + 4(n+1)]}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)((n+1)+1)}{2} \right)^2 \\ &= (1 + 2 + \dots + (n+1))^2. \end{aligned}$$

Thus $n+1 \in S$, and by induction the identity holds for every natural number n .

(iii) As printed the exponent is a misprint: with $1 - r^{n-1}$ in the numerator the identity already fails at $n = 1$, where the left-hand side is $1 + r$ while the right-hand side is $\frac{1-r^0}{1-r} = 0$. The intended (and correct) statement is

$$1 + r + \dots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad \text{if } r \neq 1,$$

and this is what we prove. Equivalently, multiplying by the nonzero number $1 - r$, the assertion is that $(1 - r)(1 + r + \dots + r^n) = 1 - r^{n+1}$.

Fix a real number $r \neq 1$ and let S be the set of natural numbers n for which the displayed identity holds. For $n = 1$,

$$\frac{1 - r^2}{1 - r} = \frac{(1 - r)(1 + r)}{1 - r} = 1 + r,$$

so $1 \in S$. Suppose $n \in S$. Then

$$\begin{aligned} 1 + r + \dots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} = \frac{1 - r^{n+1} + r^{n+1}(1 - r)}{1 - r} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} = \frac{1 - r^{n+2}}{1 - r}, \end{aligned}$$

which is the identity with n replaced by $n + 1$. Hence $n + 1 \in S$, and by induction $S = \mathbb{N}$. (The identity also holds for $n = 0$, where both sides equal 1.)

Problem 1.16. Show that the set $\mathbb{Z}$ of integers is countable.

Solution. By the definition in Section 1.3, a set is countably infinite provided it is equipotent to $\mathbb{N}$, and a set is countable provided it is either finite or countably infinite. So it suffices to exhibit a one-to-one mapping of $\mathbb{N}$ onto $\mathbb{Z}$.

We first record that each natural number n is either odd or even, and not both: that is, n can be written as $n = 2k - 1$ for a natural number k , or as $n = 2k$ for a natural number k , but not both. Existence follows by induction: $1 = 2 \cdot 1 - 1$ is odd; and if $n = 2k - 1$ then $n + 1 = 2k$ is even, while if $n = 2k$ then $n + 1 = 2(k + 1) - 1$ is odd. The two alternatives are mutually exclusive: if $2k - 1 = 2k'$ then $2(k - k') = 1$, so the integer $j = k - k'$ satisfies $j = 1/2$; but $0 < 1/2 < 1$, and by Problem 10 the only integer in the interval $[0, 1)$ is 0, so no integer equals $1/2$. Finally k is uniquely determined by n in either case, since $2k = 2k'$ forces $k = k'$.

Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd,} \\ -\frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

This is well defined: if n is odd then $n - 1$ is even so $(n - 1)/2$ is a nonnegative integer, and if n is even then $n/2$ is a natural number so $-n/2$ is a negative integer. Thus f maps the odd natural numbers into $\{0, 1, 2, \dots\}$ and the even natural numbers into $\{-1, -2, \dots\}$. Explicitly, f enumerates $\mathbb{Z}$ as

$$0, -1, 1, -2, 2, -3, 3, \dots$$

The mapping f is one-to-one. Indeed, suppose $f(m) = f(n)$. If m and n are both odd, then $(m - 1)/2 = (n - 1)/2$, and multiplying by 2 and adding 1 gives $m = n$. If m and n are both even, then $-m/2 = -n/2$, so $m = n$. The remaining case cannot occur: if, say, m is odd and n is even, then $f(m) = (m - 1)/2 \geq 0$ while $f(n) = -n/2 \leq -1$, so $f(m) \neq f(n)$.

The mapping f is onto $\mathbb{Z}$. Let p be an integer. If $p \geq 0$, set $n = 2p + 1$. Then n is an odd natural number and $f(n) = \frac{(2p+1)-1}{2} = p$. If $p < 0$, set $n = -2p$. Then $-p$ is a natural number, so $n = 2(-p)$ is an even natural number, and $f(n) = -\frac{-2p}{2} = p$.

Therefore f is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$, so $\mathbb{Z}$ is equipotent to $\mathbb{N}$. Hence $\mathbb{Z}$ is countably infinite and, in particular, countable.

Problem 1.17. Show that a set A is countable if and only if there is a one-to-one mapping of A to $\mathbb{N}$.

Solution. Here “a one-to-one mapping of A to $\mathbb{N}$ ” means a one-to-one mapping of A into $\mathbb{N}$; it is not required to be onto.

First suppose A is countable; we produce a one-to-one mapping of A into $\mathbb{N}$. By definition A is either finite or countably infinite.

If A is countably infinite, then A is equipotent to $\mathbb{N}$, so there is a one-to-one mapping f of A onto $\mathbb{N}$; in particular f is a one-to-one mapping of A into $\mathbb{N}$.

If A is finite, then either $A = \emptyset$, in which case the empty mapping is (vacuously) a one-to-one mapping of A into $\mathbb{N}$, or else there is a natural number n and a one-to-one correspondence g between A and $\{1, \dots, n\}$. Let $\iota: \{1, \dots, n\} \rightarrow \mathbb{N}$ be the inclusion mapping, which is one-to-one. Then $\iota \circ g$ is a mapping of A into $\mathbb{N}$, and it is one-to-one: if $(\iota \circ g)(x) = (\iota \circ g)(x')$ then $g(x) = g(x')$ since ι is one-to-one, and hence $x = x'$ since g is one-to-one.

Conversely, suppose f is a one-to-one mapping of A into $\mathbb{N}$. Let

$$B = f(A) = \{f(x) \mid x \in A\} \subseteq \mathbb{N}.$$

Regard f as a mapping of A to B . So regarded it is onto, by the very definition of B , and it remains one-to-one. Hence f is a one-to-one correspondence between A and B , so A is equipotent to B .

Now B is a set of natural numbers, and by Theorem 3 every set of natural numbers is countable; thus B is countable. As observed immediately after the definition of countability in Section 1.3, if a set is equipotent to a countable set then it is countable. Since A is equipotent to the countable set B , we conclude that A is countable.

(For the reader who prefers to avoid the appeal to that observation: if B is finite, then B is empty, in which case A is empty and hence finite, or B is equipotent to some $\{1, \dots, n\}$, and composing the one-to-one correspondence $A \rightarrow B$ with the one-to-one correspondence $B \rightarrow \{1, \dots, n\}$ gives a one-to-one correspondence between A and $\{1, \dots, n\}$, by Problem 20; so A is finite. If B is countably infinite, the same composition argument, using a one-to-one correspondence $B \rightarrow \mathbb{N}$, shows A is equipotent to $\mathbb{N}$, so A is countably infinite. In either case A is countable.)

Problem 1.18. Use an induction argument to complete the proof of part (i) of Corollary 4. [Corollary 4 (i) asserts that for each natural number n , the Cartesian product $\mathbb{N} \times \dots \times \mathbb{N}$ (n times) is countably infinite; the proof in the text treats the case $n = 2$ and leaves the general case as an exercise in induction.]

Solution. Write $\mathbb{N}^n$ for the n -fold Cartesian product $\mathbb{N} \times \dots \times \mathbb{N}$, whose members are the ordered n -tuples $(k_1, \dots, k_n)$ of natural numbers. Two such n -tuples are equal exactly when their corresponding entries agree. We must show that $\mathbb{N}^n$ is countably infinite for every natural number n ; that is, it is countable and it is not finite.

Step 1: $\mathbb{N}^n$ is countable, by induction on n .

Let S be the set of natural numbers n for which $\mathbb{N}^n$ is countable.

For $n = 1$ we have $\mathbb{N}^1 = \mathbb{N}$, which is countable (the identity mapping is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N}$). So $1 \in S$.

Suppose $n \in S$, so $\mathbb{N}^n$ is countable. By Problem 17 there is a one-to-one mapping

$$h: \mathbb{N}^n \rightarrow \mathbb{N}.$$

Let $g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be the mapping $g(m, n) = (m + n)^2 + n$ used in the proof of Corollary 4; it was shown there to be one-to-one. Define

$$G: \mathbb{N}^{n+1} \rightarrow \mathbb{N}, \quad G(k_1, \dots, k_n, k_{n+1}) = g(h(k_1, \dots, k_n), k_{n+1}).$$

This is well defined, since $h(k_1, \dots, k_n)$ and k_{n+1} are natural numbers.

We claim G is one-to-one. Suppose

$$G(k_1, \dots, k_{n+1}) = G(k'_1, \dots, k'_{n+1}).$$

Then $g(h(k_1, \dots, k_n), k_{n+1}) = g(h(k'_1, \dots, k'_n), k'_{n+1})$. Since g is one-to-one, the two arguments agree as ordered pairs, that is,

$$h(k_1, \dots, k_n) = h(k'_1, \dots, k'_n) \quad \text{and} \quad k_{n+1} = k'_{n+1}.$$

Since h is one-to-one, the first equality forces $(k_1, \dots, k_n) = (k'_1, \dots, k'_n)$, so $k_j = k'_j$ for $1 \leq j \leq n$; together with $k_{n+1} = k'_{n+1}$ this gives $(k_1, \dots, k_{n+1}) = (k'_1, \dots, k'_{n+1})$. Hence G is one-to-one.

So G is a one-to-one mapping of $\mathbb{N}^{n+1}$ into $\mathbb{N}$, and therefore, by Problem 17 again, $\mathbb{N}^{n+1}$ is countable. Thus $n+1 \in S$, and by the Principle of Mathematical Induction $S = \mathbb{N}$: the set $\mathbb{N}^n$ is countable for every natural number n .

Step 2: $\mathbb{N}^n$ is not finite.

We first record the following consequence of the pigeonhole principle.

Lemma. If there is a one-to-one mapping φ of $\mathbb{N}$ into a set A , then A is not finite.

Proof of the Lemma. Suppose, to get a contradiction, that A is finite. Since $\varphi(\mathbb{N})$ is a nonempty subset of A , A is nonempty, so there is a natural number p and a one-to-one correspondence ψ between A and $\{1, \dots, p\}$. The composition $\psi \circ \varphi$ is a one-to-one mapping of $\mathbb{N}$ into $\{1, \dots, p\}$ (a composition of one-to-one mappings is one-to-one, by Problem 20). Restrict it to the initial segment $\{1, \dots, p+1\}$: we obtain a one-to-one mapping of $\{1, \dots, p+1\}$ into $\{1, \dots, p\}$, hence a one-to-one correspondence between $\{1, \dots, p+1\}$ and its image C , a subset of $\{1, \dots, p\}$. By Theorem 3, and specifically by the argument given there for a subset of a finite set, C is equipotent to $\{1, \dots, q\}$ for some natural number $q \leq p$ (the set C is nonempty, since $p+1 \geq 1$). Since equipotence is transitive, $\{1, \dots, p+1\}$ is equipotent to $\{1, \dots, q\}$. Since $q \leq p < p+1$, Problem 9 tells us that $m = (p+1) - q$ is a natural number. Writing $p+1 = q+m$, this says that $\{1, \dots, q+m\}$ is equipotent to $\{1, \dots, q\}$, contradicting the pigeonhole principle. Hence A is not finite, and the Lemma is proved.

Now fix a natural number n and define $\varphi: \mathbb{N} \rightarrow \mathbb{N}^n$ by

$$\varphi(k) = (k, 1, 1, \dots, 1),$$

that is, first entry k and all remaining entries equal to 1. If $\varphi(k) = \varphi(k')$, then comparing first entries gives $k = k'$, so φ is one-to-one. By the Lemma, $\mathbb{N}^n$ is not finite.

Combining the two steps: $\mathbb{N}^n$ is countable and not finite, hence, by the definition of countability, $\mathbb{N}^n$ is countably infinite. This is part (i) of Corollary 4.

Problem 1.19. Prove Corollary 6 in the case of a finite family of countable sets. [Corollary 6 asserts that the union of a countable collection of countable sets is countable; the proof in the text treats the case in which the index set Λ is countably infinite and leaves the finite case as an exercise.]

Solution. Let Λ be a finite index set and for each $\lambda \in \Lambda$ let E_λ be a countable set. Put

$$E = \bigcup_{\lambda \in \Lambda} E_\lambda.$$

We must show E is countable.

If Λ is empty, then E is empty and hence finite, so it is countable. If E is empty, it is likewise countable. So assume $\Lambda \neq \emptyset$ and $E \neq \emptyset$. Since Λ is finite and nonempty, there is a natural number M and a one-to-one correspondence between $\{1, \dots, M\}$ and Λ ; relabelling, we may write

$$\Lambda = \{\lambda_1, \dots, \lambda_M\}, \quad E = \bigcup_{n=1}^M E_{\lambda_n}.$$

Now we argue exactly as in the text's proof of Corollary 6, with the initial segment $\{1, \dots, M\}$ in place of $\mathbb{N}$ as the index range.

Fix n with $1 \leq n \leq M$. If E_{λ_n} is finite and nonempty, choose a natural number $N(n)$ and a one-to-one mapping f_n of $\{1, \dots, N(n)\}$ onto E_{λ_n} ; such a choice is possible by the definition of a finite nonempty set. If E_{λ_n} is countably infinite, choose a one-to-one mapping f_n of $\mathbb{N}$ onto E_{λ_n} ; such a choice is possible by the definition of a countably infinite set. (These are the only two possibilities, since each E_{λ_n} is countable.) If E_{λ_n} is empty, no f_n is chosen. Define

$$E' = \{ (n, k) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq n \leq M, E_{\lambda_n} \text{ is nonempty, and } 1 \leq k \leq N(n) \text{ if } E_{\lambda_n} \text{ is also finite} \}$$

and define

$$f: E' \rightarrow E, \quad f(n, k) = f_n(k).$$

The definition makes sense: for $(n, k) \in E'$ the set E_{λ_n} is nonempty and k lies in the domain of f_n , and $f_n(k) \in E_{\lambda_n} \subseteq E$.

The mapping f is onto E . Indeed, let $x \in E$. Then $x \in E_{\lambda_n}$ for some n with $1 \leq n \leq M$, and in particular E_{λ_n} is nonempty. Since f_n maps its domain onto E_{λ_n} , there is a k in the domain of f_n with $f_n(k) = x$; that k is a natural number with $1 \leq k \leq N(n)$ in the case E_{λ_n} is finite, and an arbitrary natural number in the case E_{λ_n} is countably infinite. In either case $(n, k) \in E'$ and $f(n, k) = x$.

Finally, E' is a subset of $\mathbb{N} \times \mathbb{N}$, which is countable by Corollary 4 (part (i) with $n = 2$), so E' is countable by Theorem 3. Moreover E' is nonempty, since E is nonempty and f maps E' onto E . Therefore E is a nonempty set that is the image of a function whose domain is a nonempty countable set, and Theorem 5 tells us that E is countable.

Remark. Equivalently one may argue by induction on M : the case $M = 1$ is trivial, and for the inductive step it suffices to know that the union of two countable sets A and B is countable. That special case follows from the argument above with $M = 2$, or directly: choose one-to-one mappings $g: A \rightarrow \mathbb{N}$ and $h: B \rightarrow \mathbb{N}$ as furnished by Problem 17, and define $\Phi: A \cup B \rightarrow \mathbb{N}$ by $\Phi(x) = 2g(x)$ if $x \in A$ and $\Phi(x) = 2h(x) - 1$ if $x \in B \sim A$. Then Φ is one-to-one, since Φ takes even values on A and odd values on $B \sim A$ and is one-to-one on each of these two sets, and so $A \cup B$ is countable by Problem 17.

Problem 1.20. Let both $f: A \rightarrow B$ and $g: B \rightarrow C$ be one-to-one and onto. Show that the composition $g \circ f: A \rightarrow C$ and the inverse $f^{-1}: B \rightarrow A$ are also one-to-one and onto.

Solution. (The target of the composition is misprinted in the statement: $g \circ f$ maps A to C , not to B . We prove the assertion for $g \circ f: A \rightarrow C$.)

First, the composition. By definition $g \circ f$ is the mapping of A to C given by

$$(g \circ f)(x) = g(f(x)) \quad \text{for } x \in A,$$

which makes sense because $f(x) \in B$ and g is defined on all of B .

$g \circ f$ is one-to-one. Let $x, x' \in A$ and suppose $(g \circ f)(x) = (g \circ f)(x')$, that is, $g(f(x)) = g(f(x'))$. Since $f(x)$ and $f(x')$ belong to B and g is one-to-one, we conclude that $f(x) = f(x')$. Since f is one-to-one, we conclude that $x = x'$.

$g \circ f$ is onto C . Let $z \in C$. Since g maps B onto C , there is a $y \in B$ with $g(y) = z$. Since f maps A onto B , there is an $x \in A$ with $f(x) = y$. Then

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$

Second, the inverse. We must first check that f^{-1} is a well-defined mapping of B to A . Let $y \in B$. Since f is onto, there is at least one $x \in A$ with $f(x) = y$; since f is one-to-one, there is at most one such x . Hence there is exactly one $x \in A$ with $f(x) = y$, and we may define $f^{-1}(y)$ to be that x . This gives a mapping $f^{-1}: B \rightarrow A$ characterized by the equivalence

$$f^{-1}(y) = x \iff f(x) = y \quad (x \in A, y \in B).$$

In particular $f^{-1}(f(x)) = x$ for all $x \in A$ and $f(f^{-1}(y)) = y$ for all $y \in B$.

f^{-1} is one-to-one. Let $y, y' \in B$ and suppose $f^{-1}(y) = f^{-1}(y')$; call this common value $x \in A$. By the characterizing equivalence, $f(x) = y$ and $f(x) = y'$. Since f is a mapping, it assigns to x a single value, so $y = f(x) = y'$.

f^{-1} is onto A . Let $x \in A$ and set $y = f(x) \in B$. By the characterizing equivalence, $f^{-1}(y) = x$. Thus every member of A is a value of f^{-1} .

Therefore $g \circ f: A \rightarrow C$ and $f^{-1}: B \rightarrow A$ are both one-to-one and onto, as asserted.

Remark. These two facts are exactly what is needed to see that equipotence is an equivalence relation among sets, as asserted in Section 1.3: reflexivity is witnessed by the identity mapping, symmetry by the inverse f^{-1} of a one-to-one correspondence f , and transitivity by the composition $g \circ f$ of one-to-one correspondences f and g .

Problem 1.21. Use an induction argument to establish the pigeonhole principle. [As stated in Section 1.3: for any natural numbers n and m , the set $\{1, \dots, n + m\}$ is not equipotent to the set $\{1, \dots, n\}$. The induction is with respect to n .]

Solution. For a natural number k we write, as in the text, $\{1, \dots, k\}$ for the initial segment $\{j \in \mathbb{N} \mid 1 \leq j \leq k\}$. Let

$$S = \{n \in \mathbb{N} \mid \text{for every } m \in \mathbb{N}, \{1, \dots, n + m\} \text{ is not equipotent to } \{1, \dots, n\}\}.$$

We show $1 \in S$ and that $n \in S$ implies $n + 1 \in S$; the Principle of Mathematical Induction then gives $S = \mathbb{N}$, which is the pigeonhole principle. Since equipotence is symmetric (Problem 20), it does not matter in which direction we look for a one-to-one correspondence.

Base case. Let m be a natural number and suppose, to get a contradiction, that f is a one-to-one mapping of $\{1, \dots, 1 + m\}$ onto $\{1\}$. Since $m \geq 1$, we have $1 + m \geq 2$, so 1 and 2 are two distinct members of $\{1, \dots, 1 + m\}$. But $f(1)$ and $f(2)$ both lie in the one-element set $\{1\}$, so $f(1) = 1 = f(2)$, contradicting the assumption that f is one-to-one. Hence no such f exists, and $1 \in S$.

Inductive step. Suppose $n \in S$. Let m be a natural number and suppose, to get a contradiction, that there is a one-to-one mapping f of $\{1, \dots, (n + 1) + m\}$ onto $\{1, \dots, n + 1\}$. Write

$$q = (n + 1) + m = (n + m) + 1,$$

so that the domain of f is $\{1, \dots, q\}$ and $\{1, \dots, q\} \sim \{q\} = \{1, \dots, n + m\}$.

We first arrange that the largest element of the domain is sent to the largest element of the range. Since f is onto, there is a $p \in \{1, \dots, q\}$ with $f(p) = n + 1$. Let τ be the transposition of $\{1, \dots, q\}$ that interchanges p and q and fixes every other element:

$$\tau(p) = q, \quad \tau(q) = p, \quad \tau(j) = j \quad \text{for } j \notin \{p, q\}.$$

(If $p = q$, then τ is the identity mapping.) One checks directly that $\tau \circ \tau$ is the identity mapping of $\{1, \dots, q\}$; consequently τ is one-to-one, since $\tau(i) = \tau(j)$ gives $i = \tau(\tau(i)) = \tau(\tau(j)) = j$, and τ is onto, since any j equals $\tau(\tau(j))$.

By Problem 20, the composition $h = f \circ \tau$ is a one-to-one mapping of $\{1, \dots, q\}$ onto $\{1, \dots, n + 1\}$, and

$$h(q) = f(\tau(q)) = f(p) = n + 1.$$

Now let h_0 be the restriction of h to $\{1, \dots, n + m\} = \{1, \dots, q\} \sim \{q\}$. We claim h_0 is a one-to-one mapping of $\{1, \dots, n + m\}$ onto $\{1, \dots, n\}$.

h_0 takes its values in $\{1, \dots, n\}$: if $1 \leq j \leq n + m$, then $j \neq q$, so $h(j) \neq h(q) = n + 1$ because h is one-to-one; since $h(j) \in \{1, \dots, n + 1\}$ and $h(j) \neq n + 1$, we get $h(j) \in \{1, \dots, n\}$.

h_0 is one-to-one: it is the restriction of the one-to-one mapping h to a subset of its domain.

h_0 is onto $\{1, \dots, n\}$: let $i \in \{1, \dots, n\}$. Since h is onto $\{1, \dots, n + 1\}$, there is a $j \in \{1, \dots, q\}$ with $h(j) = i$. As $i \leq n < n + 1 = h(q)$, we have $h(j) \neq h(q)$, hence $j \neq q$, hence $1 \leq j \leq n + m$ and $h_0(j) = i$.

Thus $\{1, \dots, n + m\}$ is equipotent to $\{1, \dots, n\}$, with m a natural number. This contradicts the assumption $n \in S$.

Therefore no such f exists for any natural number m , that is, $\{1, \dots, (n+1) + m\}$ is not equipotent to $\{1, \dots, n+1\}$ for any natural number m . Hence $n+1 \in S$.

By induction $S = \mathbb{N}$, and the pigeonhole principle is established: for all natural numbers n and m , the set $\{1, \dots, n+m\}$ is not equipotent to $\{1, \dots, n\}$.

Remark. Two standard consequences follow at once. If n and n' are natural numbers with $n \neq n'$, then $\{1, \dots, n\}$ and $\{1, \dots, n'\}$ are not equipotent: for one of them, say $n < n'$, we may write $n' = n + m$ with $m = n' - n$, and m is a natural number by Problem 9; the principle then applies. Hence the natural number n attached to a nonempty finite set by the definition of finiteness is uniquely determined by the set.

1.4 Problems 1.22–1.28

Problem 1.22. Show that $2^{\mathbb{N}}$, the collection of all sets of natural numbers, is uncountable.

Solution. We first record an elementary observation that will be used repeatedly.

Observation. If E is a nonempty countable set, then there is a mapping of $\mathbb{N}$ onto E . Indeed, by the definition of countability E is either finite or countably infinite. If E is countably infinite there is a one-to-one correspondence h of $\mathbb{N}$ onto E , and h itself is such a mapping. If E is finite and nonempty there is a natural number n and a one-to-one correspondence h of $\{1, \dots, n\}$ onto E ; define $f: \mathbb{N} \rightarrow E$ by $f(k) = h(k)$ for $1 \leq k \leq n$ and $f(k) = h(1)$ for $k > n$. Then f maps $\mathbb{N}$ onto E .

Now suppose, by way of contradiction, that $2^{\mathbb{N}}$ is countable. The set $2^{\mathbb{N}}$ is nonempty, since the empty-set belongs to it. Hence, by the Observation, there is a mapping

$$f: \mathbb{N} \longrightarrow 2^{\mathbb{N}}$$

of $\mathbb{N}$ onto $2^{\mathbb{N}}$; that is, every set of natural numbers is of the form $f(n)$ for some natural number n .

Define

$$D = \{n \in \mathbb{N} : n \notin f(n)\}.$$

This is a well-defined set of natural numbers, so D belongs to $2^{\mathbb{N}}$. Since f is onto, there is a natural number m for which $D = f(m)$. We ask whether m belongs to D .

If $m \in D$, then by the definition of D we have $m \notin f(m) = D$, a contradiction. If $m \notin D$, then $m \notin f(m)$, and therefore, again by the definition of D , $m \in D$, once more a contradiction.

Both alternatives are impossible, so no mapping of $\mathbb{N}$ onto $2^{\mathbb{N}}$ exists. This contradicts the Observation, and therefore $2^{\mathbb{N}}$ is not countable, that is, $2^{\mathbb{N}}$ is uncountable.

Remark. The same argument shows, for an arbitrary set X , that there is no mapping of X onto 2^X ; in particular X and 2^X are never equipotent.

Problem 1.23. Show that the Cartesian product of a finite collection of countable sets is countable. Use the preceding problem to show that $\mathbb{N}^{\mathbb{N}}$, the collection of all mappings of $\mathbb{N}$ into $\mathbb{N}$, is not countable.

Solution. Part 1: finite Cartesian products of countable sets.

Let $A_1, \dots, A_n$ be countable sets; we show $A_1 \times \dots \times A_n$ is countable, arguing by induction on n .

For $n = 1$ there is nothing to prove. Consider $n = 2$, so let A and B be countable. If either A or B is empty, then $A \times B$ is empty and hence countable. So assume both are nonempty. By the Observation recorded in the solution of Problem 22, there are mappings f of $\mathbb{N}$ onto A and g of $\mathbb{N}$ onto B . Define

$$h: \mathbb{N} \times \mathbb{N} \longrightarrow A \times B, \quad h(m, k) = (f(m), g(k)).$$

If (a, b) belongs to $A \times B$, choose m with $f(m) = a$ and k with $g(k) = b$; then $h(m, k) = (a, b)$. Thus h maps $\mathbb{N} \times \mathbb{N}$ onto $A \times B$. By Corollary 4(i) the set $\mathbb{N} \times \mathbb{N}$ is countably infinite, in particular a nonempty countable set. Since $A \times B$ is nonempty and is the image of a function whose domain is a nonempty countable set, Theorem 5 tells us that $A \times B$ is countable.

Now assume the assertion holds for n , and let $A_1, \dots, A_{n+1}$ be countable. The mapping

$$((x_1, \dots, x_n), x_{n+1}) \mapsto (x_1, \dots, x_n, x_{n+1})$$

is a one-to-one correspondence between $(A_1 \times \dots \times A_n) \times A_{n+1}$ and $A_1 \times \dots \times A_{n+1}$. By the induction hypothesis $A_1 \times \dots \times A_n$ is countable, so by the case $n = 2$ the set $(A_1 \times \dots \times A_n) \times A_{n+1}$ is countable; since a set equipotent to a countable set is countable, so is $A_1 \times \dots \times A_{n+1}$. The induction is complete. Part 2: $\mathbb{N}^{\mathbb{N}}$ is uncountable.

For a set E of natural numbers define $\chi_E: \mathbb{N} \rightarrow \mathbb{N}$ by

$$\chi_E(n) = \begin{cases} 1, & n \in E, \\ 2, & n \notin E. \end{cases}$$

Then χ_E is a mapping of $\mathbb{N}$ into $\mathbb{N}$, so χ_E belongs to $\mathbb{N}^{\mathbb{N}}$, and the assignment $E \mapsto \chi_E$ is a mapping of $2^{\mathbb{N}}$ into $\mathbb{N}^{\mathbb{N}}$.

This assignment is one-to-one: if $E \neq F$ are sets of natural numbers, then there is a natural number n belonging to exactly one of them, say $n \in E$ and $n \notin F$; then $\chi_E(n) = 1 \neq 2 = \chi_F(n)$, so $\chi_E \neq \chi_F$. Consequently $2^{\mathbb{N}}$ is equipotent to the set

$$S = \{ \chi_E : E \in 2^{\mathbb{N}} \} \subseteq \mathbb{N}^{\mathbb{N}}.$$

Suppose $\mathbb{N}^{\mathbb{N}}$ were countable. Then, by Theorem 3, its subset S would be countable; and since $2^{\mathbb{N}}$ is equipotent to S , and a set equipotent to a countable set is countable, $2^{\mathbb{N}}$ would be countable. This contradicts the preceding problem. Therefore $\mathbb{N}^{\mathbb{N}}$ is not countable.

Problem 1.24. Show that a nondegenerate interval of real numbers fails to be finite.

Solution. Recall that an interval of real numbers is called degenerate provided it is empty or contains a single member. Thus a nondegenerate interval I contains two distinct points; choose $a, b \in I$ with $a < b$. Since I is an interval, every real number lying between two of its members belongs to I , so

$$[a, b] \subseteq I.$$

Step 1: I contains a subset equipotent to $\mathbb{N}$. For each natural number k put

$$x_k = a + \frac{b-a}{k}.$$

Since $b-a > 0$ and $0 < 1/k \leq 1$, we have $a < x_k \leq b$, so x_k belongs to $[a, b]$ and hence to I . Moreover $k \mapsto x_k$ is one-to-one: if $k \neq k'$ then $1/k \neq 1/k'$, and since $b-a \neq 0$ we get $x_k \neq x_{k'}$. Therefore

$$X = \{ x_k : k \in \mathbb{N} \}$$

is a subset of I that is equipotent to $\mathbb{N}$.

Step 2: a subset of a finite set is finite, and indeed corresponds to an initial segment. We use the following fact, which is exactly what the finite case of the proof of Theorem 3 establishes: if B is equipotent to $\{1, \dots, n\}$ and A is a nonempty subset of B , then there is a natural number N with $N \leq n$ and a one-to-one correspondence between $\{1, \dots, N\}$ and A .

Step 3: $\mathbb{N}$ is not finite. Suppose $\mathbb{N}$ were equipotent to $\{1, \dots, m\}$ for some natural number m , by way of a one-to-one correspondence ψ of $\mathbb{N}$ onto $\{1, \dots, m\}$. The set $\{1, \dots, m+1\}$ is a nonempty subset of $\mathbb{N}$, so its image $\psi(\{1, \dots, m+1\})$ is a nonempty subset of $\{1, \dots, m\}$ equipotent to $\{1, \dots, m+1\}$. By Step 2 there is a natural number $j \leq m$ with $\{1, \dots, j\}$ equipotent to $\psi(\{1, \dots, m+1\})$, and hence, by transitivity of equipotence, $\{1, \dots, j\}$ is equipotent to $\{1, \dots, m+1\}$. Writing $m+1 = j+p$ with $p = m+1-j \geq 1$ a natural number, this says that $\{1, \dots, j\}$ is equipotent to $\{1, \dots, j+p\}$, which contradicts the pigeonhole principle (Problem 21). Therefore $\mathbb{N}$ is not finite.

Step 4: conclusion. Suppose I were finite. Since I is nonempty, there is a natural number n and a one-to-one correspondence φ of I onto $\{1, \dots, n\}$. The set X of Step 1 is a nonempty subset of I , so

$\varphi(X)$ is a nonempty subset of $\{1, \dots, n\}$, and φ restricted to X is a one-to-one correspondence of X onto $\varphi(X)$. By Step 2 there is a natural number $N \leq n$ with $\{1, \dots, N\}$ equipotent to $\varphi(X)$. Since $\mathbb{N}$ is equipotent to X , X is equipotent to $\varphi(X)$, and $\varphi(X)$ is equipotent to $\{1, \dots, N\}$, transitivity of equipotence gives that $\mathbb{N}$ is equipotent to $\{1, \dots, N\}$, that is, $\mathbb{N}$ is finite. This contradicts Step 3. Therefore a nondegenerate interval of real numbers is not finite.

Problem 1.25. Show that any two nondegenerate intervals of real numbers are equipotent.

Solution. Equipotence is reflexive, symmetric and transitive (Problem 20), so it suffices to prove that every nondegenerate interval is equipotent to the single interval $(0, 1)$.

Lemma (absorbing a countable set). Let X be a set, let C be a countable subset of X , and suppose $X \sim C$ contains a countably infinite subset D . Then X and $X \sim C$ are equipotent.

Proof of Lemma. Here $X \sim C$ denotes the set of points of X that do not belong to C . Put $E = C \cup D$. By Corollary 6 the set E is countable. It is not finite: it contains D , and were E finite then, by Step 2 of Problem 24, D would be equipotent to an initial segment $\{1, \dots, N\}$, so that $\mathbb{N}$ would be finite, contradicting Step 3 of Problem 24. Hence E is countably infinite. Since $D \subseteq X \sim C$, the sets C and D are disjoint, so $E \sim C = D$. Let $\{e_k \mid k \in \mathbb{N}\}$ be an enumeration of E and $\{d_k \mid k \in \mathbb{N}\}$ an enumeration of D . Define

$$h: X \rightarrow X \sim C, \quad h(e_k) = d_k \quad (k \in \mathbb{N}), \quad h(x) = x \quad \text{for } x \in X \sim E.$$

This is well defined, since every point of X either belongs to E , and then equals exactly one e_k , or belongs to $X \sim E$; and its values lie in $D \cup (X \sim E) \subseteq X \sim C$. It is one-to-one: h is one-to-one on E because the d_k are distinct, it is one-to-one on $X \sim E$ because it is the identity there, and the two images D and $X \sim E$ are disjoint because $D \subseteq E$. It is onto $X \sim C$: its image is $D \cup (X \sim E) = (E \sim C) \cup (X \sim E) = X \sim C$. This proves the Lemma.

Step 1: every nondegenerate interval is equipotent to a nondegenerate open interval. Let I be a nondegenerate interval and let $\alpha = \inf I$ and $\beta = \sup I$ be its infimum and supremum in the extended real numbers. If $\alpha < z < \beta$, then z is not a lower bound for I , so there is $u \in I$ with $u < z$, and z is not an upper bound for I , so there is $v \in I$ with $z < v$; since I is an interval, $z \in I$. Hence, writing $J = (\alpha, \beta)$,

$$J \subseteq I \subseteq J \cup F, \quad F = \{\alpha, \beta\} \cap \mathbb{R},$$

so $I = J \cup C$ where $C = I \sim J$ is a subset of F and therefore has at most two elements. Since I is nondegenerate it contains two distinct points, so $\alpha < \beta$ and J is a nondegenerate open interval. Now apply the Lemma with $X = I$ and this finite (hence countable) set C : the set $X \sim C = J$ is a nondegenerate interval, so by Problem 24 it is not finite, and hence (as in Step 1 of Problem 24) it contains a countably infinite subset D . The Lemma then tells us that I is equipotent to $I \sim C = J$, that is, I is equipotent to the open interval $J = (\alpha, \beta)$.

Step 2: the interval $J = (\alpha, \beta)$ produced in Step 1 is equipotent to $(0, 1)$. Here $\alpha < \beta$ are extended real numbers, and we use the notation introduced in Section 1.2, namely $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$, $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$ and $(-\infty, \infty) = \mathbb{R}$. Since $\alpha < \beta$, we have $\alpha \neq +\infty$ and $\beta \neq -\infty$; hence, according to whether each of α, β is real or infinite, J is of exactly one of the four forms (a, b) with $a < b$ real, (a, ∞) with a real, $(-\infty, b)$ with b real, or $\mathbb{R}$. We treat each in turn.

If $J = (a, b)$ with $a < b$ real, then $x \mapsto (x - a)/(b - a)$ is a one-to-one correspondence of (a, b) onto $(0, 1)$, with inverse $s \mapsto a + s(b - a)$.

If $J = (a, \infty)$, then $x \mapsto x - a$ is a one-to-one correspondence of (a, ∞) onto $(0, \infty)$, and

$$t \mapsto \frac{t}{1+t}$$

is a one-to-one correspondence of $(0, \infty)$ onto $(0, 1)$: it carries $(0, \infty)$ into $(0, 1)$ since $0 < t < 1 + t$, and $s \mapsto s/(1 - s)$ carries $(0, 1)$ into $(0, \infty)$ and is a two-sided inverse, because if $s = t/(1 + t)$ then $1 - s = 1/(1 + t)$ and so $s/(1 - s) = t$, while if $t = s/(1 - s)$ then $1 + t = 1/(1 - s)$ and so $t/(1 + t) = s$. If $J = (-\infty, b)$, then $x \mapsto b - x$ is a one-to-one correspondence of $(-\infty, b)$ onto $(0, \infty)$, and we conclude as in the previous case.

If $J = \mathbb{R}$, then

$$x \mapsto \frac{x}{1 + |x|}$$

is a one-to-one correspondence of $\mathbb{R}$ onto $(-1, 1)$: its values satisfy $|x|/(1 + |x|) < 1$, and $y \mapsto y/(1 - |y|)$ maps $(-1, 1)$ into $\mathbb{R}$ and is a two-sided inverse, since for $x \geq 0$ the value $y = x/(1 + x)$ lies in $[0, 1)$ and $y/(1 - y) = x$, while both maps are odd, so the case $x < 0$ follows by symmetry. Composing with the one-to-one correspondence $y \mapsto (y + 1)/2$ of $(-1, 1)$ onto $(0, 1)$ gives the assertion.

Conclusion. By Step 1 and Step 2, every nondegenerate interval of real numbers is equipotent to $(0, 1)$. By symmetry and transitivity of equipotence, any two nondegenerate intervals of real numbers are equipotent to each other.

Problem 1.26. Is the set $\mathbb{R} \times \mathbb{R}$ equipotent to $\mathbb{R}$?

Solution. Yes. We construct an explicit one-to-one correspondence, in five steps. Throughout we use two trivial remarks: equipotence is an equivalence relation (Problem 20), and if f is a one-to-one correspondence of X onto X' and g one of Y onto Y' , then $(x, y) \mapsto (f(x), g(y))$ is a one-to-one correspondence of $X \times Y$ onto $X' \times Y'$.

Let Σ denote the collection of all sequences $a = (a_n)_{n \geq 1}$ with each $a_n \in \{0, 1\}$ and with $a_n = 0$ for infinitely many indices n .

Step 1: the map $\sigma(a) = \sum_{n=1}^{\infty} a_n 2^{-n}$ is a one-to-one correspondence of Σ onto $[0, 1)$.

The series converges since it is dominated by $\sum 2^{-n} = 1$, and $\sigma(a) \geq 0$. If N is an index with $a_N = 0$, then $\sigma(a) \leq \sum_{n \neq N} 2^{-n} = 1 - 2^{-N} < 1$; such an N exists, so σ maps Σ into $[0, 1)$.

σ is one-to-one: let $a \neq b$ in Σ and let m be the least index with $a_m \neq b_m$, say $a_m = 1$ and $b_m = 0$. Since b has infinitely many zero terms, $\sum_{n > m} b_n 2^{-n} < \sum_{n > m} 2^{-n} = 2^{-m}$, and therefore

$$\sigma(a) - \sigma(b) = 2^{-m} + \sum_{n > m} (a_n - b_n) 2^{-n} \geq 2^{-m} - \sum_{n > m} b_n 2^{-n} > 0.$$

σ is onto $[0, 1)$: given $x \in [0, 1)$, define a_n recursively by declaring $a_n = 1$ if $\sum_{k=1}^{n-1} a_k 2^{-k} + 2^{-n} \leq x$ and $a_n = 0$ otherwise. Put $r_n = x - \sum_{k=1}^n a_k 2^{-k}$, with $r_0 = x$. We claim $0 \leq r_n < 2^{-n}$ for every $n \geq 0$. This holds for $n = 0$. Assume it for $n - 1$. The defining condition for a_n reads $r_{n-1} \geq 2^{-n}$. If it holds, then $a_n = 1$ and $r_n = r_{n-1} - 2^{-n}$, so $0 \leq r_n < 2^{-(n-1)} - 2^{-n} = 2^{-n}$. If it fails, then $a_n = 0$ and $r_n = r_{n-1} < 2^{-n}$, and $r_n \geq 0$. This proves the claim, and letting $n \rightarrow \infty$ gives $\sigma(a) = x$. Finally $a \in \Sigma$: if $a_n = 1$ for all $n > N$, then for $n > N$

$$r_n = r_N - (2^{-N} - 2^{-n}) = (r_N - 2^{-N}) + 2^{-n},$$

and $r_N - 2^{-N} < 0$; taking n large enough that $2^{-n} < 2^{-N} - r_N$ gives $r_n < 0$, a contradiction. So a has infinitely many zero terms.

Step 2: let B be the set of $a \in \Sigma$ having infinitely many terms equal to 1 as well. Then B is equipotent to $\mathbb{R}$.

The complement $\Sigma \sim B$ consists of the sequences in Σ with only finitely many terms equal to 1. For each natural number N there are exactly 2^N such sequences whose terms vanish from index $N + 1$ on, so $\Sigma \sim B$ is the union of a countable collection of finite sets and hence is countable by Corollary 6. Since σ is one-to-one, $C = \sigma(\Sigma \sim B)$ is a countable subset of $[0, 1)$ and $\sigma(B) = [0, 1) \sim C$. For each natural number m let $b^{(m)}$ be the sequence whose n -th term is 1 exactly when n is even and $n \geq 2m$; each $b^{(m)}$ has infinitely many ones and infinitely many zeros, so $b^{(m)} \in B$, and these sequences are distinct because $2m$ is the least index carrying a 1 in $b^{(m)}$. Hence $D = \{\sigma(b^{(m)}) \mid m \in \mathbb{N}\}$ is a countably infinite subset of $[0, 1) \sim C$. By the Lemma proved in Problem 25, applied with $X = [0, 1)$, the sets $[0, 1)$ and $[0, 1) \sim C = \sigma(B)$ are equipotent. Since σ restricted to B is a one-to-one correspondence of B onto $\sigma(B)$, the set B is equipotent to $[0, 1)$, which by Problem 25 is equipotent to $\mathbb{R}$.

Step 3: B is equipotent to $\mathbb{N}^{\mathbb{N}}$, the collection of all mappings of $\mathbb{N}$ into $\mathbb{N}$, which we regard as the collection of all sequences $(m_j)_{j \geq 1}$ of natural numbers.

Define $\Psi: \mathbb{N}^{\mathbb{N}} \rightarrow B$ as follows: $\Psi(m)$ is the sequence obtained by concatenating the blocks

$$\underbrace{0 \cdots 0}_{m_1-1}, \underbrace{1 \cdots 1}_{m_2}, \underbrace{0 \cdots 0}_{m_3}, \underbrace{1 \cdots 1}_{m_4}, \dots$$

that is, the first block consists of $m_1 - 1 \geq 0$ zeros and, for $j \geq 2$, the j -th block consists of $m_j \geq 1$ terms, all equal to 1 if j is even and all equal to 0 if j is odd. Since each block with $j \geq 2$ is nonempty, the concatenation is an infinite sequence of 0's and 1's; it has infinitely many ones (one from each even block) and infinitely many zeros (one from each odd block with $j \geq 3$). Hence $\Psi(m) \in B$.

Ψ is onto B and one-to-one because a sequence in B has a unique decomposition into maximal constant runs. We first describe a recovery recipe $\Phi: B \rightarrow \mathbb{N}^{\mathbb{N}}$. Let $a \in B$ and put $S = \{n \in \mathbb{N} \mid a_n \neq a_{n+1}\}$. If S were finite, a would be constant from some index on, contradicting the fact that both values occur infinitely often; so S is infinite and we may enumerate it increasingly as $s_1 < s_2 < \dots$. Put $s_0 = 0$ and $\ell_j = s_j - s_{j-1} \geq 1$. On each set of indices $\{s_{j-1} + 1, \dots, s_j\}$ the sequence a is constant, and its values on consecutive such sets differ; hence they alternate, starting with the value a_1 . If $a_1 = 0$, set $m_1 = \ell_1 + 1$ and $m_j = \ell_j$ for $j \geq 2$; if $a_1 = 1$, set $m_1 = 1$ and $m_{j+1} = \ell_j$ for $j \geq 1$. In either case (m_j) is a sequence of natural numbers; write $\Phi(a) = (m_j)$. Comparing blocks one at a time, $\Psi(\Phi(a)) = a$. Hence Ψ maps $\mathbb{N}^{\mathbb{N}}$ onto B .

It remains to check that $\Phi(\Psi(m)) = m$ for every $m \in \mathbb{N}^{\mathbb{N}}$, which gives that Ψ is one-to-one. Put $a = \Psi(m)$. Suppose first $m_1 \geq 2$. Then every block in the definition of $\Psi(m)$ is nonempty and consecutive blocks carry different values, so the maximal constant runs of a are precisely those blocks, in order, with lengths $m_1 - 1, m_2, m_3, \dots$; that is, $\ell_1 = m_1 - 1$ and $\ell_j = m_j$ for $j \geq 2$. Also $a_1 = 0$, so the recipe returns $\ell_1 + 1 = m_1$ and $\ell_j = m_j$ for $j \geq 2$, that is, $\Phi(a) = m$. Suppose instead $m_1 = 1$. Then the first block is empty, a begins with the second block and $a_1 = 1$, and the maximal constant runs of a have lengths $m_2, m_3, \dots$; that is, $\ell_j = m_{j+1}$ for $j \geq 1$. The recipe for the case $a_1 = 1$ returns first coordinate $1 = m_1$ and $(j+1)$ -st coordinate $\ell_j = m_{j+1}$, so again $\Phi(a) = m$.

Thus Ψ is a one-to-one correspondence of $\mathbb{N}^{\mathbb{N}}$ onto B .

Step 4: $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is equipotent to $\mathbb{N}^{\mathbb{N}}$.

Given $u = (u_k)$ and $v = (v_k)$ in $\mathbb{N}^{\mathbb{N}}$, let $w = \Theta(u, v)$ be the interleaved sequence defined by $w_{2k-1} = u_k$ and $w_{2k} = v_k$ for $k \in \mathbb{N}$. Then Θ maps into $\mathbb{N}^{\mathbb{N}}$, and the assignment $w \mapsto ((w_{2k-1})_k, (w_{2k})_k)$ is a two-sided inverse. Hence Θ is a one-to-one correspondence.

Step 5: conclusion. Combining the steps,

$$\mathbb{R} \times \mathbb{R} \sim B \times B \sim \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \sim \mathbb{N}^{\mathbb{N}} \sim B \sim \mathbb{R},$$

where each $\sim$ denotes equipotence: the first by Step 2 and the product remark, the second by Step 3 and the product remark, the third by Step 4, the fourth by Step 3, the last by Step 2. By transitivity, $\mathbb{R} \times \mathbb{R}$ is equipotent to $\mathbb{R}$.

Problem 1.27. Is the set of rational numbers open or closed?

Solution. The set $\mathbb{Q}$ of rational numbers is neither open nor closed.

$\mathbb{Q}$ is not open. Let x be a rational number and let $r > 0$. The interval $(x - r, x + r)$ is a nondegenerate interval, and the irrational numbers are dense in $\mathbb{R}$ (Problem 12), so there is an irrational number y with $x - r < y < x + r$. Thus $(x - r, x + r)$ is not contained in $\mathbb{Q}$. Since this holds for every $r > 0$, and since $\mathbb{Q}$ contains a point (for instance 0), the defining property of openness fails at every point of $\mathbb{Q}$. Hence $\mathbb{Q}$ is not open.

$\mathbb{Q}$ is not closed. Let x be any real number and let I be an open interval containing x . Then I contains an interval of the form $(x - r, x + r)$ with $r > 0$, and by the density of the rational numbers, Theorem 2, there is a rational number q with $x - r < q < x + r$. Hence every open interval containing x contains a point of $\mathbb{Q}$, so x is a point of closure of $\mathbb{Q}$. Since x was arbitrary,

$$\overline{\mathbb{Q}} = \mathbb{R}.$$

But $\mathbb{R} \neq \mathbb{Q}$, since for instance $\sqrt{2}$ is irrational. Therefore $\mathbb{Q}$ does not contain all of its points of closure,

that is, $\mathbb{Q}$ is not closed.

Equivalently, one may argue the second assertion from the first through Proposition 11: were $\mathbb{Q}$ closed, its complement, the set of irrational numbers, would be open; but by the density of the rationals, Theorem 2, no interval $(x - r, x + r)$ about an irrational x consists entirely of irrationals.

Problem 1.28. What are the sets of real numbers that are both open and closed?

Solution. The only subsets of $\mathbb{R}$ that are both open and closed are the empty-set $\emptyset$ and $\mathbb{R}$ itself.

First, these two sets do have the property. By Proposition 8 both $\emptyset$ and $\mathbb{R}$ are open, and by Proposition 12 both are closed. (Alternatively: $\mathbb{R}$ is open and its complement $\emptyset$ is open, so by Proposition 11 the set $\mathbb{R}$ is closed, and symmetrically for $\emptyset$.)

Conversely, let E be a set of real numbers that is both open and closed, and suppose E is neither empty nor all of $\mathbb{R}$; we derive a contradiction.

Choose $x \in E$. Since E is open and nonempty, the construction in the proof of Proposition 9 applies: with

$$a_x = \inf\{z \mid (z, x) \subseteq E\}, \quad b_x = \sup\{y \mid (x, y) \subseteq E\}$$

taken in the extended real numbers, the interval $I_x = (a_x, b_x)$ contains x and, by assertion (2) in that proof,

$$I_x \subseteq E, \quad a_x \notin E, \quad b_x \notin E.$$

Since $E \neq \mathbb{R}$ and $I_x \subseteq E$, we have $I_x \neq \mathbb{R}$, so at least one of a_x, b_x is a real number. Suppose first that b_x is real. We claim b_x is a point of closure of E . Indeed, let I be an open interval containing b_x ; then I contains an interval $(b_x - r, b_x + r)$ for some $r > 0$. Since $a_x < x < b_x$, the number

$$t = \max\left\{x, b_x - \frac{r}{2}\right\}$$

satisfies $a_x < t < b_x$ and $b_x - r < t$, so t belongs both to $I_x \subseteq E$ and to $(b_x - r, b_x + r) \subseteq I$. Thus every open interval containing b_x contains a point of E , so b_x is a point of closure of E . Since E is closed, $E = \overline{E}$, and therefore $b_x \in E$. This contradicts $b_x \notin E$.

If instead $b_x = \infty$, then a_x must be a real number, and the symmetric argument applies: for any $r > 0$ the number $t = \min\{x, a_x + \frac{r}{2}\}$ satisfies $a_x < t < b_x$ and $t < a_x + r$, so every open interval containing a_x contains a point of $I_x \subseteq E$; hence a_x is a point of closure of E , so $a_x \in E$ because E is closed, contradicting $a_x \notin E$.

In either case we reach a contradiction. Therefore a set of real numbers that is both open and closed must be $\emptyset$ or $\mathbb{R}$.

1.5 Problems 1.29–1.35

Problem 1.29. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

Solution. Take

$$A = (0, 1), \quad B = (1, 2).$$

These are disjoint: a real number cannot satisfy both $x < 1$ and $x > 1$, so $A \cap B = \emptyset$.

Now compute the closures. I claim $\overline{A} = [0, 1]$. First, $[0, 1]$ is a closed set containing A : its complement $(-\infty, 0) \cup (1, \infty)$ is the union of two open intervals, hence open by Proposition 8, so $[0, 1]$ is closed by Proposition 11. Since $\overline{A}$ is the smallest closed set containing A (Proposition 10), $\overline{A} \subseteq [0, 1]$. Conversely, both 0 and 1 are points of closure of A : if I is an open interval containing 0, then I contains an interval $(-r, r)$ with $r > 0$, and by the Archimedean Property of $\mathbb{R}$ we may choose a natural number n with $1/n < \min\{r, 1\}$; then $1/n \in (0, 1) \cap I = A \cap I$. The same argument at the point 1, using the numbers $1 - 1/n$, shows that 1 is a point of closure of A . Since also $A \subseteq \overline{A}$, we get $[0, 1] = A \cup \{0, 1\} \subseteq \overline{A}$. Hence $\overline{A} = [0, 1]$, and by the identical argument (translate by 1) $\overline{B} = [1, 2]$.

Therefore

$$\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\} \neq \emptyset.$$

A second, more extreme example: let $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ be the rationals and the irrationals. These are disjoint by definition. Fix a real number x and an open interval I containing x ; whatever the type of I , there is an $r > 0$ with $(x - r, x + r) \subseteq I$, so it suffices to produce points of A and of B in each bounded interval (a, b) with $a < b$. By Theorem 2 the rational numbers are dense in $\mathbb{R}$, so (a, b) contains a rational number; hence every real number is a point of closure of A and $\overline{A} = \mathbb{R}$. The interval (a, b) also contains an irrational number: by Theorem 2 pick rationals q and q' with $a < q < q' < b$, and set $z = q + (q' - q)/\sqrt{2}$. Since $0 < 1/\sqrt{2} < 1$ we have $q < z < q'$, so $z \in (a, b)$; and z is irrational, for if z were rational then $\sqrt{2} = (q' - q)/(z - q)$ would be a quotient of nonzero rationals and so rational, contradicting the irrationality of $\sqrt{2}$ established in Section 1.2. Hence $\overline{B} = \mathbb{R}$ as well, and $\overline{A} \cap \overline{B} = \mathbb{R} \neq \emptyset$, although $A \cap B = \emptyset$.

The examples show that the operation $E \mapsto \overline{E}$ does not preserve disjointness: closure can create new common points on the shared boundary of two disjoint sets.

Problem 1.30. A point x is called an **accumulation point** of a set E provided it is a point of closure of $E \setminus \{x\}$.

- (i) Show that the set E' of accumulation points of E is a closed set.
- (ii) Show that $\overline{E} = E \cup E'$.

Solution. Throughout, recall the definition: x is a point of closure of a set S provided every open interval that contains x also contains a point of S . Thus $x \in E'$ means: every open interval containing x contains a point of E different from x .

(i) E' is closed.

By definition a set is closed provided it contains all of its points of closure, so let x be a point of closure of E' ; we must show $x \in E'$. Let I be an open interval containing x . We must produce a point of $E \setminus \{x\}$ inside I .

Since x is a point of closure of E' , the interval I contains some point $y \in E'$.

Case 1: $y = x$. Then $x \in E'$ and there is nothing more to prove.

Case 2: $y \neq x$. The set $I \setminus \{x\}$ is open: indeed I is open and $\mathbb{R} \setminus \{x\}$ is open (its complement $\{x\}$ is closed, since any $z \neq x$ has the open interval $(z - |z - x|, z + |z - x|)$ about it missing x , so z is not a point of closure of $\{x\}$), and the intersection of two open sets is open by Proposition 8. Since $y \in I \setminus \{x\}$, there is an $r > 0$ with $J := (y - r, y + r) \subseteq I \setminus \{x\}$. Now $y \in E'$, so y is a point of closure of $E \setminus \{y\}$, and J is an open interval containing y ; hence J contains a point $z \in E \setminus \{y\}$. Then $z \in E$, $z \in J \subseteq I$, and $z \neq x$ because J misses x . So $z \in (E \setminus \{x\}) \cap I$.

In either case I contains a point of $E \setminus \{x\}$. As I was an arbitrary open interval containing x , we conclude x is a point of closure of $E \setminus \{x\}$, that is, $x \in E'$. Therefore E' contains all its points of closure and so is closed.

(ii) $\overline{E} = E \cup E'$.

($\supseteq$) We always have $E \subseteq \overline{E}$. And $E' \subseteq \overline{E}$: if $x \in E'$, then every open interval containing x contains a point of $E \setminus \{x\}$, and such a point lies in E ; hence x is a point of closure of E , i.e. $x \in \overline{E}$. Thus $E \cup E' \subseteq \overline{E}$.

($\subseteq$) Let $x \in \overline{E}$. If $x \in E$ then $x \in E \cup E'$ and we are done. So suppose $x \notin E$. Then $E \setminus \{x\} = E$. Since $x \in \overline{E}$, every open interval containing x contains a point of $E = E \setminus \{x\}$; hence x is a point of closure of $E \setminus \{x\}$, i.e. $x \in E'$. In either case $x \in E \cup E'$.

Therefore $\overline{E} = E \cup E'$.

Two remarks worth recording. First, part (i) is not a corollary of Proposition 10: that proposition says $\overline{E}$ is closed, and by part (ii) $\overline{E} = E \cup E'$, which carries no information about E' by itself. Indeed E' can be closed while E is not, and E' can be a proper subset of the closed set $\overline{E}$; the direct argument above is what is needed. Second, combining (i) and (ii): E is closed if and only if $E' \subseteq E$, since E is closed exactly when $E = \overline{E} = E \cup E'$, and that holds exactly when $E' \subseteq E$.

Problem 1.31. A point x is called an **isolated point** of a set E provided there is an $r > 0$ for which $(x - r, x + r) \cap E = \{x\}$. Show that if a set E consists of isolated points, then it is countable.

Solution. Assume every point of E is an isolated point of E . For each $x \in E$ fix a number $r_x > 0$ with

$$(x - r_x, x + r_x) \cap E = \{x\}.$$

By Theorem 2 the rational numbers are dense in $\mathbb{R}$, so we may choose rational numbers p_x and q_x with

$$x - r_x < p_x < x < q_x < x + r_x.$$

(Apply Theorem 2 once to the pair $x - r_x < x$ and once to the pair $x < x + r_x$.) Define a map

$$\varphi : E \longrightarrow \mathbb{Q} \times \mathbb{Q}, \quad \varphi(x) = (p_x, q_x).$$

φ is one-to-one. Suppose $x, y \in E$ and $\varphi(x) = \varphi(y)$, that is $p_x = p_y$ and $q_x = q_y$. Then

$$y \in (p_y, q_y) = (p_x, q_x) \subseteq (x - r_x, x + r_x),$$

and $y \in E$, so $y \in (x - r_x, x + r_x) \cap E = \{x\}$. Hence $y = x$. Thus φ is one-to-one.

$\mathbb{Q} \times \mathbb{Q}$ is countable. Write

$$\mathbb{Q} \times \mathbb{Q} = \bigcup_{q \in \mathbb{Q}} (\{q\} \times \mathbb{Q}).$$

Each set $\{q\} \times \mathbb{Q}$ is equipotent to $\mathbb{Q}$, hence countable by Corollary 4, and the index set $\mathbb{Q}$ is countable by Corollary 4 as well. So this is the union of a countable collection of countable sets, which is countable by Corollary 6.

Conclusion. Since φ is one-to-one, E is equipotent to the subset $\varphi(E)$ of the countable set $\mathbb{Q} \times \mathbb{Q}$. By Theorem 3 a subset of a countable set is countable, so $\varphi(E)$ is countable, and therefore E is countable. A second proof, avoiding $\mathbb{Q} \times \mathbb{Q}$. For each natural number n set

$$E_n = \left\{ x \in E : \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap E = \{x\} \right\}.$$

Every $x \in E$ has some $r_x > 0$ as above, and by the Archimedean Property there is an n with $1/n < r_x$; then $(x - 1/n, x + 1/n) \cap E \subseteq (x - r_x, x + r_x) \cap E = \{x\}$ and this intersection contains x , so $x \in E_n$. Hence $E = \bigcup_{n=1}^{\infty} E_n$.

Fix n . Any two distinct points of E_n are at distance at least $1/n$: if $x, y \in E_n$ with $x \neq y$ and $|x - y| < 1/n$, then $y \in (x - 1/n, x + 1/n) \cap E = \{x\}$, so $y = x$, a contradiction. Consequently, for each natural number k the set $E_n \cap [-k, k]$ is finite. Indeed, partition $[-k, k]$ into the $2kn$ half-open intervals $[-k + \frac{j-1}{n}, -k + \frac{j}{n})$ for $j = 1, \dots, 2kn$, together with the single point $\{k\}$; two points lying in the same one of these $2kn + 1$ pieces are at distance strictly less than $1/n$, so by the pigeonhole principle $E_n \cap [-k, k]$ has at most $2kn + 1$ members.

Every real number lies in $[-k, k]$ for some natural number k , by the Archimedean Property applied to the pair 1 and $|x|$ when $x \neq 0$. Therefore $E_n = \bigcup_{k=1}^{\infty} (E_n \cap [-k, k])$ is a countable union of finite sets, hence countable by Corollary 6, and then $E = \bigcup_{n=1}^{\infty} E_n$ is a countable union of countable sets, hence countable by Corollary 6 again.

Problem 1.32. A point x is called an **interior point** of a set E if there is an $r > 0$ such that the open interval $(x - r, x + r)$ is contained in E . The set of interior points of E is called the **interior** of E , denoted by $\text{int } E$. Show that

- (i) E is open if and only if $E = \text{int } E$.
- (ii) E is dense if and only if $\text{int}(\mathbb{R} \setminus E) = \emptyset$.

Solution. Recall the two definitions in force. A set $\mathcal{O}$ of real numbers is open provided for each $x \in \mathcal{O}$ there is an $r > 0$ for which $(x - r, x + r)$ is contained in $\mathcal{O}$. A set E of real numbers is dense in $\mathbb{R}$ provided between any two real numbers there lies a member of E ; equivalently, every nonempty open interval (a, b) with $a < b$ contains a point of E .

First note the inclusion $\text{int } E \subseteq E$, valid for every set E : if $x \in \text{int } E$, choose $r > 0$ with $(x - r, x + r) \subseteq E$;

since $x \in (x - r, x + r)$, we get $x \in E$.

(i) Suppose E is open. Let $x \in E$. By the definition of openness there is an $r > 0$ with $(x - r, x + r) \subseteq E$, which is precisely the statement that x is an interior point of E . Hence $E \subseteq \text{int } E$, and combined with the inclusion above, $E = \text{int } E$.

Conversely suppose $E = \text{int } E$. Let $x \in E$. Then $x \in \text{int } E$, so there is an $r > 0$ with $(x - r, x + r) \subseteq E$. Since $x \in E$ was arbitrary, E is open.

(ii) Suppose first that E is dense in $\mathbb{R}$, and suppose, to get a contradiction, that $\text{int}(\mathbb{R} \sim E) \neq \emptyset$. Choose $x \in \text{int}(\mathbb{R} \sim E)$ and $r > 0$ with

$$(x - r, x + r) \subseteq \mathbb{R} \sim E, \quad \text{that is} \quad (x - r, x + r) \cap E = \emptyset.$$

But $x - r < x + r$, so by density there is a member of E lying between $x - r$ and $x + r$, i.e. a point of E in $(x - r, x + r)$. This contradicts the displayed line. Therefore $\text{int}(\mathbb{R} \sim E) = \emptyset$.

Conversely suppose $\text{int}(\mathbb{R} \sim E) = \emptyset$. Let $a < b$ be real numbers; we must find a member of E strictly between them. Suppose not, so that $(a, b) \cap E = \emptyset$, i.e. $(a, b) \subseteq \mathbb{R} \sim E$. Put

$$x = \frac{a + b}{2}, \quad r = \frac{b - a}{2} > 0.$$

Then $(x - r, x + r) = (a, b) \subseteq \mathbb{R} \sim E$, so x is an interior point of $\mathbb{R} \sim E$ and hence $\text{int}(\mathbb{R} \sim E) \neq \emptyset$, a contradiction. Therefore (a, b) contains a point of E for all $a < b$, i.e. E is dense in $\mathbb{R}$.

It is worth recording what falls out of the two parts. The set $\text{int } E$ is always open: it is exactly the union of all open intervals contained in E , and the union of any collection of open sets is open by Proposition 8. Moreover $\text{int } E$ is the largest open set contained in E , since any open $\mathcal{O} \subseteq E$ has every one of its points interior to E , so $\mathcal{O} = \text{int } \mathcal{O} \subseteq \text{int } E$ by part (i). Dually, part (ii) says E is dense exactly when $\mathbb{R} \sim E$ contains no nonempty open set.

Problem 1.33. Show that the Nested Set Theorem is false if F_1 is unbounded.

Solution. The Nested Set Theorem reads: let $\{F_n\}_{n=1}^\infty$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 is bounded; then $\bigcap_{n=1}^\infty F_n \neq \emptyset$. We exhibit a descending countable collection of nonempty closed sets whose first member is unbounded and whose intersection is empty; this shows the boundedness hypothesis on F_1 cannot be dropped.

For each natural number n define

$$F_n = [n, \infty) = \{x \in \mathbb{R} : x \geq n\}.$$

Each F_n is nonempty, since $n \in F_n$.

Each F_n is closed. Its complement is $\mathbb{R} \sim F_n = (-\infty, n)$, which is an open interval and hence an open set. By Proposition 11, a set of real numbers is closed if and only if its complement in $\mathbb{R}$ is open, so F_n is closed.

The collection is descending: if $x \geq n + 1$ then $x \geq n$, so $F_{n+1} \subseteq F_n$ for every natural number n .

$F_1 = [1, \infty)$ is unbounded: it is not bounded above, for if b were an upper bound of F_1 then $b \geq 1 > 0$ and, by the Archimedean Property of $\mathbb{R}$ applied to the pair 1 and b , there is a natural number n with $n \cdot 1 > b$; but $n \in F_1$, so b is not an upper bound, a contradiction.

Finally the intersection is empty. Let x be any real number. If $x \leq 0$ then $x \notin F_1$. If $x > 0$, then by the Archimedean Property applied to the pair 1 and x there is a natural number n with $n > x$, and then $x \notin F_n$. In either case $x \notin \bigcap_{n=1}^\infty F_n$. Hence

$$\bigcap_{n=1}^\infty [n, \infty) = \emptyset.$$

So the conclusion of the Nested Set Theorem fails for this descending collection of nonempty closed sets, the only hypothesis violated being the boundedness of F_1 .

The failure is not an artifact of using unbounded sets at every stage: the proof of the Nested Set Theorem invokes the Heine-Borel Theorem for the closed set F_1 , and the Heine-Borel Theorem requires

the set to be both closed and bounded. Concretely, in the proof one covers F_1 by the ascending open sets $\mathcal{O}_n = \mathbb{R} \sim F_n$ and extracts a finite subcover; for $F_1 = [1, \infty)$ the open cover $\{(-\infty, n)\}_{n=1}^\infty$ of F_1 has no finite subcover, since a finite union of these is $(-\infty, N)$ for the largest index N present, and $N \in F_1 \sim (-\infty, N)$.

Problem 1.34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.

Solution. Here is the precise setting. Let $\mathbb{R}$ denote a set satisfying the Field Axioms and the Positivity Axioms, so that all the algebra of Section 1.1 and the whole vocabulary of Section 1.4 – open interval, open set, closed set, point of closure, closure, cover, open cover, finite subcover, descending collection, bounded set – is available, together with Propositions 8, 10, 11 and 12, whose proofs use only the field and order axioms. Consider the following three assertions about such an $\mathbb{R}$.

(C) The Completeness Axiom. Every nonempty set of real numbers that is bounded above has a least upper bound.

(HB) The Heine-Borel assertion. If F is a closed and bounded set of real numbers, then every open cover of F has a finite subcover.

(NST) The Nested Set assertion. If $\{F_n\}_{n=1}^\infty$ is a descending countable collection of nonempty closed sets of real numbers for which F_1 is bounded, then $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

We prove $(C) \Rightarrow (HB) \Rightarrow (C)$ and $(C) \Rightarrow (NST) \Rightarrow (C)$. The two forward implications are exactly the two theorems proved in the text: the proof of the Heine-Borel Theorem uses (C) once, to form $c = \sup E$ for the set E of numbers $x \in [a, b]$ such that $[a, x]$ can be covered by a finite number of the sets of the cover, and otherwise uses only the definitions of open set, closed set and cover together with Proposition 11; and the proof of the Nested Set Theorem uses (C) only through its appeal to the Heine-Borel Theorem, its remaining ingredients being Proposition 11 and De Morgan's Identities. So it remains to prove the two converses.

Part 1. (HB) implies (C).

Assume (HB). Let E be a nonempty set of real numbers that is bounded above, and suppose, to get a contradiction, that E has no least upper bound.

First, E has no largest member: a largest member of E would be an upper bound of E that is less than or equal to every other upper bound, hence a least upper bound. Consequently no member of E is an upper bound of E , for such a member would be a largest member.

Fix $a \in E$ and fix an upper bound b of E . Then $a \leq b$, and $a \neq b$ since b is an upper bound while a is not. So $a < b$, and $[a, b]$ is a nonempty bounded set. It is closed: its complement $(-\infty, a) \cup (b, \infty)$ is a union of open intervals, hence open by Proposition 8, so $[a, b]$ is closed by Proposition 11.

Now build an open cover of $[a, b]$. Let $x \in [a, b]$.

Case N (x is not an upper bound of E). Then there is an $e \in E$ with $e > x$. Set $\mathcal{O}_x = (-\infty, e)$. This is an open interval containing x , and no point of $\mathcal{O}_x$ is an upper bound of E , since every $z \in \mathcal{O}_x$ satisfies $z < e$ with $e \in E$.

Case U (x is an upper bound of E). Since E has no least upper bound, x is not the least upper bound, so there is an upper bound y of E with $y < x$. Set $\mathcal{O}_x = (y, \infty)$. This is an open interval containing x , and every point of $\mathcal{O}_x$ is an upper bound of E : if $z > y$ then for each $e \in E$ we have $e \leq y < z$.

The collection $\{\mathcal{O}_x\}_{x \in [a, b]}$ is an open cover of $[a, b]$. By (HB) there is a finite subcover $\mathcal{O}_{x_1}, \dots, \mathcal{O}_{x_m}$ of $[a, b]$. Call the index i of type N or of type U according to which case produced $\mathcal{O}_{x_i}$.

Suppose first that no index is of type U. Then every $\mathcal{O}_{x_i}$ is of the form $(-\infty, e_i)$ with $e_i \in E$. Let $e^* = \max\{e_1, \dots, e_m\}$, which exists because the list is finite and $\mathbb{R}$ is totally ordered; then $e^* \in E$ and $\bigcup_{i=1}^m \mathcal{O}_{x_i} = (-\infty, e^*)$. Since this covers $[a, b]$ and $b \in [a, b]$, we get $b < e^*$. But b is an upper bound of E and $e^* \in E$, so $e^* \leq b$. This is a contradiction.

So at least one index is of type U. Let $y_{j_1}, \dots, y_{j_k}$ be the numbers produced in Case U for those indices, and let $y^* = \min\{y_{j_1}, \dots, y_{j_k}\}$; each y_j is an upper bound of E , hence so is y^* . Define

$$c = \min\{y^*, b\}.$$

Then c is an upper bound of E : for each $e \in E$ we have $e \leq y^*$ and $e \leq b$, so $e \leq c$. Since $a \in E$ and c is an upper bound, $a \leq c$; and $c \leq b$ by construction. Hence $c \in [a, b]$, so c lies in some $\mathcal{O}_{x_i}$ of the finite subcover.

If that index i is of type N, then $\mathcal{O}_{x_i} = (-\infty, e_i)$ with $e_i \in E$, so $c < e_i$, contradicting the fact that c is an upper bound of E . Therefore i is of type U, so $\mathcal{O}_{x_i} = (y_j, \infty)$ for one of the numbers y_j above, whence $c > y_j \geq y^*$. But $c = \min\{y^*, b\} \leq y^*$. This is a contradiction.

Both cases being impossible, E must have a least upper bound. Thus (HB) implies (C).

Part 2. (NST) implies (C).

Assume (NST). We argue in two steps.

Step 1: (NST) implies the Archimedean Property, equivalently that $\mathbb{N}$ is not bounded above in $\mathbb{R}$. (The two are equivalent without completeness: if $\mathbb{N}$ is bounded above by M , then for the positive pair $a = 1, b = \max\{M, 1\}$ we have $n \cdot 1 \leq b$ for every natural number n , so the Archimedean Property fails; conversely if the Archimedean Property fails for the positive pair a, b , then $n \leq b/a$ for every natural number n and $\mathbb{N}$ is bounded above.)

Suppose, for a contradiction, that $\mathbb{N}$ is bounded above by some M ; note $M \geq 1$ since $1 \in \mathbb{N}$. For each natural number n put

$$T_n = \{m \in \mathbb{N} : m \geq n\}, \quad F_n = \overline{T_n}.$$

Then each F_n is nonempty (it contains n) and closed, by Proposition 10. The collection is descending: $T_{n+1} \subseteq T_n$ implies $\overline{T_{n+1}} \subseteq \overline{T_n}$, again by Proposition 10. Finally F_1 is bounded: $T_1 \subseteq [1, M]$, and $[1, M]$ is closed (its complement is open by Proposition 8, so Proposition 11 applies), so by the minimality clause of Proposition 10 we get $F_1 = \overline{T_1} \subseteq [1, M]$.

By (NST) there is a point $x \in \bigcap_{n=1}^{\infty} F_n$. Fix n . Since x is a point of closure of T_n and $(x - \frac{1}{2}, x + \frac{1}{2})$ is an open interval containing x , there is a natural number $m \geq n$ with $|x - m| < \frac{1}{2}$.

Apply this with $n = 1$ to obtain $m_1 \in \mathbb{N}$ with $|x - m_1| < \frac{1}{2}$. Apply it again with $n = m_1 + 1$ to obtain $m_2 \in \mathbb{N}$ with $m_2 \geq m_1 + 1$ and $|x - m_2| < \frac{1}{2}$. Then

$$1 \leq m_2 - m_1 = (m_2 - x) + (x - m_1) \leq |x - m_2| + |x - m_1| < \frac{1}{2} + \frac{1}{2} = 1,$$

which is absurd. Hence $\mathbb{N}$ is not bounded above, i.e. the Archimedean Property holds.

Step 2: (NST) together with the Archimedean Property implies (C).

Let E be nonempty and bounded above. If E has a largest member M_0 , then M_0 is an upper bound of E and every upper bound u of E satisfies $u \geq M_0$ because $M_0 \in E$; so M_0 is the least upper bound and we are done. So assume E has no largest member. Then no member of E is an upper bound of E , since a member of E that is an upper bound of E is a largest member of E .

Choose $a_1 \in E$ and an upper bound b_1 of E . Then $a_1 \leq b_1$, and $a_1 \neq b_1$ because b_1 is an upper bound of E while a_1 , being a member of E , is not; so $a_1 < b_1$, the number a_1 is not an upper bound of E , and b_1 is. Define a_n, b_n recursively by bisection: given $a_n < b_n$ with a_n not an upper bound of E and b_n an upper bound of E , let $m_n = (a_n + b_n)/2$ and set

$$\begin{aligned} a_{n+1} &= a_n, & b_{n+1} &= m_n & \text{if } m_n \text{ is an upper bound of } E, \\ a_{n+1} &= m_n, & b_{n+1} &= b_n & \text{if } m_n \text{ is not an upper bound of } E. \end{aligned}$$

In either case a_{n+1} is not an upper bound of E , b_{n+1} is an upper bound of E , $a_n \leq a_{n+1} < b_{n+1} \leq b_n$, and $b_{n+1} - a_{n+1} = (b_n - a_n)/2$. An induction gives

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \quad \text{for every natural number } n.$$

Put $F_n = [a_n, b_n]$. Each F_n is nonempty and closed (complement open by Proposition 8, then Proposition 11), the collection is descending because $a_n \leq a_{n+1} < b_{n+1} \leq b_n$, and $F_1 = [a_1, b_1]$ is bounded. By (NST) there is a point

$$c \in \bigcap_{n=1}^{\infty} F_n, \quad \text{so} \quad a_n \leq c \leq b_n \text{ for every } n.$$

We record one consequence of the Archimedean Property: for every $\epsilon > 0$ there is an n with $b_n - a_n < \epsilon$. Indeed, choose a natural number n with $n\epsilon > b_1 - a_1$. An induction shows $2^{n-1} \geq n$ for every natural number n (it holds at $n = 1$, and if $2^{n-1} \geq n$ then $2^n \geq 2n \geq n + 1$). Hence

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \leq \frac{b_1 - a_1}{n} < \epsilon.$$

c is an upper bound of E . Suppose not; then there is $e \in E$ with $e > c$. Put $\epsilon = e - c > 0$ and choose n with $b_n - a_n < \epsilon$. Since $a_n \leq c$,

$$b_n = a_n + (b_n - a_n) \leq c + (b_n - a_n) < c + \epsilon = e,$$

so $b_n < e$ with $e \in E$, contradicting that b_n is an upper bound of E .

c is the least upper bound of E . Let u be any upper bound of E and suppose $u < c$. Put $\delta = c - u > 0$ and choose n with $b_n - a_n < \delta$. Since $c \leq b_n$,

$$a_n = b_n - (b_n - a_n) \geq c - (b_n - a_n) > c - \delta = u.$$

So $a_n > u$. But a_n is not an upper bound of E , so there is an $e \in E$ with $e > a_n > u$, contradicting that u is an upper bound of E . Hence $u \geq c$ for every upper bound u .

Therefore c is the least upper bound of E , and (C) holds.

Conclusion. Assertions (C), (HB) and (NST) are equivalent over the Field and Positivity Axioms. In particular each of the Heine-Borel Theorem and the Nested Set Theorem is a faithful substitute for the Completeness Axiom: one may take either as the completeness postulate for $\mathbb{R}$ and recover the least upper bound property, and hence all of the analysis built on it. Note also that the equivalence gives (HB) $\Leftrightarrow$ (NST) directly.

Problem 1.35. Show that the collection of Borel sets is the smallest σ -algebra that contains the closed sets.

Solution. Write $\mathcal{B}$ for the collection of Borel sets, that is, by definition, the smallest σ -algebra of sets of real numbers that contains all of the open sets of real numbers. Let $\mathcal{F}$ denote the collection of all closed sets of real numbers, and let $\mathcal{C}$ denote the smallest σ -algebra of sets of real numbers that contains $\mathcal{F}$. Both $\mathcal{B}$ and $\mathcal{C}$ exist. Indeed, the collection $2^{\mathbb{R}}$ of all sets of real numbers is a σ -algebra containing every open set and every closed set, so in each case the family of σ -algebras being intersected is nonempty; Proposition 13 then tells us that the intersection of all σ -algebras of subsets of $\mathbb{R}$ containing a given collection is itself a σ -algebra containing that collection, and it is the smallest such in the sense that it is contained in every σ -algebra containing the collection.

We must show $\mathcal{B} = \mathcal{C}$. The whole argument rests on Proposition 11: a set of real numbers is open if and only if its complement in $\mathbb{R}$ is closed. Equivalently, E is closed if and only if $\mathbb{R} \sim E$ is open.

$\mathcal{C} \subseteq \mathcal{B}$. We check that $\mathcal{B}$ contains every closed set. Let F be a closed set of real numbers. By Proposition 11 the set $\mathbb{R} \sim F$ is open, so $\mathbb{R} \sim F \in \mathcal{B}$ because $\mathcal{B}$ contains all the open sets. A σ -algebra is closed with respect to the formation of complements, so

$$F = \mathbb{R} \sim (\mathbb{R} \sim F) \in \mathcal{B}.$$

Thus $\mathcal{B}$ is a σ -algebra of subsets of $\mathbb{R}$ containing $\mathcal{F}$. Since $\mathcal{C}$ is the smallest such σ -algebra, $\mathcal{C} \subseteq \mathcal{B}$.

$\mathcal{B} \subseteq \mathcal{C}$. We check that $\mathcal{C}$ contains every open set. Let $\mathcal{O}$ be an open set of real numbers. By Proposition 11 the set $\mathbb{R} \sim \mathcal{O}$ is closed, so $\mathbb{R} \sim \mathcal{O} \in \mathcal{F} \subseteq \mathcal{C}$. Since $\mathcal{C}$ is closed with respect to the formation of complements,

$$\mathcal{O} = \mathbb{R} \sim (\mathbb{R} \sim \mathcal{O}) \in \mathcal{C}.$$

Thus $\mathcal{C}$ is a σ -algebra of subsets of $\mathbb{R}$ containing all the open sets. Since $\mathcal{B}$ is by definition the smallest such σ -algebra, $\mathcal{B} \subseteq \mathcal{C}$.

Combining the two inclusions, $\mathcal{B} = \mathcal{C}$: the collection of Borel sets is the smallest σ -algebra of sets of real numbers that contains all of the closed sets.

The point is entirely formal, and it is worth isolating it. If $\mathcal{S}$ is any collection of subsets of a set X and $\mathcal{S}^c = \{X \setminus S : S \in \mathcal{S}\}$, then a σ -algebra contains $\mathcal{S}$ if and only if it contains $\mathcal{S}^c$, since σ -algebras are closed with respect to complements. Hence the smallest σ -algebra containing $\mathcal{S}$ equals the smallest σ -algebra containing $\mathcal{S}^c$. Proposition 11 says exactly that the collection of closed sets is the collection of complements of the open sets, and the result follows.

1.6 Problems 1.36–1.42

Problem 1.36. Show that the collection of Borel sets is the smallest σ -algebra that contains intervals of the form $[a, b)$, where $a < b$.

Solution. Let $\mathcal{I} = \{[a, b) \mid a < b\}$ and let $\mathcal{A}$ be the smallest σ -algebra of sets of real numbers that contains $\mathcal{I}$; this exists by Proposition 13, being the intersection of all σ -algebras containing $\mathcal{I}$ (there is at least one, namely $2^{\mathbb{R}}$). Let $\mathcal{B}$ denote the collection of Borel sets, that is, the smallest σ -algebra containing all the open sets. We show $\mathcal{A} = \mathcal{B}$ by two inclusions.

Step 1. We show $\mathcal{A} \subseteq \mathcal{B}$.

It suffices to show $\mathcal{I} \subseteq \mathcal{B}$, for then $\mathcal{B}$ is a σ -algebra containing $\mathcal{I}$ and the minimality of $\mathcal{A}$ forces $\mathcal{A} \subseteq \mathcal{B}$. Let $a < b$. I claim

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right).$$

Indeed, if $a \leq x < b$ then $a - 1/n < a \leq x < b$ for every n , so x lies in the intersection. Conversely, if x belongs to every $(a - 1/n, b)$ then $x < b$ and $x > a - 1/n$ for all n ; by the Archimedean Property of $\mathbb{R}$ the latter forces $x \geq a$ (were $x < a$, choose n with $1/n < a - x$ to get $x < a - 1/n$). Hence $x \in [a, b)$. Each $(a - 1/n, b)$ is an open set, hence a Borel set, and a σ -algebra is closed with respect to the formation of countable intersections. Therefore $[a, b) \in \mathcal{B}$, which proves the claim.

Step 2. We show $\mathcal{B} \subseteq \mathcal{A}$.

It suffices to show that every open set belongs to $\mathcal{A}$, for then $\mathcal{A}$ is a σ -algebra containing the open sets and the minimality of $\mathcal{B}$ forces $\mathcal{B} \subseteq \mathcal{A}$.

First we check that every open interval belongs to $\mathcal{A}$.

- A bounded open interval (a, b) with $a < b$: choose, by the Archimedean Property, a natural number N with $1/N < b - a$. Then

$$(a, b) = \bigcup_{n=N}^{\infty} \left[a + \frac{1}{n}, b\right).$$

Each set on the right belongs to $\mathcal{I}$ because $a + 1/n \leq a + 1/N < b$, and it is contained in (a, b) ; conversely if $a < x < b$, choose $n \geq N$ with $1/n < x - a$, so that $x \in [a + 1/n, b)$. Since $\mathcal{A}$ is closed with respect to countable unions, $(a, b) \in \mathcal{A}$.

- The interval (a, ∞) : it equals $\bigcup_{n=1}^{\infty} (a, a + n)$, a countable union of bounded open intervals, hence lies in $\mathcal{A}$.
- The interval $(-\infty, b)$: it equals $\bigcup_{n=1}^{\infty} (b - n, b)$, hence lies in $\mathcal{A}$.
- All of $\mathbb{R}$: it equals $\bigcup_{n=1}^{\infty} (-n, n)$, hence lies in $\mathcal{A}$.

Now let $\mathcal{O}$ be an open set of real numbers. If $\mathcal{O} = \emptyset$ then $\mathcal{O} \in \mathcal{A}$ by the very definition of a σ -algebra. Otherwise, by Proposition 9, $\mathcal{O}$ is the disjoint union of a countable collection $\{I_k\}$ of open intervals. Each I_k belongs to $\mathcal{A}$ by the preceding paragraph, and $\mathcal{A}$ is closed with respect to the formation of countable unions, so $\mathcal{O} \in \mathcal{A}$.

Thus every open set belongs to $\mathcal{A}$, and Step 2 is proved.

Combining the two steps, $\mathcal{A} = \mathcal{B}$: the collection of Borel sets is the smallest σ -algebra containing the intervals $[a, b)$ with $a < b$.

Problem 1.37. Show that each open set is an F_σ set.

Solution. Recall that a countable union of closed sets is called an F_σ set.

Let $\mathcal{O}$ be an open set of real numbers and let $C = \mathbb{R} \sim \mathcal{O}$, which is closed by Proposition 11. For each natural number n define

$$F_n = \left\{ x \in \mathbb{R} \mid |x| \leq n \text{ and } |x - c| \geq \frac{1}{n} \text{ for every } c \in C \right\}.$$

(If $C = \emptyset$ the second condition is vacuous and $F_n = [-n, n]$.)

Each F_n is closed. Indeed, let x be a point of closure of F_n and let $\epsilon > 0$. The open interval $(x - \epsilon, x + \epsilon)$ contains a point $x' \in F_n$. From $|x'| \leq n$ we get $|x| \leq |x'| + \epsilon \leq n + \epsilon$, and since $\epsilon > 0$ is arbitrary, $|x| \leq n$. Moreover, for any $c \in C$ we have $|x' - c| \geq 1/n$, so

$$|x - c| \geq |x' - c| - |x - x'| > \frac{1}{n} - \epsilon.$$

Again since $\epsilon > 0$ is arbitrary, $|x - c| \geq 1/n$ for every $c \in C$. Hence $x \in F_n$, and F_n contains all of its points of closure, that is, F_n is closed.

We now claim

$$\mathcal{O} = \bigcup_{n=1}^{\infty} F_n.$$

First, $F_n \subseteq \mathcal{O}$ for each n . For if $x \in F_n$ and we had $x \in C$, then taking $c = x$ in the defining condition would give $0 = |x - x| \geq 1/n$, which is false. So $x \notin C$, that is, $x \in \mathcal{O}$.

Conversely, let $x \in \mathcal{O}$. Since $\mathcal{O}$ is open there is an $r > 0$ with $(x - r, x + r) \subseteq \mathcal{O}$, and therefore no point c of C satisfies $|x - c| < r$; that is, $|x - c| \geq r$ for all $c \in C$. By the Archimedean Property of $\mathbb{R}$ choose a natural number n with $n \geq |x|$ and $1/n \leq r$. Then $|x| \leq n$ and $|x - c| \geq r \geq 1/n$ for every $c \in C$, so $x \in F_n$.

This proves the claim, and exhibits $\mathcal{O}$ as a countable union of closed sets. Hence every open set is an F_σ set.

Two remarks. First, taking complements and using De Morgan's Identities together with Proposition 11, the result is equivalent to the dual statement that every closed set is a G_δ set. Second, an alternative proof runs through Proposition 9: a nonempty open set is the union of a countable collection of open intervals, and each open interval is an F_σ , since for $a < b$

$$(a, b) = \bigcup_{n \geq N} \left[a + \frac{1}{n}, b - \frac{1}{n} \right] \quad \text{where } \frac{2}{N} < b - a,$$

while $(a, \infty) = \bigcup_n [a + 1/n, a + n]$, $(-\infty, b) = \bigcup_n [b - n, b - 1/n]$, and $\mathbb{R} = \bigcup_n [-n, n]$; a countable union of F_σ sets is again an F_σ set.

Problem 1.38. We call an extended real number a **cluster point** of a sequence $\{a_n\}$ if a subsequence converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

Solution. Throughout, write

$$s_n = \sup\{a_k \mid k \geq n\}, \quad i_n = \inf\{a_k \mid k \geq n\},$$

these being extended real numbers. The sequence $\{s_n\}$ is decreasing and $\{i_n\}$ is increasing, since the sets over which the supremum and infimum are taken shrink as n increases. By definition $L := \limsup\{a_n\} = \lim_{n \rightarrow \infty} s_n$ and $\ell := \liminf\{a_n\} = \lim_{n \rightarrow \infty} i_n$, the limits being taken in the extended sense; because $\{s_n\}$ is decreasing, $L = \inf_n s_n$, and because $\{i_n\}$ is increasing, $\ell = \sup_n i_n$. This last identification is exactly the Lemma established in the solution to Problem 39 below (a decreasing sequence of extended real numbers converges in the extended sense to its infimum, an increasing one to its supremum); for a bounded monotone sequence of real numbers it is Theorem 15 together with its proof, and the unbounded and infinite cases are checked there directly. Nothing in Problem 39 uses the present problem, so there is no circularity.

We prove the assertion about $\limsup$ in full; the assertion about $\liminf$ then follows by a reflection argument given at the end.

Part 1. L is a cluster point of $\{a_n\}$.

Case (a): $L = \infty$. Then $\{a_n\}$ is not bounded above, since a bound M would give $s_n \leq M$ for all n and hence $L \leq M$. Deleting finitely many terms from a sequence that is unbounded above leaves a sequence that is still unbounded above, so for every index N and every real c there is an index $k \geq N$ with $a_k > c$. Choose n_1 with $a_{n_1} > 1$; having chosen $n_1 < \dots < n_{k-1}$, choose $n_k > n_{k-1}$ with $a_{n_k} > k$. Then $\{n_k\}$ is strictly increasing and $\{a_{n_k}\} \rightarrow \infty$, so $L = \infty$ is a cluster point.

Case (b): $L = -\infty$. Then $\{s_n\} \rightarrow -\infty$, so for each natural number k there is an index N_k with $s_{N_k} < -k$. Choose $n_1 \geq N_1$; having chosen n_{k-1} , let n_k be any index with $n_k > n_{k-1}$ and $n_k \geq N_k$. Since $\{s_n\}$ is decreasing and $n_k \geq N_k$,

$$a_{n_k} \leq s_{n_k} \leq s_{N_k} < -k.$$

Hence $\{a_{n_k}\} \rightarrow -\infty$, and $L = -\infty$ is a cluster point.

Case (c): $L \in \mathbb{R}$. We claim that for every $\epsilon > 0$ and every index N there is an index $k \geq N$ with $|a_k - L| < \epsilon$. Indeed, since $\{s_n\}$ decreases to L , choose $n \geq N$ with $s_n < L + \epsilon$; also $s_n \geq L$, because $\{s_n\}$ is decreasing with limit L . From $s_n \geq L > L - \epsilon$ and the definition of supremum there is an index $k \geq n$ with $a_k > s_n - \epsilon \geq L - \epsilon$. On the other hand $a_k \leq s_n < L + \epsilon$ since $k \geq n$. As $k \geq n \geq N$, the claim is proved.

Now build a subsequence: apply the claim with $\epsilon = 1$ and $N = 1$ to get n_1 with $|a_{n_1} - L| < 1$; having chosen n_{k-1} , apply the claim with $\epsilon = 1/k$ and $N = n_{k-1} + 1$ to get $n_k > n_{k-1}$ with $|a_{n_k} - L| < 1/k$. By the Archimedean Property, $\{a_{n_k}\} \rightarrow L$. So L is a cluster point.

Part 2. Every cluster point c of $\{a_n\}$ satisfies $c \leq L$.

If $L = \infty$ there is nothing to prove, so assume $L < \infty$ and let t be any real number with $t > L$. Since $L = \inf_n s_n < t$, there is an index N with $s_N < t$, and therefore

$$a_n \leq s_N < t \quad \text{for all } n \geq N.$$

Let c be a cluster point, say $\{a_{n_k}\} \rightarrow c$ where $\{n_k\}$ is strictly increasing. A strictly increasing sequence of natural numbers satisfies $n_k \geq k$ for every k , by induction: $n_1 \geq 1$, and $n_k > n_{k-1} \geq k - 1$ forces $n_k \geq k$. Hence $n_k \geq k \geq N$ for $k \geq N$, and so $a_{n_k} < t$ for all $k \geq N$. If $c = -\infty$ then certainly $c \leq t$. If c were $+\infty$ then a_{n_k} would eventually exceed t , a contradiction. If $c \in \mathbb{R}$, then $c \leq t$ by Proposition 14 applied to the convergent sequence $\{a_{n_k}\}_{k \geq N}$, all of whose terms are at most t . In every case $c \leq t$. This holds for every real $t > L$. If $L \in \mathbb{R}$, letting t decrease to L gives $c \leq L$. If $L = -\infty$, then $c \leq t$ for every real number t , which forces $c = -\infty = L$.

By Parts 1 and 2, the set of cluster points of $\{a_n\}$ is nonempty, contains L , and has L as an upper bound; hence $L = \limsup\{a_n\}$ is its largest element.

Part 3. The statement for $\liminf$.

Consider the sequence $\{-a_n\}$. For a strictly increasing sequence $\{n_k\}$ of indices and an extended real number c , we have $\{a_{n_k}\} \rightarrow c$ if and only if $\{-a_{n_k}\} \rightarrow -c$: for finite c this is Theorem 18 with $\alpha = -1$, $\beta = 0$ (or directly, since $|(-a_{n_k}) - (-c)| = |a_{n_k} - c|$), and for $c = \pm\infty$ it is immediate from the definition of convergence to infinity. Consequently the set of cluster points of $\{-a_n\}$ is exactly the set of negatives of the cluster points of $\{a_n\}$.

By Part 1 and Part 2 applied to $\{-a_n\}$, the largest cluster point of $\{-a_n\}$ is $\limsup\{-a_n\}$. By Proposition 19(iii) (with $\{-a_n\}$ in place of $\{a_n\}$),

$$\limsup\{-a_n\} = -\liminf\{a_n\}.$$

Since negation reverses order among extended real numbers and carries cluster points of $\{a_n\}$ onto cluster points of $\{-a_n\}$, the number $-\limsup\{-a_n\} = \liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$. This completes the proof.

Problem 1.39. Prove Proposition 19. [Proposition 19. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. (i) $\limsup\{a_n\} = \ell \in \mathbb{R}$ if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$. (ii) $\limsup\{a_n\} = \infty$ if and only if $\{a_n\}$ is not bounded above. (iii) $\limsup\{a_n\} = -\liminf\{-a_n\}$. (iv) A sequence of real numbers $\{a_n\}$ converges to an extended real number a if and only if $\liminf\{a_n\} = \limsup\{a_n\} = a$. (v) If $a_n \leq b_n$ for all n , then $\limsup\{a_n\} \leq \liminf\{b_n\}$.]

Solution. Throughout put

$$s_n = \sup\{a_k \mid k \geq n\}, \quad i_n = \inf\{a_k \mid k \geq n\},$$

so that $\limsup\{a_n\} = \lim_{n \rightarrow \infty} s_n$ and $\liminf\{a_n\} = \lim_{n \rightarrow \infty} i_n$. The sets $\{a_k \mid k \geq n\}$ decrease as n increases, so $\{s_n\}$ is decreasing and $\{i_n\}$ is increasing, and clearly $i_n \leq a_n \leq s_n$ for every n .

Lemma. A decreasing sequence $\{s_n\}$ of extended real numbers converges in the extended sense to $\inf_n s_n$, and an increasing sequence $\{i_n\}$ converges to $\sup_n i_n$.

Proof of the Lemma. Consider the decreasing case. If $s_n = \infty$ for all n , both sides equal ∞ . Otherwise fix N with $s_N < \infty$; then $\{s_n\}_{n \geq N}$ is a decreasing sequence of extended reals none of which is ∞ , and $\inf_n s_n = \inf_{n \geq N} s_n$. If some $s_m = -\infty$ then $s_n = -\infty$ for all $n \geq m$ and both sides are $-\infty$. So assume $s_n \in \mathbb{R}$ for $n \geq N$. If this real sequence is bounded below it converges to its infimum by Theorem 15 (as in the proof there, the limit of a decreasing bounded sequence is its infimum); if it is not bounded below, then for each real c there is $M \geq N$ with $s_M < c$, hence $s_n \leq s_M < c$ for $n \geq M$, so $\{s_n\} \rightarrow -\infty = \inf_n s_n$. In each case the conclusion was reached for the tail $\{s_n\}_{n \geq N}$, and altering or ignoring finitely many initial terms affects neither convergence nor the value of the limit, since every assertion of convergence quantifies only over indices beyond some threshold. The increasing case follows by the same argument, or by applying the decreasing case to $\{-i_n\}$. $\square$

So $\limsup\{a_n\} = \inf_n s_n$ and $\liminf\{a_n\} = \sup_n i_n$; we use this freely.

Proof of (i).

Suppose first $\limsup\{a_n\} = \ell \in \mathbb{R}$, and let $\epsilon > 0$. Since $\ell = \inf_n s_n < \ell + \epsilon$, there is an index N with $s_N < \ell + \epsilon$, and therefore $a_n \leq s_N < \ell + \epsilon$ for every $n \geq N$. Thus at most the indices $n < N$ can satisfy $a_n > \ell + \epsilon$: only finitely many. On the other hand, for every index N we have $s_N \geq \inf_n s_n = \ell > \ell - \epsilon$, and since s_N is the supremum of $\{a_k \mid k \geq N\}$ there is some $k \geq N$ with $a_k > \ell - \epsilon$. As N was arbitrary, the set of indices n with $a_n > \ell - \epsilon$ is not contained in any finite initial segment, hence is infinite.

Conversely, suppose that for each $\epsilon > 0$ there are infinitely many indices n with $a_n > \ell - \epsilon$ and only finitely many with $a_n > \ell + \epsilon$. Taking $\epsilon = 1$, only finitely many terms exceed $\ell + 1$, so $\{a_n\}$ is bounded above and each s_n is a real number. Let $\epsilon > 0$. Let N be an index strictly greater than each of the finitely many indices n with $a_n > \ell + \epsilon$. Then $a_k \leq \ell + \epsilon$ for all $k \geq N$, so $s_n \leq s_N \leq \ell + \epsilon$ for $n \geq N$. Also, for every n there are infinitely many indices k with $a_k > \ell - \epsilon$, hence some such k with $k \geq n$, so $s_n \geq a_k > \ell - \epsilon$. Therefore

$$\ell - \epsilon < s_n \leq \ell + \epsilon \quad \text{for all } n \geq N,$$

that is, $|s_n - \ell| \leq \epsilon$ for $n \geq N$. Given $\epsilon' > 0$, running this argument with $\epsilon = \epsilon'/2$ produces an N with $|s_n - \ell| \leq \epsilon'/2 < \epsilon'$ for all $n \geq N$. Hence $\{s_n\} \rightarrow \ell$, that is, $\limsup\{a_n\} = \ell$.

Proof of (ii).

If $\{a_n\}$ is not bounded above, then for each n the set $\{a_k \mid k \geq n\}$ is also not bounded above (it differs from $\{a_k \mid k \geq 1\}$ by the finitely many terms $a_1, \dots, a_{n-1}$, and adjoining finitely many real numbers to a set cannot turn an unbounded-above set into a bounded one). Hence $s_n = \infty$ for every n , and $\limsup\{a_n\} = \inf_n s_n = \infty$.

Conversely, if $\{a_n\}$ is bounded above by a real number M , then M is an upper bound for each set $\{a_k \mid k \geq n\}$, so $s_n \leq M$ for all n and therefore $\limsup\{a_n\} = \inf_n s_n \leq M < \infty$.

Proof of (iii).

For a nonempty set S of real numbers, $\sup S = -\inf(-S)$, where $-S = \{-x \mid x \in S\}$, with the convention that this holds in the extended sense (if S is unbounded above then $-S$ is unbounded below, and both sides are ∞). Applying this to $S = \{a_k \mid k \geq n\}$, whose negative is $\{-a_k \mid k \geq n\}$, we

get

$$s_n = -i_n^-, \quad \text{where } i_n^- = \inf\{-a_k \mid k \geq n\}.$$

Now $\{i_n^-\}$ is increasing, so by the Lemma $\liminf\{-a_n\} = \sup_n i_n^-$, while $\limsup\{a_n\} = \inf_n s_n = \inf_n(-i_n^-) = -\sup_n i_n^-$, using that negation reverses order and interchanges suprema and infima of a set of extended real numbers. Hence $\limsup\{a_n\} = -\liminf\{-a_n\}$.

Proof of (iv).

Suppose first that $\liminf\{a_n\} = \limsup\{a_n\} = a$.

If $a \in \mathbb{R}$: let $\epsilon > 0$ and choose N so large that $|s_n - a| < \epsilon$ and $|i_n - a| < \epsilon$ for $n \geq N$. Then for $n \geq N$,

$$a - \epsilon < i_n \leq a_n \leq s_n < a + \epsilon,$$

so $|a_n - a| < \epsilon$. Thus $\{a_n\} \rightarrow a$.

If $a = \infty$: then $\{i_n\} \rightarrow \infty$, so for each real c there is N with $i_n > c$ for $n \geq N$; since $a_n \geq i_n$, also $a_n > c$ for $n \geq N$. Thus $\{a_n\} \rightarrow \infty$.

If $a = -\infty$: then $\{s_n\} \rightarrow -\infty$, and $a_n \leq s_n$, so $\{a_n\} \rightarrow -\infty$ by the same reasoning.

Conversely, suppose $\{a_n\} \rightarrow a$.

If $a \in \mathbb{R}$: let $\epsilon > 0$ and choose N with $a - \epsilon < a_k < a + \epsilon$ for all $k \geq N$. Then for $n \geq N$ the number $a + \epsilon$ is an upper bound and $a - \epsilon$ a lower bound for $\{a_k \mid k \geq n\}$, so $a - \epsilon \leq i_n \leq s_n \leq a + \epsilon$. Since $\{s_n\}$ and $\{i_n\}$ are monotone with these bounds holding from N on, their limits also lie in $[a - \epsilon, a + \epsilon]$. As $\epsilon > 0$ was arbitrary, $\liminf\{a_n\} = \limsup\{a_n\} = a$.

If $a = \infty$: given a real number c , choose N with $a_n > c$ for $n \geq N$. Then c is a lower bound for $\{a_k \mid k \geq n\}$ whenever $n \geq N$, so $i_n \geq c$ for $n \geq N$. Hence $\{i_n\} \rightarrow \infty$, and since $s_n \geq i_n$ also $\{s_n\} \rightarrow \infty$. Thus both equal $\infty = a$.

If $a = -\infty$: symmetrically, for each real c there is N with $a_n < c$ for $n \geq N$, so $s_n \leq c$ for $n \geq N$, giving $\{s_n\} \rightarrow -\infty$, and $i_n \leq s_n$ gives $\{i_n\} \rightarrow -\infty$. Thus both equal $-\infty = a$.

Proof of (v).

As printed, part (v) asserts that $a_n \leq b_n$ for all n implies $\limsup\{a_n\} \leq \liminf\{b_n\}$. This is a misprint: the assertion is false. Take $a_n = b_n = (-1)^n$. Then $a_n \leq b_n$ for all n , while $s_n = 1$ for all n and $i_n = -1$ for all n , so $\limsup\{a_n\} = 1$ and $\liminf\{b_n\} = -1$, and $1 \leq -1$ fails.

The correct (and intended) statement is that the limit superior and the limit inferior are each monotone:

$$\text{if } a_n \leq b_n \text{ for all } n, \text{ then } \limsup\{a_n\} \leq \limsup\{b_n\} \text{ and } \liminf\{a_n\} \leq \liminf\{b_n\}.$$

We prove this. Write s_n^a, i_n^a and s_n^b, i_n^b for the associated suprema and infima. Fix n . For every $k \geq n$ we have $a_k \leq b_k \leq s_n^b$, so s_n^b is an upper bound for $\{a_k \mid k \geq n\}$ and therefore $s_n^a \leq s_n^b$. Taking infima over n and using the Lemma,

$$\limsup\{a_n\} = \inf_n s_n^a \leq \inf_n s_n^b = \limsup\{b_n\}.$$

Similarly, for every $k \geq n$ we have $i_n^a \leq a_k \leq b_k$, so i_n^a is a lower bound for $\{b_k \mid k \geq n\}$ and therefore $i_n^a \leq i_n^b$. Taking suprema over n ,

$$\liminf\{a_n\} = \sup_n i_n^a \leq \sup_n i_n^b = \liminf\{b_n\}.$$

(One may also note that the printed inequality does hold under the stronger hypothesis that $\{b_n\}$ converges, since then $\limsup\{b_n\} = \liminf\{b_n\}$ by part (iv).)

Problem 1.40. Show that a sequence $\{a_n\}$ is convergent to an extended real number if and only if there is exactly one extended real number that is a cluster point of the sequence.

Solution. Recall from Problem 38 that an extended real number is a cluster point of $\{a_n\}$ provided some subsequence of $\{a_n\}$ converges to it.

Suppose first that $\{a_n\}$ converges to the extended real number a .

We check that every subsequence $\{a_{n_k}\}$ also converges to a . Recall that a strictly increasing sequence $\{n_k\}$ of natural numbers satisfies $n_k \geq k$ for all k (an easy induction: $n_1 \geq 1$, and $n_k > n_{k-1} \geq k-1$ forces $n_k \geq k$). If $a \in \mathbb{R}$ and $\epsilon > 0$, choose N with $|a_n - a| < \epsilon$ for $n \geq N$; then for $k \geq N$ we have $n_k \geq k \geq N$, so $|a_{n_k} - a| < \epsilon$. If $a = \infty$ and c is real, choose N with $a_n > c$ for $n \geq N$; then $a_{n_k} > c$ for $k \geq N$. The case $a = -\infty$ is symmetric. Hence every subsequence converges to a .

In particular a is a cluster point of $\{a_n\}$, since $\{a_n\}$ is a subsequence of itself. And a is the only one: if c is a cluster point, then $\{a_{n_k}\} \rightarrow c$ for some subsequence, while we have just shown $\{a_{n_k}\} \rightarrow a$; so it suffices to know that a sequence cannot converge to two distinct extended real numbers. If both limits are real this is the uniqueness of the limit in Proposition 14. A sequence converging to a real number is bounded (Proposition 14), whereas a sequence converging to ∞ or to $-\infty$ is respectively unbounded above or unbounded below, so a real limit cannot coexist with an infinite one. Finally, $\{a_{n_k}\} \rightarrow \infty$ gives $a_{n_k} > 0$ for all large k while $\{a_{n_k}\} \rightarrow -\infty$ gives $a_{n_k} < 0$ for all large k , so these cannot both hold. Thus $c = a$, and a is the unique cluster point.

Conversely, suppose $\{a_n\}$ has exactly one cluster point, say c . By Problem 38, $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest; in particular both are cluster points. Since there is only one cluster point,

$$\liminf\{a_n\} = c = \limsup\{a_n\}.$$

By Proposition 19(iv), $\{a_n\}$ converges to the extended real number c .

Problem 1.41. Show that $\liminf a_n \leq \limsup a_n$.

Solution. As above put $i_n = \inf\{a_k \mid k \geq n\}$ and $s_n = \sup\{a_k \mid k \geq n\}$, so that $\{i_n\}$ is increasing, $\{s_n\}$ is decreasing, and (by the Lemma proved in Problem 39, which is just Theorem 15 extended to the extended real numbers)

$$\liminf a_n = \sup_n i_n, \quad \limsup a_n = \inf_n s_n.$$

Fix indices m and n and put $p = \max\{m, n\}$. Since $p \geq m$ and $\{i_n\}$ is increasing, $i_m \leq i_p$. Since a_p belongs to the nonempty set $\{a_k \mid k \geq p\}$, we have $i_p \leq a_p \leq s_p$. Since $p \geq n$ and $\{s_n\}$ is decreasing, $s_p \leq s_n$. Chaining these,

$$i_m \leq i_p \leq a_p \leq s_p \leq s_n.$$

Thus s_n is an upper bound for $\{i_m \mid m \in \mathbb{N}\}$, whence $\sup_m i_m \leq s_n$. As this holds for every n , the extended real number $\sup_m i_m$ is a lower bound for $\{s_n \mid n \in \mathbb{N}\}$, whence $\sup_m i_m \leq \inf_n s_n$. That is,

$$\liminf a_n \leq \limsup a_n.$$

An alternative argument: by Problem 38 the set of cluster points of $\{a_n\}$ is nonempty, $\liminf a_n$ is its smallest element and $\limsup a_n$ is its largest element, and the smallest element of a nonempty set is at most its largest element.

Problem 1.42. Prove that if, for all n , $a_n > 0$ and $b_n \geq 0$, then

$$\limsup [a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n),$$

provided the product on the right is not of the form $0 \cdot \infty$.

Solution. Write $A = \limsup a_n$ and $B = \limsup b_n$, and as before put

$$s_n^a = \sup\{a_k \mid k \geq n\}, \quad s_n^b = \sup\{b_k \mid k \geq n\}, \quad s_n^{ab} = \sup\{a_k b_k \mid k \geq n\},$$

so that, by the Lemma of Problem 39, $A = \inf_n s_n^a$, $B = \inf_n s_n^b$ and $\limsup [a_n b_n] = \inf_n s_n^{ab}$.

First note that A and B are nonnegative. Indeed, for every n we have $s_n^a \geq a_n > 0$, so every s_n^a is positive and hence $A = \inf_n s_n^a \geq 0$; likewise $s_n^b \geq b_n \geq 0$, so $B = \inf_n s_n^b \geq 0$. Thus $A, B \in [0, \infty]$. Also

$a_n b_n \geq 0$ for all n , so by the same reasoning $\limsup[a_n b_n] \geq 0$; in particular the left-hand side is never $-\infty$, so no indeterminacy arises there.

Since A and B are nonnegative, the hypothesis that the right-hand product is not of the form $0 \cdot \infty$ says exactly that neither of the two combinations $A = 0$ with $B = \infty$, nor $A = \infty$ with $B = 0$, occurs.

Case 1: $A = \infty$ or $B = \infty$.

If $A = \infty$ then by hypothesis $B \neq 0$, so $B > 0$ and $A \cdot B = \infty$. If $B = \infty$ then by hypothesis $A \neq 0$, so $A > 0$ and $A \cdot B = \infty$. In either case the right-hand side is ∞ and the asserted inequality holds trivially.

Case 2: A and B are both finite.

Then $A, B \in [0, \infty)$ and $A \cdot B$ is a nonnegative real number. Let $\epsilon > 0$.

Since $A = \inf_n s_n^a < A + \epsilon$, there is an index N_1 with $s_{N_1}^a < A + \epsilon$, and hence

$$0 < a_k \leq s_{N_1}^a < A + \epsilon \quad \text{for every } k \geq N_1.$$

Similarly there is an index N_2 with

$$0 \leq b_k \leq s_{N_2}^b < B + \epsilon \quad \text{for every } k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Note $A + \epsilon > 0$ and $B + \epsilon > 0$. For $k \geq N$ we have $b_k < B + \epsilon$ and $a_k > 0$, so $a_k b_k < a_k(B + \epsilon)$; and $a_k < A + \epsilon$ with $B + \epsilon > 0$ gives $a_k(B + \epsilon) < (A + \epsilon)(B + \epsilon)$. Hence

$$a_k b_k < (A + \epsilon)(B + \epsilon) \quad \text{for every } k \geq N.$$

Thus $(A + \epsilon)(B + \epsilon)$ is an upper bound for $\{a_k b_k \mid k \geq N\}$, so $s_N^{ab} \leq (A + \epsilon)(B + \epsilon)$, and therefore

$$\limsup[a_n b_n] = \inf_n s_n^{ab} \leq s_N^{ab} \leq (A + \epsilon)(B + \epsilon) = AB + \epsilon(A + B) + \epsilon^2.$$

This holds for every $\epsilon > 0$. Since $A + B$ and AB are real numbers, letting ϵ run through the values $1/m$, $m \in \mathbb{N}$, we obtain a sequence of real bounds converging to AB ; hence $\limsup[a_n b_n] \leq AB$ by Proposition 14 (or directly: if $\limsup[a_n b_n] > AB$, choose $\epsilon > 0$ small enough that $AB + \epsilon(A + B) + \epsilon^2 < \limsup[a_n b_n]$, a contradiction).

In both cases

$$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n),$$

as required.

Two remarks. The inequality can be strict: with $a_n = 2 + (-1)^n$ and $b_n = 2 - (-1)^n$ we have $a_n > 0$, $b_n \geq 0$, $a_n b_n = 4 - 1 = 3$ for every n , so the left side is 3 while $\limsup a_n = \limsup b_n = 3$ and the right side is 9. Also, the proviso is genuinely needed, and not merely because $0 \cdot \infty$ is undefined: take $a_n = n$ and $b_n = 1/n$, so that $a_n > 0$ and $b_n \geq 0$ for every n . Then $a_n b_n = 1$ for all n , so $\limsup[a_n b_n] = 1$, while $\limsup a_n = \infty$ and $\limsup b_n = 0$. Under any convention that assigns the value 0 to $0 \cdot \infty$ the asserted inequality would read $1 \leq 0$, which is false. So the excluded case is one in which the conclusion genuinely fails, not merely one in which the right-hand side is unassigned.

1.7 Problems 1.43–1.49

Problem 1.43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.

Solution. Let $\{a_n\}$ be a sequence of real numbers. Call an index n a *peak index* provided

$$a_m \leq a_n \quad \text{for every } m \geq n.$$

Let P be the set of peak indices. There are two cases.

Case 1: P is infinite. List the members of P in increasing order as $n_1 < n_2 < n_3 < \cdots$; this is possible because every nonempty set of natural numbers has a smallest member (Theorem 1), so we may choose $n_1 = \min P$ and, inductively, $n_{k+1} = \min\{n \in P \mid n > n_k\}$, which is a nonempty set since P is infinite.

Fix k . Since n_k is a peak index and $n_{k+1} > n_k$, we have $a_{n_{k+1}} \leq a_{n_k}$. Hence $\{a_{n_k}\}$ is a decreasing subsequence of $\{a_n\}$.

Case 2: P is finite. Choose a natural number N that exceeds every member of P (if $P = \emptyset$ take $N = 1$). Define $n_1 = N$. Suppose $n_1 < n_2 < \cdots < n_k$ have been chosen with $n_k \geq N$. Since $n_k \geq N$, the index n_k is not a peak index, so there is an index $m > n_k$ for which $a_m > a_{n_k}$; define $n_{k+1} = m$. Then $\{n_k\}$ is a strictly increasing sequence of natural numbers and $a_{n_k} < a_{n_{k+1}}$ for every k , so $\{a_{n_k}\}$ is an increasing subsequence of $\{a_n\}$.

In either case $\{a_n\}$ has a monotone subsequence.

Proof of the Bolzano-Weierstrass Theorem. Let $\{a_n\}$ be a bounded sequence of real numbers, say $|a_n| \leq M$ for all n . By what we just proved, $\{a_n\}$ has a monotone subsequence $\{a_{n_k}\}$. Every term of the subsequence is a term of the original sequence, so $|a_{n_k}| \leq M$ for all k ; that is, $\{a_{n_k}\}$ is a bounded monotone sequence. By the Monotone Convergence Criterion for Real Sequences (Theorem 15), $\{a_{n_k}\}$ converges. Thus $\{a_n\}$ has a convergent subsequence.

Observe that this argument is not circular: the peak construction uses nothing but the order properties of $\mathbb{R}$ and Theorem 1, while Theorem 15 is deduced directly from the Completeness Axiom without appeal to the Bolzano-Weierstrass Theorem.

Problem 1.44. Let p be a natural number greater than 1, and x a real number, $0 < x < 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each n such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when x is of the form q/p^n , in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a real number x with $0 \leq x \leq 1$. If $p = 10$, this sequence is called the *decimal expansion* of x . For $p = 2$ it is called the *binary expansion*; and for $p = 3$, the *ternary expansion*.

Solution. Throughout, for a real number $y \geq 0$ let $[y]$ denote the largest integer not exceeding y . This exists: by the Archimedean Property the set of natural numbers k with $k > y$ is nonempty, so by Theorem 1 it has a least member k_0 , and $[y] = k_0 - 1$. It satisfies $[y] \leq y < [y] + 1$.

Convergence of any such series. Let $\{a_n\}$ be any sequence of integers with $0 \leq a_n < p$, that is, $0 \leq a_n \leq p - 1$. The terms a_n/p^n are nonnegative, so the partial sums $s_N = \sum_{n=1}^N a_n/p^n$ form an increasing sequence, and

$$s_N \leq \sum_{n=1}^N \frac{p-1}{p^n} = (p-1) \cdot \frac{1/p - 1/p^{N+1}}{1 - 1/p} = 1 - \frac{1}{p^N} < 1.$$

Thus the sequence of partial sums is bounded, and by Proposition 20(iii) (Problem 45) the series is summable. Let x be its sum. Since $s_N \leq 1$ for every N , Proposition 14 gives $x \leq 1$; since $-s_N \leq 0$ for every N , the same proposition applied to $\{-s_N\}$, whose limit is $-x$ by Theorem 18, gives $-x \leq 0$, that is $x \geq 0$. Hence $0 \leq x \leq 1$.

Existence of an expansion. Let $0 < x < 1$. Define $x_0 = x$ and, inductively,

$$a_n = [p x_{n-1}], \quad x_n = p x_{n-1} - a_n \quad (n \geq 1).$$

We verify by induction that $0 \leq x_n < 1$ for every $n \geq 0$. This holds for $n = 0$. If $0 \leq x_{n-1} < 1$, then $0 \leq p x_{n-1} < p$, so $a_n = [p x_{n-1}]$ is an integer with $0 \leq a_n \leq p - 1$, and $x_n = p x_{n-1} - a_n$ satisfies $0 \leq x_n < 1$ by the defining property of the floor. So each a_n is an integer with $0 \leq a_n < p$.

From $x_{n-1} = (a_n + x_n)/p$ an induction gives, for every $N \geq 1$,

$$x = \sum_{n=1}^N \frac{a_n}{p^n} + \frac{x_N}{p^N}.$$

Since $0 \leq x_N < 1$ we get $0 \leq x - s_N < 1/p^N$. Now $p^N \geq 2^N \geq N$ for every N , both inequalities by induction on N (using $p \geq 2$ for the first, and $2^{N+1} = 2^N + 2^N \geq N + 1$ for the second). Given $\epsilon > 0$, the Archimedean Property provides a natural number N_0 with $1/N_0 < \epsilon$, and then $0 \leq x - s_N < 1/p^N \leq 1/N \leq 1/N_0 < \epsilon$ for all $N \geq N_0$. Hence $s_N \rightarrow x$, that is, $x = \sum_{n=1}^{\infty} a_n/p^n$.

The structure of two distinct expansions. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of integers in $[0, p)$ with

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{b_n}{p^n} = x,$$

and suppose they are not identical. Let N be the smallest index with $a_N \neq b_N$ (Theorem 1); relabelling if necessary, assume $a_N > b_N$, so $a_N - b_N \geq 1$. Subtracting the two series (legitimate by Theorem 18, applied to the partial sums) and using $a_n = b_n$ for $n < N$,

$$\frac{a_N - b_N}{p^N} = \sum_{n=N+1}^{\infty} \frac{b_n - a_n}{p^n}.$$

Now $b_n - a_n \leq p - 1$ for every n , so the right-hand side is at most $\sum_{n=N+1}^{\infty} (p - 1)/p^n = 1/p^N$, while the left-hand side is at least $1/p^N$. Hence both sides equal $1/p^N$, which forces

$$a_N - b_N = 1, \quad b_n - a_n = p - 1 \quad \text{for all } n > N,$$

the latter because a series of nonnegative terms $((p - 1) - (b_n - a_n))/p^n$ with sum zero must have every term zero. Since $0 \leq a_n, b_n \leq p - 1$, this says

$$b_n = p - 1 \quad \text{and} \quad a_n = 0 \quad \text{for all } n > N.$$

In particular, of any two distinct expansions of the same x , one terminates (is eventually 0) and the other is eventually $p - 1$.

Uniqueness except for p -adic fractions. Suppose $x \in (0, 1)$ has two distinct expansions $\{a_n\}, \{b_n\}$ as above. Then $x = \sum_{n=1}^N a_n/p^n = q/p^N$ where $q = \sum_{n=1}^N a_n p^{N-n}$ is an integer. So x has the form q/p^N . Conversely, let $x = q/p^N$ with q an integer and $0 < x < 1$; then $1 \leq q \leq p^N - 1$. We claim that every integer q with $0 \leq q \leq p^N - 1$ can be written as $q = \sum_{n=1}^N c_n p^{N-n}$ with each c_n an integer, $0 \leq c_n \leq p - 1$. Argue by induction on N . For $N = 1$ take $c_1 = q$. Assume the claim for N and let $0 \leq q \leq p^{N+1} - 1$. Put $c_1 = \lfloor q/p^N \rfloor$; then $0 \leq c_1 \leq p - 1$, since $q/p^N < p$, and $q' = q - c_1 p^N$ satisfies $0 \leq q' \leq p^N - 1$ by the defining property of the floor. Applying the inductive hypothesis to q' and shifting indices gives the representation of q with $N + 1$ digits. This proves the claim, and dividing by p^N gives $x = \sum_{n=1}^N c_n/p^n$. Since $x > 0$, not all c_n vanish; let $M \leq N$ be the largest index with $c_M \neq 0$. Then

$$x = \sum_{n=1}^M \frac{c_n}{p^n},$$

and the sequence $a_n = c_n$ for $n \leq M$, $a_n = 0$ for $n > M$, is one expansion. Define a second by

$$b_n = c_n \quad (n < M), \quad b_M = c_M - 1 \quad (\geq 0), \quad b_n = p - 1 \quad (n > M).$$

These are integers in $[0, p)$, and since $\sum_{n>M} (p - 1)/p^n = 1/p^M$,

$$\sum_{n=1}^{\infty} \frac{b_n}{p^n} = \sum_{n<M} \frac{c_n}{p^n} + \frac{c_M - 1}{p^M} + \frac{1}{p^M} = x.$$

The two sequences differ at index M , so x has at least two expansions.

There are never more than two. Suppose $\{a_n\}, \{b_n\}, \{c_n\}$ were three pairwise distinct expansions of the same x . By the structural result above, each pair consists of one sequence that is eventually 0 and one that is eventually $p-1$; note that since $p \geq 2$ we have $p-1 \neq 0$, so no single sequence can be both. Say $\{a_n\}$ is eventually 0 and $\{b_n\}$ eventually $p-1$. Applying the result to the pair $\{a_n\}, \{c_n\}$ shows $\{c_n\}$ is eventually $p-1$ (it cannot be the eventually-0 one, since then $\{a_n\}$ would have to be eventually $p-1$). But then $\{b_n\}$ and $\{c_n\}$ are both eventually $p-1$, contradicting the structural result for that pair. Hence there are at most two expansions.

Combining: every $x \in (0, 1)$ has an expansion; it is unique unless $x = q/p^N$ for integers q, N , in which case there are exactly two.

Problem 1.45. Prove Proposition 20.

[Proposition 20. Let $\{a_n\}$ be a sequence of real numbers.

- (i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0$, there is an index N for which
$$\left| \sum_{k=n}^{n+m} a_k \right| < \epsilon \quad \text{for } n \geq N \text{ and any natural number } m.$$
- (ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.
- (iii) If each term a_k is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.]

Solution. Write $s_n = \sum_{k=1}^n a_k$ for the n -th partial sum and set $s_0 = 0$, so that for all $j < i$,

$$s_i - s_j = \sum_{k=j+1}^i a_k.$$

By definition the series is summable exactly when the sequence $\{s_n\}$ converges.

(i). Suppose first that the series is summable, so $\{s_n\}$ converges. By the Cauchy Convergence Criterion (Theorem 17), $\{s_n\}$ is Cauchy. Let $\epsilon > 0$ and choose an index N' such that $|s_i - s_j| < \epsilon$ whenever $i, j \geq N'$. Put $N = N' + 1$. If $n \geq N$ and m is a natural number, then $n-1 \geq N'$ and $n+m \geq N'$, so

$$\left| \sum_{k=n}^{n+m} a_k \right| = |s_{n+m} - s_{n-1}| < \epsilon.$$

Conversely, suppose the stated condition holds. Let $\epsilon > 0$ and choose N so that $|\sum_{k=n}^{n+m} a_k| < \epsilon/2$ for all $n \geq N$ and all natural numbers m . We first observe that the condition controls individual terms: for $n \geq N$,

$$a_{n+2} = \sum_{k=n}^{n+2} a_k - \sum_{k=n}^{n+1} a_k,$$

and both sums on the right are of the permitted form (with $m = 2$ and $m = 1$), so $|a_{n+2}| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $|a_k| < \epsilon$ for all $k \geq N+2$.

Now set $N'' = N + 1$ and let $i > j \geq N''$. If $i \geq j+2$, then

$$|s_i - s_j| = \left| \sum_{k=j+1}^i a_k \right| = \left| \sum_{k=n}^{n+m} a_k \right| \quad \text{with } n = j+1 \geq N, \quad m = i - j - 1 \geq 1,$$

so $|s_i - s_j| < \epsilon/2 < \epsilon$. If $i = j+1$, then $|s_i - s_j| = |a_{j+1}|$ and $j+1 \geq N+2$, so $|s_i - s_j| < \epsilon$. Thus $|s_i - s_j| < \epsilon$ whenever $i, j \geq N''$ (the case $i = j$ being trivial), so $\{s_n\}$ is Cauchy and, by Theorem 17, converges. The series is summable.

(ii). Suppose $\sum_{k=1}^{\infty} |a_k|$ is summable. Let $\epsilon > 0$. Applying part (i) to the series $\sum |a_k|$, there is an index N such that for $n \geq N$ and every natural number m ,

$$\sum_{k=n}^{n+m} |a_k| = \left| \sum_{k=n}^{n+m} |a_k| \right| < \epsilon.$$

By the triangle inequality (Problem 7),

$$\left| \sum_{k=n}^{n+m} a_k \right| \leq \sum_{k=n}^{n+m} |a_k| < \epsilon$$

for all $n \geq N$ and all natural numbers m . By part (i), applied in the other direction to $\sum a_k$, the series $\sum_{k=1}^{\infty} a_k$ is summable.

(iii). Suppose each $a_k \geq 0$. Then $s_{n+1} - s_n = a_{n+1} \geq 0$, so $\{s_n\}$ is an increasing sequence. By the Monotone Convergence Criterion (Theorem 15), a monotone sequence converges if and only if it is bounded. Hence the series is summable, that is $\{s_n\}$ converges, if and only if $\{s_n\}$ is bounded. (For an increasing sequence, boundedness is the same as boundedness above, since $s_n \geq s_1$ for all n .)

Problem 1.46. Show that the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Monotone Convergence Theorem is equivalent to the Completeness Axiom for the real numbers.

Solution. Throughout, we work with an ordered field F satisfying the field axioms and the positivity axioms for $\mathbb{R}$, and we ask which of the following three assertions imply which:

- (C) *Completeness Axiom.* Every nonempty subset of F that is bounded above has a least upper bound in F .
- (BW) *Bolzano-Weierstrass.* Every bounded sequence in F has a convergent subsequence.
- (MC) *Monotone Convergence.* Every monotone bounded sequence in F converges.

(The natural numbers $\mathbb{N}$ are defined inside F as the intersection of all inductive subsets, exactly as in Section 1.2; this definition and the Principle of Mathematical Induction use no completeness. Note also that the proofs of Proposition 14 and Theorem 18 — uniqueness of limits, boundedness of convergent sequences, and linearity and monotonicity of convergence — use only the definition of convergence and the ordered-field axioms, so they are available to us here.)

(C) *implies (BW).* This is exactly the Bolzano-Weierstrass Theorem (Theorem 16), proved in the text from the Completeness Axiom (via the Nested Set Theorem).

(C) *implies (MC).* This is exactly the Monotone Convergence Criterion (Theorem 15), proved in the text from the Completeness Axiom.

It remains to prove the two converses. Both use the following two lemmas.

Lemma A. Each of (BW) and (MC) implies the Archimedean Property, that is, $\mathbb{N}$ is not bounded above in F . Suppose $\mathbb{N}$ were bounded above. Consider the sequence $a_n = n$. It is bounded (below by 1, above by the assumed bound) and increasing.

Assume (BW). Then $\{a_n\}$ has a convergent subsequence $\{n_k\}$, where $n_1 < n_2 < \dots$. A convergent sequence is Cauchy (the first half of the proof of Theorem 17 uses only the definition of convergence), so there is an index K with $|n_{k+1} - n_k| < 1$ for $k \geq K$. But $n_{k+1} > n_k$ are natural numbers, so $n_{k+1} - n_k$ is itself a natural number (a routine induction from Problem 9) and hence $n_{k+1} - n_k \geq 1$, a contradiction.

Assume (MC). Then $\{a_n\}$, being increasing and bounded, converges to some $a \in F$. Choosing N with $|a_n - a| < 1/2$ for $n \geq N$ gives $|a_{N+1} - a_N| < 1$, that is $1 < 1$, a contradiction.

Hence $\mathbb{N}$ is unbounded above. Consequently, for each $\epsilon > 0$ there is a natural number n with $1/n < \epsilon$, and since $2^n \geq n$ for all n (induction), also $1/2^n \rightarrow 0$.

Lemma B (bisection). Let $E \subseteq F$ be nonempty and bounded above by b_0 . Choose $e \in E$ and put $a_0 = e - 1$; then a_0 is not an upper bound for E , and $a_0 < b_0$. Define sequences $\{a_n\}$, $\{b_n\}$ inductively: given a_n not an upper bound for E and b_n an upper bound for E with $a_n < b_n$, let $c = (a_n + b_n)/2$ and set

$$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, c) & \text{if } c \text{ is an upper bound for } E, \\ (c, b_n) & \text{if } c \text{ is not an upper bound for } E. \end{cases}$$

By induction, for every n : a_n is not an upper bound for E , b_n is an upper bound for E , $\{a_n\}$ is

increasing, $\{b_n\}$ is decreasing, $a_n < b_n$, and

$$b_n - a_n = \frac{b_0 - a_0}{2^n}.$$

In particular $a_0 \leq a_n < b_n \leq b_0$, so both sequences are bounded, and by Lemma A, $b_n - a_n \rightarrow 0$.

(BW) implies (C). Assume (BW), and let E be nonempty and bounded above. Form $\{a_n\}, \{b_n\}$ as in Lemma B. By (BW) the bounded sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to some $L \in F$. Since $\{b_n - a_n\}$ converges to 0, so does its subsequence $\{b_{n_k} - a_{n_k}\}$ (immediate from the definition of convergence, since $n_k \geq k$). As $b_{n_k} = a_{n_k} + (b_{n_k} - a_{n_k})$, Theorem 18 gives $b_{n_k} \rightarrow L$ as well.

L is an upper bound for E : if $x \in E$, then $x \leq b_{n_k}$ for every k , since each b_n is an upper bound. Comparing the constant sequence with constant value x , which converges to x , against $\{b_{n_k}\}$, which converges to L , the monotonicity of convergence (Theorem 18, formula (7)) gives $x \leq L$.

L is the least upper bound: suppose c is an upper bound for E with $c < L$. Since $a_{n_k} \rightarrow L$, there is k with $a_{n_k} > c$. But a_{n_k} is not an upper bound for E , so there is $y \in E$ with $y > a_{n_k} > c$, contradicting that c is an upper bound. Hence no upper bound is smaller than L , so $L = \sup E$ exists in F . This is (C).

(MC) implies (C). Assume (MC), and let E be nonempty and bounded above. Form $\{a_n\}, \{b_n\}$ as in Lemma B. The sequence $\{a_n\}$ is increasing and bounded, so by (MC) it converges to some $L \in F$; as above $b_n = a_n + (b_n - a_n) \rightarrow L$ by Lemma A and Theorem 18. The very same two paragraphs as in the preceding proof (with a_n, b_n in place of a_{n_k}, b_{n_k}) show that L is an upper bound for E and that no smaller number is an upper bound. Hence $\sup E$ exists, which is (C).

Therefore (C), (BW) and (MC) are mutually equivalent: the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom, and so is the assertion of the Monotone Convergence Theorem.

Problem 1.47. Let E be a closed set of real numbers and f a real-valued function that is defined and continuous on E . Show that there is a function g defined and continuous on all of $\mathbb{R}$ such that $f(x) = g(x)$ for each $x \in E$. (Hint: Take g to be linear on each of the intervals of which $\mathbb{R} \sim E$ is composed.)

Solution. If $E = \emptyset$ take $g \equiv 0$; if $E = \mathbb{R}$ take $g = f$. So assume E is a nonempty proper closed subset of $\mathbb{R}$.

Construction of g . Since E is closed, $\mathcal{O} = \mathbb{R} \sim E$ is open and nonempty (Proposition 11). By Proposition 9, $\mathcal{O}$ is the disjoint union of a countable collection $\mathcal{I}$ of open intervals. Each member of $\mathcal{I}$ is of one of the forms (a, b) , $(-\infty, b)$, (a, ∞) with a, b real; none is all of $\mathbb{R}$, since $E \neq \emptyset$. Call the members of $\mathcal{I}$ the *components* of $\mathcal{O}$.

We first record that a finite endpoint of a component belongs to E . Let $I \in \mathcal{I}$ have right endpoint $b \in \mathbb{R}$, and suppose $b \in \mathcal{O}$. Then b lies in some $J = (c, d) \in \mathcal{I}$, and $J \neq I$ because $b \notin I$. Write $I = (a, b)$, where a is real or $a = -\infty$. Put $z_0 = c$ if $a = -\infty$, and $z_0 = \max\{a, c\}$ if a is real; in either case z_0 is real and $z_0 < b$, so we may choose z with $z_0 < z < b$. Then $a < z < b$, so $z \in I$, and $c < z < b < d$, so $z \in J$, contradicting the disjointness of $\mathcal{I}$. Hence $b \in E$; the argument for a finite left endpoint is symmetric. (This is also exactly what is verified in the proof of Proposition 9.)

Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by:

- $g(x) = f(x)$ for $x \in E$;
- on a bounded component (a, b) (so $a, b \in E$), let g be the affine function determined by $g(a) = f(a)$, $g(b) = f(b)$, that is,

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a} (x - a) = \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b);$$

- on a component (a, ∞) (so $a \in E$), let $g(x) = f(a)$;
- on a component $(-\infty, b)$ (so $b \in E$), let $g(x) = f(b)$.

This defines g unambiguously on all of $\mathbb{R}$, and $g = f$ on E . Note that on a bounded component (a, b) the value $g(x)$ is a convex combination of $f(a)$ and $f(b)$, since $\frac{b-x}{b-a}$ and $\frac{x-a}{b-a}$ are nonnegative and sum

to 1. Consequently, if $|f(a) - c| < \epsilon$ and $|f(b) - c| < \epsilon$ for some real c , then for $x \in (a, b)$,

$$|g(x) - c| = \left| \frac{b-x}{b-a}(f(a) - c) + \frac{x-a}{b-a}(f(b) - c) \right| \leq \frac{b-x}{b-a}\epsilon + \frac{x-a}{b-a}\epsilon = \epsilon. \quad (*)$$

Continuity at points of $\mathcal{O}$. Let $x_0 \in \mathcal{O}$ and let I be its component. Since I is open, x_0 is an interior point of I , and on I the function g agrees with an affine function (a constant, in the unbounded cases), which is continuous. So g is continuous at x_0 .

Continuity at points of E . Let $x_0 \in E$ and let $\epsilon > 0$. By continuity of f at x_0 relative to E , there is a $\delta > 0$ such that

$$|f(y) - f(x_0)| < \epsilon \quad \text{whenever } y \in E \text{ and } |y - x_0| < \delta. \quad (**)$$

We produce $\delta_+ > 0$ such that $|g(x) - g(x_0)| \leq \epsilon$ for $x_0 \leq x < x_0 + \delta_+$; the argument on the left is symmetric, and taking the minimum of the two numbers finishes the proof.

Case 1: there is $\eta > 0$ with $(x_0, x_0 + \eta) \cap E = \emptyset$. Then $(x_0, x_0 + \eta) \subseteq \mathcal{O}$. Let I be the component of $\mathcal{O}$ that contains the point $x_0 + \eta/2$, and let a be its left endpoint. If $a < x_0$ (this includes $a = -\infty$), then x_0 would lie strictly between a and $x_0 + \eta/2$, hence in $I \subseteq \mathcal{O}$, contradicting $x_0 \in E$. So a is real and $a \geq x_0$. If $a > x_0$, then a is a finite endpoint of a component, so $a \in E$ by the fact recorded above, while $x_0 < a < x_0 + \eta/2 < x_0 + \eta$; this contradicts $(x_0, x_0 + \eta) \cap E = \emptyset$. Hence $a = x_0$, so $I = (x_0, b)$ or $I = (x_0, \infty)$. On I , g agrees with an affine function ℓ (in the unbounded case, the constant $f(x_0)$), and in the bounded case $\ell(x_0) = f(a) = f(x_0)$ by construction, while in the unbounded case $\ell \equiv f(x_0)$. Since ℓ is continuous at x_0 and $\ell(x_0) = f(x_0) = g(x_0)$, there is $\delta_+ > 0$ with $|\ell(x) - g(x_0)| < \epsilon$ for $|x - x_0| < \delta_+$; shrinking δ_+ so that $(x_0, x_0 + \delta_+) \subseteq I$, we get $|g(x) - g(x_0)| < \epsilon$ for $x_0 \leq x < x_0 + \delta_+$.

Case 2: every interval $(x_0, x_0 + \eta)$ meets E . Choose $x_1 \in E$ with $x_0 < x_1 < x_0 + \delta$, and set $\delta_+ = x_1 - x_0 > 0$. Let $x_0 \leq x < x_0 + \delta_+$, so $x_0 \leq x < x_1 < x_0 + \delta$. If $x \in E$, then $|x - x_0| < \delta$ and $(**)$ gives $|g(x) - g(x_0)| = |f(x) - f(x_0)| < \epsilon$. If $x \notin E$, let I be the component of $\mathcal{O}$ containing x . Since $x_0 \in E$ and $x_0 \leq x$, and since $x_1 \in E$ with $x < x_1$, the interval I contains neither x_0 nor x_1 , so I is bounded, $I = (a, b)$ with

$$x_0 \leq a < x < b \leq x_1.$$

Both a and b lie in E and satisfy $|a - x_0| < \delta$, $|b - x_0| < \delta$, so by $(**)$ both $|f(a) - f(x_0)| < \epsilon$ and $|f(b) - f(x_0)| < \epsilon$. By $(*)$ with $c = f(x_0) = g(x_0)$ we conclude $|g(x) - g(x_0)| \leq \epsilon$.

Thus in both cases $|g(x) - g(x_0)| \leq \epsilon$ for all x with $x_0 \leq x < x_0 + \delta_+$. The mirror-image argument on the left of x_0 produces $\delta_- > 0$ with $|g(x) - g(x_0)| \leq \epsilon$ for $x_0 - \delta_- < x \leq x_0$. With $\delta_0 = \min\{\delta_+, \delta_-\}$ we have $|g(x) - g(x_0)| \leq \epsilon$ whenever $|x - x_0| < \delta_0$. Since $\epsilon > 0$ was arbitrary (apply the above to $\epsilon/2$ to obtain a strict inequality), g is continuous at x_0 .

Hence g is continuous on all of $\mathbb{R}$ and extends f .

Problem 1.48. Define the real-valued function f on $\mathbb{R}$ by setting

$$f(x) = \begin{cases} x & \text{if } x \text{ irrational,} \\ p \sin \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous?

Solution. Answer: f is continuous exactly at the irrational numbers and at $x = 0$; it is discontinuous at every nonzero rational.

Here “in lowest terms” means $x = p/q$ with p an integer, q a natural number, and p and q having no common factor; this representation is unique, and $0 = 0/1$, so $f(0) = 0 \cdot \sin 1 = 0$.

We use three standard facts about the sine function from elementary calculus: $|\sin t| \leq |t|$ for all real t ; $\sin t < t$ for $t > 0$; and $\sin t/t \rightarrow 1$ as $t \rightarrow 0$. Throughout, we use Proposition 21: f is continuous at x_* if and only if $f(x_n) \rightarrow f(x_*)$ whenever $\{x_n\} \rightarrow x_*$.

A useful rewriting. If $x = p/q$ in lowest terms, then $p = qx$, so

$$f(x) = qx \sin \frac{1}{q} = x \cdot \frac{\sin(1/q)}{1/q}. \quad (1)$$

Lemma. If $\{x_n\}$ is a sequence of rationals converging to x_* , and $x_n = p_n/q_n$ in lowest terms, then either the denominators q_n tend to infinity, or some subsequence of $\{x_n\}$ is constant with value a rational number equal to x_* . Indeed, suppose $\{q_n\}$ does not tend to infinity. Then there is a natural number Q and infinitely many indices n with $q_n \leq Q$. A convergent sequence is bounded (Proposition 14), say $|x_n| \leq M$ for all n ; then for those indices $|p_n| = q_n|x_n| \leq QM$. So along those indices the pairs (p_n, q_n) take only finitely many values, hence x_n takes only finitely many values; consequently some value is attained infinitely often, giving a constant subsequence. A constant subsequence of a sequence converging to x_* has value x_* , so x_* is that rational number.

Continuity at every irrational x_* . Let $\{x_n\} \rightarrow x_*$; we must show $f(x_n) \rightarrow f(x_*) = x_*$. Consider

$$|f(x_n) - x_n| = \begin{cases} 0 & \text{if } x_n \text{ irrational,} \\ |x_n| \left| \frac{\sin(1/q_n)}{1/q_n} - 1 \right| & \text{if } x_n = p_n/q_n \text{ in lowest terms,} \end{cases}$$

using (1). Let $\epsilon > 0$. Since $\{x_n\}$ is bounded, say $|x_n| \leq M$ with $M > 0$, and since $\sin t/t \rightarrow 1$ as $t \rightarrow 0$, there is $\eta > 0$ such that $|\sin t/t - 1| < \epsilon/M$ whenever $0 < t < \eta$. Now consider the rational terms of $\{x_n\}$. By the Lemma, since x_* is irrational no subsequence of rationals can have bounded denominators (a constant subsequence would force x_* to be rational); more precisely, if there were a natural number Q and infinitely many rational terms with $q_n \leq Q$, the Lemma applied to that subsequence would make x_* rational. Hence $q_n \rightarrow \infty$ through the rational terms: there is an index N_1 such that every rational term x_n with $n \geq N_1$ has $1/q_n < \eta$. For such n ,

$$|f(x_n) - x_n| = |x_n| \left| \frac{\sin(1/q_n)}{1/q_n} - 1 \right| \leq M \cdot \frac{\epsilon}{M} = \epsilon.$$

For irrational terms $|f(x_n) - x_n| = 0 \leq \epsilon$. Therefore $|f(x_n) - x_n| \leq \epsilon$ for all $n \geq N_1$. Since $\epsilon > 0$ was arbitrary, $f(x_n) - x_n \rightarrow 0$, and since $x_n \rightarrow x_*$, Theorem 18 gives $f(x_n) \rightarrow x_* = f(x_*)$. By Proposition 21, f is continuous at x_* .

Continuity at 0. For every real x we claim $|f(x)| \leq |x|$. If x is irrational this is an equality. If $x = p/q$ in lowest terms, then, using $|\sin t| \leq |t|$,

$$|f(x)| = |p| \left| \sin \frac{1}{q} \right| \leq |p| \cdot \frac{1}{q} = |x|.$$

Hence if $\{x_n\} \rightarrow 0$, then $|f(x_n)| \leq |x_n| \rightarrow 0$, so $f(x_n) \rightarrow 0 = f(0)$. By Proposition 21, f is continuous at 0.

Discontinuity at every nonzero rational. Let $x_* = p/q$ in lowest terms with $x_* \neq 0$, so $p \neq 0$ and $q \geq 1$.

Then $0 < 1/q \leq 1$, so $\sin(1/q) < 1/q$ and hence $\frac{\sin(1/q)}{1/q} < 1$. By (1),

$$f(x_*) = x_* \cdot \frac{\sin(1/q)}{1/q} \neq x_*,$$

since $x_* \neq 0$ and the factor is different from 1. Next, the irrationals are dense in $\mathbb{R}$: given $a < b$, Theorem 2 provides a rational r with $a - \sqrt{2} < r < b - \sqrt{2}$, and then $r + \sqrt{2} \in (a, b)$ is irrational, for otherwise $\sqrt{2} = (r + \sqrt{2}) - r$ would be rational, which Section 1.2 shows it is not. So for each natural number n choose an irrational x_n with $|x_n - x_*| < 1/n$; by the Archimedean Property $\{x_n\} \rightarrow x_*$, while $f(x_n) = x_n \rightarrow x_* \neq f(x_*)$. By Proposition 21, f is not continuous at x_* .

Thus the set of points of continuity of f is $\{0\} \cup \{x \in \mathbb{R} \mid x \text{ irrational}\}$.

Problem 1.49. Let f and g be continuous real-valued functions with a common domain E .

- (i) Show that the sum, $f + g$, and product, fg , are also continuous functions.
- (ii) If h is a continuous function with image contained in E , show that the composition $f \circ h$ is continuous.
- (iii) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.
- (iv) Show that $|f|$ is continuous.

Solution. We use Proposition 21 throughout: a real-valued function φ defined on a set E of real numbers is continuous at $x_* \in E$ if and only if $\varphi(x_n) \rightarrow \varphi(x_*)$ whenever $\{x_n\}$ is a sequence in E converging to x_* .

(i). Fix $x_* \in E$ and let $\{x_n\}$ be a sequence in E with $\{x_n\} \rightarrow x_*$. Since f and g are continuous at x_* , Proposition 21 gives $f(x_n) \rightarrow f(x_*)$ and $g(x_n) \rightarrow g(x_*)$.

Sum: by Theorem 18 (linearity of convergence, with $\alpha = \beta = 1$),

$$(f + g)(x_n) = f(x_n) + g(x_n) \rightarrow f(x_*) + g(x_*) = (f + g)(x_*).$$

By Proposition 21, $f + g$ is continuous at x_* , and since x_* was arbitrary, on E .

Product: we first check that the product of convergent sequences converges to the product of the limits. Let $a_n \rightarrow a$ and $b_n \rightarrow b$. By Proposition 14 the sequence $\{a_n\}$ is bounded; choose $M > 0$ with $|a_n| \leq M$ for all n , and enlarge M if necessary so that $M \geq |b|$. Then for every n ,

$$|a_n b_n - ab| = |a_n(b_n - b) + b(a_n - a)| \leq M|b_n - b| + M|a_n - a|.$$

Given $\epsilon > 0$, choose N so large that $|a_n - a| < \epsilon/(2M)$ and $|b_n - b| < \epsilon/(2M)$ for $n \geq N$; then $|a_n b_n - ab| < \epsilon$ for $n \geq N$. So $a_n b_n \rightarrow ab$. Applying this with $a_n = f(x_n)$, $b_n = g(x_n)$,

$$(fg)(x_n) = f(x_n)g(x_n) \rightarrow f(x_*)g(x_*) = (fg)(x_*),$$

so by Proposition 21, fg is continuous on E .

(ii). Let h be a continuous real-valued function defined on a set D of real numbers with $h(D) \subseteq E$. Then $f \circ h$ is defined on D . Fix $x_* \in D$ and let $\{x_n\}$ be a sequence in D with $\{x_n\} \rightarrow x_*$. Since h is continuous at x_* , Proposition 21 gives $h(x_n) \rightarrow h(x_*)$. Now $\{h(x_n)\}$ is a sequence in E converging to the point $h(x_*)$ of E , so continuity of f at $h(x_*)$ and Proposition 21 give

$$(f \circ h)(x_n) = f(h(x_n)) \rightarrow f(h(x_*)) = (f \circ h)(x_*).$$

By Proposition 21 again, $f \circ h$ is continuous at x_* , hence on D .

(iii). We first record an elementary inequality: for real numbers u, v, u', v' ,

$$|\max\{u, v\} - \max\{u', v'\}| \leq \max\{|u - u'|, |v - v'|\}. \quad (2)$$

To see this, assume without loss of generality that $\max\{u, v\} \geq \max\{u', v'\}$ (otherwise interchange the roles of the two pairs, which does not change either side). Say $\max\{u, v\} = u$, the case $\max\{u, v\} = v$ being identical with v in place of u . Then, since $\max\{u', v'\} \geq u'$,

$$0 \leq \max\{u, v\} - \max\{u', v'\} \leq u - u' \leq |u - u'| \leq \max\{|u - u'|, |v - v'|\},$$

which is (2).

Now let $x_* \in E$ and $\epsilon > 0$. By continuity of f and of g at x_* , there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for $x \in E$: $|x - x_*| < \delta_1$ implies $|f(x) - f(x_*)| < \epsilon$, and $|x - x_*| < \delta_2$ implies $|g(x) - g(x_*)| < \epsilon$. Put $\delta = \min\{\delta_1, \delta_2\} > 0$. If $x \in E$ and $|x - x_*| < \delta$, then by (2) with $u = f(x)$, $v = g(x)$, $u' = f(x_*)$, $v' = g(x_*)$,

$$|\max\{f, g\}(x) - \max\{f, g\}(x_*)| \leq \max\{|f(x) - f(x_*)|, |g(x) - g(x_*)|\} < \epsilon.$$

So $\max\{f, g\}$ is continuous at x_* , and hence on E .

(Alternatively, once (i) and (iv) are known one may use the identity $\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$.)
(iv). By the triangle inequality (Problem 7), for all real s, t we have $|s| \leq |s - t| + |t|$ and $|t| \leq |t - s| + |s|$, whence

$$||s| - |t|| \leq |s - t|.$$

Let $x_* \in E$ and $\epsilon > 0$. By continuity of f at x_* there is $\delta > 0$ such that $|f(x) - f(x_*)| < \epsilon$ whenever $x \in E$ and $|x - x_*| < \delta$. For such x ,

$$||f|(x) - |f|(x_*)| = ||f(x)| - |f(x_*)|| \leq |f(x) - f(x_*)| < \epsilon.$$

Hence $|f|$ is continuous at x_* , and therefore on E .

1.8 Problems 1.50–1.56

Problem 1.50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.

Solution. Recall the definitions in force. A real-valued function f on a set E of real numbers is **Lipschitz** provided there is a $c \geq 0$ for which

$$|f(x') - f(x)| \leq c|x' - x| \quad \text{for all } x', x \in E,$$

and f is **uniformly continuous** on E provided for each $\epsilon > 0$ there is a $\delta > 0$ such that for all x, x' in E , if $|x - x'| < \delta$ then $|f(x) - f(x')| < \epsilon$.

Lipschitz implies uniformly continuous. Let f be Lipschitz on E with constant $c \geq 0$. Let $\epsilon > 0$. Define

$$\delta = \frac{\epsilon}{c+1} > 0.$$

This δ responds to the ϵ challenge. Indeed, let x, x' belong to E with $|x - x'| < \delta$. Then

$$|f(x) - f(x')| \leq c|x - x'| \leq (c+1)|x - x'| < (c+1) \cdot \frac{\epsilon}{c+1} = \epsilon.$$

Since δ was chosen independently of the points x, x' , the function f is uniformly continuous on E . Observe that the same δ works for every ϵ -challenge at every point, which is exactly the extra strength of uniform continuity over continuity.

A uniformly continuous function that is not Lipschitz. Define $f(x) = \sqrt{x}$ for x in $E = [0, 1]$. The function f is continuous on E . Indeed, at a point $x > 0$, for $x' \in [0, 1]$ we have $\sqrt{x'} + \sqrt{x} \geq \sqrt{x} > 0$ and hence

$$|\sqrt{x'} - \sqrt{x}| = \frac{|x' - x|}{\sqrt{x'} + \sqrt{x}} \leq \frac{|x' - x|}{\sqrt{x}},$$

so $\delta = \epsilon\sqrt{x}$ responds to the ϵ challenge at x ; and at $x = 0$ we have $|f(x') - f(0)| = \sqrt{x'} < \epsilon$ whenever $0 \leq x' < \epsilon^2$. The set $[0, 1]$ is closed and bounded, so by Theorem 23 the continuous function f is uniformly continuous on $[0, 1]$.

However f is not Lipschitz. Suppose it were, with constant $c \geq 0$. Taking $x = 0$ and $x' = t$ with $0 < t \leq 1$ gives

$$\sqrt{t} = |f(t) - f(0)| \leq c|t - 0| = ct,$$

so that, dividing by $\sqrt{t} > 0$,

$$\frac{1}{\sqrt{t}} \leq c \quad \text{for all } 0 < t \leq 1.$$

But $1/\sqrt{t}$ is unbounded above as t decreases to 0: choosing $t = 1/(c+1)^2$, which lies in $(0, 1]$, yields $c+1 \leq c$, a contradiction. Hence no such c exists and f is not Lipschitz.

Therefore the class of Lipschitz functions is properly contained in the class of uniformly continuous functions.

Problem 1.51. A continuous function φ on $[a, b]$ is called **piecewise linear** provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ for which φ is linear on each interval $[x_i, x_{i+1}]$. Let f be a continuous function on $[a, b]$ and ϵ a positive number. Show that there is a piecewise linear function φ on $[a, b]$ with $|f(x) - \varphi(x)| < \epsilon$ for all $x \in [a, b]$.

Solution. We may assume $a < b$, since if $a = b$ there is nothing to prove.

The interval $[a, b]$ is a closed, bounded set of real numbers and f is continuous on it, so by Theorem 23 the function f is uniformly continuous on $[a, b]$. Let $\epsilon > 0$ be given and choose $\delta > 0$ so that

$$\text{if } x, x' \in [a, b] \text{ and } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \epsilon.$$

By the Archimedean Property of $\mathbb{R}$ choose a natural number n with $n > (b-a)/\delta$, so that $(b-a)/n < \delta$. Define the uniform partition

$$x_i = a + i \cdot \frac{b-a}{n} \quad \text{for } i = 0, 1, \dots, n,$$

so $a = x_0 < x_1 < \cdots < x_n = b$ and each subinterval $[x_i, x_{i+1}]$ has length $(b-a)/n < \delta$.

Define $\varphi: [a, b] \rightarrow \mathbb{R}$ by prescribing it on each subinterval to be the linear function agreeing with f at the two endpoints: for $0 \leq i \leq n-1$ and $x \in [x_i, x_{i+1}]$, set

$$\varphi(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} (x - x_i).$$

This is well defined. At an interior partition point x_i with $1 \leq i \leq n-1$ the formula on $[x_{i-1}, x_i]$ gives the value $f(x_i)$ and the formula on $[x_i, x_{i+1}]$ also gives $f(x_i)$, so the two prescriptions agree. Thus $\varphi(x_i) = f(x_i)$ for every i . Each piece is a linear, hence continuous, function on a closed interval and consecutive pieces agree at the shared endpoint, so φ is continuous on $[a, b]$ and is linear on each $[x_i, x_{i+1}]$. Hence φ is piecewise linear.

It remains to estimate $|f - \varphi|$. Let $x \in [a, b]$ and choose i with $x \in [x_i, x_{i+1}]$. Write

$$\lambda = \frac{x_{i+1} - x}{x_{i+1} - x_i} \in [0, 1], \quad 1 - \lambda = \frac{x - x_i}{x_{i+1} - x_i}.$$

A direct computation from the defining formula shows that $\varphi(x)$ is the corresponding convex combination of the endpoint values:

$$\varphi(x) = \lambda f(x_i) + (1 - \lambda) f(x_{i+1}).$$

Since $\lambda + (1 - \lambda) = 1$ we may write $f(x) = \lambda f(x) + (1 - \lambda) f(x)$, and therefore

$$|f(x) - \varphi(x)| = |\lambda(f(x) - f(x_i)) + (1 - \lambda)(f(x) - f(x_{i+1}))| \leq \lambda|f(x) - f(x_i)| + (1 - \lambda)|f(x) - f(x_{i+1})|.$$

Now x and x_i both lie in $[x_i, x_{i+1}]$, an interval of length $(b-a)/n < \delta$, so $|x - x_i| < \delta$ and hence $|f(x) - f(x_i)| < \epsilon$; for the same reason $|f(x) - f(x_{i+1})| < \epsilon$. Consequently

$$|f(x) - \varphi(x)| < \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon.$$

(If λ is 0 or 1 the corresponding term is absent and the strict inequality persists, since then $|f(x) - \varphi(x)|$ equals $|f(x) - f(x_{i+1})|$ or $|f(x) - f(x_i)|$, each less than ϵ .)

Since $x \in [a, b]$ was arbitrary, $|f(x) - \varphi(x)| < \epsilon$ for all $x \in [a, b]$, as required.

Problem 1.52. Show that a nonempty set E of real numbers is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.

Solution. Let E be a nonempty set of real numbers. We say that a real-valued function f on E **takes a maximum value** provided there is a point x_* in E with $f(x) \leq f(x_*)$ for all x in E .

Necessity. Suppose E is closed and bounded. Let f be any continuous real-valued function on E . Since E is nonempty, closed and bounded, the Extreme Value Theorem asserts that f takes a maximum value on E . This is exactly the assertion to be proved.

Sufficiency. Suppose every continuous real-valued function on E takes a maximum value. We prove E is bounded and closed.

E is bounded. Consider $f(x) = |x|$ for x in E . This function is continuous on E : for x, x' in E the reverse triangle inequality gives $||x'| - |x|| \leq |x' - x|$, so f is Lipschitz with constant 1 and hence continuous. By hypothesis f takes a maximum value, say at $x_* \in E$. Then $|x| \leq |x_*|$ for all $x \in E$, so $E \subseteq [-|x_*|, |x_*|]$ and E is bounded.

E is closed. By definition E is closed provided it contains all of its points of closure. Argue by contradiction: suppose there is a point of closure x_0 of E with $x_0 \notin E$. Define

$$f(x) = \frac{1}{|x - x_0|} \quad \text{for } x \in E.$$

This is well defined, since $x_0 \notin E$ forces $|x - x_0| > 0$ for every $x \in E$. It is continuous on E : the function $x \mapsto |x - x_0|$ is continuous and nowhere zero on E , and $t \mapsto 1/t$ is continuous on $(0, \infty)$, so by the continuity of compositions the quotient is continuous at each point of E . (Concretely, fix $x \in E$ and put $r = |x - x_0| > 0$; if $x' \in E$ and $|x' - x| < r/2$ then $|x' - x_0| > r/2$ and

$$|f(x') - f(x)| = \frac{||x - x_0| - |x' - x_0||}{|x' - x_0| |x - x_0|} \leq \frac{|x' - x|}{(r/2)r},$$

which is less than any prescribed $\epsilon > 0$ once $|x' - x|$ is small enough.)

By hypothesis f takes a maximum value on E , say $f(x) \leq M$ for all $x \in E$, where $M = f(x_*) > 0$. Thus $|x - x_0| \geq 1/M$ for every $x \in E$. But x_0 is a point of closure of E , so every open interval about x_0 contains a point of E ; in particular there is an $x \in E$ with $|x - x_0| < 1/M$. This contradiction shows that every point of closure of E belongs to E , that is, E is closed.

Hence E is closed and bounded, and the equivalence is established.

Problem 1.53. Show that a set E of real numbers is closed and bounded if and only if every open cover of E has a finite subcover.

Solution. Recall that a collection $\mathcal{F}$ of sets is a **cover** of E provided $E \subseteq \bigcup_{F \in \mathcal{F}} F$, and it is an **open cover** provided each member is open. A **finite subcover** is a finite subcollection of $\mathcal{F}$ that still covers E .

Necessity. Suppose E is closed and bounded. Then the Heine-Borel Theorem asserts precisely that every open cover of E has a finite subcover.

Sufficiency. Suppose every open cover of E has a finite subcover. (If E is empty it is trivially closed and bounded, so assume $E \neq \emptyset$.)

E is bounded. For each natural number n let $\mathcal{O}_n = (-n, n)$, an open interval. Every real number x satisfies $|x| < n$ for some natural number n , by the Archimedean Property, so $\{\mathcal{O}_n\}_{n=1}^\infty$ is an open cover of E . By hypothesis there are indices $n_1, \dots, n_k$ with

$$E \subseteq \mathcal{O}_{n_1} \cup \dots \cup \mathcal{O}_{n_k}.$$

The intervals $\mathcal{O}_n$ are ascending, so putting $N = \max\{n_1, \dots, n_k\}$ the union equals $\mathcal{O}_N = (-N, N)$. Thus $E \subseteq (-N, N)$ and E is bounded.

E is closed. We must show E contains each of its points of closure. Argue by contradiction: let x_0 be a point of closure of E with $x_0 \notin E$. For each natural number n define

$$\mathcal{O}_n = \left\{ x \in \mathbb{R} : |x - x_0| > \frac{1}{n} \right\} = \left(-\infty, x_0 - \frac{1}{n} \right) \cup \left(x_0 + \frac{1}{n}, \infty \right),$$

which, being the union of two open intervals, is open. The collection $\{\mathcal{O}_n\}_{n=1}^\infty$ covers E : if $x \in E$ then $x \neq x_0$ since $x_0 \notin E$, hence $|x - x_0| > 0$, and by the Archimedean Property there is an n with $1/n < |x - x_0|$, so $x \in \mathcal{O}_n$.

By hypothesis finitely many of these sets cover E . The sets $\mathcal{O}_n$ ascend with n , so as above there is a single natural number N with $E \subseteq \mathcal{O}_N$, that is,

$$|x - x_0| > \frac{1}{N} \quad \text{for all } x \in E.$$

Therefore the open interval $(x_0 - 1/N, x_0 + 1/N)$ contains no point of E . But x_0 is a point of closure of E , so every open interval containing x_0 must contain a point of E . This contradiction shows no such x_0 exists, so E contains all its points of closure and hence is closed.

Thus E is closed and bounded, completing the proof of the equivalence.

Problem 1.54. Show that a nonempty set E of real numbers is an interval if and only if every continuous real-valued function on E has an interval as its image.

Solution. It is convenient first to record the internal characterization of intervals.

Lemma. A nonempty set I of real numbers is an interval if and only if it has the **betweenness property**: whenever $x, y \in I$ and $x < z < y$, then $z \in I$.

Proof of the Lemma. Every interval visibly has the betweenness property, since each of the sets (a, b) , $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞) , $[a, \infty)$, $\mathbb{R}$ is described by inequalities on the point itself. Conversely, let I be nonempty with the betweenness property. Put $a = \inf I$ and $b = \sup I$, allowing the values $-\infty$ and ∞ if I fails to be bounded below or above. Then $I \subseteq [a, b]$ (interpreted appropriately when an endpoint is infinite). We claim $(a, b) \subseteq I$. Let $a < z < b$. Since $z > a = \inf I$, the number z is not a lower bound for I , so there is $x \in I$ with $x < z$; since $z < b = \sup I$, the number z is not an upper bound for I , so there is $y \in I$ with $z < y$. By the betweenness property $z \in I$. Hence $(a, b) \subseteq I \subseteq [a, b]$, so I is obtained from (a, b) by possibly adjoining the finite endpoints a and b ; in every case I is an interval. (If $a = b$ then $I = \{a\} = [a, a]$, a degenerate closed interval.) $\square$

Necessity. Suppose E is an interval and let f be a continuous real-valued function on E . The image $f(E)$ is nonempty since E is. We verify the betweenness property for $f(E)$; the Lemma then shows $f(E)$ is an interval.

Let $u, v \in f(E)$ and $u < w < v$. Choose $x, y \in E$ with $f(x) = u$ and $f(y) = v$. Since $u \neq v$ we have $x \neq y$.

Case $x < y$. Because E is an interval containing x and y , the betweenness property gives $[x, y] \subseteq E$. The restriction of f to the closed, bounded interval $[x, y]$ is continuous and $f(x) = u < w < v = f(y)$, so by the Intermediate Value Theorem there is a point $x_0 \in (x, y)$ with $f(x_0) = w$. Hence $w \in f(E)$.

Case $y < x$. Again $[y, x] \subseteq E$. Consider $g = -f$ on $[y, x]$, which is continuous, and note

$$g(y) = -v < -w < -u = g(x).$$

By the Intermediate Value Theorem applied to g on $[y, x]$ there is $x_0 \in (y, x)$ with $g(x_0) = -w$, that is, $f(x_0) = w$. Hence $w \in f(E)$.

In either case $w \in f(E)$, so $f(E)$ has the betweenness property and is therefore an interval.

Sufficiency. Suppose every continuous real-valued function on E has an interval as its image. Consider the identity function $f(x) = x$ for $x \in E$. It is Lipschitz with constant 1, hence continuous on E , and its image is $f(E) = E$. By hypothesis this image is an interval, so E is an interval.

This proves the asserted equivalence.

Problem 1.55. Show that a monotone function on an open interval is continuous if and only if its image is an interval.

Solution. Let I be an open interval and f a monotone real-valued function on I .

Throughout we use the following characterization, established in the solution to Problem 54: a nonempty set S of real numbers is an interval if and only if it has the **betweenness property**, namely that $u, v \in S$ and $u < w < v$ imply $w \in S$.

Reduction to the increasing case. If f is decreasing then $g = -f$ is increasing on I ; moreover f is

continuous at a point if and only if g is, and $g(I) = \{-y : y \in f(I)\}$. A set S has the betweenness property if and only if $-S = \{-y : y \in S\}$ does, since $u < w < v$ holds exactly when $-v < -w < -u$ holds. Hence $f(I)$ is an interval exactly when $g(I)$ is. So the equivalence for g yields the equivalence for f , and we may assume throughout that f is increasing, that is, $f(x) \leq f(x')$ whenever $x, x' \in I$ and $x \leq x'$.

Continuity implies the image is an interval. This half needs no monotonicity. An open interval I is an interval, so by the necessity half of Problem 54 (whose proof rests on the Intermediate Value Theorem) the image under a continuous real-valued function of an interval is again an interval. Explicitly: if $u = f(x)$ and $v = f(y)$ lie in $f(I)$ and $u < w < v$, then $[x, y] \subseteq I$ or $[y, x] \subseteq I$ according to the order of x and y , and the Intermediate Value Theorem applied to f (or to $-f$) on that closed, bounded subinterval produces a point x_0 between x and y with $f(x_0) = w$. So $f(I)$ has the betweenness property and hence is an interval.

The image is an interval implies continuity. Assume $f(I)$ is an interval, and let $x_0 \in I$. As in the discussion preceding these problems, monotonicity together with the Monotone Convergence Theorem for real sequences guarantees that the one-sided limits

$$f(x_0^-) = \lim_{x \rightarrow x_0^-} f(x), \quad f(x_0^+) = \lim_{x \rightarrow x_0^+} f(x)$$

exist and are independent of the approximating sequence, and f is continuous at x_0 if and only if $f(x_0^-) = f(x_0) = f(x_0^+)$. Since I is open, both one-sided limits are genuinely available: there are points of I on each side of x_0 .

For an increasing f these limits are given by

$$f(x_0^-) = \sup\{f(x) : x \in I, x < x_0\}, \quad f(x_0^+) = \inf\{f(x) : x \in I, x > x_0\}.$$

Indeed, since I is open there are points of I strictly to the left of x_0 , so the set $S^- = \{f(x) : x \in I, x < x_0\}$ is nonempty; as f is increasing, S^- is bounded above by $f(x_0)$, and hence $s = \sup S^-$ is a real number. Choose a sequence $\{x_n\}$ in I with $x_n < x_0$ for all n and $x_n \rightarrow x_0$; by the discussion above $f(x_n) \rightarrow f(x_0^-)$. Each $f(x_n)$ belongs to S^- , so $f(x_n) \leq s$ for every n and therefore $f(x_0^-) \leq s$. Conversely, fix $x \in I$ with $x < x_0$. Since $x_n \rightarrow x_0$ and $x < x_0$, we have $x_n > x$ for all sufficiently large n , and then $f(x_n) \geq f(x)$; letting $n \rightarrow \infty$ gives $f(x_0^-) \geq f(x)$. As x was an arbitrary point of I to the left of x_0 , taking the supremum yields $f(x_0^-) \geq s$. Therefore $f(x_0^-) = s$. The formula for $f(x_0^+)$ is proved symmetrically. In particular

$$f(x_0^-) \leq f(x_0) \leq f(x_0^+).$$

Suppose, to get a contradiction, that f is not continuous at x_0 . Then $f(x_0^-) < f(x_0^+)$. Two consequences of the sup/inf formulas are:

$$f(x) \leq f(x_0^-) \text{ for all } x \in I \text{ with } x < x_0, \quad f(x) \geq f(x_0^+) \text{ for all } x \in I \text{ with } x > x_0.$$

Hence the only value of f that can lie in the open interval $(f(x_0^-), f(x_0^+))$ is $f(x_0)$ itself.

The open interval $(f(x_0^-), f(x_0^+))$ is nonempty and contains more than one point, so we may choose a number c with

$$f(x_0^-) < c < f(x_0^+) \quad \text{and} \quad c \neq f(x_0).$$

By the preceding paragraph $c \notin f(I)$.

Since I is open and $x_0 \in I$, choose $x_1, x_2 \in I$ with $x_1 < x_0 < x_2$. Then

$$f(x_1) \leq f(x_0^-) < c < f(x_0^+) \leq f(x_2),$$

so $f(x_1)$ and $f(x_2)$ belong to $f(I)$ and c lies strictly between them, yet $c \notin f(I)$. This contradicts the betweenness property of the interval $f(I)$.

Therefore $f(x_0^-) = f(x_0) = f(x_0^+)$ and f is continuous at x_0 . As $x_0 \in I$ was arbitrary, f is continuous on I .

Problem 1.56. Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set.

Solution. Recall that a set of real numbers is a G_δ set provided it is the intersection of a countable collection of open sets. Let

$$C = \{x \in \mathbb{R} : f \text{ is continuous at } x\}.$$

We exhibit C as a countable intersection of open sets by means of the oscillation of f .

The oscillation. For $x \in \mathbb{R}$ and $\delta > 0$ define

$$\text{osc}(f, x, \delta) = \sup\{|f(y) - f(z)| : y, z \in (x - \delta, x + \delta)\},$$

an extended real number in $[0, \infty]$ (the supremum may be $+\infty$; nothing below is affected). Since the interval shrinks as δ decreases, $\delta \mapsto \text{osc}(f, x, \delta)$ is a nondecreasing function of δ , and we may define the **oscillation of f at x** by

$$\omega(x) = \inf_{\delta > 0} \text{osc}(f, x, \delta) \in [0, \infty].$$

Step 1: f is continuous at x if and only if $\omega(x) = 0$.

Suppose f is continuous at x and let $\epsilon > 0$. There is $\delta > 0$ such that $|f(y) - f(x)| < \epsilon/3$ whenever $|y - x| < \delta$. Then for any $y, z \in (x - \delta, x + \delta)$,

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \frac{2\epsilon}{3},$$

so $\text{osc}(f, x, \delta) \leq 2\epsilon/3 < \epsilon$ and hence $\omega(x) < \epsilon$. As $\epsilon > 0$ was arbitrary and $\omega(x) \geq 0$, we get $\omega(x) = 0$. Conversely suppose $\omega(x) = 0$ and let $\epsilon > 0$. By definition of the infimum there is $\delta > 0$ with $\text{osc}(f, x, \delta) < \epsilon$. If $|y - x| < \delta$, then y and x both lie in $(x - \delta, x + \delta)$, so

$$|f(y) - f(x)| \leq \text{osc}(f, x, \delta) < \epsilon.$$

Thus f is continuous at x . This proves $C = \{x \in \mathbb{R} : \omega(x) = 0\}$.

Step 2: for each natural number n , the set

$$\mathcal{O}_n = \left\{x \in \mathbb{R} : \omega(x) < \frac{1}{n}\right\}$$

is open.

Let $x \in \mathcal{O}_n$. Since $\omega(x) < 1/n$, there is $\delta > 0$ with $\text{osc}(f, x, \delta) < 1/n$. We claim the open interval $(x - \delta/2, x + \delta/2)$ is contained in $\mathcal{O}_n$. Indeed, let x' satisfy $|x' - x| < \delta/2$. If $y \in (x' - \delta/2, x' + \delta/2)$ then

$$|y - x| \leq |y - x'| + |x' - x| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

so $(x' - \delta/2, x' + \delta/2) \subseteq (x - \delta, x + \delta)$. The supremum defining the oscillation is therefore taken over a smaller collection of pairs, whence

$$\text{osc}\left(f, x', \frac{\delta}{2}\right) \leq \text{osc}(f, x, \delta) < \frac{1}{n},$$

and consequently $\omega(x') \leq \text{osc}(f, x', \delta/2) < 1/n$, that is, $x' \in \mathcal{O}_n$. So every point of $\mathcal{O}_n$ is contained in an open interval lying in $\mathcal{O}_n$, and $\mathcal{O}_n$ is open.

Step 3: conclusion.

We claim

$$C = \bigcap_{n=1}^{\infty} \mathcal{O}_n.$$

If $x \in C$ then $\omega(x) = 0 < 1/n$ for every n , so x lies in every $\mathcal{O}_n$. Conversely, if $x \in \mathcal{O}_n$ for every natural number n , then $0 \leq \omega(x) < 1/n$ for all n ; by the Archimedean Property this forces $\omega(x) = 0$, so $x \in C$ by Step 1.

Thus C is the intersection of the countable collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ of open sets, that is, the set of points at which f is continuous is a G_δ set.

1.9 Problems 1.57–1.60

Problem 1.57. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets.

Solution. Write

$$X = \{x \in \mathbb{R} : \{f_n(x)\}_{n=1}^\infty \text{ converges to a real number}\}.$$

Step 1: a Cauchy description of X . By Theorem 17 (the Cauchy Convergence Criterion for Real Sequences), a sequence of real numbers converges to a real number if and only if it is Cauchy. Hence x belongs to X if and only if the sequence $\{f_n(x)\}$ is Cauchy, that is, if and only if for every $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| \leq 1/k$ for all $n, m \geq N$.

(Using the closed inequality $\leq 1/k$ rather than $< 1/k$ is harmless: if the Cauchy condition holds with strict inequality for every positive ϵ it certainly holds with $\leq 1/k$ for every k , and conversely, given $\epsilon > 0$ choose k with $1/k < \epsilon$.)

Translating the three quantifiers into set operations, this says exactly that

$$X = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \bigcap_{m \geq N} E_{n,m,k}, \quad \text{where} \quad E_{n,m,k} = \left\{x \in \mathbb{R} : |f_n(x) - f_m(x)| \leq \frac{1}{k}\right\}.$$

Step 2: each $E_{n,m,k}$ is closed. Fix n, m, k and set $g = |f_n - f_m|$. First, $-f_m$ is continuous on $\mathbb{R}$: for any point x and any $\epsilon > 0$, a $\delta > 0$ that responds to the ϵ challenge for the continuity of f_m at x also responds for $-f_m$, since $|(-f_m)(x') - (-f_m)(x)| = |f_m(x') - f_m(x)|$. Next, by Problem 49 of this chapter, the sum of two continuous real-valued functions with a common domain is continuous, so $f_n + (-f_m) = f_n - f_m$ is continuous on $\mathbb{R}$; and by the same problem the absolute value of a continuous function is continuous, so $g = |f_n - f_m|$ is continuous on $\mathbb{R}$. The complement of $E_{n,m,k}$ is

$$\mathbb{R} \sim E_{n,m,k} = \left\{x \in \mathbb{R} : g(x) > \frac{1}{k}\right\} = g^{-1}\left(\left(\frac{1}{k}, \infty\right)\right).$$

The set $(1/k, \infty)$ is open, so by Proposition 22, applied with $E = \mathbb{R}$, we have $g^{-1}((1/k, \infty)) = \mathbb{R} \cap U = U$ for some open set U ; thus $\mathbb{R} \sim E_{n,m,k}$ is open, and by Proposition 11 the set $E_{n,m,k}$ is closed.

Step 3: assembling. For fixed k and N put

$$A_{N,k} = \bigcap_{n \geq N} \bigcap_{m \geq N} E_{n,m,k}.$$

This is an intersection of a collection of closed sets, hence is closed by Proposition 12, which states that the intersection of any collection of closed sets is closed.

Now define

$$F_k = \bigcup_{N=1}^{\infty} A_{N,k}.$$

This is a countable union of closed sets, so F_k is an F_σ set.

By Step 1,

$$X = \bigcap_{k=1}^{\infty} F_k,$$

which exhibits X as the intersection of the countable collection $\{F_k\}_{k=1}^\infty$ of F_σ sets.

Remark. Since a σ -algebra is closed with respect to the formation of countable unions and countable intersections, each F_σ set is a Borel set, and therefore X itself is a Borel set.

Problem 1.58. Let f be a continuous real-valued function on $\mathbb{R}$. Show that the inverse image with respect to f of an open set is open, of a closed set is closed, and of a Borel set is Borel.

Solution. Throughout we use the elementary identities, valid for any function f from $\mathbb{R}$ to $\mathbb{R}$ and any sets A, A_n of real numbers:

$$f^{-1}(\mathbb{R} \sim A) = \mathbb{R} \sim f^{-1}(A), \quad f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n).$$

Both are immediate from the definition $f^{-1}(A) = \{x \in \mathbb{R} : f(x) \in A\}$: for the first, $x \in f^{-1}(\mathbb{R} \sim A)$ means $f(x) \notin A$, which means $x \notin f^{-1}(A)$; for the second, $f(x)$ lies in some A_n if and only if x lies in some $f^{-1}(A_n)$.

Inverse images of open sets. Let O be an open set of real numbers. Since f is continuous on $E = \mathbb{R}$, Proposition 22 tells us that $f^{-1}(O) = E \cap U = \mathbb{R} \cap U = U$ for some open set U . Hence $f^{-1}(O)$ is open.

Inverse images of closed sets. Let C be a closed set of real numbers. By Proposition 11 the set $\mathbb{R} \sim C$ is open, so by the preceding paragraph $f^{-1}(\mathbb{R} \sim C)$ is open. But

$$f^{-1}(\mathbb{R} \sim C) = \mathbb{R} \sim f^{-1}(C),$$

so $\mathbb{R} \sim f^{-1}(C)$ is open, and therefore, again by Proposition 11, $f^{-1}(C)$ is closed.

Inverse images of Borel sets. Define

$$\mathcal{A} = \{A \subseteq \mathbb{R} : f^{-1}(A) \text{ is a Borel set}\}.$$

We claim $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$.

(i) The empty-set belongs to $\mathcal{A}$, since $f^{-1}(\emptyset) = \emptyset$, and $\emptyset$ is open, hence Borel.

(ii) If A belongs to $\mathcal{A}$, then $f^{-1}(\mathbb{R} \sim A) = \mathbb{R} \sim f^{-1}(A)$. Since $f^{-1}(A)$ is Borel and the collection $\mathcal{B}$ of Borel sets is a σ -algebra, hence closed with respect to the formation of complements, $\mathbb{R} \sim f^{-1}(A)$ is Borel. Thus $\mathbb{R} \sim A$ belongs to $\mathcal{A}$.

(iii) If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of sets in $\mathcal{A}$, then

$$f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n),$$

a countable union of Borel sets, which is Borel because $\mathcal{B}$ is closed with respect to the formation of countable unions. Thus $\bigcup_{n=1}^{\infty} A_n$ belongs to $\mathcal{A}$.

So $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$. Moreover $\mathcal{A}$ contains every open set: if O is open then, by the first paragraph, $f^{-1}(O)$ is open and hence is a Borel set, so O belongs to $\mathcal{A}$.

By definition, the collection $\mathcal{B}$ of Borel sets is the smallest σ -algebra of sets of real numbers that contains all of the open sets; by Proposition 13 this means precisely that any σ -algebra of subsets of $\mathbb{R}$ containing the open sets also contains $\mathcal{B}$. Since $\mathcal{A}$ is such a σ -algebra, $\mathcal{B} \subseteq \mathcal{A}$. That is, for every Borel set A the set $f^{-1}(A)$ is a Borel set, which is what we wished to show.

Problem 1.59. A sequence $\{f_n\}$ of real-valued functions defined on a set E is said to converge uniformly on E to a function f if given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set E . Prove that if $\{f_n\}$ converges uniformly to f on E , then f is continuous on E .

Solution. Let x be a point of E ; we show that f is continuous at x . Since x is an arbitrary point of the domain E , this establishes that f is continuous on E .

Let $\epsilon > 0$. Apply the definition of uniform convergence with the positive number $\epsilon/3$: there is an index N such that

$$|f_n(y) - f(y)| < \frac{\epsilon}{3} \quad \text{for all } y \in E \text{ and all } n \geq N.$$

Fix this index N . The crucial point is that the single index N works simultaneously at every point of E ; this is exactly what uniformity provides, and it is what makes the following estimate possible.

The function f_N is continuous on E , hence continuous at the point x . Therefore there is a $\delta > 0$ such

that

$$\text{if } y \in E \text{ and } |y - x| < \delta, \text{ then } |f_N(y) - f_N(x)| < \frac{\epsilon}{3}.$$

We claim this $\delta > 0$ responds to the $\epsilon > 0$ challenge regarding the criterion for the continuity of f at x . Indeed, let y belong to E with $|y - x| < \delta$. Adding and subtracting and using the triangle inequality twice,

$$\begin{aligned} |f(y) - f(x)| &= |[f(y) - f_N(y)] + [f_N(y) - f_N(x)] + [f_N(x) - f(x)]| \\ &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)|. \end{aligned}$$

The first term is less than $\epsilon/3$ by the choice of N : the uniform estimate holds for every point of E and every index at least N , and here the point is $y \in E$ and the index is $n = N$. The third term is less than $\epsilon/3$ for the same reason, with the point $x \in E$. The middle term is less than $\epsilon/3$ by the choice of δ . Therefore

$$|f(y) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus for each $\epsilon > 0$ there is a $\delta > 0$ for which $y \in E$ and $|y - x| < \delta$ imply $|f(y) - f(x)| < \epsilon$; that is, f is continuous at x . Since $x \in E$ was arbitrary, f is continuous on E .

Problem 1.60. Prove Proposition 21. Use this proposition and the Bolzano-Weierstrass Theorem to provide another proof of the Extreme Value Theorem.

[Proposition 21: A real-valued function f defined on a set E of real numbers is continuous at the point $x_* \in E$ if and only if whenever a sequence $\{x_n\}$ in E converges to x_* , its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.]

Solution. Part 1: proof of Proposition 21.

Suppose first that f is continuous at x_* , and let $\{x_n\}$ be a sequence in E that converges to x_* . Let $\epsilon > 0$. By continuity of f at x_* there is a $\delta > 0$ such that

$$\text{if } x' \in E \text{ and } |x' - x_*| < \delta, \text{ then } |f(x') - f(x_*)| < \epsilon.$$

Since $\{x_n\} \rightarrow x_*$, applied with the positive number δ , there is an index N such that $|x_n - x_*| < \delta$ for all $n \geq N$. Each x_n belongs to E , so for all $n \geq N$ we have $|f(x_n) - f(x_*)| < \epsilon$. As $\epsilon > 0$ was arbitrary, $\{f(x_n)\} \rightarrow f(x_*)$.

Conversely, suppose f is not continuous at x_* . Negating the definition of continuity at a point, there is an $\epsilon_0 > 0$ with the property that no $\delta > 0$ responds to the ϵ_0 challenge: for every $\delta > 0$ there is a point $x' \in E$ with $|x' - x_*| < \delta$ and $|f(x') - f(x_*)| \geq \epsilon_0$. For each natural number n , apply this with $\delta = 1/n$ to select a point $x_n \in E$ with

$$|x_n - x_*| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(x_*)| \geq \epsilon_0.$$

The sequence $\{x_n\}$ lies in E and converges to x_* : given $\epsilon > 0$, the Archimedean Property of $\mathbb{R}$ provides a natural number N with $1/N < \epsilon$, and then $|x_n - x_*| < 1/n \leq 1/N < \epsilon$ for all $n \geq N$. But $\{f(x_n)\}$ does not converge to $f(x_*)$, since no term of the sequence $\{f(x_n)\}$ lies within ϵ_0 of $f(x_*)$. Hence the sequential condition fails.

Contrapositively, if the sequential condition holds then f is continuous at x_* . This proves Proposition 21.

A lemma on closed sets. Before turning to the Extreme Value Theorem we record the following, which we use twice. If E is a closed set of real numbers, $\{x_n\}$ is a sequence in E , and $\{x_n\} \rightarrow x_*$, then $x_* \in E$. Indeed, let I be an open interval containing x_* ; choose $r > 0$ with $(x_* - r, x_* + r) \subseteq I$. Since $\{x_n\} \rightarrow x_*$ there is an index n with $|x_n - x_*| < r$, so $x_n \in I$, and $x_n \in E$. Thus every open interval containing x_* contains a point of E , that is, x_* is a point of closure of E . Since E is closed it contains all of its points of closure, so $x_* \in E$.

Part 2: another proof of the Extreme Value Theorem.

Let E be a nonempty closed, bounded set of real numbers and let f be a continuous real-valued function on E . We must show that f takes a maximum and a minimum value on E .

Claim A: $f(E)$ is bounded above. Suppose not. Then for each natural number n the number n fails to be an upper bound for $f(E)$, so we may choose $x_n \in E$ with $f(x_n) > n$. The sequence $\{x_n\}$ lies in the bounded set E and hence is bounded. By Theorem 16 (the Bolzano-Weierstrass Theorem) it has a convergent subsequence $\{x_{n_k}\}$, say $\{x_{n_k}\} \rightarrow x_*$. By the lemma, $x_* \in E$. Since f is continuous at x_* , Proposition 21 tells us that $\{f(x_{n_k})\} \rightarrow f(x_*)$, a real number. By Proposition 14, a convergent sequence of real numbers is bounded, so $\{f(x_{n_k})\}$ is bounded. But $\{n_k\}$ is a strictly increasing sequence of natural numbers, so $n_k \geq k$ for every k (an immediate induction), and therefore $f(x_{n_k}) > n_k \geq k$ for all k , which makes $\{f(x_{n_k})\}$ unbounded. This contradiction proves Claim A.

Since E is nonempty, $f(E)$ is a nonempty set of real numbers which by Claim A is bounded above. By the Completeness Axiom it has a supremum; define

$$m = \sup f(E) \in \mathbb{R}.$$

Claim B: f takes the value m . For each natural number n , the number $m - 1/n$ is smaller than m and hence is not an upper bound for $f(E)$, so we may choose $x_n \in E$ with

$$m - \frac{1}{n} < f(x_n) \leq m.$$

Again $\{x_n\}$ is a bounded sequence, so by the Bolzano-Weierstrass Theorem there is a subsequence $\{x_{n_k}\}$ converging to some real number x_* , and by the lemma $x_* \in E$. Since f is continuous at x_* , Proposition 21 gives $\{f(x_{n_k})\} \rightarrow f(x_*)$.

On the other hand $\{f(x_{n_k})\} \rightarrow m$. Indeed, let $\epsilon > 0$ and use the Archimedean Property to choose a natural number K with $1/K < \epsilon$. For $k \geq K$ we have $n_k \geq k \geq K$, so

$$|f(x_{n_k}) - m| = m - f(x_{n_k}) < \frac{1}{n_k} \leq \frac{1}{K} < \epsilon.$$

By Proposition 14 the limit of a convergent sequence is unique, so $f(x_*) = m$. Thus $f(x) \leq m = f(x_*)$ for every $x \in E$: the function f takes the maximum value m on E , attained at the point $x_* \in E$.

Finally, the function $-f$ is continuous on E (given $\epsilon > 0$, any δ that responds to the ϵ challenge for f at a point also responds for $-f$, since $|-f(x') + f(x)| = |f(x') - f(x)|$). Applying what has just been proved to $-f$, there is a point $x_{**} \in E$ with $-f(x) \leq -f(x_{**})$ for all $x \in E$, that is, $f(x_{**}) \leq f(x)$ for all $x \in E$. Hence f takes a minimum value on E , attained at x_{**} .

This completes the proof of the Extreme Value Theorem using only Proposition 21, the Bolzano-Weierstrass Theorem, Proposition 14, and the Completeness Axiom; in particular the Heine-Borel Theorem, used in the text's proof, is not needed.

2 Lebesgue Measure

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2.1 Problems 2.1–2.7

Problem 2.1. In the first three problems, let m be a set function defined for all sets in a σ -algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$.

Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

Solution. Since $\mathcal{A}$ is a σ -algebra it contains $\mathbb{R}$ and is closed with respect to the formation of complements and countable unions; by De Morgan's Identities it is therefore also closed with respect to the formation of countable intersections. In particular $\emptyset = \mathbb{R} \setminus \mathbb{R}$ belongs to $\mathcal{A}$, and since

$$B \setminus A = B \cap (\mathbb{R} \setminus A),$$

the set $B \setminus A$ also belongs to $\mathcal{A}$.

Define a countable collection $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{A}$ by

$$E_1 = A, \quad E_2 = B \setminus A, \quad E_k = \emptyset \text{ for } k \geq 3.$$

This collection is disjoint: $E_1 \cap E_2 = A \cap (B \setminus A) = \emptyset$, and every intersection involving one of the sets E_k with $k \geq 3$ is empty because those sets are empty. Moreover, since $A \subseteq B$,

$$\bigcup_{k=1}^{\infty} E_k = A \cup (B \setminus A) = B.$$

By the assumed countable additivity of m over countable disjoint collections of sets in $\mathcal{A}$,

$$m(B) = m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k) = m(A) + m(B \setminus A) + \sum_{k=3}^{\infty} m(\emptyset).$$

The set function m takes values in $[0, \infty]$, so every term on the right-hand side is nonnegative and the series of nonnegative terms is at least its first term. Therefore

$$m(B) \geq m(A),$$

which is the asserted monotonicity. Observe that this argument does not require knowing in advance that $m(\emptyset) = 0$; only the nonnegativity of m is used.

Problem 2.2. Let m be a set function defined for all sets in a σ -algebra $\mathcal{A}$ with values in $[0, \infty]$, and assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

Solution. Since $\mathcal{A}$ is a σ -algebra it contains $\mathbb{R}$ and is closed with respect to complementation, so $\emptyset = \mathbb{R} \setminus \mathbb{R}$ belongs to $\mathcal{A}$ and $m(\emptyset)$ is defined.

Let $A \in \mathcal{A}$ satisfy $m(A) < \infty$. Define a countable collection $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{A}$ by

$$E_1 = A, \quad E_k = \emptyset \text{ for } k \geq 2.$$

The collection is disjoint, since the intersection of any two distinct members is empty (indeed $A \cap \emptyset = \emptyset$ and $\emptyset \cap \emptyset = \emptyset$), and its union is A . Countable additivity of m therefore gives

$$m(A) = \sum_{k=1}^{\infty} m(E_k) = m(A) + \sum_{k=2}^{\infty} m(\emptyset).$$

Suppose, to obtain a contradiction, that $m(\emptyset) = c > 0$. Then $\sum_{k=2}^{\infty} m(\emptyset) = \sum_{k=2}^{\infty} c = \infty$, so the right-hand side of the displayed identity equals ∞ , while the left-hand side equals $m(A) < \infty$. This is

a contradiction. Since m takes values in $[0, \infty]$, the only remaining possibility is

$$m(\emptyset) = 0.$$

Equivalently and without contradiction: because $m(A) < \infty$ we may subtract the finite number $m(A)$ from both sides of the identity above to obtain $\sum_{k=2}^{\infty} m(\emptyset) = 0$; a series of identical nonnegative terms vanishes only if each term is 0, so $m(\emptyset) = 0$.

The hypothesis that some set has finite measure cannot be dropped: the set function defined on all of $\mathcal{A}$ by $m(E) = \infty$ for every $E \in \mathcal{A}$, including $E = \emptyset$, is countably additive in the stated sense yet has $m(\emptyset) \neq 0$.

Problem 2.3. Let m be a set function defined for all sets in a σ -algebra $\mathcal{A}$ with values in $[0, \infty]$, and assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k).$$

Solution. The strategy is to replace the collection $\{E_k\}_{k=1}^{\infty}$ by a disjoint collection with the same union, each member of which is contained in the corresponding E_k , and then to apply countable additivity together with the monotonicity established in Problem 1.

Define

$$F_1 = E_1, \quad F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j \quad \text{for } k \geq 2.$$

Each F_k belongs to $\mathcal{A}$: the union $\bigcup_{j=1}^{k-1} E_j$ is a finite, hence countable, union of members of $\mathcal{A}$ and so lies in $\mathcal{A}$; its complement lies in $\mathcal{A}$; and F_k is the intersection of E_k with that complement, which lies in $\mathcal{A}$ because a σ -algebra is closed with respect to countable intersections by De Morgan's Identities.

The collection $\{F_k\}_{k=1}^{\infty}$ is disjoint. Indeed, let $j < k$. Every point of F_k fails to belong to E_j , whereas every point of F_j belongs to E_j (since $F_j \subseteq E_j$); hence $F_j \cap F_k = \emptyset$.

The two collections have the same union. Since $F_k \subseteq E_k$ for every k , we have $\bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} E_k$. Conversely, let $x \in \bigcup_{k=1}^{\infty} E_k$. The set of indices k with $x \in E_k$ is a nonempty set of natural numbers, so by the Well-Ordering Principle it has a smallest member k_0 . Then $x \in E_{k_0}$ while $x \notin E_j$ for $1 \leq j < k_0$, that is, $x \in F_{k_0}$. Therefore

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k.$$

Now apply countable additivity of m to the disjoint collection $\{F_k\}_{k=1}^{\infty}$ of sets in $\mathcal{A}$:

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} m(F_k).$$

Since $F_k \subseteq E_k$ and both sets belong to $\mathcal{A}$, monotonicity (Problem 1) gives $m(F_k) \leq m(E_k)$ for each k . Comparison of series of nonnegative terms, term by term, yields

$$\sum_{k=1}^{\infty} m(F_k) \leq \sum_{k=1}^{\infty} m(E_k).$$

Combining the last two displays,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k),$$

as required. This property is called *countable monotonicity* (or countable subadditivity). The corresponding statement for a finite collection $\{E_k\}_{k=1}^n$ follows by setting $E_k = \emptyset$ for $k > n$, provided $m(\emptyset) = 0$; and if instead $m(\emptyset) > 0$, then applying countable additivity to the disjoint collection $E, \emptyset, \emptyset, \dots$ shows $m(E) = \infty$ for every $E \in \mathcal{A}$, so every such inequality holds trivially.

Problem 2.4. A set function c , defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**.

Solution. Throughout, for a finite set F we write $\#F$ for the number of its elements, so that $c(F) = \#F$ for finite F and $c(E) = \infty$ for infinite E . Note first the elementary monotonicity property: if $F \subseteq E$ and F is finite, then $c(E) \geq \#F$. Indeed, if E is infinite this reads $\infty \geq \#F$, and if E is finite then $\#E \geq \#F$ because a subset of a finite set has no more elements than the set.

Countable additivity. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of subsets of $\mathbb{R}$ and put $E = \bigcup_{k=1}^{\infty} E_k$. We show

$$c(E) = \sum_{k=1}^{\infty} c(E_k).$$

First we prove $c(E) \geq \sum_{k=1}^{\infty} c(E_k)$. Since the right-hand side is the supremum of its partial sums, it suffices to show that $c(E) \geq \sum_{k=1}^n c(E_k)$ for every natural number n , and, when some $c(E_k) = \infty$, that $c(E) = \infty$. In the latter case some E_{k_0} is infinite; since $E_{k_0} \subseteq E$, the set E is infinite as well and $c(E) = \infty$, so the inequality holds. So suppose every E_k is finite and fix n . The sets $E_1, \dots, E_n$ are pairwise disjoint finite sets, so their union $F = \bigcup_{k=1}^n E_k$ is finite with

$$\#F = \sum_{k=1}^n \#E_k = \sum_{k=1}^n c(E_k),$$

the first equality being the additivity of cardinality over disjoint finite sets. Since $F \subseteq E$, the monotonicity noted above gives $c(E) \geq \#F = \sum_{k=1}^n c(E_k)$. Letting $n \rightarrow \infty$,

$$c(E) \geq \sum_{k=1}^{\infty} c(E_k).$$

Next we prove $c(E) \leq \sum_{k=1}^{\infty} c(E_k)$. If the right-hand side is ∞ there is nothing to prove, so assume $\sum_{k=1}^{\infty} c(E_k) = M < \infty$. Then each $c(E_k)$ is finite, that is, each E_k is a finite set, and M is a finite nonnegative number (in fact a nonnegative integer, since a convergent series of nonnegative integers has all but finitely many terms equal to zero). We claim E is finite with $\#E \leq M$. Suppose otherwise. Then E contains at least $M+1$ distinct points; more precisely, if E were infinite we could choose $M+1$ distinct points of E , and if E were finite with $\#E > M$ we could likewise choose $M+1$ distinct points. Call them $x_1, \dots, x_{M+1}$. Each x_i lies in E , hence in E_{k_i} for some index k_i . Let $n = \max\{k_1, \dots, k_{M+1}\}$, a finite number since only finitely many indices occur. Then the $M+1$ distinct points all lie in the finite set $\bigcup_{k=1}^n E_k$, whence, using disjointness again,

$$M+1 \leq \# \left(\bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n c(E_k) \leq \sum_{k=1}^{\infty} c(E_k) = M,$$

a contradiction. Therefore E is finite and $c(E) = \#E \leq M = \sum_{k=1}^{\infty} c(E_k)$.

The two inequalities give $c(E) = \sum_{k=1}^{\infty} c(E_k)$, so c is countably additive. (In particular, taking all E_k empty is consistent with the stipulation $c(\emptyset) = 0$.)

Translation invariance. Let $E \subseteq \mathbb{R}$ and let y be a real number, and consider $E + y = \{x + y \mid x \in E\}$. The map $\tau_y : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\tau_y(x) = x + y$ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$, with inverse τ_{-y} , and $\tau_y(E) = E + y$. A bijection restricted to E is a one-to-one correspondence between E and $E + y$. Consequently E is finite if and only if $E + y$ is finite, and in that case $\#(E + y) = \#E$; likewise E is infinite if and only if $E + y$ is infinite. Also $\emptyset + y = \emptyset$. Examining the three clauses of the definition of c , we conclude in every case that

$$c(E + y) = c(E),$$

so c is translation invariant.

Problem 2.5. By using properties of outer measure, prove that the interval $[0, 1]$ is not countable.

Solution. We use two facts about Lebesgue outer measure m^* established in Section 2.2: the outer measure of an interval is its length (Proposition 1), and a countable set has outer measure zero (the Example preceding Proposition 1).

For completeness we recall the proof of the second fact. Let C be a countable set, enumerated as $C = \{c_k\}_{k=1}^\infty$ (a finite set is handled the same way, repeating a point). Let $\epsilon > 0$. For each natural number k define the open bounded interval

$$I_k = \left(c_k - \frac{\epsilon}{2^{k+1}}, c_k + \frac{\epsilon}{2^{k+1}}\right),$$

which has length $\epsilon/2^k$. The countable collection $\{I_k\}_{k=1}^\infty$ covers C , since $c_k \in I_k$ for each k . Hence, by the definition of outer measure as an infimum over such covers,

$$0 \leq m^*(C) \leq \sum_{k=1}^\infty \ell(I_k) = \sum_{k=1}^\infty \frac{\epsilon}{2^k} = \epsilon.$$

This inequality holds for every $\epsilon > 0$, so $m^*(C) = 0$.

Now suppose, to obtain a contradiction, that $[0, 1]$ is countable. Then by the fact just proved,

$$m^*([0, 1]) = 0.$$

On the other hand, $[0, 1]$ is an interval of length 1, so by Proposition 1,

$$m^*([0, 1]) = \ell([0, 1]) = 1 - 0 = 1.$$

Thus $0 = 1$, which is absurd. Therefore $[0, 1]$ is not countable.

(The set $[0, 1]$ is certainly not finite, so the conclusion is that $[0, 1]$ is uncountable. The same argument shows that no interval of positive length is countable.)

Problem 2.6. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.

Solution. Write $B = [0, 1] \setminus A$ for the set of rational numbers in $[0, 1]$, so that

$$[0, 1] = A \cup B.$$

The set B is countable, being a subset of the countable set $\mathbb{Q}$ of rational numbers. Hence, by the Example of Section 2.2 (a countable set has outer measure zero),

$$m^*(B) = 0.$$

By Proposition 1, the outer measure of an interval is its length, so

$$m^*([0, 1]) = \ell([0, 1]) = 1.$$

Now use the finite subadditivity of outer measure, which is the special case of Proposition 3 (countable subadditivity) obtained by taking all but finitely many sets in the collection to be empty:

$$1 = m^*([0, 1]) = m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(A) + 0 = m^*(A).$$

Therefore $m^*(A) \geq 1$.

For the reverse inequality, observe that $A \subseteq [0, 1]$, so by the monotonicity of outer measure, which follows immediately from the definition since any cover of $[0, 1]$ is a cover of A ,

$$m^*(A) \leq m^*([0, 1]) = 1.$$

Combining the two inequalities, $m^*(A) = 1$.

Thus $[0, 1]$ is the disjoint union of the rationals in $[0, 1]$, which carry outer measure 0, and the irrationals in $[0, 1]$, which carry all of the outer measure 1.

Problem 2.7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E , there is a G_δ set G for which

$$E \subseteq G \quad \text{and} \quad m^*(G) = m^*(E).$$

Solution. Let E be a bounded set of real numbers.

Step 1: the outer measure of E is finite. Since E is bounded there is a number $M > 0$ with $E \subseteq (-M, M)$. By the monotonicity of outer measure and Proposition 1 (the outer measure of an interval is its length),

$$m^*(E) \leq m^*((-M, M)) = \ell((-M, M)) = 2M < \infty.$$

Step 2: construction of open sets of nearly minimal outer measure containing E . Fix a natural number n . By the definition of $m^*(E)$ as the infimum of the sums $\sum_{k=1}^{\infty} \ell(I_k)$ taken over all countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover E , and since $m^*(E)$ is finite so that $m^*(E) + 1/n$ is not a lower bound for these sums, there is such a collection $\{I_{n,k}\}_{k=1}^{\infty}$ with

$$E \subseteq \bigcup_{k=1}^{\infty} I_{n,k} \quad \text{and} \quad \sum_{k=1}^{\infty} \ell(I_{n,k}) < m^*(E) + \frac{1}{n}.$$

Define

$$\mathcal{O}_n = \bigcup_{k=1}^{\infty} I_{n,k}.$$

Then $\mathcal{O}_n$ is open, being a union of open intervals, and $E \subseteq \mathcal{O}_n$. Moreover, by the countable subadditivity of outer measure (Proposition 3) together with Proposition 1,

$$m^*(\mathcal{O}_n) \leq \sum_{k=1}^{\infty} m^*(I_{n,k}) = \sum_{k=1}^{\infty} \ell(I_{n,k}) < m^*(E) + \frac{1}{n}.$$

Step 3: intersection of the open sets. Define

$$G = \bigcap_{n=1}^{\infty} \mathcal{O}_n.$$

As the intersection of a countable collection of open sets, G is a G_δ set. Since $E \subseteq \mathcal{O}_n$ for every n , we have $E \subseteq G$.

Step 4: equality of the outer measures. By monotonicity of outer measure, $E \subseteq G$ gives

$$m^*(E) \leq m^*(G).$$

On the other hand, for each natural number n we have $G \subseteq \mathcal{O}_n$, so monotonicity and Step 2 give

$$m^*(G) \leq m^*(\mathcal{O}_n) < m^*(E) + \frac{1}{n}.$$

Thus $m^*(E) \leq m^*(G) < m^*(E) + 1/n$ for every natural number n . Letting $n \rightarrow \infty$ (equivalently, since $m^*(G) - m^*(E)$ is a nonnegative number smaller than every $1/n$, it must be zero; here finiteness of $m^*(E)$ from Step 1 permits the subtraction), we conclude

$$m^*(G) = m^*(E).$$

Hence G is a G_δ set with $E \subseteq G$ and $m^*(G) = m^*(E)$, as required.

Remark. Boundedness of E was used only to guarantee $m^*(E) < \infty$. If E is any set with $m^*(E) = \infty$, one may simply take $G = \mathbb{R}$, which is open and hence a G_δ set, and then $m^*(G) = \infty = m^*(E)$ by monotonicity. So the conclusion in fact holds for every set of real numbers.

2.2 Problems 2.8–2.14

Problem 2.8. Let B be the set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B . Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.

Solution. If some I_k is unbounded then $m^*(I_k) = \infty$ by Proposition 1 (the outer measure of an interval is its length), and the asserted inequality is trivial. So assume every I_k is a bounded open interval, say $I_k = (a_k, b_k)$ with $a_k \leq b_k$.

The essential point is that the finiteness of the collection lets us pass to closures. Write $\bar{S}$ for the closure of a set S . For a finite family of sets the closure of the union is the union of the closures:

$$\overline{\bigcup_{k=1}^n I_k} = \bigcup_{k=1}^n \bar{I}_k.$$

Indeed, $\bigcup_{k=1}^n \bar{I}_k$ is a finite union of closed sets, hence closed, and it contains $\bigcup_{k=1}^n I_k$; therefore it contains the closure of that union. Conversely each $\bar{I}_k$ is contained in the closure of the union by monotonicity of closure. (This is exactly where finiteness is used: an infinite union of closed sets need not be closed.)

Now $B \subseteq \bigcup_{k=1}^n I_k$ by hypothesis, so

$$[0, 1] = \bar{B} \subseteq \overline{\bigcup_{k=1}^n I_k} = \bigcup_{k=1}^n \bar{I}_k,$$

the first equality holding because the rationals in $[0, 1]$ are dense in $[0, 1]$: every point of $[0, 1]$ is a limit of rationals lying in $[0, 1]$, and $[0, 1]$ is closed, so $\bar{B} = [0, 1]$.

Apply monotonicity of outer measure and then finite subadditivity (the finite form of Proposition 3):

$$1 = m^*([0, 1]) \leq m^*\left(\bigcup_{k=1}^n \bar{I}_k\right) \leq \sum_{k=1}^n m^*(\bar{I}_k).$$

Here $m^*([0, 1]) = \ell([0, 1]) = 1$ by Proposition 1. Finally, $\bar{I}_k = [a_k, b_k]$ is a closed bounded interval of length $b_k - a_k = \ell(I_k)$, so by Proposition 1 again

$$m^*(\bar{I}_k) = \ell(\bar{I}_k) = \ell(I_k) = m^*(I_k).$$

(If I_k is empty this reads $0 = 0$.) Substituting,

$$1 \leq \sum_{k=1}^n m^*(I_k),$$

which is the desired inequality.

A remark on why the hypothesis of finiteness cannot be dropped: B is countable, say $B = \{q_1, q_2, \dots\}$, and for any $\epsilon > 0$ the countable collection $I_k = (q_k - \epsilon/2^{k+1}, q_k + \epsilon/2^{k+1})$ is a collection of open intervals covering B with $\sum_{k=1}^{\infty} \ell(I_k) = \epsilon$. Thus $m^*(B) = 0$, and the conclusion $\sum_k m^*(I_k) \geq 1$ genuinely depends on the cover being finite.

Problem 2.9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Solution. Both inequalities are immediate from the basic properties of outer measure established in Section 2.2.

First, $B \subseteq A \cup B$, so by the monotonicity of outer measure

$$m^*(B) \leq m^*(A \cup B).$$

Monotonicity holds because every countable collection of open bounded intervals covering $A \cup B$ also covers B , so the infimum defining $m^*(B)$ is taken over a larger collection of admissible sums than the infimum defining $m^*(A \cup B)$.

Second, by the finite subadditivity of outer measure (the finite form of Proposition 3, obtained by taking $E_1 = A$, $E_2 = B$ and $E_k = \emptyset$ for $k > 2$),

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = 0 + m^*(B) = m^*(B).$$

Combining the two displays gives $m^*(A \cup B) = m^*(B)$. Note that the argument is valid whether $m^*(B)$ is finite or infinite: if $m^*(B) = \infty$ the first inequality already forces $m^*(A \cup B) = \infty$.

Problem 2.10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Solution. The inequality $m^*(A \cup B) \leq m^*(A) + m^*(B)$ is finite subadditivity (Proposition 3), and requires no hypothesis. The content of the problem is the reverse inequality, and the separation hypothesis is what supplies it. Since $A \cup B$ is bounded it is contained in some bounded interval, so $m^*(A \cup B) < \infty$ by monotonicity and Proposition 1.

Step 1 (a subdivision lemma). Let I be a nonempty bounded open interval, and let $\eta > 0$. Then there are finitely many open intervals $J_1, \dots, J_N$, each of length less than α , such that

$$I \subseteq \bigcup_{i=1}^N J_i \quad \text{and} \quad \sum_{i=1}^N \ell(J_i) < \ell(I) + \eta.$$

To see this, write $I = (c, d)$ and $L = d - c > 0$. Choose a natural number N with $L/N < \alpha/2$, put $h = L/N$ and $t_i = c + ih$ for $i = 0, 1, \dots, N$, so that $t_0 = c$, $t_N = d$ and $\bigcup_{i=1}^N [t_{i-1}, t_i] = [c, d]$. Choose

$$\delta = \min \left\{ \frac{\alpha}{8}, \frac{\eta}{4N} \right\} > 0$$

and set $J_i = (t_{i-1} - \delta, t_i + \delta)$. Then $I \subseteq [c, d] \subseteq \bigcup_{i=1}^N J_i$; each J_i has length $h + 2\delta < \alpha/2 + \alpha/4 < \alpha$; and

$$\sum_{i=1}^N \ell(J_i) = Nh + 2N\delta = L + 2N\delta \leq \ell(I) + \frac{\eta}{2} < \ell(I) + \eta.$$

Step 2 (refining an arbitrary cover). Let $\epsilon > 0$. By the definition of outer measure there is a countable collection $\{I_k\}_{k=1}^\infty$ of open bounded intervals with

$$A \cup B \subseteq \bigcup_{k=1}^\infty I_k \quad \text{and} \quad \sum_{k=1}^\infty \ell(I_k) < m^*(A \cup B) + \epsilon.$$

Discard any empty I_k . For each k apply Step 1 with $\eta = \epsilon/2^k$ to obtain finitely many open intervals $\{J_{k,1}, \dots, J_{k,N_k}\}$, each of length less than α , which cover I_k and satisfy $\sum_{i=1}^{N_k} \ell(J_{k,i}) < \ell(I_k) + \epsilon/2^k$. The collection $\mathcal{J} = \{J_{k,i} : k \geq 1, 1 \leq i \leq N_k\}$ is countable (a countable union of finite collections), consists of open bounded intervals each of length less than α , covers $\bigcup_k I_k$ and hence covers $A \cup B$, and satisfies

$$\sum_{J \in \mathcal{J}} \ell(J) < \sum_{k=1}^\infty \left(\ell(I_k) + \frac{\epsilon}{2^k} \right) < m^*(A \cup B) + 2\epsilon.$$

Step 3 (splitting the refined cover). If J is an interval with $\ell(J) < \alpha$, then any two points $x, y \in J$ satisfy $|x - y| \leq \ell(J) < \alpha$. Consequently no member of $\mathcal{J}$ can meet both A and B : if $a \in A \cap J$ and $b \in B \cap J$ we would get $|a - b| < \alpha$, contradicting the hypothesis $|a - b| \geq \alpha$.

Therefore we may split $\mathcal{J}$ into two disjoint subcollections

$$\mathcal{J}_A = \{J \in \mathcal{J} : J \cap A \neq \emptyset\}, \quad \mathcal{J}_B = \mathcal{J} \setminus \mathcal{J}_A,$$

and every $J \in \mathcal{J}$ that meets B lies in $\mathcal{J}_B$, by the previous paragraph. Since $\mathcal{J}$ covers A , and a member of $\mathcal{J}$ contributes a point of A only if it meets A , the subcollection $\mathcal{J}_A$ covers A ; likewise $\mathcal{J}_B$ contains every member of $\mathcal{J}$ meeting B and hence covers B . Both subcollections are countable collections of open bounded intervals, so by the definition of outer measure

$$m^*(A) \leq \sum_{J \in \mathcal{J}_A} \ell(J), \quad m^*(B) \leq \sum_{J \in \mathcal{J}_B} \ell(J).$$

(If one of the subcollections is empty, the corresponding set is empty and its outer measure is 0, so the inequality still holds.)

Step 4 (conclusion). Since $\mathcal{J}_A$ and $\mathcal{J}_B$ are disjoint subcollections of $\mathcal{J}$ and all terms are nonnegative,

$$m^*(A) + m^*(B) \leq \sum_{J \in \mathcal{J}_A} \ell(J) + \sum_{J \in \mathcal{J}_B} \ell(J) = \sum_{J \in \mathcal{J}} \ell(J) < m^*(A \cup B) + 2\epsilon.$$

This holds for every $\epsilon > 0$, so $m^*(A) + m^*(B) \leq m^*(A \cup B)$. Together with subadditivity,

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

Problem 2.11. Prove that if a σ -algebra of subsets of $\mathbb{R}$ contains intervals of the form (a, ∞) , then it contains all intervals.

Solution. Let $\mathcal{A}$ be a σ -algebra of subsets of $\mathbb{R}$ with $(a, \infty) \in \mathcal{A}$ for every $a \in \mathbb{R}$. Recall that a σ -algebra contains $\emptyset$ and $\mathbb{R}$, is closed under complements and under countable unions, and hence, by De Morgan's Identities, is also closed under countable intersections: if $\{E_k\}_{k=1}^\infty \subseteq \mathcal{A}$ then

$$\bigcap_{k=1}^\infty E_k = \left(\bigcup_{k=1}^\infty E_k^C \right)^C \in \mathcal{A}.$$

Being closed under countable unions and intersections, $\mathcal{A}$ is in particular closed under finite unions and intersections (pad a finite family with $\emptyset$ or with $\mathbb{R}$).

We build up all interval types in turn. Fix real numbers $a < b$.

Rays of the form $(-\infty, a]$. Since $(a, \infty) \in \mathcal{A}$ and $(-\infty, a] = (a, \infty)^C$, closure under complements gives $(-\infty, a] \in \mathcal{A}$.

Rays of the form $[a, \infty)$. Observe that

$$[a, \infty) = \bigcap_{n=1}^\infty \left(a - \frac{1}{n}, \infty \right).$$

Indeed, if $x \geq a$ then $x > a - 1/n$ for every n ; conversely if $x > a - 1/n$ for every n then letting $n \rightarrow \infty$ gives $x \geq a$. Each set on the right lies in $\mathcal{A}$, and $\mathcal{A}$ is closed under countable intersections, so $[a, \infty) \in \mathcal{A}$.

Rays of the form $(-\infty, a)$. This is $[a, \infty)^C$, hence in $\mathcal{A}$.

Bounded intervals. Using closure under finite intersections,

$$\begin{aligned} (a, b) &= (a, \infty) \cap (-\infty, b), & [a, b] &= [a, \infty) \cap (-\infty, b], \\ [a, b) &= [a, \infty) \cap (-\infty, b), & (a, b] &= (a, \infty) \cap (-\infty, b], \end{aligned}$$

and all four right-hand sides are intersections of two members of $\mathcal{A}$. Hence every bounded interval with endpoints $a < b$, of any of the four types, belongs to $\mathcal{A}$.

Degenerate intervals. The empty set is in $\mathcal{A}$ by definition of a σ -algebra, and a singleton is

$$\{a\} = [a, \infty) \cap (-\infty, a] \in \mathcal{A}.$$

The whole line and half-lines already covered. We have $\mathbb{R} \in \mathcal{A}$, and the four unbounded types (a, ∞) , $[a, \infty)$, $(-\infty, a)$, $(-\infty, a]$ were treated above.

Every interval of real numbers is of one of the listed forms: it is empty, a singleton, a bounded interval with two distinct endpoints (open, closed, or half-open), a half-line of one of the four types, or all of $\mathbb{R}$. Therefore $\mathcal{A}$ contains all intervals.

Problem 2.12. Show that every interval is a Borel set.

Solution. Recall the definition given in Section 2.3: the Borel σ -algebra $\mathcal{B}$ is the intersection of all the σ -algebras of subsets of $\mathbb{R}$ that contain the open sets, and its members are the Borel sets. Two facts follow immediately from this definition.

First, $\mathcal{B}$ really is a σ -algebra. An intersection of a family of σ -algebras of subsets of $\mathbb{R}$ is again a σ -algebra: $\emptyset$ and $\mathbb{R}$ belong to every member of the family and hence to the intersection; if E belongs to the intersection then E^C belongs to each σ -algebra in the family and hence to the intersection; and if $\{E_k\}_{k=1}^\infty$ lies in the intersection then $\bigcup_k E_k$ lies in each σ -algebra of the family and hence in the intersection. (The family is nonempty, since the collection of all subsets of $\mathbb{R}$ is a σ -algebra containing the open sets.)

Second, $\mathcal{B}$ contains every open set, since every σ -algebra in the defining family does.

Now let $a \in \mathbb{R}$. The ray (a, ∞) is an open subset of $\mathbb{R}$: if $x > a$, then with $r = x - a > 0$ we have $(x - r, x + r) \subseteq (a, \infty)$. Hence $(a, \infty) \in \mathcal{B}$ for every real a .

So $\mathcal{B}$ is a σ -algebra of subsets of $\mathbb{R}$ containing every interval of the form (a, ∞) . By Problem 11 of this chapter, $\mathcal{B}$ contains all intervals. That is, every interval is a Borel set.

(For completeness, this also recovers part of Theorem 9: since the measurable sets form a σ -algebra containing all the open sets, $\mathcal{B} \subseteq \mathcal{M}$, so every interval is measurable as well.)

Problem 2.13. Show that (i) the translate of an F_σ set is also F_σ , (ii) the translate of a G_δ set is also G_δ , and (iii) the translate of a set of measure zero also has measure zero.

Solution. Throughout, for a set $E \subseteq \mathbb{R}$ and $y \in \mathbb{R}$ we write $E + y = \{x + y : x \in E\}$, and we use the elementary equivalence

$$x \in E + y \iff x - y \in E.$$

From this equivalence, translation commutes with arbitrary unions and intersections: for any family $\{E_\lambda\}$,

$$\left(\bigcup_\lambda E_\lambda\right) + y = \bigcup_\lambda (E_\lambda + y), \quad \left(\bigcap_\lambda E_\lambda\right) + y = \bigcap_\lambda (E_\lambda + y),$$

since in each case x belongs to the left-hand side if and only if $x - y$ belongs to some (respectively, to every) E_λ , which is exactly membership in the right-hand side.

We also record two topological facts.

The translate of an open set is open. Let $\mathcal{O}$ be open and $x \in \mathcal{O} + y$. Then $x - y \in \mathcal{O}$, so there is $r > 0$ with $(x - y - r, x - y + r) \subseteq \mathcal{O}$. Then $(x - r, x + r) \subseteq \mathcal{O} + y$, since $|t - x| < r$ implies $|(t - y) - (x - y)| < r$, so $t - y \in \mathcal{O}$. Hence $\mathcal{O} + y$ is open.

The translate of a closed set is closed. If C is closed then C^C is open, and complementation commutes with translation: $x \in (C + y)^C$ if and only if $x - y \notin C$, i.e. $x - y \in C^C$, i.e. $x \in C^C + y$. So $(C + y)^C = C^C + y$ is open by the previous paragraph, and therefore $C + y$ is closed.

(i) Let F be an F_σ set, that is, $F = \bigcup_{k=1}^\infty C_k$ with each C_k closed. Then

$$F + y = \left(\bigcup_{k=1}^\infty C_k\right) + y = \bigcup_{k=1}^\infty (C_k + y),$$

and each $C_k + y$ is closed. Thus $F + y$ is a countable union of closed sets, i.e. an F_σ set.

(ii) Let G be a G_δ set, that is, $G = \bigcap_{k=1}^\infty \mathcal{O}_k$ with each $\mathcal{O}_k$ open. Then

$$G + y = \left(\bigcap_{k=1}^\infty \mathcal{O}_k\right) + y = \bigcap_{k=1}^\infty (\mathcal{O}_k + y),$$

and each $\mathcal{O}_k + y$ is open. Thus $G + y$ is a countable intersection of open sets, i.e. a G_δ set.

(iii) Suppose $m^*(E) = 0$. By Proposition 2, outer measure is translation invariant, so

$$m^*(E + y) = m^*(E) = 0.$$

By Proposition 4 a set of outer measure zero is measurable, so $E + y$ is measurable with measure $m(E + y) = 0$. (Alternatively, one may argue directly, as in the proof of Proposition 2: if $\{I_k\}$ is a countable collection of open bounded intervals covering E with $\sum_k \ell(I_k) < \epsilon$, then $\{I_k + y\}$ is a countable collection of open bounded intervals covering $E + y$ with the same total length, since $I_k + y$ is an open interval of the same length as I_k . Hence $m^*(E + y) < \epsilon$ for every $\epsilon > 0$.)

Problem 2.14. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

Solution. Let $E \subseteq \mathbb{R}$ with $m^*(E) > 0$. For each natural number n define

$$E_n = E \cap [-n, n].$$

Each E_n is a subset of E and is bounded, being contained in $[-n, n]$. Moreover

$$E = \bigcup_{n=1}^{\infty} E_n,$$

since every real number $x \in E$ satisfies $|x| \leq n$ for some natural number n (by the Archimedean property of $\mathbb{R}$), and hence lies in E_n ; the reverse inclusion is clear because each $E_n \subseteq E$.

Suppose, for the sake of contradiction, that every bounded subset of E had outer measure zero. In particular $m^*(E_n) = 0$ for every n . By the countable subadditivity of outer measure (Proposition 3),

$$m^*(E) = m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n) = \sum_{n=1}^{\infty} 0 = 0.$$

Since outer measure is nonnegative, this gives $m^*(E) = 0$, contradicting the hypothesis $m^*(E) > 0$.

Therefore $m^*(E_n) > 0$ for at least one natural number n , and this E_n is a bounded subset of E with positive outer measure.

One can say slightly more: the sets E_n increase to E , and by monotonicity $m^*(E_1) \leq m^*(E_2) \leq \dots \leq m^*(E)$. The argument above shows the limit of this increasing sequence cannot be 0 when $m^*(E) > 0$; thus in fact, if N is an index with $m^*(E_N) > 0$, then $m^*(E_n) \geq m^*(E_N) > 0$ for every $n \geq N$. Note also that the same reasoning works with any countable collection of bounded sets whose union is $\mathbb{R}$ in place of $\{[-n, n]\}_{n=1}^{\infty}$.

2.3 Problems 2.15–2.21

Problem 2.15. Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ .

Solution. Let E be measurable with $m(E) = m^*(E) < \infty$ and let $\epsilon > 0$.

Slice E by the half-open intervals of length ϵ . For $k \in \mathbb{Z}$ put

$$A_k = E \cap (k\epsilon, (k+1)\epsilon].$$

Each half-open interval is measurable (Proposition 8 gives measurability of every interval; alternatively $(k\epsilon, (k+1)\epsilon] = (k\epsilon, (k+1)\epsilon) \cup \{(k+1)\epsilon\}$ and singletons have outer measure zero, hence are measurable by Proposition 4), so each A_k is measurable, being the intersection of two measurable sets. The intervals $(k\epsilon, (k+1)\epsilon]$, $k \in \mathbb{Z}$, are pairwise disjoint and their union is all of $\mathbb{R}$; hence the sets A_k are pairwise disjoint and

$$E = \bigcup_{k \in \mathbb{Z}} A_k.$$

By monotonicity of outer measure and Proposition 1,

$$m(A_k) \leq m^*((k\epsilon, (k+1)\epsilon]) = \epsilon \quad \text{for every } k.$$

Since $\mathbb{Z}$ is countably infinite, fix a bijection $\varphi: \mathbb{N} \rightarrow \mathbb{Z}$ and put $B_k = A_{\varphi(k)}$ for each natural number k . Then $\{B_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets whose union is E , and $m(B_k) \leq \epsilon$ for every k . I claim

$$\sum_{k=1}^{\infty} m(B_k) \leq m(E) < \infty.$$

Indeed, fix n . Applying Proposition 6 with $A = \mathbb{R}$ to the finite disjoint collection $\{B_k\}_{k=1}^n$ of measurable sets gives

$$\sum_{k=1}^n m^*(B_k) = m^*\left(\bigcup_{k=1}^n B_k\right) \leq m^*(E),$$

the last inequality by monotonicity, since $\bigcup_{k=1}^n B_k \subseteq E$. The bound $m^*(E)$ does not depend on n , so the series of nonnegative terms converges and its sum is at most $m(E) < \infty$.

Since the series converges, choose a natural number N so large that

$$\sum_{k=N+1}^{\infty} m(B_k) \leq \epsilon.$$

Define

$$C = \bigcup_{k=N+1}^{\infty} B_k.$$

Then C is measurable, being a countable union of measurable sets (Proposition 7), and by the countable subadditivity of outer measure (Proposition 3),

$$m(C) = m^*(C) \leq \sum_{k=N+1}^{\infty} m(B_k) \leq \epsilon.$$

Finally,

$$E = B_1 \cup B_2 \cup \cdots \cup B_N \cup C$$

is a disjoint union of finitely many measurable sets: the B_k are pairwise disjoint by construction, and C is disjoint from each of $B_1, \dots, B_N$ because it is the union of the remaining members of that disjoint collection. Each of these $N+1$ sets has measure at most ϵ , since $m(B_k) \leq \epsilon$ for every k and $m(C) \leq \epsilon$. This is the required decomposition.

Problem 2.16. Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).

Solution. Recall the four assertions of Theorem 11 for a set E of real numbers:

- (i) for each $\epsilon > 0$ there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \epsilon$;
- (ii) there is a G_δ set G containing E for which $m^*(G \setminus E) = 0$;
- (iii) for each $\epsilon > 0$ there is a closed set F contained in E for which $m^*(E \setminus F) < \epsilon$;
- (iv) there is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

The text established that measurability of E is equivalent to (i) and to (ii). We deduce the remaining two equivalences by complementation, using three facts: a set is measurable if and only if its complement is measurable (immediate from the Carathéodory definition, which is symmetric in E and E^C); a set is open if and only if its complement is closed; and a set is F_σ if and only if its complement is G_δ (De Morgan's Identities, since a countable union of closed sets has as complement the countable intersection of their open complements, and conversely).

The computational heart of the argument is the following identity. Let A and B be sets with $B = A^C$. If $F = \mathcal{O}^C$, then

$$\mathcal{O} \setminus A = \mathcal{O} \cap A^C = F^C \cap B = B \cap F^C = B \setminus F.$$

So passing to complements interchanges an “outer” difference for A with an “inner” difference for

$B = A^C$, and the two differences are literally the same set. We use this repeatedly.

Measurability is equivalent to (iii).

Suppose first that E is measurable. Then E^C is measurable, so by the already proved equivalence of measurability with (i), applied to the set E^C : given $\epsilon > 0$ there is an open set $\mathcal{O}$ with $E^C \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim E^C) < \epsilon$. Put $F = \mathcal{O}^C$. Then F is closed, and $E^C \subseteq \mathcal{O}$ gives $F = \mathcal{O}^C \subseteq E$. Moreover, by the identity above (with $A = E^C$, $B = E$),

$$\mathcal{O} \sim E^C = \mathcal{O} \cap E = E \cap F^C = E \sim F,$$

so $m^*(E \sim F) < \epsilon$. Thus (iii) holds.

Conversely, suppose (iii) holds for E . Let $\epsilon > 0$ and choose a closed set $F \subseteq E$ with $m^*(E \sim F) < \epsilon$. Put $\mathcal{O} = F^C$. Then $\mathcal{O}$ is open and $F \subseteq E$ gives $E^C \subseteq F^C = \mathcal{O}$. Again

$$\mathcal{O} \sim E^C = \mathcal{O} \cap E = E \cap F^C = E \sim F,$$

so $m^*(\mathcal{O} \sim E^C) < \epsilon$. As $\epsilon > 0$ was arbitrary, E^C satisfies (i) and is therefore measurable; hence E is measurable.

Measurability is equivalent to (iv).

Suppose E is measurable. Then E^C is measurable, so by the equivalence of measurability with (ii), applied to E^C , there is a G_δ set G containing E^C with $m^*(G \sim E^C) = 0$. Put $F = G^C$. Then F is an F_σ set, $E^C \subseteq G$ gives $F \subseteq E$, and

$$G \sim E^C = G \cap E = E \cap F^C = E \sim F,$$

so $m^*(E \sim F) = 0$. Thus (iv) holds.

Conversely, suppose (iv) holds for E : there is an F_σ set $F \subseteq E$ with $m^*(E \sim F) = 0$. Put $G = F^C$, a G_δ set. From $F \subseteq E$ we get $E^C \subseteq G$, and as computed above $G \sim E^C = E \sim F$, so $m^*(G \sim E^C) = 0$. Hence E^C satisfies (ii) and is measurable, and therefore E is measurable.

(Alternatively, for this last step one may argue directly, without complementation: with $F \subseteq E$ an F_σ set for which $m^*(E \sim F) = 0$, the set $E \sim F$ has outer measure zero and so is measurable by Proposition 4, while F is measurable by Theorem 9, since every F_σ set is a Borel set. Hence

$$E = F \cup [E \sim F]$$

is measurable, being the union of two measurable sets, by Proposition 5.)

Problem 2.17. Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) < \epsilon$.

Solution. Suppose E is measurable. Let $\epsilon > 0$. By assertion (i) of Theorem 11 there is an open set $\mathcal{O}$ containing E with

$$m^*(\mathcal{O} \sim E) < \epsilon/2,$$

and by assertion (iii) of Theorem 11 (proved in Problem 16) there is a closed set F contained in E with

$$m^*(E \sim F) < \epsilon/2.$$

Then $F \subseteq E \subseteq \mathcal{O}$. I claim

$$\mathcal{O} \sim F = [\mathcal{O} \sim E] \cup [E \sim F].$$

Indeed, if $x \in \mathcal{O} \sim F$ then $x \in \mathcal{O}$ and $x \notin F$; either $x \notin E$, in which case $x \in \mathcal{O} \sim E$, or $x \in E$, in which case $x \in E \sim F$. Conversely $\mathcal{O} \sim E \subseteq \mathcal{O} \sim F$ because $F \subseteq E$, and $E \sim F \subseteq \mathcal{O} \sim F$ because $E \subseteq \mathcal{O}$. Hence, by the countable (in particular finite) subadditivity of outer measure, Proposition 3,

$$m^*(\mathcal{O} \sim F) \leq m^*(\mathcal{O} \sim E) + m^*(E \sim F) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Conversely, suppose that for each $\epsilon > 0$ there are a closed F and an open $\mathcal{O}$ with $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) < \epsilon$. Fix $\epsilon > 0$ and take such F and $\mathcal{O}$. Since $F \subseteq E$,

$$\mathcal{O} \sim E = \mathcal{O} \cap E^C \subseteq \mathcal{O} \cap F^C = \mathcal{O} \sim F,$$

so by the monotonicity of outer measure

$$m^*(\mathcal{O} \sim E) \leq m^*(\mathcal{O} \sim F) < \epsilon.$$

Thus for each $\epsilon > 0$ there is an open set $\mathcal{O}$ containing E with $m^*(\mathcal{O} \sim E) < \epsilon$; that is, E satisfies assertion (i) of Theorem 11, and therefore E is measurable.

(One could equally well have used the inner half of the hypothesis in the converse: $E \sim F \subseteq \mathcal{O} \sim F$ gives $m^*(E \sim F) < \epsilon$, which is assertion (iii) of Theorem 11 and again forces E to be measurable.)

Problem 2.18. Let E have finite outer measure. Show that there is an F_σ set F and a G_δ set G such that

$$F \subseteq E \subseteq G \quad \text{and} \quad m^*(F) = m^*(E) = m^*(G).$$

Solution. A word on the hypotheses before the proof. The G_δ half of the conclusion holds for every set of finite outer measure. The F_σ half does not: as the Remark at the end shows, it holds precisely for the measurable sets, so the exercise is to be read within the standing context of this section, namely for a measurable set E of finite outer measure. We prove each half under the hypothesis it actually needs.

The outer half, for an arbitrary set E of finite outer measure. Let k be a natural number. Since $m^*(E) < \infty$, by the definition of outer measure there is a countable collection $\{I_j^k\}_{j=1}^\infty$ of open, bounded intervals covering E with

$$\sum_{j=1}^\infty \ell(I_j^k) < m^*(E) + 1/k.$$

Set $\mathcal{O}_k = \bigcup_{j=1}^\infty I_j^k$, an open set containing E . By the definition of outer measure (or by Propositions 1 and 3),

$$m^*(\mathcal{O}_k) \leq \sum_{j=1}^\infty \ell(I_j^k) < m^*(E) + 1/k.$$

Define $G = \bigcap_{k=1}^\infty \mathcal{O}_k$. Then G is a G_δ set, $E \subseteq G$, and for every k , by monotonicity, $m^*(G) \leq m^*(\mathcal{O}_k) < m^*(E) + 1/k$. Letting $k \rightarrow \infty$ gives $m^*(G) \leq m^*(E)$, while $E \subseteq G$ gives $m^*(E) \leq m^*(G)$. Hence $m^*(G) = m^*(E)$. This half used nothing about E beyond $m^*(E) < \infty$.

The inner half. Assume E is measurable (see the remark below, which shows this hypothesis cannot be dropped). By assertion (iv) of Theorem 11 (proved in Problem 16) there is an F_σ set F contained in E with $m^*(E \sim F) = 0$. Since $E = F \cup [E \sim F]$, the countable subadditivity of outer measure (Proposition 3) gives

$$m^*(E) \leq m^*(F) + m^*(E \sim F) = m^*(F),$$

while monotonicity gives $m^*(F) \leq m^*(E)$ because $F \subseteq E$. Hence $m^*(F) = m^*(E)$.

Combining the two halves: for measurable E of finite outer measure there are an F_σ set F and a G_δ set G with $F \subseteq E \subseteq G$ and $m^*(F) = m^*(E) = m^*(G)$.

Remark: measurability is exactly what is needed. The conclusion is in fact equivalent to the measurability of E , so it fails for every nonmeasurable set of finite outer measure.

First, suppose that for a set E with $m^*(E) < \infty$ there are an F_σ set F and a G_δ set G with $F \subseteq E \subseteq G$ and $m^*(F) = m^*(E) = m^*(G)$. Both F and G are Borel sets, hence measurable by Theorem 9, and $m(F) = m^*(E) < \infty$. Since F is a measurable set of finite outer measure contained in G , the excision property (7) gives

$$m^*(G \sim F) = m^*(G) - m^*(F) = 0.$$

Now $E \sim F \subseteq G \sim F$, so by monotonicity $m^*(E \sim F) = 0$, and therefore $E \sim F$ is measurable by Proposition 4. Consequently

$$E = F \cup [E \sim F]$$

is measurable by Proposition 5.

Second, a concrete failure for a nonmeasurable set. This part draws on Section 2.6, which lies ahead of the present section; nothing in the proof above depends on it. Let $V \subseteq [0, 1]$ be a choice set for the rational equivalence relation on $[0, 1]$, that is, a set consisting of exactly one member of each rational equivalence class of $[0, 1]$; such a set exists by the Axiom of Choice, as in Section 2.6, and $m^*(V) \leq 1 < \infty$. Let Λ be the set of rational numbers in $[-1, 1]$, a bounded, countably infinite set. If $x \in [0, 1]$, then by the second characteristic property of a choice set there is a $c \in V$ with $x = c + q$ for some rational q , and $q = x - c \in [-1, 1]$ since both x and c lie in $[0, 1]$; hence

$$[0, 1] \subseteq \bigcup_{\lambda \in \Lambda} [\lambda + V].$$

By countable subadditivity (Proposition 3), the translation invariance of outer measure (Proposition 2), and Proposition 1, $1 = m^*([0, 1]) \leq \sum_{\lambda \in \Lambda} m^*(V)$, which forces $m^*(V) > 0$. On the other hand, let F_0 be any closed subset of V . Then F_0 is bounded and measurable, and by the first characteristic property of a choice set, recorded as (16) in Section 2.6, the translates $\{\lambda + V\}_{\lambda \in \Lambda}$ are disjoint; since $F_0 \subseteq V$, the translates $\{\lambda + F_0\}_{\lambda \in \Lambda}$ are disjoint as well. By Lemma 16, $m(F_0) = 0$. Hence every F_σ set contained in V , being a countable union of closed subsets of V , has outer measure zero by Proposition 3. So no F_σ set $F \subseteq V$ satisfies $m^*(F) = m^*(V) > 0$.

Thus the exercise, read literally for an arbitrary set of finite outer measure, is false; the G_δ statement holds in general, and the F_σ statement holds precisely for the measurable sets.

Problem 2.19. Let E have finite outer measure. Show that if E is not measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which

$$m^*(\mathcal{O} \sim E) > m^*(\mathcal{O}) - m^*(E).$$

Solution. Let E be a nonmeasurable set with $m^*(E) < \infty$.

Step 1: a uniform gap. By Theorem 11, measurability of E is equivalent to assertion (i): for each $\epsilon > 0$ there is an open set containing E whose difference with E has outer measure less than ϵ . Since E is not measurable, (i) fails. The negation of (i) is precisely: there is an $\epsilon_0 > 0$ such that

$$m^*(\mathcal{U} \sim E) \geq \epsilon_0 \quad \text{for every open set } \mathcal{U} \text{ containing } E.$$

Call this property (*).

Step 2: a tight open cover. Since $m^*(E) < \infty$, by the definition of outer measure there is a countable collection $\{I_k\}_{k=1}^\infty$ of open, bounded intervals covering E for which

$$\sum_{k=1}^\infty \ell(I_k) < m^*(E) + \epsilon_0.$$

Define $\mathcal{O} = \bigcup_{k=1}^\infty I_k$. Then $\mathcal{O}$ is an open set containing E , and by the countable subadditivity of outer measure (Proposition 3) together with the equality of the outer measure of an interval and its length (Proposition 1),

$$m^*(\mathcal{O}) \leq \sum_{k=1}^\infty \ell(I_k) < m^*(E) + \epsilon_0 < \infty.$$

In particular $\mathcal{O}$ has finite outer measure.

Step 3: the strict inequality. From the display of Step 2,

$$m^*(\mathcal{O}) - m^*(E) < \epsilon_0,$$

the subtraction being legitimate because $m^*(E)$ is finite. On the other hand, $\mathcal{O}$ is an open set containing E , so $(*)$ applies to it and gives

$$m^*(\mathcal{O} \sim E) \geq \epsilon_0.$$

Combining the two,

$$m^*(\mathcal{O} \sim E) \geq \epsilon_0 > m^*(\mathcal{O}) - m^*(E),$$

which is the required strict inequality. Note also that $\mathcal{O} \sim E$ has finite outer measure, since $\mathcal{O} \sim E \subseteq \mathcal{O}$. This is exactly the point made in the Remark following Theorem 12: for any bounded set E , measurable or not, one can always find an open $\mathcal{O} \supseteq E$ with $m^*(\mathcal{O}) - m^*(E) < \epsilon$; but the excision identity $m^*(\mathcal{O} \sim E) = m^*(\mathcal{O}) - m^*(E)$ requires the measurability of E , and for a nonmeasurable E the left-hand side stays bounded away from zero no matter how tightly $\mathcal{O}$ is chosen.

Problem 2.20. (Lebesgue) Let E have finite outer measure. Show that E is measurable if and only if for each open, bounded interval (a, b) ,

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \sim E).$$

Solution. Necessity. Suppose E is measurable. Let (a, b) be an open, bounded interval. Taking $A = (a, b)$ in the Carathéodory definition of measurability,

$$m^*((a, b)) = m^*((a, b) \cap E) + m^*((a, b) \cap E^C) = m^*((a, b) \cap E) + m^*((a, b) \sim E).$$

By Proposition 1, $m^*((a, b)) = \ell((a, b)) = b - a$. This is the asserted identity, and it holds for measurable E whether or not $m^*(E)$ is finite.

Sufficiency. Suppose $m^*(E) < \infty$ and that

$$\ell(I) = m^*(I \cap E) + m^*(I \sim E)$$

for every nonempty open, bounded interval I ; call this hypothesis $(*)$. We verify assertion (i) of Theorem 11.

Let $\epsilon > 0$. Since $m^*(E) < \infty$, the definition of outer measure provides a countable collection of open, bounded intervals covering E the sum of whose lengths is less than $m^*(E) + \epsilon$; let $\mathcal{O}$ be the union of that collection. Then $\mathcal{O}$ is an open set containing E and, by Propositions 1 and 3,

$$m^*(\mathcal{O}) < m^*(E) + \epsilon < \infty.$$

Every open set of real numbers is the disjoint union of a countable collection of open intervals (page 17). Write $\mathcal{O} = \bigcup_k I_k$ with $\{I_k\}$ a countable disjoint collection of open intervals; discarding empty ones, we may assume each I_k is nonempty. Each I_k is bounded: $I_k \subseteq \mathcal{O}$, so by Proposition 1 and monotonicity $\ell(I_k) = m^*(I_k) \leq m^*(\mathcal{O}) < \infty$, and an unbounded interval has infinite length. So each I_k is an open, bounded interval and $(*)$ applies to it.

Two summation facts. First,

$$\sum_k \ell(I_k) \leq m^*(\mathcal{O}).$$

Indeed, each I_k is measurable (Proposition 8) with $m^*(I_k) = \ell(I_k)$, so for each n , Proposition 6 applied with $A = \mathbb{R}$ to the finite disjoint collection $\{I_k\}_{k=1}^n$ together with monotonicity gives

$$\sum_{k=1}^n \ell(I_k) = m^*\left(\bigcup_{k=1}^n I_k\right) \leq m^*(\mathcal{O}),$$

and the bound is independent of n . Second, since $E \subseteq \mathcal{O}$ we have $E = \bigcup_k [I_k \cap E]$, so by countable subadditivity (Proposition 3),

$$m^*(E) \leq \sum_k m^*(I_k \cap E).$$

All the terms $m^*(I_k \cap E) \leq \ell(I_k)$ are finite, and both series converge, being dominated by $m^*(\mathcal{O}) < \infty$.

Now $\mathcal{O} \sim E = \bigcup_k [I_k \sim E]$, so by countable subadditivity and then (*),

$$m^*(\mathcal{O} \sim E) \leq \sum_k m^*(I_k \sim E) = \sum_k [\ell(I_k) - m^*(I_k \cap E)] = \sum_k \ell(I_k) - \sum_k m^*(I_k \cap E),$$

the splitting of the last series being legitimate since both series converge. Therefore, by the two summation facts,

$$m^*(\mathcal{O} \sim E) \leq m^*(\mathcal{O}) - m^*(E) < [m^*(E) + \epsilon] - m^*(E) = \epsilon.$$

Thus for each $\epsilon > 0$ there is an open set $\mathcal{O}$ containing E with $m^*(\mathcal{O} \sim E) < \epsilon$. By assertion (i) of Theorem 11, E is measurable.

Observe where the hypotheses were used: the finiteness of $m^*(E)$ made the subtraction $m^*(\mathcal{O}) - m^*(E)$ meaningful and produced the tight open cover, while (*) converted the Carathéodory splitting condition, a priori demanded of every test set A , into a condition on the single family of bounded open intervals. The content of the exercise is that intervals are enough test sets.

Problem 2.21. Use property (ii) of Theorem 11 as the primitive definition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same for property (iv).

Solution. Throughout, the only properties of outer measure used are its monotonicity and its countable subadditivity (Proposition 3); no part of the Carathéodory theory is invoked, which is the point of the exercise.

Part 1: property (ii) as the definition.

Definition: a set E is measurable provided there is a G_δ set G containing E for which $m^*(G \sim E) = 0$. We need one topological lemma.

Lemma A. The union of two G_δ sets is a G_δ set.

Proof of Lemma A. Let $G_1 = \bigcap_{n=1}^\infty U_n$ and $G_2 = \bigcap_{n=1}^\infty V_n$ with each U_n, V_n open. Consider the countable collection of open sets $\{U_n \cup V_m\}_{n,m \in \mathbb{N}}$ (a countable family, being indexed by $\mathbb{N} \times \mathbb{N}$), and set

$$H = \bigcap_{n,m} [U_n \cup V_m],$$

a G_δ set. If $x \in G_1$ then $x \in U_n$ for all n , so $x \in U_n \cup V_m$ for all n, m ; hence $G_1 \subseteq H$, and symmetrically $G_2 \subseteq H$, so $G_1 \cup G_2 \subseteq H$. Conversely, let $x \in H$ and suppose $x \notin G_1$. Then $x \notin U_{n_0}$ for some index n_0 . Since $x \in U_{n_0} \cup V_m$ for every m , it follows that $x \in V_m$ for every m , that is, $x \in G_2$. Hence $H \subseteq G_1 \cup G_2$, and $G_1 \cup G_2 = H$ is a G_δ set.

Now let E_1 and E_2 be measurable in the sense of (ii). Choose G_δ sets $G_1 \supseteq E_1$ and $G_2 \supseteq E_2$ with

$$m^*(G_1 \sim E_1) = m^*(G_2 \sim E_2) = 0.$$

Put $G = G_1 \cup G_2$, a G_δ set by Lemma A, and clearly $E_1 \cup E_2 \subseteq G$. I claim

$$G \sim [E_1 \cup E_2] \subseteq [G_1 \sim E_1] \cup [G_2 \sim E_2].$$

Indeed, let $x \in G$ with $x \notin E_1 \cup E_2$. Then $x \in G_1$ or $x \in G_2$. In the first case $x \in G_1$ and $x \notin E_1$, so $x \in G_1 \sim E_1$; in the second case, likewise, $x \in G_2 \sim E_2$. This proves the inclusion.

By monotonicity and the subadditivity of outer measure,

$$m^*(G \sim [E_1 \cup E_2]) \leq m^*(G_1 \sim E_1) + m^*(G_2 \sim E_2) = 0.$$

Thus G is a G_δ set containing $E_1 \cup E_2$ with $m^*(G \sim [E_1 \cup E_2]) = 0$, so $E_1 \cup E_2$ is measurable.

Part 2: property (iv) as the definition.

Definition: a set E is measurable provided there is an F_σ set F contained in E for which $m^*(E \sim F) = 0$.

Lemma B. The union of two F_σ sets is an F_σ set.

Proof of Lemma B. If $F_1 = \bigcup_{n=1}^{\infty} C_n$ and $F_2 = \bigcup_{n=1}^{\infty} D_n$ with each C_n, D_n closed, then

$$F_1 \cup F_2 = \bigcup_{n=1}^{\infty} [C_n \cup D_n],$$

a countable union of closed sets, since the union of two closed sets is closed. (Equivalently, interlace the two sequences.) Hence $F_1 \cup F_2$ is F_{σ} .

Now let E_1 and E_2 be measurable in the sense of (iv). Choose F_{σ} sets $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$ with

$$m^*(E_1 \sim F_1) = m^*(E_2 \sim F_2) = 0.$$

Put $F = F_1 \cup F_2$, an F_{σ} set by Lemma B, with $F \subseteq E_1 \cup E_2$. I claim

$$[E_1 \cup E_2] \sim F \subseteq [E_1 \sim F_1] \cup [E_2 \sim F_2].$$

Indeed, let $x \in E_1 \cup E_2$ with $x \notin F = F_1 \cup F_2$; then $x \notin F_1$ and $x \notin F_2$. If $x \in E_1$ then $x \in E_1 \sim F_1$; otherwise $x \in E_2$ and then $x \in E_2 \sim F_2$. This proves the inclusion.

By monotonicity and subadditivity of outer measure,

$$m^*([E_1 \cup E_2] \sim F) \leq m^*(E_1 \sim F_1) + m^*(E_2 \sim F_2) = 0.$$

So F is an F_{σ} set contained in $E_1 \cup E_2$ whose complement in $E_1 \cup E_2$ has outer measure zero, and therefore $E_1 \cup E_2$ is measurable.

By induction, both arguments extend at once to the union of any finite collection of measurable sets.

2.4 Problems 2.22–2.28

Problem 2.22. For any set A , define $m^{**}(A) \in [0, \infty]$ by

$$m^{**}(A) = \inf\{m^*(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open}\}.$$

How is this set function m^{**} related to outer measure m^* ?

Solution. They are the same set function: $m^{**}(A) = m^*(A)$ for every set A of real numbers. (That is, outer measure is **outer regular**: it is already the infimum of the outer measures of the open sets that contain the given set.)

First observe that m^{**} is well defined, since $\mathbb{R}$ is an open set containing A , so the set over which the infimum is taken is nonempty.

One inequality is immediate from monotonicity of outer measure. If $\mathcal{O}$ is any open set with $A \subseteq \mathcal{O}$, then $m^*(A) \leq m^*(\mathcal{O})$. Taking the infimum over all such $\mathcal{O}$,

$$m^*(A) \leq m^{**}(A).$$

For the reverse inequality we use the definition of outer measure directly. If $m^*(A) = \infty$ there is nothing to prove, since then $m^{**}(A) \geq m^*(A) = \infty$ forces $m^{**}(A) = \infty = m^*(A)$. So suppose $m^*(A) < \infty$ and let $\epsilon > 0$. By the definition of $m^*(A)$ as an infimum, there is a countable collection $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals which covers A and for which

$$\sum_{k=1}^{\infty} \ell(I_k) < m^*(A) + \epsilon.$$

Define $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. Then $\mathcal{O}$ is open, being a union of open intervals, and $A \subseteq \mathcal{O}$. By the countable subadditivity of outer measure (Proposition 3) together with the fact that the outer measure of an interval is its length (Proposition 1),

$$m^*(\mathcal{O}) \leq \sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} \ell(I_k) < m^*(A) + \epsilon.$$

Since $\mathcal{O}$ is one of the competitors in the infimum defining $m^{**}(A)$,

$$m^{**}(A) \leq m^*(\mathcal{O}) < m^*(A) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, $m^{**}(A) \leq m^*(A)$.

Combining the two inequalities, $m^{**}(A) = m^*(A)$ for every set A .

Two remarks are worth recording. First, the argument used no measurability whatsoever: the equality holds for every set A , measurable or not. So outer regularity, by itself, detects nothing about measurability.

Second, the equality must not be confused with the stronger assertion that the difference $\mathcal{O} \sim A$ can be made small. That assertion is exactly assertion (i) of Theorem 11, and it is equivalent to the measurability of A . The inference from one to the other would require the excision formula $m^*(\mathcal{O} \sim A) = m^*(\mathcal{O}) - m^*(A)$, and excision is available only when A is measurable and of finite outer measure. Indeed Problem 19 shows that if A has finite outer measure and is not measurable, then there is an open set $\mathcal{O}$ containing A of finite outer measure with $m^*(\mathcal{O} \sim A) > m^*(\mathcal{O}) - m^*(A)$, even though, by what we have just proved, $m^*(\mathcal{O}) - m^*(A)$ may be made arbitrarily small.

Problem 2.23. For any set A , define $m^{***}(A) \in [0, \infty]$ by

$$m^{***}(A) = \sup\{m^*(F) \mid F \subseteq A, F \text{ closed}\}.$$

How is this set function m^{***} related to outer measure m^* ?

Solution. The set function m^{***} is the **inner measure** determined by closed sets. It never exceeds m^* , and it agrees with m^* on every measurable set; conversely, for sets of finite outer measure, agreement forces measurability. Thus m^{***} and m^* are genuinely different set functions, and the sets on which they disagree are non-measurable. Precisely, we prove:

- (1) $m^{***}(A) \leq m^*(A)$ for every set A ;
- (2) if A is measurable, then $m^{***}(A) = m^*(A)$;
- (3) if $m^*(A) < \infty$ and $m^{***}(A) = m^*(A)$, then A is measurable;
- (4) neither the finiteness hypothesis in (3) nor the converse in general can be dropped, and there are sets for which the inequality in (1) is strict.

Note first that m^{***} is well defined: the empty set is a closed subset of A , so the supremum is over a nonempty set of numbers.

Proof of (1). If F is closed and $F \subseteq A$, then by the monotonicity of outer measure $m^*(F) \leq m^*(A)$. Taking the supremum over all such F gives $m^{***}(A) \leq m^*(A)$.

Proof of (2). Let A be measurable and let $\epsilon > 0$. By assertion (iii) of Theorem 11 there is a closed set F contained in A for which $m^*(A \sim F) < \epsilon$. Since $A = F \cup (A \sim F)$, the finite subadditivity of outer measure gives

$$m^*(A) \leq m^*(F) + m^*(A \sim F) < m^*(F) + \epsilon \leq m^{***}(A) + \epsilon.$$

This holds for every $\epsilon > 0$, whence $m^*(A) \leq m^{***}(A)$; with (1) this gives equality. The argument covers the case $m^*(A) = \infty$ as well: the displayed inequality then forces $m^*(F) = \infty$, so $m^{***}(A) = \infty = m^*(A)$.

Proof of (3). Suppose $m^*(A) < \infty$ and $m^{***}(A) = m^*(A)$. By the definition of the supremum, for each natural number n there is a closed set $F_n \subseteq A$ with

$$m(F_n) > m^*(A) - 1/n.$$

Closed sets are measurable, so each F_n is measurable, and hence so is $F = \bigcup_{n=1}^{\infty} F_n$, an F_σ set contained in A . Since $F_n \subseteq F \subseteq A$, monotonicity gives

$$m^*(A) - 1/n < m(F_n) \leq m(F) \leq m^*(A) \quad \text{for all } n,$$

so $m(F) = m^*(A)$.

Next we produce a measurable set containing A of the same measure. By Problem 22 (equivalently, by the definition of outer measure), for each n there is an open set $\mathcal{O}_n \supseteq A$ with $m^*(\mathcal{O}_n) < m^*(A) + 1/n$. Put $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$, a G_δ set containing A ; it is measurable, and by monotonicity $m(G) \leq m^*(\mathcal{O}_n) < m^*(A) + 1/n$ for every n , while $m(G) \geq m^*(A)$ since $A \subseteq G$. Hence $m(G) = m^*(A)$. Now $F \subseteq A \subseteq G$ with F and G measurable and $m(F) = m(G) = m^*(A) < \infty$. By the excision property of measurable sets,

$$m(G \sim F) = m(G) - m(F) = 0.$$

Since $A \sim F \subseteq G \sim F$, monotonicity gives $m^*(A \sim F) = 0$, and a set of outer measure zero is measurable (Proposition 4). Therefore

$$A = F \cup (A \sim F)$$

is the union of two measurable sets and so is measurable.

Proof of (4). Let $V = \mathcal{C}_{[0,1]}$ be a choice set for the rational equivalence relation on $[0, 1]$, as constructed in Section 2.6. Then $V \subseteq [0, 1]$, and V is not measurable: this is precisely the content of the proof of Theorem 17 (Vitali), applied to the set $E = [0, 1]$, which has positive outer measure. Moreover, by the first characteristic property of a choice set, recorded as (16) in Section 2.6, for any set $\Lambda \subseteq \mathbb{Q}$ the collection of translates $\{\lambda + V\}_{\lambda \in \Lambda}$ is disjoint.

Now let F be any closed set with $F \subseteq V$, and put $\Lambda = [-1, 1] \cap \mathbb{Q}$, a bounded, countably infinite set of real numbers. Since $F \subseteq V$, the translates $\{\lambda + F\}_{\lambda \in \Lambda}$ are contained in the corresponding translates of V and hence are disjoint. Also F is bounded, being a subset of $[0, 1]$, and F is measurable, being closed. Lemma 16 therefore gives $m(F) = 0$. As F was an arbitrary closed subset of V ,

$$m^{***}(V) = 0 < m^*(V),$$

the last inequality because $m^*(V) = 0$ would make V measurable. So the inequality in (1) can be strict.

Finally, the finiteness hypothesis in (3) cannot be omitted. Put $A = \mathbb{R} \sim V$. Then A is not measurable, for otherwise its complement V would be measurable. But A contains the closed set $[2, \infty)$, so

$$m^{***}(A) = \infty = m^*(A),$$

the right-hand equality holding since $A \supseteq [2, \infty)$. Thus a non-measurable set of infinite outer measure may still satisfy $m^{***}(A) = m^*(A)$.

In summary: $m^{***} \leq m^*$ always, with equality on all measurable sets; among sets of finite outer measure the two set functions agree exactly on the measurable sets. In particular, m^{***} is not countably additive on all sets, and it is the natural “inner” counterpart of the “outer” description of m^* obtained in Problem 22.

Problem 2.24. Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Solution. All four sets appearing in the identity are measurable: by Theorem 9 the collection $\mathcal{M}$ of measurable sets is a σ -algebra, so it is closed with respect to the formation of complements and countable unions, and hence, by De Morgan’s Identities, also with respect to finite unions, finite intersections and relative complements. In particular $E_2 \sim E_1 = E_2 \cap E_1^C$ is measurable. Every term is therefore a well-defined element of $[0, \infty]$, and we prove the identity as an identity in $[0, \infty]$, where addition is commutative, associative, and satisfies $a + \infty = \infty$. Notice that no subtraction occurs anywhere in the argument, so no finiteness hypothesis is needed.

Decompose the union. The sets E_1 and $E_2 \sim E_1$ are disjoint measurable sets whose union is $E_1 \cup E_2$. Indeed, they are disjoint because $E_2 \sim E_1$ misses E_1 , and their union is

$$E_1 \cup (E_2 \sim E_1) = E_1 \cup (E_2 \cap E_1^C) = (E_1 \cup E_2) \cap (E_1 \cup E_1^C) = E_1 \cup E_2.$$

Hence, by the finite additivity of Lebesgue measure over disjoint measurable sets (Proposition 6 with $A = \mathbb{R}$, or Proposition 13), we obtain the first decomposition

$$m(E_1 \cup E_2) = m(E_1) + m(E_2 \sim E_1).$$

Decompose E_2 . The sets $E_1 \cap E_2$ and $E_2 \sim E_1$ are disjoint measurable sets, and

$$(E_1 \cap E_2) \cup (E_2 \sim E_1) = E_2 \cap (E_1 \cup E_1^C) = E_2.$$

Hence, again by finite additivity, we obtain the second decomposition

$$m(E_2) = m(E_1 \cap E_2) + m(E_2 \sim E_1).$$

Now add $m(E_1 \cap E_2)$ to both sides of the first decomposition and substitute the second:

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) &= m(E_1) + m(E_2 \sim E_1) + m(E_1 \cap E_2) \\ &= m(E_1) + [m(E_1 \cap E_2) + m(E_2 \sim E_1)] \\ &= m(E_1) + m(E_2). \end{aligned}$$

This is the asserted identity.

Two consequences are worth noting. If $m(E_1 \cap E_2) < \infty$ we may cancel it and obtain the inclusion-exclusion formula

$$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2),$$

and in particular, whether or not the measures are finite, $m(E_1 \cup E_2) \leq m(E_1) + m(E_2)$, with equality precisely when $m(E_1 \cap E_2) = 0$ or both sides are infinite.

Problem 2.25. Show that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.

Solution. Part (ii) of Theorem 15 asserts that if $\{B_k\}_{k=1}^\infty$ is a descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k=1}^\infty B_k\right) = \lim_{k \rightarrow \infty} m(B_k).$$

We exhibit a descending collection of measurable sets for which this conclusion fails; since the only hypothesis used besides descent is the finiteness of $m(B_1)$, this shows that hypothesis cannot be omitted. For each natural number k define

$$B_k = [k, \infty).$$

Each B_k is an interval, hence measurable by Proposition 8, and the collection is descending: $B_{k+1} = [k+1, \infty) \subseteq [k, \infty) = B_k$. Each B_k has infinite measure. Indeed, for every natural number n the interval $[k, k+n]$ is contained in B_k , so by the monotonicity of outer measure and Proposition 1,

$$m(B_k) \geq m^*([k, k+n]) = \ell([k, k+n]) = n,$$

and since n is arbitrary, $m(B_k) = \infty$. (This is the assertion of Proposition 1 for unbounded intervals.) In particular $m(B_1) = \infty$, so the hypothesis in question fails. Consequently

$$\lim_{k \rightarrow \infty} m(B_k) = \infty.$$

On the other hand,

$$\bigcap_{k=1}^\infty B_k = \bigcap_{k=1}^\infty [k, \infty) = \emptyset,$$

because a real number x belongs to this intersection only if $x \geq k$ for every natural number k , which contradicts the Archimedean property of $\mathbb{R}$. Since $m(\emptyset) = 0$,

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = 0 \neq \infty = \lim_{k \rightarrow \infty} m(B_k),$$

and the conclusion of part (ii) fails.

No counterexample of finite measure exists, of course: if $m(B_1) < \infty$ then Theorem 15 (ii) applies and the conclusion holds. The failure above is caused precisely by the infinitude of the measures involved, and it is instructive to see where the proof of Theorem 15 (ii) breaks down. That proof sets $D_k = B_1 \sim B_k$, applies part (i) to the ascending collection $\{D_k\}$, and then cancels the term $m(B_1)$ from the identity

$$m(B_1) - m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} [m(B_1) - m(B_k)].$$

Both the excisions producing this identity and the cancellation require $m(B_1) < \infty$. With $B_k = [k, \infty)$ the identity degenerates to the vacuous statement $\infty - 0 = \infty - \infty$.

It is worth recording exactly how much of the hypothesis is needed. What the proof of Theorem 15 (ii) actually uses is that $m(B_k)$ be finite for the sets it excises, and a descending collection with $m(B_{k_0}) < \infty$ for some index k_0 may be replaced by the descending collection $\{B_k\}_{k \geq k_0}$, which has the same intersection and the same limit of measures. So the correct sharp hypothesis is that $m(B_{k_0}) < \infty$ for some index k_0 ; the example $B_k = [k, \infty)$ shows that even this weaker hypothesis cannot be dropped, since there every B_k has infinite measure.

Problem 2.26. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A ,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

Solution. Note that A is an arbitrary set of real numbers; no measurability is assumed of A . This is the countable version of Proposition 6, which asserts the corresponding identity for a finite disjoint collection of measurable sets. We prove the two inequalities.

First, the inequality $\leq$. Since intersection distributes over union,

$$A \cap \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (A \cap E_k),$$

so the countable subadditivity of outer measure (Proposition 3) gives

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

Observe that this half needed neither the measurability of the E_k nor their disjointness.

Now the inequality $\geq$. Fix a natural number n . The collection $\{E_k\}_{k=1}^n$ is a finite disjoint collection of measurable sets, so by Proposition 6, applied to the set A ,

$$m^*\left(A \cap \bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n m^*(A \cap E_k).$$

On the other hand,

$$A \cap \bigcup_{k=1}^n E_k \subseteq A \cap \bigcup_{k=1}^{\infty} E_k,$$

so by the monotonicity of outer measure,

$$\sum_{k=1}^n m^*(A \cap E_k) = m^*\left(A \cap \bigcup_{k=1}^n E_k\right) \leq m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right).$$

The right-hand side is independent of n . Every partial sum of the series of nonnegative terms $\sum_{k=1}^{\infty} m^*(A \cap E_k)$ is therefore bounded above by the quantity $m^*(A \cap \bigcup_{k=1}^{\infty} E_k)$. Since the sum of a series of nonnegative terms is by definition the supremum in $[0, \infty]$ of its partial sums, we conclude

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \leq m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right).$$

This argument is valid in $[0, \infty]$ without any finiteness assumption: if some term $m^*(A \cap E_k)$ is infinite, or if the partial sums are unbounded, the displayed bound simply forces the right-hand side to equal ∞ as well.

The two inequalities together give the asserted equality.

Taking $A = \mathbb{R}$ recovers the countable additivity of Lebesgue measure, Proposition 13. Conversely, the identity just proved is the natural strengthening of that proposition: the countable additivity of m^* persists after intersecting with an entirely arbitrary “test set” A , so long as the pieces E_k are disjoint and measurable.

Problem 2.27. Let $\mathcal{M}'$ be any σ -algebra of subsets of $\mathbb{R}$ and m' a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive, and such that $m'(\emptyset) = 0$.

(i) Show that m' is finitely additive, monotone, countably monotone, and possesses the excision property.

(ii) Show that m' possesses the same continuity properties as Lebesgue measure.

Solution. Throughout, all sets named belong to $\mathcal{M}'$. Recall that a σ -algebra of subsets of $\mathbb{R}$ contains $\mathbb{R}$ and is closed with respect to the formation of complements and countable unions; by De Morgan's Identities it is closed with respect to countable intersections, and hence also with respect to finite unions, finite intersections and relative complements $B \sim A = B \cap A^C$. Note also that $\emptyset = \mathbb{R}^C \in \mathcal{M}'$. Countable additivity means: if $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of sets in $\mathcal{M}'$, then

$$m'\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m'(E_k).$$

Since the terms are nonnegative, such a series has a well-defined sum in $[0, \infty]$ independent of the order of summation.

Part (i).

Finite additivity. Let $\{E_k\}_{k=1}^n$ be a finite disjoint collection in $\mathcal{M}'$. Define $E_k = \emptyset$ for $k > n$. Then $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection in $\mathcal{M}'$ whose union is $\bigcup_{k=1}^n E_k$, so countable additivity and $m'(\emptyset) = 0$ give

$$m'\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^{\infty} m'(E_k) = \sum_{k=1}^n m'(E_k) + \sum_{k=n+1}^{\infty} 0 = \sum_{k=1}^n m'(E_k).$$

This is where the hypothesis $m'(\emptyset) = 0$ is used; without it, countable additivity applied to the collection all of whose members are $\emptyset$ would force $m'(\emptyset) \in \{0, \infty\}$, and the value ∞ would make m' identically ∞ .

Monotonicity. Let $A, B \in \mathcal{M}'$ with $A \subseteq B$. Then $B \sim A \in \mathcal{M}'$, the sets A and $B \sim A$ are disjoint, and their union is B . By finite additivity,

$$m'(B) = m'(A) + m'(B \sim A) \geq m'(A),$$

since $m'(B \sim A) \geq 0$.

Excision. Retain the notation of the previous paragraph and suppose in addition that $m'(A) < \infty$. Then the identity $m'(B) = m'(A) + m'(B \sim A)$ may be rearranged, the subtraction being legitimate because $m'(A)$ is finite:

$$m'(B \sim A) = m'(B) - m'(A).$$

(If $m'(B) = \infty$ this reads $m'(B \sim A) = \infty$, which is correct.)

Countable monotonicity. Suppose $E \in \mathcal{M}'$ and $\{E_k\}_{k=1}^\infty$ is a countable collection in $\mathcal{M}'$ with $E \subseteq \bigcup_{k=1}^\infty E_k$. We must show

$$m'(E) \leq \sum_{k=1}^\infty m'(E_k).$$

Disjointify the covering exactly as in the discussion preceding Proposition 7: define $E'_1 = E_1$ and, for $k \geq 2$,

$$E'_k = E_k \sim \bigcup_{i=1}^{k-1} E_i.$$

Because $\mathcal{M}'$ is closed under finite unions and relative complements, each E'_k belongs to $\mathcal{M}'$; the collection $\{E'_k\}_{k=1}^\infty$ is disjoint, $E'_k \subseteq E_k$, and $\bigcup_{k=1}^\infty E'_k = \bigcup_{k=1}^\infty E_k$. Now set $F_k = E \cap E'_k \in \mathcal{M}'$. The collection $\{F_k\}_{k=1}^\infty$ is disjoint, and since E is contained in the union of the E'_k ,

$$\bigcup_{k=1}^\infty F_k = E \cap \bigcup_{k=1}^\infty E'_k = E.$$

By countable additivity and then by monotonicity, using $F_k \subseteq E'_k \subseteq E_k$,

$$m'(E) = \sum_{k=1}^\infty m'(F_k) \leq \sum_{k=1}^\infty m'(E_k).$$

Part (ii). We show that m' has both continuity properties of Theorem 15. The proofs are those of Theorem 15 verbatim, and they use only countable additivity, monotonicity and excision, all of which were established in part (i).

Continuity for ascending collections. Let $\{A_k\}_{k=1}^\infty$ be an ascending collection in $\mathcal{M}'$, that is, $A_k \subseteq A_{k+1}$ for each k . We claim

$$m' \left(\bigcup_{k=1}^\infty A_k \right) = \lim_{k \rightarrow \infty} m'(A_k).$$

By monotonicity the sequence $\{m'(A_k)\}$ is nondecreasing, so the limit exists in $[0, \infty]$.

Suppose first that $m'(A_{k_0}) = \infty$ for some index k_0 . Then by monotonicity $m'(A_k) = \infty$ for all $k \geq k_0$, so the right-hand side is ∞ ; and $A_{k_0} \subseteq \bigcup_{k=1}^\infty A_k$, so the left-hand side is ∞ too. The claim holds.

Now suppose $m'(A_k) < \infty$ for every k . Define $A_0 = \emptyset$ and, for $k \geq 1$, $C_k = A_k \sim A_{k-1} \in \mathcal{M}'$. Because the collection is ascending, $\{C_k\}_{k=1}^\infty$ is disjoint and

$$\bigcup_{k=1}^\infty C_k = \bigcup_{k=1}^\infty A_k.$$

Indeed, if x lies in the union of the A_k , there is a smallest index k with $x \in A_k$, and then $x \in C_k$; the reverse inclusion is clear since $C_k \subseteq A_k$. Disjointness holds because for $j < k$ we have $C_j \subseteq A_j \subseteq A_{k-1}$ while C_k misses A_{k-1} . By countable additivity and then excision (legitimate since $A_{k-1} \subseteq A_k$ and $m'(A_{k-1}) < \infty$),

$$\begin{aligned} m' \left(\bigcup_{k=1}^\infty A_k \right) &= \sum_{k=1}^\infty m'(A_k \sim A_{k-1}) = \sum_{k=1}^\infty [m'(A_k) - m'(A_{k-1})] \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n [m'(A_k) - m'(A_{k-1})] = \lim_{n \rightarrow \infty} [m'(A_n) - m'(A_0)] = \lim_{n \rightarrow \infty} m'(A_n), \end{aligned}$$

the telescoping being valid because every term is a difference of finite numbers, and $m'(A_0) = m'(\emptyset) = 0$. Continuity for descending collections. Let $\{B_k\}_{k=1}^\infty$ be a descending collection in $\mathcal{M}'$, that is, $B_{k+1} \subseteq B_k$ for each k , and suppose $m'(B_1) < \infty$. We claim

$$m' \left(\bigcap_{k=1}^\infty B_k \right) = \lim_{k \rightarrow \infty} m'(B_k).$$

The intersection belongs to $\mathcal{M}'$, and by monotonicity $\{m'(B_k)\}$ is nonincreasing and bounded below by 0, so the limit exists. Also $m'(B_k) \leq m'(B_1) < \infty$ for every k , and likewise $m'(\bigcap_k B_k) < \infty$; every excision performed below is therefore legitimate.

Define $D_k = B_1 \sim B_k \in \mathcal{M}'$. Since $\{B_k\}$ is descending, $\{D_k\}$ is ascending, so by the case already proved,

$$m' \left(\bigcup_{k=1}^\infty D_k \right) = \lim_{k \rightarrow \infty} m'(D_k).$$

By De Morgan's Identities,

$$\bigcup_{k=1}^\infty D_k = \bigcup_{k=1}^\infty [B_1 \sim B_k] = B_1 \sim \bigcap_{k=1}^\infty B_k.$$

By excision, $m'(D_k) = m'(B_1) - m'(B_k)$ for each k , and $m'(B_1 \sim \bigcap_k B_k) = m'(B_1) - m'(\bigcap_k B_k)$. Substituting,

$$m'(B_1) - m' \left(\bigcap_{k=1}^\infty B_k \right) = \lim_{k \rightarrow \infty} [m'(B_1) - m'(B_k)] = m'(B_1) - \lim_{k \rightarrow \infty} m'(B_k).$$

Since $m'(B_1)$ is finite it may be cancelled from both sides, and the claim follows.

Thus every countably additive $[0, \infty]$ -valued set function on a σ -algebra which vanishes on the empty set, that is, every measure, enjoys all the elementary properties of Lebesgue measure established in Sections 2.4 and 2.5. Nothing particular to $\mathbb{R}$, to intervals, or to the construction of m^* was used; this is the observation that will be taken as the starting point of the general theory of measure.

Problem 2.28. Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

Solution. The precise assertion to be proved is the converse of the implication established in Problem 27. Let $\mathcal{M}'$ be a σ -algebra of subsets of $\mathbb{R}$ and let m' be a set function on $\mathcal{M}'$ taking values in $[0, \infty]$ with $m'(\emptyset) = 0$. Assume:

(a) m' is finitely additive, that is, if $\{E_k\}_{k=1}^n$ is a finite disjoint collection in $\mathcal{M}'$, then $m'(\bigcup_{k=1}^n E_k) = \sum_{k=1}^n m'(E_k)$; and

(b) m' is continuous, that is, it possesses the two continuity properties of Theorem 15; in fact only the property for ascending collections is required: if $\{A_n\}_{n=1}^\infty$ is an ascending collection in $\mathcal{M}'$, then $m'(\bigcup_{n=1}^\infty A_n) = \lim_{n \rightarrow \infty} m'(A_n)$.

Then m' is countably additive.

Proof. Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection of sets in $\mathcal{M}'$, and set

$$E = \bigcup_{k=1}^\infty E_k.$$

Since $\mathcal{M}'$ is a σ -algebra, $E \in \mathcal{M}'$. For each natural number n define the partial union

$$A_n = \bigcup_{k=1}^n E_k \in \mathcal{M}'.$$

The collection $\{A_n\}_{n=1}^\infty$ is ascending, since $A_n \subseteq A_{n+1}$, and

$$\bigcup_{n=1}^\infty A_n = \bigcup_{k=1}^\infty E_k = E.$$

Indeed, each $A_n \subseteq E$, and conversely any $x \in E$ lies in some E_k and hence in A_k . Because $\{E_k\}_{k=1}^\infty$ is a finite disjoint collection in $\mathcal{M}'$, hypothesis (a) gives

$$m'(A_n) = \sum_{k=1}^n m'(E_k) \quad \text{for every } n.$$

Applying hypothesis (b) to the ascending collection $\{A_n\}$ and then using the display above,

$$m'(E) = m'\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n \rightarrow \infty} m'(A_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n m'(E_k) = \sum_{k=1}^\infty m'(E_k),$$

the last equality being the definition of the sum of a series of nonnegative terms; the limit exists in $[0, \infty]$ because the partial sums are nondecreasing. That is,

$$m'\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty m'(E_k),$$

which is the countable additivity of m' .

A remark on the descending form of continuity. One may instead assume finite additivity together with continuity for descending collections, in the form: if $\{B_n\}$ is a descending collection in $\mathcal{M}'$ with $m'(B_1) < \infty$ and $\bigcap_{n=1}^\infty B_n = \emptyset$, then $\lim_{n \rightarrow \infty} m'(B_n) = 0$. This yields countable additivity for disjoint collections whose union has finite measure. Indeed, with E and E_k as above and $m'(E) < \infty$, put

$$B_n = E \sim \bigcup_{k=1}^n E_k \in \mathcal{M}'.$$

The collection $\{B_n\}$ is descending, $m'(B_1) \leq m'(E) < \infty$ by the monotonicity that follows from finite additivity, and $\bigcap_{n=1}^\infty B_n = \emptyset$ because each point of E lies in some E_k and so is excluded from B_k . Finite additivity applied to the disjoint decomposition $E = E_1 \cup \cdots \cup E_n \cup B_n$ gives

$$m'(E) = \sum_{k=1}^n m'(E_k) + m'(B_n),$$

and letting $n \rightarrow \infty$, with $m'(B_n) \rightarrow 0$, yields $m'(E) = \sum_{k=1}^\infty m'(E_k)$. This second route needs the finiteness of $m'(E)$, which is why the ascending form of continuity is the one that gives the theorem in full generality.

Taken together with Problem 27, the conclusion is that for a $[0, \infty]$ -valued set function on a σ -algebra which vanishes on the empty set, countable additivity is equivalent to the conjunction of finite additivity and continuity. This is the sense in which Theorem 15 and Proposition 13 carry the same information about Lebesgue measure.

2.5 Problems 2.29–2.35

- Problem 2.29.** (i) Show that rational equivalence defines an equivalence relation on any set.
(ii) Explicitly find a choice set for the rational equivalence relation on $\mathbb{Q}$.
(iii) Define two numbers to be irrationally equivalent provided their difference is irrational. Is this an equivalence relation on $\mathbb{R}$? Is this an equivalence relation on $\mathbb{Q}$?

Solution. Throughout, for a nonempty set E of real numbers, two points $x, y \in E$ are called rationally equivalent provided $x - y \in \mathbb{Q}$; write $x \sim y$.

Part (i). Let E be any nonempty set of real numbers.

Reflexivity: for $x \in E$ we have $x - x = 0$, and $0 \in \mathbb{Q}$. Hence $x \sim x$.

Symmetry: suppose $x \sim y$, so that $q := x - y \in \mathbb{Q}$. Then $y - x = -q$, and $\mathbb{Q}$ is closed under negation, so $y - x \in \mathbb{Q}$ and therefore $y \sim x$.

Transitivity: suppose $x \sim y$ and $y \sim z$, say $x - y = q \in \mathbb{Q}$ and $y - z = r \in \mathbb{Q}$. Then

$$x - z = (x - y) + (y - z) = q + r,$$

and $\mathbb{Q}$ is closed under addition, so $x - z \in \mathbb{Q}$ and therefore $x \sim z$.

Thus $\sim$ is reflexive, symmetric and transitive on E ; that is, it is an equivalence relation on E . Observe that the only properties of $\mathbb{Q}$ used are that it is a subgroup of the additive group $\mathbb{R}$: it contains 0 and is closed under negation and addition.

Part (ii). Take $E = \mathbb{Q}$. If $x, y \in \mathbb{Q}$ then $x - y \in \mathbb{Q}$, so $x \sim y$. Hence any two rationals are rationally equivalent, and the rational equivalence relation on $\mathbb{Q}$ has exactly one equivalence class, namely $\mathbb{Q}$ itself. A choice set consists of exactly one member of each equivalence class, so

$$\mathcal{C}_{\mathbb{Q}} = \{0\}$$

is a choice set (as is $\{q\}$ for any single rational q ; these singletons are precisely the choice sets).

It is worth checking the two characteristic properties listed in the text. Property (i), that the difference of two (distinct) points of the choice set is not rational, holds vacuously, since $\{0\}$ contains no two distinct points. Property (ii), that each $x \in \mathbb{Q}$ may be written $x = c + q$ with $c \in \mathcal{C}_{\mathbb{Q}}$ and q rational, holds with $c = 0$ and $q = x$.

Note that no appeal to the Axiom of Choice is needed here: there is only one equivalence class, and we have named a member of it explicitly.

Part (iii). Define $x \approx y$ provided $x - y$ is irrational.

On $\mathbb{R}$ this is not an equivalence relation. It fails reflexivity at every point: for each $x \in \mathbb{R}$, $x - x = 0$ is rational, so $x \not\approx x$. (The relation is symmetric, since $x - y$ is irrational if and only if $y - x = -(x - y)$ is irrational. But it also fails transitivity: take $x = 0$, $y = \sqrt{2}$, $z = 2$. Then $x - y = -\sqrt{2}$ and $y - z = \sqrt{2} - 2$ are both irrational, so $x \approx y$ and $y \approx z$; but $x - z = -2$ is rational, so $x \not\approx z$.)

On $\mathbb{Q}$ this is not an equivalence relation either. Indeed, if $x, y \in \mathbb{Q}$ then $x - y \in \mathbb{Q}$ is rational, so no pair of rationals is irrationally equivalent: the relation $\approx$ restricted to $\mathbb{Q}$ is the empty relation. It is symmetric and vacuously transitive, but it is not reflexive, since $\mathbb{Q}$ is nonempty and $x \not\approx x$ for every $x \in \mathbb{Q}$. Hence it is not an equivalence relation.

Problem 2.30. Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.

Solution. Let E be a set of real numbers with $m^*(E) > 0$, and let $\mathcal{C}_E$ be a choice set for the rational equivalence relation on E . We argue by contradiction: suppose $\mathcal{C}_E$ is countable (recall that in this book “countable” includes “finite”, so this covers both the finite and the countably infinite case).

We first observe that

$$E \subseteq \bigcup_{q \in \mathbb{Q}} (q + \mathcal{C}_E).$$

Indeed, let $x \in E$. By the second characteristic property of a choice set, there is a point $c \in \mathcal{C}_E$ and a rational number q with $x = c + q$. Thus $x \in q + \mathcal{C}_E$, which proves the inclusion.

Now $\mathbb{Q}$ is countable, so the union above is a countable union of sets. Each set $q + \mathcal{C}_E$ is a translate of the countable set $\mathcal{C}_E$, hence is itself countable, and a countable set of real numbers has outer measure zero (the Example following the statement of monotonicity in Section 2.2). Therefore, by the countable

subadditivity of outer measure (Proposition 3 of this chapter) together with its monotonicity,

$$m^*(E) \leq m^*\left(\bigcup_{q \in \mathbb{Q}} (q + \mathcal{C}_E)\right) \leq \sum_{q \in \mathbb{Q}} m^*(q + \mathcal{C}_E) = \sum_{q \in \mathbb{Q}} 0 = 0.$$

Hence $m^*(E) = 0$, contradicting the assumption that $m^*(E) > 0$.

We conclude that $\mathcal{C}_E$ is not countable. In particular $\mathcal{C}_E$ is not finite, so it is uncountably infinite, as asserted.

Two remarks. First, no measurability of $\mathcal{C}_E$ is needed anywhere in this argument; only outer measure, which is defined for every set, is used. Second, translation invariance of outer measure was not even required, since a translate of a countable set is countable, and countable sets have outer measure zero directly.

Problem 2.31. Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that E is bounded.

Solution. The assertion to be justified is the opening sentence of the proof of Theorem 17: "By the countable subadditivity of outer measure, we may suppose E is bounded." Precisely, what must be shown is the following implication.

Claim. Suppose it has been proved that every bounded set of positive outer measure contains a nonmeasurable subset. Then every set of positive outer measure contains a nonmeasurable subset.

Proof of the claim. Let E be a set of real numbers with $m^*(E) > 0$. For each natural number n put

$$E_n = E \cap [-n, n].$$

Each E_n is bounded, and since every real number lies in $[-n, n]$ for n sufficiently large,

$$E = \bigcup_{n=1}^{\infty} E_n.$$

We assert that $m^*(E_n) > 0$ for some n . Otherwise $m^*(E_n) = 0$ for every n , and then, by the countable subadditivity of outer measure (Proposition 3 of this chapter),

$$m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n) = 0,$$

so that $m^*(E) = 0$, contrary to hypothesis.

Fix an index n with $m^*(E_n) > 0$. Then E_n is a bounded set of positive outer measure, so by the assumed bounded case, E_n contains a subset W that fails to be measurable. Since $W \subseteq E_n \subseteq E$, the set E contains the nonmeasurable subset W . This proves the claim, and hence justifies the reduction. It is worth recording explicitly why the reduction is desirable: boundedness of E is used twice in the proof of Theorem 17. It guarantees that $\mathcal{C}_E$ is bounded, so that Lemma 16 (which requires E bounded) may be applied to $\mathcal{C}_E$; and it supplies an interval $[-b, b] \supseteq E$, which is what allows the bounded, countably infinite set of rationals $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$ to be chosen so that the inclusion $E \subseteq \bigcup_{\lambda \in \Lambda_0} (\lambda + \mathcal{C}_E)$ holds. As Problem 32 shows, Lemma 16 genuinely fails without boundedness of Λ , so this reduction is not a mere convenience.

Problem 2.32. Does Lemma 16 remain true if Λ is allowed to be finite or to be uncountably infinite? Does it remain true if Λ is allowed to be unbounded?

Solution. Recall Lemma 16: if E is a bounded measurable set of real numbers and there is a bounded, countably infinite set Λ of real numbers for which the collection of translates $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint, then $m(E) = 0$. We examine the three proposed relaxations in turn. The answers are: no for finite Λ ,

yes for uncountably infinite (bounded) Λ , and no for unbounded Λ .

Finite Λ : the lemma fails. Let $E = [0, 1]$ and $\Lambda = \{0, 1\}$. Then E is bounded and measurable, Λ is bounded and finite, and the translates are $0 + E = [0, 1]$ and $1 + E = [1, 2]$, which are disjoint. Yet $m(E) = 1 \neq 0$. Even more simply, one may take $\Lambda = \{0\}$: the collection consisting of the single set E is trivially disjoint, so with $E = [0, 1]$ we again get a bounded measurable set of positive measure. The reason the proof of Lemma 16 breaks down is transparent from equation (15): with Λ finite, the right-hand sum $\sum_{\lambda \in \Lambda} m(\lambda + E) = (\#\Lambda) \cdot m(E)$ is finite whatever the value of $m(E)$, so no contradiction arises. It is exactly the infinitude of Λ that forces the sum of infinitely many equal positive numbers to be infinite.

Uncountably infinite Λ (still bounded): the lemma remains true. Suppose E is bounded and measurable, Λ is bounded and uncountably infinite, and $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint. If $E = \emptyset$ then $m(E) = 0$ and there is nothing to prove, so assume $E \neq \emptyset$. Since Λ is infinite, it possesses a countably infinite subset $\Lambda_0 = \{\lambda_1, \lambda_2, \dots\}$: choose $\lambda_1 \in \Lambda$, and having chosen the distinct points $\lambda_1, \dots, \lambda_n$, the set $\Lambda \setminus \{\lambda_1, \dots, \lambda_n\}$ is nonempty because Λ is infinite, so a point λ_{n+1} may be chosen from it. Now Λ_0 is countably infinite, and it is bounded because Λ is. Moreover $\{\lambda + E\}_{\lambda \in \Lambda_0}$ is a subcollection of a disjoint collection and hence is disjoint. Lemma 16, applied to E and Λ_0 , yields $m(E) = 0$.

So allowing Λ to be uncountable does not change the conclusion; it merely weakens the hypothesis in an inessential way, since only a countably infinite bounded piece of Λ is ever used.

Unbounded Λ : the lemma fails. Let $E = [0, 1]$ and $\Lambda = \mathbb{Z}$. Then E is bounded and measurable, and Λ is countably infinite but unbounded. The translates are $n + E = [n, n + 1]$ for $n \in \mathbb{Z}$, and these are pairwise disjoint. Yet $m(E) = 1 \neq 0$. Here the proof of Lemma 16 breaks down at the following point: the argument used boundedness of Λ (together with boundedness of E) to conclude that $\bigcup_{\lambda \in \Lambda} (\lambda + E)$ is bounded and therefore of finite measure, which is what makes the left-hand side of (15) finite. With $\Lambda = \mathbb{Z}$ the union is all of $\mathbb{R}$, of infinite measure, and (15) reads $\infty = \sum_{n \in \mathbb{Z}} 1$, a true statement carrying no contradiction.

In summary, the two hypotheses on Λ play different roles and only one of them can be relaxed: Λ must be infinite (uncountable is fine), and Λ must be bounded.

Problem 2.33. Let E be a nonmeasurable set of finite outer measure. Show that there is a G_δ set G that contains E for which

$$m^*(E) = m^*(G), \quad \text{while } m^*(G \setminus E) > 0.$$

Solution. Write $\alpha = m^*(E)$, so that $0 \leq \alpha < \infty$ by hypothesis.

Construction of G . Fix a natural number n . By the definition of outer measure as an infimum, there is a countable collection $\{I_{n,k}\}_k$ of nonempty open, bounded intervals which covers E and for which

$$\sum_k \ell(I_{n,k}) < \alpha + \frac{1}{n}.$$

(The infimum is finite since $\alpha < \infty$, so such a collection exists.) Set $\mathcal{O}_n = \bigcup_k I_{n,k}$. Then $\mathcal{O}_n$ is an open set containing E , and by the countable subadditivity and monotonicity of measure together with $m(I) = \ell(I)$ for an interval I ,

$$m(\mathcal{O}_n) \leq \sum_k m(I_{n,k}) = \sum_k \ell(I_{n,k}) < \alpha + \frac{1}{n}.$$

Now define

$$G = \bigcap_{n=1}^{\infty} \mathcal{O}_n.$$

Being a countable intersection of open sets, G is a G_δ set; in particular G is measurable, since by Theorem 9 of this chapter the measurable sets form a σ -algebra containing the Borel sets, and every G_δ set is measurable. Each $\mathcal{O}_n$ contains E , so $E \subseteq G$.

Verification that $m^*(E) = m^*(G)$. By monotonicity of outer measure and $E \subseteq G$ we have $\alpha = m^*(E) \leq m^*(G)$. On the other hand, $G \subseteq \mathcal{O}_n$ for each n , so by monotonicity

$$m^*(G) \leq m(\mathcal{O}_n) < \alpha + \frac{1}{n} \quad \text{for every } n.$$

Letting $n \rightarrow \infty$ gives $m^*(G) \leq \alpha$. Hence $m^*(G) = \alpha = m^*(E)$, as required. Note in particular that $m(G) = \alpha < \infty$.

Verification that $m^*(G \sim E) > 0$. Suppose, to the contrary, that $m^*(G \sim E) = 0$. A set of outer measure zero is measurable (Proposition 4 of this chapter), so $G \sim E$ is measurable. But G is measurable, and the measurable sets form a σ -algebra, so

$$E = G \sim (G \sim E)$$

is measurable. (The displayed identity holds because $E \subseteq G$: a point of G lies outside $G \sim E$ precisely when it lies in E .) This contradicts the hypothesis that E fails to be measurable. Therefore $m^*(G \sim E) > 0$.

Thus G is a G_δ set containing E with $m^*(G) = m^*(E)$ and $m^*(G \sim E) > 0$, which is what was to be shown.

A remark on what this exhibits. Since E and $G \sim E$ are disjoint with union G , we have

$$m^*(E) + m^*(G \sim E) > m^*(E) = m(G) = m^*(E \cup (G \sim E)),$$

where the strict inequality uses $m^*(G \sim E) > 0$ together with the finiteness of $m^*(E)$. So the pair $E, G \sim E$ is an explicit disjoint pair for which the subadditivity inequality $m^*(A \cup B) \leq m^*(A) + m^*(B)$ is strict, exactly as promised by Theorem 18. The hypothesis $m^*(E) < \infty$ enters twice: it is what makes the covers used in the construction of G available at all, and it is what makes the last display a strict inequality rather than the vacuous $\infty > \infty$.

Problem 2.34. Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero.

Solution. Let φ denote the Cantor-Lebesgue function on $[0, 1]$, let $\mathbf{C}$ be the Cantor set, and let $\mathcal{O} = [0, 1] \setminus \mathbf{C}$ be the union of the countable disjoint collection $\{I_k\}_{k=1}^\infty$ of open intervals removed in the Cantor deletion process. By Proposition 19, $m(\mathbf{C}) = 0$, and hence $m(\mathcal{O}) = 1$; since the I_k are disjoint open intervals, countable additivity gives

$$\sum_{k=1}^{\infty} \ell(I_k) = m(\mathcal{O}) = 1.$$

By Proposition 20, φ is increasing and continuous and maps $[0, 1]$ onto $[0, 1]$; in particular $\varphi(0) = 0$ and $\varphi(1) = 1$. Moreover, by the very definition of φ on the removed intervals, φ is constant on each I_k .

Define ψ on $[0, 1]$ by

$$\psi(x) = \frac{\varphi(x) + x}{2}.$$

Step 1: ψ is a continuous, strictly increasing map of $[0, 1]$ onto $[0, 1]$. It is continuous as a linear combination of the continuous functions φ and the identity. If $0 \leq u < v \leq 1$ then $\varphi(u) \leq \varphi(v)$ because φ is increasing, while $u < v$; adding and halving gives $\psi(u) < \psi(v)$. Thus ψ is strictly increasing, and in particular one-to-one. Finally $\psi(0) = (\varphi(0) + 0)/2 = 0$ and $\psi(1) = (1 + 1)/2 = 1$; since ψ is increasing, its values on $[0, 1]$ lie in $[0, 1]$, and since ψ is continuous, the Intermediate Value Theorem shows every point of $[0, 1]$ is attained. Hence ψ is a one-to-one map of $[0, 1]$ onto $[0, 1]$.

Step 2: $m(\psi(\mathcal{O})) = 1/2$. Fix k and write $I_k = (a_k, b_k)$. Since φ is constant on I_k , say $\varphi \equiv c_k$ there, we

have $\psi(x) = (c_k + x)/2$ for $x \in I_k$, so

$$\psi(I_k) = \left(\frac{c_k + a_k}{2}, \frac{c_k + b_k}{2} \right),$$

an open interval of length $\ell(I_k)/2$. Because ψ is one-to-one and the I_k are disjoint, the collection $\{\psi(I_k)\}_{k=1}^{\infty}$ is disjoint. Therefore $\psi(\mathcal{O}) = \bigcup_{k=1}^{\infty} \psi(I_k)$ is open, hence measurable, and by countable additivity

$$m(\psi(\mathcal{O})) = \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \frac{1}{2} \sum_{k=1}^{\infty} \ell(I_k) = \frac{1}{2}.$$

Step 3: $m(\psi(\mathbf{C})) = 1/2$. The set $\mathbf{C}$ is closed and bounded, and ψ is continuous, so $\psi(\mathbf{C})$ is compact, hence closed, hence measurable. Since ψ is a one-to-one map of $[0, 1]$ onto $[0, 1]$ and $[0, 1] = \mathbf{C} \cup \mathcal{O}$ is a disjoint decomposition, its image

$$[0, 1] = \psi(\mathbf{C}) \cup \psi(\mathcal{O})$$

is also a disjoint decomposition. By additivity of measure and Step 2,

$$m(\psi(\mathbf{C})) = m([0, 1]) - m(\psi(\mathcal{O})) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Step 4: the function. Let $f = \psi^{-1}$, defined on $[0, 1]$ with values in $[0, 1]$; this makes sense by Step 1. As noted in the proof of Proposition 21 (and established in Problem 45), a strictly increasing continuous function defined on an interval has a continuous inverse, so f is continuous on $[0, 1]$. Moreover f is strictly increasing: if $0 \leq u < v \leq 1$, write $u = \psi(s)$ and $v = \psi(t)$ with $s, t \in [0, 1]$; were $s \geq t$ we would get $u = \psi(s) \geq \psi(t) = v$, a contradiction, so $s < t$, that is, $f(u) < f(v)$.

Now set $A = \psi(\mathbf{C}) \subseteq [0, 1]$. By Step 3, A is measurable with $m(A) = 1/2 > 0$, and since ψ is one-to-one and onto $[0, 1]$,

$$f(A) = \psi^{-1}(\psi(\mathbf{C})) = \mathbf{C},$$

which has measure zero by Proposition 19.

Thus f is a continuous, strictly increasing function on $[0, 1]$ carrying the set A of measure $1/2$ onto the set $\mathbf{C}$ of measure zero, as required.

Problem 2.35. Let f be an increasing function on the open interval I . For $x_0 \in I$ show that f is continuous at x_0 if and only if there are sequences $\{a_n\}$ and $\{b_n\}$ in I such that for each n , $a_n < x_0 < b_n$, and $\lim_{n \rightarrow \infty} [f(b_n) - f(a_n)] = 0$.

Solution. Recall that f increasing means $f(u) \leq f(v)$ whenever $u \leq v$ in I .

Necessity. Suppose f is continuous at x_0 . Since I is open and $x_0 \in I$, there is an $r > 0$ with $(x_0 - r, x_0 + r) \subseteq I$. For each natural number n define

$$a_n = x_0 - \frac{r}{n+1}, \quad b_n = x_0 + \frac{r}{n+1}.$$

Then a_n and b_n belong to I and $a_n < x_0 < b_n$ for every n , while $a_n \rightarrow x_0$ and $b_n \rightarrow x_0$. Continuity of f at x_0 (in its sequential form) gives $f(a_n) \rightarrow f(x_0)$ and $f(b_n) \rightarrow f(x_0)$, and therefore

$$\lim_{n \rightarrow \infty} [f(b_n) - f(a_n)] = f(x_0) - f(x_0) = 0.$$

Sufficiency. Suppose there are sequences $\{a_n\}$ and $\{b_n\}$ in I with $a_n < x_0 < b_n$ for each n and $f(b_n) - f(a_n) \rightarrow 0$. Note first that, since f is increasing and $a_n < x_0 < b_n$,

$$f(a_n) \leq f(x_0) \leq f(b_n) \quad \text{for every } n,$$

so each difference $f(b_n) - f(a_n)$ is nonnegative.

Let $\varepsilon > 0$. Choose an index N for which

$$f(b_N) - f(a_N) < \varepsilon,$$

and define

$$\delta = \min\{x_0 - a_N, b_N - x_0\} > 0.$$

Since a_N and b_N lie in the interval I , the whole interval (a_N, b_N) is contained in I ; and $(x_0 - \delta, x_0 + \delta) \subseteq (a_N, b_N)$ by the choice of δ . Hence every x with $|x - x_0| < \delta$ belongs to I and satisfies $a_N < x < b_N$. For such an x , monotonicity of f gives

$$f(a_N) \leq f(x) \leq f(b_N).$$

Since also $f(a_N) \leq f(x_0) \leq f(b_N)$, both $f(x)$ and $f(x_0)$ lie in the interval $[f(a_N), f(b_N)]$, whose length is less than ε . Consequently

$$|f(x) - f(x_0)| \leq f(b_N) - f(a_N) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0 .

Remark. The hypothesis that f be increasing is essential in the sufficiency direction: it is what forces $f(x)$ to be trapped between $f(a_N)$ and $f(b_N)$ for all x between a_N and b_N , so that the smallness of a single difference $f(b_N) - f(a_N)$ controls the oscillation of f on an entire neighborhood of x_0 . The hypothesis that I be open is what guarantees, in the necessity direction, that points of I strictly on both sides of x_0 exist.

2.6 Problems 2.36–2.42

Problem 2.36. Show that if f is any increasing function on $[0, 1]$ that agrees with the Cantor-Lebesgue function φ on the complement of the Cantor set, then $f = \varphi$ on all of $[0, 1]$.

Solution. Write $\mathbf{C}$ for the Cantor set and $\mathcal{O} = [0, 1] \setminus \mathbf{C}$ for its complement in $[0, 1]$. We are given that f is increasing on $[0, 1]$ and that $f = \varphi$ on $\mathcal{O}$.

Step 1: $\mathcal{O}$ is dense in $[0, 1]$. By Proposition 19, $m(\mathbf{C}) = 0$. If some nondegenerate interval $J \subseteq [0, 1]$ were contained in $\mathbf{C}$, then by monotonicity of measure and Proposition 1 we would have $0 < \ell(J) = m(J) \leq m(\mathbf{C}) = 0$, a contradiction. Hence every nondegenerate subinterval of $[0, 1]$ contains a point of $\mathcal{O}$.

(Alternatively and without measure theory: $\mathbf{C} \subseteq C_k$, and C_k is a disjoint union of closed intervals of length 3^{-k} , so $\mathbf{C}$ contains no interval of length exceeding 3^{-k} for any k .)

Step 2: $f = \varphi$ at every interior point. Let $x \in \mathbf{C}$ with $0 < x < 1$, and let N be a natural number with $1/N < \min\{x, 1 - x\}$. For each $n \geq N$ the intervals $(x - 1/n, x)$ and $(x, x + 1/n)$ are nondegenerate subintervals of $[0, 1]$, so by Step 1 we may select points

$$u_n \in \mathcal{O} \cap (x - \frac{1}{n}, x), \quad v_n \in \mathcal{O} \cap (x, x + \frac{1}{n}).$$

Then $u_n < x < v_n$ and $u_n \rightarrow x$, $v_n \rightarrow x$.

Since f is increasing and agrees with φ on $\mathcal{O}$,

$$\varphi(u_n) = f(u_n) \leq f(x) \leq f(v_n) = \varphi(v_n).$$

Since φ is increasing (Proposition 20),

$$\varphi(u_n) \leq \varphi(x) \leq \varphi(v_n).$$

Thus both $f(x)$ and $\varphi(x)$ lie in the interval $[\varphi(u_n), \varphi(v_n)]$, so

$$|f(x) - \varphi(x)| \leq \varphi(v_n) - \varphi(u_n).$$

By Proposition 20 the function φ is continuous, so $\varphi(u_n) \rightarrow \varphi(x)$ and $\varphi(v_n) \rightarrow \varphi(x)$, whence the right-hand side tends to 0. Therefore $f(x) = \varphi(x)$.

Since $f = \varphi$ on $\mathcal{O}$ by hypothesis, and every point of $\mathbf{C} \cap (0, 1)$ has just been handled, we conclude that $f = \varphi$ on all of $(0, 1)$.

Step 3: the endpoints. Since φ is increasing and maps $[0, 1]$ onto $[0, 1]$ (Proposition 20), $\varphi(0) = 0$ and $\varphi(1) = 1$. The two points 0 and 1 lie in $\mathbf{C}$, and the argument of Step 2 is only one-sided there, so it

yields inequalities rather than equalities. By Step 1 we may choose $v_n \in \mathcal{O} \cap (0, 1/n)$, so that $v_n \downarrow 0$, and then

$$f(0) \leq f(v_n) = \varphi(v_n) \rightarrow \varphi(0) = 0,$$

so $f(0) \leq 0 = \varphi(0)$; symmetrically $f(1) \geq 1 = \varphi(1)$. With the natural normalization that f , like φ , takes its values in $[0, 1]$ (equivalently, that $f(0) \geq 0$ and $f(1) \leq 1$), these inequalities become equalities and $f = \varphi$ on all of $[0, 1]$.

Some such normalization is genuinely needed: the function g defined by $g(0) = -1$ and $g = \varphi$ on $(0, 1]$ is increasing on $[0, 1]$ and agrees with φ on $\mathcal{O}$ (note $0 \notin \mathcal{O}$, since $0 \in \mathbf{C}$), yet $g \neq \varphi$. The substance of the problem, namely that the values of φ on the measure-zero set $\mathbf{C}$ are completely determined by its values on the complement $\mathcal{O}$ together with monotonicity, is exactly what Step 2 establishes.

Problem 2.37. Let f be a continuous function defined on E . Is it true that $f^{-1}(A)$ is always measurable if A is measurable?

Solution. No. We exhibit a continuous function f defined on a closed bounded interval E and a measurable set A for which $f^{-1}(A)$ is not measurable.

Let φ be the Cantor-Lebesgue function and, as in Proposition 21, define

$$\psi(x) = \varphi(x) + x \quad \text{for } x \in [0, 1].$$

By Proposition 21, ψ is a strictly increasing continuous function that maps $[0, 1]$ onto $[0, 2]$, and, by assertion (ii) of that proposition, there is a measurable set A , a subset of the Cantor set $\mathbf{C}$, for which $\psi(A)$ is nonmeasurable.

Set $E = [0, 2]$ and let

$$f = \psi^{-1} : [0, 2] \rightarrow [0, 1].$$

This is well defined because ψ , being strictly increasing, is one-to-one, and it maps $[0, 1]$ onto $[0, 2]$.

The function f is continuous. This is Problem 45, and the proof is short: f is strictly increasing on $[0, 2]$ and its range is all of $[0, 1]$. If f failed to be continuous at some $y_0 \in [0, 2]$, then, f being increasing and bounded, the one-sided limits

$$\lambda = \sup_{y < y_0} f(y) \quad \text{and} \quad \mu = \inf_{y > y_0} f(y)$$

exist (interpreting the sup as $f(y_0)$ if $y_0 = 0$, and the inf as $f(y_0)$ if $y_0 = 2$) and satisfy $\lambda \leq f(y_0) \leq \mu$ with $\lambda < \mu$. But then f takes no value in (λ, μ) other than possibly $f(y_0)$, so its range omits points of the interval $(\lambda, \mu) \subseteq [0, 1]$, contradicting the fact that the range of f is all of $[0, 1]$.

Now compute the inverse image of A under f . For $y \in [0, 2]$,

$$y \in f^{-1}(A) \iff f(y) = \psi^{-1}(y) \in A \iff y \in \psi(A),$$

the last equivalence because ψ is a bijection of $[0, 1]$ onto $[0, 2]$. Hence

$$f^{-1}(A) = \psi(A).$$

The set A is measurable: it is a subset of $\mathbf{C}$, which has measure zero by Proposition 19, so $m^*(A) = 0$ by the monotonicity of outer measure, and therefore A is measurable by Proposition 4. But $\psi(A)$ is nonmeasurable by the choice of A . Therefore f is continuous on $E = [0, 2]$, the set A is measurable, and $f^{-1}(A)$ is not measurable.

Remark. What is true is that $f^{-1}(A)$ is measurable whenever f is continuous and A is a Borel set: the collection of sets B for which $f^{-1}(B)$ is measurable is a σ -algebra (inverse images preserve complements and countable unions), and it contains the open sets because f is continuous, hence it contains the smallest σ -algebra containing the open sets, namely the Borel sets. Compare Problem 46, where the same argument gives the sharper conclusion that $f^{-1}(A)$ is itself Borel. The example above shows this cannot be improved to Lebesgue measurable sets A ; this is consistent with Proposition 22, which asserts that there are Lebesgue measurable sets which are not Borel.

Problem 2.38. Let the function $f: [a, b] \rightarrow \mathbb{R}$ be Lipschitz, that is, there is a constant $c \geq 0$ such that for all $u, v \in [a, b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that f maps a set of measure zero onto a set of measure zero. Show that f maps an F_σ set onto an F_σ set. Conclude that f maps a measurable set to a measurable set.

Solution. Throughout, all sets considered are subsets of $[a, b]$. Observe first that f is continuous, since $|f(u) - f(v)| \leq c|u - v|$ forces $f(v) \rightarrow f(u)$ as $v \rightarrow u$.

A basic estimate. Let J be any subset of $[a, b]$ that is contained in an interval of length ℓ . If J is empty the estimate below is trivial, so assume $J \neq \emptyset$. For all $u, v \in J$ we have $|f(u) - f(v)| \leq c|u - v| \leq c\ell$, so the set $f(J)$ is bounded and

$$\sup f(J) - \inf f(J) = \sup_{u, v \in J} |f(u) - f(v)| \leq c\ell.$$

Consequently $f(J)$ is contained in the closed bounded interval $[\inf f(J), \inf f(J) + c\ell]$, whose length is $c\ell$ (a degenerate one-point interval if $c = 0$ or $\ell = 0$). By the monotonicity of outer measure and Proposition 1,

$$m^*(f(J)) \leq c\ell.$$

Call this estimate (*).

Part 1: f maps a set of measure zero onto a set of measure zero.

Let $E \subseteq [a, b]$ with $m^*(E) = 0$, and let $\epsilon > 0$. By the definition of outer measure there is a countable collection $\{I_k\}_{k=1}^\infty$ of open intervals which covers E and for which

$$\sum_{k=1}^\infty \ell(I_k) < \epsilon.$$

For each k put $J_k = I_k \cap [a, b]$. Then $\{J_k\}$ covers E , each J_k is contained in an interval of length at most $\ell(I_k)$, and

$$f(E) \subseteq \bigcup_{k=1}^\infty f(J_k).$$

By the countable subadditivity of outer measure (Proposition 3), monotonicity, and the estimate (*),

$$m^*(f(E)) \leq \sum_{k=1}^\infty m^*(f(J_k)) \leq \sum_{k=1}^\infty c\ell(I_k) = c \sum_{k=1}^\infty \ell(I_k) \leq c\epsilon.$$

Since $\epsilon > 0$ was arbitrary, $m^*(f(E)) = 0$. In particular, by Proposition 4, $f(E)$ is measurable and $m(f(E)) = 0$.

Part 2: f maps an F_σ set onto an F_σ set.

Let $F \subseteq [a, b]$ be an F_σ set, say $F = \bigcup_{k=1}^\infty F_k$ with each F_k closed. Replacing F_k by $F_k \cap [a, b]$ (which is again closed, and does not change the union since $F \subseteq [a, b]$), we may assume each F_k is a closed subset of $[a, b]$. Then each F_k is closed and bounded, and f is continuous, so $f(F_k)$ is a compact, hence closed, subset of $\mathbb{R}$. (Concretely: $f(F_k)$ is bounded by the estimate (*) applied to $[a, b]$, and it is closed because if $y_n = f(x_n) \rightarrow y$ with $x_n \in F_k$, then by the Bolzano-Weierstrass Theorem a subsequence x_{n_j} converges to some x , which lies in F_k since F_k is closed, and then $y = \lim f(x_{n_j}) = f(x)$ by continuity, so $y \in f(F_k)$.)

Since images commute with unions,

$$f(F) = \bigcup_{k=1}^\infty f(F_k),$$

a countable union of closed sets, that is, an F_σ set.

Part 3: f maps a measurable set to a measurable set.

Let $E \subseteq [a, b]$ be measurable. By assertion (iv) of Theorem 11 there is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$. Write

$$E = F \cup (E \setminus F), \quad \text{so} \quad f(E) = f(F) \cup f(E \setminus F).$$

By Part 2, $f(F)$ is an F_σ set; an F_σ set is a Borel set and therefore measurable by Theorem 9. By Part 1, $m^*(f(E \sim F)) = 0$, so $f(E \sim F)$ is measurable by Proposition 4. Since the measurable sets form a σ -algebra (Theorem 9), the union $f(E)$ of these two measurable sets is measurable.

Problem 2.39. Let F be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the n th deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that F is a closed set, $[0, 1] \sim F$ dense in $[0, 1]$, and $m(F) = 1 - \alpha$. Such a set F is called a generalized Cantor set.

Solution. The construction. Set $F_0 = [0, 1]$. Having obtained F_{n-1} as a disjoint union of 2^{n-1} closed intervals, each of the same length ℓ_{n-1} , form F_n by removing from each of these 2^{n-1} intervals the open middle interval of length $\alpha 3^{-n}$; what remains is a disjoint union of 2^n closed intervals, each of the common length

$$\ell_n = \frac{\ell_{n-1} - \alpha 3^{-n}}{2}, \quad \ell_0 = 1.$$

Then $F = \bigcap_{n=0}^{\infty} F_n$.

The lengths ℓ_n . We claim

$$\ell_n = (1 - \alpha) 2^{-n} + \alpha 3^{-n} \quad \text{for all } n \geq 0.$$

Indeed this holds for $n = 0$, since $(1 - \alpha) + \alpha = 1$; and if it holds for $n - 1$, then

$$\frac{\ell_{n-1} - \alpha 3^{-n}}{2} = \frac{(1 - \alpha) 2^{-(n-1)} + \alpha 3^{-(n-1)} - \alpha 3^{-n}}{2} = (1 - \alpha) 2^{-n} + \frac{\alpha 3^{-n}(3 - 1)}{2} = (1 - \alpha) 2^{-n} + \alpha 3^{-n},$$

which is ℓ_n . In particular $\ell_n > 0$ for every n , since $0 < \alpha < 1$; so the construction never runs out of room and is legitimate. Also $\ell_n \rightarrow 0$.

F is closed. Each F_n is a finite union of closed intervals, hence closed. Therefore $F = \bigcap_{n=0}^{\infty} F_n$, being an intersection of closed sets, is closed.

$m(F) = 1 - \alpha$. Each F_n is the disjoint union of 2^n closed intervals of length ℓ_n , so by the finite additivity of Lebesgue measure and Proposition 1,

$$m(F_n) = 2^n \ell_n = 2^n [(1 - \alpha) 2^{-n} + \alpha 3^{-n}] = (1 - \alpha) + \alpha \left(\frac{2}{3}\right)^n.$$

The collection $\{F_n\}_{n=0}^{\infty}$ is a descending sequence of measurable sets with $m(F_0) = 1 < \infty$. Hence by the continuity of measure (Theorem 15),

$$m(F) = m\left(\bigcap_{n=0}^{\infty} F_n\right) = \lim_{n \rightarrow \infty} m(F_n) = \lim_{n \rightarrow \infty} [(1 - \alpha) + \alpha (2/3)^n] = 1 - \alpha.$$

(As a check, the total length removed is $\sum_{n=1}^{\infty} 2^{n-1} \cdot \alpha 3^{-n} = \frac{\alpha}{2} \sum_{n=1}^{\infty} (2/3)^n = \frac{\alpha}{2} \cdot 2 = \alpha$.)

$[0, 1] \sim F$ is dense in $[0, 1]$. It suffices to show that every nondegenerate interval $J \subseteq [0, 1]$ contains a point of $[0, 1] \sim F$; equivalently, that F contains no nondegenerate interval.

Suppose $J \subseteq F$ is an interval with $\ell(J) > 0$. Since $\ell_n \rightarrow 0$, choose n with $\ell_n < \ell(J)$. Then $J \subseteq F \subseteq F_n$. The set F_n is a disjoint union of 2^n closed intervals separated by nonempty open gaps, so a connected subset of F_n must lie inside a single one of those intervals: indeed, if J met two distinct components of F_n it would, by the intermediate value property of intervals, contain a point of one of the removed open gaps lying between them, contradicting $J \subseteq F_n$. Hence J is contained in a single component of F_n , and therefore

$$\ell(J) \leq \ell_n < \ell(J),$$

a contradiction. So F contains no nondegenerate interval, and every nondegenerate subinterval of $[0, 1]$ meets $[0, 1] \sim F$. Thus $[0, 1] \sim F$ is dense in $[0, 1]$.

(Note that this density statement is not a consequence of $m(F) = 1 - \alpha < 1$ alone; it uses the geometry of the construction. Conversely, F is a closed set with empty interior that nonetheless has positive measure $1 - \alpha > 0$, that is, a nowhere dense set of positive measure.)

Problem 2.40. Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: Consider the complement of the generalized Cantor set of the preceding problem.)

Solution. Fix α with $0 < \alpha < 1$ and let $F \subseteq [0, 1]$ be the generalized Cantor set of Problem 39, so that F is closed, $[0, 1] \sim F$ is dense in $[0, 1]$, and $m(F) = 1 - \alpha > 0$. Define

$$\mathcal{O} = (0, 1) \sim F = (0, 1) \cap (\mathbb{R} \sim F).$$

Since F is closed, $\mathbb{R} \sim F$ is open, and therefore $\mathcal{O}$, an intersection of two open sets, is an open set of real numbers.

Claim: $F \cap (0, 1) \subseteq \partial\mathcal{O}$.

Recall that $\partial\mathcal{O} = \overline{\mathcal{O}} \sim \text{int } \mathcal{O}$, and that $\text{int } \mathcal{O} = \mathcal{O}$ because $\mathcal{O}$ is open. So we must show that each $x \in F \cap (0, 1)$ is a point of closure of $\mathcal{O}$ but does not belong to $\mathcal{O}$.

That $x \notin \mathcal{O}$ is immediate from $x \in F$.

That x is a point of closure of $\mathcal{O}$: let $\epsilon > 0$. Since $0 < x < 1$, we may shrink ϵ so that $(x - \epsilon, x + \epsilon) \subseteq (0, 1)$. This is a nondegenerate subinterval of $[0, 1]$, so by Problem 39 the dense set $[0, 1] \sim F$ meets it; that is, there is a point

$$y \in (x - \epsilon, x + \epsilon) \quad \text{with} \quad y \notin F.$$

Since also $y \in (0, 1)$, we have $y \in \mathcal{O}$. Thus every neighborhood of x contains a point of $\mathcal{O}$, so $x \in \overline{\mathcal{O}}$. This proves the claim.

Conclusion. The boundary $\partial\mathcal{O}$ is a closed set, hence measurable by Theorem 9. Since 0 and 1 are never removed in the construction of F , we have $F \cap (0, 1) = F \sim \{0, 1\}$, and a two-point set has measure zero by Proposition 4, so by the excision property of measure

$$m(F \cap (0, 1)) = m(F) = 1 - \alpha.$$

Therefore, by the monotonicity of measure and the claim,

$$m(\partial\mathcal{O}) \geq m(F \cap (0, 1)) = 1 - \alpha > 0.$$

So $\mathcal{O}$ is an open set whose boundary has positive measure. (Taking α small makes $m(\partial\mathcal{O})$ as close to 1 as we please, while $\mathcal{O}$ itself is an open subset of $(0, 1)$ of measure α ; in fact here $\partial\mathcal{O} = F$, since $\overline{\mathcal{O}} = [0, 1]$.)

Problem 2.41. A nonempty subset X of $\mathbb{R}$ is called perfect provided it is closed and each neighborhood of any point in X contains infinitely many points of X . Show that the Cantor set is perfect. (Hint: The endpoints of all of the subintervals occurring in the Cantor construction belong to $\mathbf{C}$.)

Solution. Recall the construction: $C_0 = [0, 1]$, and C_k is the disjoint union of 2^k closed intervals each of length 3^{-k} , obtained from C_{k-1} by removing the open middle third of each of its components; $\mathbf{C} = \bigcap_{k=1}^{\infty} C_k$.

By Proposition 19 the set $\mathbf{C}$ is closed, and it is nonempty since (as we verify below) $0 \in \mathbf{C}$. It remains to show that each neighborhood of each point of $\mathbf{C}$ contains infinitely many points of $\mathbf{C}$.

Step 1: every endpoint of every component of every C_k belongs to $\mathbf{C}$.

Let e be an endpoint of a component J of C_k . We show by induction on $n \geq k$ that e is an endpoint of a component of C_n . This holds for $n = k$. Suppose e is an endpoint of a component J' of C_n . Passing to C_{n+1} deletes the open middle third of J' and leaves the two closed outer thirds of J' as components of C_{n+1} ; the two endpoints of J' are endpoints of these two outer thirds. Hence e is an endpoint of a component of C_{n+1} , completing the induction.

In particular $e \in C_n$ for every $n \geq k$, and since $C_0 \supseteq C_1 \supseteq \cdots$ we also have $e \in C_n$ for $n < k$. Therefore $e \in \bigcap_{n=1}^{\infty} C_n = \mathbf{C}$. (Applied to $C_0 = [0, 1]$, this gives $0, 1 \in \mathbf{C}$.)

Step 2: for each k , each component J of C_k meets $\mathbf{C}$ in an infinite set.

Fix $j \geq 1$. The set $J \cap C_{k+j}$ consists of exactly 2^j pairwise disjoint closed intervals of length $3^{-(k+j)}$: this is immediate by induction on j , since each of the components of C_{k+j-1} lying inside J splits into two components of C_{k+j} lying inside J .

Each of these 2^j intervals is a component of C_{k+j} , so by Step 1 both of its endpoints lie in $\mathbf{C}$; and both endpoints lie in J . Since the 2^j intervals are pairwise disjoint and nondegenerate, these 2^{j+1} endpoints are distinct. Hence $J \cap \mathbf{C}$ contains at least 2^{j+1} points. As $j \geq 1$ was arbitrary, $J \cap \mathbf{C}$ is infinite.

Step 3: conclusion.

Let $x \in \mathbf{C}$ and let $\mathcal{U}$ be a neighborhood of x ; choose $\epsilon > 0$ with $(x - \epsilon, x + \epsilon) \subseteq \mathcal{U}$. Choose a natural number k with $3^{-k} < \epsilon$. Since $x \in \mathbf{C} \subseteq C_k$, the point x belongs to some component J of C_k , and $\ell(J) = 3^{-k}$. For every $y \in J$ we then have

$$|y - x| \leq \ell(J) = 3^{-k} < \epsilon,$$

so $J \subseteq (x - \epsilon, x + \epsilon) \subseteq \mathcal{U}$.

By Step 2 the set $J \cap \mathbf{C}$ is infinite, and it is contained in $\mathcal{U} \cap \mathbf{C}$. Hence $\mathcal{U}$ contains infinitely many points of $\mathbf{C}$.

Since $\mathbf{C}$ is nonempty and closed, and every neighborhood of every point of $\mathbf{C}$ contains infinitely many points of $\mathbf{C}$, the Cantor set is perfect.

Problem 2.42. Prove that every perfect subset X of $\mathbb{R}$ is uncountable. (Hint: If X is countable, construct a descending sequence of bounded, closed subsets of X whose intersection is empty.)

Solution. Let $X \subseteq \mathbb{R}$ be perfect: nonempty, closed, and such that every neighborhood of every point of X contains infinitely many points of X .

First note that X is infinite. Indeed, pick $x \in X$; the neighborhood $(x - 1, x + 1)$ of x already contains infinitely many points of X .

Suppose, to get a contradiction, that X is countable. Being infinite and countable, it may be enumerated as $X = \{x_n\}_{n=1}^{\infty}$. For $y \in \mathbb{R}$ and $r > 0$ write $B(y, r) = (y - r, y + r)$ and $\bar{B}(y, r) = [y - r, y + r]$.

Construction. We construct points $y_n \in X$ and radii $r_n > 0$ such that, setting

$$F_n = X \cap \bar{B}(y_n, r_n),$$

we have for every n :

1. $y_n \in X$, so $F_n \neq \emptyset$;
2. $x_n \notin F_n$;
3. $\bar{B}(y_{n+1}, r_{n+1}) \subseteq B(y_n, r_n)$, so that $F_{n+1} \subseteq F_n$.

Base step. Since X is infinite, choose $y_1 \in X$ with $y_1 \neq x_1$, and put $r_1 = \frac{1}{2}|y_1 - x_1| > 0$. Then $y_1 \in F_1$ and $x_1 \notin \bar{B}(y_1, r_1)$, so $x_1 \notin F_1$.

Inductive step. Suppose $y_n \in X$ and $r_n > 0$ have been chosen. Since $y_n \in X$, the neighborhood $B(y_n, r_n)$ of y_n contains infinitely many points of X . At most one of them equals x_{n+1} , so we may choose

$$y_{n+1} \in X \cap B(y_n, r_n) \quad \text{with} \quad y_{n+1} \neq x_{n+1}.$$

Now choose $r_{n+1} > 0$ with

$$r_{n+1} < \min \left\{ r_n - |y_{n+1} - y_n|, \frac{1}{2}|y_{n+1} - x_{n+1}| \right\},$$

both quantities being strictly positive by the choice of y_{n+1} . The first constraint gives $\bar{B}(y_{n+1}, r_{n+1}) \subseteq B(y_n, r_n)$: if $|z - y_{n+1}| \leq r_{n+1}$ then

$$|z - y_n| \leq |z - y_{n+1}| + |y_{n+1} - y_n| \leq r_{n+1} + |y_{n+1} - y_n| < r_n.$$

The second gives $x_{n+1} \notin \bar{B}(y_{n+1}, r_{n+1})$. Properties 1, 2, 3 hold, and the construction proceeds.

The contradiction. Each $F_n = X \cap \bar{B}(y_n, r_n)$ is the intersection of two closed sets (X is closed by hypothesis, $\bar{B}(y_n, r_n)$ is closed), hence closed; it is bounded, being contained in $\bar{B}(y_n, r_n)$; and it is

nonempty, since $y_n \in X \cap \bar{B}(y_n, r_n)$. By property 3,

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots,$$

so $\{F_n\}_{n=1}^\infty$ is a descending countable collection of nonempty closed sets of real numbers with F_1 bounded. By the Nested Set Theorem,

$$\bigcap_{n=1}^\infty F_n \neq \emptyset.$$

On the other hand, let $z \in \bigcap_{n=1}^\infty F_n$. Then $z \in F_1 \subseteq X$, so $z = x_N$ for some natural number N . But $z \in F_N$, while property 2 says $x_N \notin F_N$ - a contradiction. Hence the intersection is empty.

These two conclusions are incompatible, so X cannot be countable. Therefore every perfect subset of $\mathbb{R}$ is uncountable.

2.7 Problems 2.43–2.47

Problem 2.43. Use the preceding two problems to provide another proof of the uncountability of the Cantor set.

Solution. The two preceding problems are the following. Problem 41: a nonempty subset X of $\mathbb{R}$ is called *perfect* provided it is closed and each neighborhood of any point in X contains infinitely many points of X ; the Cantor set $\mathbf{C}$ is perfect. Problem 42: every perfect subset of $\mathbb{R}$ is uncountable. Granting these two assertions, the Cantor set is uncountable, since by Problem 41 it is perfect and by Problem 42 every perfect set is uncountable. For completeness we carry out both arguments, using only the description of the Cantor set given in Section 2.7 and results from Chapter 1.

Recall the construction: $C_0 = [0, 1]$, and C_{k+1} is obtained from C_k by removing the open middle third of each of the 2^k disjoint closed intervals of length 3^{-k} whose union is C_k ; then

$$\mathbf{C} = \bigcap_{k=1}^\infty C_k.$$

By Proposition 19 the set $\mathbf{C}$ is closed, and it is nonempty since $0 \in C_k$ for every k .

Step 1: the endpoints of every interval occurring in the construction belong to $\mathbf{C}$.

Let $J = [a, b]$ be one of the 2^k closed intervals whose union is C_k . In passing from C_k to C_{k+1} the open middle third of J is deleted, so the two intervals of C_{k+1} contained in J are $[a, a + (b - a)/3]$ and $[b - (b - a)/3, b]$. Thus a is the left endpoint of an interval of C_{k+1} and b is the right endpoint of an interval of C_{k+1} . Iterating this observation, for every $j \geq k$ the points a and b are endpoints of intervals of C_j , and in particular $a, b \in C_j$. Since the collection $\{C_j\}$ is descending, $a, b \in C_j$ for $j < k$ as well. Hence $a, b \in \bigcap_{j \geq 1} C_j = \mathbf{C}$.

Step 2: $\mathbf{C}$ is perfect.

The set $\mathbf{C}$ is nonempty and closed. Let $x \in \mathbf{C}$ and let $\mathcal{O}$ be a neighborhood of x ; choose $r > 0$ with $(x - r, x + r) \subseteq \mathcal{O}$. Choose an index k with $3^{-k} < r$. Since $x \in \mathbf{C} \subseteq C_k$ and C_k is the disjoint union of 2^k closed intervals of length 3^{-k} , the point x belongs to one of them, say J . Every point y of J satisfies $|y - x| \leq 3^{-k} < r$, so

$$J \subseteq (x - r, x + r) \subseteq \mathcal{O}.$$

Now fix $j > k$. Exactly 2^{j-k} of the intervals whose disjoint union is C_j are contained in J (each interval of C_i inside J splits into two intervals of C_{i+1} inside J , and this is an induction on i starting at $i = k$). These 2^{j-k} intervals are pairwise disjoint, and by Step 1 the left endpoint of each of them belongs to $\mathbf{C}$. Hence $J \cap \mathbf{C}$ contains at least 2^{j-k} points. As j may be taken arbitrarily large, $J \cap \mathbf{C}$ is infinite, and therefore $\mathcal{O} \cap \mathbf{C}$ is infinite. Thus $\mathbf{C}$ is perfect.

Step 3: a perfect subset of $\mathbb{R}$ is uncountable.

Let X be perfect and suppose, to get a contradiction, that X is countable. Since X is nonempty it may be enumerated as $X = \{x_1, x_2, x_3, \dots\}$ (a finite list is handled by repetition, but in fact the perfectness of X already forces X to be infinite). We construct inductively points $y_n \in X$ and radii $r_n > 0$ such

that, writing

$$I_n = [y_n - r_n, y_n + r_n], \quad F_n = X \cap I_n,$$

we have for every n :

(i) $F_n \neq \emptyset$; (ii) $I_{n+1} \subseteq (y_n - r_n, y_n + r_n)$, hence $F_{n+1} \subseteq F_n$; (iii) $x_n \notin I_n$, hence $x_n \notin F_n$.

To begin, pick any point $y_0 \in X$ and put $r_0 = 1$. Suppose $y_{n-1} \in X$ and $r_{n-1} > 0$ have been chosen ($n \geq 1$). The interval $(y_{n-1} - r_{n-1}, y_{n-1} + r_{n-1})$ is a neighborhood of the point y_{n-1} of X , so it contains infinitely many points of X ; in particular we may choose

$$y_n \in X \cap (y_{n-1} - r_{n-1}, y_{n-1} + r_{n-1}) \quad \text{with} \quad y_n \neq x_n.$$

Since $y_n \neq x_n$ and y_n lies in the open interval $(y_{n-1} - r_{n-1}, y_{n-1} + r_{n-1})$, we may choose $r_n > 0$ so small that

$$r_n < |y_n - x_n| \quad \text{and} \quad [y_n - r_n, y_n + r_n] \subseteq (y_{n-1} - r_{n-1}, y_{n-1} + r_{n-1}).$$

Then (iii) holds because $|x_n - y_n| > r_n$, and (ii) holds by the second requirement; (i) holds because $y_n \in F_n$.

Each $F_n = X \cap I_n$ is the intersection of two closed sets, hence closed, and it is bounded since I_n is bounded. By (i) and (ii), $\{F_n\}_{n=1}^{\infty}$ is a descending countable collection of nonempty bounded closed sets of real numbers. By the Nested Set Theorem there is a point

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

Since $F_1 \subseteq X$, we have $x \in X$, so $x = x_N$ for some index N . But $x \in F_N$, contradicting (iii), which says $x_N \notin F_N$. This contradiction shows X is uncountable.

Conclusion. By Step 2 the Cantor set $\mathbf{C}$ is a perfect subset of $\mathbb{R}$, and by Step 3 every perfect subset of $\mathbb{R}$ is uncountable. Therefore $\mathbf{C}$ is uncountable, which is a second proof of the uncountability asserted in Proposition 19.

Problem 2.44. A subset A of $\mathbb{R}$ is said to be *nowhere dense* in $\mathbb{R}$ provided that for every open set $\mathcal{O}$, $\mathcal{O}$ has an open subset that is disjoint from A . Show that the Cantor set is nowhere dense in $\mathbb{R}$.

Solution. Of course the empty set is an open subset of every open set and is disjoint from A , so the definition is to be read as requiring a *nonempty* open subset; we prove this stronger statement for the Cantor set $\mathbf{C}$.

Let $\mathcal{O}$ be a nonempty open set of real numbers. Choose a point $x \in \mathcal{O}$. Since $\mathcal{O}$ is open there is an $r > 0$ for which

$$(x - r, x + r) \subseteq \mathcal{O}.$$

We first observe that the open interval $(x - r, x + r)$ is not contained in $\mathbf{C}$. Indeed, by Proposition 19 the Cantor set has measure zero, $m(\mathbf{C}) = 0$, whereas the measure of an interval is its length, so

$$m((x - r, x + r)) = 2r > 0.$$

If we had $(x - r, x + r) \subseteq \mathbf{C}$, then by the monotonicity of Lebesgue measure we would get $2r = m((x - r, x + r)) \leq m(\mathbf{C}) = 0$, a contradiction. Hence there is a point

$$z \in (x - r, x + r) \quad \text{with} \quad z \notin \mathbf{C}.$$

By Proposition 19 the set $\mathbf{C}$ is closed, so its complement $\mathbb{R} \sim \mathbf{C}$ is open. Define

$$\mathcal{U} = (x - r, x + r) \cap (\mathbb{R} \sim \mathbf{C}).$$

Then $\mathcal{U}$ is open, being the intersection of two open sets; it is nonempty, since $z \in \mathcal{U}$; it is contained in $\mathcal{O}$, since $\mathcal{U} \subseteq (x - r, x + r) \subseteq \mathcal{O}$; and it is disjoint from $\mathbf{C}$ by its very definition. Thus every nonempty open set $\mathcal{O}$ has a nonempty open subset disjoint from $\mathbf{C}$, that is, $\mathbf{C}$ is nowhere dense in $\mathbb{R}$.

Two remarks. First, closedness of $\mathbf{C}$ was used only to make $\mathcal{U}$ open. The same two paragraphs prove, for an arbitrary *closed* set A , that A is nowhere dense if and only if A has empty interior: if A contains no nonempty open set, then for a given nonempty open $\mathcal{O}$ we may choose $z \in \mathcal{O} \sim A$, and $\mathcal{O} \cap (\mathbb{R} \sim A)$ is then a nonempty open subset of $\mathcal{O}$ disjoint from A ; conversely, if some nonempty open $\mathcal{O}_0$ is contained in A , then every nonempty open subset of $\mathcal{O}_0$ meets A , so A is not nowhere dense. Since $\mathbf{C}$ is closed and, as shown above, contains no open interval, $\mathbf{C}$ indeed has empty interior.

Second, the appeal to measure can be replaced by a direct appeal to the construction. Given the interval $(x - r, x + r)$, if it fails to meet $[0, 1]$ it is itself a nonempty open subset of $\mathcal{O}$ that is disjoint from $\mathbf{C} \subseteq [0, 1]$, and we are done. Otherwise pick $y \in (x - r, x + r) \cap [0, 1]$; since $|y - x| < r$ we may then choose an index k with

$$3^{-k} < r - |y - x|.$$

If $y \notin \mathbf{C}$ we may take $z = y$ in the argument above. If $y \in \mathbf{C}$, then y lies in one of the 2^k closed intervals J of length 3^{-k} whose disjoint union is C_k , and every point w of J satisfies $|w - x| \leq |w - y| + |y - x| \leq 3^{-k} + |y - x| < r$, so $J \subseteq (x - r, x + r) \subseteq \mathcal{O}$. The open middle third of J is deleted in passing from C_k to C_{k+1} and so is disjoint from $\mathbf{C}$; it is therefore a nonempty open subset of $\mathcal{O}$ disjoint from $\mathbf{C}$.

Problem 2.45. Show that a strictly increasing function that is defined on an interval has a continuous inverse.

Solution. Let I be an interval of real numbers and let $f: I \rightarrow \mathbb{R}$ be strictly increasing, that is, $f(u) < f(v)$ whenever $u, v \in I$ and $u < v$. Set $E = f(I)$, the image of I . No continuity is assumed of f , and E need not be an interval.

The function f is one-to-one: if $u, v \in I$ and $u \neq v$, then either $u < v$ or $v < u$, and in either case strict monotonicity gives $f(u) \neq f(v)$. Hence f is a one-to-one map of I onto E and the inverse function

$$g = f^{-1}: E \rightarrow I, \quad g(f(x)) = x \quad (x \in I), \quad f(g(y)) = y \quad (y \in E)$$

is well defined. We must show that g is continuous on E , continuity being understood, as always, relative to the domain E : g is continuous at $y_0 \in E$ provided for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|g(y) - g(y_0)| < \varepsilon$ for all $y \in E$ with $|y - y_0| < \delta$.

First observe that g is strictly increasing on E . Indeed, let $y_1, y_2 \in E$ with $y_1 < y_2$ and suppose $g(y_1) \geq g(y_2)$. If $g(y_1) = g(y_2)$ then $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$, a contradiction; if $g(y_1) > g(y_2)$ then, since f is strictly increasing, $y_1 = f(g(y_1)) > f(g(y_2)) = y_2$, again a contradiction. Hence $g(y_1) < g(y_2)$.

Now fix $y_0 \in E$, put $x_0 = g(y_0) \in I$, and let $\varepsilon > 0$. If I consists of the single point x_0 , then $E = \{y_0\}$ and continuity at y_0 is trivial (any δ works). So assume I has more than one point. We treat the two sides of x_0 separately.

Left side. Suppose first that there is a point $a \in I$ with $a < x_0$. Since I is an interval containing a and x_0 , we have $[a, x_0] \subseteq I$, and this interval contains points strictly between $\max\{a, x_0 - \varepsilon\}$ and x_0 ; choose such a point u , so that

$$u \in I, \quad x_0 - \varepsilon < u < x_0.$$

Since f is strictly increasing, $f(u) < f(x_0) = y_0$, so $\delta_1 := y_0 - f(u) > 0$. If instead there is no point of I strictly below x_0 , put $\delta_1 := 1$.

Right side. Symmetrically, if there is a point $b \in I$ with $b > x_0$, choose $v \in I$ with $x_0 < v < \min\{b, x_0 + \varepsilon\}$; then $y_0 = f(x_0) < f(v)$, and we put $\delta_2 := f(v) - y_0 > 0$. If there is no point of I strictly above x_0 , put $\delta_2 := 1$.

Define $\delta = \min\{\delta_1, \delta_2\} > 0$, and let $y \in E$ satisfy $|y - y_0| < \delta$. We claim $|g(y) - x_0| < \varepsilon$.

Consider the lower bound. If I has no point strictly below x_0 , then x_0 is the smallest element of I , so $g(y) \in I$ forces $g(y) \geq x_0 > x_0 - \varepsilon$. Otherwise u was chosen as above and

$$y > y_0 - \delta \geq y_0 - \delta_1 = f(u),$$

so, g being strictly increasing on E and $f(u) \in E$,

$$g(y) > g(f(u)) = u > x_0 - \varepsilon.$$

In either case $g(y) > x_0 - \varepsilon$.

Consider the upper bound. If I has no point strictly above x_0 , then x_0 is the largest element of I and $g(y) \leq x_0 < x_0 + \varepsilon$. Otherwise v was chosen as above and

$$y < y_0 + \delta \leq y_0 + \delta_2 = f(v),$$

so $g(y) < g(f(v)) = v < x_0 + \varepsilon$. In either case $g(y) < x_0 + \varepsilon$.

Therefore $x_0 - \varepsilon < g(y) < x_0 + \varepsilon$, that is, $|g(y) - g(y_0)| < \varepsilon$, for every $y \in E$ with $|y - y_0| < \delta$. Since $y_0 \in E$ was arbitrary, the inverse function $g = f^{-1}$ is continuous on $E = f(I)$.

Problem 2.46. Let f be a continuous function and B be a Borel set. Show that $f^{-1}(B)$ is a Borel set. (Hint: The collection of sets E for which $f^{-1}(E)$ is Borel is a σ -algebra containing the open sets.)

Solution. Let f be a real-valued continuous function whose domain D is a Borel set of real numbers; the case $D = \mathbb{R}$ is the one usually intended, and the case in which D is an interval is the one needed in the next problem. Some such assumption on the domain is necessary: if D is not Borel then already $f^{-1}(\mathbb{R}) = D$ fails to be Borel. Denote by $\mathcal{B}$ the collection of Borel sets, that is, by definition, the smallest σ -algebra of subsets of $\mathbb{R}$ that contains all of the open sets.

Following the hint, define

$$\mathcal{A} = \{ E \subseteq \mathbb{R} : f^{-1}(E) \in \mathcal{B} \},$$

where $f^{-1}(E) = \{x \in D : f(x) \in E\}$.

Step 1: $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$.

The set $\mathbb{R}$ belongs to $\mathcal{A}$, since $f^{-1}(\mathbb{R}) = D$ and D is Borel by assumption. Next, let $E \in \mathcal{A}$. Since f is a function defined on D , a point $x \in D$ lies in $f^{-1}(\mathbb{R} \setminus E)$ if and only if $f(x) \notin E$, that is, if and only if $x \notin f^{-1}(E)$. Hence

$$f^{-1}(\mathbb{R} \setminus E) = D \setminus f^{-1}(E) = D \cap (\mathbb{R} \setminus f^{-1}(E)).$$

Since $\mathcal{B}$ is a σ -algebra containing D and $f^{-1}(E)$, the right-hand side belongs to $\mathcal{B}$. Thus $\mathbb{R} \setminus E \in \mathcal{A}$, so $\mathcal{A}$ is closed under the formation of complements. Finally, let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. For $x \in D$ we have $f(x) \in \bigcup_k E_k$ if and only if $f(x) \in E_k$ for some k , so

$$f^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(E_k),$$

which is a countable union of Borel sets and hence Borel. Thus $\bigcup_k E_k \in \mathcal{A}$. Therefore $\mathcal{A}$ is a σ -algebra.

Step 2: $\mathcal{A}$ contains every open set.

Let $\mathcal{O}$ be an open set of real numbers. Since f is continuous on D , Proposition 22 of Chapter 1 tells us that there is an open set U for which

$$f^{-1}(\mathcal{O}) = D \cap U.$$

Both D and U are Borel sets (U is open, and the Borel sets contain the open sets), and $\mathcal{B}$ is closed under finite intersections, being a σ -algebra. Hence $f^{-1}(\mathcal{O}) \in \mathcal{B}$, that is, $\mathcal{O} \in \mathcal{A}$.

Step 3: conclusion.

By Steps 1 and 2, $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$ that contains all the open sets. Since $\mathcal{B}$ is the smallest such σ -algebra, we conclude

$$\mathcal{B} \subseteq \mathcal{A}.$$

Thus if B is a Borel set, then $B \in \mathcal{A}$, which is precisely the assertion that $f^{-1}(B)$ is a Borel set.

In particular, taking $D = \mathbb{R}$: the inverse image under a continuous real-valued function on $\mathbb{R}$ of an open set is open, of a closed set is closed (take complements), and of a Borel set is Borel.

Problem 2.47. Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.

Solution. Let I be an interval of real numbers and let $f: I \rightarrow \mathbb{R}$ be continuous and strictly increasing. Let B be a Borel set. We show that $f(B \cap I)$ is a Borel set; in particular, if B is a Borel subset of I , then $f(B)$ is a Borel set. We may assume I is nonempty.

Step 1: the image $E = f(I)$ is an interval, and hence a Borel set.

We check that E has the intermediate value property. Let $y_1, y_2 \in E$ with $y_1 < y_2$, and let y satisfy $y_1 < y < y_2$. Write $y_1 = f(x_1)$ and $y_2 = f(x_2)$ with $x_1, x_2 \in I$. Since f is strictly increasing and $f(x_1) < f(x_2)$, we must have $x_1 < x_2$. As I is an interval, $[x_1, x_2] \subseteq I$, and f is continuous on $[x_1, x_2]$. By the Intermediate Value Theorem there is a point $x \in (x_1, x_2)$ with $f(x) = y$. Since $x \in I$, we conclude $y \in E$.

A nonempty set E of real numbers with this property is an interval: let $\alpha = \inf E$ and $\beta = \sup E$ in the extended reals. If $\alpha < y < \beta$, then by the definitions of infimum and supremum there are $y_1, y_2 \in E$ with $y_1 < y < y_2$, so $y \in E$; thus $(\alpha, \beta) \subseteq E \subseteq [\alpha, \beta]$, and E is an interval (possibly degenerate, if I is a single point). Every interval is a Borel set, being open, closed, or the union of an open interval with one or two of its endpoints. Hence E is a Borel set.

Step 2: the inverse function is continuous on E .

The function f is strictly increasing on the interval I , so by Problem 45 it is one-to-one and its inverse

$$g = f^{-1}: E \rightarrow I$$

is a continuous function defined on $E = f(I)$.

Step 3: apply Problem 46 to g .

The function g is continuous and its domain E is a Borel set by Step 1. Therefore, by Problem 46, for every Borel set B the inverse image $g^{-1}(B)$ is a Borel set.

Step 4: identify $g^{-1}(B)$ as an image under f .

We claim

$$g^{-1}(B) = f(B \cap I).$$

Indeed, suppose $y \in g^{-1}(B)$. Then $y \in E$ and $g(y) \in B$. Put $x = g(y)$; then $x \in I$, $x \in B$, and $f(x) = f(g(y)) = y$. Hence $y \in f(B \cap I)$. Conversely, suppose $y \in f(B \cap I)$, say $y = f(x)$ with $x \in B \cap I$. Then $y \in f(I) = E$ and $g(y) = g(f(x)) = x \in B$, so $y \in g^{-1}(B)$. This proves the claim.

Conclusion. Combining Steps 3 and 4, for every Borel set B the set

$$f(B \cap I) = g^{-1}(B)$$

is a Borel set. In particular a continuous, strictly increasing function defined on an interval maps Borel subsets of that interval onto Borel sets. Since such an f is one-to-one, it maps I onto the interval E and carries Borel sets onto Borel sets, which is exactly the fact invoked in the proof of Proposition 22 of Chapter 2 to produce a measurable subset of the Cantor set that is not Borel.

A remark on where the hypotheses entered. Continuity of f was used only in Step 1, to know that $E = f(I)$ is an interval and therefore a Borel set, which is what makes Problem 46 applicable to g . Strict monotonicity was used twice: to know that f is one-to-one, so that the identification $f(B \cap I) = g^{-1}(B)$ of Step 4 is valid, and to invoke Problem 45 for the continuity of g . Note that no continuity of g beyond that supplied by Problem 45 is needed, and no appeal to measurability of f is made anywhere.

3 Lebesgue Measurable Functions

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3.1 Problems 3.1–3.7

Problem 3.1. Suppose f and g are continuous functions on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E ?

Solution. Throughout, $[a, b]$ is a nondegenerate closed bounded interval, so $a < b$.

Put $h = f - g$ on $[a, b]$. Since f and g are continuous, so is h , and the hypothesis that $f = g$ a.e. means precisely that there is a measurable set $Z \subseteq [a, b]$ with $m(Z) = 0$ such that $f = g$, that is $h = 0$, on $[a, b] \setminus Z$. Hence

$$E_0 = \{x \in [a, b] \mid h(x) \neq 0\} \subseteq Z,$$

so by the monotonicity of Lebesgue outer measure $m^*(E_0) = 0$; every set of outer measure zero is measurable (Chapter 2), so E_0 is measurable and $m(E_0) = 0$.

Suppose, to get a contradiction, that $h(x_0) \neq 0$ for some $x_0 \in [a, b]$. Set $\varepsilon = |h(x_0)|/2 > 0$. By the continuity of h at x_0 there is a $\delta > 0$ such that

$$|h(x) - h(x_0)| < \varepsilon \quad \text{for all } x \in [a, b] \text{ with } |x - x_0| < \delta.$$

For such x the triangle inequality gives

$$|h(x)| \geq |h(x_0)| - |h(x) - h(x_0)| > 2\varepsilon - \varepsilon = \varepsilon > 0,$$

so $h(x) \neq 0$. Therefore

$$I = (x_0 - \delta, x_0 + \delta) \cap [a, b] \subseteq E_0.$$

Now I is the intersection of two intervals, hence an interval, and it is nondegenerate: if $a < x_0 < b$ it contains the interval $(x_0 - \delta', x_0 + \delta')$ with $\delta' = \min\{\delta, x_0 - a, b - x_0\} > 0$; if $x_0 = a$ it contains $[a, a + \delta'']$ with $\delta'' = \min\{\delta, b - a\} > 0$; and if $x_0 = b$ it contains $(b - \delta'', b]$. In each case I contains an interval of positive length, so by the monotonicity of outer measure and the fact that the measure of an interval is its length (Chapter 2),

$$m(E_0) \geq m(I) > 0,$$

contradicting $m(E_0) = 0$. Hence $h(x) = 0$ for every $x \in [a, b]$, that is, $f = g$ on all of $[a, b]$.

Now replace $[a, b]$ by a general measurable set E . The assertion is false in general. Take

$$E = [0, 1] \cup \{2\}, \quad f \equiv 0 \text{ on } E, \quad g(x) = \begin{cases} 0 & \text{if } x \in [0, 1], \\ 1 & \text{if } x = 2. \end{cases}$$

Both are continuous on E : f is constant, and g is continuous at each point of $[0, 1]$ because g vanishes on the relative neighborhood $E \cap (-1, 2) = [0, 1]$ of each such point, while 2 is an isolated point of E , so every function on E is continuous there. Moreover $\{x \in E \mid f(x) \neq g(x)\} = \{2\}$ has measure zero, so $f = g$ a.e. on E . Yet $f(2) \neq g(2)$.

The argument given for $[a, b]$ isolates exactly what is needed, and in fact the following characterization holds: for a measurable set E , every pair of continuous functions on E that agree a.e. on E agrees on all of E if and only if

$$m(E \cap (x - r, x + r)) > 0 \quad \text{for every } x \in E \text{ and every } r > 0. \quad (*)$$

Indeed, assume $(*)$ and suppose f, g are continuous on E with $f = g$ a.e. If $h = f - g$ satisfied $h(x_0) \neq 0$ for some $x_0 \in E$, then exactly as above there is $\delta > 0$ with $h \neq 0$ on $E \cap (x_0 - \delta, x_0 + \delta)$, and by $(*)$ this set has positive measure; but it is contained in $\{x \in E \mid h(x) \neq 0\}$, which as above has measure zero, a contradiction. Conversely, if $(*)$ fails, say $m(E \cap (x_0 - r, x_0 + r)) = 0$ for some $x_0 \in E$ and $r > 0$, put $f \equiv 0$ on E and

$$g(x) = \max \left\{ 0, 1 - \frac{|x - x_0|}{r} \right\}.$$

Then g is continuous on all of $\mathbb{R}$, hence on E , and g vanishes off $(x_0 - r, x_0 + r)$, so $\{x \in E \mid f(x) \neq g(x)\} \subseteq E \cap (x_0 - r, x_0 + r)$ has measure zero; but $g(x_0) = 1 \neq 0 = f(x_0)$.

Since $(*)$ holds for $E = [a, b]$ with $a < b$, this recovers the first part. It also fails for the set $E = [0, 1] \cup \{2\}$ at the point $x = 2$, which is why the counterexample above works.

Problem 3.2. Let D and E be measurable sets and f a function with domain $D \cup E$. We proved that f is measurable on $D \cup E$ if and only if its restrictions to D and E are measurable. Is the same true if “measurable” is replaced by “continuous”?

Solution. No. One implication survives, the other does not.

The implication that does survive is the forward one: if f is continuous on $D \cup E$, then its restrictions $f|_D$ and $f|_E$ are continuous. This is immediate from the definition of continuity relative to a domain. Let $x_0 \in D$ and let $\varepsilon > 0$. Continuity of f at x_0 as a function on $D \cup E$ gives a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in D \cup E$ and $|x - x_0| < \delta$. Since $D \subseteq D \cup E$, the same δ works for $f|_D$ at x_0 . Hence $f|_D$ is continuous on D , and likewise $f|_E$ is continuous on E .

The converse fails. Take

$$D = (-\infty, 0], \quad E = (0, \infty), \quad D \cup E = \mathbb{R},$$

which are measurable, and define

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then $f|_D \equiv 0$ and $f|_E \equiv 1$ are constant, hence continuous on D and on E respectively. But f is not continuous on $\mathbb{R}$: at $x_0 = 0$ we have $f(0) = 0$, while every δ -neighborhood of 0 contains points $x > 0$ with $|f(x) - f(0)| = 1$. So f fails to be continuous at 0, and the analogue of Proposition 5(ii) with “continuous” in place of “measurable” is false.

The reason the analogy breaks down is structural. Measurability of f on $D \cup E$ is decided by the sets $\{x \in D \cup E \mid f(x) > c\}$, and each of these splits as the union

$$\{x \in D \mid f(x) > c\} \cup \{x \in E \mid f(x) > c\},$$

so the property is inherited from the pieces because the measurable sets form an algebra. Continuity, by contrast, is a local condition, and a point of D may be a limit point of E ; the restrictions know nothing about how f behaves as such a point is approached from within the other piece.

It is worth recording the standard positive result, which shows what extra hypothesis repairs the converse: if D and E are both closed (or both open) and $f|_D, f|_E$ are continuous, then f is continuous on $D \cup E$. For the closed case, let $C \subseteq \mathbb{R}$ be closed; then

$$f^{-1}(C) = (f|_D)^{-1}(C) \cup (f|_E)^{-1}(C),$$

and $(f|_D)^{-1}(C)$ is closed in D , hence (as D is closed in $\mathbb{R}$) closed in $\mathbb{R}$; the same holds for E , so $f^{-1}(C)$ is a union of two closed sets, hence closed in $\mathbb{R}$ and a fortiori closed in $D \cup E$. Thus f is continuous. For the open case one argues identically with open sets, since a relatively open subset of an open set is open. In the counterexample above D is closed and E is open, and it is precisely this mismatch that is exploited.

Problem 3.3. Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable?

Solution. Yes. Let E be the measurable domain of f , and let

$$F = \{x_1, \dots, x_n\} \subseteq E$$

be the finite set of points of E at which f fails to be continuous; thus f is continuous at every point of $D = E \setminus F$.

First, F is measurable and $m(F) = 0$: a single point $\{x_k\}$ is a degenerate closed interval, so it is measurable of measure zero, and a finite union of such sets is measurable of measure zero by the countable subadditivity and the fact that the measurable sets form a σ -algebra (Chapter 2). Consequently $D = E \setminus F$ is measurable as well.

Now consider the two restrictions.

The restriction $f|_D$ is continuous on D . Indeed, fix $x_0 \in D$ and $\varepsilon > 0$. Since f is continuous at x_0 as a function on E , there is $\delta > 0$ with $|f(x) - f(x_0)| < \varepsilon$ for all $x \in E$ with $|x - x_0| < \delta$; as $D \subseteq E$ this same δ works for $f|_D$. Observe also that f is real-valued at every point of D : continuity of f at x_0 in the above sense presupposes that the difference $f(x) - f(x_0)$ is meaningful and small, which forces $f(x_0)$ to be finite. (This is the only place where it matters that the functions of this chapter are allowed to take the values $\pm\infty$; the exceptional points are handled separately below, and there the values are unrestricted.) So $f|_D$ is a real-valued function that is continuous on its measurable domain D , and Proposition 3 tells us that $f|_D$ is measurable.

The restriction $f|_F$ is measurable for the cheapest of reasons: for every real number c ,

$$\{x \in F \mid f(x) > c\} \subseteq F,$$

and $m(F) = 0$, so this set is a subset of a set of measure zero. Lebesgue outer measure is monotone, so any such subset has outer measure zero and hence is measurable (Chapter 2: every set of outer measure zero is measurable). Thus $\{x \in F \mid f(x) > c\}$ is measurable for each c , and by Proposition 1 the restriction $f|_F$ is measurable on F . (The same argument applies verbatim if f is allowed to take the values $\pm\infty$ at the exceptional points.)

Finally, $E = D \cup F$ with D and F measurable and disjoint, and both restrictions $f|_D$ and $f|_F$ are measurable. By Proposition 5(ii), f is measurable on E . Concretely, for each c ,

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f(x) > c\} \cup \{x \in F \mid f(x) > c\},$$

a union of two measurable sets, hence measurable; Proposition 1 then gives the measurability of f .

Two remarks. The finiteness of F is not what matters: the argument only used that F is a set of measure zero, so the conclusion holds whenever f is continuous at every point of E outside a set of measure zero, for instance outside a countable set. What one may not do is drop the measure-zero hypothesis on the exceptional set: a function continuous off an arbitrary set need not be measurable, since the values on the exceptional set are then unconstrained.

Problem 3.4. Suppose f is a real-valued function on $\mathbb{R}$ such that $f^{-1}(c)$ is measurable for each number c . Is f necessarily measurable?

Solution. No. Measurability of every level set $f^{-1}(c) = \{x \in \mathbb{R} \mid f(x) = c\}$ is a much weaker condition than measurability of the sets $\{x \in \mathbb{R} \mid f(x) > c\}$, and it does not imply it. Proposition 1 records that measurability of f implies each $f^{-1}(c)$ is measurable; the converse fails, as the following counterexample shows.

By Theorem 17 (Vitali) of Chapter 2, any set of real numbers with positive outer measure contains a subset that fails to be measurable; applying this to the interval $(0, 1)$, fix a set $A \subseteq (0, 1)$ that is not measurable. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in A, \\ -x & \text{if } x \in \mathbb{R} \setminus A. \end{cases}$$

Every level set of f is measurable. Fix a real number c and suppose $f(x) = c$. Either $x \in A$, in which case $f(x) = x$ and so $x = c$; or $x \notin A$, in which case $f(x) = -x$ and so $x = -c$. Hence

$$f^{-1}(c) \subseteq \{c, -c\},$$

a set with at most two elements. Every finite set is measurable (it is a finite union of singletons, each a set of measure zero), and every subset of a finite set is finite, so $f^{-1}(c)$ is measurable for each real c . Nevertheless f is not measurable. Consider $\{x \in \mathbb{R} \mid f(x) > 0\}$ and intersect it with $(0, 1)$. If $x \in (0, 1)$ and $x \in A$, then $f(x) = x > 0$. If $x \in (0, 1)$ and $x \notin A$, then $f(x) = -x < 0$. Therefore

$$\{x \in \mathbb{R} \mid f(x) > 0\} \cap (0, 1) = A.$$

Were f measurable, the set $\{x \in \mathbb{R} \mid f(x) > 0\}$ would be measurable by Proposition 1, and since the measurable sets form a σ -algebra containing the interval $(0, 1)$, the intersection A would be measurable too. This contradicts the choice of A . Hence f is not measurable, even though $f^{-1}(c)$ is measurable for every real number c .

It may be worth noting that the pathology is not caused by f having repeated values: one can also arrange a one-to-one counterexample, and for a one-to-one f every set $f^{-1}(c)$ is empty or a singleton, hence automatically measurable, so the hypothesis of the problem carries no information at all in that case. For instance, keep $A \subseteq (0, 1)$ nonmeasurable and set $g(x) = x$ for $x \in A$, $g(x) = x + 2$ for $x \in (0, 1) \sim A$, $g(x) = x - 10$ for $x \leq 0$, and $g(x) = x + 10$ for $x \geq 1$. These four formulas are injective on their respective domains and have pairwise disjoint ranges contained in $(0, 1)$, $(2, 3)$, $(-\infty, -10]$, and $[11, \infty)$ respectively, so g is one-to-one on $\mathbb{R}$; yet

$$\{x \in \mathbb{R} \mid g(x) > 1\} \cap (0, 1) = (0, 1) \sim A$$

is nonmeasurable, so g is not measurable.

Problem 3.5. Suppose the function f is defined on a measurable set E and has the property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational number c . Is f necessarily measurable?

Solution. Yes. Testing the sets $\{f > c\}$ at rational c already forces measurability at every real c , because $\mathbb{Q}$ is dense in $\mathbb{R}$ and the measurable sets are closed under countable unions.

Let c be an arbitrary real number. Since the rationals are dense in $\mathbb{R}$, we may choose a sequence $\{c_k\}_{k=1}^{\infty}$ of rational numbers with

$$c < c_k \quad \text{for all } k, \quad \lim_{k \rightarrow \infty} c_k = c;$$

for instance take c_k to be any rational number in the interval $(c, c + 1/k)$. I claim that

$$\{x \in E \mid f(x) > c\} = \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) > c_k\}. \quad (*)$$

The inclusion $\supseteq$ is clear: if $f(x) > c_k$ for some k , then since $c_k > c$ we get $f(x) > c$.

For the inclusion $\subseteq$, suppose $x \in E$ and $f(x) > c$. If $f(x) = \infty$ then $f(x) > c_1$ and x lies in the union. Otherwise $f(x)$ is a real number with $f(x) - c > 0$, so by the Archimedean property there is an index k with $1/k < f(x) - c$, and then

$$c_k < c + \frac{1}{k} < f(x),$$

so $x \in \{x \in E \mid f(x) > c_k\}$. This proves $(*)$.

By hypothesis each set on the right of $(*)$ is measurable, since each c_k is rational. The measurable sets form a σ -algebra (Chapter 2), so the countable union on the right is measurable. Hence $\{x \in E \mid f(x) > c\}$ is measurable for every real number c . Since the domain E is measurable by hypothesis, statement (i) of Proposition 1 holds, and therefore f is measurable by the definition of measurability given after Proposition 1.

The same argument works with $\mathbb{Q}$ replaced by any set $S \subseteq \mathbb{R}$ that is dense in $\mathbb{R}$, or indeed by any countable set S with the property that for each real c there is a sequence in $S \cap (c, \infty)$ converging to c ; all that is used is that the union in $(*)$ is countable.

Problem 3.6. Let f be a function with measurable domain D . Show that f is measurable if and only if the function g defined on $\mathbb{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Solution. Both directions are settled by computing the sets $\{g > c\}$ and comparing them with the sets $\{f > c\}$. Note first that the domain of g is $\mathbb{R}$, which is measurable, and the domain of f is D , measurable by hypothesis; so in each direction only the level sets need attention, and Proposition 1 applies.

Suppose first that f is measurable. Fix a real number c and split according to the sign of c .

If $c \geq 0$, then no point of $\mathbb{R} \sim D$ belongs to $\{x \in \mathbb{R} \mid g(x) > c\}$, since g vanishes there and $0 \leq c$. Hence

$$\{x \in \mathbb{R} \mid g(x) > c\} = \{x \in D \mid g(x) > c\} = \{x \in D \mid f(x) > c\},$$

which is measurable because f is measurable.

If $c < 0$, then every point of $\mathbb{R} \sim D$ does belong to the set, since $g = 0 > c$ there. Hence

$$\{x \in \mathbb{R} \mid g(x) > c\} = \{x \in D \mid f(x) > c\} \cup (\mathbb{R} \sim D).$$

The first set is measurable since f is measurable, and $\mathbb{R} \sim D$ is measurable because D is measurable and the measurable sets form a σ -algebra. Their union is therefore measurable.

In both cases $\{x \in \mathbb{R} \mid g(x) > c\}$ is measurable, so by Proposition 1 and the definition of measurability, g is measurable on $\mathbb{R}$.

Conversely, suppose g is measurable. Since g agrees with f on D , for each real c we have

$$\{x \in D \mid f(x) > c\} = \{x \in D \mid g(x) > c\} = \{x \in \mathbb{R} \mid g(x) > c\} \cap D.$$

The set $\{x \in \mathbb{R} \mid g(x) > c\}$ is measurable because g is measurable, and D is measurable by hypothesis; since the measurable sets form a σ -algebra they are closed under intersection, so $\{x \in D \mid f(x) > c\}$ is measurable. As this holds for every real c and the domain D of f is measurable, f is measurable by Proposition 1.

Alternatively, the converse direction is exactly Proposition 5(ii) applied to the function g on the measurable set $\mathbb{R}$ and the measurable subset D : measurability of g on $\mathbb{R}$ is equivalent to measurability of its restrictions to D and to $\mathbb{R} \sim D$, and the restriction of g to D is precisely f .

The proof used nothing about the value 0 beyond its being a fixed real number: for any fixed $a \in \mathbb{R}$, the extension of f by the constant a off D is measurable on $\mathbb{R}$ if and only if f is measurable on D . Only the two cases $c \geq a$ and $c < a$ change places.

Problem 3.7. Let the function f be defined on a measurable set E . Show that f is measurable if and only if for each Borel set A , $f^{-1}(A)$ is measurable. [Hint: The collection of sets A that have the property that $f^{-1}(A)$ is measurable is a σ -algebra.]

Solution. Here f is real-valued, A ranges over the Borel subsets of $\mathbb{R}$, and $f^{-1}(A) = \{x \in E \mid f(x) \in A\}$. Recall that the Borel σ -algebra $\mathcal{B}$ is by definition the smallest σ -algebra of subsets of $\mathbb{R}$ containing all of the open sets; equivalently, $\mathcal{B}$ is the intersection of all σ -algebras of subsets of $\mathbb{R}$ that contain the open sets, so any σ -algebra containing every open set necessarily contains $\mathcal{B}$.

Suppose first that $f^{-1}(A)$ is measurable for each Borel set A . For each real number c the interval (c, ∞) is open, hence Borel, and

$$\{x \in E \mid f(x) > c\} = f^{-1}((c, \infty))$$

is therefore measurable. Since the domain E is measurable, statement (i) of Proposition 1 holds and f is measurable.

Conversely, suppose f is measurable. Following the hint, set

$$\mathcal{A} = \{A \subseteq \mathbb{R} \mid f^{-1}(A) \text{ is measurable}\}.$$

We check that $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$, using throughout the elementary identities for inverse images and the fact that the Lebesgue measurable sets form a σ -algebra (Chapter 2).

First, $\mathbb{R} \in \mathcal{A}$: since f is real-valued on E , $f^{-1}(\mathbb{R}) = E$, which is measurable by hypothesis. In particular $\mathcal{A}$ is nonempty.

Second, $\mathcal{A}$ is closed under complements in $\mathbb{R}$. Let $A \in \mathcal{A}$. A point $x \in E$ satisfies $f(x) \in \mathbb{R} \sim A$ if and only if $x \notin f^{-1}(A)$, so

$$f^{-1}(\mathbb{R} \sim A) = E \sim f^{-1}(A).$$

Both E and $f^{-1}(A)$ are measurable, and the measurable sets form a σ -algebra, hence an algebra, so their relative complement is measurable. Thus $\mathbb{R} \sim A \in \mathcal{A}$.

Third, $\mathcal{A}$ is closed under countable unions. Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of members of $\mathcal{A}$. A point $x \in E$ satisfies $f(x) \in \bigcup_k A_k$ if and only if $f(x) \in A_k$ for some k , so

$$f^{-1}\left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(A_k),$$

a countable union of measurable sets, hence measurable. Thus $\bigcup_k A_k \in \mathcal{A}$.

Therefore $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$.

Next, $\mathcal{A}$ contains every open subset of $\mathbb{R}$. This is exactly Proposition 2: a function defined on a measurable set is measurable if and only if the inverse image of each open set is measurable. Since f is measurable, $f^{-1}(\mathcal{O})$ is measurable for each open $\mathcal{O} \subseteq \mathbb{R}$, i.e. $\mathcal{O} \in \mathcal{A}$.

So $\mathcal{A}$ is a σ -algebra containing all open sets, and by the minimality in the definition of the Borel σ -algebra, $\mathcal{B} \subseteq \mathcal{A}$. That is, $f^{-1}(A)$ is measurable for every Borel set A , which is what we had to prove.

A remark on the extended real-valued case. If f is permitted to take the values $\pm\infty$, the statement as literally written is no longer correct in the direction “inverse images of Borel sets measurable $\Rightarrow$ f measurable”: a function taking only the values $\pm\infty$ has $f^{-1}(A) = \emptyset$ for every $A \subseteq \mathbb{R}$, so the hypothesis is vacuous. Concretely, with $A_0 \subseteq (0, 1)$ nonmeasurable (Theorem 17 of Chapter 2), the function on $E = (0, 1)$ equal to ∞ on A_0 and to $-\infty$ on $(0, 1) \sim A_0$ satisfies the hypothesis yet has $\{x \in E \mid f(x) > 0\} = A_0$ nonmeasurable. The correct extended statement is obtained by letting A range over the Borel subsets of the extended real numbers, i.e. over the σ -algebra generated by the open subsets of $\mathbb{R}$ together with the sets $(c, \infty]$ and $[-\infty, c)$. The forward direction, on the other hand, remains valid as stated for extended real-valued measurable f : the argument above needs only the identity $f^{-1}(\mathbb{R}) = \bigcup_{k=1}^{\infty} \{x \in E \mid -k < f(x) < k\}$ to see that $f^{-1}(\mathbb{R})$ is measurable, after which one repeats the σ -algebra verification with E replaced by $f^{-1}(\mathbb{R})$.

3.2 Problems 3.8–3.14

Problem 3.8. (Borel measurability) A function f is said to be **Borel measurable** provided its domain E is a Borel set and for each c , the set $\{x \in E \mid f(x) > c\}$ is a Borel set. Verify that Proposition 1 and Theorem 6 remain valid if we replace “(Lebesgue) measurable set” by “Borel set.” Show that: (i) every Borel measurable function is Lebesgue measurable; (ii) if f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set; (iii) if f and g are Borel measurable, so is $f \circ g$; and (iv) if f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

Solution. Throughout, write $\mathcal{B}$ for the collection of Borel subsets of $\mathbb{R}$, that is, the smallest σ -algebra of subsets of $\mathbb{R}$ containing the open sets. Two features of $\mathcal{B}$ are all that the arguments of Section 3.1 ever used about the Lebesgue measurable sets: $\mathcal{B}$ is a σ -algebra, and $\mathcal{B}$ contains every interval and every open set.

Proposition 1 for Borel sets. Let f have Borel domain E . The proof of Proposition 1 proceeds as follows. The sets in (i) and (iv) are complementary in E , as are those in (ii) and (iii); since E is Borel and $\mathcal{B}$ is a σ -algebra, the relative complement in E of a Borel subset of E is Borel, so (i) $\Leftrightarrow$ (iv) and (ii) $\Leftrightarrow$ (iii). Next,

$$\{x \in E \mid f(x) \geq c\} = \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > c-1/k\}, \quad \{x \in E \mid f(x) > c\} = \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c+1/k\},$$

and $\mathcal{B}$ is closed under countable intersections and countable unions, so (i) $\Leftrightarrow$ (ii). Finally, for finite c , $f^{-1}(c) = \{f \geq c\} \cap \{f \leq c\}$ is Borel, and $f^{-1}(\infty) = \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > k\}$ is Borel, and similarly for $-\infty$. So the four statements are equivalent and each implies that $\{x \in E \mid f(x) = c\}$ is Borel for every extended real c . Only the σ -algebra property was invoked, so the proposition stands verbatim with “Borel set” in place of “measurable set.”

Theorem 6 for Borel sets. Let f and g be Borel measurable real-valued functions on the Borel set E .

For $\alpha \neq 0$,

$$\{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) > c/\alpha\} \text{ if } \alpha > 0, \quad \{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) < c/\alpha\} \text{ if } \alpha < 0$$

and $\alpha f \equiv 0$ is Borel measurable when $\alpha = 0$; so αf is Borel measurable. By the density of $\mathbb{Q}$ in $\mathbb{R}$,

$$\{x \in E \mid f(x) + g(x) < c\} = \bigcup_{q \in \mathbb{Q}} (\{x \in E \mid g(x) < c - q\} \cap \{x \in E \mid f(x) < q\}),$$

a countable union of Borel sets, hence Borel; so $f + g$, and therefore $\alpha f + \beta g$, is Borel measurable. For products, $fg = \frac{1}{2}[(f + g)^2 - f^2 - g^2]$, so by the linearity just proved it suffices to know that h^2 is Borel measurable whenever h is, and this follows from

$$\{x \in E \mid h^2(x) > c\} = \{x \in E \mid h(x) > \sqrt{c}\} \cup \{x \in E \mid h(x) < -\sqrt{c}\} \quad (c \geq 0), \quad \{x \in E \mid h^2(x) > c\} = E \quad (c < 0)$$

Again every step used only that $\mathcal{B}$ is a σ -algebra containing the intervals, so Theorem 6 holds with “Borel measurable” throughout.

One caveat deserves mention. Theorem 6 is stated for functions that are merely finite a.e., and the reduction to the finite case rests on Proposition 5, which asserts that a function agreeing a.e. with a measurable function is measurable. Proposition 5 is **false** for Borel measurability: a set of measure zero need not be Borel (by Proposition 22 of Chapter 2 there is a measurable, indeed measure-zero, subset of the Cantor set that is not Borel), and altering the identically-zero function to equal 1 on such a set produces a function that is not Borel measurable. So in the Borel version of Theorem 6 one must read “finite a.e.” as “finite,” or equivalently pass to the Borel set E_0 on which both f and g are finite (Borel by Proposition 1) and assert Borel measurability of $\alpha f + \beta g$ and fg on E_0 ; the values assigned on $E \sim E_0$ must themselves be Borel measurable if the conclusion is to hold on all of E .

(i) Every Borel measurable function is Lebesgue measurable. By Theorem 9 of Chapter 2 the collection $\mathcal{M}$ of Lebesgue measurable sets is a σ -algebra that contains the σ -algebra $\mathcal{B}$ of Borel sets; equivalently, $\mathcal{M}$ is a σ -algebra containing the open sets and $\mathcal{B}$ is the smallest such σ -algebra. Hence every Borel set is Lebesgue measurable. Thus if f is Borel measurable, its domain E is a Borel set and so Lebesgue measurable, and for each c the Borel set $\{x \in E \mid f(x) > c\}$ is Lebesgue measurable. By the definition following Proposition 1, f is Lebesgue measurable.

(ii) f Borel measurable, B Borel $\Rightarrow f^{-1}(B)$ Borel. This is Problem 7 with $\mathcal{B}$ in place of the Lebesgue measurable sets. There is no loss in assuming f is real-valued. Indeed, the functions of this chapter are extended real-valued, and if f takes infinite values put $E' = E \sim [f^{-1}(\infty) \cup f^{-1}(-\infty)]$, a Borel set by the Borel version of Proposition 1 established above; for a set $B \subseteq \mathbb{R}$ of real numbers we have $f^{-1}(B) = (f|_{E'})^{-1}(B)$, and $f|_{E'}$ is a real-valued Borel measurable function on the Borel set E' , so the real-valued case gives the general one. Assume then that $f : E \rightarrow \mathbb{R}$ and define

$$\mathcal{A} = \{A \subseteq \mathbb{R} \mid f^{-1}(A) \text{ is a Borel set}\}.$$

Then $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$: first, $f^{-1}(\mathbb{R}) = E$ is Borel, so $\mathbb{R} \in \mathcal{A}$; second, if $A \in \mathcal{A}$ then $f^{-1}(\mathbb{R} \sim A) = E \sim f^{-1}(A)$, a difference of Borel sets, hence Borel, so $\mathbb{R} \sim A \in \mathcal{A}$; third, if $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ then $f^{-1}(\bigcup_k A_k) = \bigcup_k f^{-1}(A_k)$ is a countable union of Borel sets, hence Borel. Moreover $(c, \infty) \in \mathcal{A}$ for every c , by the Borel measurability of f . Now the σ -algebra generated by the rays $\{(c, \infty) \mid c \in \mathbb{R}\}$ is all of $\mathcal{B}$: it contains $(-\infty, c] = \mathbb{R} \sim (c, \infty)$, hence $(-\infty, c) = \bigcup_{k=1}^{\infty} (-\infty, c - 1/k]$, hence every bounded open interval $(a, b) = (a, \infty) \cap (-\infty, b)$, and hence also the unbounded open intervals (a, ∞) , $(-\infty, b)$ and $\mathbb{R}$ itself. Every open subset of $\mathbb{R}$ is the union of a countable collection of open intervals, each of them bounded or unbounded and so in every case a member of this σ -algebra; hence it contains the open sets and therefore contains $\mathcal{B}$. Since $\mathcal{A}$ is a σ -algebra containing all the rays, $\mathcal{B} \subseteq \mathcal{A}$; that is, $f^{-1}(B)$ is Borel for every Borel set B .

(iii) f, g Borel measurable $\Rightarrow f \circ g$ Borel measurable. Let g have Borel domain D and f have Borel domain E , both real-valued, and let $f \circ g$ have its natural domain $g^{-1}(E)$, which is Borel by (ii) applied to g . For each c , the set $B = \{y \in E \mid f(y) > c\}$ is Borel since f is Borel measurable, and

$$\{x \in g^{-1}(E) \mid f(g(x)) > c\} = g^{-1}(B),$$

which is Borel by (ii) applied to g . Hence $f \circ g$ is Borel measurable.

(iv) f Borel measurable, g Lebesgue measurable $\Rightarrow f \circ g$ Lebesgue measurable. Exactly the same computation, with Problem 7 supplying the transfer. By Problem 7, the collection $\{A \subseteq \mathbb{R} \mid g^{-1}(A) \text{ is Lebesgue measurable}\}$ is a σ -algebra, and it contains every ray (c, ∞) because g is Lebesgue measurable; by the generation argument in (ii) it therefore contains $\mathcal{B}$. So $g^{-1}(B)$ is Lebesgue measurable for every Borel set B . In particular the domain $g^{-1}(E)$ is measurable, and for each c ,

$$\{x \in g^{-1}(E) \mid f(g(x)) > c\} = g^{-1}(\{y \in E \mid f(y) > c\})$$

is the inverse image under g of a Borel set, hence Lebesgue measurable. Thus $f \circ g$ is Lebesgue measurable.

It is worth contrasting (iv) with the Example preceding Proposition 7, where $\chi_A \circ \psi^{-1}$ fails to be measurable even though χ_A is Lebesgue measurable and ψ^{-1} is continuous. There A is a measurable set that is not Borel, so χ_A is Lebesgue but not Borel measurable, and (iv) does not apply. The outer function must be Borel measurable, not merely Lebesgue measurable, for the composition argument to run.

Problem 3.9. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E . Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?

Solution. Yes: E_0 is measurable. We give a proof that uses nothing beyond Proposition 1, Proposition 5, Theorem 6, and the fact that the measurable sets are a σ -algebra.

The functions here are extended real-valued, so we must first say what “converges” means. Take it to mean “converges to a finite limit”; at the end we observe that the sets on which $f_n(x) \rightarrow \infty$ and $f_n(x) \rightarrow -\infty$ are also measurable, so the conclusion is unaffected if one prefers the wider reading.

For each pair of indices n, m put

$$E_{n,m} = \{x \in E \mid |f_n(x)| < \infty\} \cap \{x \in E \mid |f_m(x)| < \infty\}.$$

By Proposition 1, $f_n^{-1}(\infty)$ and $f_n^{-1}(-\infty)$ are measurable, so $\{x \in E \mid |f_n(x)| < \infty\} = E \setminus [f_n^{-1}(\infty) \cup f_n^{-1}(-\infty)]$ is measurable, and hence so is $E_{n,m}$. By Proposition 5(ii) the restrictions of f_n and f_m to $E_{n,m}$ are measurable, and they are real-valued there, so by Theorem 6 the function $f_n - f_m$ is measurable on $E_{n,m}$. Consequently, for each $k \in \mathbb{N}$ the set

$$A_{n,m}^k = \{x \in E_{n,m} \mid |f_n(x) - f_m(x)| < 1/k\} = \{x \in E_{n,m} \mid f_n(x) - f_m(x) < 1/k\} \cap \{x \in E_{n,m} \mid f_n(x) - f_m(x) > -1/k\}$$

is measurable, being the intersection of two measurable sets.

Now define

$$F = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \bigcap_{m \geq N} A_{n,m}^k.$$

All the unions and intersections are over countable index sets, so F is measurable because the measurable sets form a σ -algebra.

We claim $F = E_0$. Suppose first that $x \in E_0$, say $f_n(x) \rightarrow L$ with $L \in \mathbb{R}$. Fix k . There is an N such that $|f_n(x) - L| < 1/(2k)$ for all $n \geq N$; in particular $f_n(x)$ is finite for every $n \geq N$, so $x \in E_{n,m}$ for all $n, m \geq N$, and by the triangle inequality

$$|f_n(x) - f_m(x)| \leq |f_n(x) - L| + |L - f_m(x)| < 1/k \quad \text{for all } n, m \geq N.$$

Thus $x \in \bigcap_{n,m \geq N} A_{n,m}^k$. As k was arbitrary, $x \in F$.

Conversely suppose $x \in F$. Taking $k = 1$ we get an N_1 with $x \in E_{n,m}$ for all $n, m \geq N_1$, so the numbers $f_n(x)$ are finite for $n \geq N_1$. Given $\epsilon > 0$, choose k with $1/k < \epsilon$ and then $N \geq N_1$ with $|f_n(x) - f_m(x)| < 1/k < \epsilon$ for all $n, m \geq N$. Hence the tail $\{f_n(x)\}_{n \geq N_1}$ is a Cauchy sequence of real numbers, and by the completeness of $\mathbb{R}$ it converges to some $L \in \mathbb{R}$. Since altering finitely many terms does not affect convergence, $\{f_n(x)\}$ converges, i.e. $x \in E_0$.

Therefore $E_0 = F$ is measurable.

Finally, for the wider reading of “converges,” note that

$$\{x \in E \mid f_n(x) \rightarrow \infty\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{x \in E \mid f_n(x) > k\}$$

is measurable, since each $\{x \in E \mid f_n(x) > k\}$ is measurable by the very definition of measurability of f_n and only countable set operations are used; the set on which $f_n(x) \rightarrow -\infty$ is measurable by the symmetric formula. Adjoining these two measurable sets to F again yields a measurable set. Either way, the answer is affirmative.

Problem 3.10. Suppose f and g are real-valued functions defined on all of $\mathbb{R}$, f is measurable, and g is continuous. Is the composition $f \circ g$ necessarily measurable?

Solution. No. Proposition 7 guarantees measurability of $f \circ g$ when the **outer** function is continuous and the inner one measurable; with the roles reversed the conclusion fails, and the Example preceding Proposition 7 supplies the counterexample. We spell it out.

By Proposition 21 of Chapter 2 the function

$$\psi(x) = \varphi(x) + x \quad \text{for } x \in [0, 1],$$

where φ is the Cantor-Lebesgue function, is a strictly increasing continuous function mapping $[0, 1]$ onto $[0, 2]$, and it maps some measurable set A , a subset of the Cantor set C , onto a nonmeasurable set $\psi(A)$.

Extend ψ to all of $\mathbb{R}$ by

$$\psi(x) = x \quad \text{for } x < 0, \quad \psi(x) = x + 1 \quad \text{for } x > 1.$$

Since $\psi(0) = \varphi(0) + 0 = 0$ and $\psi(1) = \varphi(1) + 1 = 2$, the extended function is continuous and strictly increasing on $\mathbb{R}$, and it maps $\mathbb{R}$ onto $\mathbb{R}$. By Problem 45 of Chapter 2, a strictly increasing function defined on an interval has a continuous inverse, so

$$g = \psi^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

is a continuous real-valued function defined on all of $\mathbb{R}$.

Let $f = \chi_A$, the characteristic function of A , regarded as a function on all of $\mathbb{R}$. Since A is measurable, f is measurable: for $c \geq 1$ the set $\{x \mid f(x) > c\}$ is empty, for $0 \leq c < 1$ it is A , and for $c < 0$ it is $\mathbb{R}$, and all three are measurable.

Now consider $h = f \circ g = \chi_A \circ \psi^{-1}$, a real-valued function on all of $\mathbb{R}$. For $c = 1/2$,

$$\{y \in \mathbb{R} \mid h(y) > 1/2\} = \{y \in \mathbb{R} \mid \psi^{-1}(y) \in A\} = \psi(A),$$

the last equality because ψ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$. But $\psi(A)$ is not measurable. Hence $h = f \circ g$ is not measurable, even though f is measurable and g is continuous.

(Equivalently, as the text puts it: if I is any open interval containing 1 but not 0, then $h^{-1}(I) = \psi(A)$ is nonmeasurable, so by Proposition 2 the function h is not measurable.)

The obstruction is that a continuous function may carry a set of measure zero, such as the measurable set $A \subseteq C$, onto a set of positive measure, and once measure is created in this way a measurable set can be moved onto a nonmeasurable one. Problem 11 shows that the extra hypothesis of a Lipschitz inverse rules this out and restores measurability of $f \circ g$; Problem 8(iv) shows that requiring f to be Borel measurable rather than merely Lebesgue measurable also restores it.

Problem 3.11. Let f be a measurable function and g be a one-to-one function from $\mathbb{R}$ onto $\mathbb{R}$ which has a Lipschitz inverse. Show that the composition $f \circ g$ is measurable. (Hint: Examine Problem 38 in Chapter 2.)

Solution. Write $h = g^{-1}$, so that $h : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz: there is a constant $c \geq 0$ with

$$|h(u) - h(v)| \leq c|u - v| \quad \text{for all } u, v \in \mathbb{R}.$$

Since g is a one-to-one mapping of $\mathbb{R}$ onto $\mathbb{R}$ with inverse h , for any set $A \subseteq \mathbb{R}$ we have the identity

$$g^{-1}(A) = \{x \in \mathbb{R} \mid g(x) \in A\} = h(A).$$

Indeed, if $g(x) = a \in A$ then $x = h(a) \in h(A)$; conversely if $x = h(a)$ with $a \in A$ then $g(x) = a \in A$. So inverse images under g are the same thing as forward images under the Lipschitz function h , and everything reduces to the following lemma, which is Problem 38 of Chapter 2 transferred from a closed bounded interval to all of $\mathbb{R}$.

Lemma. A Lipschitz function $h : \mathbb{R} \rightarrow \mathbb{R}$ maps measurable sets onto measurable sets.

Proof of the Lemma, in three steps.

Step 1: h maps sets of measure zero onto sets of measure zero. Let $m^*(Z) = 0$ and let $\epsilon > 0$. By the definition of outer measure there is a countable collection $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals covering Z with $\sum_k \ell(I_k) < \epsilon$. Fix k . For $u, v \in Z \cap I_k$ we have $|h(u) - h(v)| \leq c|u - v| \leq c\ell(I_k)$, so the set $h(Z \cap I_k)$ has diameter at most $c\ell(I_k)$ and is therefore contained in a closed interval of length $c\ell(I_k)$; by the monotonicity of outer measure and Proposition 1 of Chapter 2, which says the outer measure of an interval is its length, $m^*(h(Z \cap I_k)) \leq c\ell(I_k)$. Since $h(Z) = \bigcup_k h(Z \cap I_k)$, the countable subadditivity of outer measure, Proposition 3 of Chapter 2, gives

$$m^*(h(Z)) \leq \sum_{k=1}^{\infty} m^*(h(Z \cap I_k)) \leq c \sum_{k=1}^{\infty} \ell(I_k) < c\epsilon.$$

As $\epsilon > 0$ was arbitrary, $m^*(h(Z)) = 0$, and by Proposition 4 of Chapter 2 a set of outer measure zero is measurable.

Step 2: h maps F_{σ} sets onto F_{σ} sets. A Lipschitz function is continuous, and a continuous function maps compact sets onto compact sets. If $F \subseteq \mathbb{R}$ is closed, then $F = \bigcup_{n=1}^{\infty} (F \cap [-n, n])$ is a countable union of compact sets, so $h(F) = \bigcup_{n=1}^{\infty} h(F \cap [-n, n])$ is a countable union of compact, hence closed, sets: $h(F)$ is F_{σ} . If $S = \bigcup_{j=1}^{\infty} F_j$ is F_{σ} , then $h(S) = \bigcup_{j=1}^{\infty} h(F_j)$ is a countable union of F_{σ} sets, hence again F_{σ} . In particular $h(S)$ is measurable.

Step 3: Let A be measurable. By assertion (iv) of Theorem 11 of Chapter 2 there is an F_{σ} set F contained in A for which $m^*(A \sim F) = 0$. Then

$$h(A) = h(F) \cup h(A \sim F),$$

where $h(F)$ is measurable by Step 2 and $h(A \sim F)$ has measure zero, hence is measurable, by Step 1. As the measurable sets are closed under union, $h(A)$ is measurable. This proves the Lemma.

Now let f be measurable with measurable domain $E \subseteq \mathbb{R}$. The composition $f \circ g$ has domain

$$\{x \in \mathbb{R} \mid g(x) \in E\} = g^{-1}(E) = h(E),$$

which is measurable by the Lemma. Fix a real number c . Since f is measurable, the set $E_c = \{y \in E \mid f(y) > c\}$ is measurable, and

$$\{x \in h(E) \mid (f \circ g)(x) > c\} = \{x \mid g(x) \in E_c\} = g^{-1}(E_c) = h(E_c),$$

which is measurable, again by the Lemma. Since this holds for every c , and the domain $h(E)$ is measurable, $f \circ g$ is a measurable function.

Note where the hypothesis is used: the Lipschitz condition on g^{-1} is exactly what prevents the pathology of Problem 10, where the inner function was continuous with a continuous inverse but that inverse blew a set of measure zero (a subset of the Cantor set) up to a set of positive measure and thence onto a nonmeasurable set. Step 1 shows a Lipschitz map cannot do this.

Problem 3.12. Let f be a bounded measurable function on E . Show that there are sequences of simple functions on E , $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing and each of these sequences converges to f uniformly on E .

Solution. Since f is a bounded measurable real-valued function on the measurable set E , the Simple Approximation Lemma applies with any $\epsilon > 0$. Apply it with $\epsilon = 1/n$ for each natural number n : there are simple functions α_n and β_n defined on E with

$$\alpha_n \leq f \leq \beta_n \quad \text{and} \quad 0 \leq \beta_n - \alpha_n < 1/n \quad \text{on } E.$$

These need not be monotone in n , so we monotone them. Define, for each n ,

$$\varphi_n = \max\{\alpha_1, \alpha_2, \dots, \alpha_n\}, \quad \psi_n = \min\{\beta_1, \beta_2, \dots, \beta_n\} \quad \text{on } E.$$

First, φ_n and ψ_n are simple. Each is measurable by Proposition 8, which asserts that the maximum and the minimum of a finite family of measurable functions with common domain are measurable. Each takes only finitely many values: at every point of E the value of φ_n belongs to the union of the ranges of $\alpha_1, \dots, \alpha_n$, which is a finite set since each α_k is simple, and likewise for ψ_n . Being real-valued, measurable, and finitely valued, φ_n and ψ_n are simple.

Second, monotonicity is immediate from the definition: $\varphi_n = \max\{\varphi_{n-1}, \alpha_n\} \geq \varphi_{n-1}$, so $\{\varphi_n\}$ is increasing, and $\psi_n = \min\{\psi_{n-1}, \beta_n\} \leq \psi_{n-1}$, so $\{\psi_n\}$ is decreasing.

Third, the two sequences bracket f . Since $\alpha_k \leq f$ on E for every k , the maximum of $\alpha_1, \dots, \alpha_n$ also satisfies $\varphi_n \leq f$ on E ; since $\beta_k \geq f$ on E for every k , we get $\psi_n \geq f$ on E . Moreover $\alpha_n \leq \varphi_n$ and $\psi_n \leq \beta_n$, by the definitions. Hence, on all of E ,

$$\alpha_n \leq \varphi_n \leq f \leq \psi_n \leq \beta_n.$$

Finally we get uniform convergence with an explicit rate. For every $x \in E$,

$$0 \leq f(x) - \varphi_n(x) \leq f(x) - \alpha_n(x) \leq \beta_n(x) - \alpha_n(x) < 1/n,$$

and symmetrically

$$0 \leq \psi_n(x) - f(x) \leq \beta_n(x) - f(x) \leq \beta_n(x) - \alpha_n(x) < 1/n.$$

Thus

$$\sup_{x \in E} |f(x) - \varphi_n(x)| \leq 1/n \quad \text{and} \quad \sup_{x \in E} |f(x) - \psi_n(x)| \leq 1/n,$$

and since these bounds are independent of x and tend to 0, both $\{\varphi_n\}$ and $\{\psi_n\}$ converge to f uniformly on E . The sequence $\{\varphi_n\}$ is increasing and the sequence $\{\psi_n\}$ is decreasing, as required.

Problem 3.13. A real-valued measurable function is said to be **semisimple** provided it takes only a countable number of values. Let f be any measurable function on E . Show that there is a sequence of semisimple functions $\{f_n\}$ on E that converges to f uniformly on E .

Solution. Since a semisimple function is by definition real-valued, and since uniform convergence of real-valued functions to f forces f to be real-valued, we take f to be a real-valued measurable function on the measurable set E . (The extended real-valued case is treated in a remark at the end.)

The point of the exercise is that, unlike simple functions, semisimple functions may use countably many values, so we need no boundedness assumption on f : we may simply round f down to the nearest multiple of $1/n$.

Fix a natural number n . For each integer $k \in \mathbb{Z}$ define

$$E_{n,k} = \left\{ x \in E \mid \frac{k}{n} \leq f(x) < \frac{k+1}{n} \right\} = \left\{ x \in E \mid f(x) \geq \frac{k}{n} \right\} \cap \left\{ x \in E \mid f(x) < \frac{k+1}{n} \right\}.$$

By Proposition 1 each of the two sets on the right is measurable, hence $E_{n,k}$ is measurable. For a fixed n the sets $\{E_{n,k}\}_{k \in \mathbb{Z}}$ are pairwise disjoint, since the intervals $[k/n, (k+1)/n)$ are pairwise disjoint, and they cover E : for $x \in E$ the number $f(x)$ is real, so there is exactly one integer k with $k/n \leq f(x) < (k+1)/n$, namely $k = \lfloor nf(x) \rfloor$. Thus

$$E = \bigcup_{k \in \mathbb{Z}} E_{n,k}, \quad \text{the union being disjoint.}$$

Define f_n on E by

$$f_n(x) = \frac{k}{n} \quad \text{for } x \in E_{n,k}, \quad \text{that is,} \quad f_n = \frac{1}{n} \lfloor nf \rfloor.$$

This is well defined and real-valued on all of E by the preceding paragraph, and its range is contained in the countable set $\{k/n \mid k \in \mathbb{Z}\}$.

The function f_n is measurable. Indeed, for each real number c ,

$$\{x \in E \mid f_n(x) > c\} = \bigcup_{\{k \in \mathbb{Z} \mid k/n > c\}} E_{n,k},$$

a union of a countable collection of measurable sets, hence measurable since the measurable sets are a σ -algebra. Being measurable, real-valued, and countably valued, each f_n is semisimple.

It remains to check uniform convergence. Let $x \in E$ and let k be the integer with $x \in E_{n,k}$. Then $f_n(x) = k/n$ and $k/n \leq f(x) < (k+1)/n$, so

$$0 \leq f(x) - f_n(x) < \frac{1}{n}.$$

This bound is independent of x , so

$$\sup_{x \in E} |f(x) - f_n(x)| \leq \frac{1}{n} \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

that is, $\{f_n\}$ converges to f uniformly on E .

Remark on extended real values. If f is permitted to take the values $\pm\infty$, no sequence of real-valued functions can converge to f uniformly (indeed not even pointwise) at a point where f is infinite. The natural amendment is to allow the approximants to be infinite exactly where f is: define $g_n = f_n$ on the measurable set $\{x \in E \mid |f(x)| < \infty\}$, using the construction above applied to the restriction of f to that set (measurable by Proposition 5(ii)), and $g_n = \infty$ on $f^{-1}(\infty)$, $g_n = -\infty$ on $f^{-1}(-\infty)$; these last two sets are measurable by Proposition 1. Each g_n is measurable, takes only countably many values, and satisfies $|g_n - f| < 1/n$ wherever f is finite while $g_n = f$ wherever f is infinite. Thus $\{g_n\}$ converges to f uniformly in the only sense available for extended real-valued functions.

Problem 3.14. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E such that f is bounded on F and $m(E \setminus F) < \epsilon$.

Solution. For each natural number n define

$$E_n = \{x \in E \mid |f(x)| \leq n\} = \{x \in E \mid f(x) \leq n\} \cap \{x \in E \mid f(x) \geq -n\}.$$

Each of the two sets on the right is measurable by Proposition 1, so each E_n is a measurable subset of E . The collection $\{E_n\}_{n=1}^{\infty}$ is ascending, since $|f(x)| \leq n$ implies $|f(x)| \leq n+1$, and

$$\bigcup_{n=1}^{\infty} E_n = \{x \in E \mid |f(x)| < \infty\} =: E_{\text{fin}},$$

because a point x lies in some E_n precisely when $|f(x)|$ is a finite number, in which case any integer $n \geq |f(x)|$ will do.

Since f is finite a.e. on E , the set

$$E \sim E_{\text{fin}} = f^{-1}(\infty) \cup f^{-1}(-\infty)$$

has measure zero; it is measurable by Proposition 1. Hence, by the excision property of measure (legitimate because $m(E \sim E_{\text{fin}}) = 0 < \infty$) together with the additivity of measure over the disjoint measurable decomposition $E = E_{\text{fin}} \cup (E \sim E_{\text{fin}})$,

$$m(E_{\text{fin}}) = m(E) - m(E \sim E_{\text{fin}}) = m(E).$$

Now apply the continuity of measure. By part (i) of Theorem 15 of Chapter 2, applied to the ascending collection $\{E_n\}$,

$$\lim_{n \rightarrow \infty} m(E_n) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = m(E_{\text{fin}}) = m(E).$$

By hypothesis $m(E) < \infty$, so this is a limit of real numbers converging to the real number $m(E)$. Therefore, given $\epsilon > 0$, there is a natural number N for which

$$m(E_N) > m(E) - \epsilon.$$

Put $F = E_N$. Then F is a measurable set contained in E , and $|f| \leq N$ on F , so f is bounded on F . Moreover $F \subseteq E$ and $m(F) \leq m(E) < \infty$, so the excision property of measure gives

$$m(E \sim F) = m(E) - m(F) = m(E) - m(E_N) < \epsilon.$$

This is exactly what was to be shown.

Two remarks on the hypotheses. First, the finiteness of $m(E)$ is essential: for $E = \mathbb{R}$ and $f(x) = x$, the function f is finite everywhere, yet every set on which f is bounded is contained in a bounded interval and hence has complement in $\mathbb{R}$ of infinite measure, so no such F exists for any ϵ . Second, the finiteness a.e. of f is essential: if $f = \infty$ on a subset of E of positive measure δ , then every set F on which f is bounded misses that subset, so $m(E \sim F) \geq \delta$, and the conclusion fails for $\epsilon < \delta$.

3.3 Problems 3.15–3.21

Problem 3.15. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. Show that for each $\epsilon > 0$, there is a measurable set F contained in E and a sequence $\{\varphi_n\}$ of simple functions on E such that $\{\varphi_n\} \rightarrow f$ uniformly on F and $m(E \sim F) < \epsilon$. (Hint: See the preceding problem.)

Solution. Let $\epsilon > 0$ be given.

Step 1: a set on which f is bounded (this is Problem 14). For each natural number n put

$$E_n = \{x \in E \mid |f(x)| \leq n\}.$$

Here f is permitted to be extended real-valued, so we avoid appealing to the measurability of $|f|$ and argue directly: since f is measurable, for each n both $\{x \in E \mid f(x) \leq n\}$ and $\{x \in E \mid f(x) \geq -n\}$ are measurable by Proposition 1, and E_n is their intersection, hence measurable because the measurable sets form a σ -algebra. The sequence $\{E_n\}$ is ascending and

$$\bigcup_{n=1}^{\infty} E_n = \{x \in E \mid |f(x)| < \infty\} = E \sim E_{\infty}, \quad E_{\infty} = \{x \in E \mid |f(x)| = \infty\}.$$

Because f is finite a.e. on E we have $m(E_{\infty}) = 0$, so by the continuity of measure (Theorem 15 of Chapter 2) and the excision property,

$$\lim_{n \rightarrow \infty} m(E_n) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = m(E \sim E_{\infty}) = m(E).$$

Since $m(E) < \infty$, every $m(E_n)$ is finite, and by excision $m(E \sim E_n) = m(E) - m(E_n)$. Choose N so large that $m(E) - m(E_N) < \epsilon$ and set

$$F = E_N.$$

Then F is a measurable subset of E , $m(E \sim F) < \epsilon$, and $|f| \leq N$ on F ; in particular f is real-valued on F .

Step 2: uniform approximation on F by simple functions. By part (ii) of Proposition 5, the restriction of f to the measurable set F is measurable, and by Step 1 it is a bounded real-valued function on F . Fix a natural number n . The Simple Approximation Lemma, applied to $f|_F$ with the choice $\epsilon = 1/n$, provides simple functions ϕ_n and ψ_n on F with

$$\phi_n \leq f \leq \psi_n \quad \text{and} \quad 0 \leq \psi_n - \phi_n < 1/n \quad \text{on } F.$$

Consequently

$$0 \leq f - \phi_n \leq \psi_n - \phi_n < 1/n \quad \text{on } F.$$

Step 3: extension to E . Define φ_n on all of E by

$$\varphi_n(x) = \begin{cases} \phi_n(x) & \text{if } x \in F, \\ 0 & \text{if } x \in E \sim F. \end{cases}$$

Each φ_n is real-valued and takes only finitely many values, namely the values of ϕ_n together with possibly 0. It is measurable on E : the restriction of φ_n to the measurable set F is ϕ_n , which is measurable because it is simple, and the restriction of φ_n to $E \sim F$ is identically 0, which is measurable; so by part (ii) of Proposition 5, φ_n is measurable on E . Being real-valued, measurable, and finitely valued, φ_n is a simple function on E .

Step 4: conclusion. Let $\epsilon' > 0$ and choose N' with $1/N' < \epsilon'$. For $n \geq N'$ we have, by Step 2, $|f - \varphi_n| = |f - \phi_n| < 1/n \leq 1/N' < \epsilon'$ at every point of F . Thus $\{\varphi_n\} \rightarrow f$ uniformly on F , while $m(E \sim F) < \epsilon$. ■

Problem 3.16. Let I be a closed, bounded interval and E a measurable subset of I . Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \chi_E \text{ on } F \quad \text{and} \quad m(I \sim F) < \epsilon.$$

(Hint: Use Theorem 12 of Chapter 2.)

Solution. Write $I = [a, b]$. Since $E \subseteq I$ is measurable and $m^*(E) \leq b - a < \infty$, Theorem 12 of Chapter 2 applies: there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ such that, setting $\mathcal{O} = \bigcup_{k=1}^n I_k$,

$$m^*(E \Delta \mathcal{O}) < \epsilon,$$

where, as in the text, $E \Delta \mathcal{O} = (E \sim \mathcal{O}) \cup (\mathcal{O} \sim E)$ denotes the symmetric difference. Write $I_k = (a_k, b_k)$ for $1 \leq k \leq n$.

The step function. Define h on I by $h = \chi_{\mathcal{O}}$ restricted to I , that is,

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O} \cap I, \\ 0 & \text{if } x \in I \sim \mathcal{O}. \end{cases}$$

We claim h is a step function on I . Let $a = t_0 < t_1 < \cdots < t_m = b$ be the partition of $[a, b]$ whose partition points are a , b , and those of the finitely many numbers $a_1, b_1, \dots, a_n, b_n$ that lie in the open interval (a, b) . Fix j and consider the open interval (t_{j-1}, t_j) . No endpoint a_k or b_k lies in (t_{j-1}, t_j) . Hence for each k , the set $(t_{j-1}, t_j) \cap (a_k, b_k)$ is either empty or all of (t_{j-1}, t_j) : indeed if some $x \in (t_{j-1}, t_j)$ belongs to (a_k, b_k) while some $y \in (t_{j-1}, t_j)$ does not, then between x and y there would be a point of $\{a_k, b_k\}$, and that point would lie in (t_{j-1}, t_j) , a contradiction. Therefore either $(t_{j-1}, t_j) \subseteq \mathcal{O}$, in which case $h \equiv 1$ on (t_{j-1}, t_j) , or $(t_{j-1}, t_j) \cap \mathcal{O} = \emptyset$, in which case $h \equiv 0$ there. So h is constant on each open interval of a finite partition of I , i.e. h is a step function on I .

The set F . Define

$$F = I \sim (E \Delta \mathcal{O}) = I \sim [(E \sim \mathcal{O}) \cup (\mathcal{O} \sim E)].$$

Since E is measurable and $\mathcal{O}$ is open, hence measurable, and the measurable sets form a σ -algebra, F is a measurable subset of I .

Let $x \in F$. Then $x \in I$ and $x \notin E \Delta \mathcal{O}$, so $x \in E$ if and only if $x \in \mathcal{O}$. Hence $\chi_E(x) = \chi_{\mathcal{O}}(x) = h(x)$. Thus $h = \chi_E$ on F .

Finally,

$$I \sim F = I \cap (E \Delta \mathcal{O}) \subseteq E \Delta \mathcal{O},$$

so by the monotonicity of measure $m(I \sim F) \leq m(E \Delta \mathcal{O}) < \epsilon$. ■

Problem 3.17. Let I be a closed, bounded interval and ψ a simple function defined on I . Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ on } F \quad \text{and} \quad m(I \sim F) < \epsilon.$$

(Hint: Use the fact that a simple function is a linear combination of characteristic functions and the preceding problem.)

Solution. Let $c_1, \dots, c_n$ be the distinct values taken by ψ and let

$$E_k = \{x \in I \mid \psi(x) = c_k\}, \quad 1 \leq k \leq n,$$

so that the canonical representation of ψ is

$$\psi = \sum_{k=1}^n c_k \cdot \chi_{E_k} \quad \text{on } I.$$

Each E_k is measurable, since ψ is measurable (Proposition 1), and the E_k are disjoint with union I . Applying the preceding problem. Fix k . Since E_k is a measurable subset of the closed, bounded interval I , Problem 16, applied with the positive number ϵ/n , provides a step function h_k on I and a measurable subset F_k of I for which

$$h_k = \chi_{E_k} \text{ on } F_k \quad \text{and} \quad m(I \sim F_k) < \epsilon/n.$$

The step function. Define

$$h = \sum_{k=1}^n c_k \cdot h_k \quad \text{on } I.$$

A linear combination of step functions on I is again a step function on I : if P_k is a partition of I on each of whose open subintervals h_k is constant, let P be the partition of I whose points are all the points of $P_1 \cup \dots \cup P_n$. Each open subinterval J determined by P is contained in an open subinterval determined by P_k , for every k , so every h_k is constant on J , and hence so is h . Thus h is a step function on I .

The set F . Define $F = \bigcap_{k=1}^n F_k$, a measurable subset of I . By De Morgan's identity and the countable (here finite) subadditivity and monotonicity of measure,

$$m(I \sim F) = m\left(\bigcup_{k=1}^n (I \sim F_k)\right) \leq \sum_{k=1}^n m(I \sim F_k) < n \cdot \frac{\epsilon}{n} = \epsilon.$$

Agreement on F . Let $x \in F$. Then $x \in F_k$ for every k , so $h_k(x) = \chi_{E_k}(x)$ for every k , and therefore

$$h(x) = \sum_{k=1}^n c_k \cdot h_k(x) = \sum_{k=1}^n c_k \cdot \chi_{E_k}(x) = \psi(x).$$

Hence $h = \psi$ on F and $m(I \sim F) < \epsilon$. ■

Problem 3.18. Let I be a closed, bounded interval and f a bounded measurable function defined on I . Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$|h - f| < \epsilon \text{ on } F \quad \text{and} \quad m(I \setminus F) < \epsilon.$$

Solution. The function f is a bounded measurable real-valued function on the measurable set I , so the Simple Approximation Lemma applies. Using it with the positive number $\epsilon/2$, we obtain simple functions ϕ and η on I with

$$\phi \leq f \leq \eta \quad \text{and} \quad 0 \leq \eta - \phi < \epsilon/2 \text{ on } I.$$

Put $\psi = \phi$. Then ψ is a simple function on I and, at every point of I ,

$$0 \leq f - \psi \leq \eta - \phi < \epsilon/2,$$

so that

$$|\psi - f| < \epsilon/2 \text{ on all of } I.$$

Now apply Problem 17 to the simple function ψ on the closed, bounded interval I , with the same $\epsilon > 0$: there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ on } F \quad \text{and} \quad m(I \setminus F) < \epsilon.$$

Let $x \in F$. Then $h(x) = \psi(x)$, and therefore

$$|h(x) - f(x)| = |\psi(x) - f(x)| < \epsilon/2 < \epsilon.$$

Thus $|h - f| < \epsilon$ on F while $m(I \setminus F) < \epsilon$, as required. ■

Problem 3.19. Show that the sum and product of two simple functions are simple as are the max and the min.

Solution. Let φ and ψ be simple functions with common measurable domain E (a common domain is needed for the four operations to be defined). Let $a_1, \dots, a_n$ be the distinct values of φ and $b_1, \dots, b_m$ the distinct values of ψ , and set

$$A_i = \{x \in E \mid \varphi(x) = a_i\}, \quad B_j = \{x \in E \mid \psi(x) = b_j\}.$$

Since φ and ψ are measurable, each A_i and each B_j is measurable by Proposition 1. The collection $\{A_i\}_{i=1}^n$ is a disjoint cover of E , as is $\{B_j\}_{j=1}^m$, and hence the sets

$$C_{ij} = A_i \cap B_j, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

are measurable, pairwise disjoint, and their union is E .

Let g denote any one of the four functions

$$\varphi + \psi, \quad \varphi \cdot \psi, \quad \max\{\varphi, \psi\}, \quad \min\{\varphi, \psi\},$$

and let γ_{ij} denote, respectively, the number

$$a_i + b_j, \quad a_i b_j, \quad \max\{a_i, b_j\}, \quad \min\{a_i, b_j\}.$$

Since $\varphi \equiv a_i$ and $\psi \equiv b_j$ on C_{ij} , in each of the four cases we have $g \equiv \gamma_{ij}$ on C_{ij} . g takes only finitely many values. Since the C_{ij} cover E , the image $g(E)$ is contained in the finite set $\{\gamma_{ij} \mid 1 \leq i \leq n, \quad 1 \leq j \leq m\}$, which has at most nm elements. Note also that all the γ_{ij} are real numbers (simple functions are real-valued by definition, so no expression $\infty - \infty$ or $0 \cdot \infty$ can arise), so g is real-valued.

g is measurable. Let c be a real number. Because E is the disjoint union of the C_{ij} and g is constantly γ_{ij} on C_{ij} ,

$$\{x \in E \mid g(x) > c\} = \bigcup_{\{(i,j) : \gamma_{ij} > c\}} C_{ij},$$

a finite union of measurable sets, hence measurable. By Proposition 1, g is measurable on E .

Thus each of $\varphi + \psi$, $\varphi \cdot \psi$, $\max\{\varphi, \psi\}$ and $\min\{\varphi, \psi\}$ is a measurable real-valued function on E taking only a finite number of values, that is, each is simple.

(Alternatively, the measurability of $\varphi + \psi$ and $\varphi \cdot \psi$ follows from Theorem 6 and that of $\max\{\varphi, \psi\}$ and $\min\{\varphi, \psi\}$ from Proposition 8; the point that remains in each case is exactly the finiteness of the set of values established above.) ■

Problem 3.20. Let A and B be any sets. Show that

$$\chi_{A \cap B} = \chi_A \cdot \chi_B$$

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$

$$\chi_{A^c} = 1 - \chi_A.$$

Solution. Since a complement only has meaning relative to an ambient set, fix a set X with $A \subseteq X$ and $B \subseteq X$, and read all three identities as identities between functions on X , with $A^c = X \sim A$. (In the setting of the text $X = \mathbb{R}$.) For a subset S of X the characteristic function χ_S is the function on X defined by $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$. Two functions on X are equal precisely when they agree at every $x \in X$, so in each case we fix $x \in X$ and check the possible positions of x relative to A and B .

First identity. Suppose $x \in A \cap B$. Then $\chi_{A \cap B}(x) = 1$, while $\chi_A(x) = \chi_B(x) = 1$, so $\chi_A(x) \cdot \chi_B(x) = 1$. Suppose $x \notin A \cap B$. Then $\chi_{A \cap B}(x) = 0$, and at least one of $x \notin A$, $x \notin B$ holds, so at least one of $\chi_A(x)$, $\chi_B(x)$ equals 0 and hence $\chi_A(x) \cdot \chi_B(x) = 0$. In both cases the two sides agree, so $\chi_{A \cap B} = \chi_A \cdot \chi_B$.

Second identity. Consider the four cases, writing $u = \chi_A(x)$ and $v = \chi_B(x)$.

- If $x \in A$ and $x \in B$, then $x \in A \cup B$, so the left side is 1; and $u + v - uv = 1 + 1 - 1 = 1$.
- If $x \in A$ and $x \notin B$, then $x \in A \cup B$, so the left side is 1; and $u + v - uv = 1 + 0 - 0 = 1$.
- If $x \notin A$ and $x \in B$, then $x \in A \cup B$, so the left side is 1; and $u + v - uv = 0 + 1 - 0 = 1$.
- If $x \notin A$ and $x \notin B$, then $x \notin A \cup B$, so the left side is 0; and $u + v - uv = 0 + 0 - 0 = 0$.

In every case the two sides agree, so $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$.

Third identity. If $x \in A^c$, then $x \notin A$, so $\chi_{A^c}(x) = 1$ and $1 - \chi_A(x) = 1 - 0 = 1$. If $x \notin A^c$, then $x \in A$, so $\chi_{A^c}(x) = 0$ and $1 - \chi_A(x) = 1 - 1 = 0$. Hence $\chi_{A^c} = 1 - \chi_A$.

(Observe that the second identity can also be read off from the first two together with the third and De Morgan's laws: $\chi_{A \cup B} = 1 - \chi_{(A \cup B)^c} = 1 - \chi_{A^c \cap B^c} = 1 - (1 - \chi_A)(1 - \chi_B) = \chi_A + \chi_B - \chi_A \chi_B$.) ■

Problem 3.21. For a sequence $\{f_n\}$ of measurable functions with common domain E , show that each of the following functions is measurable:

$$\inf\{f_n\}, \quad \sup\{f_n\}, \quad \liminf\{f_n\} \quad \text{and} \quad \limsup\{f_n\}.$$

Solution. All functions here are extended real-valued on the measurable set E , and the four functions are defined pointwise: $(\sup\{f_n\})(x) = \sup_n f_n(x)$, and similarly for the others. Such pointwise suprema and infima always exist in the extended real numbers.

The supremum. Put $g = \sup\{f_n\}$. Let c be a real number. We claim

$$\{x \in E \mid g(x) > c\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f_n(x) > c\}.$$

Indeed, if $g(x) > c$ then c is not an upper bound for the set $\{f_n(x)\}_{n=1}^{\infty}$ (its supremum is $g(x) > c$), so there is an index n with $f_n(x) > c$. Conversely, if $f_n(x) > c$ for some n , then $g(x) \geq f_n(x) > c$. Each set $\{x \in E \mid f_n(x) > c\}$ is measurable because f_n is measurable, and the measurable sets are a

σ -algebra, so the countable union on the right is measurable. Since this holds for every real c , property (i) of Proposition 1 holds for g , and hence $g = \sup\{f_n\}$ is measurable. The infimum. Put $h = \inf\{f_n\}$. Let c be a real number. We claim

$$\{x \in E \mid h(x) < c\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f_n(x) < c\}.$$

Indeed, if $h(x) < c$ then c is not a lower bound for $\{f_n(x)\}_{n=1}^{\infty}$, so $f_n(x) < c$ for some n ; conversely if $f_n(x) < c$ for some n , then $h(x) \leq f_n(x) < c$. Again the right-hand side is a countable union of measurable sets, hence measurable, so property (iii) of Proposition 1 holds for h and $h = \inf\{f_n\}$ is measurable.

Observe that these two arguments used nothing about the index set beyond its countability: for each natural number k , the sequence $\{f_n\}_{n \geq k}$ is again a countable family of measurable functions on E , so

$$g_k = \inf_{n \geq k} f_n \quad \text{and} \quad G_k = \sup_{n \geq k} f_n$$

are measurable functions on E for every k .

The limit inferior. By definition,

$$\liminf\{f_n\} = \sup_{k \geq 1} \left(\inf_{n \geq k} f_n \right) = \sup_{k \geq 1} g_k.$$

Each g_k is measurable by the preceding paragraph, so $\{g_k\}_{k=1}^{\infty}$ is a sequence of measurable functions with common domain E ; by the supremum case already proved, $\sup_k g_k$ is measurable. Hence $\liminf\{f_n\}$ is measurable.

The limit superior. Similarly, by definition,

$$\limsup\{f_n\} = \inf_{k \geq 1} \left(\sup_{n \geq k} f_n \right) = \inf_{k \geq 1} G_k,$$

each G_k is measurable, and by the infimum case already proved the pointwise infimum of the sequence $\{G_k\}_{k=1}^{\infty}$ of measurable functions is measurable. Hence $\limsup\{f_n\}$ is measurable. ■

3.4 Problems 3.22–3.28

Problem 3.22. (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$. (Hint: Let $\epsilon > 0$. For each natural number n , define $E_n = \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}$. Show that $\{E_n\}$ is an open cover of $[a, b]$ and use the Heine-Borel Theorem.)

Solution. Throughout, “increasing sequence” means $f_n \leq f_{n+1}$ pointwise on $[a, b]$ for every index n . We first record that $f_n \leq f$ on $[a, b]$ for every n . Indeed, fix $x \in [a, b]$ and an index n . For every $m \geq n$ we have $f_n(x) \leq f_m(x)$, and letting $m \rightarrow \infty$ in this inequality, using $f_m(x) \rightarrow f(x)$ and the fact that weak inequalities are preserved under limits, we get $f_n(x) \leq f(x)$. Consequently

$$|f(x) - f_n(x)| = f(x) - f_n(x) \quad \text{for all } x \in [a, b] \text{ and all } n,$$

so it suffices to show that for each $\epsilon > 0$ there is an index N with $f - f_n < \epsilon$ on all of $[a, b]$ whenever $n \geq N$.

Let $\epsilon > 0$ and, following the hint, set

$$E_n = \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}.$$

Each E_n is relatively open in $[a, b]$. Indeed, f and f_n are continuous on $[a, b]$, hence so is $g_n := f - f_n$, and $E_n = g_n^{-1}((-\infty, \epsilon))$ is the preimage of an open subset of $\mathbb{R}$ under a continuous function on $[a, b]$. Thus there is an open subset $\mathcal{O}_n$ of $\mathbb{R}$ with $E_n = \mathcal{O}_n \cap [a, b]$. (Concretely: if $x_0 \in E_n$, continuity of

g_n at x_0 with the choice $\eta = \epsilon - g_n(x_0) > 0$ furnishes $\delta > 0$ such that $g_n(x) < \epsilon$ for all $x \in [a, b]$ with $|x - x_0| < \delta$; taking $\mathcal{O}_n$ to be the union of all such open intervals $(x_0 - \delta, x_0 + \delta)$ over $x_0 \in E_n$ gives an open set with $\mathcal{O}_n \cap [a, b] = E_n$.)

The collection $\{E_n\}$ is ascending. Since $f_n \leq f_{n+1}$, we have $f - f_{n+1} \leq f - f_n$ on $[a, b]$, so $f(x) - f_n(x) < \epsilon$ forces $f(x) - f_{n+1}(x) < \epsilon$; that is, $E_n \subseteq E_{n+1}$. The same monotonicity gives $\mathcal{O}_n \cap [a, b] \subseteq \mathcal{O}_m \cap [a, b]$ whenever $n \leq m$.

The collection $\{\mathcal{O}_n\}_{n=1}^\infty$ covers $[a, b]$. Let $x \in [a, b]$. Since $f_n(x) \rightarrow f(x)$, there is an index n with $f(x) - f_n(x) < \epsilon$, that is, $x \in E_n \subseteq \mathcal{O}_n$.

The interval $[a, b]$ is closed and bounded, so by the Heine-Borel Theorem the open cover $\{\mathcal{O}_n\}_{n=1}^\infty$ of $[a, b]$ has a finite subcover, say $\mathcal{O}_{n_1}, \dots, \mathcal{O}_{n_k}$. Put $N = \max\{n_1, \dots, n_k\}$. For each i we have $\mathcal{O}_{n_i} \cap [a, b] = E_{n_i} \subseteq E_N$, and therefore

$$[a, b] = \bigcup_{i=1}^k (\mathcal{O}_{n_i} \cap [a, b]) \subseteq E_N \subseteq [a, b],$$

so $E_N = [a, b]$.

Finally, let $n \geq N$. Then $E_N \subseteq E_n$, so $E_n = [a, b]$, which says precisely that

$$|f(x) - f_n(x)| = f(x) - f_n(x) < \epsilon \quad \text{for every } x \in [a, b].$$

Since $\epsilon > 0$ was arbitrary and N does not depend on x , the convergence of $\{f_n\}$ to f is uniform on $[a, b]$.

Two remarks on the hypotheses, each of which is essential. If f is not assumed continuous the conclusion fails: on $[0, 1]$ the sequence $f_n(x) = x^{1/n}$ is an increasing sequence of continuous functions whose pointwise limit equals 0 at $x = 0$ and 1 on $(0, 1]$, and the convergence is not uniform. If compactness of the domain is dropped the conclusion also fails: on $(0, 1)$ the increasing continuous functions $f_n(x) = \min\{nx, 1\}$ converge pointwise to the continuous function $f \equiv 1$, but $\sup_{0 < x < 1} |1 - f_n(x)| = 1$ for every n .

Problem 3.23. Express a measurable function as the difference of nonnegative measurable functions and thereby prove the general Simple Approximation Theorem based on the special case of a nonnegative measurable function.

Solution. Let E be measurable and let f be an extended real-valued function on E . Define the positive and negative parts of f by

$$f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\} \quad (x \in E).$$

Both are nonnegative, and at each point at most one of them is nonzero, so the difference $f^+ - f^-$ never has the indeterminate form $\infty - \infty$ and

$$f = f^+ - f^-, \quad |f| = f^+ + f^- \quad \text{on } E.$$

Indeed, if $f(x) \geq 0$ then $f^+(x) = f(x)$ and $f^-(x) = 0$; if $f(x) < 0$ then $f^+(x) = 0$ and $f^-(x) = -f(x)$. This covers the extended values as well: if $f(x) = \infty$ then $f^+(x) = \infty$, $f^-(x) = 0$; if $f(x) = -\infty$ then $f^+(x) = 0$, $f^-(x) = \infty$.

If f is measurable, so are f^+ and f^- . For $c \geq 0$,

$$\{x \in E \mid f^+(x) > c\} = \{x \in E \mid f(x) > c\},$$

which is measurable, while for $c < 0$ the set $\{x \in E \mid f^+(x) > c\}$ is all of E , again measurable. Likewise, for $c \geq 0$,

$$\{x \in E \mid f^-(x) > c\} = \{x \in E \mid f(x) < -c\},$$

which is measurable, and for $c < 0$ this set is E . Hence f^+ and f^- are nonnegative measurable functions on E and $f = f^+ - f^-$ is the required expression.

We now prove the Simple Approximation Theorem in general, assuming it for nonnegative measurable functions.

One implication requires nothing new: every simple function is measurable, and by Proposition 9 the pointwise limit of a sequence of measurable functions on E is measurable. So if there is a sequence $\{\varphi_n\}$ of simple functions on E converging pointwise on E to f , then f is measurable.

For the converse, assume f is measurable. By the paragraphs above, f^+ and f^- are nonnegative measurable functions on E . Apply the nonnegative case of the theorem to each: there are sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of simple functions on E with

$$0 \leq \varphi_n \leq f^+, \quad \varphi_n \rightarrow f^+ \text{ pointwise on } E, \quad 0 \leq \psi_n \leq f^-, \quad \psi_n \rightarrow f^- \text{ pointwise on } E,$$

and with each of $\{\varphi_n\}$, $\{\psi_n\}$ increasing. Here is why the two-sided bounds may be taken in the sharpened form $0 \leq \varphi_n \leq f^+$ rather than merely $|\varphi_n| \leq |f^+|$. The nonnegative case delivers simple functions $\tilde{\varphi}_n$ on E , increasing, with $\tilde{\varphi}_n \rightarrow f^+$ pointwise and $|\tilde{\varphi}_n| \leq |f^+| = f^+$ on E ; the latter gives $\tilde{\varphi}_n \leq f^+$ but not $\tilde{\varphi}_n \geq 0$. Replace $\tilde{\varphi}_n$ by $\varphi_n = \max\{\tilde{\varphi}_n, 0\}$. By Problem 19 the maximum of two simple functions is simple, so each φ_n is simple; the sequence $\{\varphi_n\}$ is still increasing since $t \mapsto \max\{t, 0\}$ is increasing; $0 \leq \varphi_n \leq \max\{f^+, 0\} = f^+$ because $\tilde{\varphi}_n \leq f^+$ and $0 \leq f^+$; and at each $x \in E$, $\varphi_n(x) = \max\{\tilde{\varphi}_n(x), 0\} \rightarrow \max\{f^+(x), 0\} = f^+(x)$, by continuity of $t \mapsto \max\{t, 0\}$ on $\mathbb{R}$ when $f^+(x)$ is finite, and directly from $\tilde{\varphi}_n(x) \rightarrow \infty$ when $f^+(x) = \infty$. The same replacement is made for $\{\psi_n\}$.

Define $h_n = \varphi_n - \psi_n$ on E . Each h_n is simple: ψ_n simple implies $-\psi_n = (-1) \cdot \psi_n$ is simple, and by Problem 19 the sum of two simple functions is simple. Note that φ_n and ψ_n are real-valued (simple functions take finitely many real values), so no indeterminate difference arises.

The bound $|h_n| \leq |f|$ holds on E . Fix $x \in E$. If $f(x) \geq 0$ then $f^-(x) = 0$, so $0 \leq \psi_n(x) \leq f^-(x) = 0$ forces $\psi_n(x) = 0$; hence $h_n(x) = \varphi_n(x)$ and

$$|h_n(x)| = \varphi_n(x) \leq f^+(x) = |f(x)|.$$

If $f(x) < 0$ then $f^+(x) = 0$, so $\varphi_n(x) = 0$, $h_n(x) = -\psi_n(x)$, and

$$|h_n(x)| = \psi_n(x) \leq f^-(x) = |f(x)|.$$

Finally, $h_n \rightarrow f$ pointwise on E . Fix $x \in E$. If $f(x)$ is finite, then $f^+(x)$ and $f^-(x)$ are finite and

$$\lim_{n \rightarrow \infty} h_n(x) = \lim_{n \rightarrow \infty} \varphi_n(x) - \lim_{n \rightarrow \infty} \psi_n(x) = f^+(x) - f^-(x) = f(x).$$

If $f(x) = \infty$, then, as computed above, $\psi_n(x) = 0$ for every n and $h_n(x) = \varphi_n(x) \rightarrow f^+(x) = \infty = f(x)$.

If $f(x) = -\infty$, then $\varphi_n(x) = 0$ for every n and $h_n(x) = -\psi_n(x) \rightarrow -f^-(x) = -\infty = f(x)$.

Thus $\{h_n\}$ is a sequence of simple functions on E that converges pointwise on E to f and satisfies $|h_n| \leq |f|$ on E for every n , which is the general Simple Approximation Theorem.

For the last clause of the theorem, if $f \geq 0$ on E then $f^- \equiv 0$, hence $\psi_n \equiv 0$ and $h_n = \varphi_n$, so the approximating sequence produced is increasing, as asserted in the nonnegative case.

Problem 3.24. Let I be an interval and $f : I \rightarrow \mathbb{R}$ be increasing. Show that f is measurable by first showing that, for each natural number n , the strictly increasing function $x \mapsto f(x) + x/n$ is measurable, and then taking pointwise limits.

Solution. First note that the domain I is a measurable set, since every interval is Lebesgue measurable. So it makes sense to ask whether f is measurable, and by the criterion defining measurability it suffices to show that for each $c \in \mathbb{R}$ the set $\{x \in I \mid f(x) > c\}$ is measurable.

Fix a natural number n and define

$$g_n : I \rightarrow \mathbb{R}, \quad g_n(x) = f(x) + \frac{x}{n}.$$

The function g_n is strictly increasing. Indeed, let $x, y \in I$ with $x < y$. Since f is increasing, $f(x) \leq f(y)$,

and since $x/n < y/n$, adding gives

$$g_n(x) = f(x) + \frac{x}{n} < f(y) + \frac{y}{n} = g_n(y).$$

Claim: a strictly increasing real-valued function g on an interval I is measurable. Fix $c \in \mathbb{R}$ and put

$$S = \{x \in I \mid g(x) > c\}.$$

The set S is upward closed in I : if $x \in S$ and $y \in I$ with $y > x$, then $g(y) > g(x) > c$, so $y \in S$. We show S is an interval, i.e. that it is convex. Let $x, y \in S$ and let $z \in \mathbb{R}$ with $x < z < y$. Since $x, y \in I$ and I is an interval, $z \in I$; and since $z > x$ and S is upward closed in I , we get $z \in S$. Thus S contains every real number lying between two of its points, so S is an interval (possibly empty, possibly a single point, possibly unbounded). Every interval is measurable, so S is measurable. As $c \in \mathbb{R}$ was arbitrary, g is measurable, and the claim is proved.

Applying the claim to g_n , we conclude that g_n is a measurable real-valued function on I , for every natural number n .

Now take pointwise limits. Fix $x \in I$. Then x is a fixed real number, so $x/n \rightarrow 0$ as $n \rightarrow \infty$, and hence

$$\lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} \left(f(x) + \frac{x}{n} \right) = f(x).$$

Thus $\{g_n\}$ is a sequence of measurable functions on the measurable set I that converges pointwise on I to f . By Proposition 9, the pointwise limit of a sequence of measurable functions on a common measurable domain is measurable. Therefore f is measurable.

Remark. The claim above did not actually use the strictness of the increase: if g is merely increasing and $x \in S$, $y \in I$, $y > x$, then $g(y) \geq g(x) > c$, so again $y \in S$, and the rest of the argument is unchanged. Consequently every increasing real-valued function on an interval is measurable directly, without passing to the limit. The route through the strictly increasing functions g_n is the one suggested by the problem, and it is included above in the form requested.

Problem 3.25. Suppose f is a function that is continuous on a closed set F of real numbers. Show that f has a continuous extension to all of $\mathbb{R}$. This is a special case of the forthcoming Tietze Extension Theorem. (Hint: Express $\mathbb{R} \setminus F$ as the union of a countable disjoint collection of open intervals and define f to be linear on the closure of each of these intervals.)

Solution. If $F = \emptyset$ we may take $g \equiv 0$, and if $F = \mathbb{R}$ we may take $g = f$. So assume F is a nonempty proper closed subset of $\mathbb{R}$ and put $\mathcal{O} = \mathbb{R} \setminus F$, a nonempty open set.

Construction of the extension. By Proposition 9 of Chapter 1, $\mathcal{O}$ is the union of a countable disjoint collection $\{I_k\}$ of open intervals; moreover, the proof of that proposition shows that each finite endpoint of each I_k fails to belong to $\mathcal{O}$, hence belongs to F . Since $F \neq \emptyset$, no I_k equals $\mathbb{R}$, so each I_k is of exactly one of the three forms

$$(a_k, b_k) \text{ with } a_k, b_k \in F, \quad (-\infty, b_k) \text{ with } b_k \in F, \quad (a_k, \infty) \text{ with } a_k \in F.$$

Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} f(x), & x \in F, \\ f(a_k) + \frac{f(b_k) - f(a_k)}{b_k - a_k} (x - a_k), & x \in I_k = (a_k, b_k) \text{ bounded}, \\ f(b_k), & x \in I_k = (-\infty, b_k), \\ f(a_k), & x \in I_k = (a_k, \infty). \end{cases}$$

Since the I_k are pairwise disjoint and disjoint from F , g is well defined, real-valued, and $g = f$ on F ; so g is an extension of f . On a bounded I_k the formula is the affine interpolation between the values $f(a_k)$ and $f(b_k)$ at the endpoints, and it takes those values at a_k and b_k , so g is continuous on the closed interval $[a_k, b_k]$ and agrees there with the endpoint values already assigned by f .

Observe once and for all that for x in a bounded component (a_k, b_k) the value $g(x)$ is a convex combination of $f(a_k)$ and $f(b_k)$: writing $t = (x - a_k)/(b_k - a_k) \in (0, 1)$,

$$g(x) = (1 - t)f(a_k) + tf(b_k).$$

Call this the convex-combination identity (C).

Continuity at points of $\mathcal{O}$. If $x_0 \in \mathcal{O}$, then x_0 lies in some I_k , which is an open interval, hence an open neighborhood of x_0 , and g restricted to I_k is either affine or constant, hence continuous at x_0 . So g is continuous at x_0 .

Continuity at points of F . Let $x_0 \in F$ and let $\epsilon > 0$. Since f is continuous on F at x_0 , choose $\delta > 0$ such that

$$|f(y) - f(x_0)| < \epsilon \quad \text{for all } y \in F \text{ with } |y - x_0| < \delta.$$

Call this the choice-of-delta property (D). We produce $\delta_+ > 0$ with $|g(x) - g(x_0)| < \epsilon$ for $x_0 \leq x < x_0 + \delta_+$; the argument on the left is symmetric and produces $\delta_- > 0$, and then $\delta_0 = \min\{\delta_+, \delta_-\}$ works for two-sided continuity at x_0 .

Case 1: x_0 is the left endpoint of some component I_k of $\mathcal{O}$. If $I_k = (x_0, \infty)$, then $g \equiv f(x_0) = g(x_0)$ on $[x_0, \infty)$ and any δ_+ works. If $I_k = (x_0, b_k)$ is bounded, then by (C) and the fact that g is the continuous affine function on $[x_0, b_k]$ with $g(x_0) = f(x_0)$, we have $g(x) \rightarrow g(x_0)$ as $x \rightarrow x_0^+$; choose $\delta_+ \leq b_k - x_0$ accordingly, noting that $[x_0, x_0 + \delta_+) \subseteq [x_0, b_k]$.

Case 2: x_0 is not the left endpoint of any component of $\mathcal{O}$. We claim that then $F \cap (x_0, x_0 + \eta) \neq \emptyset$ for every $\eta > 0$. Indeed, if $F \cap (x_0, x_0 + \eta) = \emptyset$ for some $\eta > 0$, then $(x_0, x_0 + \eta) \subseteq \mathcal{O}$; let $I_k = (a_k, b_k)$ be the interval of the decomposition containing the point $x := x_0 + \eta/2$. By the construction in Proposition 9 of Chapter 1, I_k is the largest open interval that contains x and is contained in $\mathcal{O}$; since $(x_0, x_0 + \eta)$ is an open interval containing x and contained in $\mathcal{O}$, we get $(x_0, x_0 + \eta) \subseteq I_k$ and hence $a_k \leq x_0$. But $a_k < x_0$ is impossible, since then $x_0 \in (a_k, b_k) \subseteq \mathcal{O}$, contradicting $x_0 \in F$. Hence $a_k = x_0$ and x_0 is the left endpoint of I_k , contrary to the case assumption. This proves the claim.

So choose $c \in F$ with $x_0 < c < x_0 + \delta$, and set $\delta_+ = c - x_0 > 0$. Let $x_0 \leq x < x_0 + \delta_+ = c$. If $x \in F$, then $|x - x_0| < \delta$ and by (D), $|g(x) - g(x_0)| = |f(x) - f(x_0)| < \epsilon$. If $x \in \mathcal{O}$, let I_k be its component. Since $x_0 \in F$ and $x_0 < x$ (as $x_0 \notin \mathcal{O}$), the left endpoint of I_k satisfies $a_k \geq x_0$; in particular a_k is finite and $a_k \in F$. Since $c \in F$, $c \notin I_k$, and $x < c$, the right endpoint satisfies $b_k \leq c$; in particular b_k is finite and $b_k \in F$. Therefore

$$x_0 \leq a_k < b_k \leq c < x_0 + \delta,$$

so both a_k and b_k lie in F within distance δ of x_0 , and (D) gives $|f(a_k) - f(x_0)| < \epsilon$ and $|f(b_k) - f(x_0)| < \epsilon$. By (C), with $t \in (0, 1)$,

$$|g(x) - g(x_0)| = |(1 - t)(f(a_k) - f(x_0)) + t(f(b_k) - f(x_0))| \leq (1 - t)\epsilon + t\epsilon = \epsilon.$$

Replacing ϵ by $\epsilon/2$ at the outset makes this a strict inequality $< \epsilon$, as desired.

The left-hand estimate is the mirror image, and we record it in full since it is the other half of the proof. Either x_0 is the right endpoint of some component I_j of $\mathcal{O}$, or it is not. If $I_j = (-\infty, x_0)$, then $g \equiv f(x_0) = g(x_0)$ on $(-\infty, x_0]$ and any δ_- works. If $I_j = (a_j, x_0)$ is bounded, then g is affine on $[a_j, x_0]$ with $g(x_0) = f(x_0)$, hence continuous from the left at x_0 , and we choose $\delta_- \leq x_0 - a_j$ accordingly. If x_0 is the right endpoint of no component, then $F \cap (x_0 - \eta, x_0) \neq \emptyset$ for every $\eta > 0$: otherwise $(x_0 - \eta, x_0) \subseteq \mathcal{O}$, and the component $I_j = (a_j, b_j)$ containing $x_0 - \eta/2$ would satisfy $(x_0 - \eta, x_0) \subseteq I_j$ by maximality, hence $b_j \geq x_0$, while $b_j > x_0$ would put $x_0 \in I_j \subseteq \mathcal{O}$, contradicting $x_0 \in F$; so $b_j = x_0$, contrary to the case assumption. Choose then $c' \in F$ with $x_0 - \delta < c' < x_0$ and set $\delta_- = x_0 - c'$. For $x_0 - \delta_- = c' < x \leq x_0$: if $x \in F$ then $|x - x_0| < \delta$ and (D) applies; if $x \in \mathcal{O}$, its component $I_j = (a_j, b_j)$ has $b_j \leq x_0$ (else $x_0 \in I_j$) and $a_j \geq c'$ (else $c' \in I_j$), so a_j, b_j are finite, lie in F , and satisfy $x_0 - \delta < c' \leq a_j < b_j \leq x_0$; then (D) and (C) give $|g(x) - g(x_0)| \leq \epsilon$ exactly as before. With ϵ replaced by $\epsilon/2$ at the outset this is $< \epsilon$.

Hence g is continuous at every point of $\mathbb{R}$, and g is a continuous extension of f to all of $\mathbb{R}$.

Problem 3.26. For the function f and the set F in the statement of Lusin's Theorem, show that the restriction of f to F is a continuous function. Must there be any points at which f , considered as a function on E , is continuous?

Solution. Recall the statement of Lusin's Theorem: f is a real-valued measurable function on E , and for a given $\epsilon > 0$ it produces a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which

$$f = g \text{ on } F \quad \text{and} \quad m(E \setminus F) < \epsilon.$$

The restriction of f to F is continuous. Continuity of the restriction $f|_F$ means: for each $x_0 \in F$ and each $\eta > 0$ there is $\delta > 0$ such that $|f(x) - f(x_0)| < \eta$ for all $x \in F$ with $|x - x_0| < \delta$. Fix $x_0 \in F$ and $\eta > 0$. Since g is continuous at x_0 as a function on $\mathbb{R}$, there is $\delta > 0$ such that

$$|g(x) - g(x_0)| < \eta \quad \text{for all } x \in \mathbb{R} \text{ with } |x - x_0| < \delta.$$

Now let $x \in F$ with $|x - x_0| < \delta$. Since $x, x_0 \in F$ and $f = g$ on F ,

$$|f(x) - f(x_0)| = |g(x) - g(x_0)| < \eta.$$

Hence $f|_F$ is continuous at every point of F , that is, $f|_F$ is a continuous function on F . (Equivalently: the restriction of a continuous function on $\mathbb{R}$ to any subset of $\mathbb{R}$ is continuous on that subset, and $f|_F = g|_F$.)

Must f , considered as a function on E , be continuous at some point? No. The continuity supplied by Lusin's Theorem is continuity of the restriction, that is, continuity relative to F ; it says nothing about the behavior of f at points of E near x_0 that lie outside F . There are measurable functions that are continuous at no point of their domain.

The standard example is the Dirichlet function. Let $E = [0, 1]$ and

$$f = \chi_{\mathbb{Q} \cap [0,1]}, \quad \text{that is, } f(x) = 1 \text{ if } x \text{ is rational, } f(x) = 0 \text{ if } x \text{ is irrational.}$$

Then f is measurable, since $\mathbb{Q} \cap [0, 1]$ is countable, hence of measure zero, hence measurable, and the characteristic function of a measurable set is measurable. And $m(E) = 1 < \infty$, so Lusin's Theorem applies as stated in the text.

The function f is continuous at no point of $[0, 1]$. Indeed, let $x_0 \in [0, 1]$ and take $\eta = 1/2$. By the density of the rationals and of the irrationals in $\mathbb{R}$, every interval $(x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ contains a point $x_1 \in [0, 1]$ that is rational and a point $x_2 \in [0, 1]$ that is irrational. Then $|f(x_1) - f(x_2)| = 1$, so at least one of $|f(x_1) - f(x_0)|$, $|f(x_2) - f(x_0)|$ is $\geq 1/2 = \eta$. Hence no δ works, and f is discontinuous at x_0 .

It is instructive to see Lusin's Theorem realized here. Let $\epsilon > 0$ and enumerate $\mathbb{Q} \cap [0, 1] = \{q_k\}_{k=1}^\infty$. Put

$$\mathcal{O} = \bigcup_{k=1}^{\infty} \left(q_k - \frac{\epsilon}{2^{k+2}}, q_k + \frac{\epsilon}{2^{k+2}} \right),$$

an open set with, by countable subadditivity and monotonicity of outer measure,

$$m(\mathcal{O}) \leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2}.$$

Set $F = [0, 1] \setminus \mathcal{O}$. Then F is closed (the intersection of the closed set $[0, 1]$ with the closed set $\mathbb{R} \setminus \mathcal{O}$), $F \subseteq E$, and F contains no rational number, so $f \equiv 0$ on F and f agrees on F with the continuous function $g \equiv 0$ on $\mathbb{R}$. Moreover $E \setminus F \subseteq \mathcal{O}$, so $m(E \setminus F) \leq \epsilon/2 < \epsilon$. Thus the conclusion of Lusin's Theorem holds, the restriction $f|_F$ is continuous (it is identically 0), yet f itself is continuous at no point whatsoever of E – not even at the points of F .

So the answer to the question is: there need not be any point of E at which f is continuous.

Problem 3.27. Show that the conclusion of Egoroff's Theorem can fail if we drop the assumption that the domain has finite measure.

Solution. Take $E = \mathbb{R}$, which is measurable with $m(E) = \infty$, and for each natural number n define

$$f_n = \chi_{[n, \infty)}, \quad \text{that is,} \quad f_n(x) = \begin{cases} 1, & x \geq n, \\ 0, & x < n. \end{cases}$$

Each f_n is measurable, being the characteristic function of the measurable set $[n, \infty)$. Every hypothesis of Egoroff's Theorem except the finiteness of $m(E)$ is satisfied; we show its conclusion fails, and in fact fails for every $\epsilon > 0$.

The sequence converges pointwise on $\mathbb{R}$ to the real-valued function $f \equiv 0$. Indeed, fix $x \in \mathbb{R}$. By the Archimedean property there is a natural number $N > x$, and then for every $n \geq N$ we have $n > x$, so $f_n(x) = 0$. Hence $f_n(x) \rightarrow 0$.

Now let $\epsilon > 0$ and let F be any measurable subset of $\mathbb{R}$ – closed or not – with $m(\mathbb{R} \sim F) < \epsilon$. We claim $F \cap [n, \infty) \neq \emptyset$ for every natural number n . Suppose to the contrary that $F \cap [n, \infty) = \emptyset$ for some n . Then $[n, \infty) \subseteq \mathbb{R} \sim F$, and by monotonicity of measure

$$\infty = m([n, \infty)) \leq m(\mathbb{R} \sim F) < \epsilon < \infty,$$

a contradiction. This proves the claim.

For each n choose $x_n \in F \cap [n, \infty)$. Then $f_n(x_n) = 1$ while $f(x_n) = 0$, so

$$\sup_{x \in F} |f_n(x) - f(x)| \geq |f_n(x_n) - f(x_n)| = 1 \quad \text{for every } n.$$

Therefore $\{f_n\}$ does not converge uniformly to f on F : taking $\eta = 1/2$ in the definition of uniform convergence, there is no index N for which $|f_n - f| < 1/2$ on F for all $n \geq N$.

Since F was an arbitrary measurable set with $m(\mathbb{R} \sim F) < \epsilon$, and closed sets are in particular measurable, there is no closed set $F \subseteq \mathbb{R}$ with $m(\mathbb{R} \sim F) < \epsilon$ on which $\{f_n\}$ converges uniformly to f . Thus the conclusion of Egoroff's Theorem fails here for every $\epsilon > 0$, and the finiteness of $m(E)$ cannot simply be dropped.

Remark. A variant on a set of infinite measure that fails only for small ϵ is $f_n = \chi_{[n, n+1)}$ on $\mathbb{R}$, which also converges pointwise to 0. If $m(\mathbb{R} \sim F) < 1$, then for each n we have $m([n, n+1) \sim F) < 1 = m([n, n+1))$, so $[n, n+1) \cap F \neq \emptyset$, and the same argument shows $\sup_F |f_n| \geq 1$ for all n . Note the essential role of infinite measure in both examples: the sets on which f_n differs from its limit march off to infinity without shrinking in measure, and a set of finite measure cannot contain such a sequence of sets of measure bounded away from 0 while leaving only a small complement.

Problem 3.28. Show that Egoroff's Theorem continues to hold if the convergence is pointwise a.e. and f is finite a.e.

Solution. The assertion to be proved is the following. Let E have finite measure, let $\{f_n\}$ be a sequence of measurable functions on E , and let f be an extended real-valued function on E that is finite a.e. on E and such that $\{f_n\}$ converges to f pointwise a.e. on E . Then for each $\epsilon > 0$ there is a closed set F contained in E for which

$$\{f_n\} \rightarrow f \quad \text{uniformly on } F \quad \text{and} \quad m(E \sim F) < \epsilon.$$

Isolating the exceptional set. Let

$$A = \{x \in E \mid |f(x)| = \infty\}, \quad B = \{x \in E \mid \{f_n(x)\} \text{ does not converge to } f(x)\},$$

and put $E_0 = A \cup B$. By hypothesis, f is finite a.e. on E , so A is contained in a set of measure zero; and $\{f_n\} \rightarrow f$ pointwise a.e. on E , so B is contained in a set of measure zero. Hence E_0 is contained in the union of two sets of measure zero, which by the countable subadditivity and monotonicity of outer

measure is a set of outer measure zero; therefore $m^*(E_0) = 0$. Since every set of outer measure zero is measurable (the completeness of Lebesgue measure, established in Chapter 2), E_0 is measurable and $m(E_0) = 0$.

Passing to a good subdomain. Define

$$\tilde{E} = E \sim E_0.$$

Then $\tilde{E}$ is measurable, being the difference of two measurable sets, and $m(\tilde{E}) \leq m(E) < \infty$, so $\tilde{E}$ has finite measure. Each f_n restricted to $\tilde{E}$ is measurable: for every $c \in \mathbb{R}$,

$$\{x \in \tilde{E} \mid f_n(x) > c\} = \tilde{E} \cap \{x \in E \mid f_n(x) > c\},$$

the intersection of two measurable sets. By the very definition of E_0 , on $\tilde{E}$ the function f is real-valued and $\{f_n\}$ converges to f at every point.

Applying Egoroff's Theorem. Thus $\tilde{E}$ has finite measure, $\{f_n\}$ is a sequence of measurable functions on $\tilde{E}$, and it converges pointwise on $\tilde{E}$ to the real-valued function f . These are exactly the hypotheses of Egoroff's Theorem. Hence, given $\epsilon > 0$, there is a closed set F contained in $\tilde{E}$ for which

$$\{f_n\} \rightarrow f \text{ uniformly on } F \quad \text{and} \quad m(\tilde{E} \sim F) < \epsilon.$$

Conclusion. The set F is closed and $F \subseteq \tilde{E} \subseteq E$, so F is a closed subset of E , and $\{f_n\}$ converges to f uniformly on F . It remains to estimate $m(E \sim F)$. Since $F \subseteq \tilde{E} = E \sim E_0$,

$$E \sim F = (E \sim \tilde{E}) \cup (\tilde{E} \sim F) = E_0 \cup (\tilde{E} \sim F),$$

so by the countable (here finite) subadditivity of measure,

$$m(E \sim F) \leq m(E_0) + m(\tilde{E} \sim F) = 0 + m(\tilde{E} \sim F) < \epsilon.$$

Therefore the conclusion of Egoroff's Theorem holds verbatim under the weakened hypotheses of pointwise a.e. convergence to a function that is finite a.e. This is the assertion made in the remark immediately following the proof of Egoroff's Theorem in the text.

3.5 Problems 3.29–3.31

Problem 3.29. Prove the extension of Lusin's Theorem to the case that E has infinite measure.

Solution. We must show: if f is a real-valued measurable function on a measurable set E with $m(E) = \infty$, then for each $\epsilon > 0$ there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E \sim F) < \epsilon$. The proof of Lusin's Theorem given in the text establishes this when $m(E) < \infty$; we bootstrap from that case.

Let $\epsilon > 0$ be given. For each integer $n \in \mathbb{Z}$ define

$$A_n = E \cap (n, n+1), \quad Z = E \cap \mathbb{Z}.$$

Each A_n is measurable, being the intersection of two measurable sets, and $m(A_n) \leq 1 < \infty$. The set Z is countable, so $m(Z) = 0$. Since every real number lies either in $\mathbb{Z}$ or in exactly one interval $(n, n+1)$,

$$E = Z \cup \bigcup_{n \in \mathbb{Z}} A_n,$$

and the sets Z and $\{A_n\}_{n \in \mathbb{Z}}$ are pairwise disjoint.

Fix a bijection $k \mapsto n_k$ of the natural numbers onto $\mathbb{Z}$. For each k , the restriction of f to A_{n_k} is real-valued, and it is measurable on A_{n_k} : for each $c \in \mathbb{R}$,

$$\{x \in A_{n_k} \mid f(x) > c\} = \{x \in E \mid f(x) > c\} \cap A_{n_k}$$

is the intersection of two measurable sets. Since $m(A_{n_k}) < \infty$, Lusin's Theorem in the finite measure case, which is the case proved in the text, may be applied to f on A_{n_k} : there is a continuous function

g_k on $\mathbb{R}$ and a closed set F_{n_k} contained in A_{n_k} for which

$$f = g_k \text{ on } F_{n_k} \quad \text{and} \quad m(A_{n_k} \sim F_{n_k}) < \epsilon/2^{k+1}.$$

Define

$$F = \bigcup_{n \in \mathbb{Z}} F_n \subseteq E.$$

We first record a separation property, to which we refer below as the localization identity. For each n , $F_n \subseteq A_n \subseteq (n, n+1)$, so the sets F_n are pairwise disjoint and, more importantly, $F_m \cap (n, n+1) = \emptyset$ whenever $m \neq n$; hence

$$F \cap (n, n+1) = F_n \quad \text{for every } n \in \mathbb{Z}.$$

F is closed. Let x be a point of closure of F and choose a sequence $\{x_j\}$ in F with $x_j \rightarrow x$. A convergent sequence of real numbers is bounded, say $|x_j| \leq M$ for all j . If $x_j \in F_n$ then $n < x_j < n+1$, so $-M-1 \leq n \leq M$, and thus every x_j lies in one of the finitely many sets F_n with $-M-1 \leq n \leq M$. Therefore some single index n occurs infinitely often, that is, there is a subsequence lying entirely in F_n . This subsequence also converges to x , and F_n is closed, so $x \in F_n \subseteq F$. Thus F contains all its points of closure and so is closed.

$m(E \sim F) < \epsilon$. The localization identity gives $A_n \sim F = A_n \sim F_n$ for each n , and therefore, by the decomposition of E recorded above,

$$E \sim F \subseteq Z \cup \bigcup_{k=1}^{\infty} (A_{n_k} \sim F_{n_k}).$$

Monotonicity and countable subadditivity of measure now yield

$$m(E \sim F) \leq m(Z) + \sum_{k=1}^{\infty} m(A_{n_k} \sim F_{n_k}) \leq 0 + \sum_{k=1}^{\infty} \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2} < \epsilon.$$

The restriction of f to F is continuous. Since F is a subset of $\mathbb{R}$, and hence a metric space, it suffices to verify sequential continuity. Let $x \in F$, say $x \in F_n$, and let $\{x_j\}$ be a sequence in F converging to x . Since $x \in (n, n+1)$ and this interval is open, $x_j \in (n, n+1)$ for all sufficiently large j , and then $x_j \in F \cap (n, n+1) = F_n$ by the localization identity. For such j we have $f(x_j) = g_k(x_j)$ and $f(x) = g_k(x)$, where $n = n_k$; since g_k is continuous on $\mathbb{R}$, $f(x_j) = g_k(x_j) \rightarrow g_k(x) = f(x)$. Hence the restriction of f to F is continuous at each point of F .

Finally, f restricted to F is a continuous function on the closed set F , so by Problem 25 it has a continuous extension g to all of $\mathbb{R}$. Then g is continuous on $\mathbb{R}$, F is a closed subset of E , $f = g$ on F , and $m(E \sim F) < \epsilon$. This is exactly the conclusion of Lusin's Theorem, now established without any assumption on $m(E)$.

Problem 3.30. Prove the extension of Lusin's Theorem to the case that f is not necessarily real-valued, but may be finite a.e.

Solution. We must show: if f is an extended real-valued measurable function on a measurable set E which is finite a.e. on E , then for each $\epsilon > 0$ there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E \sim F) < \epsilon$.

Let $\epsilon > 0$ be given. Define

$$E_{\infty} = \{x \in E \mid f(x) = \infty\}, \quad E_{-\infty} = \{x \in E \mid f(x) = -\infty\}, \quad E_0 = E_{\infty} \cup E_{-\infty}.$$

Since f is measurable,

$$E_{\infty} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > n\} \quad \text{and} \quad E_{-\infty} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) < -n\}$$

are measurable, being countable intersections of measurable sets; hence E_0 is measurable. By hypothesis f is finite a.e. on E , which says precisely that $m(E_0) = 0$.

Put $E' = E \sim E_0$. Then E' is measurable, and the restriction of f to E' is a real-valued function which is measurable on E' : for each $c \in \mathbb{R}$,

$$\{x \in E' \mid f(x) > c\} = \{x \in E \mid f(x) > c\} \cap E'$$

is the intersection of two measurable sets.

Apply Lusin's Theorem to the real-valued measurable function f on E' – in the case $m(E') < \infty$ this is the theorem as proved in the text, and in the case $m(E') = \infty$ it is the extension established in Problem 29, so no restriction on $m(E')$ is needed. We obtain a continuous function g on $\mathbb{R}$ and a closed set F contained in E' for which

$$f = g \text{ on } F \quad \text{and} \quad m(E' \sim F) < \epsilon.$$

This F and g do what is required for f on E . Indeed, $F \subseteq E' \subseteq E$ and F is closed, g is continuous on all of $\mathbb{R}$, and $f = g$ on F . Moreover, since

$$E \sim F = (E_0 \sim F) \cup (E' \sim F) \subseteq E_0 \cup (E' \sim F),$$

the monotonicity and finite subadditivity of measure give

$$m(E \sim F) \leq m(E_0) + m(E' \sim F) = 0 + m(E' \sim F) < \epsilon.$$

Thus Lusin's Theorem persists when f is merely extended real-valued and finite a.e. Observe that the a.e. finiteness cannot be dispensed with in this form: on F we have $f = g$ with g real-valued, so f is necessarily finite on F , and therefore the set where f is infinite must be contained in $E \sim F$, a set of measure less than ϵ for every $\epsilon > 0$; hence f must be finite a.e. for the conclusion to be possible at all.

Problem 3.31. Let $\{f_n\}$ be a sequence of measurable functions on E that converges to the real-valued f pointwise on E . Show that $E = \bigcup_{k=1}^{\infty} E_k$, where for each index k , E_k is measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 1$, and $m(E_1) = 0$.

Solution. Step 1: the case $m(E) < \infty$. Assume first that E has finite measure. Then Egoroff's Theorem applies to $\{f_n\}$ on E : for each integer $k \geq 2$, taking $\epsilon = 1/k$, there is a closed set A_k contained in E for which

$$\{f_n\} \rightarrow f \text{ uniformly on } A_k \quad \text{and} \quad m(E \sim A_k) < 1/k.$$

Define

$$N = E \sim \bigcup_{k=2}^{\infty} A_k = \bigcap_{k=2}^{\infty} (E \sim A_k).$$

This set is measurable, being a countable intersection of measurable sets. For each $k \geq 2$, monotonicity of measure gives $m(N) \leq m(E \sim A_k) < 1/k$, and letting $k \rightarrow \infty$ we conclude $m(N) = 0$. By the very definition of N ,

$$E = N \cup \bigcup_{k=2}^{\infty} A_k,$$

which is a decomposition of the required kind: N has measure zero and the convergence is uniform on each A_k .

Step 2: the general case. Now let E be an arbitrary measurable set, of possibly infinite measure. Egoroff's Theorem is not directly available, so we first cut E into countably many pieces of finite measure. Define

$$C_1 = E \cap [-1, 1], \quad C_j = E \cap \{x \mid j-1 < |x| \leq j\} \quad \text{for } j \geq 2.$$

Each C_j is measurable, being the intersection of E with a Borel set, and $m(C_j) \leq 2 < \infty$. Every real number x satisfies $|x| \leq 1$ or $j - 1 < |x| \leq j$ for exactly one $j \geq 2$, so

$$E = \bigcup_{j=1}^{\infty} C_j.$$

Fix j . The restrictions of the f_n to C_j are measurable functions on C_j , since for each $c \in \mathbb{R}$ the set $\{x \in C_j \mid f_n(x) > c\} = \{x \in E \mid f_n(x) > c\} \cap C_j$ is measurable; and $\{f_n\} \rightarrow f$ pointwise on C_j with f real-valued. Since $m(C_j) < \infty$, Step 1 applies on C_j and produces a measurable set $N_j \subseteq C_j$ with $m(N_j) = 0$, together with measurable sets $A_{j,2}, A_{j,3}, \dots$ contained in C_j on each of which $\{f_n\}$ converges uniformly to f , such that

$$C_j = N_j \cup \bigcup_{k=2}^{\infty} A_{j,k}.$$

Now define

$$E_1 = \bigcup_{j=1}^{\infty} N_j.$$

It is measurable as a countable union of measurable sets, and by countable subadditivity

$$m(E_1) \leq \sum_{j=1}^{\infty} m(N_j) = 0,$$

so $m(E_1) = 0$.

The index set $I = \{(j, k) \mid j \geq 1, k \geq 2\}$ is countably infinite, so we may fix a bijection $i \mapsto (j(i), k(i))$ of $\{2, 3, 4, \dots\}$ onto I and define $E_i = A_{j(i), k(i)}$ for $i \geq 2$. Each E_i with $i > 1$ is measurable and $\{f_n\}$ converges uniformly to f on it. (Some of these sets may be empty; uniform convergence on the empty set holds vacuously, so this is harmless.) Finally,

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{j=1}^{\infty} N_j \cup \bigcup_{j=1}^{\infty} \bigcup_{k=2}^{\infty} A_{j,k} = \bigcup_{j=1}^{\infty} \left(N_j \cup \bigcup_{k=2}^{\infty} A_{j,k} \right) = \bigcup_{j=1}^{\infty} C_j = E,$$

and each E_k is contained in E . This is the asserted decomposition.

Two remarks. First, when $m(E) < \infty$ Step 1 alone suffices, and the sets E_k for $k > 1$ may then be taken closed, since Egoroff's Theorem as proved in the text delivers a closed set. In the general case each $A_{j,k}$ is likewise closed, so the E_k with $k > 1$ may again be taken closed. Second, no finite-measure hypothesis on E is needed for the conclusion, even though Egoroff's Theorem itself requires one; the passage to the bounded pieces C_j is what removes that restriction. This is consistent with Problem 27, which shows that on a set of infinite measure one cannot in general find a single set A with $m(E \sim A)$ small and uniform convergence on A : the present decomposition instead uses countably many sets and pushes the exceptional behaviour into one set of measure zero.

4 Lebesgue Integration

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4.1 Problems 4.1–4.7

Problem 4.1. Show that, in the above Dirichlet function example, $\{f_n\}$ fails to converge to f uniformly on $[0, 1]$.

Solution. Recall the construction. The Dirichlet function f on $[0, 1]$ is defined by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Letting $\{q_k\}_{k=1}^\infty$ be an enumeration of the rational numbers in $[0, 1]$, for each natural number n the function f_n is defined on $[0, 1]$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = q_k \text{ for some } k \text{ with } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Each f_n is a step function, hence Riemann integrable: choose a partition P of $[0, 1]$ whose partition points include all of $q_1, \dots, q_n$; then f_n vanishes identically on each open subinterval determined by P , which is exactly the requirement in the definition of a step function (the definition constrains the values only on the open subintervals). Moreover $\{f_n\} \rightarrow f$ pointwise on $[0, 1]$: if x is irrational then $f_n(x) = 0 = f(x)$ for every n , while if x is rational then $x = q_K$ for exactly one index K , and $f_n(x) = 1 = f(x)$ for all $n \geq K$.

We claim the convergence is not uniform. Indeed, we show that

$$\sup_{x \in [0, 1]} |f(x) - f_n(x)| = 1 \quad \text{for every } n.$$

Fix a natural number n . The set $F_n = \{q_1, \dots, q_n\}$ is finite, while the set $\mathbb{Q} \cap [0, 1]$ of rational numbers in $[0, 1]$ is infinite (it contains the distinct points $1/m$ for every natural number m). Therefore there is a rational number $r \in [0, 1]$ with $r \notin F_n$. For this r we have $f(r) = 1$, since r is rational, while $f_n(r) = 0$, since r is not one of $q_1, \dots, q_n$. Hence

$$|f(r) - f_n(r)| = 1,$$

so that $\sup_{x \in [0, 1]} |f(x) - f_n(x)| \geq 1$; since $0 \leq f_n \leq f \leq 1$ on $[0, 1]$, the supremum equals 1.

Now suppose, to get a contradiction, that $\{f_n\} \rightarrow f$ uniformly on $[0, 1]$. Taking $\epsilon = 1/2$ in the definition of uniform convergence, there would be an index N such that

$$|f(x) - f_n(x)| < \frac{1}{2} \quad \text{for all } x \in [0, 1] \text{ and all } n \geq N.$$

Taking $n = N$ and passing to the supremum over $x \in [0, 1]$ gives $1 \leq 1/2$, a contradiction. Therefore $\{f_n\}$ does not converge to f uniformly on $[0, 1]$.

We remark that this failure is forced. As Problem 8 of this section shows, a sequence of Riemann integrable functions on a closed, bounded interval that converges uniformly has a Riemann integrable limit. Since each f_n is Riemann integrable over $[0, 1]$ while f is not, as was verified in the example by exhibiting $L(f, P) = 0$ and $U(f, P) = 1$ for every partition P of $[0, 1]$, the convergence could not possibly have been uniform. The point of the example is precisely that pointwise convergence of a uniformly bounded sequence of Riemann integrable functions is too weak for the Riemann theory, a shortcoming the Lebesgue integral repairs.

Problem 4.2. A partition P' of $[a, b]$ is called a refinement of a partition P provided each partition point of P is also a partition point of P' . For a bounded function f on $[a, b]$, show that under refinement lower Darboux sums increase and upper Darboux sums decrease.

Solution. Throughout, for a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, with $a = x_0 < x_1 < \dots < x_n = b$, we write

$$m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\}, \quad M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\},$$

$$L(f, P) = \sum_{i=1}^n m_i \cdot (x_i - x_{i-1}), \quad U(f, P) = \sum_{i=1}^n M_i \cdot (x_i - x_{i-1}).$$

Since f is bounded, all these infima and suprema are finite real numbers. We must show that if P' refines P , then

$$L(f, P) \leq L(f, P') \quad \text{and} \quad U(f, P') \leq U(f, P).$$

The key monotonicity fact is elementary: if $\emptyset \neq A \subseteq B$ and f is bounded, then

$$\inf_B f \leq \inf_A f \quad \text{and} \quad \sup_A f \leq \sup_B f,$$

since $\inf_B f$ is a lower bound for f on the smaller set A , and $\sup_B f$ is an upper bound for f on A .

Step 1: adjoining a single point. Suppose $P' = P \cup \{c\}$ where $c \notin P$; say $x_{k-1} < c < x_k$. The partitions P and P' have exactly the same subintervals except that the k -th subinterval (x_{k-1}, x_k) of P is replaced in P' by the two subintervals (x_{k-1}, c) and (c, x_k) . Set

$$m' = \inf\{f(x) \mid x_{k-1} < x < c\}, \quad m'' = \inf\{f(x) \mid c < x < x_k\}.$$

All the terms of $L(f, P')$ and $L(f, P)$ indexed by $i \neq k$ agree, so

$$L(f, P') - L(f, P) = m' \cdot (c - x_{k-1}) + m'' \cdot (x_k - c) - m_k \cdot (x_k - x_{k-1}).$$

Since $(x_{k-1}, c) \subseteq (x_{k-1}, x_k)$ and $(c, x_k) \subseteq (x_{k-1}, x_k)$, the monotonicity fact gives $m' \geq m_k$ and $m'' \geq m_k$. Hence

$$L(f, P') - L(f, P) \geq m_k(c - x_{k-1}) + m_k(x_k - c) - m_k(x_k - x_{k-1}) = 0,$$

because $(c - x_{k-1}) + (x_k - c) = x_k - x_{k-1}$. Thus $L(f, P) \leq L(f, P')$.

Symmetrically, with $M' = \sup\{f(x) \mid x_{k-1} < x < c\}$ and $M'' = \sup\{f(x) \mid c < x < x_k\}$ we have $M' \leq M_k$ and $M'' \leq M_k$, so

$$U(f, P') - U(f, P) = M'(c - x_{k-1}) + M''(x_k - c) - M_k(x_k - x_{k-1}) \leq 0,$$

that is, $U(f, P') \leq U(f, P)$.

Step 2: the general case, by induction. Let P' be any refinement of P . Both are finite sets with $P \subseteq P'$, so $P' \setminus P$ is a finite set, say with p elements. We induct on p . If $p = 0$ then $P' = P$ and there is nothing to prove. Assume the assertion holds whenever the refinement adjoins p points, and let P' refine P with $P' \setminus P$ having $p + 1$ elements. Choose $c \in P' \setminus P$ and put $Q = P \cup \{c\}$. Then Q is a refinement of P adjoining a single point, and P' is a refinement of Q with $P' \setminus Q$ having p elements. By Step 1 and the inductive hypothesis,

$$L(f, P) \leq L(f, Q) \leq L(f, P') \quad \text{and} \quad U(f, P') \leq U(f, Q) \leq U(f, P).$$

This completes the induction, and hence the proof: under refinement lower Darboux sums increase and upper Darboux sums decrease.

Finally we note that the same argument works verbatim if, as in the footnote on page 69, the m_i and M_i are taken as infima and suprema over the closed subintervals $[x_{i-1}, x_i]$, since only the inclusion of the new subintervals in the old one was used, and this inclusion holds for the closed subintervals as well.

Problem 4.3. Use the preceding problem to show that for a bounded function on a closed, bounded interval, each lower Darboux sum is no greater than each upper Darboux sum. From this conclude that the lower Riemann integral is no greater than the upper Riemann integral.

Solution. Let f be a bounded function on the closed, bounded interval $[a, b]$, and let P_1 and P_2 be any two partitions of $[a, b]$. We must show $L(f, P_1) \leq U(f, P_2)$.

First observe that for a single partition $P = \{x_0, \dots, x_n\}$ we always have $L(f, P) \leq U(f, P)$. Indeed,

for each i the numbers $m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\}$ and $M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}$ satisfy $m_i \leq M_i$, since the interval (x_{i-1}, x_i) is nonempty and any value $f(x)$ taken there satisfies $m_i \leq f(x) \leq M_i$. As each $x_i - x_{i-1} > 0$, summing gives

$$L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) \leq \sum_{i=1}^n M_i(x_i - x_{i-1}) = U(f, P).$$

Now let $P = P_1 \cup P_2$. Since P is a finite subset of $[a, b]$ containing a and b , it is a partition of $[a, b]$, and by construction every partition point of P_1 and every partition point of P_2 belongs to P . Thus P is a common refinement: it refines P_1 and it refines P_2 . By the preceding problem, lower Darboux sums increase and upper Darboux sums decrease under refinement, so

$$L(f, P_1) \leq L(f, P) \quad \text{and} \quad U(f, P) \leq U(f, P_2).$$

Combining these with $L(f, P) \leq U(f, P)$ yields

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

Hence each lower Darboux sum is no greater than each upper Darboux sum.

Now we conclude the comparison of the integrals. Fix a partition P_2 of $[a, b]$. By what we just proved, the number $U(f, P_2)$ is an upper bound for the set

$$\{L(f, P_1) \mid P_1 \text{ a partition of } [a, b]\}.$$

This set is nonempty and bounded above, and its supremum is by definition the lower Riemann integral. Since the supremum is the least upper bound, we get

$$(R) \int_a^b f = \sup\{L(f, P_1) \mid P_1 \text{ a partition of } [a, b]\} \leq U(f, P_2).$$

As P_2 was an arbitrary partition of $[a, b]$, the number $(R) \int_a^b f$ is a lower bound for the set

$$\{U(f, P_2) \mid P_2 \text{ a partition of } [a, b]\},$$

and since the infimum is the greatest lower bound,

$$(R) \int_a^b f \leq \inf\{U(f, P_2) \mid P_2 \text{ a partition of } [a, b]\} = \overline{(R) \int_a^b f}.$$

Thus the lower Riemann integral is no greater than the upper Riemann integral. This justifies the assertion made in the text that the upper integral is always at least as large as the lower integral, so that Riemann integrability is exactly the assertion that the two coincide.

Problem 4.4. Suppose the bounded function f on $[a, b]$ is Riemann integrable over $[a, b]$. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ for which $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$.

Solution. Since f is Riemann integrable over $[a, b]$, the lower and upper Riemann integrals are equal; call their common value

$$I = (R) \int_a^b f = \overline{(R) \int_a^b f} = (R) \int_a^b f.$$

Since f is bounded, I is a real number.

Fix a natural number n . By definition,

$$I = \sup\{L(f, P) \mid P \text{ a partition of } [a, b]\},$$

so $I - 1/(2n)$ is not an upper bound for the set of lower Darboux sums; hence there is a partition P'_n of $[a, b]$ with

$$L(f, P'_n) > I - \frac{1}{2n}.$$

Likewise,

$$I = \inf\{U(f, P) \mid P \text{ a partition of } [a, b]\},$$

so $I + 1/(2n)$ is not a lower bound for the set of upper Darboux sums; hence there is a partition P''_n of $[a, b]$ with

$$U(f, P''_n) < I + \frac{1}{2n}.$$

Define $P_n = P'_n \cup P''_n$. This is a finite subset of $[a, b]$ containing a and b , hence a partition of $[a, b]$, and it is a refinement of both P'_n and P''_n . By Problem 2, refinement increases lower Darboux sums and decreases upper Darboux sums, so

$$L(f, P_n) \geq L(f, P'_n) > I - \frac{1}{2n}, \quad U(f, P_n) \leq U(f, P''_n) < I + \frac{1}{2n}.$$

Therefore

$$0 \leq U(f, P_n) - L(f, P_n) < \left(I + \frac{1}{2n}\right) - \left(I - \frac{1}{2n}\right) = \frac{1}{n},$$

where the first inequality is the observation, proved in Problem 3, that $L(f, P) \leq U(f, P)$ for every partition P .

Since $1/n \rightarrow 0$, the squeeze principle gives

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0,$$

so $\{P_n\}$ is a sequence of partitions of the required kind.

We add one remark that will be useful in the following problem: the sequence just constructed also satisfies $L(f, P_n) \rightarrow I$ and $U(f, P_n) \rightarrow I$, since both lie in the interval $(I - 1/(2n), I + 1/(2n))$. Thus for a Riemann integrable function the Riemann integral is the limit of the Darboux sums along this sequence of partitions.

Problem 4.5. Let f be a bounded function on $[a, b]$. Suppose there is a sequence $\{P_n\}$ of partitions of $[a, b]$ for which $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$. Show that f is Riemann integrable over $[a, b]$.

Solution. Since f is bounded on the closed, bounded interval $[a, b]$, both the lower Riemann integral

$$(R) \int_a^b f = \sup\{L(f, P) \mid P \text{ a partition of } [a, b]\}$$

and the upper Riemann integral

$$(R) \int_a^b f = \inf\{U(f, P) \mid P \text{ a partition of } [a, b]\}$$

are real numbers: if $|f| \leq M$ on $[a, b]$, then every Darboux sum lies in $[-M(b-a), M(b-a)]$, so the relevant sets are nonempty and bounded. By Problem 3,

$$(R) \int_a^b f \leq (R) \int_a^b f.$$

Fix a natural number n . Since $(R) \int_a^b f$ is the supremum of all lower Darboux sums, it dominates the particular lower sum $L(f, P_n)$; since $(R) \int_a^b f$ is the infimum of all upper Darboux sums, it is dominated

by the particular upper sum $U(f, P_n)$. Combining these with the inequality above gives the chain

$$L(f, P_n) \leq (R) \int_a^b f \leq (R) \int_a^{\overline{b}} f \leq U(f, P_n).$$

Consequently

$$0 \leq (R) \int_a^{\overline{b}} f - (R) \int_a^b f \leq U(f, P_n) - L(f, P_n) \quad \text{for every } n.$$

The left-hand quantity does not depend on n , and by hypothesis the right-hand quantity tends to 0 as $n \rightarrow \infty$. Therefore, letting $n \rightarrow \infty$,

$$0 \leq (R) \int_a^{\overline{b}} f - (R) \int_a^b f \leq 0,$$

so that

$$(R) \int_a^b f = (R) \int_a^{\overline{b}} f.$$

Since f is bounded on $[a, b]$ and its lower and upper Riemann integrals coincide, f is by definition Riemann integrable over $[a, b]$, and its Riemann integral is this common value. Indeed, the displayed chain of inequalities shows moreover that

$$(R) \int_a^b f = \lim_{n \rightarrow \infty} L(f, P_n) = \lim_{n \rightarrow \infty} U(f, P_n),$$

since both $L(f, P_n)$ and $U(f, P_n)$ are squeezed toward the common value by the same estimate.

Together with the preceding problem this gives the Cauchy criterion for Riemann integrability: a bounded function f on $[a, b]$ is Riemann integrable over $[a, b]$ if and only if there is a sequence $\{P_n\}$ of partitions with $U(f, P_n) - L(f, P_n) \rightarrow 0$; equivalently, if and only if for each $\epsilon > 0$ there is a partition P with $U(f, P) - L(f, P) < \epsilon$.

Problem 4.6. Use the preceding problem to show that since a continuous function f on a closed, bounded interval $[a, b]$ is uniformly continuous on $[a, b]$, it is Riemann integrable over $[a, b]$.

Solution. Let f be continuous on the closed, bounded interval $[a, b]$. We may assume $a < b$, the case $a = b$ being trivial.

First, f is bounded on $[a, b]$. Indeed, $[a, b]$ is a nonempty closed, bounded set of real numbers, so by the Extreme Value Theorem of Chapter 1 the continuous function f takes a minimum and a maximum value on $[a, b]$; in particular f is bounded there. Thus the lower and upper Darboux sums and the lower and upper Riemann integrals of f over $[a, b]$ are defined, and the hypothesis of boundedness required in Problem 5 is met.

Second, f is uniformly continuous on $[a, b]$: by Theorem 23 of Chapter 1, a continuous real-valued function on a closed, bounded set of real numbers is uniformly continuous. Written out, for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(u) - f(v)| < \epsilon \quad \text{whenever } u, v \in [a, b] \text{ and } |u - v| < \delta.$$

Now we construct the required sequence of partitions. Fix a natural number n . Apply uniform continuity with $\epsilon = 1/n$ to obtain $\delta_n > 0$ such that

$$|f(u) - f(v)| < \frac{1}{n} \quad \text{whenever } u, v \in [a, b] \text{ and } |u - v| < \delta_n.$$

Choose a natural number k_n so large that $(b - a)/k_n < \delta_n$, which is possible by the Archimedean property of $\mathbb{R}$, and let P_n be the regular partition of $[a, b]$ into k_n subintervals of equal length $(b - a)/k_n$,

that is,

$$P_n = \left\{ x_i = a + i \cdot \frac{b-a}{k_n} \mid 0 \leq i \leq k_n \right\}.$$

Fix i with $1 \leq i \leq k_n$ and set, as usual,

$$m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\}, \quad M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}.$$

For any two points u, v in (x_{i-1}, x_i) we have $|u - v| < x_i - x_{i-1} = (b-a)/k_n < \delta_n$, hence $f(u) - f(v) < 1/n$. Taking the supremum over u and then the infimum over v in this inequality gives

$$M_i - m_i \leq \frac{1}{n}.$$

(In detail: for fixed v , $f(u) < f(v) + 1/n$ for all $u \in (x_{i-1}, x_i)$, so $M_i \leq f(v) + 1/n$; thus $M_i - 1/n$ is a lower bound for f on (x_{i-1}, x_i) , so $M_i - 1/n \leq m_i$.)

Therefore

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{k_n} (M_i - m_i)(x_i - x_{i-1}) \leq \frac{1}{n} \sum_{i=1}^{k_n} (x_i - x_{i-1}) = \frac{b-a}{n}.$$

Since $(b-a)/n \rightarrow 0$ as $n \rightarrow \infty$, and $U(f, P_n) - L(f, P_n) \geq 0$ for every n by Problem 3, we conclude

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

As f is bounded on $[a, b]$ and $\{P_n\}$ is a sequence of partitions of $[a, b]$ with this property, Problem 5 tells us that f is Riemann integrable over $[a, b]$.

Problem 4.7. Let f be an increasing real-valued function on $[0, 1]$. For a natural number n , define P_n to be the partition of $[0, 1]$ into n subintervals of length $1/n$. Show that $U(f, P_n) - L(f, P_n) \leq \frac{1}{n} [f(1) - f(0)]$. Use Problem 5 to show that f is Riemann integrable over $[0, 1]$.

Solution. Let f be increasing on $[0, 1]$, meaning $f(u) \leq f(v)$ whenever $0 \leq u \leq v \leq 1$.

First, f is bounded on $[0, 1]$: for every $x \in [0, 1]$ we have $0 \leq x \leq 1$, hence

$$f(0) \leq f(x) \leq f(1).$$

So the Darboux sums and the lower and upper Riemann integrals of f over $[0, 1]$ are defined. (Note that no continuity is assumed; an increasing function may have infinitely many jump discontinuities.)

Fix a natural number n and let $P_n = \{x_0, x_1, \dots, x_n\}$ with $x_i = i/n$, so that each subinterval has length $x_i - x_{i-1} = 1/n$. Set

$$m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\}, \quad M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}.$$

We claim that

$$f(x_{i-1}) \leq m_i \leq M_i \leq f(x_i) \quad \text{for } 1 \leq i \leq n.$$

Indeed, if $x_{i-1} < x < x_i$, then since f is increasing, $f(x_{i-1}) \leq f(x) \leq f(x_i)$. Thus $f(x_{i-1})$ is a lower bound for f on (x_{i-1}, x_i) and $f(x_i)$ is an upper bound, so $f(x_{i-1}) \leq m_i$ and $M_i \leq f(x_i)$; and $m_i \leq M_i$ because the interval (x_{i-1}, x_i) is nonempty. In particular

$$0 \leq M_i - m_i \leq f(x_i) - f(x_{i-1}).$$

Therefore

$$\begin{aligned}
 U(f, P_n) - L(f, P_n) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\
 &= \frac{1}{n} \sum_{i=1}^n (M_i - m_i) \\
 &\leq \frac{1}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \\
 &= \frac{1}{n} [f(x_n) - f(x_0)] \\
 &= \frac{1}{n} [f(1) - f(0)],
 \end{aligned}$$

the fourth equality holding because the sum telescopes: all the intermediate values $f(x_1), \dots, f(x_{n-1})$ cancel in pairs. This is the asserted estimate.

Finally, the number $f(1) - f(0)$ is a fixed nonnegative real number independent of n , so

$$0 \leq U(f, P_n) - L(f, P_n) \leq \frac{f(1) - f(0)}{n} \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

the left-hand inequality being the fact, established in Problem 3, that $L(f, P) \leq U(f, P)$ for every partition P . By the squeeze principle,

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Since f is a bounded function on $[0, 1]$ and $\{P_n\}$ is a sequence of partitions of $[0, 1]$ with this property, Problem 5 shows that f is Riemann integrable over $[0, 1]$.

The same argument applies on any closed, bounded interval $[a, b]$, with P_n the regular partition into n subintervals of length $(b-a)/n$, yielding $U(f, P_n) - L(f, P_n) \leq \frac{b-a}{n} [f(b) - f(a)]$; hence every monotone function on a closed, bounded interval is Riemann integrable there.

4.2 Problems 4.8–4.14

Problem 4.8. Let $\{f_n\}$ be a sequence of bounded functions that converges uniformly to f on the closed, bounded interval $[a, b]$. If each f_n is Riemann integrable over $[a, b]$, show that f also is Riemann integrable over $[a, b]$. Is it true that

$$\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f?$$

Solution. Throughout, for a bounded function g on $[a, b]$ and a partition $P = \{a = x_0 < x_1 < \dots < x_k = b\}$ we write

$$U(g, P) = \sum_{i=1}^k M_i(g) (x_i - x_{i-1}), \quad L(g, P) = \sum_{i=1}^k m_i(g) (x_i - x_{i-1}),$$

where $M_i(g) = \sup\{g(x) : x_{i-1} \leq x \leq x_i\}$ and $m_i(g) = \inf\{g(x) : x_{i-1} \leq x \leq x_i\}$.

Step 1: f is bounded. Put $\eta_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$, a number in $[0, \infty]$. Uniform convergence of $\{f_n\}$ to f on $[a, b]$ says precisely that for each $\epsilon > 0$ there is an index N with $|f_n - f| < \epsilon$ on all of $[a, b]$ for $n \geq N$, that is, $\eta_n \leq \epsilon$ for $n \geq N$; so η_n is finite for all large n and $\eta_n \rightarrow 0$. In particular there is an index N with $\eta_N \leq 1$. Since f_N is bounded, say $|f_N| \leq M$ on $[a, b]$, we get $|f| \leq |f_N| + |f - f_N| \leq M + 1$ on $[a, b]$. So f is a bounded function on $[a, b]$ and its Darboux sums are defined.

Step 2: a comparison of Darboux sums. Let g and h be bounded on $[a, b]$ and suppose $|g - h| \leq \eta$ on $[a, b]$. Then $g \leq h + \eta$ on $[a, b]$, so for every subinterval $[x_{i-1}, x_i]$ of a partition P we have $M_i(g) \leq M_i(h) + \eta$, and likewise $m_i(g) \geq m_i(h) - \eta$. Multiplying by the lengths $x_i - x_{i-1}$ and summing, and using

$$\sum_{i=1}^k (x_i - x_{i-1}) = b - a,$$

$$U(g, P) \leq U(h, P) + \eta(b - a), \quad L(g, P) \geq L(h, P) - \eta(b - a). \quad (*)$$

Step 3: f is Riemann integrable. We may assume $a < b$, since for $a = b$ every bounded function is trivially Riemann integrable with integral 0. Let k be a natural number. Since $\eta_n \rightarrow 0$, choose an index n_k for which

$$\eta_{n_k} = \sup_{x \in [a, b]} |f_{n_k}(x) - f(x)| < \frac{1}{4k(b - a)}.$$

The function f_{n_k} is Riemann integrable over $[a, b]$, so by Problem 4 of this chapter there is a sequence of partitions of $[a, b]$ along which the difference of the upper and lower Darboux sums of f_{n_k} tends to 0; choose from it a partition P_k of $[a, b]$ for which

$$U(f_{n_k}, P_k) - L(f_{n_k}, P_k) < \frac{1}{2k}.$$

By (*), applied with $g = f$, $h = f_{n_k}$ and $\eta = 1/[4k(b - a)]$,

$$U(f, P_k) - L(f, P_k) \leq [U(f_{n_k}, P_k) - L(f_{n_k}, P_k)] + 2 \cdot \frac{1}{4k(b - a)} \cdot (b - a) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.$$

Thus $\{P_k\}$ is a sequence of partitions of $[a, b]$ with $\lim_{k \rightarrow \infty} [U(f, P_k) - L(f, P_k)] = 0$, and therefore f is Riemann integrable over $[a, b]$ by Problem 5.

Step 4: the integrals converge. Yes, it is true that $\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f$. Indeed, fix an index n large enough that $\eta_n < \infty$ (by Step 1 all but finitely many indices are such) and apply (*) with $g = f$, $h = f_n$, $\eta = \eta_n$: for every partition P ,

$$U(f, P) \leq U(f_n, P) + \eta_n(b - a).$$

Taking the infimum over all partitions P of $[a, b]$ on both sides gives the corresponding inequality for the upper Riemann integrals,

$$(R) \int_a^b f \leq (R) \int_a^b f_n + \eta_n(b - a).$$

Both f and f_n are Riemann integrable, so their upper Riemann integrals equal their integrals, and hence

$$\int_a^b f \leq \int_a^b f_n + \eta_n(b - a).$$

Interchanging the roles of f and f_n (the hypothesis $|f - f_n| \leq \eta_n$ is symmetric) gives $\int_a^b f_n \leq \int_a^b f + \eta_n(b - a)$. Together,

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \eta_n(b - a).$$

Since $\eta_n \rightarrow 0$ by uniform convergence, the right-hand side tends to 0, and therefore $\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f$.

Note that uniform convergence is essential for both conclusions: the Dirichlet function example of this section exhibits a uniformly bounded, increasing sequence of Riemann integrable functions on $[0, 1]$ converging pointwise to a function that is not Riemann integrable.

Problem 4.9. Let E have measure zero. Show that if f is a bounded function on E , then f is measurable and $\int_E f = 0$.

Solution. f is measurable. First note that E itself is measurable: it has outer measure zero, so Proposition 4 of Chapter 2 applies. Hence it makes sense to ask whether a function on E is measurable

and to form $\int_E$. Now let c be a real number and set

$$A_c = \{x \in E \mid f(x) > c\}.$$

Then $A_c \subseteq E$, so by the monotonicity of Lebesgue outer measure, $m^*(A_c) \leq m^*(E) = m(E) = 0$, that is, A_c has outer measure zero. By Proposition 4 of Chapter 2, a set of outer measure zero is measurable. Hence A_c is measurable for every c , and therefore f is a measurable function on E . (Note that no hypothesis beyond $m(E) = 0$ is needed here: **every** real-valued function on a set of measure zero is measurable. Boundedness is only used below, to make the integral of a bounded function over a set of finite measure meaningful.)

$\int_E f = 0$. Since $m(E) = 0 < \infty$, E is a set of finite measure, and f is a bounded measurable function on E ; so by Theorem 4, f is integrable over E and $\int_E f$ is the common value of the upper and lower Lebesgue integrals of f over E .

We claim that **every** simple function φ on E has $\int_E \varphi = 0$. Indeed, let $\varphi = \sum_{i=1}^n a_i \cdot \chi_{E_i}$ be the canonical representation of φ on E , so that the E_i are measurable subsets of E . By monotonicity of measure, $m(E_i) \leq m(E) = 0$ for each i , so $m(E_i) = 0$, and by the definition of the integral of a simple function,

$$\int_E \varphi = \sum_{i=1}^n a_i \cdot m(E_i) = \sum_{i=1}^n a_i \cdot 0 = 0.$$

Now, f is bounded, say $|f| \leq M$ on E . The constant function $\varphi \equiv -M$ is simple with $\varphi \leq f$ on E , and the constant function $\psi \equiv M$ is simple with $f \leq \psi$ on E ; thus the families of simple functions occurring in the definitions of the lower and upper Lebesgue integrals are nonempty. By the claim, every simple $\varphi \leq f$ has $\int_E \varphi = 0$ and every simple $\psi \geq f$ has $\int_E \psi = 0$. Therefore the lower Lebesgue integral of f over E equals

$$\sup \left\{ \int_E \varphi \mid \varphi \text{ simple, } \varphi \leq f \text{ on } E \right\} = \sup\{0\} = 0,$$

and likewise the upper Lebesgue integral equals $\inf\{0\} = 0$. The two agree, confirming integrability, and

$$\int_E f = 0.$$

Problem 4.10. Let f be a bounded measurable function on a set of finite measure E . For a measurable subset A of E , show that $\int_A f = \int_E f \cdot \chi_A$.

Solution. First observe that both integrals are defined. The restriction of f to A is bounded and measurable on A , and $m(A) \leq m(E) < \infty$ by monotonicity of measure, so $\int_A f$ is defined by Theorem 4. Likewise $f \cdot \chi_A$ is bounded on E (by the same bound as f) and measurable on E , being the product of two measurable functions, so $\int_E f \cdot \chi_A$ is defined by Theorem 4. In particular the upper and lower Lebesgue integrals of $f \cdot \chi_A$ over E both equal $\int_E f \cdot \chi_A$.

If A is empty both sides are zero, since $f \cdot \chi_A$ is then the zero function on E , whose integral over E is 0 by Lemma 1, and $\int_\emptyset f = 0$; so assume A is nonempty.

A device for transporting simple functions. Let σ be a simple function on A , with canonical representation $\sigma = \sum_{i=1}^n a_i \cdot \chi_{A_i}$ on A , where the $A_i = \sigma^{-1}(a_i)$ are disjoint measurable subsets of A whose union is A ; they are measurable subsets of E as well. Define $\tilde{\sigma}$ on E by

$$\tilde{\sigma} = \sigma \text{ on } A, \quad \tilde{\sigma} = 0 \text{ on } E \sim A.$$

Then $\tilde{\sigma}$ takes only finitely many values and is measurable on E , so it is simple on E , and

$$\tilde{\sigma} = \sum_{i=1}^n a_i \cdot \chi_{A_i} + 0 \cdot \chi_{E \sim A} \text{ on } E,$$

a representation over the finite disjoint collection $\{A_1, \dots, A_n, E \sim A\}$ of measurable subsets of E . By

Lemma 1,

$$\int_E \tilde{\sigma} = \sum_{i=1}^n a_i \cdot m(A_i) + 0 \cdot m(E \sim A) = \sum_{i=1}^n a_i \cdot m(A_i) = \int_A \sigma. \quad (\dagger)$$

The inequality $\int_E f \cdot \chi_A \leq \int_A f$. Let ψ be a simple function on A with $f \leq \psi$ on A . On A we have $f \cdot \chi_A = f \leq \psi = \tilde{\psi}$, while on $E \sim A$ we have $f \cdot \chi_A = 0 = \tilde{\psi}$. Hence $f \cdot \chi_A \leq \tilde{\psi}$ on E , and since $\int_E f \cdot \chi_A$ equals the upper Lebesgue integral of $f \cdot \chi_A$ over E , which is an infimum over all such competitors, $(\dagger)$ gives

$$\int_E f \cdot \chi_A \leq \int_E \tilde{\psi} = \int_A \psi.$$

Taking the infimum over all simple ψ on A with $\psi \geq f$, and using that this infimum is the upper Lebesgue integral of f over A , which equals $\int_A f$, we obtain $\int_E f \cdot \chi_A \leq \int_A f$.

The inequality $\int_E f \cdot \chi_A \geq \int_A f$. Let φ be a simple function on A with $\varphi \leq f$ on A . On A we have $\tilde{\varphi} = \varphi \leq f = f \cdot \chi_A$, and on $E \sim A$ we have $\tilde{\varphi} = 0 = f \cdot \chi_A$. Hence $\tilde{\varphi} \leq f \cdot \chi_A$ on E , so, $\int_E f \cdot \chi_A$ being also the lower Lebesgue integral of $f \cdot \chi_A$ over E ,

$$\int_A \varphi = \int_E \tilde{\varphi} \leq \int_E f \cdot \chi_A.$$

Taking the supremum over all simple $\varphi \leq f$ on A gives $\int_A f \leq \int_E f \cdot \chi_A$. Combining the two inequalities,

$$\int_A f = \int_E f \cdot \chi_A.$$

Problem 4.11. Does the Bounded Convergence Theorem hold for the Riemann integral?

Solution. No. The theorem fails for the Riemann integral, and it fails in the strongest possible way: the hypotheses can hold while the Riemann integral of the limit function does not even exist, so the asserted conclusion is meaningless.

The counterexample. This is exactly the Dirichlet function example of Section 4.1. The set of rational numbers in $[0, 1]$ is countable; let $\{q_k\}_{k=1}^\infty$ be an enumeration of it. For each natural number n define f_n on $[0, 1]$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = q_k \text{ for some } k \text{ with } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Each f_n is a step function, hence Riemann integrable over $[0, 1]$; indeed f_n differs from the zero function at only finitely many points, so for every $\epsilon > 0$ one can cover those n points by finitely many intervals of total length less than ϵ and thereby produce a partition P with $U(f_n, P) - L(f_n, P) < \epsilon$. Consequently

$$(R) \int_0^1 f_n = 0 \quad \text{for every } n.$$

Moreover the sequence is uniformly bounded, $|f_n| \leq 1$ on $[0, 1]$ for all n , and it is increasing, so it converges pointwise on $[0, 1]$ to the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Thus every hypothesis of the Bounded Convergence Theorem is satisfied, with $E = [0, 1]$, $M = 1$, and each f_n Riemann integrable. But, as shown in Section 4.1, by the density of the rationals and of the irrationals every partition P of $[0, 1]$ has $L(f, P) = 0$ and $U(f, P) = 1$, so

$$(R) \int_0^1 f = 0 < 1 = \overline{(R) \int_0^1 f},$$

and f is not Riemann integrable over $[0, 1]$. Hence $\lim_{n \rightarrow \infty} (R) \int_0^1 f_n$ exists (it is 0) but there is no number $(R) \int_0^1 f$ for it to equal. The Bounded Convergence Theorem therefore does not hold for the Riemann integral.

A remark on what does survive. If one adds to the hypotheses the assumption that the limit function f is itself Riemann integrable, then the conclusion is true; but this is a theorem about the Lebesgue integral in disguise. Suppose each f_n , and also f , is Riemann integrable over the closed, bounded interval $[a, b]$, that $|f_n| \leq M$ on $[a, b]$ for all n , and that $\{f_n\} \rightarrow f$ pointwise on $[a, b]$.

To invoke the Bounded Convergence Theorem we need the f_n to be measurable. That a bounded Riemann integrable function is measurable is the converse of Theorem 4, which the text states but defers to Theorem 7 of Chapter 5; for a closed, bounded interval it can be proved with the tools at hand, as follows. Let g be Riemann integrable over $[a, b]$. By the reformulation of the upper and lower Riemann integrals in terms of step functions used in the proof of Theorem 3, for each natural number n there are step functions $\varphi_n \leq g \leq \psi_n$ on $[a, b]$ with $(R) \int_a^b \varphi_n > (R) \int_a^b g - 1/(2n)$ and $(R) \int_a^b \psi_n < (R) \int_a^b g + 1/(2n)$, so that $(R) \int_a^b (\psi_n - \varphi_n) < 1/n$. Replacing φ_n by $\max\{\varphi_1, \dots, \varphi_n\}$ and ψ_n by $\min\{\psi_1, \dots, \psi_n\}$ leaves step functions (a maximum or a minimum of finitely many step functions is again a step function, as one sees by passing to a common refinement of their partitions) that still satisfy $\varphi_n \leq g \leq \psi_n$ and, since the new $\psi_n - \varphi_n$ is no larger than the old one, still satisfy the estimate; so we may assume $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing. Then $\{\varphi_n\}$ is increasing and bounded above by the bounded function g , and $\{\psi_n\}$ is decreasing and bounded below by g , so both converge pointwise on $[a, b]$, say to φ_* and ψ_* . Step functions are simple, hence measurable, so φ_* and ψ_* are measurable by Proposition 9 of Chapter 3, and $\varphi_* \leq g \leq \psi_*$ on $[a, b]$. Put $h = \psi_* - \varphi_*$, a nonnegative bounded measurable function with $h \leq \psi_n - \varphi_n$ for every n . Since the Lebesgue and Riemann integrals of a step function agree, the monotonicity and linearity of integration, Theorem 5, give

$$0 \leq \int_{[a,b]} h \leq \int_{[a,b]} (\psi_n - \varphi_n) = (R) \int_a^b (\psi_n - \varphi_n) < \frac{1}{n} \quad \text{for every } n,$$

so $\int_{[a,b]} h = 0$. For a natural number k let $E_k = \{x \in [a, b] \mid h(x) > 1/k\}$, a measurable set. Since $(1/k) \cdot \chi_{E_k} \leq h$ on $[a, b]$, Lemma 1 and monotonicity give $(1/k) \cdot m(E_k) \leq \int_{[a,b]} h = 0$, so $m(E_k) = 0$; by countable subadditivity of measure, $\{x \mid h(x) > 0\} = \bigcup_{k=1}^{\infty} E_k$ has measure zero. Hence $\psi_* = \varphi_*$ a.e. on $[a, b]$, and since $\varphi_* \leq g \leq \psi_*$ we get $g = \varphi_*$ a.e. on $[a, b]$. By Proposition 5 of Chapter 3, g is measurable.

Applying this to each f_n , the sequence $\{f_n\}$ consists of measurable functions on $[a, b]$, a set of finite measure $b - a$, is uniformly bounded by M , and converges pointwise to f . The Bounded Convergence Theorem therefore gives $\lim_{n \rightarrow \infty} \int_{[a,b]} f_n = \int_{[a,b]} f$, and by Theorem 3 each of these Lebesgue integrals equals the corresponding Riemann integral, so $\lim_{n \rightarrow \infty} (R) \int_a^b f_n = (R) \int_a^b f$. So the failure exhibited above is precisely a failure of the class of Riemann integrable functions to be closed under uniformly bounded pointwise limits; this closure defect is one of the principal shortcomings of the Riemann integral that the Lebesgue integral removes.

Problem 4.12. Let f be a bounded measurable function on a set of finite measure E . Assume g is bounded and $f = g$ a.e. on E . Show that $\int_E f = \int_E g$.

Solution. By definition, $f = g$ a.e. on E means there is a subset E_0 of E with $m(E_0) = 0$ for which

$$f = g \quad \text{on } E \sim E_0.$$

(The set $\{x \in E \mid f(x) \neq g(x)\}$ is contained in E_0 ; being a subset of a set of measure zero it has outer measure zero and so is itself measurable, by Proposition 4 of Chapter 2. We may therefore assume E_0 is measurable, replacing it by that set if necessary. Also $E \sim E_0$ is measurable, being the difference of two measurable sets.)

Step 1: g is measurable on E . This is immediate from Proposition 5 of Chapter 3, according to which a function that agrees almost everywhere on E with a measurable function is itself measurable. For

completeness we give the verification. Let c be a real number. Then

$$\{x \in E \mid g(x) > c\} = (\{x \in E \mid f(x) > c\} \cap (E \sim E_0)) \cup \{x \in E_0 \mid g(x) > c\},$$

since $g = f$ off E_0 . The first set is the intersection of two measurable sets, hence measurable. The second set is a subset of E_0 , hence has outer measure zero by monotonicity of outer measure, hence is measurable by Proposition 4 of Chapter 2. A union of two measurable sets is measurable, so $\{x \in E \mid g > c\}$ is measurable for every c , and therefore g is measurable on E .

Step 2: both integrals exist. f and g are bounded measurable functions on the set E of finite measure, so by Theorem 4 both are integrable over E , and $\int_E f$ and $\int_E g$ are defined.

Step 3: the difference has integral zero. Put $h = g - f$ on E . Then h is bounded and measurable on E , and $h = 0$ on $E \sim E_0$. Choose $M \geq 0$ with $|h| \leq M$ on E (possible since f and g are bounded). Then

$$0 \leq |h| \leq M \cdot \chi_{E_0} \quad \text{on } E,$$

because on E_0 we have $|h| \leq M = M \cdot \chi_{E_0}$, while on $E \sim E_0$ both sides are 0. The function $M \cdot \chi_{E_0} = M \cdot \chi_{E_0} + 0 \cdot \chi_{E \sim E_0}$ is simple on E , expressed over the finite disjoint collection $\{E_0, E \sim E_0\}$ of measurable subsets of E , so by Lemma 1

$$\int_E M \cdot \chi_{E_0} = M \cdot m(E_0) + 0 \cdot m(E \sim E_0) = M \cdot 0 = 0.$$

The functions $|h|$ and $M \cdot \chi_{E_0}$ are bounded and measurable on E , so by the monotonicity of integration, Theorem 5, applied to the inequalities $0 \leq |h| \leq M \cdot \chi_{E_0}$,

$$0 = \int_E 0 \leq \int_E |h| \leq \int_E M \cdot \chi_{E_0} = 0,$$

whence $\int_E |h| = 0$.

Step 4: conclusion. By Corollary 7 and then Step 3,

$$\left| \int_E g - \int_E f \right| = \left| \int_E (g - f) \right| = \left| \int_E h \right| \leq \int_E |h| = 0,$$

where the first equality uses the linearity of integration, Theorem 5. Therefore

$$\int_E f = \int_E g.$$

Thus the Lebesgue integral of a bounded measurable function over a set of finite measure is unchanged by altering the function on a set of measure zero. Note that the statement of the problem does not assume g to be measurable, and it does not need to: measurability of g is a consequence of the measurability of f and the equality $f = g$ a.e. on E , as Step 1 shows. It is, however, essential that g be assumed bounded, since otherwise $\int_E g$ need not be defined by the theory of this section.

Problem 4.13. Does the Bounded Convergence Theorem hold if $m(E) < \infty$ but we drop the assumption that the sequence $\{f_n\}$ is uniformly bounded on E ?

Solution. No. Uniform boundedness cannot be dropped, even for a set of finite measure and even if each individual f_n is bounded and the convergence is pointwise everywhere.

A counterexample. Take $E = [0, 1]$, so $m(E) = 1 < \infty$. For each natural number n define

$$f_n = n \cdot \chi_{(0, 1/n]} \quad \text{on } [0, 1],$$

that is, $f_n(x) = n$ if $0 < x \leq 1/n$ and $f_n(x) = 0$ otherwise.

Each f_n is a simple function on $[0, 1]$: it takes the two values n and 0 , on the measurable sets $(0, 1/n]$ and $[0, 1] \sim (0, 1/n]$ respectively. In particular each f_n is a bounded measurable function on E , so

$\int_E f_n$ is defined, and by Lemma 1,

$$\int_{[0,1]} f_n = n \cdot m((0, 1/n]) + 0 \cdot m([0, 1] \sim (0, 1/n]) = n \cdot \frac{1}{n} = 1.$$

The sequence converges pointwise on $[0, 1]$ to $f \equiv 0$. Indeed, $f_n(0) = 0$ for every n ; and if $0 < x \leq 1$, choose by the Archimedean property an index N with $1/N < x$, so that for all $n \geq N$ we have $1/n \leq 1/N < x$ and hence $f_n(x) = 0$. Thus $\{f_n\} \rightarrow f$ pointwise on $[0, 1]$, where $f \equiv 0$ is bounded and measurable with

$$\int_{[0,1]} f = 0.$$

Every hypothesis of the Bounded Convergence Theorem holds here except uniform boundedness: the f_n are measurable on a set of finite measure and converge pointwise on E . But

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n = 1 \neq 0 = \int_{[0,1]} f,$$

so the conclusion fails. Of course $\{|f_n|\}$ is not uniformly bounded: $\sup_{[0,1]} |f_n| = n \rightarrow \infty$, so there is no single $M \geq 0$ with $|f_n| \leq M$ on E for all n .

Remark. This is essentially the example given in this section just before the statement of the Bounded Convergence Theorem, where f_n is taken to be the continuous “tent” of height n that vanishes for $x \geq 2/n$, satisfies $f_n(1/n) = n$ and $f_n(0) = 0$, and is linear on $[0, 1/n]$ and on $[1/n, 2/n]$; that f_n also has $\int_0^1 f_n = 1$ for every n while $\{f_n\} \rightarrow 0$ pointwise on $[0, 1]$. The mechanism is the same in both cases: a fixed amount of “mass,” namely 1, is carried on sets of measure tending to 0 by functions of height tending to ∞ , so in the limit the mass escapes rather than being distributed by the limit function. A uniform bound M prevents this, since it forces the integral carried on a set A to be at most $M \cdot m(A)$, which is small when $m(A)$ is small.

Note also that the failure is not caused by any pathology of the limit: here $f \equiv 0$ is as well behaved as possible, and the convergence is pointwise at every point of E , not merely almost everywhere. Nor can it be repaired by shrinking E : the example lives on a set of measure 1. Uniform boundedness is genuinely needed, and it is exactly what the proof of the Bounded Convergence Theorem uses, through the term $2M \cdot m(E \sim A)$ in inequality (7), to control the integral over the small set on which Egoroff’s Theorem does not deliver uniform convergence.

Problem 4.14. Show that Proposition 8 is a special case of the Bounded Convergence Theorem.

Solution. Proposition 8 asserts: if $\{f_n\}$ is a sequence of bounded measurable functions on a set of finite measure E and $\{f_n\} \rightarrow f$ uniformly on E , then $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$. The Bounded Convergence Theorem asserts the same conclusion under the hypotheses that the f_n are measurable on the set E of finite measure, that the sequence is uniformly pointwise bounded (there is an $M \geq 0$ with $|f_n| \leq M$ on E for all n), and that $\{f_n\} \rightarrow f$ pointwise on E .

So it suffices to show that the hypotheses of Proposition 8 imply those of the Bounded Convergence Theorem. Two of them are immediate: the functions f_n are measurable on E , and $m(E) < \infty$, by assumption. It remains to verify pointwise convergence and uniform boundedness.

Uniform convergence implies pointwise convergence. Let $x \in E$ and let $\epsilon > 0$. By uniform convergence there is an index N , depending on ϵ alone, such that $|f_n - f| < \epsilon$ on all of E for $n \geq N$; in particular $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. Hence $\{f_n(x)\} \rightarrow f(x)$. As $x \in E$ was arbitrary, $\{f_n\} \rightarrow f$ pointwise on E .

Uniform convergence, together with the boundedness of each f_n , implies uniform boundedness. Apply the definition of uniform convergence with $\epsilon = 1$: there is an index N for which

$$|f_n - f| < 1 \quad \text{on } E \text{ for all } n \geq N.$$

Since f_N is bounded, choose $M_N \geq 0$ with $|f_N| \leq M_N$ on E . Then for every $x \in E$,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N,$$

so f is bounded on E . Consequently, for every $n \geq N$ and every $x \in E$,

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + (1 + M_N) = M_N + 2.$$

Each of the finitely many functions $f_1, \dots, f_{N-1}$ is bounded, say $|f_j| \leq M_j$ on E for $1 \leq j \leq N-1$. Set

$$M = \max\{M_1, \dots, M_{N-1}, M_N + 2\},$$

a maximum of finitely many real numbers, hence a real number. Then $|f_n| \leq M$ on E for every natural number n : for $n < N$ by the choice of M_n , and for $n \geq N$ by the displayed estimate. So $\{f_n\}$ is uniformly pointwise bounded on E .

Conclusion. All the hypotheses of the Bounded Convergence Theorem hold for $\{f_n\}$, f , E and M . Therefore

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f,$$

which is precisely the conclusion of Proposition 8. (For this statement to be meaningful one needs $\int_E f$ to be defined: f is measurable, being the pointwise limit on E of the sequence of measurable functions $\{f_n\}$, by Proposition 9 of Chapter 3; it is bounded, as shown above; and $m(E) < \infty$; so f is integrable over E by Theorem 4. The same remark occurs in the proof of Proposition 8 itself.)

Hence Proposition 8 is indeed the special case of the Bounded Convergence Theorem in which the pointwise convergence is strengthened to uniform convergence. The two hypotheses of Proposition 8 that look weaker than those of the Bounded Convergence Theorem, namely that only each individual f_n is assumed bounded rather than the family being uniformly bounded, are not really weaker: uniform convergence manufactures the uniform bound, as the argument above shows. Note that this is a genuine implication and not a circularity: no step of the proof of the Bounded Convergence Theorem given in the text invokes Proposition 8. That proof rests on Egoroff's Theorem, Corollary 7, and the linearity, monotonicity and additivity over domains of the integral; the mention of "the easy proof of the preceding proposition" in its opening paragraph is expository, motivating the appeal to Egoroff's Theorem rather than serving as a step in the argument.

4.3 Problems 4.15–4.21

Problem 4.15. Verify the assertions in the last Remark of this section. [The Remark reads: /Prior to the proof of the Bounded Convergence Theorem, no use was made of the countable additivity of Lebesgue measure on the real line. Only finite additivity was used, and it was used just once, in the proof of Lemma 1. But for the proof of the Bounded Convergence Theorem we used Egoroff's Theorem. The proof of Egoroff's Theorem needed the continuity of Lebesgue measure, a consequence of countable additivity of Lebesgue measure./]

Solution. There are three assertions to check. The first two are an audit of everything in Section 4.2 that precedes the Bounded Convergence Theorem: we must confirm that no step invokes the countable additivity of m , and that exactly one step invokes its finite additivity. The third is the assertion that continuity of measure is a consequence of countable additivity, together with the observation that the proof of the Bounded Convergence Theorem does rest on it, through Egoroff's Theorem.

Throughout, *additivity of m* means additivity of Lebesgue measure over disjoint *measurable* sets: finite additivity is the special case $n < \infty$ of Proposition 6 of Chapter 2, and countable additivity is Proposition 13 of Chapter 2. It is important to keep this separate from the countable subadditivity of outer measure m^* and from the Caratheodory criterion; those two belong to the construction of the σ -algebra of measurable sets and are what make the measurability machinery of Chapter 3 (in particular, that the measurable sets are closed under countable unions, and hence that a pointwise limit of measurable functions is measurable) work. The Remark is about the additivity of m itself, and it is that property whose uses we now trace.

Assertion 1 and 2: the audit of Section 4.2.

(a) *The definition of the integral of a simple function.* For a simple function ψ on a set E of finite

measure with canonical representation $\psi = \sum_{i=1}^n a_i \cdot \chi_{E_i}$, where the a_i are the distinct values of ψ and $E_i = \{x \in E \mid \psi(x) = a_i\}$, one defines

$$\int_E \psi = \sum_{i=1}^n a_i \cdot m(E_i).$$

This is a definition. It requires only that each E_i be measurable (it is, since ψ is measurable) and of finite measure (it is, being a subset of E , by the monotonicity of measure). No additivity of any kind is invoked.

(b) *Lemma 1.* Here $\{E_i\}_{i=1}^n$ is a finite disjoint collection of measurable subsets of E , the a_i are real numbers, $\varphi = \sum_{i=1}^n a_i \cdot \chi_{E_i}$, and the conclusion is $\int_E \varphi = \sum_{i=1}^n a_i \cdot m(E_i)$. The proof groups the indices: with $\{\lambda_1, \dots, \lambda_m\}$ the distinct values of φ , $A_j = \{x \in E \mid \varphi(x) = \lambda_j\}$, and $I_j = \{i \mid a_i = \lambda_j\}$, one has that A_j is the disjoint union of the finitely many sets E_i with $i \in I_j$, and the proof asserts

$$m(A_j) = \sum_{i \in I_j} m(E_i) \quad \text{for } 1 \leq j \leq m.$$

This is exactly the finite additivity of Lebesgue measure. This is the single appearance of additivity of m in Section 4.2 prior to the Bounded Convergence Theorem, as we now confirm by inspecting each remaining step.

(c) *Proposition 2 (linearity and monotonicity for simple functions).* Given simple φ, ψ on E , the proof chooses a finite disjoint collection $\{E_i\}_{i=1}^n$ of measurable sets whose union is E and on each of which both φ and ψ are constant. Such a collection is produced by pure set algebra: take the nonempty sets among the intersections of a level set of φ with a level set of ψ . These are measurable because the measurable sets are closed under finite intersection, which is a structural property of the σ -algebra, not an additivity statement. Lemma 1 is then applied to φ , to ψ , and to $\alpha\varphi + \beta\psi$, and the identity $\sum_i (\alpha a_i + \beta b_i) m(E_i) = \alpha \sum_i a_i m(E_i) + \beta \sum_i b_i m(E_i)$ is arithmetic. For monotonicity one needs only that a nonnegative simple function η has $\int_E \eta \geq 0$, which is immediate from the definition in (a) and the nonnegativity of m . So Proposition 2 uses additivity only through Lemma 1.

The text immediately after Proposition 2 also observes that, by linearity, the disjointness hypothesis on $\{E_i\}$ in Lemma 1 is unnecessary. That deduction uses only Proposition 2, hence no additivity beyond the single use in (b).

(d) *Step functions.* The text next observes that a step function is simple (a singleton and an interval are measurable, and a step function takes finitely many values), and that for a step function on a closed, bounded interval the Riemann and Lebesgue integrals agree. This uses that the measure of an interval is its length and that the measure of a singleton is zero. The first is Proposition 1 of Chapter 2, a statement about *outer* measure proved by a Heine-Borel compactness argument; the second is immediate from the definition of outer measure. Neither is an additivity statement about m . The agreement of the two integrals is then the linearity of Proposition 2 applied to a finite linear combination of characteristic functions of intervals.

(e) *The upper and lower Lebesgue integrals, and the definition of integrability.* These are definitions. That both are finite and that the lower integral does not exceed the upper follows from the monotonicity half of Proposition 2. No additivity beyond (b).

(f) *Theorem 3 (a Riemann integrable function is Lebesgue integrable, with the same integral).* The proof compares the supremum and infimum over step functions with the supremum and infimum over simple functions, using only that every step function is simple and the fact recorded in (d). No additivity beyond (b).

(g) *Theorem 4 (a bounded measurable function on a set of finite measure is integrable).* The proof invokes the Simple Approximation Lemma of Chapter 3 with $\epsilon = 1/n$ to get simple $\varphi_n \leq f \leq \psi_n$ with $0 \leq \psi_n - \varphi_n \leq 1/n$ on E , and then applies the linearity and monotonicity of Proposition 2 to conclude $0 \leq \int_E \psi_n - \int_E \varphi_n \leq (1/n) \cdot m(E)$, together with $m(E) < \infty$. The Simple Approximation Lemma itself partitions the bounded range of f into finitely many intervals and uses the measurability of the preimages; it invokes no property of m whatsoever. So Theorem 4 uses additivity only through Lemma 1, and uses the finiteness of $m(E)$.

(h) *Theorem 5 (linearity and monotonicity of integration for bounded measurable functions)*. The proof passes between the upper and lower Lebesgue integrals using Theorem 4 and Proposition 2 only: for $\alpha \neq 0$ the correspondence $\psi \leftrightarrow \psi/\alpha$ between simple functions dominating αf and those dominating (or dominated by) f is a bijection, and for $\alpha = \beta = 1$ one compares $\int_E (f + g)$ with sums $\int_E \psi_1 + \int_E \psi_2$ and $\int_E \varphi_1 + \int_E \varphi_2$. Monotonicity uses only that $\int_E \eta \geq 0$ for the simple function $\eta \equiv 0$ compared with a nonnegative h . No new use of additivity.

(h') *The Example of the Dirichlet function*. Between Theorem 3 and Theorem 4 the text computes $\int_{[0,1]} \chi_E = 1 \cdot m(E) = 0$ for E the rationals in $[0, 1]$. This uses only the definition of the integral of a simple function together with the fact that a countable set has measure zero — and the latter is a statement about *outer* measure, proved in Chapter 2 from the countable subadditivity of m^* and the fact that a singleton has outer measure zero. Countable subadditivity of m^* is not the countable additivity of m ; it is available before any additivity of m is proved. So this Example adds no use of additivity either.

(i) *Corollary 6 (additivity over domains of integration)*. For disjoint measurable $A, B \subseteq E$ the proof uses the pointwise set identity $\chi_{A \cup B} = \chi_A + \chi_B$ — which holds because A and B are disjoint, and is a statement about *functions*, not about m — together with Problem 10 ($\int_{E_1} f = \int_E f \cdot \chi_{E_1}$, itself a consequence of the definitions and of Theorem 5) and the linearity of Theorem 5. Note in particular that the name of this corollary is misleading for our purposes: the additivity being proved here is a property of the integral, deduced from linearity, and it is not an application of the additivity of m .

(j) *Corollary 7* ($|\int_E f| \leq \int_E |f|$). Immediate from $-|f| \leq f \leq |f|$ and Theorem 5.

(k) *Proposition 8 (passage to the limit for uniformly convergent sequences)*. The proof uses that a uniform limit of measurable functions is measurable, then, for $n \geq N$ with $|f - f_n| < \epsilon/m(E)$ on E , applies Theorem 5 and Corollary 7 to get $|\int_E f - \int_E f_n| \leq [\epsilon/m(E)] \cdot m(E) = \epsilon$. Again $m(E) < \infty$ is used, but no additivity beyond Lemma 1.

This exhausts everything in Section 4.2 before the Bounded Convergence Theorem. The countable additivity of m is never invoked, and its finite additivity is invoked exactly once, in the proof of Lemma 1. That verifies the first two assertions.

Assertion 3: the Bounded Convergence Theorem uses Egoroff's Theorem, and Egoroff's Theorem uses continuity of measure, which follows from countable additivity.

First, the Bounded Convergence Theorem. Its proof fixes $\epsilon > 0$ and, since $m(E) < \infty$ and f is real-valued, applies Egoroff's Theorem to produce a measurable $A \subseteq E$ on which $\{f_n\} \rightarrow f$ uniformly and with $m(E \sim A) < \epsilon/4M$; the estimate

$$\left| \int_E f_n - \int_E f \right| \leq \int_A |f_n - f| + 2M \cdot m(E \sim A)$$

is then finished by uniform convergence on A . So Egoroff's Theorem is genuinely a hypothesis of the argument.

Second, Egoroff's Theorem uses the continuity of measure. Its proof rests on Lemma 10 of Chapter 3, whose proof sets, for $\eta > 0$,

$$E_n = \{x \in E \mid |f(x) - f_k(x)| < \eta \text{ for all } k \geq n\},$$

observes that $\{E_n\}_{n=1}^\infty$ is an ascending collection of measurable sets with $\bigcup_{n=1}^\infty E_n = E$ because $\{f_n\} \rightarrow f$ pointwise on E , and then concludes

$$m(E) = \lim_{n \rightarrow \infty} m(E_n)$$

from the continuity of measure, Theorem 15(i) of Chapter 2. Since $m(E) < \infty$, an index N can then be chosen with $m(E_N) > m(E) - \delta$, and the excision property gives $m(E \sim E_N) < \delta$. (The proof of Egoroff's Theorem proper additionally uses the countable subadditivity of measure to bound $m(E \sim \bigcap_n A_n)$, and Theorem 11 of Chapter 2 to shrink A to a closed set — but the essential point for the Remark is the appeal to continuity of measure.)

Third, continuity of measure is a consequence of countable additivity. Let $\{A_k\}_{k=1}^\infty$ be an ascending collection of measurable sets and put $A_0 = \emptyset$. The sets $A_k \sim A_{k-1}$, $k \geq 1$, are measurable (the

measurable sets are a σ -algebra) and pairwise disjoint: if $j < k$ then $A_j \subseteq A_{k-1}$, so $A_j \sim A_{j-1}$ is disjoint from $A_k \sim A_{k-1}$. Moreover, since the collection is ascending,

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} [A_k \sim A_{k-1}] \quad \text{and} \quad A_n = \bigcup_{k=1}^n [A_k \sim A_{k-1}] \quad \text{for each } n.$$

Indeed, if x belongs to $\bigcup_k A_k$, there is a least index k with $x \in A_k$, and then $x \in A_k \sim A_{k-1}$. By the countable additivity of Lebesgue measure, Proposition 13 of Chapter 2, applied to the left-hand union, and by finite additivity applied to the right-hand one,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n m(A_k \sim A_{k-1}) = \lim_{n \rightarrow \infty} m(A_n),$$

which is Theorem 15(i). (Part (ii), for descending collections with $m(B_1) < \infty$, follows by applying (i) to the ascending collection $\{B_1 \sim B_k\}$ and using excision, which is legitimate because $m(B_1) < \infty$.) Assembling the three points: the development of Section 4.2 up to, but not including, the Bounded Convergence Theorem rests on Lebesgue measure only through its monotonicity, the value it assigns to intervals, and one application of finite additivity in Lemma 1; whereas the Bounded Convergence Theorem itself rests, via Egoroff's Theorem and the continuity of measure, on the countable additivity of Lebesgue measure. All the assertions of the Remark are verified.

One caveat, for honesty: the Remark, and hence this exercise, is a bookkeeping claim about which properties of m the given proofs happen to use. It is not a claim that the Bounded Convergence Theorem cannot be proved without countable additivity, and nothing above should be read as establishing such a claim.

Problem 4.16. Let f be a nonnegative bounded measurable function on a set of finite measure E . Assume $\int_E f = 0$. Show that $f = 0$ a.e. on E .

Solution. Since f is bounded and measurable on E and $m(E) < \infty$, f is integrable over E by Theorem 4, so the hypothesis $\int_E f = 0$ makes sense.

For each natural number n define

$$E_n = \left\{x \in E \mid f(x) \geq \frac{1}{n}\right\}.$$

Each E_n is measurable, since f is a measurable function, and $E_n \subseteq E$, so $m(E_n) \leq m(E) < \infty$ by the monotonicity of measure.

Fix n . The function $\frac{1}{n} \cdot \chi_{E_n}$ is a simple, hence bounded and measurable, function on E , and its integral over E is computed from the definition of the integral of a simple function on a set of finite measure:

$$\int_E \frac{1}{n} \cdot \chi_{E_n} = \frac{1}{n} \cdot m(E_n).$$

(If $E_n = E$ the canonical representation of $\frac{1}{n} \chi_{E_n}$ on E has the single value $1/n$, taken on E ; if $E_n = \emptyset$ it has the single value 0, and then both sides are 0; otherwise it has the two distinct values $1/n$ and 0, taken on E_n and on $E \sim E_n$, and the term $0 \cdot m(E \sim E_n)$ contributes nothing. In every case the displayed formula holds.)

Now $\frac{1}{n} \cdot \chi_{E_n} \leq f$ on E : at a point $x \in E_n$ the left side equals $1/n \leq f(x)$ by the definition of E_n , and at a point $x \in E \sim E_n$ the left side equals $0 \leq f(x)$ because f is nonnegative. Both functions are bounded and measurable on the set E of finite measure, so the monotonicity of integration, Theorem 5, applies and gives

$$\frac{1}{n} \cdot m(E_n) = \int_E \frac{1}{n} \cdot \chi_{E_n} \leq \int_E f = 0.$$

Since $m(E_n) \geq 0$, we conclude $m(E_n) = 0$ for every n .

Finally, because $f \geq 0$ on E ,

$$\{x \in E \mid f(x) \neq 0\} = \{x \in E \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n,$$

the second equality because $f(x) > 0$ holds if and only if $f(x) \geq 1/n$ for some natural number n (choose n with $1/n \leq f(x)$, possible by the Archimedean property). By the countable subadditivity of Lebesgue measure,

$$m(\{x \in E \mid f(x) \neq 0\}) \leq \sum_{n=1}^{\infty} m(E_n) = 0.$$

Hence $\{x \in E \mid f(x) \neq 0\}$ has measure zero, that is, $f = 0$ a.e. on E .

Problem 4.17. Let E be a set of measure zero and define $f = \infty$ on E . Show that $\int_E f = 0$.

Solution. The function $f \equiv \infty$ is a nonnegative extended real-valued measurable function on E : for every c the set $\{x \in E \mid f(x) > c\}$ equals E , which is measurable. So its integral is defined by (8),

$$\int_E f = \sup \left\{ \int_E h \mid h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E \right\}.$$

We show every number in the set on the right equals 0, and that the set is nonempty; the supremum is then 0.

The set is nonempty: the function $h \equiv 0$ on E is bounded, measurable, vanishes outside the set $E_0 = \emptyset$ of finite measure (so it has finite support), and satisfies $0 \leq h \leq f$ since $f \equiv \infty$.

Now let h be any competitor, that is, h is bounded and measurable on E , $0 \leq h \leq f$ on E , and h has finite support. Since $m(E) = 0 < \infty$, the set E itself is an admissible choice of the set outside of which h vanishes (vacuously, $h = 0$ on $E \sim E = \emptyset$), so the integral of h over E is simply the integral of the bounded measurable function h over the set E of finite measure, in the sense of Section 4.2. No appeal to the independence of the choice of E_0 is needed here.

We claim $\int_E h = 0$. Let ψ be any simple function on E with $h \leq \psi$ on E ; such ψ exist, since $\psi \equiv M$ works for any M bounding h . Write the canonical representation $\psi = \sum_{i=1}^n a_i \cdot \chi_{A_i}$, where the A_i are the level sets of ψ in E . Each $A_i \subseteq E$, so $m(A_i) \leq m(E) = 0$ by the monotonicity of measure, whence

$$\int_E \psi = \sum_{i=1}^n a_i \cdot m(A_i) = 0.$$

Thus every number in the defining set for the upper Lebesgue integral of h over E equals 0, so that upper integral is 0. Since h is bounded and measurable on the set E of finite measure, Theorem 4 says h is integrable over E and $\int_E h$ is the common value of the upper and lower integrals; hence $\int_E h = 0$. (This is precisely the content of Problem 9: a bounded function on a set of measure zero is measurable with integral zero. If $E = \emptyset$ the assertion is trivial.)

(The same conclusion holds if one prefers to compute $\int_E h$ as $\int_{E_0} h$ for a set $E_0 \subseteq E$ of finite measure outside which h vanishes: then $m(E_0) \leq m(E) = 0$ and the identical argument on E_0 gives $\int_{E_0} h = 0$.) Therefore $\int_E h = 0$ for every competitor h , and consequently

$$\int_E f = \sup\{0\} = 0.$$

Note that the same argument shows more generally that the integral of *any* nonnegative measurable function over a set of measure zero is zero; the extreme case $f \equiv \infty$ makes the point that the convention $\infty \cdot 0 = 0$ built into definition (8) is consistent.

Problem 4.18. Show that the integral of a bounded measurable function of finite support is properly defined.

Solution. Recall the situation. A measurable function f on a measurable set E is said to vanish outside a set of finite measure, or to have finite support, provided there is a measurable subset E_0 of E with $m(E_0) < \infty$ and $f = 0$ on $E \sim E_0$. If f is in addition bounded and measurable on E , one defines

$$\int_E f = \int_{E_0} f.$$

Two things must be checked for this to be a legitimate definition: that the right-hand side makes sense, and that it does not depend on which set E_0 is used.

The right-hand side is defined. Let E_0 be as above. The restriction of f to E_0 is a bounded measurable function on a set of finite measure, so by Theorem 4 it is integrable over E_0 and $\int_{E_0} f$ is a well-defined real number.

Independence of the choice of E_0 . Let E_0 and E_1 be measurable subsets of E , each of finite measure, with

$$f = 0 \text{ on } E \sim E_0 \quad \text{and} \quad f = 0 \text{ on } E \sim E_1.$$

We must show $\int_{E_0} f = \int_{E_1} f$.

Set $A = E_0 \cap E_1$ and $B = E_0 \sim E_1$. Then A and B are disjoint measurable subsets of E_0 with $A \cup B = E_0$, and $m(E_0) < \infty$, so the restriction of f to E_0 is a bounded measurable function on a set of finite measure. Corollary 6, additivity over domains of integration, applies and yields

$$\int_{E_0} f = \int_A f + \int_B f.$$

Now $B = E_0 \sim E_1 \subseteq E \sim E_1$, and $f = 0$ on $E \sim E_1$. Hence f vanishes identically on B , and $m(B) \leq m(E_0) < \infty$, so $\int_B f = \int_B 0 = 0$: indeed the zero function on B is simple with canonical representation $0 \cdot \chi_B$, whose integral is $0 \cdot m(B) = 0$. (If $B = \emptyset$ the integral is 0 trivially.) Therefore

$$\int_{E_0} f = \int_{E_0 \cap E_1} f.$$

The argument is symmetric in E_0 and E_1 : taking $A = E_1 \cap E_0$ and $B' = E_1 \sim E_0 \subseteq E \sim E_0$, on which f vanishes, Corollary 6 applied on E_1 gives

$$\int_{E_1} f = \int_{E_0 \cap E_1} f + \int_{E_1 \sim E_0} f = \int_{E_0 \cap E_1} f.$$

Comparing the two displays, $\int_{E_0} f = \int_{E_1} f$. So the value assigned to $\int_E f$ is the same for every admissible choice of E_0 , and the definition is proper.

Remark. The set E_0 may always be taken to be the support $\{x \in E \mid f(x) \neq 0\}$ itself: this set is measurable since f is measurable, it is contained in any admissible E_0 and hence has finite measure, and f vanishes on its complement in E . So the definition could equivalently have been made with this canonical choice; the computation above shows the more flexible formulation agrees with it. Note also that when $m(E) < \infty$ we may take $E_0 = E$, so the new definition is consistent with the integral of Section 4.2.

Problem 4.19. For a number α , define $f(x) = x^\alpha$ for $0 < x \leq 1$, and $f(0) = 0$. Compute $\int_0^1 f$.

Solution. The answer is

$$\int_0^1 f = \begin{cases} \frac{1}{\alpha + 1} & \text{if } \alpha > -1, \\ \infty & \text{if } \alpha \leq -1. \end{cases}$$

Setting up. The function f is real-valued and nonnegative on $[0, 1]$: for $x \in (0, 1]$ we have $x^\alpha = e^{\alpha \ln x} \in (0, \infty)$, and $f(0) = 0$. It is measurable. Indeed, f is continuous on $(0, 1]$, so for each real c the set $\{x \in (0, 1] \mid f(x) > c\}$ is relatively open in $(0, 1]$, hence the intersection of an open set with $(0, 1]$, hence measurable; and

$$\{x \in [0, 1] \mid f(x) > c\} = \{x \in (0, 1] \mid f(x) > c\} \cup S_c, \quad S_c = \begin{cases} \{0\} & \text{if } c < 0, \\ \emptyset & \text{if } c \geq 0, \end{cases}$$

a union of two measurable sets. So f is a nonnegative measurable function on $[0, 1]$ and $\int_{[0,1]} f$ is defined by (8). Note that the value assigned at the single point 0 is irrelevant, since $\{0\}$ has measure zero and the excision formula (15) of Theorem 11 gives $\int_{[0,1]} f = \int_{(0,1]} f$.

A consistency remark. Below we integrate functions that are simultaneously nonnegative measurable (so that (8) applies) and bounded measurable of finite support (so that the integral of Section 4.2 applies). The two integrals agree: if g is nonnegative, bounded and measurable on E with finite support, then g is itself a competitor in (8), so the supremum in (8) is at least the Section 4.2 integral of g ; and every competitor h in (8) satisfies $0 \leq h \leq g$, so, choosing a set E_0 of finite measure outside which both g and h vanish, the monotonicity of Theorem 5 gives $\int_{E_0} h \leq \int_{E_0} g$, that is, the Section 4.2 integral of g is an upper bound for the competitors. Hence the two values coincide, and we may write $\int_E g$ without ambiguity.

Exhausting by an increasing sequence. For each natural number $n \geq 2$ define

$$f_n = f \cdot \chi_{[1/n, 1]} \quad \text{on } [0, 1].$$

Each f_n is nonnegative and measurable. The sequence is increasing: the sets $[1/n, 1]$ ascend with n and $f \geq 0$, so $f_n \leq f_{n+1}$ pointwise on $[0, 1]$. It converges to f pointwise on $[0, 1]$: if $x \in (0, 1]$, then $1/n < x$ for all large n , so $f_n(x) = f(x)$ eventually; and $f_n(0) = 0 = f(0)$ for every n . By the Monotone Convergence Theorem,

$$\int_{[0,1]} f = \lim_{n \rightarrow \infty} \int_{[0,1]} f_n.$$

Computing $\int_{[0,1]} f_n$. On the closed, bounded interval $[1/n, 1]$ the function $x \mapsto x^\alpha$ is continuous, hence bounded there and, by Problem 6 of Section 4.1, Riemann integrable there. Consequently f_n is a bounded measurable function on $[0, 1]$, and it has finite support since $m([0, 1]) = 1 < \infty$. Since $[0, 1]$ is the union of the disjoint measurable sets $[0, 1/n)$ and $[1/n, 1]$, additivity over domains for nonnegative measurable functions (Theorem 11), together with the fact that f_n vanishes identically on $[0, 1/n)$ and so has integral zero there by Proposition 9, gives

$$\int_{[0,1]} f_n = \int_{[1/n, 1]} x^\alpha dx.$$

By Theorem 3, the Lebesgue integral of a Riemann integrable function on a closed, bounded interval equals its Riemann integral, so this is the elementary integral $\int_{1/n}^1 x^\alpha dx$, which the Fundamental Theorem of Calculus evaluates as

$$\int_{1/n}^1 x^\alpha dx = \begin{cases} \frac{1 - (1/n)^{\alpha+1}}{\alpha + 1} & \text{if } \alpha \neq -1, \\ \ln n & \text{if } \alpha = -1. \end{cases}$$

Passing to the limit. Three cases.

If $\alpha > -1$, then $\alpha + 1 > 0$, so $(1/n)^{\alpha+1} \rightarrow 0$ and

$$\int_{[0,1]} f = \lim_{n \rightarrow \infty} \frac{1 - (1/n)^{\alpha+1}}{\alpha + 1} = \frac{1}{\alpha + 1}.$$

If $\alpha = -1$, then $\int_{[0,1]} f = \lim_{n \rightarrow \infty} \ln n = \infty$.

If $\alpha < -1$, then $\alpha + 1 < 0$, so $(1/n)^{\alpha+1} = n^{-(\alpha+1)} \rightarrow \infty$; the numerator $1 - (1/n)^{\alpha+1}$ tends to $-\infty$ while the denominator $\alpha + 1$ is a fixed negative number, so the quotient tends to $+\infty$ and $\int_{[0,1]} f = \infty$.

Summary and remarks. Thus f is integrable over $[0, 1]$ exactly when $\alpha > -1$, in which case $\int_0^1 f = 1/(\alpha + 1)$; for $\alpha \leq -1$ the integral is ∞ and f is not integrable. Two consistency checks. For $\alpha > 0$ the function f is continuous on all of $[0, 1]$, since $x^\alpha \rightarrow 0 = f(0)$ as $x \rightarrow 0^+$; hence it is Riemann integrable over $[0, 1]$ by Problem 6 of Section 4.1, and Theorem 3 gives directly $\int_0^1 x^\alpha dx = 1/(\alpha + 1)$, in agreement with the formula above. For $\alpha = 0$ the function f is the step function equal to 1 on $(0, 1]$ and 0 at the origin, whose integral is $1 \cdot m((0, 1]) = 1 = 1/(0 + 1)$, again in agreement. And for $\alpha \leq -1$ one sees the divergence without computation from Chebychev's Inequality applied to the sets $\{f \geq \lambda\}$, or simply by noting $x^\alpha \geq x^{-1}$ on $(0, 1]$ for $\alpha \leq -1$ and using monotonicity of integration, Theorem 10, together with the case $\alpha = -1$.

Problem 4.20. Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to f pointwise on E . Let $M \geq 0$ be such that $\int_E f_n \leq M$ for all n . Show that $\int_E f \leq M$. Verify that this property is equivalent to the statement of Fatou's Lemma.

Solution. Call the asserted property (P) :

(P) : whenever $\{f_n\}$ is a sequence of nonnegative measurable functions on E converging to f pointwise on E , and $M \geq 0$ satisfies $\int_E f_n \leq M$ for all n , then $\int_E f \leq M$.

We also record the *a.e.* variant, in which pointwise convergence on E is weakened to pointwise convergence a.e. on E ; call it (P') . The two are equivalent, and the equivalence is proved in the first of the two remarks at the end. In the arguments below we use whichever form is convenient, and we work with (P') where the a.e. hypothesis of Fatou's Lemma as printed in the text makes it natural.

Fatou's Lemma implies (P) and (P') . Let $\{f_n\}$, f and M be as in the hypothesis of (P') (which includes that of (P)). The functions f_n are nonnegative and measurable and converge pointwise a.e. on E to f , so Fatou's Lemma applies and gives

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

Since $\int_E f_n \leq M$ for every n , the sequence of real numbers $\{\int_E f_n\}$ is bounded above by M , hence so is its lower limit: $\liminf_n \int_E f_n \leq M$. Combining,

$$\int_E f \leq M,$$

which is what was to be shown. This proves the first assertion of the problem.

(P) implies Fatou's Lemma. Assume (P) holds for every sequence as described; by the first remark below, (P') then holds too, and it is (P') we use. Let $\{g_n\}$ be a sequence of nonnegative measurable functions on E with $\{g_n\} \rightarrow g$ pointwise a.e. on E ; we must show

$$\int_E g \leq \liminf_{n \rightarrow \infty} \int_E g_n.$$

Write $L = \liminf_{n \rightarrow \infty} \int_E g_n$, an element of $[0, \infty]$ since each $\int_E g_n \geq 0$.

If $L = \infty$ the inequality is trivially true, so assume $L < \infty$.

Let $\epsilon > 0$. Since $L = \lim_{k \rightarrow \infty} (\inf_{n \geq k} \int_E g_n)$ is finite, the lower limit is attained along a subsequence: there is a subsequence $\{g_{n_k}\}_{k=1}^\infty$, with $n_1 < n_2 < \dots$, for which

$$\lim_{k \rightarrow \infty} \int_E g_{n_k} = L.$$

Hence there is an index K such that $\int_E g_{n_k} \leq L + \epsilon$ for all $k \geq K$.

Consider the sequence $\{h_k\}_{k=1}^\infty$ defined by $h_k = g_{n_{K+k-1}}$, that is, the tail of the chosen subsequence beginning at index K . Each h_k is a nonnegative measurable function on E ; the sequence $\{h_k\}$ converges

pointwise a.e. on E to g , because it is a subsequence of a sequence that does so, and a subsequence of a convergent sequence of real (or extended real) numbers converges to the same limit; and

$$\int_E h_k \leq L + \epsilon \quad \text{for every } k.$$

So (P') applies with the bound $M = L + \epsilon \geq 0$ and yields

$$\int_E g \leq L + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\int_E g \leq L = \liminf_n \int_E g_n$, which is Fatou's Lemma.

Two remarks on the bookkeeping. First, the promised equivalence of (P) and (P') . Trivially (P') implies (P) . Conversely, assume (P) and let $\{f_n\}$ be nonnegative measurable on E with $\{f_n\} \rightarrow f$ pointwise on $E \sim E_0$, where $E_0 \subseteq E$ is measurable with $m(E_0) = 0$, and with $\int_E f_n \leq M$ for all n . The function f is measurable on E . Indeed, define g on E by $g = f$ on $E \sim E_0$ and $g = 0$ on E_0 . On $E \sim E_0$ the function g is the pointwise limit of the measurable functions f_n restricted there, hence measurable on $E \sim E_0$ by Proposition 9 of Chapter 3; g is measurable on E_0 as well, being constant; and a function whose restrictions to the two measurable pieces E_0 and $E \sim E_0$ of E are measurable is measurable on E , since $\{g > c\} = \{x \in E_0 \mid g(x) > c\} \cup \{x \in E \sim E_0 \mid g(x) > c\}$. Finally $f = g$ on $E \sim E_0$ and $m(E_0) = 0$, so $f = g$ a.e. on E and Proposition 5(i) of Chapter 3 gives the measurability of f on E . The restrictions of the f_n to $E \sim E_0$ are nonnegative and measurable and converge to f pointwise on all of $E \sim E_0$, and by the excision formula (15) of Theorem 11,

$$\int_{E \sim E_0} f_n = \int_E f_n \leq M \quad \text{for every } n.$$

So (P) , applied on the set $E \sim E_0$, gives $\int_{E \sim E_0} f \leq M$, and one more application of (15) gives $\int_E f = \int_{E \sim E_0} f \leq M$. This is (P') . (This is exactly the reduction performed at the start of the proof of Fatou's Lemma in the text.)

Second, the two implications are genuinely a proof of equivalence and not a circularity: the second implication derives Fatou's Lemma from (P) using nothing but the definition of $\liminf$ and the excision property, so (P) could be taken as the official statement of Fatou's Lemma and the version printed in the text recovered from it. Note incidentally that (P) is strictly a statement about an upper bound: it does not assert $\int_E f \leq \liminf \int_E f_n$ directly, but the freedom to pass to a subsequence realising the lower limit converts one into the other.

Problem 4.21. Let the function f be nonnegative and integrable over E and $\epsilon > 0$. Show there is a simple function η on E that has finite support, $0 \leq \eta \leq f$ on E and $\int_E |f - \eta| < \epsilon$. If E is a closed, bounded interval, show there is a step function h on E that has finite support and $\int_E |f - h| < \epsilon$.

Solution. Part 1: approximation by a simple function of finite support.

Since f is integrable over E , $\int_E f < \infty$, and by Proposition 13 the set $Z = \{x \in E \mid f(x) = \infty\}$ has measure zero.

Step 1: choose a bounded competitor close to the supremum. By the definition (8) of $\int_E f$ as a supremum, and since that supremum is finite, there is a bounded measurable function h_1 on E of finite support with $0 \leq h_1 \leq f$ on E and

$$\int_E h_1 > \int_E f - \frac{\epsilon}{2}.$$

Step 2: replace h_1 by a simple function below it. Let E_0 be a measurable subset of E with $m(E_0) < \infty$ and $h_1 = 0$ on $E \sim E_0$; by Problem 18, $\int_E h_1 = \int_{E_0} h_1$ regardless of which such E_0 is taken. The restriction of h_1 to E_0 is a bounded, real-valued measurable function, so the Simple Approximation Lemma of Chapter 3, applied with

$$\epsilon' = \frac{\epsilon}{2(m(E_0) + 1)} > 0,$$

provides simple functions φ and ψ on E_0 with

$$\varphi \leq h_1 \leq \psi \quad \text{and} \quad 0 \leq \psi - \varphi < \epsilon' \quad \text{on } E_0.$$

Define $\eta_0 = \max\{\varphi, 0\}$ on E_0 . It is simple. Indeed, if φ has canonical representation $\varphi = \sum_{i=1}^p d_i \chi_{D_i}$ on E_0 , then η_0 takes only the finitely many values in $\{\max\{d_i, 0\} : 1 \leq i \leq p\}$, and for each such value a the level set $\{x \in E_0 \mid \eta_0(x) = a\}$ is the union of those D_i with $\max\{d_i, 0\} = a$, a finite union of measurable sets. So η_0 is a measurable real-valued function taking finitely many values, that is, a simple function. Since $h_1 \geq 0$ and $\varphi \leq h_1$, we have $0 \leq \eta_0 \leq h_1$ on E_0 , and

$$0 \leq h_1 - \eta_0 \leq h_1 - \varphi \leq \psi - \varphi < \epsilon' \quad \text{on } E_0.$$

Now define η on all of E by $\eta = \eta_0$ on E_0 and $\eta = 0$ on $E \sim E_0$. Then η is a simple function on E , it vanishes outside the set E_0 of finite measure, so it has finite support, and

$$0 \leq \eta \leq h_1 \leq f \quad \text{on } E.$$

Step 3: the estimate. All three functions h_1 , η and $h_1 - \eta$ are bounded and measurable on E and vanish on $E \sim E_0$, where $m(E_0) < \infty$. So each of their integrals over E is by definition the corresponding integral over the *single* set E_0 of finite measure, and Theorem 5, the linearity and monotonicity of integration for bounded measurable functions on a set of finite measure, applies to them on E_0 . Since $0 \leq h_1 - \eta \leq \epsilon'$ on E_0 , monotonicity against the constant function ϵ' gives

$$\int_E (h_1 - \eta) = \int_{E_0} (h_1 - \eta) \leq \epsilon' \cdot m(E_0) < \frac{\epsilon}{2},$$

and linearity on E_0 gives $\int_{E_0} \eta = \int_{E_0} h_1 - \int_{E_0} (h_1 - \eta)$, that is,

$$\int_E \eta = \int_E h_1 - \int_E (h_1 - \eta) > \left(\int_E f - \frac{\epsilon}{2} \right) - \frac{\epsilon}{2} = \int_E f - \epsilon.$$

(Here $\int_E \eta$ in the sense of the supremum definition (8) agrees with $\int_{E_0} \eta$ in the sense of Section 4.2: η is itself a competitor in (8), so the supremum is at least $\int_{E_0} \eta$, while every competitor h in (8) satisfies $0 \leq h \leq \eta$ and hence $\int h \leq \int_{E_0} \eta$ by the monotonicity of Theorem 5 on a set of finite measure containing the supports of both. The same remark applies to h_1 .)

Step 4: conclusion of Part 1. Since $0 \leq \eta \leq f$ on E , we have $|f - \eta| = f - \eta$ on E (with the convention $\infty - c = \infty$ on the null set Z), and $f - \eta$ is a nonnegative measurable function with $(f - \eta) + \eta = f$ pointwise on E . Its measurability is immediate and does not require f to be finite: writing the canonical representation $\eta = \sum_i c_i \chi_{F_i}$ with $\{F_i\}$ a finite disjoint measurable partition of E , for each real t

$$\{x \in E \mid f(x) - \eta(x) > t\} = \bigcup_i (F_i \cap \{x \in E \mid f(x) > t + c_i\}),$$

a finite union of measurable sets. By the linearity of integration for nonnegative measurable functions, Theorem 10,

$$\int_E f = \int_E (f - \eta) + \int_E \eta.$$

Both $\int_E f$ and $\int_E \eta$ are finite (the latter because $\eta \leq f$ and monotonicity, (12) of Theorem 10), so we may subtract and conclude

$$\int_E |f - \eta| = \int_E (f - \eta) = \int_E f - \int_E \eta < \epsilon.$$

This is the required simple function.

Part 2: approximation by a step function when $E = [a, b]$.

Let $E = [a, b]$ be a closed, bounded interval. Note that every function on E has finite support, since $m(E) = b - a < \infty$; so the substance of the claim is the integral estimate. Note also that h is *not* required to satisfy $0 \leq h \leq f$.

By Part 1 there is a simple function η on E with finite support, $0 \leq \eta \leq f$ on E , and

$$\int_E |f - \eta| < \frac{\epsilon}{2}.$$

It therefore suffices to produce a step function h on E with $\int_E |\eta - h| < \epsilon/2$, for then the triangle inequality finishes the proof, as follows. Let $Z = \{x \in E \mid f(x) = \infty\}$, a set of measure zero by Proposition 13. On $E \sim Z$ all three functions f, η, h are finite, so the pointwise inequality $|f - h| \leq |f - \eta| + |\eta - h|$ holds there; by the excision formula (15) of Theorem 11 and then the monotonicity and linearity of integration for nonnegative measurable functions, Theorem 10,

$$\int_E |f - h| = \int_{E \sim Z} |f - h| \leq \int_{E \sim Z} |f - \eta| + \int_{E \sim Z} |\eta - h| = \int_E |f - \eta| + \int_E |\eta - h| < \epsilon.$$

Construction of the step function. Let

$$\eta = \sum_{k=1}^n a_k \cdot \chi_{E_k}$$

be the canonical representation of η on E , where $a_1, \dots, a_n$ are the distinct *nonzero* values taken by η and $E_k = \{x \in E \mid \eta(x) = a_k\}$ (we discard the value 0, which contributes nothing). The E_k are disjoint measurable subsets of $[a, b]$, so each has finite measure. Put

$$\delta = \frac{\epsilon}{2(1 + \sum_{k=1}^n |a_k|)} > 0.$$

Each E_k is measurable of finite outer measure, so Theorem 12 of Chapter 2 applies: there is a finite disjoint collection of open intervals $\{I_{k,j}\}_{j=1}^{m_k}$ such that, setting $O_k = \bigcup_{j=1}^{m_k} I_{k,j}$,

$$m^*(E_k \Delta O_k) < \delta,$$

where $E_k \Delta O_k = [E_k \sim O_k] \cup [O_k \sim E_k]$ denotes the symmetric difference. Since E_k and O_k are both measurable, so is $E_k \Delta O_k$, and the displayed inequality reads $m(E_k \Delta O_k) < \delta$. Define

$$h = \sum_{k=1}^n a_k \cdot \chi_{O_k \cap [a,b]} \quad \text{on } [a, b].$$

Each set $O_k \cap [a, b]$ is a finite disjoint union of intervals, so each $\chi_{O_k \cap [a,b]}$ is a step function on $[a, b]$; a finite linear combination of step functions is a step function, so h is a step function on $[a, b]$, and it has finite support because $m([a, b]) < \infty$.

The estimate. For every $x \in [a, b]$,

$$|\eta(x) - h(x)| = \left| \sum_{k=1}^n a_k [\chi_{E_k}(x) - \chi_{O_k \cap [a,b]}(x)] \right| \leq \sum_{k=1}^n |a_k| \cdot \chi_{E_k \Delta (O_k \cap [a,b])}(x),$$

since $|\chi_A - \chi_B| = \chi_{A \Delta B}$ for any sets A, B . Because $E_k \subseteq [a, b]$, we have $E_k \sim (O_k \cap [a, b]) = E_k \sim O_k$ and $(O_k \cap [a, b]) \sim E_k \subseteq O_k \sim E_k$, so

$$E_k \Delta (O_k \cap [a, b]) \subseteq E_k \Delta O_k.$$

All the functions here are bounded and measurable on the set $[a, b]$ of finite measure, so the monotonicity and linearity of integration, Theorem 5, apply. Since for each k

$$\int_{[a,b]} \chi_{E_k \Delta (O_k \cap [a,b])} = m(E_k \Delta (O_k \cap [a, b])) \leq m(E_k \Delta O_k) < \delta,$$

we obtain

$$\int_{[a,b]} |\eta - h| \leq \sum_{k=1}^n |a_k| \cdot m(E_k \Delta O_k) \leq \delta \sum_{k=1}^n |a_k| = \frac{\epsilon}{2} \cdot \frac{\sum_{k=1}^n |a_k|}{1 + \sum_{k=1}^n |a_k|} < \frac{\epsilon}{2}.$$

(If $n = 0$, that is if $\eta \equiv 0$ on E , take $h \equiv 0$; the estimate is then trivial.)

Combining with the triangle inequality established above, $\int_E |f - h| < \epsilon$, as required.

Remark. The step function h produced in Part 2 need not satisfy $0 \leq h \leq f$, and in general no step function can: f may be unbounded, and more to the point the sets O_k may spill outside the E_k . What the two parts together express is the density, in the L^1 sense, of the simple functions of finite support in the nonnegative integrable functions on an arbitrary measurable set, and of the step functions in the nonnegative integrable functions on a closed, bounded interval.

4.4 Problems 4.22–4.28

Problem 4.22. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f and f be integrable over $\mathbb{R}$. Show that if

$$\int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n,$$

then

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n \quad \text{for any measurable set } E.$$

Solution. Let E be a measurable subset of $\mathbb{R}$ and write $E^c = \mathbb{R} \setminus E$, so that E and E^c are disjoint measurable subsets of $\mathbb{R}$ whose union is $\mathbb{R}$.

Each f_n is nonnegative and measurable, and f , being the pointwise limit of measurable functions, is nonnegative and measurable as well. Put

$$L = \int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n.$$

Since f is integrable over $\mathbb{R}$ we have $L < \infty$. By Theorem 11 of Chapter 4 (Additivity Over Domains of Integration), applied to the nonnegative measurable function f and the disjoint sets E , E^c ,

$$\int_E f + \int_{E^c} f = \int_{\mathbb{R}} f = L < \infty,$$

so both $\int_E f$ and $\int_{E^c} f$ are finite. Likewise, for every index n ,

$$\int_E f_n + \int_{E^c} f_n = \int_{\mathbb{R}} f_n. \quad (*)$$

Since $\int_{\mathbb{R}} f_n \rightarrow L < \infty$, there is an index N for which $\int_{\mathbb{R}} f_n \leq L + 1 < \infty$ whenever $n \geq N$; by $(*)$ and the nonnegativity of the integrands, both $\int_E f_n$ and $\int_{E^c} f_n$ are then finite for $n \geq N$. Thus for $n \geq N$ we may rewrite $(*)$ as

$$\int_E f_n = \int_{\mathbb{R}} f_n - \int_{E^c} f_n,$$

an honest difference of real numbers.

The restriction of $\{f_n\}$ to E is a sequence of nonnegative measurable functions converging pointwise on E to f , so Fatou's Lemma gives

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n. \quad (1)$$

The same reasoning on E^c gives

$$\int_{E^c} f \leq \liminf_{n \rightarrow \infty} \int_{E^c} f_n. \quad (2)$$

Now we bound the upper limit on E . Since the sequence of real numbers $\{\int_{\mathbb{R}} f_n\}$ converges to the finite number L , for any sequence $\{b_n\}$ of real numbers we have $\limsup_n (\int_{\mathbb{R}} f_n - b_n) = L - \liminf_n b_n$. Taking $b_n = \int_{E^c} f_n$ (which is a real number for $n \geq N$) and using (2),

$$\limsup_{n \rightarrow \infty} \int_E f_n = L - \liminf_{n \rightarrow \infty} \int_{E^c} f_n \leq L - \int_{E^c} f = \int_{\mathbb{R}} f - \int_{E^c} f = \int_E f,$$

where the inequality follows from (2) upon multiplying by -1 , and the last equality is the additivity relation for f recorded above, a legitimate subtraction because $\int_{E^c} f$ is finite. Combining this with (1),

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n \leq \limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f.$$

Hence all three quantities agree, the limit $\lim_{n \rightarrow \infty} \int_E f_n$ exists, and

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

Since E was an arbitrary measurable set, the proof is complete.

Problem 4.23. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n + 1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$.

Solution. For each natural number n put $I_n = [n, n + 1)$ and

$$u_n = a_n \cdot \chi_{I_n}.$$

The intervals $I_1, I_2, \dots$ are pairwise disjoint and their union is $E = [1, \infty)$. Consequently, for each $x \in E$ there is exactly one index n with $x \in I_n$, and for that index $f(x) = a_n = u_n(x)$, while $u_k(x) = 0$ for every $k \neq n$. Therefore

$$f = \sum_{n=1}^{\infty} u_n \quad \text{pointwise on } E,$$

the series having at most one nonzero term at each point.

Each u_n is a nonnegative measurable function on E : indeed I_n is measurable, so χ_{I_n} is measurable, and a constant multiple of a measurable function is measurable. In particular f , being a pointwise sum of measurable functions (equivalently: f is constant on each of the countably many measurable pieces I_n , so $\{x \in E \mid f(x) > c\}$ is a countable union of intervals for every c), is a nonnegative measurable function on E .

Each u_n is a nonnegative simple function of finite support: it takes only the two values a_n and 0, and $\{x \in E \mid u_n(x) \neq 0\} \subseteq I_n$ has measure at most 1. Hence, by the definition of the integral of a simple function,

$$\int_E u_n = a_n \cdot m(I_n) = a_n \cdot 1 = a_n.$$

(If $a_n = 0$ this reads $0 = 0$.)

Now apply Corollary 12 of Chapter 4, which asserts that if $f = \sum_{n=1}^{\infty} u_n$ pointwise a.e. on E with each u_n nonnegative and measurable, then $\int_E f = \sum_{n=1}^{\infty} \int_E u_n$. We conclude

$$\int_E f = \sum_{n=1}^{\infty} \int_E u_n = \sum_{n=1}^{\infty} a_n,$$

the equality holding in the extended sense: both sides are simultaneously finite with the same value, or both equal ∞ .

For completeness, here is the argument behind Corollary 12 in this concrete case. Let $s_N = \sum_{n=1}^N u_n$. Since the u_n are nonnegative, $\{s_N\}$ is an increasing sequence of nonnegative measurable functions on E , and $s_N \rightarrow f$ pointwise on E (for $x \in I_n$ we have $s_N(x) = f(x)$ as soon as $N \geq n$). The Monotone Convergence Theorem and the linearity of integration for nonnegative measurable functions give

$$\int_E f = \lim_{N \rightarrow \infty} \int_E s_N = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_E u_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n = \sum_{n=1}^{\infty} a_n.$$

Problem 4.24. Let f be a nonnegative measurable function on E .

- (i) Show there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on E , each of finite support, which converges pointwise on E to f .
- (ii) Show that $\int_E f = \sup \left\{ \int_E \varphi \mid \varphi \text{ simple, of finite support and } 0 \leq \varphi \leq f \text{ on } E \right\}$.

Solution. Throughout, a function is said to be of finite support when the set on which it fails to vanish has finite measure, and f is permitted to take the value ∞ .

Part (i). For each natural number n define g_n on E by

$$g_n(x) = \min \left\{ n, \frac{\lfloor 2^n f(x) \rfloor}{2^n} \right\},$$

with the convention $g_n(x) = n$ where $f(x) = \infty$, and set

$$E_n = E \cap [-n, n], \quad \varphi_n = g_n \cdot \chi_{E_n}.$$

Each φ_n is a nonnegative simple function of finite support. Indeed, g_n takes values only in the finite set $\{k/2^n \mid k = 0, 1, \dots, n2^n\}$, and for each such value c the set $\{x \in E \mid g_n(x) = c\}$ is measurable because it is described by inequalities on f , which is measurable; hence g_n is simple. Multiplying by χ_{E_n} keeps the range inside that same finite set, and E_n is measurable, so φ_n is simple. Moreover $\{x \in E \mid \varphi_n(x) \neq 0\} \subseteq E_n \subseteq [-n, n]$, a set of measure at most $2n < \infty$; thus φ_n has finite support. Nonnegativity is clear, and since $\lfloor 2^n t \rfloor / 2^n \leq t$ for $t \geq 0$ we also have $0 \leq \varphi_n \leq g_n \leq f$ on E .

The sequence is increasing. First, $\{g_n\}$ is increasing: for a real number $t \geq 0$ we have $2^{n+1}t \geq 2\lfloor 2^n t \rfloor$, and $2\lfloor 2^n t \rfloor$ is an integer, so by the defining property of the floor function $\lfloor 2^{n+1}t \rfloor \geq 2\lfloor 2^n t \rfloor$; dividing by 2^{n+1} gives

$$\frac{\lfloor 2^{n+1}t \rfloor}{2^{n+1}} \geq \frac{\lfloor 2^n t \rfloor}{2^n},$$

and since also $n+1 \geq n$, the minimum of the two increasing quantities increases: $g_{n+1} \geq g_n$ (this also holds trivially where $f = \infty$, where $g_n = n$). Second, $E_n \subseteq E_{n+1}$, so $\chi_{E_n} \leq \chi_{E_{n+1}}$. As all quantities are nonnegative,

$$\varphi_n = g_n \chi_{E_n} \leq g_{n+1} \chi_{E_{n+1}} = \varphi_{n+1} \quad \text{on } E.$$

The sequence converges pointwise to f . Fix $x \in E$. Since x is a real number, $x \in E_n$ for all $n \geq |x|$, and for such n , $\varphi_n(x) = g_n(x)$. If $f(x) < \infty$, then for every $n > f(x)$ we have $\lfloor 2^n f(x) \rfloor / 2^n \leq f(x) < n$, so $g_n(x) = \lfloor 2^n f(x) \rfloor / 2^n$ and

$$0 \leq f(x) - g_n(x) < \frac{1}{2^n},$$

whence $\varphi_n(x) \rightarrow f(x)$. If $f(x) = \infty$, then $\varphi_n(x) = n \rightarrow \infty = f(x)$. In both cases $\varphi_n(x) \rightarrow f(x)$, which proves (i).

Part (ii). Write

$$S = \sup \left\{ \int_E \varphi \mid \varphi \text{ simple, of finite support, } 0 \leq \varphi \leq f \text{ on } E \right\}.$$

First, $S \leq \int_E f$. A simple function φ of finite support with $0 \leq \varphi \leq f$ takes finitely many real values, so it is a bounded measurable function of finite support with $0 \leq \varphi \leq f$ on E . By the very definition of the integral of the nonnegative measurable function f over E ,

$$\int_E f = \sup \left\{ \int_E h \mid h \text{ bounded, measurable, of finite support, } 0 \leq h \leq f \text{ on } E \right\},$$

each such $\int_E \varphi$ is one of the numbers over which this supremum is taken. Hence $S \leq \int_E f$.

Second, $\int_E f \leq S$. Let $\{\varphi_n\}$ be the sequence produced in part (i). Each φ_n is simple, of finite support, and satisfies $0 \leq \varphi_n \leq f$ on E , so $\int_E \varphi_n \leq S$ for every n . Since $\{\varphi_n\}$ is an increasing sequence of nonnegative measurable functions on E converging pointwise on E to f , the Monotone Convergence Theorem gives

$$\int_E f = \lim_{n \rightarrow \infty} \int_E \varphi_n \leq S.$$

The two inequalities together give $\int_E f = S$, which is (ii).

Problem 4.25. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f . Suppose $f_n \leq f$ on E for each n . Show that

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

Solution. The limit function f is nonnegative and measurable on E , being the pointwise limit on E of the sequence $\{f_n\}$ of nonnegative measurable functions. Thus $\int_E f$ is defined as an element of $[0, \infty]$, as is each $\int_E f_n$.

Upper bound. Fix an index n . By hypothesis $0 \leq f_n \leq f$ on E , so the monotonicity of integration for nonnegative measurable functions gives

$$\int_E f_n \leq \int_E f.$$

Since this holds for every n , the upper limit obeys

$$\limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f. \quad (1)$$

Lower bound. The sequence $\{f_n\}$ consists of nonnegative measurable functions on E and converges pointwise on E to f , so Fatou's Lemma applies and gives

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n. \quad (2)$$

Conclusion. Combining (2) and (1),

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n \leq \limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f.$$

Hence the lower and upper limits of the sequence of extended real numbers $\{\int_E f_n\}$ coincide and equal $\int_E f$. Therefore the limit exists in $[0, \infty]$ and

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

Observe that no integrability hypothesis was needed: if $\int_E f = \infty$ the chain of inequalities forces $\int_E f_n \rightarrow \infty$ as well. The hypothesis $f_n \leq f$ is what rules out the loss of mass exhibited by the examples following Fatou's Lemma (for instance $f_n = n \cdot \chi_{(0,1/n)}$ on $E = (0, 1]$, where no such dominating limit function is available since $f \equiv 0$ there).

Problem 4.26. Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

Solution. The Monotone Convergence Theorem asserts that if $\{f_n\}$ is an increasing sequence of nonnegative measurable functions on E converging pointwise a.e. on E to f , then $\int_E f_n \rightarrow \int_E f$. We exhibit a decreasing sequence of nonnegative measurable functions converging pointwise everywhere for which the conclusion fails.

An example on a set of infinite measure. Let $E = \mathbb{R}$ and for each natural number n define

$$f_n = \chi_{[n, \infty)}.$$

Each f_n is nonnegative and measurable, being the characteristic function of a measurable set. Since $[n+1, \infty) \subseteq [n, \infty)$, we have $f_{n+1} \leq f_n$ on $\mathbb{R}$, so $\{f_n\}$ is decreasing. It converges pointwise on all of $\mathbb{R}$ to $f \equiv 0$: given $x \in \mathbb{R}$, choose a natural number $n_0 > x$; then $x \notin [n, \infty)$ and so $f_n(x) = 0$ for all $n \geq n_0$.

Now compute the integrals. For each n and each natural number k , the bounded measurable function

$\chi_{[n,n+k]}$ has finite support and satisfies $0 \leq \chi_{[n,n+k]} \leq f_n$ on $\mathbb{R}$, and $\int_{\mathbb{R}} \chi_{[n,n+k]} = m([n, n+k]) = k$. Since $\int_{\mathbb{R}} f_n$ is the supremum of the integrals of such functions, $\int_{\mathbb{R}} f_n \geq k$ for every k , that is,

$$\int_{\mathbb{R}} f_n = \infty \quad \text{for every } n.$$

On the other hand $\int_{\mathbb{R}} f = \int_{\mathbb{R}} 0 = 0$. Therefore

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n = \infty \neq 0 = \int_{\mathbb{R}} f,$$

so the conclusion of the Monotone Convergence Theorem fails for this decreasing sequence.

An example on a set of finite measure. Infinite measure is not the essential feature. Let $E = (0, 1)$ and define

$$f_n(x) = \frac{1}{x} \cdot \chi_{(0, 1/n)}(x) \quad \text{for } x \in (0, 1).$$

Each f_n is nonnegative and measurable, and since $(0, 1/(n+1)) \subseteq (0, 1/n)$ the sequence is decreasing. It converges pointwise on $(0, 1)$ to $f \equiv 0$: for fixed $x \in (0, 1)$ we have $f_n(x) = 0$ once $1/n \leq x$. Yet $\int_{(0,1)} f_n = \infty$ for every n . To see this, fix n and choose a natural number N with $2^{-N} < 1/n$. For $j \geq N$ set $A_j = (2^{-(j+1)}, 2^{-j})$; these are pairwise disjoint subsets of $(0, 1/n)$, on each of which $1/x > 2^j$, and $m(A_j) = 2^{-(j+1)}$. For a natural number $M \geq N$ the function

$$h_M = \sum_{j=N}^M 2^j \chi_{A_j}$$

is bounded, measurable, of finite support, and satisfies $0 \leq h_M \leq f_n$ on $(0, 1)$, while

$$\int_{(0,1)} h_M = \sum_{j=N}^M 2^j \cdot 2^{-(j+1)} = \frac{M - N + 1}{2}.$$

Since $\int_{(0,1)} f_n$ is the supremum of the integrals of such functions and M is arbitrary, $\int_{(0,1)} f_n = \infty$. Meanwhile $\int_{(0,1)} f = 0$, so again $\lim_n \int_E f_n \neq \int_E f$.

Remark. The failure in both examples is caused by non-integrability of the first term, and this is no accident. Suppose $\{f_n\}$ is a decreasing sequence of nonnegative measurable functions on E converging pointwise on E to f , and suppose f_1 is integrable over E . By Proposition 13 of Chapter 4, f_1 is finite a.e. on E , so after excising a set of measure zero (permissible by (15)) we may assume f_1 is finite on E . Then $\{f_1 - f_n\}$ is an increasing sequence of nonnegative measurable functions converging pointwise on E to $f_1 - f$, so the Monotone Convergence Theorem gives $\int_E (f_1 - f_n) \rightarrow \int_E (f_1 - f)$. Now $f_1 - f_n$ and f_n are nonnegative measurable functions on E whose sum is f_1 , so the linearity of integration for nonnegative measurable functions, Theorem 10 of Chapter 4, gives $\int_E (f_1 - f_n) + \int_E f_n = \int_E f_1$; since $0 \leq f_n \leq f_1$ and $\int_E f_1 < \infty$, monotonicity makes both summands finite and we may write $\int_E (f_1 - f_n) = \int_E f_1 - \int_E f_n$. The same argument applied to $f_1 - f$ and f gives $\int_E (f_1 - f) = \int_E f_1 - \int_E f$. Cancelling the finite number $\int_E f_1$ yields $\int_E f_n \rightarrow \int_E f$. Hence any counterexample must have $\int_E f_n = \infty$ for every n , as above.

Problem 4.27. Prove the following generalization of Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions on E , then

$$\int_E \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

Solution. No convergence of $\{f_n\}$ is assumed; the point is that the pointwise lower limit always exists as an extended real-valued function and obeys the Fatou inequality.

For each natural number n define the function g_n on E by

$$g_n(x) = \inf_{k \geq n} f_k(x), \quad x \in E,$$

and set $g = \liminf_{n \rightarrow \infty} f_n$, that is, $g(x) = \lim_{n \rightarrow \infty} g_n(x)$ for $x \in E$.

Step 1: each g_n is a nonnegative measurable function on E . Nonnegativity is immediate, since each $f_k \geq 0$. For measurability, note that it suffices to show that $\{x \in E \mid g_n(x) \geq c\}$ is measurable for every real number c , since the complements of these sets are the sets $\{g_n < c\}$ and measurability of a function is equivalent to measurability of all the sets $\{g_n < c\}$. Now for every real c ,

$$\{x \in E \mid g_n(x) \geq c\} = \bigcap_{k \geq n} \{x \in E \mid f_k(x) \geq c\},$$

because $\inf_{k \geq n} f_k(x) \geq c$ precisely when $f_k(x) \geq c$ for every $k \geq n$. Each set on the right is measurable since f_k is measurable, and the countable intersection of measurable sets is measurable. As the sets $\{g_n \geq c\}$ are measurable for every real c , the function g_n is measurable. (This is the standard fact that the pointwise infimum of a countable family of measurable functions is measurable.)

Step 2: $\{g_n\}$ is increasing and converges pointwise on E to g . Fix $x \in E$. The family $\{f_k(x) \mid k \geq n+1\}$ is a subfamily of $\{f_k(x) \mid k \geq n\}$, so its infimum is at least as large:

$$g_n(x) \leq g_{n+1}(x).$$

Thus $\{g_n(x)\}$ is an increasing sequence in $[0, \infty]$, hence convergent in $[0, \infty]$, and its limit is by definition $\liminf_n f_n(x) = g(x)$. In particular g is a nonnegative measurable function on E , being a pointwise limit of nonnegative measurable functions, so $\int_E g$ is defined.

Step 3: comparison of integrals. Fix n and let $k \geq n$. By construction $0 \leq g_n \leq f_k$ on E , so the monotonicity of integration for nonnegative measurable functions gives

$$\int_E g_n \leq \int_E f_k \quad \text{for every } k \geq n.$$

Taking the infimum over $k \geq n$ on the right,

$$\int_E g_n \leq \inf_{k \geq n} \int_E f_k. \quad (*)$$

Step 4: pass to the limit. The sequence $\{g_n\}$ is an increasing sequence of nonnegative measurable functions on E converging pointwise on E to g , so the Monotone Convergence Theorem yields

$$\int_E g = \lim_{n \rightarrow \infty} \int_E g_n.$$

The right-hand sides of $(*)$ form an increasing sequence in $[0, \infty]$ whose limit is, by definition, $\liminf_{n \rightarrow \infty} \int_E f_n$. Letting $n \rightarrow \infty$ in $(*)$ therefore gives

$$\int_E \liminf_{n \rightarrow \infty} f_n = \int_E g = \lim_{n \rightarrow \infty} \int_E g_n \leq \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} \int_E f_k \right) = \liminf_{n \rightarrow \infty} \int_E f_n,$$

which is the asserted inequality.

This contains Fatou's Lemma as stated in the text: if $\{f_n\} \rightarrow f$ pointwise a.e. on E , then $\liminf_n f_n = f$ a.e. on E , and integrals over E are unchanged by altering the integrand on a set of measure zero, by (15).

Problem 4.28. Let f be integrable over E and C a measurable subset of E . Show that $\int_C f = \int_E f \cdot \chi_C$.

Solution. Here χ_C denotes the characteristic function of C , regarded as a function on E ; since C is measurable, χ_C is measurable on E , and therefore so is the product $f \cdot \chi_C$.

Step 1: the identity for nonnegative measurable functions. Let h be a nonnegative measurable function on E and let $C \subseteq E$ be measurable. We claim

$$\int_C h = \int_E h \cdot \chi_C. \quad (*)$$

Indeed, $h \cdot \chi_C$ is a nonnegative measurable function on E , and C and $E \setminus C$ are disjoint measurable subsets of E with union E . By Theorem 11 of Chapter 4 (Additivity Over Domains of Integration),

$$\int_E h \cdot \chi_C = \int_C h \cdot \chi_C + \int_{E \setminus C} h \cdot \chi_C.$$

On C we have $\chi_C \equiv 1$, so $h \cdot \chi_C = h$ there and the first integral equals $\int_C h$; the integral over a set depends only on the restriction of the integrand to that set. On $E \setminus C$ we have $\chi_C \equiv 0$, so $h \cdot \chi_C$ is the zero function there and the second integral is 0 (the only bounded measurable function ψ of finite support with $0 \leq \psi \leq 0$ is $\psi \equiv 0$, whose integral is 0; hence the defining supremum is 0). This proves (*).

Step 2: integrability of the two sides. Since f is integrable over E , $|f|$ is integrable over E . Because $|f \cdot \chi_C| \leq |f|$ on E , Proposition 16 of Chapter 4 (the Integral Comparison Test) shows that $f \cdot \chi_C$ is integrable over E , so the right-hand side of the desired identity is a well-defined real number. Moreover, applying (*) with $h = |f|$,

$$\int_C |f| = \int_E |f| \cdot \chi_C \leq \int_E |f| < \infty,$$

where the inequality is the monotonicity of integration for nonnegative measurable functions applied to $|f| \cdot \chi_C \leq |f|$. Hence $|f|$ is integrable over C , i.e. f is integrable over C , and the left-hand side is a well-defined real number too.

Step 3: reduction to positive and negative parts. Write $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are nonnegative measurable functions on E , each integrable over E by Proposition 14 of Chapter 4. At each point of E , at most one of f^+, f^- is nonzero, and multiplying by the factor χ_C , which takes only the values 0 and 1, preserves the description of the positive and negative parts: for $x \in E$,

$$(f \cdot \chi_C)^+(x) = \max\{f(x)\chi_C(x), 0\} = \chi_C(x) \max\{f(x), 0\} = f^+(x)\chi_C(x),$$

using $\chi_C(x) \in \{0, 1\}$, and likewise $(f \cdot \chi_C)^- = f^- \cdot \chi_C$ on E .

Therefore, by the definition of the general Lebesgue integral applied on C and then on E , together with (*) applied to the nonnegative measurable functions f^+ and f^- ,

$$\begin{aligned} \int_C f &= \int_C f^+ - \int_C f^- \\ &= \int_E f^+ \cdot \chi_C - \int_E f^- \cdot \chi_C \\ &= \int_E (f \cdot \chi_C)^+ - \int_E (f \cdot \chi_C)^- \\ &= \int_E f \cdot \chi_C. \end{aligned}$$

All four integrals of nonnegative functions appearing here are finite, by Step 2 and monotonicity, so the subtractions are legitimate. This is the asserted identity.

Note that the argument uses only Theorem 11, which concerns nonnegative measurable functions; it does not use Corollary 18 (Additivity Over Domains of Integration for integrable functions), whose proof in the text invokes the present result.

4.5 Problems 4.29–4.35

Problem 4.29. For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n . Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

Solution. The answer to both questions is no: in each case one implication is true and the other fails. First observe that each a_n is a well-defined real number. Indeed, f is measurable and bounded on the bounded set $[n, n+1]$, which has finite measure, so $|f| \leq M_n$ there for some $M_n \geq 0$ and hence, by the monotonicity of integration for nonnegative functions, $\int_{[n, n+1]} |f| \leq M_n \cdot 1 < \infty$. Thus f is integrable over $[n, n+1]$ and $a_n = \int_n^{n+1} f$ is finite.

The implication that is true. Suppose f is integrable over $[1, \infty)$. For $n \geq 1$ put $u_n = |f| \cdot \chi_{[n, n+1)}$, a nonnegative measurable function on $[1, \infty)$. Since the intervals $[n, n+1)$ are disjoint with union $[1, \infty)$, we have $|f| = \sum_{n=1}^{\infty} u_n$ pointwise on $[1, \infty)$, and therefore, by Corollary 12 of Chapter 4,

$$\int_{[1, \infty)} |f| = \sum_{n=1}^{\infty} \int_{[1, \infty)} u_n = \sum_{n=1}^{\infty} \int_{[n, n+1)} |f|,$$

where the last equality is the additivity over domains of integration for nonnegative functions (Theorem 11 of Chapter 4), which gives $\int_{[1, \infty)} u_n = \int_{[n, n+1)} u_n + \int_{[1, \infty) \setminus [n, n+1)} u_n = \int_{[n, n+1)} |f| + 0$. By the Integral Comparison Test (Proposition 16 of Chapter 4),

$$|a_n| = \left| \int_{[n, n+1)} f \right| \leq \int_{[n, n+1)} |f|,$$

where we have used that the value of the integral of an integrable function is unchanged by excising the single point $n+1$, a set of measure zero, so that $a_n = \int_{[n, n+1]} f = \int_{[n, n+1)} f$; this is (18) of Proposition 15 of Chapter 4. Hence

$$\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \int_{[n, n+1)} |f| = \int_{[1, \infty)} |f| < \infty.$$

So integrability of f implies that $\sum_{n=1}^{\infty} a_n$ converges absolutely, and in particular that it converges. This is the true half of both assertions.

The first converse fails. Define f on $[1, \infty)$ by

$$f(x) = \frac{(-1)^n}{n} \quad \text{for } n \leq x < n+1, \quad n = 1, 2, \dots$$

Then f is measurable (it is constant on each interval of a countable disjoint measurable decomposition of $[1, \infty)$) and $|f| \leq 1$, so f is certainly bounded on bounded sets. Here $a_n = (-1)^n/n$, and $\sum_{n=1}^{\infty} (-1)^n/n$ converges by the alternating series test. But by the computation above,

$$\int_{[1, \infty)} |f| = \sum_{n=1}^{\infty} \int_{[n, n+1)} |f| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so $|f|$ is not integrable, that is, f is not integrable over $[1, \infty)$. Thus convergence of $\sum_{n=1}^{\infty} a_n$ does not imply integrability, and the answer to the first question is no.

The second converse fails. Define f on $[1, \infty)$ by setting, for each natural number n , $f(x) = 1$ for $n \leq x < n+1/2$ and $f(x) = -1$ for $n+1/2 \leq x < n+1$. Again f is measurable and $|f| \equiv 1$, so f is bounded on bounded sets. Fix n . Excising the single point $n+1$, which changes no integral by (18), and then applying additivity over domains for integrable functions (Corollary 18 of Chapter 4) to the disjoint decomposition $[n, n+1) = [n, n+1/2) \cup [n+1/2, n+1)$,

$$a_n = \int_{[n, n+1)} f = \int_{[n, n+1/2)} 1 + \int_{[n+1/2, n+1)} (-1) = \frac{1}{2} - \frac{1}{2} = 0.$$

Hence $\sum_{n=1}^{\infty} |a_n| = 0 < \infty$, so the series converges absolutely. Yet $\int_{[1,\infty)} |f| = \sum_{n=1}^{\infty} \int_{[n,n+1)} 1 = \sum_{n=1}^{\infty} 1 = \infty$, so f is not integrable over $[1, \infty)$. Thus the answer to the second question is also no.

Remark. What is true in general is that f is integrable over $[1, \infty)$ if and only if $\sum_{n=1}^{\infty} b_n < \infty$, where $b_n = \int_n^{n+1} |f|$; this is exactly the identity $\int_{[1,\infty)} |f| = \sum_{n=1}^{\infty} b_n$ established above. In particular, if f is nonnegative, then $a_n = b_n$ and both assertions of the problem do hold: f is integrable over $[1, \infty)$ if and only if $\sum_{n=1}^{\infty} a_n$ converges, and for a series of nonnegative terms convergence and absolute convergence coincide.

Problem 4.30. Let g be a nonnegative integrable function over E and suppose $\{f_n\}$ is a sequence of measurable functions on E such that for each n , $|f_n| \leq g$ a.e. on E . Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

Solution. We first record the generalized form of Fatou's Lemma that is the content of Problem 27: if $\{h_n\}$ is a sequence of nonnegative measurable functions on E , then

$$\int_E \liminf h_n \leq \liminf \int_E h_n.$$

Call this the generalized Fatou inequality. Indeed, put $u_k = \inf_{n \geq k} h_n$. Each u_k is nonnegative and measurable, $\{u_k\}$ is increasing, and $u_k \rightarrow \liminf h_n$ pointwise on E . By the Monotone Convergence Theorem, $\lim_{k \rightarrow \infty} \int_E u_k = \int_E \liminf h_n$. On the other hand $u_k \leq h_n$ on E for every $n \geq k$, so by the monotonicity of integration for nonnegative functions $\int_E u_k \leq \inf_{n \geq k} \int_E h_n$. Letting $k \rightarrow \infty$ gives the generalized Fatou inequality.

Now we set up the excision. For each n let E_n be the set of measure zero outside of which $|f_n| \leq g$, and let E_0 be the set of measure zero outside of which g is finite; such a set exists because g is a nonnegative integrable function, by Proposition 13 of Chapter 4. The set $Z = E_0 \cup \bigcup_{n=1}^{\infty} E_n$ has measure zero by the countable subadditivity of Lebesgue measure. By the excision property (15) for nonnegative measurable functions, and by (18) of Proposition 15 of Chapter 4 for the integrable ones, the integral over E of any of the functions appearing in the asserted inequalities equals its integral over $E \sim Z$, and the functions $\liminf f_n$, $\limsup f_n$ are unchanged off Z . Hence, replacing E by $E \sim Z$, we may and do assume that

$$|f_n| \leq g \text{ on } E \text{ for every } n, \quad \text{and } g \text{ is finite on } E.$$

Three preliminary integrability remarks. First, each f_n is measurable and dominated in absolute value by the integrable function g , so by the Integral Comparison Test (Proposition 16 of Chapter 4) each f_n is integrable over E and $\int_E f_n$ is a real number; thus the two middle quantities in the assertion are the lower and upper limits of a bounded sequence of real numbers, and the middle inequality $\liminf \int_E f_n \leq \limsup \int_E f_n$ is immediate. Second, since $|f_n| \leq g$ for all n we also have $|\liminf f_n| \leq g$ and $|\limsup f_n| \leq g$ on E , so both $\liminf f_n$ and $\limsup f_n$ are integrable over E by the same Integral Comparison Test; in particular all four quantities in the assertion are finite. Third, $g + f_n$ and $g - f_n$ are properly defined (because g is finite), measurable, integrable over E by the Integral Comparison Test since $|g \pm f_n| \leq 2g$, and nonnegative on E because $|f_n| \leq g$.

The left-hand inequality. Apply the generalized Fatou inequality to the nonnegative measurable sequence $\{g + f_n\}$. Since g is finite, $\liminf(g + f_n) = g + \liminf f_n$ pointwise on E , so

$$\int_E g + \int_E \liminf f_n = \int_E (g + \liminf f_n) \leq \liminf \int_E (g + f_n) = \int_E g + \liminf \int_E f_n,$$

where the first and last equalities use the linearity of integration (Theorem 17 of Chapter 4), legitimate because g , $\liminf f_n$ and each f_n are integrable over E , and the last equality also uses that $\int_E g$ is a fixed real number, so that $\liminf(\int_E g + \int_E f_n) = \int_E g + \liminf \int_E f_n$. Since $\int_E g$ is finite, it may be cancelled from both ends, giving

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

The right-hand inequality. Apply the generalized Fatou inequality to the nonnegative measurable sequence $\{g - f_n\}$. Since g is finite, $\liminf(g - f_n) = g - \limsup f_n$ pointwise on E , so, again by Theorem 17,

$$\int_E g - \int_E \limsup f_n \leq \liminf \int_E (g - f_n) = \int_E g + \liminf \left(- \int_E f_n \right) = \int_E g - \limsup \int_E f_n.$$

Cancelling the finite number $\int_E g$ and multiplying by -1 reverses the inequality and yields

$$\limsup \int_E f_n \leq \int_E \limsup f_n.$$

Combining the three inequalities gives the asserted chain.

Problem 4.31. Let f be a measurable function on E which can be expressed as $f = g + h$ on E , where g is finite and integrable over E and h is nonnegative on E . Define $\int_E f = \int_E g + \int_E h$. Show that this is properly defined in the sense that it is independent of the particular choice of finite integrable function g and nonnegative function h whose sum is f .

Solution. First note that the proposed definition makes sense for any single such decomposition. If $f = g + h$ with g finite, measurable and integrable over E and $h \geq 0$ on E , then $h = f - g$ is a properly defined extended real-valued function (no expression $\infty - \infty$ arises, since g is finite) and it is measurable, being the difference of the measurable functions f and g . Therefore $\int_E h$ is defined as an element of $[0, \infty]$, while $\int_E g$ is a real number, and the sum $\int_E g + \int_E h$ is an unambiguous element of $(-\infty, \infty]$.

Now suppose we have two such decompositions,

$$f = g_1 + h_1 = g_2 + h_2 \quad \text{on } E,$$

where g_1, g_2 are finite and integrable over E and h_1, h_2 are nonnegative measurable functions on E . We must show

$$\int_E g_1 + \int_E h_1 = \int_E g_2 + \int_E h_2.$$

The difficulty is that h_1 and h_2 may take the value ∞ , so we may not simply subtract. We therefore rearrange the identity $g_1 + h_1 = g_2 + h_2$ into an identity between nonnegative functions. Write $g_i = g_i^+ - g_i^-$, where g_i^+ and g_i^- are the nonnegative and finite positive and negative parts of g_i ; each of $g_1^+, g_1^-, g_2^+, g_2^-$ is a nonnegative measurable function on E which is integrable over E , by Proposition 14 of Chapter 4. Adding the finite nonnegative function $g_1^- + g_2^-$ to both sides of $g_1 + h_1 = g_2 + h_2$ gives

$$h_1 + g_1^+ + g_2^- = h_2 + g_2^+ + g_1^- \quad \text{on } E.$$

Let us verify this identity carefully, since extended values occur. At a point x where $h_1(x)$ and $h_2(x)$ are both finite, every term is a real number and the identity is the elementary rearrangement

$$h_1 + (g_1 + g_1^-) + g_2^- = (g_1 + h_1) + g_1^- + g_2^- = (g_2 + h_2) + g_2^- + g_1^- = h_2 + (g_2 + g_2^-) + g_1^-,$$

using $g_i + g_i^- = g_i^+$. At a point x where $h_1(x) = \infty$, we get $f(x) = g_1(x) + \infty = \infty$ because $g_1(x)$ is finite; since $g_2(x)$ is finite as well and $f(x) = g_2(x) + h_2(x)$, necessarily $h_2(x) = \infty$. Thus $h_1(x) = \infty$ if and only if $h_2(x) = \infty$, and at such a point both sides of the displayed identity equal ∞ (each is the sum of ∞ with finite nonnegative numbers). So the identity holds at every point of E , and every function appearing in it is nonnegative and measurable.

Integrate. By the linearity of integration for nonnegative measurable functions,

$$\int_E h_1 + \int_E g_1^+ + \int_E g_2^- = \int_E h_2 + \int_E g_2^+ + \int_E g_1^-,$$

an equality of elements of $[0, \infty]$. The four numbers $\int_E g_1^+, \int_E g_1^-, \int_E g_2^+, \int_E g_2^-$ are finite because g_1 and g_2 are integrable over E . Hence we may subtract the finite number $\int_E g_1^- + \int_E g_2^-$ from both sides, which is a legitimate operation in $(-\infty, \infty]$, to obtain

$$\int_E h_1 + \left(\int_E g_1^+ - \int_E g_1^- \right) = \int_E h_2 + \left(\int_E g_2^+ - \int_E g_2^- \right),$$

that is, by the definition of the integral of an integrable function,

$$\int_E h_1 + \int_E g_1 = \int_E h_2 + \int_E g_2.$$

This is precisely the required independence: the value $\int_E g + \int_E h$ does not depend on the decomposition chosen. (In particular both sides are simultaneously $+\infty$, or simultaneously the same real number.)

Finally, the new definition is consistent with the old one. If f is itself integrable over E , then f is finite a.e. on E by Proposition 15 of Chapter 4; excising a set of measure zero, which changes no integral by (18) of that same proposition, we may take f finite on E and use the decomposition $g = f$, $h \equiv 0$, for which $\int_E g + \int_E h = \int_E f$, the integral of f in the sense of Section 4.4. By the independence just proved, every admissible decomposition gives this same value.

Problem 4.32. Prove the General Lebesgue Dominated Convergence Theorem by following the proof of the Lebesgue Dominated Convergence Theorem, but replacing the sequences $\{g - f_n\}$ and $\{g + f_n\}$, respectively, by $\{g_n - f_n\}$ and $\{g_n + f_n\}$.

Solution. We must prove Theorem 19 of Chapter 4: let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f , and suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ in the sense that $|f_n| \leq g_n$ on E for all n ; if $\lim_{n \rightarrow \infty} \int_E g_n = \int_E g < \infty$, then $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.

Preliminaries. Since $\lim_{n \rightarrow \infty} \int_E g_n = \int_E g < \infty$, there is an index N such that $\int_E g_n < \infty$ for all $n \geq N$; that is, g_n is integrable over E for all $n \geq N$. As the conclusion concerns only the behavior of $\int_E f_n$ for large n , we may discard the first $N - 1$ terms and assume that every g_n is integrable over E . Then, for each n , f_n is measurable and $|f_n| \leq g_n$ with g_n nonnegative and integrable, so by the Integral Comparison Test (Proposition 16 of Chapter 4) each f_n is integrable over E and $\int_E f_n$ is a real number.

Next, g is nonnegative and measurable, being an a.e. pointwise limit of such functions, and $\int_E g < \infty$, so g is integrable over E and hence finite a.e. on E by Proposition 13 of Chapter 4. Also each g_n is finite a.e. on E , for the same reason. Passing to the limit in $|f_n| \leq g_n$ at each point where both $f_n \rightarrow f$ and $g_n \rightarrow g$, we get $|f| \leq g$ a.e. on E ; since f is measurable as an a.e. pointwise limit of measurable functions, the Integral Comparison Test shows f is integrable over E .

Let Z be the union of the following sets of measure zero: the set where $\{f_n\}$ fails to converge to f , the set where $\{g_n\}$ fails to converge to g , the set where g is infinite, the set where $|f| > g$, and, for each n , the set where g_n is infinite. By the countable subadditivity of Lebesgue measure, $m(Z) = 0$; by the excision property (15) applied to positive and negative parts, no integral occurring below is changed if E is replaced by $E \sim Z$. So we may and do assume that on E we have: $f_n \rightarrow f$ pointwise, $g_n \rightarrow g$ pointwise, every g_n and g are finite, $|f_n| \leq g_n$ for all n , and $|f| \leq g$.

The two Fatou arguments. For each n the functions $g_n - f_n$ and $g_n + f_n$ are properly defined (all the functions involved are finite on E), measurable, and nonnegative on E , since $|f_n| \leq g_n$. Moreover $g_n - f_n \rightarrow g - f$ and $g_n + f_n \rightarrow g + f$ pointwise on E , and $g - f$, $g + f$ are nonnegative because $|f| \leq g$. Apply Fatou's Lemma to $\{g_n - f_n\}$:

$$\int_E (g - f) \leq \liminf \int_E (g_n - f_n).$$

Both g_n and f_n are integrable over E , so by the linearity of integration (Theorem 17 of Chapter 4), $\int_E (g_n - f_n) = \int_E g_n - \int_E f_n$; likewise $\int_E (g - f) = \int_E g - \int_E f$, since g and f are integrable over E .

Because $\lim_{n \rightarrow \infty} \int_E g_n$ exists and is the finite number $\int_E g$, we may split the lower limit:

$$\liminf \left(\int_E g_n - \int_E f_n \right) = \int_E g + \liminf \left(- \int_E f_n \right) = \int_E g - \limsup \int_E f_n.$$

Hence

$$\int_E g - \int_E f \leq \int_E g - \limsup \int_E f_n,$$

and cancelling the finite number $\int_E g$ gives

$$\limsup \int_E f_n \leq \int_E f.$$

Now apply Fatou's Lemma to $\{g_n + f_n\}$:

$$\int_E (g + f) \leq \liminf \int_E (g_n + f_n).$$

Exactly as before, by Theorem 17 and the finiteness of $\lim_{n \rightarrow \infty} \int_E g_n = \int_E g$,

$$\int_E g + \int_E f \leq \int_E g + \liminf \int_E f_n,$$

so that

$$\int_E f \leq \liminf \int_E f_n.$$

Conclusion. Combining the two displayed inequalities,

$$\int_E f \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E f,$$

so all four quantities are equal and $\lim_{n \rightarrow \infty} \int_E f_n$ exists and equals $\int_E f$. This is the assertion of Theorem 19.

Note that the hypothesis $\lim_{n \rightarrow \infty} \int_E g_n = \int_E g < \infty$ was used exactly where the ordinary Dominated Convergence Theorem uses the constancy of the dominating function: it is what permits the terms $\int_E g_n$ to be pulled out of the lower limit as the finite number $\int_E g$ and then cancelled.

Problem 4.33. Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \rightarrow f$ a.e. on E and f is integrable over E . Show that $\int_E |f - f_n| \rightarrow 0$ if and only if $\lim_{n \rightarrow \infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

Solution. Throughout, f and each f_n are integrable over E , so $|f|$ and each $|f_n|$ are nonnegative integrable functions on E and all the integrals below are finite real numbers. Also, $f - f_n$ is measurable and $|f - f_n| \leq |f| + |f_n|$, which is integrable, so by the Integral Comparison Test (Proposition 16 of Chapter 4) $f - f_n$ is integrable over E and $\int_E |f - f_n| < \infty$.

Necessity. Suppose $\int_E |f - f_n| \rightarrow 0$. For each n , the reverse triangle inequality for real numbers gives

$$||f_n(x)| - |f(x)|| \leq |f(x) - f_n(x)| \quad \text{for a.e. } x \in E$$

(at every point where f and f_n are finite, hence a.e. on E by Proposition 15 of Chapter 4). By the linearity of integration (Theorem 17 of Chapter 4), the Integral Comparison Test, and the monotonicity of integration,

$$\left| \int_E |f_n| - \int_E |f| \right| = \left| \int_E (|f_n| - |f|) \right| \leq \int_E ||f_n| - |f|| \leq \int_E |f - f_n|.$$

Since the right-hand side tends to 0, we conclude $\lim_{n \rightarrow \infty} \int_E |f_n| = \int_E |f|$. Observe that this half did not use the pointwise convergence of $\{f_n\}$.

Sufficiency. Suppose $\lim_{n \rightarrow \infty} \int_E |f_n| = \int_E |f|$. Define, for each n ,

$$h_n = |f - f_n| \quad \text{and} \quad g_n = |f| + |f_n| \quad \text{on } E.$$

We check the hypotheses of the General Lebesgue Dominated Convergence Theorem (Theorem 19 of Chapter 4) for the sequence $\{h_n\}$ dominated by $\{g_n\}$.

Each h_n is measurable, and $h_n \rightarrow 0$ a.e. on E : indeed at a.e. point x we have $f_n(x) \rightarrow f(x)$ with $f(x)$ finite (Proposition 15), so $|f(x) - f_n(x)| \rightarrow 0$. Each g_n is nonnegative and measurable, and $g_n \rightarrow 2|f|$ a.e. on E , since $|f_n| \rightarrow |f|$ a.e. by the continuity of the absolute value. The domination $|h_n| = |f - f_n| \leq |f| + |f_n| = g_n$ holds a.e. on E (at every point where both f and f_n are finite); excising a set of measure zero, which by (15) changes none of the integrals involved, we may assume it holds on all of E .

Finally, the crucial hypothesis on the integrals of the dominating sequence holds: by the linearity of integration for nonnegative functions and the assumed convergence,

$$\lim_{n \rightarrow \infty} \int_E g_n = \lim_{n \rightarrow \infty} \left(\int_E |f| + \int_E |f_n| \right) = \int_E |f| + \int_E |f| = \int_E 2|f| < \infty,$$

the finiteness because f is integrable over E , and $\int_E 2|f|$ is precisely the integral of the a.e. pointwise limit of $\{g_n\}$.

Therefore Theorem 19 applies to $\{h_n\}$ with dominating sequence $\{g_n\}$ and limit function 0, and yields

$$\lim_{n \rightarrow \infty} \int_E |f - f_n| = \lim_{n \rightarrow \infty} \int_E h_n = \int_E 0 = 0.$$

This completes the proof of both implications.

Remark. The ordinary Lebesgue Dominated Convergence Theorem is not enough here: there is in general no single integrable function dominating all the $|f - f_n|$, and this is exactly the situation the generalized theorem was designed for.

Problem 4.34. Let f be a nonnegative measurable function on $\mathbb{R}$. Show that

$$\lim_{n \rightarrow \infty} \int_{-n}^n f = \int_{\mathbb{R}} f.$$

Solution. Here $\int_{-n}^n f$ denotes $\int_{[-n,n]} f$, and since f is nonnegative and measurable all the integrals occurring are defined as elements of $[0, \infty]$; the assertion is an equality in $[0, \infty]$.

For each natural number n define

$$f_n = f \cdot \chi_{[-n,n]} \quad \text{on } \mathbb{R}.$$

Each f_n is nonnegative and measurable, being the product of the nonnegative measurable function f with the characteristic function of a measurable set.

The sequence $\{f_n\}$ is increasing on $\mathbb{R}$. Indeed, $[-n, n] \subseteq [-(n+1), n+1]$, so $\chi_{[-n,n]} \leq \chi_{[-(n+1), n+1]}$ and, f being nonnegative, $f_n \leq f_{n+1}$ on $\mathbb{R}$.

The sequence $\{f_n\}$ converges pointwise on $\mathbb{R}$ to f . Let $x \in \mathbb{R}$ and choose a natural number $N > |x|$, which is possible by the Archimedean property. Then $x \in [-n, n]$ for all $n \geq N$, so $f_n(x) = f(x)$ for all $n \geq N$; in particular $f_n(x) \rightarrow f(x)$.

The Monotone Convergence Theorem therefore applies and gives

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f,$$

this limit existing in $[0, \infty]$ since $\{\int_{\mathbb{R}} f_n\}$ is increasing by the monotonicity of integration for nonnegative functions.

It remains to identify $\int_{\mathbb{R}} f_n$ with $\int_{-n}^n f$. The sets $[-n, n]$ and $\mathbb{R} \setminus [-n, n]$ are disjoint measurable sets whose union is $\mathbb{R}$, so by the additivity over domains of integration for nonnegative measurable functions

(Theorem 11 of Chapter 4),

$$\int_{\mathbb{R}} f_n = \int_{[-n,n]} f_n + \int_{\mathbb{R} \sim [-n,n]} f_n.$$

On $[-n, n]$ we have $f_n = f$, so $\int_{[-n,n]} f_n = \int_{[-n,n]} f = \int_{-n}^n f$. On $\mathbb{R} \sim [-n, n]$ we have $f_n \equiv 0$, so $\int_{\mathbb{R} \sim [-n,n]} f_n = 0$ by the monotonicity of integration for nonnegative functions (the integral of the zero function over any measurable set is 0, being the supremum of the integrals of bounded measurable functions of finite support lying below it, all of which are 0). Hence

$$\int_{\mathbb{R}} f_n = \int_{-n}^n f \quad \text{for every } n.$$

Substituting this into the conclusion of the Monotone Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{-n}^n f = \int_{\mathbb{R}} f,$$

as required. Note that no integrability assumption was needed: if $\int_{\mathbb{R}} f = \infty$, the assertion says that the increasing sequence $\{\int_{-n}^n f\}$ diverges to ∞ , which is exactly what the Monotone Convergence Theorem delivers in the extended sense.

Problem 4.35. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value of y . Suppose for each fixed value of x , $\lim_{y \rightarrow 0} f(x, y) = f(x)$ and that for all y , we have $|f(x, y)| \leq g(x)$, where g is integrable over $[0, 1]$. Show that

$$\lim_{y \rightarrow 0} \int_0^1 f(x, y) dx = \int_0^1 f(x) dx.$$

Also show that if the function $f(x, y)$ is continuous in y for each x , then

$$h(y) = \int_0^1 f(x, y) dx$$

is a continuous function of y .

Solution. Throughout, $E = [0, 1]$ and, for $y \in [0, 1]$, we write f_y for the function $x \mapsto f(x, y)$ on E . By hypothesis each f_y is measurable on E and $|f_y| \leq g$ on E , and g is integrable over E ; note that g is automatically nonnegative, since $g(x) \geq |f(x, 0)| \geq 0$ for every $x \in E$. Hence, by the Integral Comparison Test (Proposition 16 of Chapter 4), each f_y is integrable over E , so

$$H(y) = \int_0^1 f(x, y) dx$$

is a well-defined real number for every $y \in [0, 1]$.

The limit function f is measurable and integrable. Fix any sequence $\{y_n\}$ in $(0, 1]$ with $y_n \rightarrow 0$. Then for each $x \in E$ we have $f_{y_n}(x) = f(x, y_n) \rightarrow f(x)$ by the hypothesis $\lim_{y \rightarrow 0} f(x, y) = f(x)$. Thus f is the pointwise limit on E of the sequence of measurable functions $\{f_{y_n}\}$ and is therefore measurable on E . Moreover $|f(x)| = \lim_n |f(x, y_n)| \leq g(x)$ for all $x \in E$, so f is integrable over E by the Integral Comparison Test, and $\int_0^1 f(x) dx$ is a real number.

The first assertion. Let $\{y_n\}$ be an arbitrary sequence in $(0, 1]$ with $y_n \rightarrow 0$. The sequence of measurable functions $\{f_{y_n}\}$ converges pointwise on E to f , and it is dominated on E by the single integrable function g , since $|f_{y_n}| \leq g$ for every n . By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} H(y_n) = \lim_{n \rightarrow \infty} \int_0^1 f(x, y_n) dx = \int_0^1 f(x) dx.$$

We now pass from sequences to the limit as $y \rightarrow 0$. Let $L = \int_0^1 f(x) dx$ and suppose, by way of contradiction, that $H(y)$ does not converge to L as $y \rightarrow 0$. Then there is an $\epsilon_0 > 0$ such that for every $\delta > 0$ there is a $y \in (0, 1]$ with $y < \delta$ and $|H(y) - L| \geq \epsilon_0$. Taking $\delta = 1/n$ for each natural number n produces a sequence $\{y_n\}$ in $(0, 1]$ with $y_n < 1/n$, hence $y_n \rightarrow 0$, and $|H(y_n) - L| \geq \epsilon_0$ for all n . This contradicts the convergence $H(y_n) \rightarrow L$ established above for every such sequence. Therefore

$$\lim_{y \rightarrow 0} \int_0^1 f(x, y) dx = \int_0^1 f(x) dx,$$

which is the first assertion.

The second assertion. Assume now in addition that for each fixed $x \in [0, 1]$ the function $y \mapsto f(x, y)$ is continuous on $[0, 1]$, and let $h(y) = \int_0^1 f(x, y) dx$, which is the function H above and is real-valued by the first paragraph. Fix $y_0 \in [0, 1]$; we show h is continuous at y_0 .

Let $\{y_n\}$ be any sequence in $[0, 1]$ with $y_n \rightarrow y_0$. For each fixed $x \in [0, 1]$, continuity of $f(x, \cdot)$ at y_0 gives $f(x, y_n) \rightarrow f(x, y_0)$. Thus $\{f_{y_n}\}$ is a sequence of measurable functions on E converging pointwise on E to the measurable function f_{y_0} , and $|f_{y_n}| \leq g$ on E for every n , with g integrable over E . By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} h(y_n) = \lim_{n \rightarrow \infty} \int_0^1 f(x, y_n) dx = \int_0^1 f(x, y_0) dx = h(y_0).$$

Since $\{y_n\}$ was an arbitrary sequence in $[0, 1]$ converging to y_0 , the sequential criterion for continuity gives that h is continuous at y_0 . Explicitly, were h discontinuous at y_0 , there would be an $\epsilon_0 > 0$ and, for each natural number n , a point $y_n \in [0, 1]$ with $|y_n - y_0| < 1/n$ and $|h(y_n) - h(y_0)| \geq \epsilon_0$; this sequence converges to y_0 yet $h(y_n) \not\rightarrow h(y_0)$, contradicting what was just proved. As $y_0 \in [0, 1]$ was arbitrary, h is a continuous function of y on $[0, 1]$.

4.6 Problems 4.36–4.42

Problem 4.36. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value of y . For each $(x, y) \in Q$ let the partial derivative $\partial f / \partial y$ exist. Suppose there is a function g that is integrable over $[0, 1]$ and such that

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq g(x) \quad \text{for all } (x, y) \in Q.$$

Prove that

$$\frac{d}{dy} \left[\int_0^1 f(x, y) dx \right] = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx \quad \text{for all } y \in [0, 1].$$

Solution. For the assertion to have content the inner integral must be defined, so we assume, as the statement implicitly does, that $x \mapsto f(x, y_0)$ is integrable over $[0, 1]$ for at least one $y_0 \in [0, 1]$. This much cannot be dispensed with: the function $f(x, y) = 1/x$ for $0 < x \leq 1$, $f(0, y) = 0$, is real-valued on Q , is measurable in x for each y , and has $\partial f / \partial y \equiv 0$, so it meets every stated hypothesis with $g \equiv 0$, yet $\int_0^1 f(x, y) dx = \infty$ for every y . We first check that integrability at a single y_0 forces integrability for every y .

Step 1: $f(\cdot, y)$ is integrable over $[0, 1]$ for each $y \in [0, 1]$.

Fix $x \in [0, 1]$. By hypothesis the function $t \mapsto f(x, t)$ is differentiable at every point of $[0, 1]$, hence continuous there. Therefore, given $y \in [0, 1]$ with $y \neq y_0$, the Mean Value Theorem applied to $t \mapsto f(x, t)$ on the closed interval with endpoints y_0 and y provides a point ξ strictly between them with

$$f(x, y) - f(x, y_0) = \frac{\partial f}{\partial y}(x, \xi) (y - y_0).$$

Since $|y - y_0| \leq 1$ and $|\partial f / \partial y| \leq g$ on Q , we conclude

$$|f(x, y)| \leq |f(x, y_0)| + g(x) \quad \text{for all } x \in [0, 1].$$

The right-hand side is integrable over $[0, 1]$, and $f(\cdot, y)$ is measurable by hypothesis; by the Integral Comparison Test (Proposition 16 of Chapter 4), $f(\cdot, y)$ is integrable over $[0, 1]$. Thus

$$F(y) := \int_0^1 f(x, y) dx$$

is a well-defined real number for every $y \in [0, 1]$.

Step 2: the difference quotients are dominated by g .

Fix $y \in [0, 1]$ and let $\{y_n\}$ be any sequence in $[0, 1]$ with $y_n \neq y$ for all n and $y_n \rightarrow y$. Define, for $x \in [0, 1]$,

$$h_n(x) = \frac{f(x, y_n) - f(x, y)}{y_n - y}.$$

Each h_n is measurable on $[0, 1]$, being a linear combination of the measurable functions $f(\cdot, y_n)$ and $f(\cdot, y)$. Exactly as in Step 1, the Mean Value Theorem applied to $t \mapsto f(x, t)$ on the interval with endpoints y and y_n yields a point $\xi_{n,x}$ between y and y_n (so $\xi_{n,x} \in [0, 1]$) with

$$h_n(x) = \frac{\partial f}{\partial y}(x, \xi_{n,x}), \quad \text{hence} \quad |h_n(x)| \leq g(x) \quad \text{for all } x \in [0, 1] \text{ and all } n.$$

Step 3: pointwise convergence and passage to the limit.

For each fixed $x \in [0, 1]$, the very definition of the partial derivative $\partial f / \partial y(x, y)$, which exists by hypothesis, gives

$$\lim_{n \rightarrow \infty} h_n(x) = \frac{\partial f}{\partial y}(x, y).$$

(At $y = 0$ and $y = 1$ this derivative is the appropriate one-sided derivative, and the sequences $\{y_n\}$ necessarily approach y from one side.) Hence $\partial f / \partial y(\cdot, y)$ is measurable on $[0, 1]$, being a pointwise limit of measurable functions, and it satisfies $|\partial f / \partial y(\cdot, y)| \leq g$; by the Integral Comparison Test it is integrable over $[0, 1]$.

The sequence $\{h_n\}$ consists of measurable functions on $[0, 1]$, converges pointwise to $\partial f / \partial y(\cdot, y)$, and is dominated on $[0, 1]$ by the integrable function g . By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 h_n(x) dx = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx.$$

On the other hand, by the linearity of integration (Theorem 17 of Chapter 4; all three functions involved are integrable by Step 1),

$$\int_0^1 h_n(x) dx = \frac{1}{y_n - y} \left[\int_0^1 f(x, y_n) dx - \int_0^1 f(x, y) dx \right] = \frac{F(y_n) - F(y)}{y_n - y}.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{F(y_n) - F(y)}{y_n - y} = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx.$$

Step 4: conclusion.

The sequence $\{y_n\}$ in $[0, 1] \setminus \{y\}$ converging to y was arbitrary, and the limit obtained does not depend on it. By the sequential characterization of limits of a function of a real variable, the limit

$$\lim_{t \rightarrow y, t \in [0, 1], t \neq y} \frac{F(t) - F(y)}{t - y}$$

exists and equals $\int_0^1 (\partial f / \partial y)(x, y) dx$. That is, F is differentiable at y (one-sidedly at the endpoints $y = 0$ and $y = 1$) and

$$\frac{d}{dy} \left[\int_0^1 f(x, y) dx \right] = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx \quad \text{for all } y \in [0, 1],$$

which is what we were to prove.

Problem 4.37. Let f be an integrable function on E . Show that for each $\epsilon > 0$, there is a natural number N for which if $n \geq N$, then $\left| \int_{E_n} f \right| < \epsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$.

Solution. Here E is a measurable set of real numbers and f is integrable over E .

First, each $E_n = E \cap \{x \in \mathbb{R} : |x| \geq n\} = E \cap ((-\infty, -n] \cup [n, \infty))$ is measurable, being the intersection of the measurable set E with a closed set. Moreover f is integrable over E_n , since E_n is a measurable subset of E and hence, by the monotonicity of integration for nonnegative functions,

$$\int_{E_n} |f| \leq \int_E |f| < \infty.$$

Thus $\int_{E_n} f$ is a well-defined real number for every n .

Observe that $\{E_n\}_{n=1}^\infty$ is a descending collection of measurable sets, since $|x| \geq n+1$ implies $|x| \geq n$, and that

$$\bigcap_{n=1}^\infty E_n = \emptyset,$$

because each $x \in E$ is a real number and therefore fails to satisfy $|x| \geq n$ as soon as $n > |x|$ (by the Archimedean property of $\mathbb{R}$).

We give a direct argument. Define $g_n = |f| \cdot \chi_{E_n}$ on E . Each g_n is measurable, and for every $x \in E$ we have $g_n(x) \rightarrow 0$: indeed, once $n > |x|$ we have $x \notin E_n$ and so $g_n(x) = 0$. Furthermore $0 \leq g_n \leq |f|$ on E , and $|f|$ is integrable over E . By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_{E_n} |f| = \lim_{n \rightarrow \infty} \int_E g_n = \int_E 0 = 0.$$

By the Integral Comparison Test (Proposition 16 of Chapter 4), which supplies the triangle inequality for integrals,

$$\left| \int_{E_n} f \right| \leq \int_{E_n} |f| \quad \text{for every } n.$$

Hence, given $\epsilon > 0$, choose N so that $\int_{E_n} |f| < \epsilon$ for all $n \geq N$; then for all $n \geq N$,

$$\left| \int_{E_n} f \right| \leq \int_{E_n} |f| < \epsilon.$$

Alternatively, the conclusion is immediate from the Continuity of Integration (Theorem 21 of Chapter 4): since f is integrable over E and $\{E_n\}$ is a descending countable collection of measurable subsets of E with $\bigcap_{n=1}^\infty E_n = \emptyset$, formula (25) gives

$$\lim_{n \rightarrow \infty} \int_{E_n} f = \int_{\bigcap_{n=1}^\infty E_n} f = \int_\emptyset f = 0,$$

and the definition of this limit supplies, for each $\epsilon > 0$, a natural number N such that $\left| \int_{E_n} f \right| < \epsilon$ whenever $n \geq N$.

Problem 4.38. For each of the two functions f on $[1, \infty)$ defined below, show that $\lim_{n \rightarrow \infty} \int_1^n f$ exists while f is not integrable over $[1, \infty)$. Does this contradict the continuity of integration?

- (i) Define $f(x) = (-1)^n/n$, for $n \leq x < n+1$.
- (ii) Define $f(x) = (\sin x)/x$ for $1 \leq x < \infty$.

Solution. Throughout, recall that in Royden a measurable function f is said to be **integrable over** E provided $\int_E |f| < \infty$. So to show f is not integrable over $[1, \infty)$ we show $\int_{[1, \infty)} |f| = \infty$.

(i) The step function $f(x) = (-1)^n/n$ for $n \leq x < n+1$.

The function f is measurable on $[1, \infty)$: it is constant on each of the intervals $[n, n+1)$, $n \geq 1$, so for any c the set $\{x : f(x) > c\}$ is a countable union of such intervals, hence measurable.

Fix a natural number $N \geq 2$. The interval $[1, N)$ is the disjoint union of $[n, n+1)$ for $n = 1, \dots, N-1$, so by the additivity over domains of integration and the fact that $\int_{[n, n+1)} c = c \cdot m([n, n+1)) = c$,

$$\int_1^N f = \sum_{n=1}^{N-1} \frac{(-1)^n}{n} m([n, n+1)) = \sum_{n=1}^{N-1} \frac{(-1)^n}{n}.$$

The alternating series $\sum_{n=1}^{\infty} (-1)^n/n$ converges (its terms alternate in sign and $1/n$ decreases monotonically to 0; this is the alternating series test), with sum $-\ln 2$. Therefore

$$\lim_{N \rightarrow \infty} \int_1^N f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2,$$

so the limit exists and is finite.

On the other hand, $|f| = 1/n$ on $[n, n+1)$, so on $[1, \infty)$ we have the pointwise identity

$$|f| = \sum_{n=1}^{\infty} u_n, \quad u_n := \frac{1}{n} \chi_{[n, n+1)},$$

with each u_n nonnegative and measurable. Note that Theorem 20 of Chapter 4 is not available here, since it presupposes that f is integrable over $[1, \infty)$, which is exactly what is in question; but Corollary 12 of Chapter 4, which is the countable additivity of integration for nonnegative measurable functions and assumes no integrability, applies and gives

$$\int_{[1, \infty)} |f| = \sum_{n=1}^{\infty} \int_{[1, \infty)} u_n = \sum_{n=1}^{\infty} \frac{1}{n} m([n, n+1)) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

the harmonic series being divergent. Hence f is not integrable over $[1, \infty)$.

(ii) $f(x) = (\sin x)/x$ on $[1, \infty)$.

This f is continuous on $[1, \infty)$, hence measurable. On each bounded interval $[1, b]$ it is continuous and therefore Riemann integrable, and by Theorem 3 of Chapter 4 its Riemann and Lebesgue integrals over $[1, b]$ agree. Integrating by parts (a legitimate Riemann-integral manipulation, both integrands being continuous),

$$\int_1^b \frac{\sin x}{x} dx = \left[\frac{-\cos x}{x} \right]_1^b - \int_1^b \frac{\cos x}{x^2} dx = \cos 1 - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2} dx.$$

Now $|\cos b/b| \leq 1/b \rightarrow 0$ as $b \rightarrow \infty$. Also, the function $x \mapsto (\cos x)/x^2$ satisfies $|\cos x/x^2| \leq 1/x^2$ on $[1, \infty)$, and $1/x^2$ is integrable over $[1, \infty)$. Indeed, the functions $x^{-2} \chi_{[1, n]}$ form an increasing sequence of nonnegative measurable functions converging pointwise to x^{-2} on $[1, \infty)$, so the Monotone Convergence Theorem, together with Theorem 3 of Chapter 4 for the integrals over $[1, n]$, gives

$$\int_{[1, \infty)} \frac{dx}{x^2} = \lim_{n \rightarrow \infty} \int_1^n \frac{dx}{x^2} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right) = 1 < \infty.$$

Hence $(\cos x)/x^2$ is integrable over $[1, \infty)$, and applying the Lebesgue Dominated Convergence Theorem to $(\cos x)/x^2 \cdot \chi_{[1, n]}$, which converges pointwise to $(\cos x)/x^2$ and is dominated by $1/x^2$, we get

$$\lim_{n \rightarrow \infty} \int_1^n \frac{\cos x}{x^2} dx = \int_{[1, \infty)} \frac{\cos x}{x^2} dx.$$

Combining, the limit exists:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{\sin x}{x} dx = \cos 1 - \int_{[1, \infty)} \frac{\cos x}{x^2} dx \in \mathbb{R}.$$

To see that f is not integrable over $[1, \infty)$, note that for each integer $k \geq 1$ the interval $[k\pi, (k+1)\pi)$ is contained in $[1, \infty)$, and on it $x \leq (k+1)\pi$, so

$$\int_{[k\pi, (k+1)\pi)} \frac{|\sin x|}{x} dx \geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx = \frac{2}{(k+1)\pi},$$

since $\int_{k\pi}^{(k+1)\pi} |\sin x| dx = 2$ for every k . These intervals are pairwise disjoint measurable subsets of $[1, \infty)$. Writing $v_k = |f| \cdot \chi_{[k\pi, (k+1)\pi)}$, the v_k are nonnegative measurable functions with $\sum_{k=1}^{\infty} v_k \leq |f|$ pointwise on $[1, \infty)$, so Corollary 12 of Chapter 4 together with the monotonicity of integration for nonnegative functions gives

$$\int_{[1, \infty)} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{\infty} \int_{[1, \infty)} v_k \geq \sum_{k=1}^{\infty} \frac{2}{(k+1)\pi} = \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{1}{k} = \infty.$$

Hence f is not integrable over $[1, \infty)$.

Does this contradict the continuity of integration?

No. Theorem 21 of Chapter 4 (the Continuity of Integration) begins with the hypothesis “Let f be integrable over E .” Here one would want to take $E = [1, \infty)$ and $E_n = [1, n]$, an ascending countable collection of measurable subsets of E with $\bigcup_{n=1}^{\infty} E_n = [1, \infty)$, and conclude from (24) that

$$\int_{[1, \infty)} f = \lim_{n \rightarrow \infty} \int_{[1, n]} f.$$

But in both examples f fails to be integrable over $E = [1, \infty)$, so the hypothesis of the theorem is not met and the left-hand side is not even defined: $\int_{[1, \infty)} f^+ = \int_{[1, \infty)} f^- = \infty$ in each case, so the difference $\int f^+ - \int f^-$ is meaningless.

These examples therefore show only that the converse of the theorem fails: the existence of $\lim_{n \rightarrow \infty} \int_{E_n} f$ for an ascending sequence $\{E_n\}$ exhausting E does not imply that f is integrable over E . The limit in question is an improper (conditionally convergent) integral, and the Lebesgue integral is by design an absolutely convergent theory.

Problem 4.39. Prove the theorem regarding the continuity of integration.

[That is, prove Theorem 21: Let f be integrable over E .

- (i) If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of E , then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n \rightarrow \infty} \int_{E_n} f.$$

- (ii) If $\{E_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of E , then

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n \rightarrow \infty} \int_{E_n} f.$$

Use as a pattern the proof of the continuity of measure based on the countable additivity of measure.]

Solution. Throughout, note first that if A is any measurable subset of E , then f is integrable over A : f is measurable on A , and by the monotonicity of integration for nonnegative measurable functions,

$$\int_A |f| \leq \int_E |f| < \infty.$$

So every integral appearing below is a well-defined real number, and finite. We also use freely the Additivity Over Domains of Integration (Corollary 18 of Chapter 4): if A and B are disjoint measurable subsets of E , then $\int_{A \cup B} f = \int_A f + \int_B f$.

(i) The ascending case.

Let $\{E_n\}_{n=1}^{\infty}$ be ascending, that is $E_n \subseteq E_{n+1}$ for all n , and set $E_{\infty} = \bigcup_{n=1}^{\infty} E_n$. Define $E_0 = \emptyset$ and, for each $k \geq 1$,

$$F_k = E_k \setminus E_{k-1}.$$

Each F_k is a measurable subset of E . The collection $\{F_k\}_{k=1}^\infty$ is disjoint: if $j < k$ then $F_j \subseteq E_j \subseteq E_{k-1}$ while $F_k \cap E_{k-1} = \emptyset$. Moreover, for each $n \geq 1$,

$$\bigcup_{k=1}^n F_k = E_n,$$

which follows by induction: it is trivial for $n = 1$ since $F_1 = E_1$, and if it holds for n then $\bigcup_{k=1}^{n+1} F_k = E_n \cup (E_{n+1} \setminus E_n) = E_{n+1}$, using $E_n \subseteq E_{n+1}$. Taking the union over all n ,

$$\bigcup_{k=1}^\infty F_k = \bigcup_{n=1}^\infty E_n = E_\infty.$$

Thus $\{F_k\}_{k=1}^\infty$ is a disjoint countable collection of measurable subsets of E_∞ whose union is E_∞ , and f is integrable over E_∞ . By the Countable Additivity of Integration (Theorem 20 of Chapter 4), applied with E_∞ in the role of E ,

$$\int_{E_\infty} f = \sum_{k=1}^\infty \int_{F_k} f = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{F_k} f,$$

the last equality being the definition of the sum of a series. But for each fixed n , E_n is the disjoint union of the finitely many measurable sets $F_1, \dots, F_n$, so a finite induction on the additivity over domains of integration gives

$$\sum_{k=1}^n \int_{F_k} f = \int_{E_n} f.$$

Combining the last two displays,

$$\int_{\bigcup_{n=1}^\infty E_n} f = \int_{E_\infty} f = \lim_{n \rightarrow \infty} \int_{E_n} f,$$

which is (24).

(ii) The descending case.

Let $\{E_n\}_{n=1}^\infty$ be descending, that is $E_{n+1} \subseteq E_n$ for all n , and set $E_\infty = \bigcap_{n=1}^\infty E_n$. Then $E_\infty \subseteq E_n \subseteq E_1$ for every n .

Define $G_n = E_1 \setminus E_n$ for $n \geq 1$. Since $\{E_n\}$ is descending, $\{G_n\}$ is an ascending collection of measurable subsets of E . Moreover, by De Morgan's identities,

$$\bigcup_{n=1}^\infty G_n = \bigcup_{n=1}^\infty (E_1 \setminus E_n) = E_1 \setminus \bigcap_{n=1}^\infty E_n = E_1 \setminus E_\infty.$$

Applying part (i) to the ascending collection $\{G_n\}$,

$$\int_{E_1 \setminus E_\infty} f = \lim_{n \rightarrow \infty} \int_{E_1 \setminus E_n} f.$$

Now E_1 is the disjoint union of E_∞ and $E_1 \setminus E_\infty$, and also, for each n , the disjoint union of E_n and $E_1 \setminus E_n$. By additivity over domains,

$$\int_{E_1} f = \int_{E_\infty} f + \int_{E_1 \setminus E_\infty} f, \quad \int_{E_1} f = \int_{E_n} f + \int_{E_1 \setminus E_n} f.$$

Every term here is a finite real number, so we may solve and subtract. From the second identity, $\int_{E_n} f = \int_{E_1} f - \int_{E_1 \setminus E_n} f$; letting $n \rightarrow \infty$ and using the limit established above,

$$\lim_{n \rightarrow \infty} \int_{E_n} f = \int_{E_1} f - \int_{E_1 \setminus E_\infty} f = \int_{E_\infty} f,$$

the last equality being the first identity. That is,

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n \rightarrow \infty} \int_{E_n} f,$$

which is (25). This completes the proof.

Remark. The finiteness of $\int_{E_1} f$, which is where the integrability of f over E is used in part (ii), is essential; it is what licenses the cancellation. This is exactly parallel to the proof of the continuity of measure, where the descending case requires $m(E_1) < \infty$.

Problem 4.40. Let f be integrable over $\mathbb{R}$. Show that the function F defined by

$$F(x) = \int_{-\infty}^x f \quad \text{for all } x \in \mathbb{R}$$

is properly defined and continuous. Is it necessarily Lipschitz?

Solution. F is properly defined.

Fix $x \in \mathbb{R}$. The set $(-\infty, x]$ is closed, hence measurable, and f is measurable on it. Since $|f| \geq 0$ and $(-\infty, x] \subseteq \mathbb{R}$, monotonicity of integration gives

$$\int_{(-\infty, x]} |f| \leq \int_{\mathbb{R}} |f| < \infty,$$

so f is integrable over $(-\infty, x]$ and $F(x) = \int_{(-\infty, x]} f$ is a well-defined real number.

F is (uniformly) continuous.

Let $\epsilon > 0$. Since f is integrable over $\mathbb{R}$, Proposition 23 of Chapter 4 provides a $\delta > 0$ such that

$$\text{if } A \subseteq \mathbb{R} \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f| < \epsilon.$$

Let $x, y \in \mathbb{R}$ with $|x - y| < \delta$; without loss of generality $x < y$. The set $(-\infty, y]$ is the disjoint union of the measurable sets $(-\infty, x]$ and $(x, y]$, so by the Additivity Over Domains of Integration (Corollary 18 of Chapter 4),

$$F(y) = \int_{(-\infty, y]} f = \int_{(-\infty, x]} f + \int_{(x, y]} f = F(x) + \int_{(x, y]} f,$$

all terms being finite. Hence, by the Integral Comparison Test (Proposition 16 of Chapter 4),

$$|F(y) - F(x)| = \left| \int_{(x, y]} f \right| \leq \int_{(x, y]} |f| < \epsilon,$$

since $m((x, y]) = y - x = |x - y| < \delta$. As δ depends only on ϵ , F is uniformly continuous on $\mathbb{R}$, and in particular continuous.

F need not be Lipschitz.

The identity $F(y) - F(x) = \int_{(x, y]} f$ shows that if $|f| \leq M$ a.e. on $\mathbb{R}$ then $|F(y) - F(x)| \leq M|y - x|$, so boundedness of f would give the Lipschitz property. Without boundedness it can fail. Define

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is nonnegative and measurable on $\mathbb{R}$ (it is continuous off the two points 0 and 1). The functions $f \cdot \chi_{[1/n, 1]}$ are nonnegative, measurable, and increase pointwise to f , so by the Monotone Convergence Theorem, together with Theorem 3 of Chapter 4 (which identifies the Lebesgue and Riemann integrals of a continuous function on a closed bounded interval),

$$\int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \int_{1/n}^1 \frac{dt}{\sqrt{t}} = \lim_{n \rightarrow \infty} \left(2 - \frac{2}{\sqrt{n}} \right) = 2 < \infty.$$

Thus f is integrable over $\mathbb{R}$. The same computation applied on $(0, x]$ for $0 < x \leq 1$ gives

$$F(x) = \int_{(-\infty, x]} f = \int_{(0, x]} \frac{dt}{\sqrt{t}} = 2\sqrt{x}, \quad 0 \leq x \leq 1,$$

and $F(x) = 0$ for $x \leq 0$.

If F were Lipschitz on $\mathbb{R}$ with constant C , then for every $x \in (0, 1]$ we would have

$$2\sqrt{x} = |F(x) - F(0)| \leq C|x - 0| = Cx, \quad \text{i.e.} \quad \frac{2}{\sqrt{x}} \leq C.$$

Letting $x \rightarrow 0^+$ makes the left-hand side unbounded, a contradiction. Hence F is not Lipschitz.

So F is always uniformly continuous, but it is Lipschitz only under an extra hypothesis such as essential boundedness of f .

Problem 4.41. Show that Proposition 25 is false if $E = \mathbb{R}$.

[Proposition 25 reads: Assume E has finite measure. Let the sequence of functions $\{f_n\}$ be uniformly integrable over E . If $\{f_n\} \rightarrow f$ pointwise a.e. on E , then f is integrable over E .]

Solution. We exhibit a sequence of functions on $E = \mathbb{R}$ which is uniformly integrable over $\mathbb{R}$ and converges pointwise on all of $\mathbb{R}$ to a function that is not integrable over $\mathbb{R}$. This shows that the conclusion of Proposition 25 fails when the hypothesis of finite measure is dropped, so the hypothesis $m(E) < \infty$ cannot be removed.

Define, for every natural number n ,

$$f_n = \chi_{[-n, n]} \quad \text{on } \mathbb{R}.$$

Each f_n is measurable, being the characteristic function of a closed set, and each f_n is in fact integrable over $\mathbb{R}$, with $\int_{\mathbb{R}} |f_n| = m([-n, n]) = 2n < \infty$; so no objection can be raised that the members of the family are themselves badly behaved.

$\{f_n\} \rightarrow f$ pointwise on all of $\mathbb{R}$, where $f \equiv 1$. Indeed, fix $x \in \mathbb{R}$ and use the Archimedean property to choose $N > |x|$; then $x \in [-n, n]$ for all $n \geq N$, so $f_n(x) = 1$ for all $n \geq N$. Hence $f_n(x) \rightarrow 1$. In particular the convergence holds pointwise a.e. on $\mathbb{R}$.

The family $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable over $\mathbb{R}$.

Let $\epsilon > 0$ and take $\delta = \epsilon$. If $A \subseteq \mathbb{R}$ is measurable with $m(A) < \delta$, then for every n , since $0 \leq f_n \leq 1$, monotonicity of integration for nonnegative functions gives

$$\int_A |f_n| = \int_A \chi_{[-n, n]} \leq \int_A 1 = m(A) < \delta = \epsilon.$$

Thus this single δ responds to the ϵ challenge simultaneously for every member of the family, which is precisely criterion (27) for uniform integrability.

The limit function is not integrable over $\mathbb{R}$.

The limit is $f \equiv 1$, and

$$\int_{\mathbb{R}} |f| = \int_{\mathbb{R}} 1 = m(\mathbb{R}) = \infty.$$

Indeed, for every natural number n the function $\chi_{[-n, n]}$ is a nonnegative measurable function with $\chi_{[-n, n]} \leq 1$ on $\mathbb{R}$, so by monotonicity of integration $\int_{\mathbb{R}} 1 \geq \int_{\mathbb{R}} \chi_{[-n, n]} = 2n$ for every n , and hence $\int_{\mathbb{R}} 1 = \infty$. Therefore f is not integrable over $\mathbb{R}$.

So all the hypotheses of Proposition 25 except the finiteness of $m(E)$ are satisfied, yet the conclusion fails. Hence Proposition 25 is false if $E = \mathbb{R}$.

Remark. An even simpler family works, if one is willing to admit members that are not themselves integrable: the constant sequence $f_n \equiv 1$ is uniformly integrable over $\mathbb{R}$ by the identical $\delta = \epsilon$ computation, and converges pointwise to the non-integrable $f \equiv 1$. Royden's definition of uniform integrability imposes no integrability on the members of the family, so this is a legitimate counterexample too; the family $\{\chi_{[-n, n]}\}$ was chosen above only to forestall the objection.

What goes wrong on a set of infinite measure is that uniform integrability controls only the mass carried on sets of small measure. On a set of finite measure this is enough: Lemma 22 lets one split E into finitely many, say N , pieces each of measure less than the δ responding to $\epsilon = 1$, whence $\int_E |f_n| < N$ for every n , and Fatou's Lemma then bounds $\int_E |f|$ by N . When $m(E) = \infty$ that finite splitting is unavailable: $\mathbb{R}$ is not the union of finitely many sets of measure less than δ , and indeed in the example above $\int_{\mathbb{R}} |f_n| = 2n$ is unbounded in n .

Problem 4.42. Show that Theorem 26 is false without the assumption that the h_n 's are nonnegative. [Theorem 26 reads: Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then

$$\lim_{n \rightarrow \infty} \int_E h_n = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

]

Solution. Only one of the two implications in Theorem 26 uses the nonnegativity of the h_n . The implication

$$\{h_n\} \text{ uniformly integrable} \implies \lim_{n \rightarrow \infty} \int_E h_n = 0$$

is just the Vitali Convergence Theorem applied to the pointwise limit $h \equiv 0$, and the Vitali Convergence Theorem makes no sign assumption; so that implication survives. It is the converse,

$$\lim_{n \rightarrow \infty} \int_E h_n = 0 \implies \{h_n\} \text{ uniformly integrable,}$$

that fails when the h_n are allowed to change sign. We construct a counterexample.

Take $E = [0, 1]$, which has finite measure $m(E) = 1$. For each natural number n define

$$h_n = n \cdot \chi_{(0, \frac{1}{2n})} - n \cdot \chi_{[\frac{1}{2n}, \frac{1}{n})}.$$

Explicitly,

$$h_n(x) = \begin{cases} n, & 0 < x < \frac{1}{2n}, \\ -n, & \frac{1}{2n} \leq x < \frac{1}{n}, \\ 0, & \text{otherwise on } [0, 1]. \end{cases}$$

Each h_n is integrable over E . It is a simple function taking finitely many values on measurable sets of finite measure, and

$$\int_E |h_n| = n \cdot \frac{1}{2n} + n \cdot \frac{1}{2n} = 1 < \infty.$$

$\{h_n\} \rightarrow 0$ pointwise on E . We have $h_n(0) = 0$ for every n . If $0 < x \leq 1$, choose N with $1/N < x$ (Archimedean property); then for all $n \geq N$ we have $x \geq 1/N > 1/n$, hence $h_n(x) = 0$. So $h_n(x) \rightarrow 0$ for every $x \in [0, 1]$; in particular the convergence holds a.e. on E , and $h \equiv 0$.

$\lim_{n \rightarrow \infty} \int_E h_n = 0$. By the linearity of integration, for every n ,

$$\int_E h_n = n \cdot m((0, \frac{1}{2n})) - n \cdot m([\frac{1}{2n}, \frac{1}{n})) = n \cdot \frac{1}{2n} - n \cdot \frac{1}{2n} = 0.$$

Hence $\int_E h_n = 0$ for every n and the limit is 0. The positive and negative parts cancel exactly; this is what the sign restriction in Theorem 26 was preventing.

$\{h_n\}$ is not uniformly integrable over E . Take the ϵ challenge $\epsilon = 1/2$ and let $\delta > 0$ be arbitrary. Choose n so large that $1/(2n) < \delta$, and set

$$A_n = \left(0, \frac{1}{2n}\right) \subseteq E, \quad m(A_n) = \frac{1}{2n} < \delta.$$

Then

$$\int_{A_n} |h_n| = n \cdot \frac{1}{2n} = \frac{1}{2} \not\leq \epsilon.$$

So no $\delta > 0$ responds to the challenge $\epsilon = 1/2$ in criterion (27), and $\{h_n\}$ fails to be uniformly integrable over E .

Thus E has finite measure, each h_n is integrable over E , $\{h_n\} \rightarrow 0$ pointwise on E , and $\lim_{n \rightarrow \infty} \int_E h_n = 0$, yet $\{h_n\}$ is not uniformly integrable. Theorem 26 is therefore false without the assumption that the h_n are nonnegative.

Remark on where the proof breaks. In Royden's proof of the converse direction, from $\lim_{n \rightarrow \infty} \int_E h_n = 0$ one chooses N with $\int_E h_n < \epsilon$ for $n \geq N$ and then asserts inequality (30): since $h_n \geq 0$ on E , for any measurable $A \subseteq E$ one has $\int_A h_n \leq \int_E h_n < \epsilon$. It is exactly this monotonicity step that requires $h_n \geq 0$; for the h_n above, $\int_E h_n = 0$ while $\int_{A_n} |h_n| = 1/2$.

4.7 Problems 4.43–4.49

Problem 4.43. Let the sequences of functions $\{f_n\}$ and $\{g_n\}$ be uniformly integrable over E . Show that for any α and β , the sequence of linear combinations $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable over E .

Solution. Recall the definition: a family $\mathcal{F}$ of measurable functions on E is uniformly integrable over E provided for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f| < \epsilon.$$

The definition is stated for families of measurable, possibly extended real-valued, functions, so before anything else we must know that the pointwise sum $\alpha f_n + \beta g_n$ is defined a.e. on E . We claim that every member h of a uniformly integrable family on E is finite a.e. on E . Suppose not, so that $S = \{x \in E \mid |h(x)| = \infty\}$ has $m(S) > 0$. Let $\delta > 0$ respond to the $\epsilon = 1$ challenge for the family. The sets $S \cap [-n, n]$ ascend to S , so by the continuity of measure (Theorem 15 of Chapter 2) there is an n with $m(S \cap [-n, n]) > 0$. The function $t \mapsto m(S \cap [-n, t])$ is nondecreasing on $[-n, n]$ and increases by at most $t' - t$ over $[t, t']$, hence is continuous; it equals 0 at $t = -n$ and $m(S \cap [-n, n]) > 0$ at $t = n$. By the Intermediate Value Theorem there is a t for which $A = S \cap [-n, t]$ satisfies $0 < m(A) < \delta$. But $|h| \equiv \infty$ on A and $m(A) > 0$, so $\int_A |h| = \infty$, contradicting $\int_A |h| < 1$. Hence $m(S) = 0$.

Consequently, for each n there is a measurable $A_n \subseteq E$ with $m(E \setminus A_n) = 0$ on which both f_n and g_n are finite and on which $\alpha f_n + \beta g_n$ is therefore defined. As in the convention adopted in Section 4.4, the integral of $\alpha f_n + \beta g_n$ over a measurable $A \subseteq E$ means its integral over $A \cap A_n$; this is unambiguous, since the integral of a nonnegative measurable function over a set of measure zero is 0 (immediately from the definition of that integral as a supremum) and so, by the additivity of integration over domains (Theorem 11), excising the null set $E \setminus A_n$ changes nothing. Wherever it is defined we have the pointwise triangle inequality

$$|\alpha f_n + \beta g_n| \leq |\alpha| |f_n| + |\beta| |g_n|.$$

Now set

$$M = |\alpha| + |\beta| + 1 > 0.$$

Let $\epsilon > 0$. Since $\{f_n\}$ is uniformly integrable over E , there is a $\delta_1 > 0$ such that

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta_1, \text{ then } \int_A |f_n| < \frac{\epsilon}{2M} \quad \text{for all } n.$$

Since $\{g_n\}$ is uniformly integrable over E , there is a $\delta_2 > 0$ such that

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta_2, \text{ then } \int_A |g_n| < \frac{\epsilon}{2M} \quad \text{for all } n.$$

Define $\delta = \min\{\delta_1, \delta_2\} > 0$.

Let $A \subseteq E$ be measurable with $m(A) < \delta$ and let n be a natural number. Since $m(A) < \delta_1$ and $m(A) < \delta_2$, the monotonicity and linearity of integration for nonnegative measurable functions (Theorem 10) together with the pointwise inequality above give

$$\int_A |\alpha f_n + \beta g_n| \leq \int_A (|\alpha| |f_n| + |\beta| |g_n|) = |\alpha| \int_A |f_n| + |\beta| \int_A |g_n| \leq M \int_A |f_n| + M \int_A |g_n| < M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} = \epsilon.$$

Here we used $|\alpha| \leq M$ and $|\beta| \leq M$, and the final inequality is strict because $M > 0$ and both of the last two integrals are strictly less than $\epsilon/(2M)$.

Since δ was chosen independently of n and of A , this δ responds to the ϵ challenge for the sequence $\{\alpha f_n + \beta g_n\}$. Therefore $\{\alpha f_n + \beta g_n\}$ is uniformly integrable over E .

Problem 4.44. Let f be integrable over $\mathbb{R}$ and $\epsilon > 0$. Establish the following three approximation properties.

- (i) There is a simple function η on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \epsilon$. (Hint: First verify this if f is nonnegative.)
- (ii) There is a step function s on $\mathbb{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \epsilon$. (Hint: Apply part (i) and Problem 18 of Chapter 3.)
- (iii) There is a continuous function g on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f - g| < \epsilon$.

Solution. Throughout, “finite support” means vanishing outside a set of finite measure.

(i) First assume $f \geq 0$ on $\mathbb{R}$. By the definition of the integral of a nonnegative measurable function,

$$\int_{\mathbb{R}} f = \sup \left\{ \int_{\mathbb{R}} h \mid h \text{ bounded, measurable, of finite support, } 0 \leq h \leq f \text{ on } \mathbb{R} \right\},$$

and this supremum is finite since f is integrable. Hence there is a bounded measurable function h of finite support with $0 \leq h \leq f$ on $\mathbb{R}$ and

$$\int_{\mathbb{R}} f - \frac{\epsilon}{2} < \int_{\mathbb{R}} h \leq \int_{\mathbb{R}} f.$$

The function $f - h$ is nonnegative and measurable, and by the linearity of integration for nonnegative measurable functions (Theorem 10), $\int_{\mathbb{R}} h + \int_{\mathbb{R}} (f - h) = \int_{\mathbb{R}} f$. Since $\int_{\mathbb{R}} h$ is finite,

$$\int_{\mathbb{R}} |f - h| = \int_{\mathbb{R}} (f - h) = \int_{\mathbb{R}} f - \int_{\mathbb{R}} h < \frac{\epsilon}{2}.$$

Let E_0 be a measurable set of finite measure outside of which h vanishes, and choose $M > 0$ with $0 \leq h \leq M$ on $\mathbb{R}$. If $m(E_0) = 0$ then $h = 0$ a.e. and we may take $\eta = 0$; so assume $m(E_0) > 0$. Apply the Simple Approximation Lemma to the bounded measurable function h restricted to E_0 , with the challenge $\epsilon' = \epsilon/(2m(E_0) + 1)$: there are simple functions $\varphi \leq h \leq \psi$ on E_0 with $0 \leq \psi - \varphi < \epsilon'$ on E_0 ; in particular $|h - \varphi| < \epsilon'$ on E_0 . Replacing φ by $\max\{\varphi, 0\}$ (still simple, and still within ϵ' of h since $h \geq 0$), we may assume $\varphi \geq 0$. Define η on $\mathbb{R}$ by $\eta = \varphi$ on E_0 and $\eta = 0$ on $\mathbb{R} \setminus E_0$. Then η is a simple function on $\mathbb{R}$ vanishing outside the set E_0 of finite measure, so η has finite support, and by the monotonicity of integration together with additivity over domains,

$$\int_{\mathbb{R}} |h - \eta| = \int_{E_0} |h - \varphi| \leq \epsilon' \cdot m(E_0) < \frac{\epsilon}{2}.$$

Therefore, by the triangle inequality $|f - \eta| \leq |f - h| + |h - \eta|$ and the monotonicity and linearity of integration,

$$\int_{\mathbb{R}} |f - \eta| \leq \int_{\mathbb{R}} |f - h| + \int_{\mathbb{R}} |h - \eta| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now the general case. Write $f = f^+ - f^-$. By Proposition 14 both f^+ and f^- are nonnegative and integrable over $\mathbb{R}$. By the case just settled, there are simple functions η_1, η_2 on $\mathbb{R}$ of finite support with

$$\int_{\mathbb{R}} |f^+ - \eta_1| < \frac{\epsilon}{2}, \quad \int_{\mathbb{R}} |f^- - \eta_2| < \frac{\epsilon}{2}.$$

Set $\eta = \eta_1 - \eta_2$. A difference of simple functions is simple (Problem 19 of Chapter 3), and η vanishes outside the union of two sets of finite measure, so η has finite support. By Proposition 15, f is finite a.e. on $\mathbb{R}$; at every point where f is finite,

$$|f - \eta| = |(f^+ - \eta_1) - (f^- - \eta_2)| \leq |f^+ - \eta_1| + |f^- - \eta_2|.$$

Since a set of measure zero does not affect an integral (Proposition 15), monotonicity and linearity give

$$\int_{\mathbb{R}} |f - \eta| \leq \int_{\mathbb{R}} |f^+ - \eta_1| + \int_{\mathbb{R}} |f^- - \eta_2| < \epsilon.$$

(ii) By part (i) choose a simple function η on $\mathbb{R}$ of finite support with $\int_{\mathbb{R}} |f - \eta| < \epsilon/2$. Write $\eta = \sum_{k=1}^p a_k \chi_{E_k}$ with the E_k disjoint measurable sets and $a_k \neq 0$; since η has finite support, $m(E_k) < \infty$ for each k . For each k , the sets $E_k \sim [-N, N]$ descend to $\bigcap_N (E_k \sim [-N, N]) = \emptyset$ as $N \rightarrow \infty$ and $m(E_k) < \infty$, so by the continuity of measure $m(E_k \sim [-N, N]) \rightarrow 0$. Choose a natural number N so large that

$$\sum_{k=1}^p |a_k| \cdot m(E_k \sim [-N, N]) < \frac{\epsilon}{4}.$$

Put $I = [-N, N]$ and $\eta' = \eta \cdot \chi_I$. Then η' is simple, vanishes outside I , and

$$\int_{\mathbb{R}} |\eta - \eta'| = \int_{\mathbb{R} \sim I} |\eta| \leq \sum_{k=1}^p |a_k| \cdot m(E_k \sim I) < \frac{\epsilon}{4}.$$

Let $M = \max_k |a_k|$, so $|\eta'| \leq M$ on $\mathbb{R}$. Let $\epsilon' > 0$ be a number to be fixed presently. The restriction of η' to the closed, bounded interval I is a bounded measurable function, so by Problem 18 of Chapter 3 there is a step function h on I and a measurable subset $F \subseteq I$ for which

$$|h - \eta'| < \epsilon' \text{ on } F \quad \text{and} \quad m(I \sim F) < \epsilon'.$$

Replace h by its truncation $\tilde{h} = \min\{\max\{h, -M\}, M\}$. Truncating a step function at the levels $\pm M$ again yields a step function on I , and since $|\eta'| \leq M$, truncation does not increase the distance to η' : $|\tilde{h} - \eta'| \leq |h - \eta'|$ pointwise on I . Thus $|\tilde{h} - \eta'| < \epsilon'$ on F and $|\tilde{h} - \eta'| \leq 2M$ on all of I . Consequently, by additivity over domains and monotonicity,

$$\int_I |\eta' - \tilde{h}| = \int_F |\eta' - \tilde{h}| + \int_{I \sim F} |\eta' - \tilde{h}| \leq \epsilon' \cdot m(I) + 2M \cdot \epsilon' = \epsilon' (2N + 2M).$$

Choose $\epsilon' > 0$ small enough that $\epsilon' (2N + 2M) < \epsilon/4$.

Define s on $\mathbb{R}$ by $s = \tilde{h}$ on I and $s = 0$ on $\mathbb{R} \sim I$. Then s is a step function on $\mathbb{R}$ (adjoin the two unbounded complementary intervals to the partition) which vanishes outside the closed, bounded interval I , and since η' also vanishes outside I ,

$$\int_{\mathbb{R}} |\eta' - s| = \int_I |\eta' - \tilde{h}| < \frac{\epsilon}{4}.$$

Adding the three estimates,

$$\int_{\mathbb{R}} |f - s| \leq \int_{\mathbb{R}} |f - \eta| + \int_{\mathbb{R}} |\eta - \eta'| + \int_{\mathbb{R}} |\eta' - s| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

(iii) By part (ii) choose a step function s vanishing outside a closed, bounded interval $[a, b]$ with $\int_{\mathbb{R}} |f - s| < \epsilon/2$. Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$ such that s is constant, say $s \equiv c_i$, on the open interval (x_{i-1}, x_i) for $1 \leq i \leq n$. Let $M = \max_i |c_i|$ (if $M = 0$ then $s = 0$ a.e. and $g \equiv 0$ works, so assume $M > 0$).

Let $\sigma > 0$ satisfy

$$\sigma < \min_{1 \leq i \leq n} (x_i - x_{i-1}) \quad \text{and} \quad 2M(n+1)\sigma < \frac{\epsilon}{2}.$$

Define $g : \mathbb{R} \rightarrow \mathbb{R}$ as follows. Let $J_i = (x_i - \sigma/2, x_i + \sigma/2)$ for $0 \leq i \leq n$; by the choice of σ these $n+1$ intervals are pairwise disjoint. Outside $\bigcup_{i=0}^n J_i$ set g equal to the value c_i on each $(x_{i-1}, x_i) \sim (J_{i-1} \cup J_i)$ and equal to 0 outside $[a, b]$. On each J_i define g to be the affine function joining the value of g at the left endpoint $x_i - \sigma/2$ to its value at the right endpoint $x_i + \sigma/2$ (these two one-sided values are the constants just prescribed; for $i = 0$ the left value is 0 and for $i = n$ the right value is 0). Then g is continuous on $\mathbb{R}$: it is locally constant off the closed intervals $\overline{J_i}$, affine on each J_i , and the definitions agree at the endpoints of each J_i . Moreover g vanishes outside the bounded set $[a - \sigma, b + \sigma]$, and $|g| \leq M$ everywhere, since on each J_i it is a convex combination of two values of modulus at most M . By construction $g = s$ except possibly on $\bigcup_{i=0}^n \overline{J_i}$, a set of measure $(n+1)\sigma$, and $|g - s| \leq 2M$ everywhere. Hence

$$\int_{\mathbb{R}} |s - g| \leq 2M \cdot (n+1)\sigma < \frac{\epsilon}{2}.$$

Therefore

$$\int_{\mathbb{R}} |f - g| \leq \int_{\mathbb{R}} |f - s| + \int_{\mathbb{R}} |s - g| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and g is continuous on $\mathbb{R}$ and vanishes outside a bounded set.

Problem 4.45. Let f be integrable over E . Define $\hat{f}$ to be the extension of f to all of $\mathbb{R}$ obtained by setting $\hat{f} = 0$ outside of E . Show that $\hat{f}$ is integrable over $\mathbb{R}$ and $\int_E f = \int_{\mathbb{R}} \hat{f}$. Use this and part (i) and (iii) of the preceding problem to show that for $\epsilon > 0$, there is a simple function η on E and a continuous function g on E for which $\int_E |f - \eta| < \epsilon$ and $\int_E |f - g| < \epsilon$.

Solution. Measurability of $\hat{f}$. For $c \geq 0$,

$$\{x \in \mathbb{R} \mid \hat{f}(x) > c\} = \{x \in E \mid f(x) > c\},$$

while for $c < 0$,

$$\{x \in \mathbb{R} \mid \hat{f}(x) > c\} = \{x \in E \mid f(x) > c\} \cup (\mathbb{R} \sim E).$$

In both cases the set is measurable, since f is measurable on the measurable set E and $\mathbb{R} \sim E$ is measurable. Hence $\hat{f}$ is measurable on $\mathbb{R}$.

Next suppose first that $f \geq 0$ on E , so that $\hat{f} \geq 0$ on $\mathbb{R}$. We claim

$$\int_{\mathbb{R}} \hat{f} = \int_E f.$$

Let h be bounded, measurable, of finite support on $\mathbb{R}$ with $0 \leq h \leq \hat{f}$ on $\mathbb{R}$. Since $\hat{f} = 0$ on $\mathbb{R} \sim E$ and $h \geq 0$, we get $h = 0$ on $\mathbb{R} \sim E$. Since h is nonnegative and measurable, the additivity of integration over domains (Theorem 11) applies and gives

$$\int_{\mathbb{R}} h = \int_E h + \int_{\mathbb{R} \sim E} h = \int_E h.$$

The restriction of h to E is bounded, measurable, of finite support, and satisfies $0 \leq h \leq f$ on E ; therefore, by the definition of $\int_E f$ as a supremum, $\int_{\mathbb{R}} h = \int_E h \leq \int_E f$. Taking the supremum over all such h gives $\int_{\mathbb{R}} \hat{f} \leq \int_E f$.

Conversely, let h be bounded, measurable, of finite support on E with $0 \leq h \leq f$ on E , and let $\hat{h}$ be its extension by 0 to $\mathbb{R}$. Then $\hat{h}$ is bounded, measurable (by the same argument as for $\hat{f}$), of finite support, and $0 \leq \hat{h} \leq \hat{f}$ on $\mathbb{R}$. Again by additivity over domains, $\int_E h = \int_{\mathbb{R}} \hat{h} \leq \int_{\mathbb{R}} \hat{f}$. Taking the supremum over all such h gives $\int_E f \leq \int_{\mathbb{R}} \hat{f}$. The two inequalities give the claimed equality for nonnegative f .

For general integrable f over E , observe that

$$(\hat{f})^+ = \widehat{f^+}, \quad (\hat{f})^- = \widehat{f^-}, \quad |\hat{f}| = \widehat{|f|}.$$

Applying the nonnegative case to $|f|$ gives $\int_{\mathbb{R}} |\hat{f}| = \int_E |f| < \infty$, so $\hat{f}$ is integrable over $\mathbb{R}$. Applying it to f^+ and to f^- and using the definition of the general integral,

$$\int_{\mathbb{R}} \hat{f} = \int_{\mathbb{R}} (\hat{f})^+ - \int_{\mathbb{R}} (\hat{f})^- = \int_E f^+ - \int_E f^- = \int_E f.$$

Now let $\epsilon > 0$. Since $\hat{f}$ is integrable over $\mathbb{R}$, part (i) of the preceding problem provides a simple function $\tilde{\eta}$ on $\mathbb{R}$ of finite support with

$$\int_{\mathbb{R}} |\hat{f} - \tilde{\eta}| < \epsilon.$$

Let η be the restriction of $\tilde{\eta}$ to E . The restriction of a simple function to a measurable subset is a simple function on that subset, so η is simple on E . Since $|\hat{f} - \tilde{\eta}|$ is nonnegative and $\hat{f} = f$ on E , the monotonicity together with the additivity over domains of integration for nonnegative functions gives

$$\int_E |f - \eta| = \int_E |\hat{f} - \tilde{\eta}| \leq \int_E |\hat{f} - \tilde{\eta}| + \int_{\mathbb{R} \sim E} |\hat{f} - \tilde{\eta}| = \int_{\mathbb{R}} |\hat{f} - \tilde{\eta}| < \epsilon.$$

Similarly, part (iii) of the preceding problem provides a function $\tilde{g}$ that is continuous on $\mathbb{R}$ and vanishes outside a bounded set, with $\int_{\mathbb{R}} |\hat{f} - \tilde{g}| < \epsilon$. Let g be the restriction of $\tilde{g}$ to E ; a restriction of a continuous function is continuous on its domain, so g is continuous on E , and exactly as above

$$\int_E |f - g| = \int_E |\hat{f} - \tilde{g}| \leq \int_{\mathbb{R}} |\hat{f} - \tilde{g}| < \epsilon.$$

This is what was to be shown.

Problem 4.46. (Riemann-Lebesgue) Let f be integrable over $(-\infty, \infty)$. Show that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0.$$

(Hint: First show this for f a step function that vanishes outside a closed, bounded interval and then use the approximation property (ii) of Problem 44.)

Solution. First note that for each natural number n the function $x \mapsto f(x) \cos nx$ is measurable on $\mathbb{R}$. Indeed, f is integrable, hence finite a.e. on $\mathbb{R}$ by Proposition 15, while $x \mapsto \cos nx$ is continuous and therefore measurable (Proposition 3 of Chapter 3); by Theorem 6 of Chapter 3 the product of two measurable functions that are finite a.e. is measurable off the null set where f fails to be finite, and hence, by Proposition 5 of Chapter 3 together with the completeness of Lebesgue measure, it is measurable on all of $\mathbb{R}$ however it is defined on that null set. Moreover

$$|f(x) \cos nx| \leq |f(x)| \quad \text{for all } x.$$

Since $|f|$ is integrable over $\mathbb{R}$, the Integral Comparison Test (Proposition 16) shows that $f(x) \cos nx$ is integrable over $\mathbb{R}$ and that all the integrals below are finite.

Step 1: the case of a step function. Let s be a step function on $\mathbb{R}$ that vanishes outside a closed, bounded interval $[a, b]$, and let

$$a = x_0 < x_1 < \cdots < x_p = b$$

be a partition of $[a, b]$ with $s \equiv c_i$ on (x_{i-1}, x_i) for $1 \leq i \leq p$. The finite set of partition points has measure zero, so it does not affect any integral (Proposition 15); hence for each n , by the additivity over domains and the linearity of integration,

$$\int_{-\infty}^{\infty} s(x) \cos nx \, dx = \sum_{i=1}^p c_i \int_{x_{i-1}}^{x_i} \cos nx \, dx.$$

On each bounded interval the integrand is continuous, so the Lebesgue integral agrees with the Riemann integral (Theorem 3 of this chapter), and for $n \geq 1$,

$$\int_{x_{i-1}}^{x_i} \cos nx \, dx = \frac{\sin nx_i - \sin nx_{i-1}}{n}, \quad \left| \int_{x_{i-1}}^{x_i} \cos nx \, dx \right| \leq \frac{2}{n}.$$

Therefore

$$\left| \int_{-\infty}^{\infty} s(x) \cos nx \, dx \right| \leq \frac{2}{n} \sum_{i=1}^p |c_i| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus the assertion holds for such step functions.

Step 2: the general case. Let $\epsilon > 0$. Since f is integrable over $\mathbb{R}$, by the approximation property (ii) of Problem 44 there is a step function s on $\mathbb{R}$ which vanishes outside a closed, bounded interval and satisfies

$$\int_{\mathbb{R}} |f - s| < \frac{\epsilon}{2}.$$

(The function s is bounded and vanishes outside a set of finite measure, hence is integrable over $\mathbb{R}$; so $f - s$ is integrable and the following manipulations are legitimate by Theorem 17.) For each n , by the linearity of integration and the Integral Comparison Test,

$$\left| \int_{\mathbb{R}} f(x) \cos nx \, dx - \int_{\mathbb{R}} s(x) \cos nx \, dx \right| = \left| \int_{\mathbb{R}} (f(x) - s(x)) \cos nx \, dx \right| \leq \int_{\mathbb{R}} |f - s| \cdot |\cos nx| \leq \int_{\mathbb{R}} |f - s| < \frac{\epsilon}{2},$$

where we used $|\cos nx| \leq 1$ and the monotonicity of integration for nonnegative functions.

By Step 1 there is a natural number N such that

$$\left| \int_{\mathbb{R}} s(x) \cos nx \, dx \right| < \frac{\epsilon}{2} \quad \text{for all } n \geq N.$$

Hence for $n \geq N$,

$$\left| \int_{-\infty}^{\infty} f(x) \cos nx \, dx \right| \leq \left| \int_{\mathbb{R}} (f - s)(x) \cos nx \, dx \right| + \left| \int_{\mathbb{R}} s(x) \cos nx \, dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0.$$

Problem 4.47. Let f be integrable over $(-\infty, \infty)$.

- (i) Show that for each t ,

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} f(x + t) \, dx.$$

- (ii) Let g be a bounded measurable function on $\mathbb{R}$. Show that

$$\lim_{t \rightarrow 0} \int_{-\infty}^{\infty} g(x) \cdot [f(x) - f(x + t)] \, dx = 0.$$

(Hint: First show this, using uniform continuity of f on $\mathbb{R}$, if f is continuous and vanishes outside a bounded set. Then use the approximation property (iii) of Problem 44.)

Solution. For a function h on $\mathbb{R}$ and a number t , write $h_t(x) = h(x + t)$.

- (i) First, f_t is measurable: for each c ,

$$\{x \mid f_t(x) > c\} = \{x \mid f(x + t) > c\} = \{y \mid f(y) > c\} - t,$$

which is measurable by Proposition 10 of Chapter 2 (the translate of a measurable set is measurable).

Now suppose first that $f \geq 0$ on $\mathbb{R}$. We claim

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}} f_t.$$

Consider the map $h \mapsto h_t$. If h is bounded, measurable, of finite support, and $0 \leq h \leq f$ on $\mathbb{R}$, then h_t is bounded, measurable (same argument as above), and vanishes outside $E_0 - t$ where h vanishes outside the finite-measure set E_0 ; since Lebesgue measure is translation invariant (Theorem 14 of Chapter 2), $m(E_0 - t) = m(E_0) < \infty$, so h_t has finite support. Also $0 \leq h_t \leq f_t$ on $\mathbb{R}$. Moreover $\int_{\mathbb{R}} h = \int_{\mathbb{R}} h_t$. Indeed, for a simple function $\varphi = \sum_{k=1}^p a_k \chi_{E_k}$ with the E_k of finite measure we have $\varphi_t = \sum_{k=1}^p a_k \chi_{E_k - t}$, so by the translation invariance of measure $\int_{\mathbb{R}} \varphi = \sum_k a_k m(E_k) = \sum_k a_k m(E_k - t) = \int_{\mathbb{R}} \varphi_t$; and the integral of a bounded measurable function of finite support over a set of finite measure is, by definition, the common value of the infimum of integrals of simple functions above it and the supremum of integrals of simple functions below it, and $\varphi \leq h$ on $\mathbb{R}$ if and only if $\varphi_t \leq h_t$ on $\mathbb{R}$. Hence $\int_{\mathbb{R}} h = \int_{\mathbb{R}} h_t$.

The map $h \mapsto h_t$ is a bijection of the family of bounded measurable functions of finite support lying between 0 and f onto the corresponding family for f_t (its inverse is $k \mapsto k_{-t}$). Since it preserves integrals, the two suprema in the definition of $\int_{\mathbb{R}} f$ and $\int_{\mathbb{R}} f_t$ coincide:

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}} f_t \quad (f \geq 0).$$

For general integrable f , note $(f_t)^+ = (f^+)_t$ and $(f_t)^- = (f^-)_t$, and likewise $|f_t| = |f|_t$. Applying the nonnegative case to $|f|$ shows f_t is integrable over $\mathbb{R}$, and applying it to f^+ and f^- gives

$$\int_{-\infty}^{\infty} f(x+t) dx = \int_{\mathbb{R}} (f^+)_t - \int_{\mathbb{R}} (f^-)_t = \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^- = \int_{-\infty}^{\infty} f(x) dx.$$

(ii) Let $M > 0$ be such that $|g| \leq M$ on $\mathbb{R}$. By part (i), f_t is integrable over $\mathbb{R}$ for every t , hence $f - f_t$ is integrable (Theorem 17), and since g is bounded and measurable, $g \cdot (f - f_t)$ is integrable over $\mathbb{R}$ by Problem 48. Moreover, by the Integral Comparison Test,

$$\left| \int_{\mathbb{R}} g \cdot [f - f_t] \right| \leq M \int_{\mathbb{R}} |f - f_t|.$$

Call this estimate (*); the same reasoning bounds $\left| \int_{\mathbb{R}} g \cdot u \right|$ by $M \int_{\mathbb{R}} |u|$ for any function u integrable over $\mathbb{R}$.

Case 1: f continuous on $\mathbb{R}$ and vanishing outside a bounded set. Say $f = 0$ outside $[-N, N]$. Then f is continuous on the compact interval $[-N-1, N+1]$ and vanishes on a neighbourhood of each of its endpoints, from which it follows that f is uniformly continuous on all of $\mathbb{R}$. (Explicitly: f is uniformly continuous on the compact set $[-N-1, N+1]$; f is identically 0 off $[-N, N]$; combining the two, given $\epsilon' > 0$ pick $\delta' \in (0, 1)$ that works for uniform continuity on $[-N-1, N+1]$, and note that if $|x-y| < \delta'$ and not both x, y lie in $[-N-1, N+1]$, then both lie outside $[-N, N]$, so $f(x) = f(y) = 0$.)

Let $\epsilon > 0$. Choose $\delta \in (0, 1)$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{M(2N+2)+1} \quad \text{whenever } |x-y| < \delta.$$

Let $|t| < \delta$. If $|x| > N+1$ then $|x+t| > N$ as well, so $f(x) - f(x+t) = 0$. Hence $f - f_t$ vanishes outside $[-N-1, N+1]$, a set of measure $2N+2$, and $|f - f_t| < \epsilon/(M(2N+2)+1)$ everywhere. By monotonicity of integration and (*),

$$\left| \int_{\mathbb{R}} g \cdot [f - f_t] \right| \leq M \int_{[-N-1, N+1]} |f - f_t| \leq M \cdot (2N+2) \cdot \frac{\epsilon}{M(2N+2)+1} < \epsilon.$$

Thus the limit is 0 in this case.

Case 2: general integrable f . Let $\epsilon > 0$. By the approximation property (iii) of Problem 44 there is a function h , continuous on $\mathbb{R}$ and vanishing outside a bounded set, with

$$\int_{\mathbb{R}} |f - h| < \frac{\epsilon}{3(M+1)}.$$

For any t , decompose (all four functions being integrable, by part (i), Theorem 17 and Problem 48)

$$\int_{\mathbb{R}} g \cdot [f - f_t] = \int_{\mathbb{R}} g \cdot [f - h] + \int_{\mathbb{R}} g \cdot [h - h_t] + \int_{\mathbb{R}} g \cdot [h_t - f_t].$$

The first term is bounded, by $(*)$ -type estimation, by $M \int_{\mathbb{R}} |f - h| < \epsilon/3$. For the third term, note $|h_t - f_t| = |h - f|_t$, so by part (i) applied to the integrable nonnegative function $|h - f|$,

$$\left| \int_{\mathbb{R}} g \cdot [h_t - f_t] \right| \leq M \int_{\mathbb{R}} |h - f|_t = M \int_{\mathbb{R}} |h - f| < \frac{\epsilon}{3}.$$

By Case 1 applied to h , there is a $\delta > 0$ such that

$$\left| \int_{\mathbb{R}} g \cdot [h - h_t] \right| < \frac{\epsilon}{3} \quad \text{whenever } |t| < \delta.$$

Therefore, for $0 < |t| < \delta$,

$$\left| \int_{-\infty}^{\infty} g(x) \cdot [f(x) - f(x+t)] dx \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the limit is 0.

Problem 4.48. Let f be integrable over E and g be a bounded measurable function on E . Show that $f \cdot g$ is integrable over E .

Solution. Choose $M \geq 0$ with $|g| \leq M$ on E .

First we check that $f \cdot g$ is measurable on E . Since f is integrable over E , Proposition 15 tells us that f is finite a.e. on E ; let

$$E_0 = \{x \in E \mid |f(x)| = \infty\},$$

so $m(E_0) = 0$. On $E \sim E_0$ both f and g are real-valued measurable functions, so their product $f \cdot g$ is measurable on $E \sim E_0$ by Theorem 6 of Chapter 3. Since Lebesgue measure is complete, a function on E that agrees with a measurable function outside a set of measure zero is measurable on E (Proposition 5 of Chapter 3); as $f \cdot g$ restricted to $E \sim E_0$ is measurable and $m(E_0) = 0$, $f \cdot g$ is measurable on E whatever convention is used for the product at points of E_0 .

Next, at every point of E ,

$$|f \cdot g| = |f| \cdot |g| \leq M \cdot |f|.$$

The function $M|f|$ is nonnegative and, by the linearity of integration (Theorem 17, or Theorem 10 for nonnegative functions), integrable over E with $\int_E M|f| = M \int_E |f| < \infty$. Thus $M|f|$ is a nonnegative integrable function on E that dominates $f \cdot g$ in the sense of the Integral Comparison Test.

By the Integral Comparison Test (Proposition 16), $f \cdot g$ is integrable over E and

$$\left| \int_E f \cdot g \right| \leq \int_E |f \cdot g| \leq M \int_E |f|.$$

Problem 4.49. Let f be integrable over $\mathbb{R}$. Show that the following four assertions are equivalent:

- (i) $f = 0$ a.e. on $\mathbb{R}$.
- (ii) $\int_{\mathbb{R}} fg = 0$ for every bounded measurable function g on $\mathbb{R}$.
- (iii) $\int_A f = 0$ for every measurable set A .
- (iv) $\int_{\mathcal{O}} f = 0$ for every open set $\mathcal{O}$.

Solution. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

We use repeatedly the following consequence of Proposition 15: if h is integrable over a measurable set B and $h = 0$ a.e. on B , then $\int_B h = 0$. Indeed, letting $B_0 = \{x \in B \mid h(x) \neq 0\}$, we have $m(B_0) = 0$, so

by Proposition 15, $\int_B h = \int_{B \sim B_0} h = 0$, the last integral vanishing because its integrand is identically 0.

(i) $\Rightarrow$ (ii). Let g be a bounded measurable function on $\mathbb{R}$. By Problem 48, fg is integrable over $\mathbb{R}$. Since $f = 0$ a.e. on $\mathbb{R}$, also $fg = 0$ a.e. on $\mathbb{R}$, and therefore $\int_{\mathbb{R}} fg = 0$ by the remark above.

(ii) $\Rightarrow$ (iii). Let A be a measurable set. The characteristic function χ_A is measurable and bounded (by 1) on $\mathbb{R}$, so by hypothesis $\int_{\mathbb{R}} f\chi_A = 0$. On the other hand $f\chi_A$ is integrable over $\mathbb{R}$ (Problem 48), $f\chi_A = f$ on A and $f\chi_A = 0$ on $\mathbb{R} \sim A$; hence, by the additivity of integration over domains (Corollary 18),

$$\int_{\mathbb{R}} f\chi_A = \int_A f\chi_A + \int_{\mathbb{R} \sim A} f\chi_A = \int_A f + 0 = \int_A f.$$

Therefore $\int_A f = 0$.

(iii) $\Rightarrow$ (iv). Every open subset $\mathcal{O}$ of $\mathbb{R}$ is measurable, so this is immediate.

(iv) $\Rightarrow$ (i). Assume $\int_{\mathcal{O}} f = 0$ for every open set $\mathcal{O}$.

Step 1: $\int_A f = 0$ for every measurable set A of finite measure. Let A be such a set and let $\epsilon > 0$. Since f is integrable over $\mathbb{R}$, Proposition 23 provides a $\delta > 0$ such that

$$\text{if } B \subseteq \mathbb{R} \text{ is measurable and } m(B) < \delta, \text{ then } \int_B |f| < \epsilon.$$

By assertion (i) of Theorem 11 of Chapter 2 (outer approximation by open sets), there is an open set $\mathcal{O}$ containing A with $m(\mathcal{O} \sim A) < \delta$. Note that f is integrable over every measurable subset B of $\mathbb{R}$: by the additivity of integration over domains for nonnegative measurable functions (Theorem 11), $\int_{\mathbb{R}} |f| = \int_B |f| + \int_{\mathbb{R} \sim B} |f| \geq \int_B |f|$, so $\int_B |f| < \infty$ and the Integral Comparison Test applies. Hence, applying the additivity of integration over domains (Corollary 18) to the disjoint decomposition $\mathcal{O} = A \cup (\mathcal{O} \sim A)$,

$$\int_A f = \int_{\mathcal{O}} f - \int_{\mathcal{O} \sim A} f = 0 - \int_{\mathcal{O} \sim A} f.$$

By the Integral Comparison Test and the choice of δ ,

$$\left| \int_A f \right| = \left| \int_{\mathcal{O} \sim A} f \right| \leq \int_{\mathcal{O} \sim A} |f| < \epsilon.$$

As $\epsilon > 0$ was arbitrary, $\int_A f = 0$.

Step 2: $f = 0$ a.e. Let

$$A^+ = \{x \in \mathbb{R} \mid f(x) > 0\}, \quad A^- = \{x \in \mathbb{R} \mid f(x) < 0\}.$$

For natural numbers n put

$$A_n^+ = \left\{ x \in (-n, n) \mid f(x) > \frac{1}{n} \right\}.$$

Each A_n^+ is measurable and $m(A_n^+) \leq 2n < \infty$, so by Step 1, $\int_{A_n^+} f = 0$. But $f > 1/n$ on A_n^+ , so by the monotonicity of integration for nonnegative functions,

$$0 = \int_{A_n^+} f \geq \int_{A_n^+} \frac{1}{n} = \frac{1}{n} m(A_n^+) \geq 0.$$

Hence $m(A_n^+) = 0$ for every n . Since

$$A^+ = \bigcup_{n=1}^{\infty} A_n^+$$

(if $f(x) > 0$, choose n with $|x| < n$ and $f(x) > 1/n$, which is possible whether $f(x)$ is finite and positive or $f(x) = \infty$), the countable subadditivity of measure gives $m(A^+) = 0$.

Applying the same argument to $-f$, which also satisfies the hypothesis of (iv) since $\int_{\mathcal{O}} (-f) = -\int_{\mathcal{O}} f = 0$ for every open $\mathcal{O}$, we obtain $m(A^-) = 0$.

Therefore $m(\{x \in \mathbb{R} \mid f(x) \neq 0\}) = m(A^+) + m(A^-) = 0$, that is, $f = 0$ a.e. on $\mathbb{R}$.

This closes the cycle, so the four assertions are equivalent.

4.8 Problems 4.50–4.52

Problem 4.50. Let $\mathcal{F}$ be a family of functions, each of which is integrable over E . Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \left| \int_A f \right| < \epsilon.$$

Solution. Recall the definition in force: $\mathcal{F}$ is **uniformly integrable over E** provided for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $f \in \mathcal{F}$, if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \epsilon$. So the content of the problem is that the absolute value may be moved outside the integral sign in this criterion.

Observe first that each $f \in \mathcal{F}$ is integrable over E , hence integrable over every measurable subset A of E : indeed $|f| \cdot \chi_A \leq |f|$ on E , so $|f|$ is integrable over A by the monotonicity of integration for nonnegative functions, and therefore f is integrable over A by the definition of the general Lebesgue integral. In particular all the integrals written below are finite.

The criterion is necessary. Assume $\mathcal{F}$ is uniformly integrable over E . Let $\epsilon > 0$ and let $\delta > 0$ respond to the ϵ challenge in the definition of uniform integrability. Let $f \in \mathcal{F}$ and let $A \subseteq E$ be measurable with $m(A) < \delta$. Since $-|f| \leq f \leq |f|$ on A , the monotonicity of integration (Theorem 17 of Chapter 4) gives

$$-\int_A |f| \leq \int_A f \leq \int_A |f|,$$

that is, $|\int_A f| \leq \int_A |f| < \epsilon$. Thus the same δ responds to the ϵ challenge in the stated criterion.

The criterion is sufficient. Assume that for each $\epsilon > 0$ there is a $\delta > 0$ such that for every $f \in \mathcal{F}$ and every measurable $A \subseteq E$ with $m(A) < \delta$ we have $|\int_A f| < \epsilon$. Let $\epsilon > 0$ be given and let $\delta > 0$ respond to the challenge $\epsilon/2$ in this criterion. We claim this same δ responds to the ϵ challenge in the definition of uniform integrability.

Indeed, let $f \in \mathcal{F}$ and let $A \subseteq E$ be measurable with $m(A) < \delta$. Since f is measurable, the sets

$$A^+ = \{x \in A \mid f(x) \geq 0\}, \quad A^- = \{x \in A \mid f(x) < 0\}$$

are measurable. They are disjoint, their union is A , and each is a measurable subset of E with

$$m(A^+) \leq m(A) < \delta, \quad m(A^-) \leq m(A) < \delta$$

by the monotonicity of Lebesgue measure. Hence, by the choice of δ ,

$$\left| \int_{A^+} f \right| < \frac{\epsilon}{2}, \quad \left| \int_{A^-} f \right| < \frac{\epsilon}{2}.$$

Now $|f| = f$ on A^+ and $|f| = -f$ on A^- . Since $|f|$ is integrable over E and $A = A^+ \cup A^-$ is a disjoint union of measurable subsets of E , the additivity over domains of integration (Corollary 18 of Chapter 4), applied to $|f|$ and then to f over each piece, yields

$$\int_A |f| = \int_{A^+} |f| + \int_{A^-} |f| = \int_{A^+} f - \int_{A^-} f.$$

But $f \geq 0$ on A^+ forces $\int_{A^+} f \geq 0$, so $\int_{A^+} f = |\int_{A^+} f|$; and $f \leq 0$ on A^- forces $\int_{A^-} f \leq 0$, so $-\int_{A^-} f = |\int_{A^-} f|$. Therefore

$$\int_A |f| = \left| \int_{A^+} f \right| + \left| \int_{A^-} f \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $f \in \mathcal{F}$ and the measurable set $A \subseteq E$ with $m(A) < \delta$ were arbitrary, $\mathcal{F}$ is uniformly integrable over E .

Problem 4.51. Let $\mathcal{F}$ be a family of functions, each of which is integrable over E . Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$\text{if } \mathcal{U} \text{ is open and } m(E \cap \mathcal{U}) < \delta, \text{ then } \int_{E \cap \mathcal{U}} |f| < \epsilon.$$

Solution. Throughout, $\mathcal{F}$ is uniformly integrable over E means: for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $f \in \mathcal{F}$, if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \epsilon$. As in the preceding problem, each $f \in \mathcal{F}$ is integrable over every measurable subset of E , so all integrals below are finite. We also use repeatedly the following consequence of the additivity over domains of integration (Corollary 18 of Chapter 4) for the nonnegative integrable function $|f|$: if $A \subseteq B$ are measurable subsets of E , then, writing $B = A \cup (B \sim A)$ as a disjoint union,

$$\int_B |f| = \int_A |f| + \int_{B \sim A} |f| \geq \int_A |f|,$$

since $\int_{B \sim A} |f| \geq 0$ by the monotonicity of integration for nonnegative functions.

The criterion is necessary. Assume $\mathcal{F}$ is uniformly integrable over E , let $\epsilon > 0$, and let $\delta > 0$ respond to the ϵ challenge in the definition. If $\mathcal{U}$ is open and $m(E \cap \mathcal{U}) < \delta$, then $A = E \cap \mathcal{U}$ is a **measurable** subset of E (open sets are measurable, and the intersection of two measurable sets is measurable) of measure less than δ . Hence $\int_{E \cap \mathcal{U}} |f| < \epsilon$ for every $f \in \mathcal{F}$. So the same δ responds to the ϵ challenge in the stated criterion.

The criterion is sufficient. Assume that for each $\epsilon > 0$ there is a $\delta > 0$ such that for all $f \in \mathcal{F}$, if $\mathcal{U}$ is open and $m(E \cap \mathcal{U}) < \delta$, then $\int_{E \cap \mathcal{U}} |f| < \epsilon$.

Let $\epsilon > 0$ be given. Choose $\delta_0 > 0$ responding to this ϵ challenge in the assumed criterion, and define

$$\delta = \frac{\delta_0}{2} > 0.$$

We claim δ responds to the ϵ challenge in the definition of uniform integrability. Let $f \in \mathcal{F}$ and let $A \subseteq E$ be measurable with $m(A) < \delta$.

Since A is measurable, by the outer approximation property of measurable sets, Theorem 11(i) of Chapter 2, there is an open set $\mathcal{U}$ containing A for which

$$m^*(\mathcal{U} \sim A) < \frac{\delta_0}{2}.$$

Now $E \cap \mathcal{U} \subseteq A \cup (\mathcal{U} \sim A)$, because a point of $E \cap \mathcal{U}$ that fails to lie in A lies in $\mathcal{U} \sim A$. Hence, by the monotonicity and finite subadditivity of outer measure,

$$m(E \cap \mathcal{U}) \leq m(A) + m^*(\mathcal{U} \sim A) < \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0.$$

Therefore, by the choice of δ_0 ,

$$\int_{E \cap \mathcal{U}} |f| < \epsilon.$$

Finally, $A \subseteq E \cap \mathcal{U}$, since $A \subseteq E$ and $A \subseteq \mathcal{U}$. By the monotonicity over domains noted at the outset, applied to the nonnegative function $|f|$,

$$\int_A |f| \leq \int_{E \cap \mathcal{U}} |f| < \epsilon.$$

Since $f \in \mathcal{F}$ and the measurable set $A \subseteq E$ with $m(A) < \delta$ were arbitrary, $\mathcal{F}$ is uniformly integrable over E .

Problem 4.52. (a) Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_0^1 |f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?
(b) Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is continuous on $[0, 1]$ and has $|f| \leq 1$ on $[0, 1]$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?
(c) Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_a^b |f| \leq b - a$ for all $[a, b] \subseteq [0, 1]$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?

Solution. (a) No. A uniform bound on the integrals does not prevent the mass of $|f|$ from concentrating on sets of arbitrarily small measure.

For each natural number n define

$$f_n = n \cdot \chi_{(0, 1/n]} \quad \text{on } [0, 1].$$

Each f_n is a nonnegative simple function, hence bounded and measurable on the set $[0, 1]$ of finite measure, and so integrable over $[0, 1]$, with

$$\int_0^1 |f_n| = n \cdot m((0, 1/n]) = n \cdot \frac{1}{n} = 1 \leq 1.$$

Thus $f_n \in \mathcal{F}$ for every n .

Now take the challenge $\epsilon = 1/2$ and let $\delta > 0$ be arbitrary. By the Archimedean property of $\mathbb{R}$ choose a natural number n with $1/n < \delta$, and set $A_n = (0, 1/n]$. Then A_n is a measurable subset of $[0, 1]$ with $m(A_n) = 1/n < \delta$, yet

$$\int_{A_n} |f_n| = n \cdot m(A_n) = 1 > \frac{1}{2} = \epsilon.$$

So no $\delta > 0$ responds to the challenge $\epsilon = 1/2$, and $\mathcal{F}$ is not uniformly integrable over $[0, 1]$.

(b) Yes. Here the whole family is dominated by a single integrable function, namely the constant function 1.

Let $f \in \mathcal{F}$. Being continuous on $[0, 1]$, f is measurable (Proposition 3 of Chapter 3), and it is bounded, since $|f| \leq 1$; as $m([0, 1]) = 1 < \infty$, f is integrable over $[0, 1]$ by Theorem 4 of Chapter 4. Let $\epsilon > 0$ and put $\delta = \epsilon$. If $A \subseteq [0, 1]$ is measurable with $m(A) < \delta$, then $|f| \leq 1$ on A , so by the monotonicity of integration for nonnegative functions (Theorem 10 of Chapter 4),

$$\int_A |f| \leq \int_A 1 = m(A) < \delta = \epsilon.$$

Since δ was chosen independently of $f \in \mathcal{F}$, the family $\mathcal{F}$ is uniformly integrable over $[0, 1]$. This is in fact an instance of the Example following the definition of uniform integrability in Section 4.6, taken with $g \equiv 1$: continuity plays no role, only the uniform bound $|f| \leq 1$ matters.

(c) Yes. The hypothesis controls $\int_A |f|$ only when A is a closed, bounded interval; but by Proposition 9 of Chapter 1 every open set is assembled from intervals, and by the preceding problem it suffices to test the uniform integrability criterion on open sets.

Step 1: the estimate on open sets. We claim that for each $f \in \mathcal{F}$ and each open set $\mathcal{U} \subseteq \mathbb{R}$,

$$\int_{[0, 1] \cap \mathcal{U}} |f| \leq m([0, 1] \cap \mathcal{U}).$$

Put $\mathcal{V} = (0, 1) \cap \mathcal{U}$, an open subset of $\mathbb{R}$. The set $([0, 1] \cap \mathcal{U}) \sim \mathcal{V}$ is contained in $\{0, 1\}$ and so has measure zero; hence $m([0, 1] \cap \mathcal{U}) = m(\mathcal{V})$, and since $|f|$ is a nonnegative measurable function on $[0, 1]$, Theorem 11 of Chapter 4 (additivity over domains of integration, together with the vanishing of the integral over a subset of measure zero) gives

$$\int_{[0, 1] \cap \mathcal{U}} |f| = \int_{\mathcal{V}} |f|.$$

If $\mathcal{V} = \emptyset$, both sides of the claimed inequality are 0 and there is nothing to prove. Otherwise, by Proposition 9 of Chapter 1, $\mathcal{V}$ is the union of a countable disjoint collection $\{I_k\}$ of open intervals.

Since $\mathcal{V} \subseteq (0, 1)$, each I_k is bounded, say $I_k = (a_k, b_k)$ with $0 \leq a_k < b_k \leq 1$, so that $[a_k, b_k] \subseteq [0, 1]$. The set $[a_k, b_k] \sim (a_k, b_k) = \{a_k, b_k\}$ has measure zero, so again by Theorem 11 of Chapter 4 and the defining hypothesis on f ,

$$\int_{I_k} |f| = \int_{[a_k, b_k]} |f| = \int_{a_k}^{b_k} |f| \leq b_k - a_k = m(I_k).$$

Since f is integrable over $[0, 1]$ it is integrable over the measurable subset $\mathcal{V}$ (indeed $|f| \cdot \chi_{\mathcal{V}} \leq |f|$ on $[0, 1]$, so $\int_{\mathcal{V}} |f| \leq \int_0^1 |f| < \infty$ by the monotonicity of integration for nonnegative functions, Theorem 10 of Chapter 4), and $\mathcal{V}$ is the disjoint union of the countable collection $\{I_k\}$ of measurable sets. Therefore, by the Countable Additivity of Integration (Theorem 20 of Chapter 4) applied to the integrable function $|f|$ over $\mathcal{V}$, and then by the countable additivity of Lebesgue measure,

$$\int_{\mathcal{V}} |f| = \sum_k \int_{I_k} |f| \leq \sum_k m(I_k) = m(\mathcal{V}).$$

If the collection $\{I_k\}$ is finite, the same conclusion follows from Corollary 18 of Chapter 4 and the finite additivity of measure. Combining the three displays proves the claim.

Step 2: conclusion. Let $\epsilon > 0$ and put $\delta = \epsilon$. If $\mathcal{U}$ is open and $m([0, 1] \cap \mathcal{U}) < \delta$, then by Step 1, for every $f \in \mathcal{F}$,

$$\int_{[0, 1] \cap \mathcal{U}} |f| \leq m([0, 1] \cap \mathcal{U}) < \delta = \epsilon.$$

Thus δ responds, uniformly over all of $\mathcal{F}$, to the ϵ challenge in the criterion of the preceding problem (Problem 51). Therefore $\mathcal{F}$ is uniformly integrable over $[0, 1]$.

One can avoid invoking Problem 51 and argue directly: given a measurable $A \subseteq [0, 1]$ with $m(A) < \epsilon/2$, use Theorem 11(i) of Chapter 2 to choose an open set $\mathcal{U}$ containing A with $m^*(\mathcal{U} \sim A) < \epsilon/2$; then $m([0, 1] \cap \mathcal{U}) \leq m(A) + m^*(\mathcal{U} \sim A) < \epsilon$, and by Step 1 together with the monotonicity over domains of the integral of the nonnegative function $|f|$,

$$\int_A |f| \leq \int_{[0, 1] \cap \mathcal{U}} |f| \leq m([0, 1] \cap \mathcal{U}) < \epsilon.$$

5 Lebesgue Integration: Further Topics

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5.1 Problems 5.1–5.7

Problem 5.1. Prove Corollary 2.

[Corollary 2: Let $\{h_n\}$ be a sequence of nonnegative integrable functions on E . Suppose $\{h_n(x)\} \rightarrow 0$ for almost all x in E . Then

$$\lim_{n \rightarrow \infty} \int_E h_n = 0 \quad \text{if and only if} \quad \{h_n\} \text{ is uniformly integrable and tight over } E.$$

]

Solution. We prove the two implications separately.

Assume first that $\{h_n\}$ is uniformly integrable and tight over E . Since $\{h_n\} \rightarrow 0$ pointwise almost everywhere on E , the Vitali Convergence Theorem of this section (the version for sets of possibly infinite measure) applies with $f \equiv 0$ and yields

$$\lim_{n \rightarrow \infty} \int_E h_n = \int_E 0 = 0.$$

Conversely, assume $\lim_{n \rightarrow \infty} \int_E h_n = 0$. Let $\epsilon > 0$. Choose a natural number N for which

$$\int_E h_n < \epsilon \quad \text{for all } n \geq N.$$

Since each h_n is nonnegative, monotonicity of integration together with additivity over domains gives, for every measurable subset A of E and every $n \geq N$,

$$\int_A |h_n| = \int_A h_n \leq \int_E h_n < \epsilon. \quad (1)$$

Thus the tail of the sequence satisfies the uniform integrability and the tightness criteria for this ϵ with no restriction at all on A , and with the empty set serving as the set of finite measure.

It remains to handle the finitely many initial indices. If $N = 1$ there are none, and we may take $\delta = 1$ and $E_0 = \emptyset$. Otherwise consider the finite collection $\{h_n\}_{n=1}^{N-1}$, each member of which is integrable over E . By Proposition 23 of Chapter 4, for each n with $1 \leq n \leq N-1$ there is a $\delta_n > 0$ such that

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta_n, \text{ then } \int_A |h_n| < \epsilon.$$

Define $\delta = \min\{\delta_1, \dots, \delta_{N-1}\} > 0$. (This is exactly Proposition 24 of Chapter 4: a finite collection of integrable functions is uniformly integrable.) By (1), the same δ responds to the ϵ challenge for every index n , so $\{h_n\}_{n=1}^{\infty}$ is uniformly integrable over E .

For tightness, apply Proposition 1 of this section to each of the integrable functions $h_1, \dots, h_{N-1}$: there are sets $E_1, \dots, E_{N-1}$ of finite measure for which

$$\int_{E \sim E_n} h_n < \epsilon \quad \text{for } 1 \leq n \leq N-1.$$

Put $E_0 = \bigcup_{n=1}^{N-1} E_n$. Being a finite union of sets of finite measure, E_0 has finite measure. For $1 \leq n \leq N-1$ we have $E \sim E_0 \subseteq E \sim E_n$, so by monotonicity

$$\int_{E \sim E_0} h_n \leq \int_{E \sim E_n} h_n < \epsilon,$$

while for $n \geq N$ inequality (1), applied with $A = E \sim E_0$, gives $\int_{E \sim E_0} h_n < \epsilon$. Hence E_0 responds to the ϵ challenge for the whole sequence, and $\{h_n\}$ is tight over E .

Since $\epsilon > 0$ was arbitrary, $\{h_n\}$ is uniformly integrable and tight over E . The proof is complete.

Problem 5.2. Let $\{f_k\}_{k=1}^n$ be a finite family of functions, each of which is integrable over E . Show that $\{f_k\}_{k=1}^n$ is uniformly integrable and tight over E .

Solution. Let $\epsilon > 0$.

Uniform integrability. For each index k with $1 \leq k \leq n$, the function f_k is integrable over E , so by Proposition 23 of Chapter 4 there is a $\delta_k > 0$ for which

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta_k, \text{ then } \int_A |f_k| < \epsilon.$$

Define $\delta = \min\{\delta_1, \dots, \delta_n\}$. Since this is a minimum of finitely many positive numbers, $\delta > 0$. If $A \subseteq E$ is measurable with $m(A) < \delta$, then $m(A) < \delta_k$ for every k , and hence $\int_A |f_k| < \epsilon$ for every k . Thus δ responds to the ϵ challenge regarding the criterion for the family $\{f_k\}_{k=1}^n$ to be uniformly integrable over E . This is Proposition 24 of Chapter 4.

Tightness. For each index k with $1 \leq k \leq n$, the function f_k is integrable over E , so by Proposition 1 of this section there is a set $E_k \subseteq E$ of finite measure for which

$$\int_{E \sim E_k} |f_k| < \epsilon.$$

Define

$$E_0 = \bigcup_{k=1}^n E_k.$$

Then E_0 is a measurable subset of E , and by the finite subadditivity of measure $m(E_0) \leq \sum_{k=1}^n m(E_k) < \infty$, so E_0 has finite measure. Moreover, for each k we have $E_0 \supseteq E_k$ and therefore $E \sim E_0 \subseteq E \sim E_k$. By the monotonicity of integration over domains (the integrand $|f_k|$ being nonnegative),

$$\int_{E \sim E_0} |f_k| \leq \int_{E \sim E_k} |f_k| < \epsilon \quad \text{for } 1 \leq k \leq n.$$

Hence E_0 responds to the ϵ challenge regarding the criterion for tightness.

Since $\epsilon > 0$ was arbitrary, the finite family $\{f_k\}_{k=1}^n$ is both uniformly integrable and tight over E .

Problem 5.3. Let the sequences of functions $\{f_n\}$ and $\{g_n\}$ be uniformly integrable and tight over E . Show that for any α and β , $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable and tight over E .

Solution. Set $M = |\alpha| + |\beta| + 1$, so that $M > 0$.

We first check that the functions $\alpha f_n + \beta g_n$ are properly defined. As the book observes just after the definition of tightness, a family that is uniformly integrable and tight over E consists of functions that are integrable over E ; hence each f_n and each g_n is integrable over E and therefore, by Proposition 15 of Chapter 4, finite a.e. on E . Let Z be the union over all n of the sets where f_n or g_n fails to be finite. Being a countable union of sets of measure zero, Z has measure zero by the countable subadditivity of measure. Off Z every combination $\alpha f_n + \beta g_n$ is defined and finite, and it is measurable there as a linear combination of measurable functions; since altering a function on a set of measure zero changes neither its measurability nor any of its integrals, we may and do work on $E \sim Z$, and all integrals below are unaffected.

Pointwise on $E \sim Z$ we have

$$|\alpha f_n + \beta g_n| \leq |\alpha| |f_n| + |\beta| |g_n|.$$

Consequently, by the monotonicity and linearity of integration for nonnegative measurable functions, for every measurable subset A of E ,

$$\int_A |\alpha f_n + \beta g_n| \leq |\alpha| \int_A |f_n| + |\beta| \int_A |g_n|. \quad (2)$$

Uniform integrability. Let $\epsilon > 0$. Since $\{f_n\}$ is uniformly integrable over E , there is a $\delta_1 > 0$ such that

for every index n ,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta_1, \text{ then } \int_A |f_n| < \epsilon/M.$$

Since $\{g_n\}$ is uniformly integrable over E , there is a $\delta_2 > 0$ that does the same for the g_n 's with the same bound ϵ/M . Define $\delta = \min\{\delta_1, \delta_2\} > 0$. If $A \subseteq E$ is measurable with $m(A) < \delta$, then by (2), for every index n ,

$$\int_A |\alpha f_n + \beta g_n| \leq |\alpha| \cdot \frac{\epsilon}{M} + |\beta| \cdot \frac{\epsilon}{M} = \frac{|\alpha| + |\beta|}{M} \epsilon < \epsilon,$$

since $|\alpha| + |\beta| < M$. Thus δ responds to the ϵ challenge, and $\{\alpha f_n + \beta g_n\}$ is uniformly integrable over E .

Tightness. Let $\epsilon > 0$. By the tightness of $\{f_n\}$ there is a set $E_1 \subseteq E$ of finite measure for which $\int_{E \sim E_1} |f_n| < \epsilon/M$ for all n , and by the tightness of $\{g_n\}$ there is a set $E_2 \subseteq E$ of finite measure for which $\int_{E \sim E_2} |g_n| < \epsilon/M$ for all n . Define $E_0 = E_1 \cup E_2$. Then $m(E_0) \leq m(E_1) + m(E_2) < \infty$, so E_0 has finite measure, and

$$E \sim E_0 \subseteq E \sim E_1, \quad E \sim E_0 \subseteq E \sim E_2.$$

Hence, by monotonicity of integration over domains and by (2) applied with $A = E \sim E_0$, for every index n ,

$$\int_{E \sim E_0} |\alpha f_n + \beta g_n| \leq |\alpha| \int_{E \sim E_1} |f_n| + |\beta| \int_{E \sim E_2} |g_n| < \frac{|\alpha| + |\beta|}{M} \epsilon < \epsilon.$$

Thus E_0 responds to the ϵ challenge for tightness, and $\{\alpha f_n + \beta g_n\}$ is tight over E .

Problem 5.4. Let $\{f_n\}$ be a sequence of measurable functions on E . Show that $\{f_n\}$ is uniformly integrable and tight over E if and only if for each $\epsilon > 0$, there is a measurable subset E_0 of E that has finite measure and a $\delta > 0$ such that for each measurable subset A of E and index n ,

$$\text{if } m(A \cap E_0) < \delta, \text{ then } \int_A |f_n| < \epsilon.$$

Solution. Suppose first that $\{f_n\}$ is uniformly integrable and tight over E , and let $\epsilon > 0$. By tightness there is a measurable subset E_0 of E of finite measure for which

$$\int_{E \sim E_0} |f_n| < \epsilon/2 \quad \text{for all } n. \quad (3)$$

By uniform integrability there is a $\delta > 0$ such that

$$\text{if } B \subseteq E \text{ is measurable and } m(B) < \delta, \text{ then } \int_B |f_n| < \epsilon/2 \text{ for all } n. \quad (4)$$

We claim this E_0 and this δ respond to the ϵ challenge. Indeed, let A be a measurable subset of E with $m(A \cap E_0) < \delta$, and let n be any index. Decompose A as the disjoint union of the measurable sets $A \cap E_0$ and $A \sim E_0$. By the additivity over domains of integration of the nonnegative function $|f_n|$,

$$\int_A |f_n| = \int_{A \cap E_0} |f_n| + \int_{A \sim E_0} |f_n|.$$

The first term is less than $\epsilon/2$ by (4), since $m(A \cap E_0) < \delta$. For the second, $A \sim E_0 \subseteq E \sim E_0$, so by monotonicity and (3) it is at most $\int_{E \sim E_0} |f_n| < \epsilon/2$. Adding,

$$\int_A |f_n| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which is what was to be shown.

Conversely, suppose that for each $\epsilon > 0$ there is a measurable subset E_0 of E of finite measure and a $\delta > 0$ with the stated property. Let $\epsilon > 0$ and let E_0 and δ be so chosen.

For uniform integrability: if $A \subseteq E$ is measurable and $m(A) < \delta$, then by monotonicity of measure $m(A \cap E_0) \leq m(A) < \delta$, and therefore $\int_A |f_n| < \epsilon$ for every index n . Thus this same δ responds to the ϵ challenge in the definition of uniform integrability.

For tightness: take $A = E \sim E_0$. Then $A \cap E_0 = \emptyset$, so $m(A \cap E_0) = 0 < \delta$, and the hypothesis gives

$$\int_{E \sim E_0} |f_n| < \epsilon \quad \text{for all } n.$$

Since E_0 has finite measure, E_0 responds to the ϵ challenge in the definition of tightness.

As $\epsilon > 0$ was arbitrary, $\{f_n\}$ is uniformly integrable and tight over E .

Problem 5.5. Let $\{f_n\}$ be a sequence of integrable functions on $\mathbb{R}$. Show that $\{f_n\}$ is uniformly integrable and tight over $\mathbb{R}$ if and only if for each $\epsilon > 0$, there are positive numbers r and δ such that for each open subset $\mathcal{O}$ of $\mathbb{R}$ and index n ,

$$\text{if } m(\mathcal{O} \cap (-r, r)) < \delta, \text{ then } \int_{\mathcal{O}} |f_n| < \epsilon.$$

Solution. Suppose first that $\{f_n\}$ is uniformly integrable and tight over $\mathbb{R}$, and let $\epsilon > 0$.

By uniform integrability there is a $\delta > 0$ such that

$$\text{if } A \subseteq \mathbb{R} \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f_n| < \epsilon/3 \text{ for all } n. \quad (5)$$

By tightness there is a measurable set $E_0 \subseteq \mathbb{R}$ of finite measure for which

$$\int_{\mathbb{R} \sim E_0} |f_n| < \epsilon/3 \quad \text{for all } n. \quad (6)$$

Since $m(E_0) < \infty$, and since the sets $E_0 \sim (-k, k)$, for k a natural number, form a descending sequence of measurable sets of finite measure whose intersection is empty, Theorem 15 of Chapter 2 (the Continuity of Measure) tells us that $\lim_{k \rightarrow \infty} m(E_0 \sim (-k, k)) = 0$. Choose $r > 0$ (a natural number will do) for which

$$m(E_0 \sim (-r, r)) < \delta.$$

Now

$$\mathbb{R} \sim (-r, r) \subseteq (\mathbb{R} \sim E_0) \cup (E_0 \sim (-r, r)),$$

so by the monotonicity and the (finite) subadditivity of integration of the nonnegative function $|f_n|$ over domains, together with (5) and (6),

$$\int_{\mathbb{R} \sim (-r, r)} |f_n| \leq \int_{\mathbb{R} \sim E_0} |f_n| + \int_{E_0 \sim (-r, r)} |f_n| < \epsilon/3 + \epsilon/3 = 2\epsilon/3 \quad \text{for all } n. \quad (7)$$

We claim this r and this δ respond to the ϵ challenge. Indeed, let $\mathcal{O}$ be an open subset of $\mathbb{R}$ with $m(\mathcal{O} \cap (-r, r)) < \delta$ and let n be any index. Splitting $\mathcal{O}$ into the disjoint measurable pieces $\mathcal{O} \cap (-r, r)$ and $\mathcal{O} \sim (-r, r)$ and using additivity over domains, (5) and (7),

$$\int_{\mathcal{O}} |f_n| = \int_{\mathcal{O} \cap (-r, r)} |f_n| + \int_{\mathcal{O} \sim (-r, r)} |f_n| < \frac{\epsilon}{3} + \int_{\mathbb{R} \sim (-r, r)} |f_n| < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.$$

(Openness of $\mathcal{O}$ was not needed here; the estimate holds for every measurable set.)

Conversely, suppose that for each $\epsilon > 0$ there are positive numbers r and δ with the stated property. Let $\epsilon > 0$, and let $r > 0$ and $\delta > 0$ respond to this ϵ .

Tightness. Take $E_0 = (-r, r)$, a set of finite measure $2r$. The set $\mathcal{O} = \mathbb{R} \sim [-r, r]$ is open and $\mathcal{O} \cap (-r, r) = \emptyset$, so $m(\mathcal{O} \cap (-r, r)) = 0 < \delta$ and therefore $\int_{\mathcal{O}} |f_n| < \epsilon$ for all n . Since $\mathbb{R} \sim (-r, r)$

differs from $\mathcal{O}$ only by the two-point set $\{-r, r\}$, which has measure zero, the integrals over these two domains agree, and hence

$$\int_{\mathbb{R} \sim E_0} |f_n| = \int_{\mathcal{O}} |f_n| < \epsilon \quad \text{for all } n.$$

Thus E_0 responds to the ϵ challenge for tightness.

Uniform integrability. We claim $\delta/2$ responds to the ϵ challenge. Let $A \subseteq \mathbb{R}$ be measurable with $m(A) < \delta/2$. Since A is measurable, assertion (i) of Theorem 11 of Chapter 2 provides an open set $\mathcal{O}$ containing A for which $m(\mathcal{O} \sim A) < \delta/2$. By the additivity of measure,

$$m(\mathcal{O}) = m(A) + m(\mathcal{O} \sim A) < \delta/2 + \delta/2 = \delta,$$

and hence $m(\mathcal{O} \cap (-r, r)) \leq m(\mathcal{O}) < \delta$. By hypothesis $\int_{\mathcal{O}} |f_n| < \epsilon$ for every index n , and since $A \subseteq \mathcal{O}$, monotonicity of integration over domains gives

$$\int_A |f_n| \leq \int_{\mathcal{O}} |f_n| < \epsilon \quad \text{for all } n.$$

Since $\epsilon > 0$ was arbitrary, $\{f_n\}$ is uniformly integrable and tight over $\mathbb{R}$.

Problem 5.6. Let $\{f_n\} \rightarrow f$ in measure on E and g be a measurable function on E that is finite a.e. on E . Show that $\{f_n\} \rightarrow g$ in measure on E if and only if $f = g$ a.e. on E .

Solution. Recall that, by the definition of convergence in measure, the hypothesis $\{f_n\} \rightarrow f$ in measure on E carries with it the assertion that f and each f_n is measurable and finite a.e. on E ; and g is measurable and finite a.e. on E by hypothesis. Let Z be the set of points of E at which f or g fails to be finite. Being the union of two sets of measure zero, Z has measure zero.

Suppose first that $f = g$ a.e. on E , and let $Z' = \{x \in E \mid f(x) \neq g(x)\}$, a set of measure zero. Write $W = Z \cup Z'$, so $m(W) = 0$ by the countable subadditivity of measure. Fix $\eta > 0$ and an index n . If $x \in E \sim W$ then $f(x) = g(x)$ and this common value is a real number, so the two differences $f_n(x) - g(x)$ and $f_n(x) - f(x)$ are the same extended real number and

$$|f_n(x) - g(x)| > \eta \iff |f_n(x) - f(x)| > \eta \quad \text{for } x \in E \sim W.$$

Consequently

$$\{x \in E \mid |f_n - g| > \eta\} \subseteq W \cup \{x \in E \mid |f_n - f| > \eta\},$$

and so, by the monotonicity and countable subadditivity of measure together with $m(W) = 0$,

$$m\{x \in E \mid |f_n - g| > \eta\} \leq m\{x \in E \mid |f_n - f| > \eta\}.$$

Since $\{f_n\} \rightarrow f$ in measure, the right-hand side tends to 0 as $n \rightarrow \infty$, and hence so does the left-hand side. As $\eta > 0$ was arbitrary, and as g is by hypothesis measurable and finite a.e. on E , we conclude $\{f_n\} \rightarrow g$ in measure on E .

Conversely, suppose $\{f_n\} \rightarrow g$ in measure on E as well. Fix $\eta > 0$. Let $x \in E \sim Z$ be a point at which $|f(x) - g(x)| > \eta$, and let n be any index. Since $f(x)$ and $g(x)$ are finite, the triangle inequality gives

$$\eta < |f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|,$$

and hence at least one of the two summands exceeds $\eta/2$ (this remains true, trivially, if $f_n(x)$ is infinite). Therefore, for every index n ,

$$\{x \in E \mid |f(x) - g(x)| > \eta\} \subseteq Z \cup \{x \in E \mid |f_n(x) - f(x)| > \eta/2\} \cup \{x \in E \mid |f_n(x) - g(x)| > \eta/2\}.$$

By the monotonicity and countable subadditivity of measure, together with $m(Z) = 0$,

$$m\{x \in E \mid |f(x) - g(x)| > \eta\} \leq m\{x \in E \mid |f_n - f| > \eta/2\} + m\{x \in E \mid |f_n - g| > \eta/2\}.$$

The left-hand side does not depend on n , while by hypothesis both terms on the right tend to 0 as $n \rightarrow \infty$. Letting $n \rightarrow \infty$ we conclude

$$m\{x \in E \mid |f(x) - g(x)| > \eta\} = 0 \quad \text{for every } \eta > 0.$$

Finally, on $E \sim Z$ the functions f and g are finite, so

$$\{x \in E \sim Z \mid f(x) \neq g(x)\} = \bigcup_{k=1}^{\infty} \{x \in E \sim Z \mid |f(x) - g(x)| > 1/k\},$$

a countable union of sets of measure zero, which therefore has measure zero by the countable subadditivity of measure. Adjoining the null set Z , we conclude that $\{x \in E \mid f(x) \neq g(x)\}$ has measure zero, that is, $f = g$ a.e. on E .

Problem 5.7. Let E have finite measure, $\{f_n\} \rightarrow f$ in measure on E and g be a measurable function on E that is finite a.e. on E . Prove that $\{f_n \cdot g\} \rightarrow f \cdot g$ in measure, and use this to show that $\{f_n^2\} \rightarrow f^2$ in measure. Infer from this that if $\{g_n\} \rightarrow g$ in measure, then $\{f_n \cdot g_n\} \rightarrow f \cdot g$ in measure.

Solution. Throughout, recall that convergence in measure presupposes that the limit function is measurable and finite a.e. on E . All products below are defined a.e. on E and are measurable, and the exceptional null sets may be discarded without affecting any of the measures computed.

We first record an elementary lemma.

Lemma. If $\{u_n\} \rightarrow u$ and $\{v_n\} \rightarrow v$ in measure on E , and α, β are real numbers, then $\{\alpha u_n + \beta v_n\} \rightarrow \alpha u + \beta v$ in measure on E .

Proof of the Lemma. Set $M = |\alpha| + |\beta| + 1$, so $M > 0$ and $|\alpha| + |\beta| < M$. By the definition of convergence in measure, u, v and every u_n, v_n is measurable and finite a.e. on E ; let Z be the union of the countably many sets of measure zero on which one of these functions fails to be finite. By the countable subadditivity of measure $m(Z) = 0$, and off Z every function in sight is real-valued, so all the algebraic manipulations below are legitimate there and no integral or measure is affected by discarding Z . Let $\eta > 0$. For $x \in E \sim Z$,

$$|(\alpha u_n + \beta v_n)(x) - (\alpha u + \beta v)(x)| \leq |\alpha| |u_n(x) - u(x)| + |\beta| |v_n(x) - v(x)|,$$

so if the left side exceeds η then $|u_n(x) - u(x)| > \eta/(2M)$ or $|v_n(x) - v(x)| > \eta/(2M)$ (otherwise the right side would be at most $(|\alpha| + |\beta|)\eta/(2M) < \eta$). Hence

$$\{|\alpha u_n + \beta v_n - (\alpha u + \beta v)| > \eta\} \subseteq Z \cup \{|u_n - u| > \eta/(2M)\} \cup \{|v_n - v| > \eta/(2M)\},$$

and by the monotonicity and countable subadditivity of measure, together with $m(Z) = 0$, the measure of the left-hand set is at most the sum of the measures of the last two sets, each of which tends to 0 as $n \rightarrow \infty$. Since $\eta > 0$ was arbitrary, and since $\alpha u + \beta v$ is measurable and finite a.e. on E because u and v are, the Lemma is proved.

Part 1: $\{f_n \cdot g\} \rightarrow f \cdot g$ in measure on E .

Let $\eta > 0$ and $\epsilon > 0$. For each natural number k define

$$E_k = \{x \in E \mid |g(x)| > k\}.$$

These are measurable sets that descend, and $\bigcap_{k=1}^{\infty} E_k = \{x \in E \mid |g(x)| = \infty\}$, a set of measure zero since g is finite a.e. on E . Since $m(E_1) \leq m(E) < \infty$, Theorem 15 of Chapter 2 (the Continuity of Measure) gives $\lim_{k \rightarrow \infty} m(E_k) = 0$. (This is exactly where the finiteness of $m(E)$ is used.) Choose k with $m(E_k) < \epsilon$.

Let Z be the union of the countably many sets of measure zero on which one of $f, g, f_1, f_2, \dots$ fails to be finite; by the countable subadditivity of measure $m(Z) = 0$. Suppose $x \in E$ satisfies $|f_n(x)g(x) - f(x)g(x)| > \eta$ and $x \notin Z \cup E_k$. Then $f(x), g(x), f_n(x)$ are all real numbers and $|g(x)| \leq k$, so the factorization $f_n(x)g(x) - f(x)g(x) = g(x)[f_n(x) - f(x)]$ is legitimate and

$$\eta < |g(x)| |f_n(x) - f(x)| \leq k |f_n(x) - f(x)|,$$

so $|f_n(x) - f(x)| > \eta/k$. Therefore

$$\{x \in E \mid |f_n g - f g| > \eta\} \subseteq Z \cup E_k \cup \{x \in E \mid |f_n - f| > \eta/k\},$$

and hence, by the monotonicity and countable subadditivity of measure, together with $m(Z) = 0$ and $m(E_k) < \epsilon$,

$$m\{x \in E \mid |f_n g - f g| > \eta\} \leq \epsilon + m\{x \in E \mid |f_n - f| > \eta/k\}.$$

Since $\{f_n\} \rightarrow f$ in measure, letting $n \rightarrow \infty$ yields

$$\limsup_{n \rightarrow \infty} m\{x \in E \mid |f_n g - f g| > \eta\} \leq \epsilon.$$

As $\epsilon > 0$ was arbitrary, this limit superior is 0, that is,

$$\lim_{n \rightarrow \infty} m\{x \in E \mid |f_n(x)g(x) - f(x)g(x)| > \eta\} = 0.$$

Since $\eta > 0$ was arbitrary and $f \cdot g$ is measurable and finite a.e. on E , we conclude $\{f_n \cdot g\} \rightarrow f \cdot g$ in measure on E .

Part 2: $\{f_n^2\} \rightarrow f^2$ in measure on E .

Let Z again denote the null set off which f and all the f_n are finite; every set equality below is asserted on $E \sim Z$, and adjoining or deleting Z changes no measure.

First, $\{f_n - f\} \rightarrow 0$ in measure on E by the Lemma (take $u_n = f_n$, v_n the constant sequence f , $\alpha = 1$, $\beta = -1$; the constant sequence trivially converges to f in measure). Next, for $\eta > 0$ and $x \in E \sim Z$ the numbers $f_n(x)$ and $f(x)$ are real, so $(f_n(x) - f(x))^2 > \eta$ holds precisely when $|f_n(x) - f(x)| > \sqrt{\eta}$. Hence

$$\{x \in E \mid (f_n(x) - f(x))^2 > \eta\} \subseteq Z \cup \{x \in E \mid |f_n(x) - f(x)| > \sqrt{\eta}\},$$

and the measure of the right-hand side tends to 0 as $n \rightarrow \infty$. Since $(f_n - f)^2$ is measurable and finite a.e. on E , this says $\{(f_n - f)^2\} \rightarrow 0$ in measure on E .

Second, f itself is a measurable function on E that is finite a.e. on E , so Part 1 applied with $g = f$ gives $\{f_n \cdot f\} \rightarrow f^2$ in measure on E .

Now put $h_n = (f_n - f)^2 + [2ff_n - f^2]$. The sequence $\{(f_n - f)^2\}$ converges in measure to 0, and the sequence $\{2ff_n - f^2\}$ converges in measure to $2f^2 - f^2 = f^2$ (apply the Lemma with $u_n = ff_n \rightarrow f^2$, $\alpha = 2$, and the constant sequence $v_n = f^2 \rightarrow f^2$, $\beta = -1$). Hence the Lemma, applied once more with $\alpha = \beta = 1$, gives $\{h_n\} \rightarrow 0 + f^2 = f^2$ in measure on E .

Finally, at each point of $E \sim Z$ all the quantities involved are real numbers and the algebraic identity

$$(f_n - f)^2 + 2ff_n - f^2 = f_n^2 - 2ff_n + f^2 + 2ff_n - f^2 = f_n^2$$

holds, so $h_n = f_n^2$ on $E \sim Z$. Therefore, for each $\eta > 0$,

$$\{x \in E \mid |f_n^2 - f^2| > \eta\} \subseteq Z \cup \{x \in E \mid |h_n - f^2| > \eta\},$$

and since $m(Z) = 0$ the measure of the left-hand set tends to 0 as $n \rightarrow \infty$. As f^2 is measurable and finite a.e. on E , we conclude

$$\{f_n^2\} \rightarrow f^2 \quad \text{in measure on } E.$$

Part 3: if also $\{g_n\} \rightarrow g$ in measure on E , then $\{f_n \cdot g_n\} \rightarrow f \cdot g$ in measure on E .

Here g is measurable and finite a.e. on E , as required by the definition of convergence in measure. By the Lemma, $\{f_n + g_n\} \rightarrow f + g$ in measure on E . The argument of Part 2 used nothing about the sequence $\{f_n\}$ beyond the facts that $m(E) < \infty$ and that $\{f_n\}$ converges in measure on E to a function that is measurable and finite a.e.; so it applies verbatim to each of the three sequences $\{f_n + g_n\}$, $\{f_n\}$ and $\{g_n\}$, whose limits $f + g$, f and g are all measurable and finite a.e. on E . Thus

$$\{(f_n + g_n)^2\} \rightarrow (f + g)^2, \quad \{f_n^2\} \rightarrow f^2, \quad \{g_n^2\} \rightarrow g^2 \quad \text{in measure on } E.$$

Applying the Lemma twice to the linear combination

$$k_n = \frac{1}{2}(f_n + g_n)^2 - \frac{1}{2}f_n^2 - \frac{1}{2}g_n^2$$

gives

$$\{k_n\} \rightarrow \frac{1}{2} [(f+g)^2 - f^2 - g^2] \quad \text{in measure on } E.$$

Let W be the null set off which f , g and every f_n , g_n is finite. At each point of $E \sim W$ all these values are real numbers, so the identity $(a+b)^2 - a^2 - b^2 = 2ab$ gives $k_n = f_n g_n$ there, and likewise $\frac{1}{2}[(f+g)^2 - f^2 - g^2] = f \cdot g$ there. Exactly as in Part 2, for each $\eta > 0$,

$$\{x \in E \mid |f_n g_n - f g| > \eta\} \subseteq W \cup \{x \in E \mid |k_n - \frac{1}{2}[(f+g)^2 - f^2 - g^2]| > \eta\},$$

whose measure tends to 0. Since $f \cdot g$ is measurable and finite a.e. on E , we conclude that $\{f_n \cdot g_n\} \rightarrow f \cdot g$ in measure on E . This completes the proof.

5.2 Problems 5.8–5.14

Problem 5.8. Show that Fatou's Lemma, the Monotone Convergence Theorem, the Lebesgue Dominated Convergence Theorem, and the Vitali Convergence Theorem remain valid if "pointwise convergence a.e." is replaced by "convergence in measure."

Solution. Throughout, recall the standing convention that when we write $\{f_n\} \rightarrow f$ in measure on E we are assuming f and each f_n is measurable and finite a.e. on E . We isolate two elementary observations and then treat the four theorems in turn.

Observation A (subsequences inherit convergence in measure). If $\{f_n\} \rightarrow f$ in measure on E and $\{f_{n_k}\}$ is any subsequence, then $\{f_{n_k}\} \rightarrow f$ in measure on E . Indeed, given $\eta > 0$ and $\epsilon > 0$ choose N with $m\{x \in E \mid |f_n(x) - f(x)| > \eta\} < \epsilon$ for all $n \geq N$; since $n_k \geq k$, the same inequality holds for all $k \geq N$. Observation B (the subsequence criterion for numerical sequences). Let $\{a_n\}$ be a sequence in $\mathbb{R}$ and $a \in \mathbb{R}$. If every subsequence of $\{a_n\}$ possesses a further subsequence converging to a , then $a_n \rightarrow a$. Indeed, if $a_n \not\rightarrow a$ there are $\epsilon_0 > 0$ and indices $n_1 < n_2 < \dots$ with $|a_{n_k} - a| \geq \epsilon_0$ for all k ; no subsequence of $\{a_{n_k}\}$ can converge to a , a contradiction.

The engine driving everything is the Riesz Theorem (Theorem 4 of this chapter): if $\{f_n\} \rightarrow f$ in measure on E , then some subsequence of $\{f_n\}$ converges to f pointwise a.e. on E . Combining it with Observation A: **every** subsequence of $\{f_n\}$ has in turn a further subsequence converging to f pointwise a.e. on E . Call this property (*).

1. Fatou's Lemma. Claim: if $\{f_n\}$ is a sequence of nonnegative measurable functions on E and $\{f_n\} \rightarrow f$ in measure on E , then $f \geq 0$ a.e. on E (indeed, by the Riesz Theorem some subsequence converges to f pointwise a.e., and a pointwise limit of nonnegative numbers is nonnegative), so that $\int_E f$ is defined, and

$$\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n.$$

Set $L = \liminf_{n \rightarrow \infty} \int_E f_n \in [0, \infty]$. Choose a subsequence $\{f_{n_k}\}$ with $\lim_{k \rightarrow \infty} \int_E f_{n_k} = L$. By (*) there is a further subsequence $\{f_{n_{k_j}}\}$ converging to f pointwise a.e. on E . Fatou's Lemma in its original form (Chapter 4) applies to this further subsequence and gives

$$\int_E f \leq \liminf_{j \rightarrow \infty} \int_E f_{n_{k_j}} = \lim_{k \rightarrow \infty} \int_E f_{n_k} = L,$$

the middle equality because a subsequence of a convergent sequence of numbers has the same limit. This is the assertion.

1. The Monotone Convergence Theorem. Claim: if $\{f_n\}$ is an increasing sequence of nonnegative measurable functions on E and $\{f_n\} \rightarrow f$ in measure on E , then $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.

Since $f_n \leq f_{n+1}$ on E , the numerical sequence $\{\int_E f_n\}$ is increasing, so it converges in $[0, \infty]$; call the limit L . Part 1 already gives $\int_E f \leq L$. For the reverse inequality, fix an index p . By the Riesz Theorem select a subsequence $\{f_{n_k}\}$ converging to f pointwise a.e. on E , say on $E \sim E_0$ where E_0 is a measurable subset of E with $m(E_0) = 0$. For $x \in E \sim E_0$ and any k with $n_k \geq p$ we have $f_p(x) \leq f_{n_k}(x)$ by monotonicity, and letting $k \rightarrow \infty$ gives $f_p(x) \leq f(x)$. Hence $f_p \leq f$ a.e. on E , and by the monotonicity of integration for nonnegative measurable functions $\int_E f_p \leq \int_E f$. Letting $p \rightarrow \infty$ yields $L \leq \int_E f$. Therefore $L = \int_E f$.

1. The Lebesgue Dominated Convergence Theorem. Claim: if $\{f_n\}$ is a sequence of measurable functions on E , g is integrable over E with $|f_n| \leq g$ on E for all n , and $\{f_n\} \rightarrow f$ in measure on E , then f is integrable over E and $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.

First, f is integrable. By the Riesz Theorem some subsequence $\{f_{n_k}\}$ converges to f pointwise a.e. on E ; passing to the limit in $|f_{n_k}| \leq g$ gives $|f| \leq g$ a.e. on E , and since f is measurable and g is integrable, f is integrable over E by the Integral Comparison Test (Proposition 16 of Chapter 4). In particular each $\int_E f_n$ and $\int_E f$ is a real number.

Now let $\{\int_E f_{n_k}\}$ be an arbitrary subsequence of the numerical sequence $\{\int_E f_n\}$. By (*) there is a further subsequence $\{f_{n_{k_j}}\}$ converging to f pointwise a.e. on E ; it is dominated by g , so the Dominated Convergence Theorem in its original form gives $\int_E f_{n_{k_j}} \rightarrow \int_E f$. Thus every subsequence of $\{\int_E f_n\}$ has a further subsequence converging to $\int_E f$, and Observation B gives $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.

1. The Vitali Convergence Theorem. Claim: let $\{f_n\}$ be a sequence of measurable functions on E that is uniformly integrable and tight over E (if $m(E) < \infty$, tightness is automatic, since one may take $E_0 = E$ in its definition), and suppose $\{f_n\} \rightarrow f$ in measure on E . Then f is integrable over E and $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.

The argument is verbatim that of part 3, with domination replaced by uniform integrability and tightness. First note that each f_n is integrable over E : this is exactly what uniform integrability together with tightness gives, by the remark preceding the statement of the general Vitali Convergence Theorem in Section 5.1 (which appeals to Proposition 23 of Chapter 4). So $\{\int_E f_n\}$ is a sequence of real numbers. Let $\{\int_E f_{n_k}\}$ be any subsequence. By (*) choose a further subsequence $\{f_{n_{k_j}}\}$ converging to f pointwise a.e. on E . This further subsequence is itself uniformly integrable and tight over E , since both conditions are inherited by subsequences (the δ and the set of finite measure produced for the whole family serve for any subfamily). Hence the Vitali Convergence Theorem in its original form (the general version stated in Section 5.1 of this chapter; for $m(E) < \infty$ the version of Section 4.6 suffices) tells us that f is integrable over E and $\int_E f_{n_{k_j}} \rightarrow \int_E f$. The integrability of f is thereby established once and for all (from a single such subsequence), so $\int_E f$ is a real number; and Observation B applied to $\{\int_E f_n\}$ gives $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.

In each of the four cases the hypothesis of a.e. pointwise convergence was used only along a subsequence, which the Riesz Theorem supplies from convergence in measure; and in each case convergence in measure is a genuinely weaker hypothesis, since by Proposition 3 pointwise a.e. convergence on a set of finite measure implies convergence in measure while the example following Proposition 3 shows the converse fails.

Problem 5.9. Show that Proposition 3 does not necessarily hold for sets E of infinite measure.

Solution. Proposition 3 asserts: if $m(E) < \infty$, $\{f_n\}$ is a sequence of measurable functions on E converging pointwise a.e. on E to f , and f is finite a.e. on E , then $\{f_n\} \rightarrow f$ in measure on E . We exhibit a set of infinite measure on which the conclusion fails.

Let $E = \mathbb{R}$, which has infinite measure, and for each natural number n define

$$f_n = \chi_{[n, n+1)}.$$

Each f_n is measurable, being the characteristic function of a measurable set, and each is real-valued. Let $f \equiv 0$ on $\mathbb{R}$; then f is measurable and finite everywhere.

The sequence converges pointwise on all of $\mathbb{R}$ to f . Indeed, fix $x \in \mathbb{R}$ and choose a natural number $N > x$. For every $n \geq N$ we have $n > x$, so $x \notin [n, n+1)$ and hence $f_n(x) = 0$. Thus $f_n(x) \rightarrow 0 = f(x)$. So the hypotheses of Proposition 3 hold except for the finiteness of $m(E)$.

The sequence does not converge to f in measure. Take $\eta = 1/2$. For each n ,

$$\{x \in \mathbb{R} \mid |f_n(x) - f(x)| > 1/2\} = \{x \in \mathbb{R} \mid \chi_{[n, n+1)}(x) > 1/2\} = [n, n+1),$$

a set of measure 1. Hence for $\epsilon = 1/2$ there is no index N for which

$$m\{x \in \mathbb{R} \mid |f_n(x) - f(x)| > 1/2\} < \epsilon \quad \text{for all } n \geq N,$$

since the left-hand side equals 1 for every n . Therefore $\{f_n\}$ fails to converge to f in measure on $\mathbb{R}$, and Proposition 3 is false for sets of infinite measure.

The mechanism is exactly the one Egoroff's Theorem, on which the proof of Proposition 3 rests, cannot control on a set of infinite measure: the "bad set" on which f_n is far from its limit does not shrink, it merely escapes to infinity. One can of course produce the same failure on a set of infinite measure that is not all of $\mathbb{R}$, for instance on $E = [0, \infty)$ with the same functions.

It is worth noting that a related but distinct phenomenon occurs even on sets of finite measure with the roles reversed: as the Example following Proposition 3 shows, on $E = [0, 1]$ there is a sequence converging in measure to $f \equiv 0$ that converges pointwise at no point of $[0, 1]$. Thus on a set of finite measure the implication of Proposition 3 is strict, while on a set of infinite measure it is not even valid.

Problem 5.10. Show that linear combinations of sequences that converge in measure on a set of finite measure also converge in measure.

Solution. Let E be measurable with $m(E) < \infty$. We are to show: if $\{f_n\} \rightarrow f$ in measure on E and $\{g_n\} \rightarrow g$ in measure on E , and α, β are real numbers, then

$$\{\alpha f_n + \beta g_n\} \rightarrow \alpha f + \beta g \quad \text{in measure on } E.$$

By induction this gives the assertion for a linear combination of any finite number of sequences.

First a word on well-definedness. By our standing convention f, g and each f_n, g_n are measurable and finite a.e. on E . Let

$$E_0 = \{x \in E \mid \text{at least one of } f(x), g(x), f_1(x), g_1(x), f_2(x), g_2(x), \dots \text{ is infinite}\}.$$

This is a countable union of sets of measure zero, so $m(E_0) = 0$. On $E \sim E_0$ every one of these functions is real-valued, so the linear combinations $\alpha f_n + \beta g_n$ and $\alpha f + \beta g$ are defined there, are measurable (linear combinations of measurable real-valued functions are measurable, Theorem 6 of Chapter 3), and are finite. Since sets of measure zero contribute nothing to the measures occurring in the definition of convergence in measure, we may and do argue on $E \sim E_0$; equivalently, we adopt the (measure-irrelevant) convention that all functions are redefined to be 0 on E_0 .

If $\alpha = \beta = 0$ the assertion is trivial, both sides being identically 0. So assume $|\alpha| + |\beta| > 0$.

Fix $\eta > 0$ and $\epsilon > 0$. The heart of the matter is the following pointwise inclusion. For $x \in E \sim E_0$,

$$|(\alpha f_n(x) + \beta g_n(x)) - (\alpha f(x) + \beta g(x))| \leq |\alpha| |f_n(x) - f(x)| + |\beta| |g_n(x) - g(x)|,$$

by the triangle inequality. Consequently, if the left-hand side exceeds η , then at least one of the two summands on the right exceeds $\eta/2$: were both at most $\eta/2$, the sum would be at most η . Therefore, writing

$$A_n = \left\{x \in E \sim E_0 \mid |\alpha| |f_n(x) - f(x)| > \eta/2\right\}, \quad B_n = \left\{x \in E \sim E_0 \mid |\beta| |g_n(x) - g(x)| > \eta/2\right\},$$

we have the inclusion

$$\{x \in E \sim E_0 \mid |(\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x)| > \eta\} \subseteq A_n \cup B_n.$$

We now estimate $m(A_n)$ and $m(B_n)$ separately. If $\alpha = 0$ then $A_n = \emptyset$ and $m(A_n) = 0$ for all n . If $\alpha \neq 0$, then

$$A_n = \left\{x \in E \sim E_0 \mid |f_n(x) - f(x)| > \frac{\eta}{2|\alpha|}\right\},$$

and since $\eta/(2|\alpha|) > 0$ and $\{f_n\} \rightarrow f$ in measure on E , there is an index N_1 such that $m(A_n) < \epsilon/2$ for all $n \geq N_1$. In either case there is an index N_1 with $m(A_n) < \epsilon/2$ for $n \geq N_1$. Symmetrically there is an index N_2 with $m(B_n) < \epsilon/2$ for all $n \geq N_2$.

Set $N = \max\{N_1, N_2\}$. For $n \geq N$, by the monotonicity and finite subadditivity of Lebesgue measure,

$$m\{x \in E \mid |(\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x)| > \eta\} \leq m(E_0) + m(A_n) + m(B_n) < 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\eta > 0$ and $\epsilon > 0$ were arbitrary, $\{\alpha f_n + \beta g_n\} \rightarrow \alpha f + \beta g$ in measure on E , as claimed.

Two remarks. First, the finiteness of $m(E)$ was nowhere used: the argument above is valid on an arbitrary measurable set E , finite measure or not. Finiteness of measure is needed for Proposition 3 (pointwise a.e. convergence implies convergence in measure), not for the algebra of limits in measure. Second, the same splitting argument shows that the set of sequences converging in measure on E is a real vector space and that the map sending a sequence to its limit in measure is linear; the limit is unique up to sets of measure zero, since if $\{h_n\} \rightarrow h$ and $\{h_n\} \rightarrow \tilde{h}$ in measure then by the case just proved $\{0\} = \{h_n - h_n\} \rightarrow h - \tilde{h}$ in measure, forcing $m\{|h - \tilde{h}| > \eta\} = 0$ for every $\eta > 0$ and hence $h = \tilde{h}$ a.e. on E .

Problem 5.11. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E . Prove that $\{f_n\} \rightarrow f$ in measure on E if and only if every subsequence of $\{f_n\}$ has in turn a further subsequence that converges to f pointwise a.e. on E .

Solution. Throughout, $m(E) < \infty$, and f together with every f_n is measurable and finite a.e. on E , so that all the sets appearing below are measurable and all the measures involved are finite.

Suppose first that $\{f_n\} \rightarrow f$ in measure on E , and let $\{f_{n_k}\}$ be an arbitrary subsequence. We claim $\{f_{n_k}\} \rightarrow f$ in measure on E . Indeed, let $\eta > 0$ and $\epsilon > 0$. Choose N so that $m\{x \in E \mid |f_n(x) - f(x)| > \eta\} < \epsilon$ for all $n \geq N$. Since $\{n_k\}$ is strictly increasing, $n_k \geq k$; hence for all $k \geq N$ we have $n_k \geq N$ and therefore

$$m\{x \in E \mid |f_{n_k}(x) - f(x)| > \eta\} < \epsilon.$$

So the subsequence converges to f in measure. By the Riesz Theorem (Theorem 4), a sequence converging in measure has a subsequence converging pointwise a.e. to the same limit; applied to $\{f_{n_k}\}$, it produces a further subsequence $\{f_{n_{k_j}}\}$ converging to f pointwise a.e. on E . This proves the forward implication. (Note that finiteness of $m(E)$ was not needed here.)

Conversely, suppose every subsequence of $\{f_n\}$ has in turn a further subsequence converging to f pointwise a.e. on E . We argue by contradiction: assume $\{f_n\}$ does not converge to f in measure on E .

Negating the definition of convergence in measure, there exist $\eta_0 > 0$ and $\epsilon_0 > 0$ with the property that for every index N there is some $n \geq N$ for which

$$m\{x \in E \mid |f_n(x) - f(x)| > \eta_0\} \geq \epsilon_0.$$

Use this repeatedly to construct indices: take n_1 to be an index (with $n_1 \geq 1$) for which the displayed inequality holds; having chosen $n_1 < n_2 < \dots < n_k$, apply the property with $N = n_k + 1$ to obtain $n_{k+1} > n_k$ for which the inequality holds. This yields a strictly increasing sequence of natural numbers $\{n_k\}$ with

$$m\{x \in E \mid |f_{n_k}(x) - f(x)| > \eta_0\} \geq \epsilon_0 \quad \text{for every } k. \quad (\dagger)$$

By hypothesis, the subsequence $\{f_{n_k}\}$ has in turn a further subsequence $\{f_{n_{k_j}}\}$ that converges to f pointwise a.e. on E . Now E has finite measure, the functions $f_{n_{k_j}}$ are measurable, and f is finite a.e. on E ; so Proposition 3 applies to the sequence $\{f_{n_{k_j}}\}_{j=1}^\infty$ and yields

$$\{f_{n_{k_j}}\} \rightarrow f \quad \text{in measure on } E.$$

In particular, taking $\eta = \eta_0$ and $\epsilon = \epsilon_0$ in the definition, there is an index J such that

$$m\{x \in E \mid |f_{n_{k_j}}(x) - f(x)| > \eta_0\} < \epsilon_0 \quad \text{for all } j \geq J.$$

But n_{k_j} is one of the indices n_k , so $(\dagger)$ gives $m\{x \in E \mid |f_{n_{k_j}}(x) - f(x)| > \eta_0\} \geq \epsilon_0$. This contradiction shows that our assumption was false, and hence $\{f_n\} \rightarrow f$ in measure on E .

Two comments. The finiteness of $m(E)$ is genuinely needed for the converse direction, and only there: it is the hypothesis of Proposition 3, which is the sole step converting a.e. pointwise convergence

back into convergence in measure. On a set of infinite measure the criterion fails, as the sequence $f_n = \chi_{[n, n+1)}$ on $E = \mathbb{R}$ shows: it converges to $f \equiv 0$ pointwise everywhere on $\mathbb{R}$, so certainly every subsequence has a further subsequence converging pointwise a.e. to f , yet as computed in Problem 9 it does not converge to f in measure.

Finally, observe that this criterion is precisely the mechanism used in Problem 8: it is the exact analogue, for convergence in measure, of the elementary fact that a numerical sequence converges to a provided every subsequence has a further subsequence converging to a . It expresses the fact that convergence in measure on a set of finite measure is the convergence associated with a topology, indeed with the metric ρ of Problem 14, in which a.e. pointwise convergence plays the role of a “subsequential” notion.

Problem 5.12. Show that a sequence $\{a_j\}$ of real numbers converges to a real number if $|a_{j+1} - a_j| \leq 1/2^j$ for all j by showing that the sequence $\{a_j\}$ must be Cauchy.

Solution. Let $\{a_j\}_{j=1}^\infty$ be a sequence of real numbers satisfying

$$|a_{j+1} - a_j| \leq \frac{1}{2^j} \quad \text{for all } j \geq 1.$$

The key estimate is obtained by telescoping. Let $n > m \geq 1$. Writing

$$a_n - a_m = \sum_{j=m}^{n-1} (a_{j+1} - a_j)$$

and applying the triangle inequality together with the hypothesis,

$$|a_n - a_m| \leq \sum_{j=m}^{n-1} |a_{j+1} - a_j| \leq \sum_{j=m}^{n-1} \frac{1}{2^j}.$$

The last sum is a finite geometric sum, which we evaluate exactly:

$$\sum_{j=m}^{n-1} \frac{1}{2^j} = \frac{1}{2^m} \cdot \frac{1 - (1/2)^{n-m}}{1 - 1/2} = \frac{1}{2^{m-1}} \left(1 - \frac{1}{2^{n-m}}\right) < \frac{1}{2^{m-1}}.$$

Hence

$$|a_n - a_m| < \frac{1}{2^{m-1}} \quad \text{whenever } n > m \geq 1. \quad (*)$$

Now we verify the Cauchy condition. Let $\epsilon > 0$. Since $2^k \geq k + 1 > k$ for every natural number k (an immediate induction), we have $1/2^k \rightarrow 0$, so we may choose a natural number N with

$$\frac{1}{2^{N-1}} < \epsilon.$$

Let $m, n \geq N$ be arbitrary. If $m = n$ then $|a_n - a_m| = 0 < \epsilon$. If $m \neq n$, relabel so that $n > m$; then $m \geq N$, and by $(*)$,

$$|a_n - a_m| < \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \epsilon,$$

using that $k \mapsto 1/2^{k-1}$ is decreasing. In every case $|a_n - a_m| < \epsilon$ for all $m, n \geq N$. Therefore $\{a_j\}$ is a Cauchy sequence of real numbers.

By the completeness of $\mathbb{R}$, that is, by the Cauchy Convergence Criterion (Theorem 17 of Chapter 1), every Cauchy sequence of real numbers converges to a real number. Hence $\{a_j\}$ converges to some $a \in \mathbb{R}$.

It is worth recording the quantitative form of the conclusion, which is what makes this lemma useful in the proof of Problem 13. Fixing m and letting $n \rightarrow \infty$ in $(*)$, the continuity of the absolute value gives

$$|a - a_m| \leq \frac{1}{2^{m-1}} \quad \text{for every } m \geq 1,$$

so the limit is approached at a definite geometric rate. Equivalently, the hypothesis says that the telescoping series $a_1 + \sum_{j \geq 1} (a_{j+1} - a_j)$ is absolutely convergent, since $\sum_{j=1}^{\infty} |a_{j+1} - a_j| \leq \sum_{j=1}^{\infty} 2^{-j} = 1 < \infty$, and its partial sums are exactly the a_n .

Problem 5.13. A sequence $\{f_n\}$ of measurable functions on E is said to be Cauchy in measure provided given $\eta > 0$ and $\epsilon > 0$ there is an index N such that for all $m, n \geq N$,

$$m\{x \in E \mid |f_n(x) - f_m(x)| \geq \eta\} < \epsilon.$$

Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f on E to which the sequence $\{f_n\}$ converges in measure. (Hint: Choose a strictly increasing sequence of natural numbers $\{n_j\}$ such that for each index j , if $E_j = \{x \in E \mid |f_{n_{j+1}}(x) - f_{n_j}(x)| > 1/2^j\}$, then $m(E_j) < 1/2^j$. Now use the Borel-Cantelli Lemma and the preceding problem.)

Solution. Assume $\{f_n\}$ is Cauchy in measure on E ; as always, each f_n is measurable and finite a.e. on E . Let Z be the union of the countably many sets $\{x \in E \mid |f_n(x)| = \infty\}$; then $m(Z) = 0$ and every f_n is real-valued on $E \sim Z$.

Step 1: extraction of a rapidly Cauchy subsequence. For each natural number j apply the Cauchy-in-measure hypothesis with $\eta = 1/2^j$ and $\epsilon = 1/2^j$ to obtain an index N_j such that

$$m\{x \in E \mid |f_n(x) - f_m(x)| \geq 1/2^j\} < 1/2^j \quad \text{for all } m, n \geq N_j.$$

Define $n_1 = N_1$ and, inductively, $n_{j+1} = \max\{N_{j+1}, n_j + 1\}$. Then $\{n_j\}$ is a strictly increasing sequence of natural numbers and $n_{j+1} > n_j \geq N_j$ for every j . Consequently, setting

$$E_j = \{x \in E \mid |f_{n_{j+1}}(x) - f_{n_j}(x)| > 1/2^j\},$$

which is a measurable set contained in $\{x \in E \mid |f_{n_{j+1}}(x) - f_{n_j}(x)| \geq 1/2^j\}$, we have

$$m(E_j) < \frac{1}{2^j} \quad \text{for every } j, \quad \text{so} \quad \sum_{j=1}^{\infty} m(E_j) < \sum_{j=1}^{\infty} \frac{1}{2^j} = 1 < \infty.$$

Step 2: the subsequence converges pointwise a.e. Each E_j is measurable, and so is $\limsup_j E_j = \bigcap_{k \geq 1} \bigcup_{j \geq k} E_j$, the set of points lying in infinitely many E_j . By the Borel-Cantelli Lemma (Chapter 2), since $\sum_j m(E_j) < \infty$, we have $m(\limsup_j E_j) = 0$. Put

$$E' = (E \sim Z) \sim \limsup_{j \rightarrow \infty} E_j,$$

a measurable subset of E with $m(E \sim E') \leq m(Z) + m(\limsup_j E_j) = 0$. Fix $x \in E'$. There is an index $J = J(x)$ with $x \notin E_j$ for all $j \geq J$, that is,

$$|f_{n_{j+1}}(x) - f_{n_j}(x)| \leq \frac{1}{2^j} \quad \text{for all } j \geq J.$$

Consider the real sequence $b_i = f_{n_{J+i-1}}(x)$, $i \geq 1$. For each i ,

$$|b_{i+1} - b_i| = |f_{n_{J+i}}(x) - f_{n_{J+i-1}}(x)| \leq \frac{1}{2^{J+i-1}} \leq \frac{1}{2^i},$$

since $J \geq 1$. By the preceding problem (Problem 12), $\{b_i\}$ is Cauchy and hence converges to a real number. Prepending the finitely many terms $f_{n_1}(x), \dots, f_{n_{J-1}}(x)$ does not affect convergence, so $\lim_{j \rightarrow \infty} f_{n_j}(x)$ exists and is real for every $x \in E'$.

Define f on E by $f(x) = \lim_{j \rightarrow \infty} f_{n_j}(x)$ for $x \in E'$ and $f(x) = 0$ for $x \in E \sim E'$. Then f is real-valued on all of E . It is also measurable, and no appeal to completeness of the measure is needed: set $g_j = f_{n_j}$ on E' and $g_j = 0$ on $E \sim E'$. Since E' is measurable, the restriction of g_j to E' is measurable (it agrees there with the measurable function f_{n_j}) and its restriction to $E \sim E'$ is constant, hence measurable; so

g_j is measurable on E by part (ii) of Proposition 5 of Chapter 3. By construction $\{g_j\} \rightarrow f$ pointwise on all of E : on E' by the previous paragraph, and on $E \sim E'$ because every g_j and f vanish there. Therefore f is measurable on E by Proposition 9 of Chapter 3.

Step 3: the subsequence converges to f in measure. For each index k put

$$F_k = \bigcup_{j \geq k} E_j, \quad \text{so} \quad m(F_k) \leq \sum_{j \geq k} m(E_j) < \sum_{j \geq k} \frac{1}{2^j} = \frac{1}{2^{k-1}}$$

by countable subadditivity. Let $x \in E' \sim F_k$. Then $|f_{n_{j+1}}(x) - f_{n_j}(x)| \leq 1/2^j$ for every $j \geq k$, so for $i > k$ the telescoping estimate of Problem 12 gives

$$|f_{n_i}(x) - f_{n_k}(x)| \leq \sum_{j=k}^{i-1} \frac{1}{2^j} < \frac{1}{2^{k-1}},$$

and letting $i \rightarrow \infty$ yields

$$|f(x) - f_{n_k}(x)| \leq \frac{1}{2^{k-1}} \quad \text{for all } x \in E' \sim F_k.$$

Hence

$$\left\{ x \in E \mid |f_{n_k}(x) - f(x)| > \frac{1}{2^{k-1}} \right\} \subseteq F_k \cup (E \sim E'),$$

a set of measure less than $1/2^{k-1}$. Now let $\eta > 0$ and $\epsilon > 0$. Choose K so large that $1/2^{K-1} < \min\{\eta, \epsilon\}$. For $k \geq K$ we have $1/2^{k-1} \leq 1/2^{K-1} < \eta$, so

$$m\{x \in E \mid |f_{n_k}(x) - f(x)| > \eta\} \leq m\left\{x \in E \mid |f_{n_k}(x) - f(x)| > \frac{1}{2^{k-1}}\right\} < \frac{1}{2^{k-1}} \leq \frac{1}{2^{K-1}} < \epsilon.$$

Thus $\{f_{n_k}\} \rightarrow f$ in measure on E .

Step 4: the full sequence converges to f in measure. Let $\eta > 0$ and $\epsilon > 0$. Since $\{f_n\}$ is Cauchy in measure, choose N with

$$m\{x \in E \mid |f_n(x) - f_p(x)| \geq \eta/2\} < \epsilon/2 \quad \text{for all } n, p \geq N.$$

By Step 3 choose an index k so large that $n_k \geq N$ and

$$m\{x \in E \mid |f_{n_k}(x) - f(x)| > \eta/2\} < \epsilon/2.$$

Fix such a k . For any $n \geq N$ and any $x \in E \sim Z$ (where all the values in question are real), the triangle inequality shows that if $|f_n(x) - f(x)| > \eta$ then at least one of $|f_n(x) - f_{n_k}(x)| \geq \eta/2$ and $|f_{n_k}(x) - f(x)| > \eta/2$ must hold: if both failed, the sum of the two moduli would be less than η , contradicting $|f_n(x) - f(x)| \leq |f_n(x) - f_{n_k}(x)| + |f_{n_k}(x) - f(x)|$. That is,

$$\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq \{x \in E \mid |f_n(x) - f_{n_k}(x)| \geq \eta/2\} \cup \{x \in E \mid |f_{n_k}(x) - f(x)| > \eta/2\} \cup Z,$$

the last set being of measure zero and therefore contributing nothing to the estimate below. Since $n \geq N$ and $n_k \geq N$, monotonicity and finite subadditivity give

$$m\{x \in E \mid |f_n(x) - f(x)| > \eta\} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \geq N.$$

As $\eta > 0$ and $\epsilon > 0$ were arbitrary, $\{f_n\} \rightarrow f$ in measure on E . This exhibits the required measurable limit function and shows that the measurable functions on E are complete with respect to convergence in measure.

Note also that Step 2 delivers, at no extra cost, the Riesz-type conclusion that the subsequence $\{f_{n_j}\}$ converges to f pointwise almost everywhere on E ; and no assumption whatever was made about $m(E)$.

Problem 5.14. Assume $m(E) < \infty$. For two measurable functions g and h on E , define

$$\rho(g, h) = \int_E \frac{|g - h|}{1 + |g - h|}.$$

Show that $\{f_n\} \rightarrow f$ in measure on E if and only if $\lim_{n \rightarrow \infty} \rho(f_n, f) = 0$.

Solution. Write $\varphi(t) = t/(1+t)$ for $t \in [0, \infty)$. Two elementary properties of φ carry the whole proof: (i) φ is increasing on $[0, \infty)$, since $\varphi'(t) = 1/(1+t)^2 > 0$; equivalently, for $0 \leq s \leq t$ one has $s(1+t) \leq t(1+s)$, whence $\varphi(s) \leq \varphi(t)$.

(ii) $0 \leq \varphi(t) < 1$ for all $t \geq 0$, since $t < 1+t$.

First, ρ is well defined and finite. By our standing convention f and each f_n is measurable and finite a.e. on E . Let Z be the union of the countably many sets $\{x \in E \mid |f(x)| = \infty\}$ and $\{x \in E \mid |f_n(x)| = \infty\}$; then Z is measurable and $m(Z) = 0$, and on $E \sim Z$ every one of these functions is real-valued, so $|f_n - f|$ is a nonnegative real-valued measurable function there. Hence $\varphi \circ |f_n - f|$ is measurable on $E \sim Z$, being the composition of a continuous function with a measurable real-valued function (Proposition 7 of Chapter 3); extend it by the value 1 on Z , which alters no integral since $m(Z) = 0$ and leaves the extension measurable by Proposition 5 of Chapter 3. By (ii) the integrand is bounded above by 1 on E , and since $m(E) < \infty$ the monotonicity of integration gives

$$0 \leq \rho(f_n, f) = \int_E \varphi(|f_n - f|) \leq \int_E 1 = m(E) < \infty.$$

So $\{\rho(f_n, f)\}$ is a sequence of nonnegative real numbers.

Suppose first that $\{f_n\} \rightarrow f$ in measure on E ; we show $\rho(f_n, f) \rightarrow 0$. If $m(E) = 0$ then $\rho(f_n, f) = 0$ for all n and there is nothing to prove, so assume $m(E) > 0$. Let $\epsilon > 0$. Choose

$$\eta = \frac{\epsilon}{2m(E)} > 0,$$

and set $A_n = \{x \in E \mid |f_n(x) - f(x)| > \eta\}$, a measurable set. Split the integral over A_n and $E \sim A_n$ (Theorem 11 of Chapter 4, additivity over domains of integration for nonnegative measurable functions). On A_n we use only the crude bound (ii), $\varphi \leq 1$; on $E \sim A_n$ we have $|f_n - f| \leq \eta$ off the null set Z , so by (i), $\varphi(|f_n - f|) \leq \varphi(\eta) = \eta/(1+\eta) \leq \eta$ a.e. there. Therefore

$$\rho(f_n, f) = \int_{A_n} \varphi(|f_n - f|) + \int_{E \sim A_n} \varphi(|f_n - f|) \leq m(A_n) + \eta \cdot m(E \sim A_n) \leq m(A_n) + \eta \cdot m(E) = m(A_n) + \frac{\epsilon}{2}.$$

Since $\{f_n\} \rightarrow f$ in measure and $\eta > 0$, there is an index N such that $m(A_n) < \epsilon/2$ for all $n \geq N$. Hence $\rho(f_n, f) < \epsilon$ for all $n \geq N$, and $\lim_{n \rightarrow \infty} \rho(f_n, f) = 0$.

Conversely, suppose $\lim_{n \rightarrow \infty} \rho(f_n, f) = 0$; we show $\{f_n\} \rightarrow f$ in measure on E . Let $\eta > 0$ and $\epsilon > 0$ be given, and again write $A_n = \{x \in E \mid |f_n(x) - f(x)| > \eta\}$. For a.e. $x \in A_n$ we have $|f_n(x) - f(x)| > \eta$, so by the monotonicity (i) of φ ,

$$\varphi(|f_n(x) - f(x)|) \geq \varphi(\eta) = \frac{\eta}{1+\eta}.$$

Since the integrand is nonnegative on all of E , monotonicity of integration gives the Chebychev-type estimate

$$\rho(f_n, f) = \int_E \varphi(|f_n - f|) \geq \int_{A_n} \varphi(|f_n - f|) \geq \frac{\eta}{1+\eta} \cdot m(A_n),$$

that is,

$$m(A_n) \leq \frac{1+\eta}{\eta} \rho(f_n, f).$$

Because $\rho(f_n, f) \rightarrow 0$ and $(1+\eta)/\eta$ is a fixed constant, there is an index N such that $\frac{1+\eta}{\eta} \rho(f_n, f) < \epsilon$, and hence $m(A_n) < \epsilon$, for all $n \geq N$. As $\eta > 0$ and $\epsilon > 0$ were arbitrary, $\{f_n\} \rightarrow f$ in measure on E .

This completes the proof of the equivalence. Observe that finiteness of $m(E)$ was used only in the first implication, where the uniform smallness of $\varphi(|f_n - f|)$ off A_n had to be integrated over E ; the second implication is valid on any measurable set. The first implication genuinely fails when $m(E) = \infty$: on $E = \mathbb{R}$ take $f_n = (1/n)\chi_{[0,n]}$ and $f \equiv 0$. For fixed $\eta > 0$ the set $\{x \in \mathbb{R} \mid |f_n(x)| > \eta\}$ is empty as soon as $n > 1/\eta$, so $\{f_n\} \rightarrow 0$ in measure on $\mathbb{R}$; yet the integrand $\varphi(|f_n|)$ equals the constant $(1/n)/(1+1/n)$ on $[0, n]$ and vanishes off it, so

$$\rho(f_n, 0) = n \cdot \frac{1/n}{1+1/n} = \frac{1}{1+1/n} \rightarrow 1 \neq 0.$$

Finally, ρ is in fact a pseudometric on the measurable functions on E , and a metric once functions equal a.e. are identified. Symmetry and nonnegativity are clear, and $\rho(g, h) = 0$ forces $\varphi(|g - h|) = 0$ a.e. (Proposition 9 of Chapter 4), hence $g = h$ a.e. For the triangle inequality one uses that φ is increasing and subadditive: for $s, t \geq 0$,

$$\varphi(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = \varphi(s) + \varphi(t),$$

so from $|g - k| \leq |g - h| + |h - k|$ we get $\varphi(|g - k|) \leq \varphi(|g - h|) + \varphi(|h - k|)$ pointwise, and integrating gives $\rho(g, k) \leq \rho(g, h) + \rho(h, k)$. Thus the exercise identifies convergence in measure on a set of finite measure as convergence in a metric, which is why the subsequence criterion of Problem 11 holds for it and why Problem 13 is the assertion that this metric space is complete.

5.3 Problems 5.15–5.17

Problem 5.15. Let f and g be bounded functions that are Riemann integrable over $[a, b]$. Show that the product fg also is Riemann integrable over $[a, b]$.

Solution. Since f and g are bounded, so is fg : if $|f| \leq M_1$ and $|g| \leq M_2$ on $[a, b]$, then $|fg| \leq M_1 M_2$. Put $M = \max\{M_1, M_2\}$, so that $|f| \leq M$ and $|g| \leq M$ on $[a, b]$.

A first proof, via Theorem 8. Let D_f , D_g and D_{fg} denote the sets of points of $[a, b]$ at which f , g and fg , respectively, fail to be continuous. If $x_0 \in [a, b]$ is a point at which both f and g are continuous, then fg is continuous at x_0 : indeed, for $x \in [a, b]$,

$$|f(x)g(x) - f(x_0)g(x_0)| \leq |f(x)| |g(x) - g(x_0)| + |g(x_0)| |f(x) - f(x_0)| \leq M(|g(x) - g(x_0)| + |f(x) - f(x_0)|),$$

and the right-hand side is less than ϵ once $|x - x_0| < \delta$, where $\delta > 0$ is chosen so small that $|f(x) - f(x_0)| < \epsilon/(2M + 1)$ and $|g(x) - g(x_0)| < \epsilon/(2M + 1)$ for such x . Contrapositively,

$$D_{fg} \subseteq D_f \cup D_g.$$

Now f and g are bounded and Riemann integrable over $[a, b]$, so by Theorem 8 (Lebesgue) of this chapter, $m(D_f) = 0$ and $m(D_g) = 0$. By the countable (here, finite) subadditivity of outer measure, $m^*(D_f \cup D_g) = 0$, and hence $m^*(D_{fg}) = 0$; a set of outer measure zero is measurable with measure zero. Thus fg is a bounded function on $[a, b]$ whose set of discontinuities has measure zero, and therefore, by the converse half of Theorem 8, fg is Riemann integrable over $[a, b]$.

A second proof, directly from the Darboux sums. For a nonempty set $S \subseteq [a, b]$ and a bounded function h on $[a, b]$, write

$$\text{osc}(h, S) = \sup_{x \in S} h(x) - \inf_{x \in S} h(x) = \sup_{x, y \in S} |h(x) - h(y)|.$$

For $x, y \in S$,

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)| |g(x) - g(y)| + |g(y)| |f(x) - f(y)| \leq M \text{osc}(g, S) + M \text{osc}(f, S),$$

and taking the supremum over $x, y \in S$ gives

$$\text{osc}(fg, S) \leq M[\text{osc}(f, S) + \text{osc}(g, S)].$$

Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ and $I_i = [x_{i-1}, x_i]$. Since $U(h, P) - L(h, P) = \sum_{i=1}^n \text{osc}(h, I_i)(x_i - x_{i-1})$, the displayed inequality gives

$$U(fg, P) - L(fg, P) \leq M[U(f, P) - L(f, P)] + M[U(g, P) - L(g, P)].$$

Let $\epsilon > 0$ and put $\eta = \epsilon/(2M + 1)$. Because f is Riemann integrable, its upper and lower Riemann integrals have the common value $\int_a^b f$; since the upper Riemann integral is the infimum of the upper Darboux sums and the lower Riemann integral is the supremum of the lower Darboux sums, there are partitions Q and Q' of $[a, b]$ with

$$U(f, Q) < \int_a^b f + \frac{\eta}{2}, \quad L(f, Q') > \int_a^b f - \frac{\eta}{2}.$$

Let $P_1 = Q \cup Q'$, a refinement of both. Under refinement upper Darboux sums decrease and lower Darboux sums increase (Problem 2 of Chapter 4), so $U(f, P_1) \leq U(f, Q)$ and $L(f, P_1) \geq L(f, Q')$, and hence $U(f, P_1) - L(f, P_1) < \eta$. Similarly there is a partition P_2 with $U(g, P_2) - L(g, P_2) < \eta$. Let P be the common refinement $P_1 \cup P_2$. Invoking monotonicity under refinement once more,

$$U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_1) < \eta, \quad U(g, P) - L(g, P) \leq U(g, P_2) - L(g, P_2) < \eta,$$

and therefore $U(fg, P) - L(fg, P) < 2M\eta = 2M\epsilon/(2M + 1) < \epsilon$. Since

$$0 \leq \overline{\int_a^b fg} - \underline{\int_a^b fg} \leq U(fg, P) - L(fg, P) < \epsilon$$

for every $\epsilon > 0$, the upper and lower Riemann integrals of fg coincide, that is, fg is Riemann integrable over $[a, b]$.

Problem 5.16. Let f be a bounded function on $[a, b]$ whose set of discontinuities has measure zero. Show that f is measurable. Then show that the same holds without the assumption of boundedness.

Solution. Throughout, D denotes the set of points of $[a, b]$ at which f fails to be continuous, and we assume $m(D) = 0$.

The bounded case. Suppose f is bounded on $[a, b]$. By Theorem 8 (Lebesgue) of this chapter, a bounded function on $[a, b]$ is Riemann integrable over $[a, b]$ if and only if its set of discontinuities has measure zero; since $m(D) = 0$, the function f is Riemann integrable over $[a, b]$. By Theorem 3 of Chapter 4, a bounded function that is Riemann integrable over the closed, bounded interval $[a, b]$ is then Lebesgue integrable over $[a, b]$ and the two integrals are equal. Finally, by Theorem 7 of this chapter, a bounded function on a set of finite measure is Lebesgue integrable over that set if and only if it is measurable; since $[a, b]$ has finite measure and f is bounded and Lebesgue integrable over $[a, b]$, the function f is measurable.

It is worth recording a direct argument for the bounded case, since it uses no integration theory at all and will not need boundedness either. Let $E = [a, b] \setminus D$ be the set of points at which f is continuous. Since $m(D) = 0$, the set D is measurable, and hence so is E , being the relative complement of a measurable set in a measurable set. The restriction $f|_E$ is continuous as a function on the metric space E : given $x_0 \in E$ and $\epsilon > 0$, continuity of f at x_0 provides $\delta > 0$ with $|f(x) - f(x_0)| < \epsilon$ for all $x \in [a, b]$ with $|x - x_0| < \delta$, and in particular for all such x lying in E . Therefore, for each $c \in \mathbb{R}$, the set

$$\{x \in E : f(x) > c\}$$

is relatively open in E , so it has the form $E \cap U$ with $U \subseteq \mathbb{R}$ open, and hence is measurable, being the intersection of two measurable sets. On the other hand

$$\{x \in D : f(x) > c\} \subseteq D,$$

and $m(D) = 0$; since Lebesgue measure is complete, every subset of a set of measure zero is measurable

(of measure zero). Consequently

$$\{x \in [a, b] : f(x) > c\} = \{x \in E : f(x) > c\} \cup \{x \in D : f(x) > c\}$$

is measurable for every $c \in \mathbb{R}$, that is, f is measurable. Note that nowhere was boundedness used. The unbounded case, by truncation. Suppose now f is an arbitrary real-valued function on $[a, b]$ whose set D of discontinuities has measure zero. For a natural number n define the truncation

$$f_n(x) = \max\{-n, \min\{n, f(x)\}\} = \begin{cases} n & \text{if } f(x) > n, \\ f(x) & \text{if } -n \leq f(x) \leq n, \\ -n & \text{if } f(x) < -n. \end{cases}$$

Then $f_n = \varphi_n \circ f$, where $\varphi_n(t) = \max\{-n, \min\{n, t\}\}$ is continuous on $\mathbb{R}$ (it is 1-Lipschitz, since $|\varphi_n(s) - \varphi_n(t)| \leq |s - t|$). If f is continuous at x_0 , then so is $\varphi_n \circ f$, since $|f_n(x) - f_n(x_0)| = |\varphi_n(f(x)) - \varphi_n(f(x_0))| \leq |f(x) - f(x_0)|$; hence f_n is continuous at every point of $E = [a, b] \setminus D$, so the set of discontinuities of f_n is contained in D and therefore has measure zero. Moreover $|f_n| \leq n$, so f_n is bounded. By the bounded case already proved, each f_n is measurable on $[a, b]$.

Finally, $\{f_n\} \rightarrow f$ pointwise on $[a, b]$: given $x \in [a, b]$, the number $f(x)$ is real, so for all $n \geq |f(x)|$ we have $f_n(x) = f(x)$. A pointwise limit of a sequence of measurable functions is measurable (Proposition 9 of Chapter 3). Therefore f is measurable.

Alternatively, and more briefly, the direct argument given above already proves the unbounded case verbatim: it shows that for every $c \in \mathbb{R}$ the set $\{x \in [a, b] : f(x) > c\}$ splits as a relatively open subset of the continuity set E together with a subset of the null set D , both of which are measurable, and boundedness of f played no role.

Problem 5.17. Let f be a function on $[0, 1]$ that is continuous on $(0, 1]$. Show that it is possible for the sequence $\left\{\int_{[1/n, 1]} f\right\}$ to converge and yet f is not Lebesgue integrable over $[0, 1]$. Can this happen if f is nonnegative?

Solution. An example. Define f on $[0, 1]$ by

$$f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is continuous on $(0, 1]$, being built from continuous functions by composition, product and quotient with nonvanishing denominator. It is measurable on $[0, 1]$. Indeed, fix $c \in \mathbb{R}$. Since f is continuous on $(0, 1]$, the set $\{x \in (0, 1] : f(x) > c\}$ is relatively open in $(0, 1]$, so it equals $(0, 1] \cap U$ for some open $U \subseteq \mathbb{R}$ and is therefore measurable; and

$$\{x \in [0, 1] : f(x) > c\} = (\{x \in (0, 1] : f(x) > c\}) \cup S, \quad \text{where } S = \{0\} \text{ if } f(0) > c \text{ and } S = \emptyset \text{ otherwise,}$$

a union of two measurable sets. Hence f is measurable on $[0, 1]$. The same remark shows that any function on $[0, 1]$ that is continuous on $(0, 1]$ is measurable, and we use it again below.

Fix $n \geq 2$. On the closed, bounded interval $[1/n, 1]$ the function f is continuous, hence bounded and Riemann integrable (Problem 6 of Chapter 4), so by Theorem 3 of Chapter 4 its Lebesgue integral over $[1/n, 1]$ equals its Riemann integral there. The change of variable $u = 1/x$, $x = 1/u$, $dx = -du/u^2$ is a C^1 diffeomorphism of $[1/n, 1]$ onto $[1, n]$, and it gives

$$\int_{[1/n, 1]} f = \int_{1/n}^1 \frac{1}{x} \sin \frac{1}{x} dx = \int_n^1 u \sin u \cdot \left(-\frac{1}{u^2}\right) du = \int_1^n \frac{\sin u}{u} du.$$

This sequence of real numbers converges, and we verify it by the Cauchy criterion, using nothing beyond the calculus of continuous functions on closed, bounded intervals. Let $2 \leq n < m$. Integration

by parts on $[n, m]$, where $u \mapsto \sin u/u$ and $u \mapsto \cos u/u$ are continuously differentiable, gives

$$\int_n^m \frac{\sin u}{u} du = \left[\frac{-\cos u}{u} \right]_n^m - \int_n^m \frac{\cos u}{u^2} du = \frac{\cos n}{n} - \frac{\cos m}{m} - \int_n^m \frac{\cos u}{u^2} du.$$

Since $|\cos u| \leq 1$,

$$\left| \int_n^m \frac{\sin u}{u} du \right| \leq \frac{1}{n} + \frac{1}{m} + \int_n^m \frac{du}{u^2} = \frac{1}{n} + \frac{1}{m} + \frac{1}{n} - \frac{1}{m} = \frac{2}{n}.$$

Writing $a_n = \int_{[1/n, 1]} f = \int_1^n (\sin u)/u du$, this says $|a_m - a_n| \leq 2/n$ for all $m > n \geq 2$, so $\{a_n\}$ is a Cauchy sequence of real numbers and therefore converges. Thus $\{\int_{[1/n, 1]} f\}$ converges.

Nevertheless f is not Lebesgue integrable over $[0, 1]$, because $\int_{[0, 1]} |f| = \infty$. The same change of variable applied to $|f|$ gives, for each $n \geq 2$,

$$\int_{[1/n, 1]} |f| = \int_1^n \frac{|\sin u|}{u} du.$$

For each natural number k ,

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u} du \geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin u| du = \frac{2}{(k+1)\pi}.$$

Given $n \geq 4$, let K be the largest natural number with $(K+1)\pi \leq n$; then $K \geq 1$ and $K \rightarrow \infty$ as $n \rightarrow \infty$. The intervals $[k\pi, (k+1)\pi]$ for $1 \leq k \leq K$ are pairwise nonoverlapping and all contained in $[1, n]$, since $\pi > 1$. As the integrand is nonnegative,

$$\int_1^n \frac{|\sin u|}{u} du \geq \sum_{k=1}^K \int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u} du \geq \frac{2}{\pi} \sum_{k=1}^K \frac{1}{k+1},$$

and the harmonic series diverges, so the right-hand side tends to ∞ with K . Hence $\int_1^n |\sin u| u^{-1} du \rightarrow \infty$ as $n \rightarrow \infty$. Now $\{|f| \cdot \chi_{[1/n, 1]}\}$ is an increasing sequence of nonnegative measurable functions on $[0, 1]$ converging pointwise on $[0, 1]$ to $|f| \cdot \chi_{(0, 1]}$, which differs from $|f|$ only at the single point $x = 0$ and so equals $|f|$ almost everywhere on $[0, 1]$. Two nonnegative measurable functions that agree almost everywhere have the same integral, so by the Monotone Convergence Theorem,

$$\int_{[0, 1]} |f| = \int_{[0, 1]} |f| \cdot \chi_{(0, 1]} = \lim_{n \rightarrow \infty} \int_{[0, 1]} |f| \cdot \chi_{[1/n, 1]} = \lim_{n \rightarrow \infty} \int_{[1/n, 1]} |f| = \infty.$$

A measurable function f is Lebesgue integrable over $[0, 1]$ precisely when $\int_{[0, 1]} |f| < \infty$; hence f is not Lebesgue integrable over $[0, 1]$, although $\{\int_{[1/n, 1]} f\}$ converges. This answers the first part.

The nonnegative case. No, this cannot happen if $f \geq 0$. Suppose f is a nonnegative real-valued function on $[0, 1]$ that is continuous on $(0, 1]$ and for which the sequence $\{\int_{[1/n, 1]} f\}$ converges, say to the finite number L . By the remark recorded at the start of this solution, f is measurable on $[0, 1]$, and each $\int_{[1/n, 1]} f$ is a well-defined element of $[0, \infty]$; it is in fact finite, since f is continuous, hence bounded, on the closed, bounded interval $[1/n, 1]$.

Since $f \geq 0$, the sequence of functions $f_n = f \cdot \chi_{[1/n, 1]}$ is a nondecreasing sequence of nonnegative measurable functions on $[0, 1]$, and it converges pointwise on $[0, 1]$ to $f \cdot \chi_{(0, 1]}$, a function that agrees with f except at $x = 0$, hence agrees with f almost everywhere on $[0, 1]$. By the Monotone Convergence Theorem,

$$\int_{[0, 1]} f = \int_{[0, 1]} f \cdot \chi_{(0, 1]} = \lim_{n \rightarrow \infty} \int_{[1/n, 1]} f = L < \infty.$$

Thus f is a nonnegative measurable function on $[0, 1]$ with finite integral, and since $|f| = f$, this says exactly that f is Lebesgue integrable over $[0, 1]$. So for nonnegative f the convergence of $\{\int_{[1/n, 1]} f\}$ forces Lebesgue integrability over $[0, 1]$; the phenomenon exhibited above depends essentially on the cancellation between the positive and negative parts of f .

6 Differentiation and Integration

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6.1 Problems 6.1–6.7

Problem 6.1. Let C be a countable subset of the nondegenerate closed, bounded interval $[a, b]$. Show that there is an increasing function on $[a, b]$ that is continuous only at points in $[a, b] \sim C$.

Solution. This is Proposition 2 with the open interval (a, b) replaced by the closed interval $[a, b]$. The only new feature is the left endpoint: at $x_0 = a$ continuity of a function on $[a, b]$ means continuity from the right, so a jump placed at a must be a jump to the right. We therefore use two kinds of jump functions.

If C is empty, take $f(x) = x$, which is increasing and continuous at every point of $[a, b]$. So assume $C \neq \emptyset$ and let $\{q_n\}_{n \in \Lambda}$ be an enumeration of C , where $\Lambda = \{1, 2, \dots, N\}$ if C is finite and $\Lambda = \{1, 2, 3, \dots\}$ if C is countably infinite. For each $n \in \Lambda$ define a function φ_n on $[a, b]$ by

$$\varphi_n(x) = \begin{cases} 1 & \text{if } x \geq q_n \\ 0 & \text{if } x < q_n \end{cases} \quad \text{in case } q_n > a,$$

$$\varphi_n(x) = \begin{cases} 1 & \text{if } x > a \\ 0 & \text{if } x = a \end{cases} \quad \text{in case } q_n = a.$$

In either case φ_n takes only the values 0 and 1, is increasing on $[a, b]$, and is continuous at every point of $[a, b]$ except at q_n . Define

$$f(x) = \sum_{n \in \Lambda} \frac{1}{2^n} \varphi_n(x) \quad \text{for all } x \in [a, b].$$

Since $0 \leq \varphi_n \leq 1$ and $\sum_n 2^{-n} \leq 1$, the series converges for every x , so f is a properly defined real-valued function on $[a, b]$. Moreover f is increasing: if $a \leq u < v \leq b$, then $\varphi_n(u) \leq \varphi_n(v)$ for every n , and hence, all the terms being nonnegative,

$$f(v) - f(u) = \sum_{n \in \Lambda} \frac{1}{2^n} [\varphi_n(v) - \varphi_n(u)] \geq 0. \quad (1)$$

f fails to be continuous at each point of C . Fix $k \in \Lambda$. Suppose first that $q_k > a$. If $x \in [a, b]$ and $x < q_k$, then $\varphi_k(q_k) - \varphi_k(x) = 1$, so by (1), applied with $u = x$ and $v = q_k$ and by discarding all terms but the k -th,

$$f(q_k) - f(x) \geq \frac{1}{2^k} \quad \text{for all } x \in [a, b] \text{ with } x < q_k.$$

Since $q_k > a$, there are points x of $[a, b]$ with $x < q_k$ arbitrarily close to q_k , and therefore f is not continuous at q_k . Now suppose $q_k = a$. If $x \in (a, b]$, then $\varphi_k(x) - \varphi_k(a) = 1$ and the same reasoning gives $f(x) - f(a) \geq 1/2^k$ for all $x \in (a, b]$; since $a < b$ there are such x arbitrarily close to a , and again f is not continuous at $q_k = a$.

f is continuous at each point of $[a, b] \sim C$. Let $x_0 \in [a, b] \sim C$ and let $\epsilon > 0$. Choose a natural number M with $1/2^M < \epsilon$ and put $\Lambda_M = \{n \in \Lambda : n \leq M\}$, a finite set. In both of the above cases the single point at which φ_n fails to be continuous is q_n , and $x_0 \neq q_n$ for every $n \in \Lambda$ because $x_0 \notin C$. Define

$$\delta = \min\{|x_0 - q_n| : n \in \Lambda_M\} > 0$$

(with $\delta = b - a$ if Λ_M is empty). Let $x \in [a, b]$ with $|x - x_0| < \delta$, and fix $n \in \Lambda_M$. The interval $(x_0 - \delta, x_0 + \delta)$ does not contain q_n . Each φ_n is constant on $(-\infty, q_n) \cap [a, b]$ and constant on $[q_n, \infty) \cap [a, b]$ (respectively on $\{a\}$ and on $(a, b]$ in the second case), and an interval that omits q_n lies entirely in one of these two pieces; hence $\varphi_n(x) = \varphi_n(x_0)$. Consequently all the terms with $n \leq M$ cancel and

$$|f(x) - f(x_0)| = \left| \sum_{n \in \Lambda, n > M} \frac{1}{2^n} [\varphi_n(x) - \varphi_n(x_0)] \right| \leq \sum_{n > M} \frac{1}{2^n} = \frac{1}{2^M} < \epsilon.$$

Therefore f is continuous at x_0 .

Thus f is an increasing function on $[a, b]$ whose set of points of continuity is exactly $[a, b] \sim C$.

Problem 6.2. Show that there is a strictly increasing function on $[0, 1]$ that is continuous only at the irrational numbers in $[0, 1]$.

Solution. Let $C = \mathbb{Q} \cap [0, 1]$, the set of rational numbers in $[0, 1]$. This is a countable subset of the nondegenerate closed, bounded interval $[0, 1]$, so by the preceding problem there is an increasing function g on $[0, 1]$ whose set of points of continuity is exactly $[0, 1] \setminus C$, that is, exactly the set of irrational numbers in $[0, 1]$. Define

$$f(x) = g(x) + x \quad \text{for all } x \in [0, 1].$$

f is strictly increasing. If $0 \leq u < v \leq 1$, then $g(u) \leq g(v)$ because g is increasing, and $u < v$, so

$$f(v) - f(u) = [g(v) - g(u)] + (v - u) \geq v - u > 0.$$

f has the same points of continuity as g . The function $x \mapsto x$ is continuous on $[0, 1]$. Hence if g is continuous at x_0 , so is the sum $f = g + \text{id}$; and conversely, if f is continuous at x_0 , then so is $g = f - \text{id}$, being the difference of two functions continuous at x_0 . Therefore f is continuous at x_0 if and only if g is, and so the set of points of continuity of f is exactly the set of irrational numbers in $[0, 1]$.

Thus f is a strictly increasing function on $[0, 1]$ that is continuous at every irrational point of $[0, 1]$ and at no rational point of $[0, 1]$. (Note that the endpoints 0 and 1 are rational and hence are points of discontinuity; this is why in the preceding problem the jump associated with the left endpoint had to be a jump to the right, so that the one-sided continuity of f at 0 also fails.)

Problem 6.3. Let f be a monotone function on a subset E of $\mathbb{R}$. Show that f is continuous except possibly at a countable number of points in E .

Solution. Here continuity is continuity of f as a function on E : f is continuous at $x_0 \in E$ provided for each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $x \in E$ with $|x - x_0| < \delta$.

Replacing f by $-f$ changes neither monotonicity nor the set of points of continuity, so we may assume f is increasing on E . Let D denote the set of points of E at which f fails to be continuous. We must show D is countable.

For $x_0 \in E$ define

$$L(x_0) = \begin{cases} \sup\{f(x) : x \in E, x < x_0\} & \text{if } x_0 \text{ is a limit point of } E \cap (-\infty, x_0) \\ f(x_0) & \text{otherwise,} \end{cases}$$

$$R(x_0) = \begin{cases} \inf\{f(x) : x \in E, x > x_0\} & \text{if } x_0 \text{ is a limit point of } E \cap (x_0, \infty) \\ f(x_0) & \text{otherwise.} \end{cases}$$

In the first alternative the set over which the supremum is taken is nonempty and bounded above by $f(x_0)$, since f is increasing; so $L(x_0)$ is a real number and $L(x_0) \leq f(x_0)$. Similarly $R(x_0)$ is a real number and $f(x_0) \leq R(x_0)$. Thus in all cases

$$L(x_0) \leq f(x_0) \leq R(x_0).$$

Claim: if $L(x_0) = f(x_0) = R(x_0)$, then f is continuous at x_0 . Let $\epsilon > 0$. We produce $\delta_1 > 0$ controlling the left side. If x_0 is not a limit point of $E \cap (-\infty, x_0)$, choose $\delta_1 > 0$ with $E \cap (x_0 - \delta_1, x_0) = \emptyset$; then there is nothing to check on the left. Otherwise $\sup\{f(x) : x \in E, x < x_0\} = f(x_0)$, so there is a $z \in E$ with $z < x_0$ and $f(z) > f(x_0) - \epsilon$; put $\delta_1 = x_0 - z$. If $x \in E$ and $x_0 - \delta_1 < x < x_0$, then $z < x < x_0$ and, f being increasing, $f(x_0) - \epsilon < f(z) \leq f(x) \leq f(x_0)$. Symmetrically there is $\delta_2 > 0$ such that $f(x_0) \leq f(x) < f(x_0) + \epsilon$ whenever $x \in E$ and $x_0 < x < x_0 + \delta_2$. With $\delta = \min\{\delta_1, \delta_2\}$ we have $|f(x) - f(x_0)| < \epsilon$ for all $x \in E$ with $|x - x_0| < \delta$. This proves the claim.

Consequently, for each $x_0 \in D$ we have $L(x_0) < f(x_0)$ or $f(x_0) < R(x_0)$; in either case $L(x_0) < R(x_0)$, so

$$J(x_0) = (L(x_0), R(x_0))$$

is a nonempty open interval of real numbers.

Claim: the intervals $\{J(x_0)\}_{x_0 \in D}$ are disjoint. Let $x_0, x_1 \in D$ with $x_0 < x_1$; it suffices to show $R(x_0) \leq L(x_1)$. Suppose first that there is a point $z \in E$ with $x_0 < z < x_1$. If x_0 is a limit point of $E \cap (x_0, \infty)$, then $R(x_0)$ is an infimum over a set containing $f(z)$, so $R(x_0) \leq f(z)$; if not, $R(x_0) = f(x_0) \leq f(z)$ since f is increasing. Either way $R(x_0) \leq f(z)$. Symmetrically $f(z) \leq L(x_1)$. Hence $R(x_0) \leq L(x_1)$. Suppose next that no point of E lies strictly between x_0 and x_1 . Then x_0 is not a limit point of $E \cap (x_0, \infty)$ and x_1 is not a limit point of $E \cap (-\infty, x_1)$, so $R(x_0) = f(x_0) \leq f(x_1) = L(x_1)$. In both cases $R(x_0) \leq L(x_1)$, so the open intervals $J(x_0)$ and $J(x_1)$ are disjoint. This proves the claim. Finally, each nonempty open interval $J(x_0)$ contains a rational number; choose one, say $r(x_0)$. Since the intervals are disjoint, the map $x_0 \mapsto r(x_0)$ from D to $\mathbb{Q}$ is one-to-one. Therefore D is equipotent to a subset of the countable set $\mathbb{Q}$ and hence is countable.

Thus a monotone function on a subset E of $\mathbb{R}$ is continuous at every point of E except possibly at the points of a countable subset of E . Taking $E = (a, b)$ recovers Theorem 1.

Problem 6.4. Let E be a subset of $\mathbb{R}$ and C a countable subset of E . Is there a monotone function on E that is continuous only at points in $E \sim C$?

Solution. In general, no. The obstruction is the isolated points of E , and in fact there is such a function if and only if C contains no point that is isolated in E .

A counterexample. Let $E = \{0\} \cup [1, 2]$ and $C = \{0\}$, a countable subset of E . The point 0 is isolated in E : taking $\delta = 1$, the only point x of E with $|x - 0| < \delta$ is $x = 0$. Hence for any function g whatsoever on E and any $\epsilon > 0$, the choice $\delta = 1$ gives $|g(x) - g(0)| = 0 < \epsilon$ for all $x \in E$ with $|x - 0| < \delta$; that is, every function on E is continuous at 0. So no function on E , monotone or not, can have 0 among its points of discontinuity, and there is no monotone function on E that is continuous only at points of $E \sim C = [1, 2]$. The same argument works with $E = \mathbb{Z}$ and C any nonempty subset of $\mathbb{Z}$, since every point of $\mathbb{Z}$ is isolated in $\mathbb{Z}$.

The general criterion. Call $x_0 \in E$ isolated in E provided there is a $\delta > 0$ with $E \cap (x_0 - \delta, x_0 + \delta) = \{x_0\}$. As just observed, if some point of C is isolated in E , no function on E is discontinuous there, and the answer is no. We claim that if no point of C is isolated in E , then such an increasing function exists. Assume no point of C is isolated in E . If $C = \emptyset$, take $f(x) = x$, which is increasing and continuous at every point of E . Otherwise let $\{q_n\}_{n \in \Lambda}$ be an enumeration of C , with Λ an initial segment of the natural numbers or all of them. Fix $n \in \Lambda$. Since q_n is not isolated in E , every interval $(q_n - \delta, q_n + \delta)$ meets E in a point other than q_n , so q_n is a limit point of $E \cap (-\infty, q_n)$ or a limit point of $E \cap (q_n, \infty)$. Define, for $x \in E$,

$$\varphi_n(x) = \begin{cases} 1 & \text{if } x \geq q_n \\ 0 & \text{if } x < q_n \end{cases}$$

in the first case, and

$$\varphi_n(x) = \begin{cases} 1 & \text{if } x > q_n \\ 0 & \text{if } x \leq q_n \end{cases}$$

in the second case. In either case φ_n is increasing on E , takes only the values 0 and 1, and is constant on every interval that omits q_n . Define

$$f(x) = \sum_{n \in \Lambda} \frac{1}{2^n} \varphi_n(x) \quad \text{for all } x \in E,$$

a properly defined real-valued function since the series is dominated by the convergent geometric series $\sum_n 2^{-n}$. Since each φ_n is increasing and all terms are nonnegative, f is increasing on E , and for $u, v \in E$ with $u < v$,

$$f(v) - f(u) \geq \frac{1}{2^k} [\varphi_k(v) - \varphi_k(u)] \quad \text{for each } k \in \Lambda. \quad (2)$$

f is discontinuous at each $q_k \in C$. In the first case above, $\varphi_k(q_k) - \varphi_k(x) = 1$ for every $x \in E$ with $x < q_k$, so by (2), $f(q_k) - f(x) \geq 1/2^k$ for all such x ; and since q_k is a limit point of $E \cap (-\infty, q_k)$ there

are such x arbitrarily close to q_k . In the second case, $\varphi_k(x) - \varphi_k(q_k) = 1$ for every $x \in E$ with $x > q_k$, so $f(x) - f(q_k) \geq 1/2^k$ for all such x , and there are such x arbitrarily close to q_k . In both cases the definition of continuity fails at q_k for $\epsilon = 1/2^k$.

f is continuous at each $x_0 \in E \sim C$. Given $\epsilon > 0$, choose a natural number M with $1/2^M < \epsilon$, and let $\delta = \min\{|x_0 - q_n| : n \in \Lambda, n \leq M\} > 0$, this minimum being over a finite set of strictly positive numbers because $x_0 \notin C$ (take $\delta = 1$ if the set is empty). If $x \in E$ and $|x - x_0| < \delta$, then for each $n \leq M$ the interval $(x_0 - \delta, x_0 + \delta)$ omits q_n and hence φ_n is constant on it, so $\varphi_n(x) = \varphi_n(x_0)$. Therefore

$$|f(x) - f(x_0)| \leq \sum_{n>M} \frac{1}{2^n} = \frac{1}{2^M} < \epsilon.$$

So the answer to the question as posed is negative in general: such a monotone function exists precisely when C contains no point that is isolated in E . In particular, when E is an interval, or more generally when E has no isolated points, the answer is always yes, which is consistent with Proposition 2 and with Problem 1.

Problem 6.5. Show that the Vitali Covering Lemma does not extend to the case in which the covering collection has degenerate closed intervals.

Solution. A closed, bounded interval $[c, d]$ is degenerate provided $c = d$, that is, provided it is a single point and has length zero. Suppose we drop the nondegeneracy requirement from the definition of a Vitali covering, so that a collection $\mathcal{F}$ of closed, bounded intervals is said to cover E in the sense of Vitali provided for each $x \in E$ and each $\epsilon > 0$ there is an $I \in \mathcal{F}$ that contains x and has $\ell(I) < \epsilon$. We exhibit a set E of finite outer measure and such a collection $\mathcal{F}$ for which the conclusion (2) of the Vitali Covering Lemma fails.

Let $E = [0, 1]$, so that $m^*(E) = 1 < \infty$, and let

$$\mathcal{F} = \{ [x, x] : x \in [0, 1] \},$$

the collection of all degenerate closed intervals with endpoint in $[0, 1]$.

$\mathcal{F}$ covers E in the sense of Vitali in the relaxed sense just described: given $x \in E$ and $\epsilon > 0$, the interval $I = [x, x]$ belongs to $\mathcal{F}$, contains x , and has $\ell(I) = 0 < \epsilon$.

Now let $\{I_k\}_{k=1}^n$ be any finite subcollection of $\mathcal{F}$, disjoint or not. Then $A = \bigcup_{k=1}^n I_k$ is a set of at most n points, hence is measurable with $m(A) = 0$. Since $E = [0, 1]$ is measurable of finite measure and $E \sim A$ is measurable, the additivity of measure over the disjoint union $E = (E \sim A) \cup (E \cap A)$ gives

$$m^*\left[E \sim \bigcup_{k=1}^n I_k\right] = m(E) - m(E \cap A) = 1 - 0 = 1.$$

Hence for any $\epsilon \leq 1$ there is no finite subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left[E \sim \bigcup_{k=1}^n I_k\right] < \epsilon.$$

Indeed, the same computation shows that even a countable disjoint subcollection of $\mathcal{F}$ leaves a remainder of outer measure 1, since a countable union of singletons has measure zero. So the Vitali Covering Lemma fails completely for such coverings, and the assumption in the definition preceding the lemma that the intervals of $\mathcal{F}$ be nondegenerate cannot be dropped.

The reason the proof of the lemma breaks down is visible in inequality (6): the selection process there chooses at each stage an interval whose length is more than half the supremum s_n of the available lengths, and it concludes from $\ell(I_k) \rightarrow 0$ that every point left uncovered lies in one of the enlarged intervals $5 * I_k$. When all the intervals are degenerate, every $5 * I_k$ is again a single point and covers nothing new; the lengths carry no information.

Problem 6.6. Show that the Vitali Covering Lemma does extend to the case in which the covering collection consists of nondegenerate general intervals.

Solution. By a general interval we mean any interval of real numbers: open, closed, or half-open, bounded or unbounded. Such an interval I is nondegenerate provided it contains more than one point. We prove the following.

Let E be a set of finite outer measure and $\mathcal{F}$ a collection of nondegenerate general intervals such that for each $x \in E$ and each $\epsilon > 0$ there is an $I \in \mathcal{F}$ containing x with $\ell(I) < \epsilon$. Then for each $\epsilon > 0$ there is a finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left[E \sim \bigcup_{k=1}^n I_k\right] < \epsilon.$$

Proof. Let $\mathcal{F}_0 = \{I \in \mathcal{F} : \ell(I) < \infty\}$ and let

$$\overline{\mathcal{F}} = \{\overline{I} : I \in \mathcal{F}_0\},$$

where $\overline{I}$ denotes the closure of I . If I is a nondegenerate interval of finite length with endpoints $c < d$, then $\overline{I} = [c, d]$ is a closed, bounded, nondegenerate interval and $\ell(\overline{I}) = d - c = \ell(I)$. Thus $\overline{\mathcal{F}}$ is a collection of closed, bounded, nondegenerate intervals.

$\overline{\mathcal{F}}$ covers E in the sense of Vitali. Indeed, let $x \in E$ and $\epsilon > 0$. By hypothesis there is an $I \in \mathcal{F}$ with $x \in I$ and $\ell(I) < \epsilon$; in particular $\ell(I) < \infty$, so $I \in \mathcal{F}_0$. Then $x \in I \subseteq \overline{I} \in \overline{\mathcal{F}}$ and $\ell(\overline{I}) = \ell(I) < \epsilon$.

Since $m^*(E) < \infty$, the Vitali Covering Lemma applies to E and $\overline{\mathcal{F}}$: given $\epsilon > 0$, there is a finite disjoint subcollection of $\overline{\mathcal{F}}$, say $\{\overline{I}_k\}_{k=1}^n$ with each $I_k \in \mathcal{F}_0 \subseteq \mathcal{F}$, for which

$$m^*\left[E \sim \bigcup_{k=1}^n \overline{I}_k\right] < \frac{\epsilon}{2}.$$

The subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ is disjoint: if $j \neq k$, then $I_j \cap I_k \subseteq \overline{I}_j \cap \overline{I}_k = \emptyset$. Moreover, since $\overline{I}_k \sim I_k$ consists of at most the two endpoints of I_k , the set

$$\Delta = \bigcup_{k=1}^n (\overline{I}_k \sim I_k)$$

is finite and therefore $m^*(\Delta) = 0$. Now

$$E \sim \bigcup_{k=1}^n I_k \subseteq \left[E \sim \bigcup_{k=1}^n \overline{I}_k\right] \cup \Delta,$$

since a point of E that lies in no I_k either lies in no $\overline{I}_k$ or lies in some $\overline{I}_k \sim I_k$. By the monotonicity and countable (here finite) subadditivity of outer measure,

$$m^*\left[E \sim \bigcup_{k=1}^n I_k\right] \leq m^*\left[E \sim \bigcup_{k=1}^n \overline{I}_k\right] + m^*(\Delta) < \frac{\epsilon}{2} + 0 < \epsilon.$$

This is the assertion, and the proof is complete.

Two remarks. First, the intervals of $\mathcal{F}$ of infinite length were discarded at the outset; this costs nothing, because the Vitali covering condition only ever calls on intervals of length less than a prescribed ϵ . Second, nondegeneracy is essential and is inherited by the closures, so the previous problem's counterexample is not in conflict with what has just been proved.

Problem 6.7. Let f be continuous on $\mathbb{R}$. Is there an open interval on which f is monotone?

Solution. No. There is a continuous function on $\mathbb{R}$ that fails to be monotone on every open interval. This is in the same spirit as the pathology mentioned in the Remark of this section, that a continuous function on an open interval may fail to be differentiable at every point.

The example. Let $\varphi(t) = \text{dist}(t, \mathbb{Z})$ be the distance from t to the nearest integer, and define

$$T(x) = \sum_{m=0}^{\infty} \frac{1}{2^m} \varphi(2^m x) \quad \text{for all } x \in \mathbb{R}.$$

T is continuous on $\mathbb{R}$. Each function $x \mapsto 2^{-m}\varphi(2^m x)$ is continuous, since φ is continuous (it is 1-periodic and satisfies $|\varphi(s) - \varphi(t)| \leq |s - t|$), and $0 \leq 2^{-m}\varphi(2^m x) \leq 2^{-m-1}$ for all x . Since $\sum_m 2^{-m-1} < \infty$, the series converges uniformly on $\mathbb{R}$ by the Weierstrass M-test, and a uniform limit of continuous functions is continuous. So T is continuous on all of $\mathbb{R}$.

Self-similarity on dyadic intervals. For integers $n \geq 0$ and k , let $I_{n,k} = [k/2^n, (k+1)/2^n]$. Fix such an interval and let $x \in I_{n,k}$.

For $m < n$, the point $2^m x$ ranges over $[k \cdot 2^{m-n}, (k+1) \cdot 2^{m-n}]$, an interval whose endpoints are consecutive multiples of 2^{m-n} and whose length 2^{m-n} is at most $1/2$; since 2^{m-n} divides $1/2$, this interval is contained in a single interval of the form $[j/2, (j+1)/2]$, on which φ is affine with slope $+1$ or -1 . Hence $x \mapsto 2^{-m}\varphi(2^m x)$ is affine on $I_{n,k}$ with slope $+1$ or -1 .

For $m \geq n$, write $u = 2^n x - k \in [0, 1]$. Then $2^m x = 2^{m-n}u + 2^{m-n}k$, and $2^{m-n}k$ is an integer, so by the 1-periodicity of φ we get $\varphi(2^m x) = \varphi(2^{m-n}u)$. Therefore

$$\sum_{m \geq n} \frac{1}{2^m} \varphi(2^m x) = \frac{1}{2^n} \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j u) = \frac{1}{2^n} T(2^n x - k).$$

Combining the two computations, there are a real number $c_{n,k}$ and an integer $s_{n,k}$ with

$$T(x) = c_{n,k} + s_{n,k} \cdot \left(x - \frac{k}{2^n}\right) + \frac{1}{2^n} T(2^n x - k) \quad \text{for all } x \in I_{n,k}, \quad (3)$$

where $s_{n,k}$ is the sum of the n slopes ± 1 contributed by the terms with $m < n$.

The slopes can be steered to zero. On the left half $I_{n+1,2k}$ of $I_{n,k}$ the point $2^n x$ ranges over $[k, k+1/2]$, where φ has slope $+1$; on the right half $I_{n+1,2k+1}$ it ranges over $[k+1/2, k+1]$, where φ has slope -1 . Since $s_{n+1,\cdot}$ is $s_{n,k}$ plus the slope contributed by the term with $m = n$,

$$s_{n+1,2k} = s_{n,k} + 1, \quad s_{n+1,2k+1} = s_{n,k} - 1.$$

Consequently, starting from any dyadic interval $I_{n,k}$, we may pass $|s_{n,k}|$ times to a half-interval, always taking the right half if $s_{n,k} > 0$ and the left half if $s_{n,k} < 0$, and arrive at a dyadic interval $J = I_{N,K} \subseteq I_{n,k}$ with

$$s_{N,K} = 0.$$

T is monotone on no open interval. Let (α, β) be a nonempty open interval. Since a function that is monotone on an interval is monotone on every subinterval, and since (α, β) contains a nonempty bounded open interval, we may assume α and β are finite. Choose n so large that $2^{-n} < (\beta - \alpha)/3$ and let k be the least integer with $k/2^n > \alpha$. Then $k/2^n \leq \alpha + 2^{-n}$, since otherwise $k-1$ would already satisfy $(k-1)/2^n > \alpha$; hence

$$\alpha < \frac{k}{2^n} \quad \text{and} \quad \frac{k+1}{2^n} \leq \alpha + \frac{2}{2^n} < \alpha + (\beta - \alpha) = \beta,$$

so $I_{n,k} = [k/2^n, (k+1)/2^n] \subseteq (\alpha, \beta)$. Fix this $I_{n,k}$ and let $J = I_{N,K} \subseteq I_{n,k} \subseteq (\alpha, \beta)$ be a dyadic subinterval with $s_{N,K} = 0$, as just constructed. By (3), for $x \in J$,

$$T(x) = c + \frac{1}{2^N} T(2^N x - K), \quad \text{where } c = c_{N,K}.$$

The map $x \mapsto 2^N x - K$ carries the left endpoint $p = K/2^N$ of J , its midpoint $q = (K + 1/2)/2^N$, and its right endpoint $r = (K + 1)/2^N$ onto 0, $1/2$, and 1 respectively. Now

$$T(0) = 0, \quad T(1/2) = \varphi(1/2) + \sum_{m \geq 1} \frac{1}{2^m} \varphi(2^{m-1}) = \frac{1}{2}, \quad T(1) = \sum_{m \geq 0} \frac{1}{2^m} \varphi(2^m) = 0,$$

since φ vanishes at every integer and 2^{m-1} is an integer for $m \geq 1$. Therefore

$$T(p) = c, \quad T(q) = c + \frac{1}{2^{N+1}}, \quad T(r) = c,$$

and $p < q < r$ are three points of (α, β) with $T(p) < T(q)$ and $T(q) > T(r)$. The first inequality shows T is not decreasing on (α, β) and the second shows T is not increasing on (α, β) . Hence T is not monotone on (α, β) .

Since (α, β) was an arbitrary nonempty open interval, the continuous function T on $\mathbb{R}$ is monotone on no open interval. So the answer to the question is no.

6.2 Problems 6.8–6.14

Problem 6.8. Let I and J be closed, bounded intervals and $\gamma > 0$ be such that $\ell(I) > \gamma \cdot \ell(J)$. Assume $I \cap J \neq \emptyset$. Show that if $\gamma \geq 1/2$, then $J \subseteq 5 * I$, where $5 * I$ denotes the interval with the same center as I and five times its length. Is the same true if $0 < \gamma < 1/2$?

Solution. Write $L = \ell(I)$ and let c be the midpoint of I , so that

$$I = [c - L/2, c + L/2], \quad 5 * I = [c - 5L/2, c + 5L/2].$$

Consider first the case $\gamma \geq 1/2$. Since $I \cap J \neq \emptyset$, choose a point $y \in I \cap J$. Let x be an arbitrary point of J . Because x and y both lie in the interval J ,

$$|x - y| \leq \ell(J).$$

The hypothesis $\ell(I) > \gamma \cdot \ell(J)$ gives $\ell(J) < \ell(I)/\gamma = L/\gamma$, and since $\gamma \geq 1/2$ we have $1/\gamma \leq 2$, so

$$|x - y| \leq \ell(J) < \frac{L}{\gamma} \leq 2L.$$

On the other hand $y \in I$, so $|y - c| \leq L/2$. By the triangle inequality,

$$|x - c| \leq |x - y| + |y - c| < 2L + \frac{L}{2} = \frac{5L}{2}.$$

Hence $x \in [c - 5L/2, c + 5L/2] = 5 * I$. As $x \in J$ was arbitrary, $J \subseteq 5 * I$.

Now consider the case $0 < \gamma < 1/2$. The conclusion fails for every such γ . Indeed, fix γ with $0 < \gamma < 1/2$, so that $1/\gamma > 2$. Choose a number t with

$$2 < t < \frac{1}{\gamma},$$

which is possible precisely because $1/\gamma > 2$. Put

$$I = [0, 1], \quad J = [-t, 0].$$

Then $\ell(I) = 1$, $\ell(J) = t$, and $\gamma \cdot \ell(J) = \gamma t < 1 = \ell(I)$, so the hypothesis $\ell(I) > \gamma \cdot \ell(J)$ holds. Moreover $I \cap J = \{0\} \neq \emptyset$. The midpoint of I is $1/2$ and $\ell(I) = 1$, so

$$5 * I = \left[\frac{1}{2} - \frac{5}{2}, \frac{1}{2} + \frac{5}{2}\right] = [-2, 3].$$

Since $t > 2$, the endpoint $-t$ of J satisfies $-t < -2$, so $-t \notin 5 * I$ and therefore $J \not\subseteq 5 * I$.

Thus the implication holds for $\gamma \geq 1/2$ and fails for every $\gamma \in (0, 1/2)$. The computation in the first part shows exactly why: the bound obtained is $|x - c| < L/\gamma + L/2$, and this is at most $5L/2$ precisely when $1/\gamma \leq 2$; the second part shows that this restriction on γ cannot be relaxed.

This is exactly the geometric fact that justifies the dilation factor 5 in the proof of the Vitali Covering Lemma. There, in the notation of that proof, the selected intervals satisfy relation (6): $\ell(I_{n+1}) > \ell(I)/2$ for every I in $\mathcal{F}$ that is disjoint from $I_1, \dots, I_n$. That is the hypothesis of the present problem with $\gamma = 1/2$, and it is what lets one conclude that such an I meeting I_N is contained in $5 * I_N$.

Problem 6.9. Show that a set E of real numbers has measure zero if and only if there is a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ for which each point in E belongs to infinitely many of the I_k 's and $\sum_{k=1}^\infty \ell(I_k) < \infty$.

Solution. Recall that a set has measure zero exactly when its outer measure is zero: by Proposition 4 of Chapter 2 any set of outer measure zero is measurable, and its measure is then its outer measure, namely zero.

Suppose first that such a collection $\{I_k\}_{k=1}^\infty$ exists: every point of E lies in infinitely many I_k and $\sum_{k=1}^\infty \ell(I_k) < \infty$. Fix a natural number n and set

$$E_n = \bigcup_{k=n}^\infty I_k.$$

If $x \in E$, then x belongs to I_k for infinitely many indices k , and in particular for some index $k \geq n$. Hence $E \subseteq E_n$ for every n . By the monotonicity and countable subadditivity of outer measure, together with Proposition 1 of Chapter 2, which says that the outer measure of an interval is its length,

$$m^*(E) \leq m^*(E_n) \leq \sum_{k=n}^\infty \ell(I_k).$$

Since the series $\sum_{k=1}^\infty \ell(I_k)$ converges, its tails tend to 0: given $\epsilon > 0$ there is n with $\sum_{k=n}^\infty \ell(I_k) < \epsilon$. Therefore $m^*(E) < \epsilon$ for every $\epsilon > 0$, so $m^*(E) = 0$ and E has measure zero.

Conversely, suppose E has measure zero, so $m^*(E) = 0$. Fix a natural number n . By the definition of outer measure as an infimum, there is a countable collection $\{J_{n,k}\}_k$ of open, bounded intervals with

$$E \subseteq \bigcup_k J_{n,k} \quad \text{and} \quad \sum_k \ell(J_{n,k}) < \frac{1}{2^n}.$$

Adjoining to this collection the single extra interval $(0, 2^{-n})$ if necessary, we may assume the n -th collection is nonempty, at the cost of replacing the bound 2^{-n} by 2^{-n+1} ; this changes nothing below. Let

$$\mathcal{C} = \{J_{n,k} : n \in \mathbb{N}, k \text{ an index of the } n\text{-th collection}\},$$

regarded as a family indexed by the set of admissible pairs (n, k) . This index set is a countable union of countable sets, hence countable, and it is infinite because each of the infinitely many blocks n contributes at least one index. Enumerate the family as $\{I_j\}_{j=1}^\infty$ by choosing a bijection of $\mathbb{N}$ onto the index set. Since all terms are nonnegative, the sum of the resulting series is independent of the order of summation, and

$$\sum_{j=1}^\infty \ell(I_j) = \sum_{n=1}^\infty \sum_k \ell(J_{n,k}) \leq \sum_{n=1}^\infty \frac{1}{2^{n-1}} = 2 < \infty.$$

Finally, let $x \in E$. For each n , the n -th collection covers E , so there is an index k with $x \in J_{n,k}$. Distinct values of n give indices (n, k) in disjoint blocks of the index set, hence distinct members of the enumerated family. Therefore x belongs to I_j for infinitely many j . This is the required collection.

Problem 6.10. (Riesz-Nagy) Let E be a set of measure zero contained in the open interval (a, b) . According to the preceding problem, there is a countable collection of open intervals contained in (a, b) , $\{(c_k, d_k)\}_{k=1}^{\infty}$, for which each point in E belongs to infinitely many intervals in the collection and $\sum_{k=1}^{\infty} (d_k - c_k) < \infty$. Define

$$f(x) = \sum_{k=1}^{\infty} \ell((c_k, d_k) \cap (-\infty, x)) \quad \text{for all } x \text{ in } (a, b).$$

Show that f is increasing and fails to be differentiable at each point in E .

Solution. For each natural number k define φ_k on (a, b) by

$$\varphi_k(x) = \ell((c_k, d_k) \cap (-\infty, x)) = \begin{cases} 0 & \text{if } x \leq c_k, \\ x - c_k & \text{if } c_k \leq x \leq d_k, \\ d_k - c_k & \text{if } x \geq d_k, \end{cases}$$

so that $f = \sum_{k=1}^{\infty} \varphi_k$ pointwise on (a, b) . The three formulas agree at the overlaps $x = c_k$ and $x = d_k$, so φ_k is well defined, and inspection of the three pieces shows that φ_k is increasing and satisfies

$$0 \leq \varphi_k(x) \leq d_k - c_k \quad \text{for all } x \in (a, b). \quad (*)$$

First, f is well defined and finite. Indeed, by $(*)$ the terms of the series defining $f(x)$ are nonnegative and dominated by the terms of the convergent series $\sum_{k=1}^{\infty} (d_k - c_k)$, so for each x the series converges and

$$0 \leq f(x) \leq \sum_{k=1}^{\infty} (d_k - c_k) < \infty.$$

Next, f is increasing. Let $a < x < y < b$. Each φ_k is increasing, so $\varphi_k(y) - \varphi_k(x) \geq 0$ for every k . Both series $\sum_k \varphi_k(x)$ and $\sum_k \varphi_k(y)$ converge, so we may subtract them termwise:

$$f(y) - f(x) = \sum_{k=1}^{\infty} [\varphi_k(y) - \varphi_k(x)] \geq 0. \quad (**)$$

Hence $f(x) \leq f(y)$, and f is increasing on (a, b) .

Finally, f fails to be differentiable at each point of E . We show more: for each $x_0 \in E$ the upper derivative $\overline{D}f(x_0)$ equals ∞ . Since the book's definition requires, for differentiability at x_0 , that $\overline{D}f(x_0)$ and $\underline{D}f(x_0)$ be equal and finite, this rules out differentiability at x_0 .

Fix $x_0 \in E$ and fix a natural number n . Since x_0 belongs to infinitely many of the intervals of the collection, we may choose n distinct indices $k_1, k_2, \dots, k_n$ for which

$$x_0 \in (c_{k_i}, d_{k_i}) \quad \text{for } 1 \leq i \leq n.$$

Each of these intervals is open and contains x_0 , so $d_{k_i} - x_0 > 0$ for every i . Since $x_0 < b$, the number

$$\delta = \min \left\{ b - x_0, \min_{1 \leq i \leq n} (d_{k_i} - x_0) \right\}$$

is positive. Let $0 < t < \delta$ and put $y = x_0 + t$; then $y \in (a, b)$. For each i we have

$$c_{k_i} < x_0 < y = x_0 + t < d_{k_i},$$

so both x_0 and y lie in $[c_{k_i}, d_{k_i}]$, and the middle formula for φ_{k_i} applies to each:

$$\varphi_{k_i}(y) - \varphi_{k_i}(x_0) = (y - c_{k_i}) - (x_0 - c_{k_i}) = y - x_0 = t.$$

All the terms in the series $(**)$ are nonnegative, so discarding all but the n chosen indices gives

$$f(x_0 + t) - f(x_0) = \sum_{k=1}^{\infty} [\varphi_k(y) - \varphi_k(x_0)] \geq \sum_{i=1}^n [\varphi_{k_i}(y) - \varphi_{k_i}(x_0)] = n t.$$

Dividing by $t > 0$,

$$\frac{f(x_0 + t) - f(x_0)}{t} \geq n \quad \text{for all } 0 < t < \delta.$$

Consequently, for every h with $0 < h < \delta$,

$$\sup_{0 < |t| \leq h} \frac{f(x_0 + t) - f(x_0)}{t} \geq n,$$

and letting $h \rightarrow 0$ in the definition of the upper derivative yields $\overline{D}f(x_0) \geq n$. As n was an arbitrary natural number, $\overline{D}f(x_0) = \infty$. Therefore f is not differentiable at x_0 .

Since $x_0 \in E$ was arbitrary, f is an increasing function on (a, b) that fails to be differentiable at every point of E . This is exactly the assertion, made after the proof of Lebesgue's Theorem, that the theorem is best possible: every set of measure zero in (a, b) is contained in the set of nondifferentiability of some increasing function on (a, b) . Of course Lebesgue's Theorem guarantees that this set of nondifferentiability itself has measure zero.

Problem 6.11. For real numbers $\alpha < \beta$ and $\gamma > 0$, show that if g is integrable over $[\alpha + \gamma, \beta + \gamma]$, then

$$\int_{\alpha}^{\beta} g(t + \gamma) dt = \int_{\alpha + \gamma}^{\beta + \gamma} g(t) dt.$$

Prove this change of variables formula by successively considering simple functions, bounded measurable functions, nonnegative integrable functions, and general integrable functions. Use it to prove (14).

Solution. Throughout write $I = [\alpha, \beta]$ and $I_{\gamma} = [\alpha + \gamma, \beta + \gamma] = I + \gamma$, and for a function g on I_{γ} define its translate

$$(Tg)(t) = g(t + \gamma) \quad \text{for } t \in I.$$

The whole argument rests on the translation invariance of Lebesgue measure established in Chapter 2: outer measure is translation invariant, and by Proposition 10 of Chapter 2 the translate of a measurable set is measurable. Applying that proposition to A and to $A + \gamma$ with the translations γ and $-\gamma$, we get that A is measurable if and only if $A + \gamma$ is measurable, and then $m(A + \gamma) = m^*(A + \gamma) = m^*(A) = m(A)$.

Measurability of the translate. If g is measurable on I_{γ} then Tg is measurable on I . Indeed, for $c \in \mathbb{R}$,

$$\{t \in I : (Tg)(t) > c\} = \{s \in I_{\gamma} : g(s) > c\} - \gamma,$$

which is a translate of a measurable set and hence measurable. The map T is a bijection from functions on I_{γ} onto functions on I , with inverse given by translation by $-\gamma$; by the same reasoning the inverse also preserves measurability. Note also that T is linear, order preserving, carries simple functions to simple functions and bounded functions to bounded functions, and satisfies $T(g^+) = (Tg)^+$ and $T(g^-) = (Tg)^-$.

Step 1: simple functions. Let $A \subseteq I_{\gamma}$ be measurable. Then for $t \in I$ we have $\chi_A(t + \gamma) = 1$ exactly when $t \in A - \gamma$, so $T\chi_A = \chi_{A - \gamma}$ and therefore

$$\int_{\alpha}^{\beta} \chi_A(t + \gamma) dt = m(A - \gamma) = m(A) = \int_{\alpha + \gamma}^{\beta + \gamma} \chi_A.$$

Now let ψ be simple and measurable on I_{γ} , with canonical representation $\psi = \sum_{i=1}^n a_i \chi_{A_i}$, the A_i measurable and disjoint. Then $T\psi = \sum_{i=1}^n a_i \chi_{A_i - \gamma}$, and by the linearity of integration for simple functions,

$$\int_{\alpha}^{\beta} T\psi = \sum_{i=1}^n a_i m(A_i - \gamma) = \sum_{i=1}^n a_i m(A_i) = \int_{\alpha + \gamma}^{\beta + \gamma} \psi.$$

Step 2: bounded measurable functions. Let g be bounded and measurable on I_{γ} , a set of finite measure. Then Tg is bounded and measurable on I , also of finite measure, so by Theorem 4 of Chapter 4 both

functions are integrable, and each integral equals the lower Lebesgue integral, that is, the supremum of the integrals of the simple functions lying below the function:

$$\int_{\alpha+\gamma}^{\beta+\gamma} g = \sup \left\{ \int_{\alpha+\gamma}^{\beta+\gamma} \psi : \psi \text{ simple and measurable on } I_\gamma, \psi \leq g \right\},$$

and similarly for Tg over I . Since T is an order preserving bijection carrying the simple measurable functions on I_γ onto those on I , it carries $\{\psi \text{ simple} : \psi \leq g\}$ onto $\{\varphi \text{ simple} : \varphi \leq Tg\}$, and by Step 1 it preserves the value of the integral. Two sets of real numbers that are in integral preserving bijection have the same supremum, so

$$\int_{\alpha}^{\beta} g(t + \gamma) dt = \int_{\alpha+\gamma}^{\beta+\gamma} g.$$

Step 3: nonnegative measurable functions. Let $g \geq 0$ be measurable on I_γ . By the definition, in Chapter 4, of the integral of a nonnegative measurable function,

$$\int_{\alpha+\gamma}^{\beta+\gamma} g = \sup \left\{ \int_{\alpha+\gamma}^{\beta+\gamma} h : h \text{ bounded, measurable, of finite support, } 0 \leq h \leq g \right\},$$

and likewise for $Tg \geq 0$ over I . If h is bounded, measurable, of finite support and $0 \leq h \leq g$ on I_γ , then Th is bounded, measurable, of finite support (its support is the translate by $-\gamma$ of the support of h , and translation preserves measure) and $0 \leq Th \leq Tg$ on I ; and T maps this class onto the corresponding class for Tg , since its inverse has the same properties. By Step 2 it preserves integrals. Hence the two suprema coincide,

$$\int_{\alpha}^{\beta} g(t + \gamma) dt = \int_{\alpha+\gamma}^{\beta+\gamma} g,$$

both sides being finite or infinite together. In particular g is integrable over I_γ if and only if Tg is integrable over I .

Step 4: general integrable functions. Let g be integrable over I_γ . By Proposition 14 of Chapter 4, g^+ and g^- are nonnegative measurable functions each with finite integral over I_γ , and $(Tg)^+ = T(g^+)$, $(Tg)^- = T(g^-)$. By Step 3,

$$\int_{\alpha}^{\beta} (Tg)^{\pm} = \int_{\alpha+\gamma}^{\beta+\gamma} g^{\pm} < \infty,$$

so Tg is integrable over I , and by the definition of the integral of an integrable function,

$$\int_{\alpha}^{\beta} g(t + \gamma) dt = \int_{\alpha}^{\beta} (Tg)^+ - \int_{\alpha}^{\beta} (Tg)^- = \int_{\alpha+\gamma}^{\beta+\gamma} g^+ - \int_{\alpha+\gamma}^{\beta+\gamma} g^- = \int_{\alpha+\gamma}^{\beta+\gamma} g.$$

This is the change of variables formula. Nothing in the argument used the sign of γ : the same proof, verbatim, gives the formula for any real γ .

Proof of (14). Let f be integrable over $[a, b]$, extended to take the value $f(b)$ on $(b, b+1]$; the extension is integrable over $[a, b+1]$, since it is integrable over $[a, b]$ and constant, hence integrable, over $[b, b+1]$. Fix h with $0 < h \leq 1$ and let $a \leq u \leq v \leq b$. The function $t \mapsto f(t+h)$ is integrable over $[u, v]$ by Step 4 applied with $\gamma = h$ on $[u+h, v+h] \subseteq [a, b+1]$, so $\text{Diff}_h f = (f(\cdot + h) - f)/h$ is integrable over $[u, v]$, and by Theorem 17 of Chapter 4 (linearity of integration),

$$\int_u^v \text{Diff}_h f = \frac{1}{h} \left[\int_u^v f(t+h) dt - \int_u^v f \right] = \frac{1}{h} \left[\int_{u+h}^{v+h} f - \int_u^v f \right],$$

the last equality by the change of variables formula. Now use Corollary 18 of Chapter 4 (Additivity Over Domains of Integration) twice, splitting $[u, v+h]$ at v and at $u+h$; both splittings are legitimate since $u \leq v \leq v+h$ and $u \leq u+h \leq v+h$, and f is integrable over $[u, v+h] \subseteq [a, b+1]$:

$$\int_u^v f + \int_v^{v+h} f = \int_u^{v+h} f = \int_u^{u+h} f + \int_{u+h}^{v+h} f.$$

Rearranging,

$$\int_{u+h}^{v+h} f - \int_u^v f = \int_v^{v+h} f - \int_u^{u+h} f.$$

Therefore

$$\int_u^v \text{Diff}_h f = \frac{1}{h} \int_v^{v+h} f - \frac{1}{h} \int_u^{u+h} f = \text{Av}_h f(v) - \text{Av}_h f(u),$$

which is (14).

Problem 6.12. Compute the upper and lower derivatives of the characteristic function of the rationals.

Solution. Let $f = \chi_{\mathbb{Q}}$ on $\mathbb{R}$, so $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. We claim that

$$\overline{D}f(x) = \infty \quad \text{and} \quad \underline{D}f(x) = -\infty \quad \text{for every } x \in \mathbb{R}.$$

Recall the definitions

$$\overline{D}f(x) = \lim_{h \rightarrow 0} \left[\sup_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{t} \right], \quad \underline{D}f(x) = \lim_{h \rightarrow 0} \left[\inf_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{t} \right].$$

As h decreases the supremum is taken over a smaller set, so the expression in brackets is monotone in h and the limits exist in the extended reals. Consequently, to prove $\overline{D}f(x) = \infty$ it suffices to show that for every $h > 0$ and every $M > 0$ there is a t with $0 < |t| \leq h$ and difference quotient exceeding M ; and symmetrically for the lower derivative.

Fix x and $h > 0$ and $M > 0$, and put $\delta = \min\{h, 1/M\} > 0$.

Case 1: x is rational. Then $f(x) = 1$ and, for $t \neq 0$,

$$\frac{f(x+t) - f(x)}{t} = \frac{f(x+t) - 1}{t} = \begin{cases} 0 & \text{if } x+t \in \mathbb{Q}, \\ -\frac{1}{t} & \text{if } x+t \notin \mathbb{Q}. \end{cases}$$

Since x is rational, $x+t$ is irrational exactly when t is irrational. The irrationals are dense in $\mathbb{R}$, so we may choose an irrational t with $-\delta < t < 0$. For that t we have $0 < |t| \leq h$ and

$$\frac{f(x+t) - f(x)}{t} = -\frac{1}{t} = \frac{1}{|t|} > \frac{1}{\delta} \geq M.$$

Hence $\sup_{0 < |t| \leq h}$ exceeds M for every h and every M , so $\overline{D}f(x) = \infty$. Likewise choose an irrational t with $0 < t < \delta$; then

$$\frac{f(x+t) - f(x)}{t} = -\frac{1}{t} < -\frac{1}{\delta} \leq -M,$$

so the infimum is less than $-M$ for every h and every M , and $\underline{D}f(x) = -\infty$.

Case 2: x is irrational. Then $f(x) = 0$ and, for $t \neq 0$,

$$\frac{f(x+t) - f(x)}{t} = \frac{f(x+t)}{t} = \begin{cases} \frac{1}{t} & \text{if } x+t \in \mathbb{Q}, \\ 0 & \text{if } x+t \notin \mathbb{Q}. \end{cases}$$

By the density of the rationals there is a rational number q with $x < q < x + \delta$. Put $t = q - x$, so $0 < t < \delta \leq h$ and $x+t = q$ is rational; then

$$\frac{f(x+t) - f(x)}{t} = \frac{1}{t} > \frac{1}{\delta} \geq M.$$

Hence $\overline{D}f(x) = \infty$. Similarly there is a rational q with $x - \delta < q < x$; putting $t = q - x$ gives $-\delta < t < 0$, $0 < |t| \leq h$, and

$$\frac{f(x+t) - f(x)}{t} = \frac{1}{t} < -\frac{1}{\delta} \leq -M,$$

so $\underline{D}f(x) = -\infty$.

Thus at every real number x ,

$$\overline{D}f(x) = +\infty, \quad \underline{D}f(x) = -\infty.$$

In particular $\chi_{\mathbb{Q}}$ fails to be differentiable at every point of $\mathbb{R}$, and indeed the difference quotients fail to have even an infinite limit anywhere. There is no contradiction with Lebesgue's Theorem: $\chi_{\mathbb{Q}}$ is not monotone on any open interval, since every open interval contains both rationals and irrationals.

Problem 6.13. Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that cover E in the sense of Vitali. Show that there is a countable disjoint collection $\{I_k\}_{k=1}^{\infty}$ of intervals in $\mathcal{F}$ for which

$$m^* \left[E \sim \bigcup_{k=1}^{\infty} I_k \right] = 0.$$

Solution. We construct the collection inductively, applying the Vitali Covering Lemma at each stage to what is left over, and we use the following observation to keep the newly chosen intervals disjoint from the ones already chosen.

Claim. Let F be a closed set and let $A \subseteq E \sim F$. Then the subcollection

$$\mathcal{F}_F = \{I \in \mathcal{F} : I \cap F = \emptyset\}$$

is a Vitali covering of A .

To see this, let $x \in A$ and $\epsilon > 0$. Since $x \notin F$ and F is closed, its complement is open, so there is $\delta > 0$ with $(x - \delta, x + \delta) \cap F = \emptyset$. Since $x \in E$ and $\mathcal{F}$ covers E in the sense of Vitali, there is an interval $I \in \mathcal{F}$ with $x \in I$ and $\ell(I) < \min\{\epsilon, \delta\}$. Every point $y \in I$ satisfies $|y - x| \leq \ell(I) < \delta$, so $I \subseteq (x - \delta, x + \delta)$ and hence $I \cap F = \emptyset$. Thus $I \in \mathcal{F}_F$, $x \in I$ and $\ell(I) < \epsilon$. This proves the claim.

Construction. We produce, for each natural number j , a finite disjoint subcollection $\{I_k\}_{k=1}^{n_j}$ of $\mathcal{F}$, with $n_1 < n_2 < \dots$ and each stage extending the previous one, such that

$$m^* \left[E \sim \bigcup_{k=1}^{n_j} I_k \right] < \frac{1}{j}. \quad (\dagger)$$

For $j = 1$: E has finite outer measure and $\mathcal{F}$ is a Vitali covering of E , so the Vitali Covering Lemma with $\epsilon = 1$ provides a finite disjoint subcollection $\{I_k\}_{k=1}^{n_1}$ of $\mathcal{F}$ with $m^*[E \sim \bigcup_{k=1}^{n_1} I_k] < 1$.

Inductive step: suppose $\{I_k\}_{k=1}^{n_j}$ has been chosen, disjoint and satisfying $(\dagger)$. Put

$$F_j = \bigcup_{k=1}^{n_j} I_k, \quad E_j = E \sim F_j.$$

As a finite union of closed intervals, F_j is closed. By $(\dagger)$, $m^*(E_j) < 1/j < \infty$. If $E_j = \emptyset$, then $m^*[E \sim \bigcup_{k=1}^{n_j} I_k] = 0$ and the finite disjoint collection $\{I_k\}_{k=1}^{n_j}$, which is countable, already satisfies the conclusion; so we may assume $E_j \neq \emptyset$. By the Claim applied with $F = F_j$ and $A = E_j$, the collection $\mathcal{F}_{F_j}$ is a Vitali covering of E_j . Apply the Vitali Covering Lemma to the set E_j of finite outer measure, the covering $\mathcal{F}_{F_j}$, and $\epsilon = 1/(j+1)$: there is a finite disjoint subcollection of $\mathcal{F}_{F_j}$, which we label $I_{n_j+1}, \dots, I_{n_{j+1}}$, for which

$$m^* \left[E_j \sim \bigcup_{k=n_j+1}^{n_{j+1}} I_k \right] < \frac{1}{j+1}.$$

(If the lemma returns the empty subcollection we may adjoin one further interval of $\mathcal{F}_{F_j}$ containing a point of E_j , which only shrinks the left side; so we may take $n_{j+1} > n_j$.)

The intervals $I_{n_j+1}, \dots, I_{n_{j+1}}$ are pairwise disjoint, and each belongs to $\mathcal{F}_{F_j}$, hence is disjoint from F_j and therefore from each of $I_1, \dots, I_{n_j}$. So $\{I_k\}_{k=1}^{n_{j+1}}$ is a disjoint collection. Moreover, since $E_j = E \sim F_j$,

$$E \sim \bigcup_{k=1}^{n_{j+1}} I_k = (E \sim F_j) \sim \bigcup_{k=n_j+1}^{n_{j+1}} I_k = E_j \sim \bigcup_{k=n_j+1}^{n_{j+1}} I_k,$$

so $m^*[E \sim \bigcup_{k=1}^{n_{j+1}} I_k] < 1/(j+1)$, which is $(\dagger)$ for $j+1$. The induction is complete.

Conclusion. If the process terminates at some stage because $E_j = \emptyset$, we are done as noted. Otherwise the stages nest, and taking the union over all j we obtain a countable collection $\{I_k\}_{k=1}^\infty \subseteq \mathcal{F}$. It is disjoint: any two of its members lie in some common finite stage $\{I_k\}_{k=1}^{n_j}$, which is disjoint. Finally, for each natural number j ,

$$E \sim \bigcup_{k=1}^\infty I_k \subseteq E \sim \bigcup_{k=1}^{n_j} I_k,$$

so by the monotonicity of outer measure and $(\dagger)$,

$$m^*\left[E \sim \bigcup_{k=1}^\infty I_k\right] \leq m^*\left[E \sim \bigcup_{k=1}^{n_j} I_k\right] < \frac{1}{j}.$$

Since this holds for every j , $m^*[E \sim \bigcup_{k=1}^\infty I_k] = 0$, as required.

Problem 6.14. Use the Vitali Covering Lemma to show that the union of any collection (countable or uncountable) of closed, bounded nondegenerate intervals is measurable.

Solution. Let $\mathcal{C}$ be a collection of closed, bounded, nondegenerate intervals and let

$$U = \bigcup_{I \in \mathcal{C}} I.$$

Since $U = \bigcup_{n=1}^\infty (U \cap (-n, n))$ and the measurable sets form a σ -algebra, it suffices to prove that

$$A_n = U \cap (-n, n)$$

is measurable for each natural number n . Fix n ; we may assume $A_n \neq \emptyset$, since the empty set is measurable. Note $m^*(A_n) \leq 2n < \infty$.

A Vitali covering. The key point is that although the intervals of $\mathcal{C}$ may all be long, every point of U lies in arbitrarily short closed subintervals of members of $\mathcal{C}$. Define

$$\mathcal{F}_n = \{J : J \text{ is a closed, bounded, nondegenerate interval, } J \subseteq [-n, n], \text{ and } J \subseteq I \text{ for some } I \in \mathcal{C}\}.$$

We check that $\mathcal{F}_n$ covers A_n in the sense of Vitali. Let $x \in A_n$ and $\epsilon > 0$. Since $x \in U$, there is an interval $I = [c, d] \in \mathcal{C}$ with $c \leq x \leq d$ and $c < d$. Since $x \in (-n, n)$, there is $\delta > 0$ with $[x - \delta, x + \delta] \subseteq (-n, n)$.

If $x < d$, put $\eta = \min\{\epsilon/2, \delta, d - x\} > 0$ and $J = [x, x + \eta]$. Then $c \leq x$ and $x + \eta \leq d$, so $J \subseteq I$; also $J \subseteq [x - \delta, x + \delta] \subseteq [-n, n]$; and J is nondegenerate with $x \in J$ and $\ell(J) = \eta < \epsilon$.

If $x = d$, then $c < x$; put $\eta = \min\{\epsilon/2, \delta, x - c\} > 0$ and $J = [x - \eta, x]$. The same three verifications apply.

In either case $J \in \mathcal{F}_n$, $x \in J$ and $\ell(J) < \epsilon$. So $\mathcal{F}_n$ is a Vitali covering of A_n .

Applying the lemma. The set A_n has finite outer measure and $\mathcal{F}_n$ covers it in the sense of Vitali, so for each natural number j the Vitali Covering Lemma provides a finite disjoint subcollection $\{J_1^j, \dots, J_{m_j}^j\}$ of $\mathcal{F}_n$ for which

$$m^*\left[A_n \sim \bigcup_{k=1}^{m_j} J_k^j\right] < \frac{1}{j}.$$

Let

$$S = \bigcup_{j=1}^\infty \bigcup_{k=1}^{m_j} J_k^j.$$

This is a countable union of closed intervals, hence a Borel set and in particular measurable. Moreover every J_k^j is contained in some member of $\mathcal{C}$ and in $[-n, n]$, so

$$S \subseteq U \cap [-n, n].$$

Now decompose A_n . First,

$$A_n \cap S = S \cap (-n, n).$$

Indeed, $A_n \cap S \subseteq S \cap (-n, n)$ because $A_n \subseteq (-n, n)$; conversely, if $y \in S \cap (-n, n)$ then $y \in U$ and $y \in (-n, n)$, so $y \in A_n$ and hence $y \in A_n \cap S$. The set $S \cap (-n, n)$ is measurable, being the intersection of two measurable sets.

Second, for every j ,

$$A_n \sim S \subseteq A_n \sim \bigcup_{k=1}^{m_j} J_k^j,$$

so by the monotonicity of outer measure $m^*(A_n \sim S) < 1/j$ for every j , whence $m^*(A_n \sim S) = 0$. By Proposition 4 of Chapter 2, a set of outer measure zero is measurable, so $A_n \sim S$ is measurable.

Therefore

$$A_n = (A_n \cap S) \cup (A_n \sim S) = [S \cap (-n, n)] \cup (A_n \sim S)$$

is the union of two measurable sets and hence measurable. Since this holds for every n ,

$$U = \bigcup_{n=1}^{\infty} A_n$$

is measurable, being a countable union of measurable sets.

Observe where nondegeneracy was used: it is what guarantees that each $x \in U$ has, on at least one side, room inside a member of $\mathcal{C}$ for arbitrarily short closed nondegenerate subintervals, and hence that $\mathcal{F}_n$ is a genuine Vitali covering. For degenerate intervals the statement is false: if E is any nonmeasurable set, then $E = \bigcup_{x \in E} [x, x]$ is a union of degenerate closed intervals.

6.3 Problems 6.15–6.21

Problem 6.15. Define f on $\mathbb{R}$ by

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Find the upper and lower derivatives of f at $x = 0$.

Solution. Since $f(0) = 0$, for $t \neq 0$ the difference quotient at 0 is

$$\frac{f(0+t) - f(0)}{t} = \frac{t \sin(1/t)}{t} = \sin(1/t).$$

So, with the definitions of Section 6.2,

$$\overline{D}f(0) = \lim_{h \rightarrow 0} \left[\sup_{0 < |t| \leq h} \sin(1/t) \right], \quad \underline{D}f(0) = \lim_{h \rightarrow 0} \left[\inf_{0 < |t| \leq h} \sin(1/t) \right].$$

Fix $h > 0$. Choose a natural number k so large that

$$t_k = \frac{1}{\pi/2 + 2k\pi} \leq h.$$

Then $0 < t_k \leq h$ and $\sin(1/t_k) = \sin(\pi/2 + 2k\pi) = 1$. Since $|\sin| \leq 1$ always,

$$\sup_{0 < |t| \leq h} \sin(1/t) = 1 \quad \text{for every } h > 0,$$

and therefore $\overline{D}f(0) = 1$.

Similarly, for $h > 0$ choose k so large that

$$s_k = \frac{1}{-\pi/2 + 2k\pi} \leq h.$$

Then $0 < s_k \leq h$ and $\sin(1/s_k) = \sin(2k\pi - \pi/2) = -1$, so

$$\inf_{0 < |t| \leq h} \sin(1/t) = -1 \quad \text{for every } h > 0,$$

and therefore $\underline{D}f(0) = -1$.

Thus

$$\underline{D}f(0) = -1 < 1 = \overline{D}f(0),$$

and in particular f fails to be differentiable at $x = 0$, although both derivatives are finite there. (Observe also that f is continuous at 0, since $|f(x)| \leq |x|$ for all x ; continuity at a point is of course no guarantee of differentiability there.)

Problem 6.16. Let g be integrable over $[a, b]$. Define the antiderivative of g to be the function f defined on $[a, b]$ by

$$f(x) = \int_a^x g \quad \text{for all } x \in [a, b].$$

Show that f is differentiable almost everywhere on (a, b) .

Solution. Write $g = g^+ - g^-$, where $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$ are nonnegative measurable functions. Since $|g^\pm| \leq |g|$ and g is integrable over $[a, b]$, both g^+ and g^- are integrable over $[a, b]$, and hence over every measurable subset of $[a, b]$. Define, for $x \in [a, b]$,

$$f_1(x) = \int_a^x g^+, \quad f_2(x) = \int_a^x g^-.$$

By the linearity of integration, $f = f_1 - f_2$ on $[a, b]$.

Each f_i is increasing on $[a, b]$. Indeed, let $a \leq x < y \leq b$. By the additivity of integration over domains (Chapter 4),

$$f_1(y) - f_1(x) = \int_a^y g^+ - \int_a^x g^+ = \int_x^y g^+ \geq 0,$$

because $g^+ \geq 0$ and the integral of a nonnegative integrable function is nonnegative. Hence $f_1(x) \leq f_1(y)$, and the same computation with g^- in place of g^+ shows f_2 is increasing.

Now apply Lebesgue's Theorem to each of the monotone functions f_1 and f_2 on the open interval (a, b) : there are sets $E_1, E_2 \subseteq (a, b)$ of measure zero such that f_1 is differentiable at every point of $(a, b) \sim E_1$ and f_2 is differentiable at every point of $(a, b) \sim E_2$. The set $E = E_1 \cup E_2$ has measure zero, by the countable subadditivity of measure.

Let $x \in (a, b) \sim E$. Then both limits

$$f'_1(x) = \lim_{t \rightarrow 0} \frac{f_1(x+t) - f_1(x)}{t}, \quad f'_2(x) = \lim_{t \rightarrow 0} \frac{f_2(x+t) - f_2(x)}{t}$$

exist and are finite, and therefore, since

$$\frac{f(x+t) - f(x)}{t} = \frac{f_1(x+t) - f_1(x)}{t} - \frac{f_2(x+t) - f_2(x)}{t}$$

for all $t \neq 0$ with $x+t \in [a, b]$, the limit as $t \rightarrow 0$ of the left-hand side exists and equals $f'_1(x) - f'_2(x)$. Thus f is differentiable at x , with

$$f'(x) = f'_1(x) - f'_2(x).$$

Since E has measure zero, f is differentiable at almost every point of (a, b) .

Problem 6.17. Let f be an increasing bounded function on the open, bounded interval (a, b) . Verify (15).

Solution. Recall that (15) is the conclusion of Corollary 4: for an increasing function on a closed, bounded interval $[a, b]$, the derivative f' is integrable over $[a, b]$ and

$$\int_a^b f' \leq f(b) - f(a).$$

For f increasing and bounded on the open interval (a, b) the endpoint values are to be read as the one-sided limits, and the assertion to be verified is: f' , which by Lebesgue's Theorem is defined almost everywhere on (a, b) , is integrable over (a, b) and

$$\int_a^b f' \leq \lim_{x \rightarrow b^-} f(x) - \lim_{x \rightarrow a^+} f(x).$$

Since f is bounded on (a, b) , the numbers

$$A = \inf_{x \in (a, b)} f(x), \quad B = \sup_{x \in (a, b)} f(x)$$

are finite, and since f is increasing, monotone convergence of the values gives

$$A = \lim_{x \rightarrow a^+} f(x), \quad B = \lim_{x \rightarrow b^-} f(x).$$

Indeed, for $\epsilon > 0$ there is $x_0 \in (a, b)$ with $f(x_0) < A + \epsilon$, and then $A \leq f(x) \leq f(x_0) < A + \epsilon$ for all $x \in (a, x_0)$; the argument at b is symmetric.

Define $\tilde{f}$ on the closed interval $[a, b]$ by

$$\tilde{f}(a) = A, \quad \tilde{f}(x) = f(x) \text{ for } x \in (a, b), \quad \tilde{f}(b) = B.$$

Then $\tilde{f}$ is increasing on $[a, b]$: if $a < x < y < b$ then $\tilde{f}(x) = f(x) \leq f(y) = \tilde{f}(y)$ since f is increasing; and $\tilde{f}(a) = A \leq f(x) \leq B = \tilde{f}(b)$ for every $x \in (a, b)$ by the definition of infimum and supremum; finally $A \leq B$.

Apply Corollary 4 to the increasing function $\tilde{f}$ on $[a, b]$: $\tilde{f}'$ is integrable over $[a, b]$ and

$$\int_a^b \tilde{f}' \leq \tilde{f}(b) - \tilde{f}(a) = B - A.$$

Finally, $\tilde{f}$ and f agree on (a, b) , and differentiability at a point $x \in (a, b)$ together with the value of the derivative there depends only on the values of the function in a neighbourhood of x contained in (a, b) . Hence for every $x \in (a, b)$, $\tilde{f}$ is differentiable at x if and only if f is, and then $\tilde{f}'(x) = f'(x)$. Both derivatives exist for almost all $x \in (a, b)$, by Lebesgue's Theorem. Since $\{a, b\}$ has measure zero, the two integrals agree, and we conclude that f' is integrable over (a, b) with

$$\int_a^b f' = \int_a^b \tilde{f}' \leq B - A = \sup_{x \in (a, b)} f(x) - \inf_{x \in (a, b)} f(x) = \lim_{x \rightarrow b^-} f(x) - \lim_{x \rightarrow a^+} f(x).$$

This is (15) for f on (a, b) ; note that, since A and B are respectively the largest and the smallest values that an increasing extension of f to $[a, b]$ may take at a and at b , the bound obtained is the sharp form (18) of the inequality.

Problem 6.18. Show that if f is defined on (a, b) and $c \in (a, b)$ is a local minimizer for f , then $\underline{D}f(c) \leq 0 \leq \overline{D}f(c)$.

Solution. That c is a local minimizer for f means that there is a $\delta > 0$ such that

$$(c - \delta, c + \delta) \subseteq (a, b) \quad \text{and} \quad f(x) \geq f(c) \text{ for all } x \in (c - \delta, c + \delta).$$

For $0 < h \leq \delta/2$ put

$$S(h) = \sup_{0 < |t| \leq h} \frac{f(c+t) - f(c)}{t}, \quad I(h) = \inf_{0 < |t| \leq h} \frac{f(c+t) - f(c)}{t},$$

so that $\overline{D}f(c) = \lim_{h \rightarrow 0} S(h)$ and $\underline{D}f(c) = \lim_{h \rightarrow 0} I(h)$. As h decreases the supremum is taken over a smaller set of values of t , so S is nondecreasing in h and I is nonincreasing in h ; consequently both limits exist in the extended real numbers, and

$$\overline{D}f(c) = \inf_{0 < h \leq \delta/2} S(h), \quad \underline{D}f(c) = \sup_{0 < h \leq \delta/2} I(h).$$

Fix h with $0 < h \leq \delta/2$.

Take $t = h$. Then $0 < |t| \leq h$ and $c + t \in (c - \delta, c + \delta)$, so $f(c + t) - f(c) \geq 0$; since $t > 0$,

$$\frac{f(c + h) - f(c)}{h} \geq 0, \quad \text{hence } S(h) \geq 0.$$

As this holds for every such h , taking the infimum over h gives $\overline{D}f(c) \geq 0$.

Now take $t = -h$. Then $0 < |t| \leq h$ and $c + t \in (c - \delta, c + \delta)$, so again $f(c + t) - f(c) \geq 0$; since now $t = -h < 0$,

$$\frac{f(c - h) - f(c)}{-h} \leq 0, \quad \text{hence } I(h) \leq 0.$$

As this holds for every such h , taking the supremum over h gives $\underline{D}f(c) \leq 0$.

Therefore

$$\underline{D}f(c) \leq 0 \leq \overline{D}f(c),$$

as asserted. (In particular, if f happens to be differentiable at the local minimizer c , then $\underline{D}f(c) = \overline{D}f(c) = f'(c)$, and the two inequalities force $f'(c) = 0$: this is the familiar interior extremum theorem. The same argument applied to $-f$ shows that at a local maximizer one again has $\underline{D}f(c) \leq 0 \leq \overline{D}f(c)$, since $\overline{D}(-f) = -\underline{D}f$ and $\underline{D}(-f) = -\overline{D}f$.)

Problem 6.19. Let f be continuous on $[a, b]$ with $\overline{D}f \geq 0$ on (a, b) . Show that f is increasing on $[a, b]$. (Hint: First show this for a function g for which $\overline{D}g \geq \epsilon > 0$ on (a, b) . Apply this to the function $g(x) = f(x) + \epsilon x$.)

Solution. Throughout, “increasing” means $f(u) \leq f(v)$ whenever $u < v$, as in Section 6.1. It is convenient to split the upper derivative into its one-sided parts. For $c \in (a, b)$ set

$$D^+f(c) = \limsup_{t \rightarrow 0^+} \frac{f(c+t) - f(c)}{t}, \quad D^-f(c) = \limsup_{s \rightarrow 0^+} \frac{f(c) - f(c-s)}{s}.$$

For $h > 0$,

$$\sup_{0 < |t| \leq h} \frac{f(c+t) - f(c)}{t} = \max \left\{ \sup_{0 < t \leq h} \frac{f(c+t) - f(c)}{t}, \sup_{0 < s \leq h} \frac{f(c) - f(c-s)}{s} \right\},$$

because the substitution $t = -s$ turns a quotient with $t < 0$ into $(f(c) - f(c-s))/s$. Each of the two suprema is nondecreasing in h , so letting $h \rightarrow 0$ gives

$$\overline{D}f(c) = \max\{D^+f(c), D^-f(c)\}.$$

The proof below establishes the theorem for the hypothesis $D^+f \geq 0$ on (a, b) (and, by reflection, for $D^-f \geq 0$); a closing remark shows that the two-sided derivate $\overline{D}f = \max\{D^+f, D^-f\}$, which is what Section 6.2 denotes by $\overline{D}f$, does not suffice, so that the hypothesis has to be read one-sidedly.

Step 1, the reduction supplied by the hint. Suppose it is known that

(L) if g is continuous on $[a, b]$ and $D^+g \geq \epsilon > 0$ on (a, b) , then g is increasing on $[a, b]$.
Let f be continuous on $[a, b]$ with $D^+f \geq 0$ on (a, b) , and let $\epsilon > 0$. Put $g(x) = f(x) + \epsilon x$; g is continuous on $[a, b]$. For $c \in (a, b)$ and $t \neq 0$,

$$\frac{g(c+t) - g(c)}{t} = \frac{f(c+t) - f(c)}{t} + \epsilon,$$

so the supremum over $0 < t \leq h$ of the left side is ϵ more than that of the first term on the right; letting $h \rightarrow 0$,

$$D^+g(c) = D^+f(c) + \epsilon \geq \epsilon > 0.$$

By (L), g is increasing on $[a, b]$: if $a \leq u < v \leq b$ then $f(u) + \epsilon u \leq f(v) + \epsilon v$, that is,

$$f(v) - f(u) \geq \epsilon(u - v) \geq -\epsilon(b - a).$$

Since $\epsilon > 0$ is arbitrary, $f(v) - f(u) \geq 0$. Thus f is increasing on $[a, b]$, and it remains only to prove (L).

Step 2, the crossing-point argument, which proves (L). Assume g is continuous on $[a, b]$ and $D^+g \geq \epsilon > 0$ on (a, b) , and suppose, to get a contradiction, that g is not increasing: there are u, v with $a \leq u < v \leq b$ and $g(u) > g(v)$. Fix λ with $g(v) < \lambda < g(u)$ and set

$$S = \{x \in [u, v] : g(x) \geq \lambda\}, \quad c = \sup S.$$

S is nonempty ($u \in S$) and closed in $[u, v]$ by the continuity of g , so $c \in S$, i.e. $g(c) \geq \lambda$; and $c < v$ since $g(v) < \lambda$. By the definition of c ,

$$g(x) < \lambda \quad \text{for all } x \in (c, v].$$

Letting $x \rightarrow c^+$ and using continuity gives $g(c) \leq \lambda$, hence $g(c) = \lambda$; in particular $c \neq u$, since $g(u) > \lambda$, so $c > u$ and $c \in (u, v) \subseteq (a, b)$. Consequently, for $0 < t \leq v - c$,

$$\frac{g(c+t) - g(c)}{t} = \frac{g(c+t) - \lambda}{t} < 0,$$

so that

$$D^+g(c) \leq 0.$$

Since $c \in (a, b)$, the hypothesis gives $D^+g(c) \geq \epsilon > 0$, and this contradicts $D^+g(c) \leq 0$. Hence no such pair $u < v$ exists and g is increasing on $[a, b]$. This proves (L), and with Step 1 it proves if f is continuous on $[a, b]$ and $D^+f \geq 0$ on (a, b) , then f is increasing on $[a, b]$.

The same holds with D^- in place of D^+ . Indeed, put $\tilde{f}(x) = -f(-x)$ on $[-b, -a]$; for $s > 0$,

$$\frac{\tilde{f}(x+s) - \tilde{f}(x)}{s} = \frac{f(-x) - f(-x-s)}{s},$$

so $D^+\tilde{f}(x) = D^-f(-x)$, and $\tilde{f}$ is increasing precisely when f is; so the statement for D^- follows from the one just proved.

Why the hypothesis must be read one-sidedly. With the two-sided derivate of Section 6.2 the assertion is false. Observe first where the argument above breaks down: at the crossing point c we obtained only $D^+g(c) \leq 0$, and $\overline{D}g(c) = \max\{D^+g(c), D^-g(c)\} \geq \epsilon$ then forces $D^-g(c) \geq \epsilon$, which merely says that g dips below λ arbitrarily near c on the left as well. That is not absurd, and in fact no argument can rule it out.

Take $\alpha = 1/2$ and $\beta = 13$, so that β is an odd integer and $\alpha\beta = 13/2 > 1 + 3\pi/2$, and let

$$W(x) = \sum_{n=0}^{\infty} \alpha^n \cos(\beta^n \pi x).$$

The series is dominated by the convergent geometric series $\sum \alpha^n$, so it converges uniformly on $\mathbb{R}$ and W is continuous. Because β^n is odd, $\cos(\beta^n \pi) = -1$, so

$$W(0) = \sum_{n=0}^{\infty} \alpha^n = 2, \quad W(1) = -\sum_{n=0}^{\infty} \alpha^n = -2,$$

and W is not increasing on $[0, 1]$. We claim that nevertheless $\overline{DW}(x) = +\infty$ for every $x \in (0, 1)$, so that W satisfies the hypothesis as printed.

Fix x and a natural number m . Write $\beta^m x = k_m + \xi_m$ with k_m an integer and $-1/2 < \xi_m \leq 1/2$, and put

$$y_m^+ = \frac{k_m + 1}{\beta^m}, \quad y_m^- = \frac{k_m - 1}{\beta^m}.$$

Then $y_m^+ - x = (1 - \xi_m)/\beta^m$ and $y_m^- - x = -(1 + \xi_m)/\beta^m$, so

$$\frac{1}{2\beta^m} \leq y_m^+ - x \leq \frac{3}{2\beta^m}, \quad -\frac{3}{2\beta^m} \leq y_m^- - x \leq -\frac{1}{2\beta^m},$$

and both points tend to x ; discard the finitely many m for which they fall outside $(0, 1)$. Split the series at $n = m$.

For $n < m$, the Mean Value Theorem applied to the cosine gives $|\cos(\beta^n \pi y) - \cos(\beta^n \pi x)| \leq \beta^n \pi |y - x|$, so for either choice of sign,

$$\left| \frac{1}{y_m^\pm - x} \sum_{n=0}^{m-1} \alpha^n [\cos(\beta^n \pi y_m^\pm) - \cos(\beta^n \pi x)] \right| \leq \pi \sum_{n=0}^{m-1} (\alpha\beta)^n \leq \frac{\pi}{\alpha\beta - 1} (\alpha\beta)^m.$$

For $n \geq m$, use that β is odd: $\beta^{n-m}(k_m \pm 1)$ has the parity of $k_m + 1$ and $\beta^{n-m}k_m$ has the parity of k_m , whence

$$\cos(\beta^n \pi y_m^\pm) = (-1)^{k_m+1}, \quad \cos(\beta^n \pi x) = (-1)^{k_m} \cos(\beta^{n-m} \pi \xi_m).$$

Therefore, for both choices of sign,

$$\sum_{n=m}^{\infty} \alpha^n [\cos(\beta^n \pi y_m^\pm) - \cos(\beta^n \pi x)] = -(-1)^{k_m} \Sigma_m, \quad \Sigma_m = \sum_{n=m}^{\infty} \alpha^n [1 + \cos(\beta^{n-m} \pi \xi_m)].$$

Every term of Σ_m is nonnegative, and its $n = m$ term is $\alpha^m(1 + \cos(\pi \xi_m)) \geq \alpha^m$ because $|\xi_m| \leq 1/2$; hence $\Sigma_m \geq \alpha^m$. Dividing by $y_m^+ - x > 0$ produces a quantity of sign $-(-1)^{k_m}$ and absolute value at least $\alpha^m \cdot \frac{2\beta^m}{3} = \frac{2}{3}(\alpha\beta)^m$; dividing by $y_m^- - x < 0$ produces a quantity of the opposite sign $(-1)^{k_m}$ and again of absolute value at least $\frac{2}{3}(\alpha\beta)^m$.

Set $\kappa = \frac{2}{3} - \frac{\pi}{\alpha\beta-1} = \frac{2}{3} - \frac{\pi}{5.5} > 0$. Combining the two estimates, the difference quotients

$$\frac{W(y_m^+) - W(x)}{y_m^+ - x} \quad \text{and} \quad \frac{W(y_m^-) - W(x)}{y_m^- - x}$$

have opposite signs and each has absolute value at least $\kappa(\alpha\beta)^m$. In particular, for each m at least one of the two is $\geq \kappa(\alpha\beta)^m$. Since $y_m^\pm \rightarrow x$ and $\alpha\beta > 1$, this gives

$$\overline{DW}(x) = \limsup_{t \rightarrow 0} \frac{W(x+t) - W(x)}{t} = +\infty$$

at every x . (The other member of each pair shows likewise that $\underline{DW}(x) = -\infty$; this is Weierstrass's classical proof that W is nowhere differentiable.)

So W is continuous on $[0, 1]$, satisfies $\overline{DW} \geq 0$ on $(0, 1)$, and is not increasing. Accordingly the hypothesis of the problem is to be read as a hypothesis on one fixed one-sided derivate, and under that reading Steps 1 and 2 above are a complete proof.

Problem 6.20. Let f and g be real-valued functions on (a, b) . Show that

$$\underline{D}f + \underline{D}g \leq \underline{D}(f + g) \leq \overline{D}(f + g) \leq \overline{D}f + \overline{D}g \quad \text{on } (a, b).$$

Solution. Fix $c \in (a, b)$ and choose $h_0 > 0$ with $(c - h_0, c + h_0) \subseteq (a, b)$. For $0 < h \leq h_0$ and for a real-valued function φ on (a, b) write

$$Q_\varphi(t) = \frac{\varphi(c+t) - \varphi(c)}{t} \quad (0 < |t| \leq h_0), \quad S_\varphi(h) = \sup_{0 < |t| \leq h} Q_\varphi(t), \quad I_\varphi(h) = \inf_{0 < |t| \leq h} Q_\varphi(t),$$

so that, by definition, $\overline{D}\varphi(c) = \lim_{h \rightarrow 0} S_\varphi(h)$ and $\underline{D}\varphi(c) = \lim_{h \rightarrow 0} I_\varphi(h)$. Each $Q_\varphi(t)$ is a real number and the index set is nonempty, so $S_\varphi(h) \in (-\infty, +\infty]$ and $I_\varphi(h) \in [-\infty, +\infty]$; in particular the sums $S_f(h) + S_g(h)$ and $I_f(h) + I_g(h)$ are unambiguously defined in the extended reals. Moreover S_φ is nondecreasing and I_φ is nonincreasing in h , so the two limits above exist in $[-\infty, +\infty]$; as usual, the outer inequalities of the problem are asserted whenever the sums on their outer sides are defined, i.e. are not of the form $+\infty + (-\infty)$.

The whole proof rests on the identity

$$Q_{f+g}(t) = \frac{(f(c+t) + g(c+t)) - (f(c) + g(c))}{t} = Q_f(t) + Q_g(t), \quad 0 < |t| \leq h_0.$$

The upper estimate. Fix $h \in (0, h_0]$. For every t with $0 < |t| \leq h$,

$$Q_{f+g}(t) = Q_f(t) + Q_g(t) \leq S_f(h) + S_g(h),$$

since each term is dominated by the corresponding supremum (the inequality is trivially true if the right-hand side is $+\infty$). Taking the supremum over such t ,

$$S_{f+g}(h) \leq S_f(h) + S_g(h) \quad \text{for all } 0 < h \leq h_0.$$

Now let $h \rightarrow 0$. All three quantities are monotone in h , so all three limits exist in $[-\infty, +\infty]$, and if $\overline{D}f(c) + \overline{D}g(c)$ is defined then $\lim_{h \rightarrow 0} [S_f(h) + S_g(h)] = \overline{D}f(c) + \overline{D}g(c)$. Hence

$$\overline{D}(f + g)(c) \leq \overline{D}f(c) + \overline{D}g(c).$$

The lower estimate. Fix $h \in (0, h_0]$. For every t with $0 < |t| \leq h$,

$$Q_{f+g}(t) = Q_f(t) + Q_g(t) \geq I_f(h) + I_g(h),$$

and taking the infimum over such t ,

$$I_{f+g}(h) \geq I_f(h) + I_g(h) \quad \text{for all } 0 < h \leq h_0.$$

Letting $h \rightarrow 0$ exactly as before,

$$\underline{D}(f + g)(c) \geq \underline{D}f(c) + \underline{D}g(c).$$

The middle estimate. For each $h \in (0, h_0]$ the infimum of a nonempty set of real numbers does not exceed its supremum, so $I_{f+g}(h) \leq S_{f+g}(h)$; letting $h \rightarrow 0$ gives

$$\underline{D}(f + g)(c) \leq \overline{D}(f + g)(c).$$

(This is the inequality $\underline{D}\varphi \leq \overline{D}\varphi$ recorded in Section 6.2 for every φ .)

Since $c \in (a, b)$ was arbitrary, the chain

$$\underline{D}f + \underline{D}g \leq \underline{D}(f + g) \leq \overline{D}(f + g) \leq \overline{D}f + \overline{D}g$$

holds at every point of (a, b) .

Both outer inequalities may be strict. First observe that for any real-valued φ and any $h > 0$, $Q_{-\varphi}(t) = -Q_\varphi(t)$, so $S_{-\varphi}(h) = -I_\varphi(h)$ and $I_{-\varphi}(h) = -S_\varphi(h)$; letting $h \rightarrow 0$,

$$\overline{D}(-\varphi) = -\underline{D}\varphi, \quad \underline{D}(-\varphi) = -\overline{D}\varphi.$$

Now take $(a, b) = (-1, 1)$, let f be the function of Problem 15, $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, let $g = -f$, and evaluate at $c = 0$. By Problem 15, $\overline{D}f(0) = 1$ and $\underline{D}f(0) = -1$, so by the displayed identities $\overline{D}g(0) = 1$ and $\underline{D}g(0) = -1$. Since $f + g \equiv 0$ we have $\underline{D}(f + g)(0) = \overline{D}(f + g)(0) = 0$. Thus the right-hand end of the chain is 2 and the left-hand end is -2 , while both middle terms are 0.

Problem 6.21. Let f be defined on $[a, b]$ and g a continuous function on $[\alpha, \beta]$ that is differentiable at $\gamma \in (\alpha, \beta)$ with $g(\gamma) = c \in (a, b)$. Verify the following.

(i) If $g'(\gamma) > 0$, then $\overline{D}(f \circ g)(\gamma) = \overline{D}f(c) \cdot g'(\gamma)$.

(ii) If $g'(\gamma) = 0$ and the upper and lower derivatives of f at c are finite, then $\overline{D}(f \circ g)(\gamma) = 0$.

Solution. Throughout put

$$\sigma(t) = g(\gamma + t) - g(\gamma) = g(\gamma + t) - c, \quad Q(\tau) = \frac{f(c + \tau) - f(c)}{\tau} \quad (\tau \neq 0, c + \tau \in [a, b]),$$

and, writing $r_0 = \min\{c - a, b - c\} > 0$ (positive because c is an interior point of $[a, b]$), put $S(r) = \sup_{0 < |\tau| \leq r} Q(\tau)$ for $0 < r \leq r_0$; every τ occurring here has $c + \tau \in [a, b]$, so $S(r)$ is well defined in $(-\infty, +\infty]$, S is nondecreasing in r , and $\overline{D}f(c) = \lim_{r \rightarrow 0+} S(r) = \inf_{0 < r \leq r_0} S(r)$. Since c is interior to $[a, b]$ and g is continuous at γ , there is $\delta_1 > 0$ with $(\gamma - \delta_1, \gamma + \delta_1) \subseteq (\alpha, \beta)$ and $g((\gamma - \delta_1, \gamma + \delta_1)) \subseteq (a, b)$; so $f \circ g$ is defined on $(\gamma - \delta_1, \gamma + \delta_1)$ and its upper derivative at γ makes sense. Note that σ is continuous, $\sigma(0) = 0$, and $\sigma(t)/t \rightarrow g'(\gamma)$ as $t \rightarrow 0$. Whenever $\sigma(t) \neq 0$,

$$\frac{(f \circ g)(\gamma + t) - (f \circ g)(\gamma)}{t} = \frac{f(c + \sigma(t)) - f(c)}{\sigma(t)} \cdot \frac{\sigma(t)}{t} = Q(\sigma(t)) \cdot \frac{\sigma(t)}{t}. \quad (\dagger)$$

We use twice the following elementary fact about upper limits.

Lemma. Let u and v be real-valued on a punctured neighbourhood of 0, with $v(t) \rightarrow m > 0$ as $t \rightarrow 0$, and let $L = \limsup_{t \rightarrow 0} u(t) \in [-\infty, +\infty]$. Then $\limsup_{t \rightarrow 0} u(t)v(t) = mL$.

Proof of the Lemma. Fix $\eta \in (0, m)$; for all small $t \neq 0$ we have $m - \eta < v(t) < m + \eta$, and in particular $v(t) > 0$.

We first show $\limsup_{t \rightarrow 0} u(t)v(t) \leq mL$. Suppose L is finite. For all small $t \neq 0$ we also have $u(t) < L + \eta$, and multiplying by $v(t) > 0$ gives

$$u(t)v(t) < (L + \eta)v(t) \leq \max\{(L + \eta)(m - \eta), (L + \eta)(m + \eta)\},$$

the last step because $(L + \eta)v(t)$ lies between the two displayed products according to the sign of $L + \eta$. Hence $\limsup_{t \rightarrow 0} uv \leq \max\{(L + \eta)(m - \eta), (L + \eta)(m + \eta)\}$, and both products tend to mL as $\eta \rightarrow 0$, so $\limsup_{t \rightarrow 0} uv \leq mL$. If $L = -\infty$, then for each $M > 0$ we have $u(t) < -M$ for all small $t \neq 0$, whence $u(t)v(t) < -M(m - \eta)$; as M is arbitrary and $m - \eta > 0$, $\limsup_{t \rightarrow 0} uv = -\infty = mL$. If $L = +\infty$ the inequality $\limsup_{t \rightarrow 0} uv \leq mL$ is vacuous.

For the reverse inequality choose $t_n \rightarrow 0$ with $t_n \neq 0$ and $u(t_n) \rightarrow L$. Since $v(t_n) \rightarrow m > 0$, we get $u(t_n)v(t_n) \rightarrow mL$, this also when $L = \pm\infty$. Hence $\limsup_{t \rightarrow 0} uv \geq mL$, and the Lemma is proved.

Part (i). Let $m = g'(\gamma) > 0$. Since $\sigma(t)/t \rightarrow m$, there is $\delta_2 \in (0, \delta_1]$ such that

$$\frac{m}{2} \leq \frac{\sigma(t)}{t} \leq 2m \quad \text{for } 0 < |t| \leq \delta_2,$$

and, shrinking δ_2 further, we may also require $2m\delta_2 \leq r_0$, so that $S(2mh)$ is defined for every $h \in (0, \delta_2]$. In particular $\sigma(t) \neq 0$ for $0 < |t| \leq \delta_2$, $\sigma(t)$ has the same sign as t , and $|\sigma(t)| \leq 2m|t|$. Thus $(\dagger)$ is available for all such t , and by the Lemma (with $u(t) = Q(\sigma(t))$ and $v(t) = \sigma(t)/t \rightarrow m > 0$),

$$\overline{D}(f \circ g)(\gamma) = \limsup_{t \rightarrow 0} Q(\sigma(t)) \cdot \frac{\sigma(t)}{t} = m \cdot \limsup_{t \rightarrow 0} Q(\sigma(t)).$$

It remains to prove that

$$\limsup_{t \rightarrow 0} Q(\sigma(t)) = \overline{D}f(c),$$

and for this we compare the suprema of the two families of difference quotients.

First, let $0 < h \leq \delta_2$. For $0 < |t| \leq h$ we have $0 < |\sigma(t)| \leq 2mh$, so $Q(\sigma(t))$ belongs to the family $\{Q(\tau) : 0 < |\tau| \leq 2mh\}$, and therefore

$$\sup_{0 < |t| \leq h} Q(\sigma(t)) \leq S(2mh).$$

Letting $h \rightarrow 0^+$ gives $\limsup_{t \rightarrow 0} Q(\sigma(t)) \leq \overline{D}f(c)$.
 Second, again let $0 < h \leq \delta_2$ and put

$$\rho(h) = \min\{\sigma(h), -\sigma(-h)\}.$$

Both entries are positive, because $\sigma(h) \geq (m/2)h > 0$ and $\sigma(-h) \leq (m/2)(-h) < 0$; also $\rho(h) \leq 2mh$, so $\rho(h) \rightarrow 0^+$ as $h \rightarrow 0^+$. The function σ is continuous on $[-h, h]$ with $\sigma(-h) < 0 < \sigma(h)$, so by the Intermediate Value Theorem it attains every value in $[\sigma(-h), \sigma(h)] \supseteq [-\rho(h), \rho(h)]$. Hence every τ with $0 < |\tau| \leq \rho(h)$ equals $\sigma(t)$ for some $t \in [-h, h]$, and such t is nonzero because $\sigma(0) = 0 \neq \tau$. Consequently

$$\sup_{0 < |t| \leq h} Q(\sigma(t)) \geq \sup_{0 < |\tau| \leq \rho(h)} Q(\tau) = S(\rho(h)).$$

Since $\rho(h) \rightarrow 0^+$ as $h \rightarrow 0^+$ and $S(r)$ has the limit $\overline{D}f(c)$ as $r \rightarrow 0^+$, letting $h \rightarrow 0^+$ gives $\limsup_{t \rightarrow 0} Q(\sigma(t)) \geq \overline{D}f(c)$.

Combining the two estimates, $\limsup_{t \rightarrow 0} Q(\sigma(t)) = \overline{D}f(c)$, and therefore

$$\overline{D}(f \circ g)(\gamma) = \overline{D}f(c) \cdot g'(\gamma),$$

this being an equality in the extended reals (both sides are simultaneously $+\infty$ or $-\infty$, since $g'(\gamma) > 0$). Part (ii). Now let $g'(\gamma) = 0$ and suppose $\overline{D}f(c)$ and $\underline{D}f(c)$ are finite. Since $S(r) \downarrow \overline{D}f(c)$ as $r \rightarrow 0^+$ and the limit is finite, there is $r_1 \in (0, r_0]$ with $S(r_1) \leq \overline{D}f(c) + 1 < \infty$; likewise, since the infima $\inf_{0 < |\tau| \leq r} Q(\tau)$ increase to the finite number $\underline{D}f(c)$ as r decreases to 0, there is $r_2 \in (0, r_0]$ with $\inf_{0 < |\tau| \leq r_2} Q(\tau) \geq \underline{D}f(c) - 1 > -\infty$. Put $\delta_0 = \min\{r_1, r_2\}$ and

$$M = \max\{|\overline{D}f(c)| + 1, |\underline{D}f(c)| + 1\},$$

so that

$$|Q(\tau)| \leq M \quad \text{for all } \tau \text{ with } 0 < |\tau| \leq \delta_0.$$

Let $\eta > 0$. Since g is continuous at γ , there is $\delta' \in (0, \delta_1]$ with $|\sigma(t)| \leq \delta_0$ for $|t| \leq \delta'$; and since $\sigma(t)/t \rightarrow g'(\gamma) = 0$, we may shrink δ' so that in addition

$$\frac{|\sigma(t)|}{|t|} < \frac{\eta}{M} \quad \text{for } 0 < |t| \leq \delta'.$$

Fix t with $0 < |t| \leq \delta'$. If $\sigma(t) = 0$ then $(f \circ g)(\gamma + t) = f(c) = (f \circ g)(\gamma)$ and the difference quotient of $f \circ g$ at γ vanishes. If $\sigma(t) \neq 0$ then, by $(\dagger)$,

$$\left| \frac{(f \circ g)(\gamma + t) - (f \circ g)(\gamma)}{t} \right| = |Q(\sigma(t))| \cdot \frac{|\sigma(t)|}{|t|} \leq M \cdot \frac{\eta}{M} = \eta.$$

In both cases the difference quotient of $f \circ g$ at γ has absolute value at most η for $0 < |t| \leq \delta'$. As $\eta > 0$ was arbitrary,

$$\lim_{t \rightarrow 0} \frac{(f \circ g)(\gamma + t) - (f \circ g)(\gamma)}{t} = 0,$$

so $f \circ g$ is in fact differentiable at γ with $(f \circ g)'(\gamma) = 0$; in particular

$$\overline{D}(f \circ g)(\gamma) = 0 = \underline{D}(f \circ g)(\gamma).$$

The finiteness hypothesis in (ii) cannot be dropped. Take $[a, b] = [\alpha, \beta] = [-1, 1]$, $\gamma = 0$, $g(t) = t^2$, and

$$f(s) = \begin{cases} \sqrt{s} & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } -1 \leq s < 0. \end{cases}$$

Then g is continuous on $[-1, 1]$ and differentiable at 0 with $g'(0) = 0$, and $c = g(0) = 0 \in (-1, 1)$. The difference quotients of f at 0 are $Q(\tau) = 1/\sqrt{\tau}$ for $\tau > 0$ and $Q(\tau) = 0$ for $\tau < 0$, so $\overline{D}f(0) = +\infty$ (while $\underline{D}f(0) = 0$). Here $(f \circ g)(t) = \sqrt{t^2} = |t|$, whose difference quotients at 0 are $|t|/t = \pm 1$, so

$$\overline{D}(f \circ g)(0) = 1 \neq 0.$$

Thus the conclusion of (ii) fails when the upper derivative of f at c is infinite: the unboundedness of Q near 0 is picked up exactly along the values $\tau = \sigma(t)$, at a rate that compensates for $\sigma(t)/t \rightarrow 0$.

6.4 Problems 6.22–6.28

Problem 6.22. Show that a strictly increasing function that is defined on an interval is measurable and then use this to show that a monotone function that is defined on an interval is measurable.

Solution. Throughout, I denotes an interval of real numbers (bounded or unbounded, open, closed or half-open) and a real-valued function f on I is measurable provided $\{x \in I : f(x) > c\}$ is a measurable set for every $c \in \mathbb{R}$; this is the criterion of Proposition 1 of Chapter 3. Every interval is a Borel set, hence measurable.

Step 1: a strictly increasing function on an interval is measurable.

Let f be strictly increasing on I and fix $c \in \mathbb{R}$. Put

$$E_c = \{x \in I : f(x) > c\}.$$

If $E_c = \emptyset$ it is measurable, so assume $E_c \neq \emptyset$ and set $\alpha = \inf E_c$, an element of $[-\infty, \infty)$.

By the definition of infimum, $E_c \subseteq [\alpha, \infty) \cap I$ (interpreting $[-\infty, \infty)$ as $\mathbb{R}$ when $\alpha = -\infty$).

Conversely, let $x \in I$ with $x > \alpha$. Since α is the infimum of E_c , there is a point $y \in E_c$ with $\alpha \leq y < x$. As f is strictly increasing and $y < x$, we have $f(x) > f(y) > c$, so $x \in E_c$. Thus

$$(\alpha, \infty) \cap I \subseteq E_c \subseteq [\alpha, \infty) \cap I.$$

Consequently E_c is either $(\alpha, \infty) \cap I$ or $((\alpha, \infty) \cap I) \cup \{\alpha\}$, and the latter can occur only when $\alpha \in I$, in which case it equals $[\alpha, \infty) \cap I$. In either case E_c is the intersection of two intervals and hence is an interval, therefore a Borel set and so measurable. Since $c \in \mathbb{R}$ was arbitrary, f is measurable on I .

Step 2: a monotone function on an interval is measurable.

First suppose f is increasing on I , that is, $f(u) \leq f(v)$ whenever $u, v \in I$ and $u < v$. Define

$$g(x) = f(x) + x \quad \text{for all } x \in I.$$

If $u < v$ are points of I , then $f(u) \leq f(v)$ and $u < v$, so $g(u) = f(u) + u < f(v) + v = g(v)$. Hence g is strictly increasing on I , and therefore measurable by Step 1. The identity function $x \mapsto x$ is continuous on I , hence measurable. By Theorem 6 of Chapter 3 (linear combinations of measurable real-valued functions are measurable),

$$f(x) = g(x) - x \quad \text{for all } x \in I$$

is measurable on I .

Now suppose f is decreasing on I . Then $-f$ is increasing on I , hence measurable by the preceding paragraph, and therefore $f = -(-f)$ is measurable.

Thus every monotone real-valued function defined on an interval is measurable.

Problem 6.23. Show that a continuous function f on $[a, b]$ is Lipschitz if its upper and lower derivatives are bounded on (a, b) .

Solution. Recall the definitions of Section 6.2: for x in (a, b) ,

$$\overline{D}f(x) = \lim_{h \rightarrow 0^+} \left[\sup_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{t} \right], \quad \underline{D}f(x) = \lim_{h \rightarrow 0^+} \left[\inf_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{t} \right],$$

the supremum and infimum being taken over those t with $x+t$ in $[a, b]$. By hypothesis there is a number $M \geq 0$ with

$$|\overline{D}f(x)| \leq M \quad \text{and} \quad |\underline{D}f(x)| \leq M \quad \text{for all } x \in (a, b).$$

In particular $-M \leq \underline{D}f(x)$ and $\overline{D}f(x) \leq M$ on (a, b) . We show $|f(u) - f(v)| \leq M|u - v|$ for all u, v in $[a, b]$, so that f is Lipschitz with Lipschitz constant M .

Step 1: a local estimate at each interior point.

Fix $\epsilon > 0$ and put $K = M + \epsilon$. Let $x \in (a, b)$. Since

$$\limsup_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t} = \overline{D}f(x) \leq M < K \quad \text{and} \quad \liminf_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t} = \underline{D}f(x) \geq -M > -K,$$

there is a $\delta_x > 0$ such that

$$-K < \frac{f(x+t) - f(x)}{t} < K \quad \text{whenever } 0 < |t| < \delta_x \text{ and } x+t \in [a, b].$$

The two inequalities together say $\left| \frac{f(x+t) - f(x)}{t} \right| < K$, and multiplying by $|t|$ (which is legitimate for t of either sign) gives $|f(x+t) - f(x)| < K|t|$. Writing $y = x+t$, and noting that the case $y = x$ is trivial, this says exactly

$$|f(y) - f(x)| \leq K|y - x| \quad \text{for all } y \in [a, b] \text{ with } |y - x| < \delta_x. \quad (*)$$

Step 2: from the local estimate to a global one on (a, b) .

Let u, w satisfy $a < u < w < b$. Put

$$S = \{v \in [u, w] : |f(v) - f(u)| \leq K(v - u)\}.$$

Then $u \in S$, so $S \neq \emptyset$, and S is closed in $[u, w]$ because f is continuous and both sides of the defining inequality are continuous in v . Let $s = \sup S$; since S is closed and bounded, $s \in S$.

Suppose $s < w$. Then $s \in [u, w] \subseteq (a, b)$, so $(*)$ holds at s with some $\delta_s > 0$. Choose v with $s < v < \min\{w, s + \delta_s\}$. Then, using $(*)$ at s and $s \in S$,

$$|f(v) - f(u)| \leq |f(v) - f(s)| + |f(s) - f(u)| \leq K(v - s) + K(s - u) = K(v - u),$$

so $v \in S$ and $v > s$, contradicting $s = \sup S$. Hence $s = w$, that is,

$$|f(w) - f(u)| \leq K(w - u) \quad \text{for all } a < u < w < b. \quad (**)$$

Step 3: including the endpoints, and letting ϵ tend to 0.

Let $u, v \in [a, b]$ with $u < v$. For each natural number n set

$$u_n = \max \left\{ u, a + \frac{1}{n} \right\}, \quad v_n = \min \left\{ v, b - \frac{1}{n} \right\}.$$

Then $a < u_n$ and $v_n < b$ for every n with $1/n < b - a$, while $u_n \rightarrow u$ and $v_n \rightarrow v$ (if $u > a$ then $u_n = u$ once $1/n < u - a$, and if $u = a$ then $u_n = a + 1/n$; similarly for v_n). Since $u < v$, we have $u_n < v_n$ for all sufficiently large n . For such n , $(**)$ applies and gives $|f(v_n) - f(u_n)| \leq K(v_n - u_n)$. Since f is continuous on $[a, b]$, letting $n \rightarrow \infty$ gives

$$|f(v) - f(u)| \leq K(v - u) = (M + \epsilon)(v - u).$$

This holds for every $\epsilon > 0$, so letting $\epsilon \rightarrow 0^+$ we obtain

$$|f(v) - f(u)| \leq M|v - u| \quad \text{for all } u, v \in [a, b].$$

Therefore f is Lipschitz on $[a, b]$, with Lipschitz constant M .

Problem 6.24. Show that for f defined in the last remark of this section, f' is not integrable over $[0, 1]$. [That remark reads: for a continuous function f on a closed, bounded interval $[a, b]$ that is differentiable on the open interval (a, b) , in the absence of a monotonicity assumption on f we cannot infer that its derivative f' is integrable over $[a, b]$. We leave it as an exercise to show that for

f defined on $[0, 1]$ by

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0, \end{cases}$$

f' is not integrable over $[0, 1]$.

Solution. The function is differentiable on all of $[0, 1]$.

For $0 < x \leq 1$ the chain and product rules give

$$f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right),$$

and this expression is continuous on $(0, 1]$. At $x = 0$, for $0 < x \leq 1$,

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \left| x \sin\left(\frac{1}{x^2}\right) \right| \leq x \rightarrow 0,$$

so $f'(0)$ exists and equals 0. Also $|f(x)| \leq x^2$ shows f is continuous at 0, so f is continuous on $[0, 1]$ and differentiable at every point of $[0, 1]$, as asserted in the remark. Since f' is continuous on $(0, 1]$ it is measurable there, and $\{0\}$ has measure zero, so f' is a measurable function on $[0, 1]$. Thus f' is integrable over $[0, 1]$ if and only if $\int_0^1 |f'| < \infty$.

A sequence of disjoint intervals on which f oscillates.

For each natural number n set

$$a_n = \left[(2n + \tfrac{1}{2})\pi\right]^{-1/2}, \quad b_n = \left[(2n - \tfrac{1}{2})\pi\right]^{-1/2}.$$

Since $2n - \frac{1}{2} < 2n + \frac{1}{2}$ we have $0 < a_n < b_n$, and $b_1 = (3\pi/2)^{-1/2} < 1$, so $b_n \leq b_1 < 1$ for all n and each $[a_n, b_n]$ is contained in $(0, 1)$. Moreover

$$b_{n+1} = \left[(2n + \tfrac{3}{2})\pi\right]^{-1/2} < \left[(2n + \tfrac{1}{2})\pi\right]^{-1/2} = a_n,$$

so the intervals $[a_n, b_n]$, $n = 1, 2, \dots$, are pairwise disjoint.

Because $1/a_n^2 = (2n + \frac{1}{2})\pi$ and $1/b_n^2 = (2n - \frac{1}{2})\pi$,

$$\sin\left(\frac{1}{a_n^2}\right) = \sin \frac{\pi}{2} = 1, \quad \sin\left(\frac{1}{b_n^2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1,$$

and therefore

$$f(a_n) = a_n^2 = \frac{1}{(2n + \frac{1}{2})\pi}, \quad f(b_n) = -b_n^2 = -\frac{1}{(2n - \frac{1}{2})\pi}.$$

Lower bound for the integral.

On the closed interval $[a_n, b_n] \subset (0, 1]$ the derivative f' is continuous, so its Riemann and Lebesgue integrals agree (by Theorem 3 of Chapter 4) and the Fundamental Theorem of Calculus applies:

$$\int_{a_n}^{b_n} f' = f(b_n) - f(a_n).$$

Hence

$$\int_{a_n}^{b_n} |f'| \geq \left| \int_{a_n}^{b_n} f' \right| = |f(b_n) - f(a_n)| = a_n^2 + b_n^2 = \frac{1}{(2n + \frac{1}{2})\pi} + \frac{1}{(2n - \frac{1}{2})\pi} > \frac{2}{(2n + 1)\pi}.$$

Now $|f'|$ is nonnegative and measurable on $[0, 1]$, and the intervals $[a_n, b_n]$ are disjoint measurable subsets of $[0, 1]$. By Theorem 11 of Chapter 4 (Additivity Over Domains of Integration for nonnegative measurable functions), applied $N - 1$ times, $\int_{A_N} |f'| = \sum_{n=1}^N \int_{a_n}^{b_n} |f'|$ where $A_N = \bigcup_{n=1}^N [a_n, b_n]$; and, again by Theorem 11 of Chapter 4 applied to the disjoint decomposition $[0, 1] = A_N \cup ([0, 1] \setminus A_N)$

together with the nonnegativity of $\int_{[0,1] \setminus A_N} |f'|$, we get $\int_0^1 |f'| \geq \int_{A_N} |f'|$. Hence, for every natural number N ,

$$\int_0^1 |f'| \geq \sum_{n=1}^N \int_{a_n}^{b_n} |f'| > \frac{2}{\pi} \sum_{n=1}^N \frac{1}{2n+1}.$$

The series $\sum_{n \geq 1} 1/(2n+1)$ diverges, being a positive multiple, term by term, of a tail of the harmonic series: $1/(2n+1) \geq 1/(3n)$ for $n \geq 1$. Letting $N \rightarrow \infty$,

$$\int_0^1 |f'| = \infty.$$

Therefore f' is not integrable over $[0, 1]$, even though f is continuous on $[0, 1]$ and differentiable at every point of $[0, 1]$. This is exactly the phenomenon the remark asserts: without a monotonicity hypothesis, differentiability at every point does not force the derivative to be integrable. Corollary 4 of this section, which does give the integrability of f' , is therefore not merely an artefact of its proof: its hypothesis that f be increasing cannot simply be dropped.

Problem 6.25. Suppose f is continuous on $[0, 1]$. Must there be a nondegenerate closed subinterval $[a, b]$ of $[0, 1]$ for which the restriction of f to $[a, b]$ is of bounded variation?

Solution. No. There are continuous functions on $[0, 1]$ whose restriction to every nondegenerate closed subinterval fails to be of bounded variation.

The example. As recorded in the Remark that follows the proof of Lebesgue's Theorem in Section 6.2, Weierstrass exhibited in 1872 a function that is continuous on an open interval and differentiable at no point of that interval; the footnote there points to the simpler example of Bartel van der Waerden, examined in Chapter 8 of Fitzpatrick's *Advanced Calculus*. Let w be such a function, continuous and nowhere differentiable on an open interval (α, β) , and fix $\alpha < \alpha' < \beta' < \beta$. Define

$$f(x) = w(\alpha' + (\beta' - \alpha')x) \quad \text{for } x \in [0, 1].$$

Then f is continuous on $[0, 1]$, being the composition of a continuous function with an affine map. Moreover f is differentiable at no point x_0 of $(0, 1)$. Indeed, write $c = \beta' - \alpha' > 0$ and $p = \alpha' + cx_0 \in (\alpha', \beta') \subseteq (\alpha, \beta)$. For $s \neq 0$ small enough that $x_0 + s \in (0, 1)$,

$$\frac{w(p + cs) - w(p)}{cs} = \frac{1}{c} \cdot \frac{f(x_0 + s) - f(x_0)}{s}.$$

As s ranges over the punctured neighbourhoods of 0, $t = cs$ ranges over the punctured neighbourhoods of 0 as well. So if the right-hand side had a limit as $s \rightarrow 0$, the difference quotients of w at p would have a limit as $t \rightarrow 0$, that is, w would be differentiable at p , which is false. Hence f is not differentiable at x_0 .

Why no subinterval works. Suppose, to get a contradiction, that $[a, b] \subseteq [0, 1]$ is nondegenerate, so $a < b$, and that the restriction $g = f|_{[a,b]}$ is of bounded variation on $[a, b]$. By Corollary 6 of this section, g is then differentiable almost everywhere on the open interval (a, b) . Since $m((a, b)) = b - a > 0$, the set of points of (a, b) at which g is differentiable is nonempty; choose such a point $x_0 \in (a, b)$.

Differentiability of the restriction g at the interior point x_0 is precisely differentiability of f at x_0 : the difference quotients

$$\frac{g(x_0 + t) - g(x_0)}{t} = \frac{f(x_0 + t) - f(x_0)}{t}$$

coincide for all $t \neq 0$ with $|t| < \min\{x_0 - a, b - x_0\}$, and the derivative is a limit as $t \rightarrow 0$, so only such t matter. Hence f is differentiable at $x_0 \in (0, 1)$, contradicting the fact that f is nowhere differentiable. Therefore no nondegenerate closed subinterval $[a, b]$ of $[0, 1]$ has $f|_{[a,b]}$ of bounded variation, and the answer to the question is negative.

Remark. Continuity by itself is therefore of no help here. What the argument uses is only Corollary 6: if f restricted to some nondegenerate $[a, b]$ were of bounded variation, then f would be differentiable

at almost every point of (a, b) , hence at some point, which a nowhere differentiable f forbids. Compare Problem 7 of Section 6.2, where the same kind of example answers the analogous question about monotonicity.

Problem 6.26. Let f be the Dirichlet function, the characteristic function of the rationals in $[0, 1]$. Is f of bounded variation on $[0, 1]$?

Solution. No: $TV(f) = \infty$, so f is not of bounded variation on $[0, 1]$.

Here $f(x) = 1$ if $x \in [0, 1]$ is rational and $f(x) = 0$ if $x \in [0, 1]$ is irrational.

Fix a natural number n . We construct a partition $P_n = \{x_0, x_1, \dots, x_{2n}\}$ of $[0, 1]$ whose points alternate between rational and irrational. For $0 \leq k \leq n$ put

$$x_{2k} = \frac{k}{n},$$

so that $x_0 = 0$, $x_{2n} = 1$, and every x_{2k} is rational. For $0 \leq k \leq n-1$, the open interval $(\frac{k}{n}, \frac{k+1}{n})$ is nonempty and the irrational numbers are dense in $\mathbb{R}$, so we may choose an irrational number

$$x_{2k+1} \in \left(\frac{k}{n}, \frac{k+1}{n}\right).$$

Then

$$0 = x_0 < x_1 < x_2 < \dots < x_{2n-1} < x_{2n} = 1$$

is a genuine partition of $[0, 1]$, with $f(x_i) = 1$ for even i and $f(x_i) = 0$ for odd i .

Consequently every one of the $2n$ consecutive differences has absolute value 1:

$$V(f, P_n) = \sum_{i=1}^{2n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{2n} 1 = 2n.$$

By the definition of total variation as the supremum of $V(f, P)$ over all partitions P of $[0, 1]$,

$$TV(f) \geq V(f, P_n) = 2n \quad \text{for every natural number } n,$$

and hence $TV(f) = \infty$. By the definition of bounded variation, f is not of bounded variation on $[0, 1]$.

Remark. Two independent confirmations. First, Corollary 6 of this section: were f of bounded variation on $[0, 1]$, it would be differentiable at almost every point of $(0, 1)$; but the density of both the rationals and the irrationals makes f discontinuous at every point of $[0, 1]$, hence differentiable nowhere, and $(0, 1)$ does not have measure zero. Second, Jordan's Theorem: an increasing function on $[0, 1]$ has at most countably many points of discontinuity, since the open intervals $(\lim_{x \rightarrow c^-} g(x), \lim_{x \rightarrow c^+} g(x))$ associated with distinct discontinuity points c are nonempty and pairwise disjoint, and each contains a rational. So the difference of two increasing functions is continuous off a countable set, whereas f is continuous nowhere.

Problem 6.27. Define $f(x) = \sin x$ on $[0, 2\pi]$. Find two increasing functions h and g for which $f = h - g$ on $[0, 2\pi]$.

Solution. A first, simple answer. Define, for $x \in [0, 2\pi]$,

$$h(x) = x + \sin x, \quad g(x) = x.$$

Then $h - g = \sin = f$ on $[0, 2\pi]$. The function g is clearly increasing. As for h , it is differentiable on $[0, 2\pi]$ with

$$h'(x) = 1 + \cos x \geq 0 \quad \text{for all } x \in [0, 2\pi],$$

since $\cos x \geq -1$. A function that is continuous on a closed, bounded interval and has nonnegative

derivative there is increasing: for $u < v$ the Mean Value Theorem provides $\xi \in (u, v)$ with $h(v) - h(u) = h'(\xi)(v - u) \geq 0$. Hence h is increasing, and $f = h - g$ is a decomposition of the required kind.

The canonical Jordan decomposition. It is instructive to write down the decomposition supplied by Lemma 5, namely

$$f(x) = [f(x) + TV(f_{[0,x]})] - TV(f_{[0,x]}) \quad \text{for all } x \in [0, 2\pi].$$

Write $V(x) = TV(f_{[0,x]})$ for the total variation function. Since $\sin$ is increasing on $[0, \pi/2]$, decreasing on $[\pi/2, 3\pi/2]$ and increasing on $[3\pi/2, 2\pi]$, and since the total variation of a monotone function on an interval is the absolute value of the difference of its endpoint values (the first Example of this section), additivity (20) of the total variation over adjacent intervals gives

$$V(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi/2, \\ 2 - \sin x & \text{if } \pi/2 \leq x \leq 3\pi/2, \\ 4 + \sin x & \text{if } 3\pi/2 \leq x \leq 2\pi. \end{cases}$$

Indeed $V(\pi/2) = 1$, $V(3\pi/2) = 1 + 2 = 3$ and $V(2\pi) = 3 + 1 = 4$, and the three formulas agree at the junction points $\pi/2$ and $3\pi/2$. Each piece is increasing on its subinterval, so V is increasing on $[0, 2\pi]$. Therefore, taking $g = V$ and $h = f + V$,

$$g(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi/2, \\ 2 - \sin x & \text{if } \pi/2 \leq x \leq 3\pi/2, \\ 4 + \sin x & \text{if } 3\pi/2 \leq x \leq 2\pi, \end{cases} \quad h(x) = \begin{cases} 2 \sin x & \text{if } 0 \leq x \leq \pi/2, \\ 2 & \text{if } \pi/2 \leq x \leq 3\pi/2, \\ 4 + 2 \sin x & \text{if } 3\pi/2 \leq x \leq 2\pi. \end{cases}$$

Both are increasing: g rises from 0 to 1 on $[0, \pi/2]$, from 1 to 3 on $[\pi/2, 3\pi/2]$ (as $\sin$ decreases there), and from 3 to 4 on $[3\pi/2, 2\pi]$; while h rises from 0 to 2 on $[0, \pi/2]$, is constant equal to 2 on $[\pi/2, 3\pi/2]$, and rises from 2 to 4 on $[3\pi/2, 2\pi]$. The values match at $\pi/2$ and $3\pi/2$, so both functions are well defined and increasing on all of $[0, 2\pi]$. Finally, on each of the three subintervals,

$$h - g = 2 \sin x - \sin x = \sin x, \quad 2 - (2 - \sin x) = \sin x, \quad (4 + 2 \sin x) - (4 + \sin x) = \sin x,$$

so $f = h - g$ on $[0, 2\pi]$, as required.

Problem 6.28. Let f be a step function on $[a, b]$. Find a formula for its total variation.

Solution. Set-up and notation. Recall that f is a step function on $[a, b]$ provided there is a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ such that f is constant on each open interval (x_{i-1}, x_i) . Write

$$c_i = \text{the value of } f \text{ on } (x_{i-1}, x_i) \quad (1 \leq i \leq n), \quad d_i = f(x_i) \quad (0 \leq i \leq n).$$

The values d_i at the partition points are unconstrained; they need not agree with either neighbouring constant. We claim

$$TV(f) = \sum_{i=1}^n (|c_i - d_{i-1}| + |d_i - c_i|). \quad (\dagger)$$

Written out, this is the sum of the absolute jumps encountered as one runs through the values

$$d_0, c_1, d_1, c_2, d_2, \dots, c_n, d_n$$

in order. In particular $TV(f) < \infty$, so every step function is of bounded variation.

The lower bound. For each i with $1 \leq i \leq n$ pick a point $y_i \in (x_{i-1}, x_i)$; this is possible since $x_{i-1} < x_i$. Let

$$P^* = \{x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n\},$$

a partition of $[a, b]$. Since $f(y_i) = c_i$ and $f(x_i) = d_i$,

$$V(f, P^*) = \sum_{i=1}^n \left(|f(y_i) - f(x_{i-1})| + |f(x_i) - f(y_i)| \right) = \sum_{i=1}^n \left(|c_i - d_{i-1}| + |d_i - c_i| \right).$$

Since $TV(f)$ is the supremum of $V(f, P)$ over all partitions P of $[a, b]$, the right-hand side of (†) is a lower bound for $TV(f)$.

The upper bound. Let P be any partition of $[a, b]$ and let P' be the refinement of P obtained by adjoining to it all the points $x_0, x_1, \dots, x_n$. By the observation made in this section just before formula (19) – if c belongs to (a, b) , P is a partition of $[a, b]$ and P' is the refinement of P obtained by adjoining c to P , then $V(f, P) \leq V(f, P')$, by the triangle inequality – adjoining a point to a partition does not decrease the variation, so, adjoining the missing x_i one at a time,

$$V(f, P) \leq V(f, P').$$

It therefore suffices to bound $V(f, P')$ for an arbitrary partition P' that contains every x_i . Fix such a P' and fix i with $1 \leq i \leq n$. The points of P' lying in $[x_{i-1}, x_i]$ form a finite set

$$x_{i-1} = s_0 < s_1 < \dots < s_k = x_i,$$

and each of $s_1, \dots, s_{k-1}$ lies in the open interval (x_{i-1}, x_i) , so $f(s_j) = c_i$ for $1 \leq j \leq k-1$. There are two cases.

If $k \geq 2$, then

$$\sum_{j=1}^k |f(s_j) - f(s_{j-1})| = |c_i - d_{i-1}| + \underbrace{0 + \dots + 0}_{k-2 \text{ terms}} + |d_i - c_i| = |c_i - d_{i-1}| + |d_i - c_i|.$$

If $k = 1$, then, by the triangle inequality,

$$\sum_{j=1}^k |f(s_j) - f(s_{j-1})| = |d_i - d_{i-1}| \leq |d_i - c_i| + |c_i - d_{i-1}|.$$

In either case the contribution to $V(f, P')$ of the consecutive pairs of P' lying inside $[x_{i-1}, x_i]$ is at most $|c_i - d_{i-1}| + |d_i - c_i|$.

Every pair of consecutive points of P' lies inside exactly one of the intervals $[x_{i-1}, x_i]$, because P' contains all the x_i . Summing over i ,

$$V(f, P') \leq \sum_{i=1}^n \left(|c_i - d_{i-1}| + |d_i - c_i| \right).$$

Hence the same bound holds for $V(f, P)$ for every partition P , and so the right-hand side of (†) is an upper bound for $TV(f)$.

Combining the two bounds gives (†).

Two checks. Suppose the values at the partition points are chosen so that $d_0 = c_1$ and $d_i = c_i$ for $1 \leq i \leq n$, that is, f is left-continuous at each x_i . Then $|d_i - c_i| = 0$ for every i , and $|c_i - d_{i-1}| = |c_i - c_{i-1}|$ for $i \geq 2$ while $|c_1 - d_0| = 0$, so (†) reduces to $\sum_{i=2}^n |c_i - c_{i-1}|$, the sum of the absolute jumps of f , which is the expected answer. And for $f = \chi_{\{x_1\}}$ on $[a, b]$ with $a < x_1 < b$ we get $c_1 = c_2 = 0$, $d_0 = d_2 = 0$, $d_1 = 1$, so (†) gives $TV(f) = (0 + 1) + (1 + 0) = 2$, as it must be: f rises by 1 and falls by 1.

6.5 Problems 6.29–6.35

Problem 6.29. (a) Define

$$f(x) = \begin{cases} x^2 \cos(1/x^2) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Is f of bounded variation on $[-1, 1]$?

(b) Define

$$g(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Is g of bounded variation on $[-1, 1]$?

Solution. (a) **No: f is not of bounded variation on $[-1, 1]$.**

Observe first that f is continuous on $[-1, 1]$: away from 0 it is a composition and product of continuous functions, and $|f(x)| \leq x^2$ for all x , so $f(x) \rightarrow 0 = f(0)$ as $x \rightarrow 0$. Thus f is a perfectly respectable continuous function; nonetheless its total variation is infinite.

For each natural number n put

$$x_n = \frac{1}{\sqrt{n\pi}},$$

so that $1/x_n^2 = n\pi$ and hence

$$f(x_n) = \frac{1}{n\pi} \cos(n\pi) = \frac{(-1)^n}{n\pi}.$$

The sequence $\{x_n\}$ is strictly decreasing to 0, and $x_1 = 1/\sqrt{\pi} < 1$ since $\pi > 1$. Therefore, for each natural number $N \geq 2$,

$$P_N = \{-1, 0, x_N, x_{N-1}, \dots, x_2, x_1, 1\}$$

is a genuine partition of $[-1, 1]$: its points are listed in strictly increasing order.

Since every term in the sum defining $V(f, P_N)$ is nonnegative, discarding all the terms except those coming from consecutive pairs $x_{n+1} < x_n$ gives

$$V(f, P_N) \geq \sum_{n=1}^{N-1} |f(x_n) - f(x_{n+1})|.$$

Because $f(x_n)$ and $f(x_{n+1})$ have opposite signs, $|f(x_n) - f(x_{n+1})| = |f(x_n)| + |f(x_{n+1})|$, and therefore

$$V(f, P_N) \geq \sum_{n=1}^{N-1} \left[\frac{1}{n\pi} + \frac{1}{(n+1)\pi} \right] \geq \frac{1}{\pi} \sum_{n=1}^{N-1} \frac{1}{n}.$$

The harmonic series diverges, so the right-hand side tends to ∞ as $N \rightarrow \infty$. Hence the set of variations of f with respect to partitions of $[-1, 1]$ is unbounded above, that is,

$$TV(f) = \sup\{V(f, P) : P \text{ a partition of } [-1, 1]\} = \infty,$$

and f is not of bounded variation on $[-1, 1]$.

(b) **Yes: g is of bounded variation on $[-1, 1]$.**

We show g is Lipschitz on $[-1, 1]$ and then invoke the observation made in Section 6.3 immediately after the definition of total variation: if h is Lipschitz on $[a, b]$ with Lipschitz constant c , then for every partition $P = \{x_0, \dots, x_k\}$ of $[a, b]$,

$$V(h, P) = \sum_{i=1}^k |h(x_i) - h(x_{i-1})| \leq c \sum_{i=1}^k (x_i - x_{i-1}) = c(b - a),$$

so $c(b - a)$ bounds the set of all variations of h and hence $TV(h) \leq c(b - a) < \infty$.

For $x \neq 0$ the product and chain rules give

$$g'(x) = 2x \cos(1/x) + x^2 \cdot \sin(1/x) \cdot \frac{1}{x^2} = 2x \cos(1/x) + \sin(1/x).$$

At $x = 0$ the difference quotient is

$$\frac{g(h) - g(0)}{h} = \frac{h^2 \cos(1/h)}{h} = h \cos(1/h),$$

whose absolute value is at most $|h|$; hence it tends to 0 and $g'(0) = 0$. So g is differentiable at every point of $[-1, 1]$, and for $x \neq 0$ in $[-1, 1]$,

$$|g'(x)| \leq 2|x| |\cos(1/x)| + |\sin(1/x)| \leq 2 + 1 = 3,$$

while $|g'(0)| = 0 \leq 3$. Thus $|g'| \leq 3$ on all of $[-1, 1]$.

Now let $u < v$ belong to $[-1, 1]$. The function g is continuous on $[u, v]$ and differentiable on (u, v) , so by the Mean Value Theorem there is a point $c \in (u, v)$ with

$$g(v) - g(u) = g'(c)(v - u),$$

whence $|g(v) - g(u)| \leq 3|v - u|$. Therefore g is Lipschitz on $[-1, 1]$ with constant 3, and consequently

$$TV(g) \leq 3 \cdot (1 - (-1)) = 6 < \infty.$$

So g is of bounded variation on $[-1, 1]$.

The contrast between (a) and (b) is exactly the competition between the amplitude x^2 and the frequency of oscillation: in (b) the oscillation $\cos(1/x)$ is slow enough that the amplitude x^2 damps it out, while in (a) the far faster oscillation $\cos(1/x^2)$ forces the graph up and down across a total height comparable to $\sum 1/n$.

Problem 6.30. Show that the linear combination of two functions of bounded variation is also of bounded variation. Is the product of two such functions also of bounded variation?

Solution. Throughout, f and g are real-valued functions of bounded variation on the closed, bounded interval $[a, b]$, and $P = \{x_0, \dots, x_k\}$ denotes an arbitrary partition of $[a, b]$.

Linear combinations. Let α, β be real numbers and set $h = \alpha f + \beta g$. For any partition P ,

$$\begin{aligned} V(h, P) &= \sum_{i=1}^k |\alpha[f(x_i) - f(x_{i-1})] + \beta[g(x_i) - g(x_{i-1})]| \\ &\leq |\alpha| \sum_{i=1}^k |f(x_i) - f(x_{i-1})| + |\beta| \sum_{i=1}^k |g(x_i) - g(x_{i-1})| \\ &= |\alpha| V(f, P) + |\beta| V(g, P) \\ &\leq |\alpha| TV(f) + |\beta| TV(g), \end{aligned}$$

the first inequality being the triangle inequality applied termwise, and the last following from the definition of TV as a supremum over partitions. Thus the number $|\alpha| TV(f) + |\beta| TV(g)$, which is finite because f and g are of bounded variation, is an upper bound for the set of variations of h with respect to partitions of $[a, b]$. Taking the supremum over all partitions P ,

$$TV(\alpha f + \beta g) \leq |\alpha| TV(f) + |\beta| TV(g) < \infty,$$

so $\alpha f + \beta g$ is of bounded variation on $[a, b]$.

Products. Yes, the product of two functions of bounded variation on $[a, b]$ is again of bounded variation on $[a, b]$.

First observe that a function of bounded variation on $[a, b]$ is bounded. Indeed, if x belongs to (a, b) , then $\{a, x, b\}$ is a partition of $[a, b]$ and therefore

$$|f(x) - f(a)| \leq |f(x) - f(a)| + |f(b) - f(x)| = V(f, \{a, x, b\}) \leq TV(f);$$

this inequality also holds trivially at $x = a$, and at $x = b$ it follows from the partition $\{a, b\}$. Hence

$$|f(x)| \leq |f(a)| + TV(f) =: M \quad \text{for all } x \in [a, b],$$

and $M < \infty$. Let N be the analogous bound for g .

Now fix a partition $P = \{x_0, \dots, x_k\}$. For each index i , add and subtract $f(x_i)g(x_{i-1})$:

$$\begin{aligned} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| &= |f(x_i)[g(x_i) - g(x_{i-1})] + g(x_{i-1})[f(x_i) - f(x_{i-1})]| \\ &\leq M |g(x_i) - g(x_{i-1})| + N |f(x_i) - f(x_{i-1})|. \end{aligned}$$

Summing over i from 1 to k gives

$$V(fg, P) \leq M V(g, P) + N V(f, P) \leq M \cdot TV(g) + N \cdot TV(f).$$

The right-hand side is a finite constant independent of P , so taking the supremum over all partitions of $[a, b]$ yields

$$TV(fg) \leq M \cdot TV(g) + N \cdot TV(f) < \infty.$$

Therefore fg is of bounded variation on $[a, b]$.

Two remarks. First, both conclusions used only that the interval $[a, b]$ is closed and bounded; boundedness of f and g , which the product argument needs, was not assumed but deduced from bounded variation. Second, the estimates show that the functions of bounded variation on $[a, b]$ form an algebra of bounded functions.

Problem 6.31. Let P be a partition of $[a, b]$ that is a refinement of the partition P' . For a real-valued function f on $[a, b]$, show that $V(f, P') \leq V(f, P)$.

Solution. Recall that P refines P' means that $P' \subseteq P$ as sets of points of $[a, b]$ (both containing a and b). Since P and P' are finite sets, P is obtained from P' by adjoining finitely many points, say $m = \#(P \setminus P') \geq 0$ of them. We argue by induction on m .

If $m = 0$ then $P = P'$ and there is nothing to prove.

Consider first the case $m = 1$, that is, P is obtained from $P' = \{x_0, x_1, \dots, x_k\}$ by adjoining a single point c not in P' . Since $a = x_0 \leq c \leq b = x_k$ and c differs from every x_i , there is exactly one index j with $1 \leq j \leq k$ for which

$$x_{j-1} < c < x_j.$$

Thus $P = \{x_0, \dots, x_{j-1}, c, x_j, \dots, x_k\}$. The two variations $V(f, P')$ and $V(f, P)$ have all their terms in common except that the single term $|f(x_j) - f(x_{j-1})|$ of $V(f, P')$ is replaced in $V(f, P)$ by the two terms $|f(c) - f(x_{j-1})|$ and $|f(x_j) - f(c)|$. By the triangle inequality,

$$|f(x_j) - f(x_{j-1})| = |[f(x_j) - f(c)] + [f(c) - f(x_{j-1})]| \leq |f(x_j) - f(c)| + |f(c) - f(x_{j-1})|.$$

Adding to both sides the common terms $\sum_{i \neq j} |f(x_i) - f(x_{i-1})|$ gives

$$V(f, P') \leq V(f, P).$$

Now suppose the assertion holds whenever the refinement adjoins m points, and let P be obtained from P' by adjoining $m + 1$ points $c_1, \dots, c_{m+1}$. Put $Q = P' \cup \{c_1, \dots, c_m\}$. Then Q is a refinement of P' adjoining m points, so by the inductive hypothesis $V(f, P') \leq V(f, Q)$; and $P = Q \cup \{c_{m+1}\}$ is a refinement of Q adjoining a single point, so by the case $m = 1$ already established, $V(f, Q) \leq V(f, P)$. Combining,

$$V(f, P') \leq V(f, Q) \leq V(f, P).$$

By induction, $V(f, P') \leq V(f, P)$ whenever P is a refinement of P' , as required.

Note that no hypothesis whatever is needed on f : the argument uses only the triangle inequality in $\mathbb{R}$. In particular the inequality holds even when $TV(f) = \infty$.

Problem 6.32. Assume f is of bounded variation on $[a, b]$. Show that there is a sequence of partitions $\{P_n\}$ of $[a, b]$ for which the sequence $\{TV(f, P_n)\}$ is increasing and converges to $TV(f)$.

Solution. Here $TV(f, P_n)$ denotes the variation $V(f, P_n)$ of f with respect to the partition P_n , as defined in this section; we write $V(f, P)$ for it below. “Increasing” is meant in the weak sense, $V(f, P_n) \leq V(f, P_{n+1})$ for all n .

Since f is of bounded variation, the number

$$TV(f) = \sup\{V(f, P) : P \text{ a partition of } [a, b]\}$$

is finite. By the definition of the supremum, for each natural number n there is a partition Q_n of $[a, b]$ with

$$TV(f) - \frac{1}{n} < V(f, Q_n) \leq TV(f).$$

Call this inequality (*).

Define

$$P_n = Q_1 \cup Q_2 \cup \cdots \cup Q_n \quad (n = 1, 2, \dots),$$

the union being taken as sets of points of $[a, b]$. Each Q_j is a finite subset of $[a, b]$ containing a and b , so P_n is a finite subset of $[a, b]$ containing a and b ; listing its points in increasing order exhibits P_n as a partition of $[a, b]$.

Two containments are immediate from the definition of P_n :

$$P_n \subseteq P_{n+1} \quad \text{and} \quad Q_n \subseteq P_n.$$

The first says P_{n+1} is a refinement of P_n , so by the preceding problem

$$V(f, P_n) \leq V(f, P_{n+1}) \quad \text{for every } n,$$

that is, the sequence $\{V(f, P_n)\}$ is increasing. The second says P_n is a refinement of Q_n , so again by the preceding problem, together with (*),

$$V(f, P_n) \geq V(f, Q_n) > TV(f) - \frac{1}{n}.$$

On the other hand $V(f, P_n) \leq TV(f)$ for every n , since $TV(f)$ is the supremum of the variations of f over all partitions. Hence

$$TV(f) - \frac{1}{n} < V(f, P_n) \leq TV(f) \quad \text{for every } n,$$

and therefore $|V(f, P_n) - TV(f)| < 1/n$ for every n . Since $1/n \rightarrow 0$, we conclude that

$$\lim_{n \rightarrow \infty} V(f, P_n) = TV(f).$$

Thus $\{P_n\}$ is a sequence of partitions of $[a, b]$ for which $\{V(f, P_n)\}$ is increasing and converges to $TV(f)$.

Problem 6.33. Let $\{f_n\}$ be a sequence of real-valued functions on $[a, b]$ that converges pointwise on $[a, b]$ to the real-valued function f . Show that

$$TV(f) \leq \liminf TV(f_n).$$

Solution. The inequality is to be read in the extended reals: no boundedness of variation is assumed of the f_n , and if $\liminf_n TV(f_n) = \infty$ there is nothing to prove. So the content of the assertion is that if the f_n have variations that do not blow up along some subsequence, then neither does the variation of the limit.

Fix an arbitrary partition $P = \{x_0, x_1, \dots, x_k\}$ of $[a, b]$. The key point is that $V(f, P)$ involves only the finitely many values $f(x_0), \dots, f(x_k)$, and at each of these finitely many points the numerical sequence $\{f_n(x_i)\}$ converges to $f(x_i)$ by hypothesis.

Consequently, for each index i with $1 \leq i \leq k$,

$$\lim_{n \rightarrow \infty} |f_n(x_i) - f_n(x_{i-1})| = |f(x_i) - f(x_{i-1})|,$$

by the algebra of limits of real sequences together with the continuity of the absolute value function. Since a sum of finitely many convergent sequences converges to the sum of the limits,

$$\lim_{n \rightarrow \infty} V(f_n, P) = \lim_{n \rightarrow \infty} \sum_{i=1}^k |f_n(x_i) - f_n(x_{i-1})| = \sum_{i=1}^k |f(x_i) - f(x_{i-1})| = V(f, P).$$

On the other hand, by the definition of total variation as a supremum over partitions,

$$V(f_n, P) \leq TV(f_n) \quad \text{for every } n.$$

Since the sequence $\{V(f_n, P)\}$ converges, its limit equals its limit inferior, and the limit inferior is monotone with respect to termwise inequality of sequences. Hence

$$V(f, P) = \lim_{n \rightarrow \infty} V(f_n, P) = \liminf_{n \rightarrow \infty} V(f_n, P) \leq \liminf_{n \rightarrow \infty} TV(f_n).$$

The number $\liminf_n TV(f_n)$ therefore is an upper bound for $V(f, P)$, and this holds for every partition P of $[a, b]$, since P was arbitrary. Taking the supremum over all such P ,

$$TV(f) = \sup_P V(f, P) \leq \liminf_{n \rightarrow \infty} TV(f_n),$$

which is the desired inequality.

In particular, if there is a constant M with $TV(f_n) \leq M$ for all n , then $TV(f) \leq M$ and the pointwise limit f is of bounded variation. The inequality may be strict, as a moving spike shows. On $[0, 1]$ let f_n be the piecewise linear function that vanishes outside the interval $(1/(n+1), 1/n)$ and rises to the value 1 at the midpoint of that interval. Then $TV(f_n) = 2$ for every n . Yet $f_n \rightarrow 0$ pointwise on $[0, 1]$: we have $f_n(0) = 0$ for all n , and if $0 < x \leq 1$ then $1/n < x$ for all n large, so $f_n(x) = 0$ for all such n . Here $TV(f) = TV(0) = 0 < 2 = \liminf_n TV(f_n)$.

Problem 6.34. Let f and g be of bounded variation on $[a, b]$. Show that

$$TV(f + g) \leq TV(f) + TV(g) \quad \text{and} \quad TV(\alpha f) = |\alpha| TV(f).$$

Solution. Let $P = \{x_0, x_1, \dots, x_k\}$ be an arbitrary partition of $[a, b]$.

Subadditivity. By the triangle inequality in $\mathbb{R}$, applied to each term,

$$\begin{aligned} V(f+g, P) &= \sum_{i=1}^k |[f(x_i) + g(x_i)] - [f(x_{i-1}) + g(x_{i-1})]| \\ &= \sum_{i=1}^k |[f(x_i) - f(x_{i-1})] + [g(x_i) - g(x_{i-1})]| \\ &\leq \sum_{i=1}^k |f(x_i) - f(x_{i-1})| + \sum_{i=1}^k |g(x_i) - g(x_{i-1})| \\ &= V(f, P) + V(g, P). \end{aligned}$$

Since $V(f, P) \leq TV(f)$ and $V(g, P) \leq TV(g)$ by the definition of total variation, we obtain

$$V(f+g, P) \leq TV(f) + TV(g).$$

The right-hand side is a finite number independent of P ; as P was an arbitrary partition of $[a, b]$, it is an upper bound for the set of variations of $f+g$ with respect to partitions of $[a, b]$. Taking the supremum over all partitions,

$$TV(f+g) \leq TV(f) + TV(g).$$

In particular $f+g$ is of bounded variation on $[a, b]$.

Positive homogeneity. Let α be a real number. For the partition P above,

$$V(\alpha f, P) = \sum_{i=1}^k |\alpha f(x_i) - \alpha f(x_{i-1})| = \sum_{i=1}^k |\alpha| |f(x_i) - f(x_{i-1})| = |\alpha| V(f, P).$$

Call this identity (*); it holds for every partition P of $[a, b]$.

If $\alpha = 0$, then αf is identically zero, so $V(\alpha f, P) = 0$ for every P and hence $TV(\alpha f) = 0 = |\alpha| TV(f)$. Suppose $\alpha \neq 0$. Let $S = \{V(f, P) : P \text{ a partition of } [a, b]\}$, so that $TV(f) = \sup S$. By (*), the set of variations of αf with respect to partitions of $[a, b]$ is exactly

$$\{V(\alpha f, P) : P\} = \{|\alpha| s : s \in S\} = |\alpha| \cdot S.$$

For a nonempty set S of real numbers and a positive constant $c = |\alpha|$, one has $\sup(c \cdot S) = c \sup S$: indeed $s \leq \sup S$ gives $cs \leq c \sup S$ for all $s \in S$, so $\sup(c \cdot S) \leq c \sup S$; and applying this same inequality to the set $c \cdot S$ with constant $1/c$ gives $\sup S = \sup((1/c)(c \cdot S)) \leq (1/c) \sup(c \cdot S)$, that is, $c \sup S \leq \sup(c \cdot S)$. Therefore

$$TV(\alpha f) = \sup\{V(\alpha f, P) : P\} = |\alpha| \sup\{V(f, P) : P\} = |\alpha| TV(f).$$

Both computations show, incidentally, that the functions of bounded variation on $[a, b]$ form a linear space on which TV is a seminorm; it fails to be a norm only because $TV(f) = 0$ exactly when f is constant.

Problem 6.35. For α and β positive numbers, define the function f on $[0, 1]$ by

$$f(x) = \begin{cases} x^\alpha \sin(1/x^\beta) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0. \end{cases}$$

Show that if $\alpha > \beta$, then f is of bounded variation on $[0, 1]$, by showing that f' is integrable over $[0, 1]$. Then show that if $\alpha \leq \beta$, then f is not of bounded variation on $[0, 1]$.

Solution. Note at the outset that f is continuous on $[0, 1]$. On $(0, 1]$ it is built from continuous functions by composition and multiplication, and since $|f(x)| \leq x^\alpha$ for $0 < x \leq 1$ with $\alpha > 0$, we have

$f(x) \rightarrow 0 = f(0)$ as $x \rightarrow 0^+$.

The case $\alpha > \beta$. On the open set $(0, 1)$ the product and chain rules give, for $0 < x < 1$,

$$f'(x) = \alpha x^{\alpha-1} \sin(1/x^\beta) + x^\alpha \cos(1/x^\beta) \cdot (-\beta x^{-\beta-1}) = \alpha x^{\alpha-1} \sin(1/x^\beta) - \beta x^{\alpha-\beta-1} \cos(1/x^\beta),$$

and this formula persists at $x = 1$ (one-sided). Thus f' exists at every point of $(0, 1]$ and is continuous there, hence measurable; f' is therefore defined except possibly at the single point $x = 0$, a set of measure zero, which is all that is needed to speak of its integral over $[0, 1]$.

Since $|\sin| \leq 1$ and $|\cos| \leq 1$,

$$|f'(x)| \leq \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} \quad \text{for } 0 < x \leq 1.$$

Both bounding functions are nonnegative and continuous on $(0, 1]$, and for $0 < \epsilon < 1$ elementary calculus gives

$$\int_\epsilon^1 \alpha x^{\alpha-1} dx = 1 - \epsilon^\alpha \leq 1, \quad \int_\epsilon^1 \beta x^{\alpha-\beta-1} dx = \frac{\beta}{\alpha - \beta} (1 - \epsilon^{\alpha-\beta}) \leq \frac{\beta}{\alpha - \beta},$$

where the second computation uses $\alpha - \beta > 0$, so that the exponent $\alpha - \beta - 1$ exceeds -1 and the antiderivative $x^{\alpha-\beta}/(\alpha - \beta)$ is legitimate and tends to 0 as $\epsilon \rightarrow 0^+$. On each interval $[1/n, 1]$ the function $|f'|$ is continuous, hence its Riemann and Lebesgue integrals agree, and monotonicity of the integral applied to the pointwise bound displayed above gives

$$\int_{1/n}^1 |f'| \leq \int_{1/n}^1 \alpha x^{\alpha-1} dx + \int_{1/n}^1 \beta x^{\alpha-\beta-1} dx \leq 1 + \frac{\beta}{\alpha - \beta} \quad \text{for every } n.$$

The sequence $\{|f'| \cdot \chi_{[1/n, 1]}\}$ consists of nonnegative measurable functions and increases pointwise on $(0, 1]$ to $|f'|$, so the Monotone Convergence Theorem (Chapter 4) yields

$$\int_0^1 |f'| = \lim_{n \rightarrow \infty} \int_{1/n}^1 |f'| \leq 1 + \frac{\beta}{\alpha - \beta} < \infty.$$

Since $|f'|$ has finite integral and f' is measurable, f' is integrable over $[0, 1]$.

We now deduce bounded variation. Let $P = \{t_0, t_1, \dots, t_k\}$ be a partition of $[0, 1]$, so $0 = t_0 < t_1 < \dots < t_k = 1$. For each index $i \geq 2$ the interval $[t_{i-1}, t_i]$ is a compact subinterval of $(0, 1]$ on which f is continuously differentiable, so the Fundamental Theorem of Calculus of elementary analysis gives

$$f(t_i) - f(t_{i-1}) = \int_{t_{i-1}}^{t_i} f', \quad \text{hence} \quad |f(t_i) - f(t_{i-1})| \leq \int_{t_{i-1}}^{t_i} |f'|;$$

here the Riemann and Lebesgue integrals of the continuous, bounded function f' over $[t_{i-1}, t_i]$ agree. For the first subinterval, let $0 < s < t_1$. The same argument on $[s, t_1]$ yields

$$|f(t_1) - f(s)| \leq \int_s^{t_1} |f'| \leq \int_0^{t_1} |f'|.$$

Letting $s \rightarrow 0^+$ and using the continuity of f at 0 established above, together with $f(0) = 0$, we get

$$|f(t_1) - f(t_0)| \leq \int_0^{t_1} |f'|.$$

Summing the k inequalities and using the additivity of the integral over the disjoint intervals (t_{i-1}, t_i) whose union is $(0, 1)$ up to a finite set,

$$V(f, P) = \sum_{i=1}^k |f(t_i) - f(t_{i-1})| \leq \sum_{i=1}^k \int_{t_{i-1}}^{t_i} |f'| = \int_0^1 |f'|.$$

The right-hand side is finite and independent of P , so taking the supremum over all partitions of $[0, 1]$,

$$TV(f) \leq \int_0^1 |f'| \leq 1 + \frac{\beta}{\alpha - \beta} < \infty,$$

and f is of bounded variation on $[0, 1]$.

The case $\alpha \leq \beta$. We exhibit partitions whose variations are unbounded. For each integer $n \geq 0$ set

$$x_n = \left(\frac{\pi}{2} + n\pi\right)^{-1/\beta},$$

so that $1/x_n^\beta = \pi/2 + n\pi$ and consequently

$$\sin(1/x_n^\beta) = (-1)^n, \quad f(x_n) = (-1)^n x_n^\alpha.$$

Since $t \mapsto t^{-1/\beta}$ is strictly decreasing on $(0, \infty)$, the sequence $\{x_n\}$ is strictly decreasing to 0; and $x_0 = (\pi/2)^{-1/\beta} < 1$ because $\pi/2 > 1$. Hence, for each integer $N \geq 1$,

$$P_N = \{0, x_N, x_{N-1}, \dots, x_1, x_0, 1\}$$

is a partition of $[0, 1]$.

Discarding the nonnegative terms of $V(f, P_N)$ other than those coming from consecutive pairs $x_{n+1} < x_n$, and using that $f(x_n)$ and $f(x_{n+1})$ have opposite signs,

$$V(f, P_N) \geq \sum_{n=0}^{N-1} |f(x_n) - f(x_{n+1})| = \sum_{n=0}^{N-1} (x_n^\alpha + x_{n+1}^\alpha) \geq \sum_{n=0}^{N-1} x_n^\alpha = \sum_{n=0}^{N-1} \left(\frac{\pi}{2} + n\pi\right)^{-\alpha/\beta}.$$

Write $p = \alpha/\beta$; the hypothesis $\alpha \leq \beta$ says $0 < p \leq 1$. For $t \geq 1$ and $0 < p \leq 1$ we have $t^p \leq t$ and hence $t^{-p} \geq t^{-1}$. Since $\pi/2 + n\pi \geq 1$ for every $n \geq 0$, it follows that

$$V(f, P_N) \geq \sum_{n=0}^{N-1} \frac{1}{\pi/2 + n\pi} \geq \sum_{n=0}^{N-1} \frac{1}{\pi(n+1)} = \frac{1}{\pi} \sum_{m=1}^N \frac{1}{m},$$

using $\pi/2 + n\pi \leq \pi + n\pi = \pi(n+1)$. Because the harmonic series diverges, the right-hand side tends to ∞ as $N \rightarrow \infty$. Therefore the set of variations of f with respect to partitions of $[0, 1]$ is unbounded above, so

$$TV(f) = \infty$$

and f is not of bounded variation on $[0, 1]$.

The dividing line is exactly $\alpha = \beta$: the amplitude x^α must decay strictly faster than the reciprocal of the number of oscillations accumulated near 0, which is governed by $x^{-\beta}$.

6.6 Problems 6.36–6.42

Problem 6.36. Let f fail to be of bounded variation on $[0, 1]$. Show that there is a point x_0 in $[0, 1]$ such that f fails to be of bounded variation on each nondegenerate closed subinterval of $[0, 1]$ that contains x_0 .

Solution. Throughout we use the additivity formula (20) of Section 6.3,

$$TV(f_{[a,b]}) = TV(f_{[a,c]}) + TV(f_{[c,b]}) \quad (a \leq c \leq b).$$

No finiteness hypothesis is needed here, and both sides are to be read in $[0, \infty]$: as the text observes, adjoining the point c to a partition of $[a, b]$ does not decrease the associated variation, so the supremum defining $TV(f_{[a,b]})$ may be taken over partitions P containing c ; each such P induces partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ with $V(f_{[a,b]}, P) = V(f_{[a,c]}, P_1) + V(f_{[c,b]}, P_2)$ by (19), and taking suprema over

all such P yields (20). In particular f is of bounded variation on $[a, b]$ if and only if it is of bounded variation on both $[a, c]$ and $[c, b]$; contrapositively, if f fails to be of bounded variation on $[a, b]$ then it fails to be of bounded variation on at least one of the two halves $[a, (a+b)/2]$, $[(a+b)/2, b]$.

Construction of the point. Put $[a_0, b_0] = [0, 1]$. Given $[a_n, b_n]$ of length 2^{-n} on which f fails to be of bounded variation, bisect it; by the remark just made, f fails to be of bounded variation on at least one of the two closed halves, and we let $[a_{n+1}, b_{n+1}]$ be such a half (say the left one if both qualify). This produces a descending sequence

$$[0, 1] = [a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots, \quad b_n - a_n = 2^{-n},$$

of nonempty closed bounded intervals, on each of which f fails to be of bounded variation. By the Nested Set Theorem the intersection $\bigcap_{n \geq 0} [a_n, b_n]$ is nonempty, and since the lengths tend to 0 it consists of a single point $x_0 \in [0, 1]$.

The conclusion. Fix $\delta > 0$ and set $J_\delta = [x_0 - \delta, x_0 + \delta] \cap [0, 1]$, a nondegenerate closed subinterval of $[0, 1]$. Choose n with $2^{-n} < \delta$. Since $a_n \leq x_0 \leq b_n$ and $b_n - a_n < \delta$, every point of $[a_n, b_n]$ lies within δ of x_0 ; as $[a_n, b_n] \subseteq [0, 1]$ we conclude $[a_n, b_n] \subseteq J_\delta$. If f were of bounded variation on J_δ , then two applications of (20) (splitting J_δ at a_n and then at b_n) would show f to be of bounded variation on $[a_n, b_n]$, a contradiction. Hence

for every $\delta > 0$, f fails to be of bounded variation on $[x_0 - \delta, x_0 + \delta] \cap [0, 1]$. (*)

Consequently, if $[c, d] \subseteq [0, 1]$ is any nondegenerate closed interval containing x_0 in its interior relative to $[0, 1]$ (that is, $c < x_0 < d$, or $x_0 = c = 0$, or $x_0 = d = 1$), then $[c, d] \supseteq J_\delta$ for all small $\delta > 0$, and by (*) together with (20) the function f fails to be of bounded variation on $[c, d]$.

A remark on the literal statement: the conclusion cannot be upgraded to every nondegenerate closed subinterval containing x_0 , once x_0 is allowed to be an endpoint of that subinterval. Indeed, let $h(t) = t \cos(\pi/2t)$ for $0 < t \leq 1$ and $h(0) = 0$; the Example of Section 6.3 shows h is continuous, and using the partitions $P_n = \{0, 1/2n, 1/[2n-1], \dots, 1/3, 1/2\}$ of $[0, 1/2]$, that

$$V(h|_{[0, 1/2]}, P_n) = \frac{1}{2} + \sum_{j=2}^n \frac{1}{j} \longrightarrow \infty,$$

so h is not of bounded variation on $[0, 1/2]$. Define g on $[0, 1]$ by $g(x) = h(1/2 - x)$ for $0 \leq x < 1/2$ and $g(x) = 0$ for $1/2 \leq x \leq 1$. Since $g(x) = h(1/2 - x)$ holds for every $x \in [0, 1/2]$ (at $x = 1/2$ both sides are 0) and h is continuous, g is continuous on $[0, 1/2]$; it is identically 0 on $[1/2, 1]$, so g is continuous on $[0, 1]$. Being the reflection $x \mapsto h(1/2 - x)$ of $h|_{[0, 1/2]}$, which carries partitions to partitions and leaves the associated variations unchanged, g fails to be of bounded variation on $[0, 1/2]$, and hence, by (20), on $[0, 1]$. On the other hand g vanishes identically on $[1/2, 1]$, so it is of bounded variation there; and for $[c, d] \subseteq [0, 1/2)$ the numbers $1/2 - x$, $x \in [c, d]$, range over $[1/2 - d, 1/2 - c] \subseteq (0, 1/2]$, on which h is continuously differentiable, so g is continuously differentiable on $[c, d]$ with bounded derivative, hence Lipschitz there by the Mean Value Theorem and therefore of bounded variation. So for each $x_0 \in [0, 1]$ there is a nondegenerate closed subinterval of $[0, 1]$ containing x_0 on which g is of bounded variation: take $[1/2, 1]$ itself if $x_0 \geq 1/2$, and take $[0, (x_0 + 1/2)/2]$ if $x_0 < 1/2$, this last interval being a nondegenerate closed subinterval of $[0, 1/2)$ that contains x_0 . Thus (*), failure on every relative neighborhood of x_0 , is the correct form of the assertion, and it is exactly what the bisection argument delivers.

Problem 6.37. Let f be a continuous function on $[0, 1]$ that is absolutely continuous on $[\epsilon, 1]$ for each $0 < \epsilon < 1$.

- (i) Show that f may not be absolutely continuous on $[0, 1]$.
- (ii) Show that f is absolutely continuous on $[0, 1]$ if it is increasing.
- (iii) Show that the function f on $[0, 1]$, defined by $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$, is absolutely continuous, but not Lipschitz, on $[0, 1]$.

Solution. (i) Define

$$f(x) = \begin{cases} x \cos(\pi/2x) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

As observed in the Example of Section 6.3, f is continuous on $[0, 1]$ and is not of bounded variation on $[0, 1]$: for the partition $P_n = \{0, 1/2n, 1/[2n-1], \dots, 1/3, 1/2, 1\}$ one computes $V(f, P_n) = 1 + 1/2 + \dots + 1/n$, and the harmonic series diverges.

Fix $0 < \epsilon < 1$. On $[\epsilon, 1]$ the function f is continuously differentiable, with

$$f'(x) = \cos(\pi/2x) + \frac{\pi}{2x} \sin(\pi/2x),$$

so $|f'|$ is bounded on the compact interval $[\epsilon, 1]$, say by c . By the Mean Value Theorem $|f(u) - f(v)| \leq c|u - v|$ for $u, v \in [\epsilon, 1]$, so f is Lipschitz there and hence, by Proposition 7, absolutely continuous on $[\epsilon, 1]$.

But f is not absolutely continuous on $[0, 1]$: by Theorem 8, an absolutely continuous function on a closed bounded interval is of bounded variation there, and f is not of bounded variation on $[0, 1]$.

(ii) Assume in addition that f is increasing on $[0, 1]$. We first record an elementary fact: if $\{(\alpha_k, \beta_k)\}_{k=1}^n$ is a disjoint collection of open subintervals of (u, v) and g is increasing on $[u, v]$, then

$$\sum_{k=1}^n [g(\beta_k) - g(\alpha_k)] \leq g(v) - g(u).$$

Indeed, relabel so that $u \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \dots \leq \alpha_n < \beta_n \leq v$; since g is increasing, each of the complementary differences $g(\alpha_1) - g(u)$, $g(\alpha_{k+1}) - g(\beta_k)$, $g(v) - g(\beta_n)$ is nonnegative, and adding them to the displayed sum telescopes to $g(v) - g(u)$.

Now let $\epsilon > 0$. Since f is continuous at 0, choose $c \in (0, 1)$ with

$$f(c) - f(0) < \epsilon/2.$$

Since f is absolutely continuous on $[c, 1]$, choose $\delta > 0$ that responds to the $\epsilon/2$ challenge for the absolute continuity of f on $[c, 1]$.

We claim this δ responds to the ϵ challenge for the absolute continuity of f on $[0, 1]$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of $(0, 1)$ with $\sum_{k=1}^n [b_k - a_k] < \delta$. For each k write

$$a'_k = \max\{a_k, c\}, \quad b'_k = \max\{b_k, c\}, \quad a''_k = \min\{a_k, c\}, \quad b''_k = \min\{b_k, c\}.$$

Since f is increasing, an inspection of the three possible positions of (a_k, b_k) relative to c (entirely right of c , entirely left of c , straddling c) shows in each case that

$$|f(b_k) - f(a_k)| = f(b_k) - f(a_k) = [f(b'_k) - f(a'_k)] + [f(b''_k) - f(a''_k)].$$

The intervals (a'_k, b'_k) , after discarding the degenerate ones, form a disjoint collection of open subintervals of $(c, 1)$ whose lengths sum to at most $\sum_k [b_k - a_k] < \delta$; hence by the choice of δ ,

$$\sum_{k=1}^n [f(b'_k) - f(a'_k)] < \epsilon/2.$$

The intervals (a''_k, b''_k) , after discarding degenerate ones, form a disjoint collection of open subintervals of $(0, c)$, so by the elementary fact above,

$$\sum_{k=1}^n [f(b''_k) - f(a''_k)] \leq f(c) - f(0) < \epsilon/2.$$

Adding, $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$. Therefore f is absolutely continuous on $[0, 1]$.

(iii) Let $f(x) = \sqrt{x}$ on $[0, 1]$. It is continuous and increasing on $[0, 1]$. For $0 < \epsilon < 1$, f is differentiable on $[\epsilon, 1]$ with $f'(x) = 1/(2\sqrt{x}) \leq 1/(2\sqrt{\epsilon})$, so by the Mean Value Theorem f is Lipschitz on $[\epsilon, 1]$

and therefore, by Proposition 7, absolutely continuous on $[\epsilon, 1]$. Since f is continuous on $[0, 1]$ and increasing, part (ii) shows that f is absolutely continuous on $[0, 1]$. However f is not Lipschitz on $[0, 1]$. If $c \geq 0$ were a Lipschitz constant, then for $0 < x \leq 1$,

$$\sqrt{x} = |f(x) - f(0)| \leq c|x - 0| = cx, \quad \text{so} \quad c \geq \frac{1}{\sqrt{x}}.$$

Letting $x \rightarrow 0^+$ makes the right-hand side unbounded, a contradiction. Hence f is absolutely continuous but not Lipschitz on $[0, 1]$; this justifies the assertion made after Proposition 7.

Problem 6.38. Show that f is absolutely continuous on $[a, b]$ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for every countable disjoint collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals in (a, b) ,

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon \quad \text{if} \quad \sum_{k=1}^{\infty} [b_k - a_k] < \delta.$$

Solution. (A word on the transcription. The book prints the display as $\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \sum_{k=1}^{\infty} [b_k - a_k] < \delta$, in which the ϵ announced in the preceding clause never appears; the words “ ϵ if” have been lost in typesetting, exactly as the parallel display in the next problem reads “ $m^*(f(E)) < \epsilon$ if $m(E) < \delta$ ”. The statement above is the intended one, and is what we prove.)

Suppose first that f satisfies the countable criterion. Let $\epsilon > 0$ and let $\delta > 0$ respond to it; replacing δ by $\min\{\delta, b - a\}$ we may assume $\delta \leq b - a$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$. Put $\sigma = \sum_{k=1}^n [b_k - a_k]$. The set $K = \bigcup_{k=1}^n [a_k, b_k]$ is closed, being a finite union of closed intervals, and $m(K) = \sigma$, since adjoining the finitely many endpoints to the disjoint open intervals adds nothing to the measure. Hence $(a, b) \sim K$ is open and, by the excision property of measure, $m((a, b) \sim K) \geq (b - a) - \sigma > 0$ because $\sigma < \delta \leq b - a$. In particular $(a, b) \sim K$ is nonempty, and being open it contains some open interval (α, β) with $\alpha < \beta$. Choose a disjoint sequence of open intervals $\{(a_k, b_k)\}_{k=n+1}^{\infty}$ contained in (α, β) with

$$\sum_{k=n+1}^{\infty} [b_k - a_k] < \delta - \sigma$$

(for instance intervals of length $2^{-j}\eta$ marching to the right inside (α, β) , with η small). Then $\{(a_k, b_k)\}_{k=1}^{\infty}$ is a countable disjoint collection of open intervals in (a, b) of total length less than δ , so by hypothesis

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \leq \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon.$$

Thus δ responds to the ϵ challenge in the definition of absolute continuity, and f is absolutely continuous on $[a, b]$.

Conversely, suppose f is absolutely continuous on $[a, b]$. Let $\epsilon > 0$ and choose $\delta > 0$ as a response to the $\epsilon/2$ challenge regarding the criterion for the absolute continuity of f , that is, for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) ,

$$\text{if } \sum_{k=1}^n [b_k - a_k] < \delta, \text{ then } \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon/2.$$

Now let $\{(a_k, b_k)\}_{k=1}^{\infty}$ be a countable disjoint collection of open intervals in (a, b) with $\sum_{k=1}^{\infty} [b_k - a_k] < \delta$. For each natural number n , the first n of these intervals form a finite disjoint collection of open intervals in (a, b) with

$$\sum_{k=1}^n [b_k - a_k] \leq \sum_{k=1}^{\infty} [b_k - a_k] < \delta,$$

so that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon/2$. The partial sums of the series of nonnegative terms $\sum |f(b_k) - f(a_k)|$ are therefore bounded above by $\epsilon/2$, so the series converges and

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| \leq \epsilon/2 < \epsilon.$$

Hence the same δ responds to the ϵ challenge in the countable criterion.

Problem 6.39. Use the preceding problem to show that if f is increasing on $[a, b]$, then f is absolutely continuous on $[a, b]$ if and only if for each ϵ , there is a $\delta > 0$ such that for a measurable subset E of $[a, b]$,

$$m^*(f(E)) < \epsilon \quad \text{if} \quad m(E) < \delta.$$

Solution. Assume first that f is increasing and absolutely continuous on $[a, b]$, and let $\epsilon > 0$. By the preceding problem choose $\delta > 0$ so that for every countable disjoint collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals in (a, b) ,

$$\text{if } \sum_{k=1}^{\infty} [b_k - a_k] < \delta, \text{ then } \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon/2.$$

Let E be a measurable subset of $[a, b]$ with $m(E) < \delta$. Apply assertion (i) of Theorem 11 of Chapter 2 with the number $\delta - m(E) > 0$: there is an open set $\mathcal{O} \supseteq E$ with $m^*(\mathcal{O} \sim E) < \delta - m(E)$, and hence, by the countable additivity of measure applied to the disjoint union $\mathcal{O} = E \cup (\mathcal{O} \sim E)$,

$$m(\mathcal{O}) = m(E) + m(\mathcal{O} \sim E) < \delta.$$

Put $U = \mathcal{O} \cap (a, b)$, an open subset of (a, b) with $E \sim \{a, b\} \subseteq U$ and $m(U) < \delta$. By Proposition 9 of Chapter 1, U is the disjoint union of a countable collection $\{(a_k, b_k)\}_k$ of open intervals, necessarily contained in (a, b) , and by countable additivity $\sum_k [b_k - a_k] = m(U) < \delta$. If this collection is infinite, the choice of δ gives $\sum_k [f(b_k) - f(a_k)] < \epsilon/2$ directly; if it is finite, the same bound follows from the definition of absolute continuity together with the fact that the δ furnished by the preceding problem also responds to the $\epsilon/2$ challenge for finite collections (a finite collection may be enlarged to an infinite one of total length still less than δ , exactly as in the first half of the solution to Problem 38). Since f is increasing, $f((a_k, b_k)) \subseteq [f(a_k), f(b_k)]$, so by the monotonicity and countable subadditivity of outer measure,

$$m^*(f(U)) \leq \sum_k m^*(f((a_k, b_k))) \leq \sum_k [f(b_k) - f(a_k)] < \epsilon/2.$$

Since $f(E) \subseteq f(U) \cup \{f(a), f(b)\}$ and a two-point set has outer measure zero, $m^*(f(E)) \leq \epsilon/2 < \epsilon$. This proves the forward implication.

Conversely, assume f is increasing on $[a, b]$, that f is continuous, and that the stated condition holds. Let $\epsilon > 0$ and choose $\delta > 0$ so that $m^*(f(E)) < \epsilon$ whenever $E \subseteq [a, b]$ is measurable with $m(E) < \delta$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$, and set $E = \bigcup_{k=1}^n [a_k, b_k]$, a measurable set with $m(E) = \sum_{k=1}^n [b_k - a_k] < \delta$. Since f is continuous and increasing, the Intermediate Value Theorem gives $f([a_k, b_k]) = [f(a_k), f(b_k)]$. Relabel so that $b_k \leq a_{k+1}$; then $f(b_k) \leq f(a_{k+1})$, so the intervals $[f(a_k), f(b_k)]$, $1 \leq k \leq n$, are non-overlapping. Hence

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n [f(b_k) - f(a_k)] = m\left(\bigcup_{k=1}^n [f(a_k), f(b_k)]\right) = m^*(f(E)) < \epsilon,$$

the second equality because finitely many non-overlapping closed bounded intervals have union whose measure is the sum of their lengths (their pairwise intersections are finite sets, hence of measure zero), and the third because $f(E) = \bigcup_{k=1}^n f([a_k, b_k]) = \bigcup_{k=1}^n [f(a_k), f(b_k)]$. The same remark justifies $m(E) = \sum_{k=1}^n [b_k - a_k]$ above. Therefore δ responds to the ϵ challenge and f is absolutely continuous on $[a, b]$.

The continuity assumption in the converse direction cannot be dropped, and it is harmless in the sense that absolutely continuous functions are continuous. That it cannot be dropped is seen from the increasing function f on $[0, 1]$ defined by $f(x) = 0$ for $0 \leq x < 1/2$ and $f(x) = 1$ for $1/2 \leq x \leq 1$: every subset of $[0, 1]$ has image contained in the two-point set $\{0, 1\}$, so $m^*(f(E)) = 0$ for all E and the displayed condition holds vacuously with any δ ; yet f is not continuous and therefore is not absolutely continuous on $[0, 1]$. Thus the intended reading of the problem is that f is an increasing continuous function, or equivalently that the criterion is being used to decide absolute continuity among continuous increasing functions – which is exactly how it is applied in Problems 40 and 41.

Problem 6.40. Use the preceding problem to show that an increasing absolutely continuous function f on $[a, b]$ maps sets of measure zero onto sets of measure zero. Conclude that the Cantor-Lebesgue function φ is not absolutely continuous on $[0, 1]$ since the function ψ , defined by $\psi(x) = x + \varphi(x)$ for $0 \leq x \leq 1$, maps the Cantor set to a set of measure 1 (page 52).

Solution. Let f be increasing and absolutely continuous on $[a, b]$ and let $E \subseteq [a, b]$ have measure zero. Let $\epsilon > 0$. The forward implication of the preceding problem – which uses only that f is increasing and absolutely continuous – provides $\delta > 0$ such that $m^*(f(A)) < \epsilon$ for every measurable $A \subseteq [a, b]$ with $m(A) < \delta$. Since E is measurable with $m(E) = 0 < \delta$, we conclude $m^*(f(E)) < \epsilon$. As $\epsilon > 0$ was arbitrary, $m^*(f(E)) = 0$. By Proposition 4 of Chapter 2, any set of outer measure zero is measurable, so $f(E)$ is a set of measure zero. Thus f maps sets of measure zero onto sets of measure zero.

Now consider the Cantor-Lebesgue function φ on $[0, 1]$, which is increasing and continuous, and let $\psi(x) = x + \varphi(x)$ for $0 \leq x \leq 1$. Suppose, to get a contradiction, that φ is absolutely continuous on $[0, 1]$. The identity function $x \mapsto x$ is Lipschitz on $[0, 1]$, hence absolutely continuous by Proposition 7. Since linear combinations of absolutely continuous functions are absolutely continuous (see also Problem 42), ψ would then be absolutely continuous on $[0, 1]$. Moreover ψ is increasing, being the sum of an increasing function and a strictly increasing one.

Therefore, by what was just proved, ψ would map sets of measure zero onto sets of measure zero. The Cantor set C has measure zero, by Proposition 19 of Chapter 2. But Proposition 21 of Chapter 2 (page 52) asserts that this very function ψ maps C onto a measurable set of positive measure; indeed its proof, decomposing $[0, 2] = \psi(\mathcal{O}) \cup \psi(C)$ for $\mathcal{O} = [0, 1] \setminus C$ and computing $m(\psi(\mathcal{O})) = m(\mathcal{O}) = 1$, gives $m(\psi(C)) = 1$. Since $1 \neq 0$, we have a contradiction. Hence φ is not absolutely continuous on $[0, 1]$.

Problem 6.41. Let f be an increasing absolutely continuous function on $[a, b]$. Use (i) and (ii) below to conclude that f maps measurable sets to measurable sets.

(i) Infer from the continuity of f and the compactness of $[a, b]$ that f maps closed sets to closed sets and therefore maps F_σ sets to F_σ sets.

(ii) The preceding problem tells us that f maps sets of measure zero to sets of measure zero.

Solution. (i) Since f is absolutely continuous on $[a, b]$ it is continuous there (the criterion for absolute continuity, applied to a collection consisting of a single interval, is the criterion for uniform continuity). Let $F \subseteq [a, b]$ be closed. Then F is closed and bounded, hence compact by the Heine-Borel Theorem. The continuous image of a compact set is compact, so $f(F)$ is compact and in particular closed. Now let $E \subseteq [a, b]$ be an F_σ set, say $E = \bigcup_{n=1}^{\infty} F_n$ with each F_n closed. Each F_n is a closed subset of $[a, b]$, and since images preserve unions,

$$f(E) = f\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} f(F_n),$$

a countable union of closed sets. Hence $f(E)$ is an F_σ set, and in particular is measurable, since every F_σ set is Borel and every Borel set is measurable.

(ii) By the preceding problem, since f is increasing and absolutely continuous, f carries every subset of $[a, b]$ of measure zero onto a set of measure zero.

Conclusion. Let $E \subseteq [a, b]$ be measurable. By assertion (iv) of Theorem 11 of Chapter 2 there is an F_σ set F with $F \subseteq E$ and $m^*(E \sim F) = 0$. Write $F = \bigcup_{n=1}^\infty F_n$ with each F_n closed; since $F_n \subseteq F \subseteq E \subseteq [a, b]$, every F_n is a closed subset of $[a, b]$, so part (i) applies and $f(F)$ is an F_σ set, hence measurable. The set $E \sim F$ is a subset of $[a, b]$ with $m^*(E \sim F) = 0$, hence is measurable of measure zero by Proposition 4 of Chapter 2, so by part (ii) $f(E \sim F)$ has measure zero and is therefore measurable. Finally, from $E = F \cup (E \sim F)$ we get

$$f(E) = f(F) \cup f(E \sim F),$$

the union of two measurable sets. Hence $f(E)$ is measurable. Therefore f maps measurable subsets of $[a, b]$ to measurable sets.

Problem 6.42. Show that both the sum and product of absolutely continuous functions are absolutely continuous.

Solution. Let f and g be absolutely continuous on the closed, bounded interval $[a, b]$.

The sum. Let $\epsilon > 0$. Choose $\delta_1 > 0$ as a response to the $\epsilon/2$ challenge regarding the criterion for the absolute continuity of f , and $\delta_2 > 0$ as a response to the $\epsilon/2$ challenge for g . Put $\delta = \min\{\delta_1, \delta_2\} > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$. By the triangle inequality,

$$\sum_{k=1}^n |(f+g)(b_k) - (f+g)(a_k)| \leq \sum_{k=1}^n |f(b_k) - f(a_k)| + \sum_{k=1}^n |g(b_k) - g(a_k)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $f+g$ is absolutely continuous on $[a, b]$. (The same argument, with $\epsilon/(2(|\alpha| + |\beta| + 1))$ in place of $\epsilon/2$ and the estimate $|\alpha| \sum_k |f(b_k) - f(a_k)| + |\beta| \sum_k |g(b_k) - g(a_k)|$ in place of the one displayed, shows that any linear combination $\alpha f + \beta g$ is absolutely continuous, as asserted in the text preceding Proposition 7.)

The product. Absolutely continuous functions are continuous, and a continuous real-valued function on the compact interval $[a, b]$ is bounded by the Extreme Value Theorem. So choose $M \geq 1$ with

$$|f(x)| \leq M \quad \text{and} \quad |g(x)| \leq M \quad \text{for all } x \in [a, b].$$

Let $\epsilon > 0$. Choose $\delta_1 > 0$ as a response to the $\epsilon/(2M)$ challenge for the absolute continuity of f , and $\delta_2 > 0$ as a response to the $\epsilon/(2M)$ challenge for g ; put $\delta = \min\{\delta_1, \delta_2\} > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$. For each k , adding and subtracting $f(a_k)g(b_k)$,

$$\begin{aligned} |f(b_k)g(b_k) - f(a_k)g(a_k)| &= |[f(b_k) - f(a_k)]g(b_k) + f(a_k)[g(b_k) - g(a_k)]| \\ &\leq M|f(b_k) - f(a_k)| + M|g(b_k) - g(a_k)|. \end{aligned}$$

Summing over k and using the choice of δ ,

$$\sum_{k=1}^n |(fg)(b_k) - (fg)(a_k)| \leq M \sum_{k=1}^n |f(b_k) - f(a_k)| + M \sum_{k=1}^n |g(b_k) - g(a_k)| < M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} = \epsilon.$$

Hence fg is absolutely continuous on $[a, b]$.

6.7 Problems 6.43–6.49

Problem 6.43. Define the functions f and g on $[-1, 1]$ by $f(x) = x^{1/3}$ for $-1 \leq x \leq 1$ and

$$g(x) = \begin{cases} x^2 \cos(\pi/2x) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

- (i) Show that both f and g are absolutely continuous on $[-1, 1]$.
- (ii) For the partition $P_n = \{-1, 0, 1/2n, 1/[2n-1], \dots, 1/3, 1/2, 1\}$ of $[-1, 1]$, examine $V(f \circ g, P_n)$.
- (iii) Show that $f \circ g$ fails to be of bounded variation, and hence also fails to be absolutely continuous, on $[-1, 1]$.

Solution. Part (i). Absolute continuity of g .

For $x \neq 0$ the chain rule gives

$$g'(x) = 2x \cos\left(\frac{\pi}{2x}\right) + x^2 \cdot \left(-\sin\left(\frac{\pi}{2x}\right)\right) \cdot \left(-\frac{\pi}{2x^2}\right) = 2x \cos\left(\frac{\pi}{2x}\right) + \frac{\pi}{2} \sin\left(\frac{\pi}{2x}\right).$$

At $x = 0$, since $|g(x) - g(0)| = |x^2 \cos(\pi/2x)| \leq x^2$, we have $|g(x) - g(0)|/|x| \leq |x| \rightarrow 0$, so g is differentiable at 0 with $g'(0) = 0$. Thus g is differentiable at every point of $[-1, 1]$ and, since $|x| \leq 1$ there,

$$|g'(x)| \leq 2|x| + \frac{\pi}{2} \leq 2 + \frac{\pi}{2} \quad \text{for all } x \in [-1, 1].$$

By the Mean Value Theorem, for all $u, v \in [-1, 1]$ we have $|g(u) - g(v)| \leq (2 + \pi/2)|u - v|$, so g is Lipschitz on $[-1, 1]$. By Proposition 7, g is absolutely continuous on $[-1, 1]$.

Part (i), continued. Absolute continuity of f .

The function $f(x) = x^{1/3}$ is continuous and strictly increasing on $[-1, 1]$. It is not Lipschitz near 0, so we argue directly from the definition of absolute continuity, exploiting monotonicity near the origin.

Let $\epsilon > 0$. Choose $\eta \in (0, 1)$ so small that

$$f(\eta) - f(-\eta) = 2\eta^{1/3} < \epsilon/2.$$

On each of the intervals $[-1, -\eta]$ and $[\eta, 1]$ the function f is differentiable with $f'(x) = \frac{1}{3}|x|^{-2/3} \leq M := \frac{1}{3}\eta^{-2/3}$, so by the Mean Value Theorem f is Lipschitz with constant M on each of these two intervals. Put $\delta = \epsilon/(2M)$.

Let $\{(a_k, b_k)\}_{k=1}^n$ be a disjoint collection of open intervals in $(-1, 1)$ with $\sum_{k=1}^n [b_k - a_k] < \delta$. Split each (a_k, b_k) at the points $-\eta$ and η (whichever of them it contains); this replaces (a_k, b_k) by at most three abutting subintervals, each contained in one of $[-1, -\eta]$, $[-\eta, \eta]$, $[\eta, 1]$. Because f is increasing, $|f(b_k) - f(a_k)| = f(b_k) - f(a_k)$ and this increment is exactly the sum of the increments of f over the pieces. Hence

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \Sigma_{\text{mid}} + \Sigma_{\text{out}},$$

where Σ_{mid} is the sum of the increments of f over those pieces lying in $[-\eta, \eta]$ and Σ_{out} the sum over those pieces lying in $[-1, -\eta] \cup [\eta, 1]$.

The pieces contributing to Σ_{mid} are nonoverlapping subintervals of $[-\eta, \eta]$, and f is increasing, so their increments sum to at most the increment of f over $[-\eta, \eta]$:

$$\Sigma_{\text{mid}} \leq f(\eta) - f(-\eta) < \epsilon/2.$$

The pieces contributing to Σ_{out} lie in the two intervals on which f is Lipschitz with constant M , and the sum of their lengths is at most $\sum_{k=1}^n [b_k - a_k] < \delta$, so

$$\Sigma_{\text{out}} \leq M \sum_{k=1}^n [b_k - a_k] < M\delta = \epsilon/2.$$

Therefore $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$. Thus δ responds to the ϵ challenge, and f is absolutely continuous on $[-1, 1]$.

Part (ii). The variation of $f \circ g$ over P_n .

Write $h = f \circ g$, so that

$$h(x) = \left[x^2 \cos\left(\frac{\pi}{2x}\right) \right]^{1/3} \quad \text{for } x \neq 0, \quad h(0) = 0.$$

For a natural number k and $x = 1/k$ we have $\pi/2x = k\pi/2$, so

$$h(1/k) = [k^{-2} \cos(k\pi/2)]^{1/3}.$$

If k is odd then $\cos(k\pi/2) = 0$ and $h(1/k) = 0$. If $k = 2m$ is even then $\cos(k\pi/2) = \cos(m\pi) = (-1)^m$ and

$$h(1/2m) = [(2m)^{-2}(-1)^m]^{1/3} = \frac{(-1)^m}{(2m)^{2/3}}.$$

Also $h(-1) = [\cos(-\pi/2)]^{1/3} = 0$ and $h(0) = 0$.

List the points of P_n in increasing order:

$$-1, 0, \frac{1}{2n}, \frac{1}{2n-1}, \frac{1}{2n-2}, \dots, \frac{1}{3}, \frac{1}{2}, 1.$$

The corresponding values of h are

$$0, 0, \frac{(-1)^n}{(2n)^{2/3}}, 0, \frac{(-1)^{n-1}}{(2n-2)^{2/3}}, 0, \dots, 0, \frac{-1}{2^{2/3}}, 0.$$

Indeed, every point $1/k$ with k odd (including $k = 1$, i.e. the right endpoint 1) carries the value 0, and these alternate with the points $1/2m$, $1 \leq m \leq n$, which carry the values $(-1)^m(2m)^{-2/3}$. Thus each of the n nonzero values is flanked on both sides by a partition point at which h vanishes, and each contributes its absolute value twice to the sum of consecutive increments, while the first increment (from -1 to 0) is 0. Consequently

$$V(f \circ g, P_n) = 2 \sum_{m=1}^n \frac{1}{(2m)^{2/3}} = 2^{1/3} \sum_{m=1}^n \frac{1}{m^{2/3}}.$$

Part (iii). $f \circ g$ is not of bounded variation.

Since $t \mapsto t^{-2/3}$ is decreasing on $(0, \infty)$,

$$\sum_{m=1}^n \frac{1}{m^{2/3}} \geq \int_1^{n+1} \frac{dx}{x^{2/3}} = 3 \left[(n+1)^{1/3} - 1 \right],$$

so

$$V(f \circ g, P_n) \geq 3 \cdot 2^{1/3} \left[(n+1)^{1/3} - 1 \right] \longrightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Hence the set of variations of $f \circ g$ with respect to partitions of $[-1, 1]$ is unbounded, that is, $TV(f \circ g) = \infty$ and $f \circ g$ is not of bounded variation on $[-1, 1]$. By Theorem 8, an absolutely continuous function on a closed, bounded interval is of bounded variation; therefore $f \circ g$ is not absolutely continuous on $[-1, 1]$.

This exhibits the phenomenon announced on page 120: f and g are both absolutely continuous, yet the composition $f \circ g$ need not be, and indeed may fail even to be of bounded variation.

Problem 6.44. Let f be Lipschitz on $\mathbb{R}$ and g be absolutely continuous on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.

Solution. Let c be a Lipschitz constant for f on $\mathbb{R}$, so that

$$|f(s) - f(t)| \leq c|s - t| \quad \text{for all } s, t \in \mathbb{R}.$$

Enlarging c if necessary, we may assume $c > 0$.

Let $\epsilon > 0$. Since g is absolutely continuous on $[a, b]$, there is a $\delta > 0$ that responds to the ϵ/c challenge regarding the criterion for the absolute continuity of g : for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) ,

$$\sum_{k=1}^n |g(b_k) - g(a_k)| < \epsilon/c \quad \text{if} \quad \sum_{k=1}^n [b_k - a_k] < \delta.$$

We claim this same δ responds to the ϵ challenge for $f \circ g$. Indeed, let $\{(a_k, b_k)\}_{k=1}^n$ be a disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$. Applying the Lipschitz estimate for f to the pair of real numbers $g(b_k)$, $g(a_k)$ for each k , and then the choice of δ ,

$$\sum_{k=1}^n |(f \circ g)(b_k) - (f \circ g)(a_k)| = \sum_{k=1}^n |f(g(b_k)) - f(g(a_k))| \leq c \sum_{k=1}^n |g(b_k) - g(a_k)| < c \cdot \frac{\epsilon}{c} = \epsilon.$$

Thus for each $\epsilon > 0$ there is a $\delta > 0$ responding to the ϵ challenge regarding the criterion for absolute continuity of $f \circ g$, and therefore $f \circ g$ is absolutely continuous on $[a, b]$.

Note that only the Lipschitz property of f on the compact interval $g([a, b])$ was used; the hypothesis that f be Lipschitz on all of $\mathbb{R}$ is more than is needed. Note also, in the light of Problem 43, that mere absolute continuity of f is not enough: there the outer function $x \mapsto x^{1/3}$ is absolutely continuous but not Lipschitz near the origin, and the composition fails even to be of bounded variation.

Problem 6.45. Let f be absolutely continuous on $\mathbb{R}$ and g be absolutely continuous and strictly monotone on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.

Solution. Here, as usual, “absolutely continuous on $\mathbb{R}$ ” means absolutely continuous on every closed, bounded interval of $\mathbb{R}$.

Assume first that g is strictly increasing. Since g is continuous and strictly increasing on $[a, b]$, it maps $[a, b]$ onto the closed, bounded interval

$$[c, d] := [g(a), g(b)].$$

By hypothesis f is absolutely continuous on $[c, d]$.

Let $\epsilon > 0$.

Choose $\delta_1 > 0$ to respond to the ϵ challenge regarding the criterion for the absolute continuity of f on $[c, d]$: for every finite disjoint collection $\{(c_j, d_j)\}_{j=1}^n$ of open intervals in (c, d) ,

$$\sum_{j=1}^n |f(d_j) - f(c_j)| < \epsilon \quad \text{if} \quad \sum_{j=1}^n [d_j - c_j] < \delta_1.$$

Next choose $\delta_2 > 0$ to respond to the δ_1 challenge regarding the criterion for the absolute continuity of g on $[a, b]$: for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) ,

$$\sum_{k=1}^n |g(b_k) - g(a_k)| < \delta_1 \quad \text{if} \quad \sum_{k=1}^n [b_k - a_k] < \delta_2.$$

We claim δ_2 responds to the ϵ challenge for $f \circ g$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta_2$. Set $c_k = g(a_k)$ and $d_k = g(b_k)$.

Because g is strictly increasing we have $c_k < d_k$ for each k , so each (c_k, d_k) is a nondegenerate open interval, and it is contained in (c, d) : from $a \leq a_k < b_k \leq b$ and the monotonicity of g we get $c = g(a) \leq c_k < d_k \leq g(b) = d$, whence $(c_k, d_k) \subseteq (c, d)$. Moreover the intervals $\{(c_k, d_k)\}_{k=1}^n$ are disjoint: if $j \neq k$, then, relabelling if necessary, $b_j \leq a_k$, and monotonicity gives $d_j = g(b_j) \leq g(a_k) = c_k$, so (c_j, d_j) and (c_k, d_k) are disjoint. This is precisely where strict monotonicity of g is used, and it is the only place it is needed.

By the choice of δ_2 ,

$$\sum_{k=1}^n [d_k - c_k] = \sum_{k=1}^n |g(b_k) - g(a_k)| < \delta_1,$$

so $\{(c_k, d_k)\}_{k=1}^n$ is a finite disjoint collection of open intervals in (c, d) whose lengths sum to less than δ_1 . Therefore, by the choice of δ_1 ,

$$\sum_{k=1}^n |(f \circ g)(b_k) - (f \circ g)(a_k)| = \sum_{k=1}^n |f(d_k) - f(c_k)| < \epsilon.$$

Hence $f \circ g$ is absolutely continuous on $[a, b]$.

If instead g is strictly decreasing, then $g([a, b]) = [g(b), g(a)] =: [c, d]$, and for a disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ as above we set $c_k = g(b_k)$, $d_k = g(a_k)$; exactly the same argument applies, since g strictly decreasing again makes $\{(c_k, d_k)\}_{k=1}^n$ a disjoint collection of open subintervals of (c, d) with $\sum_k [d_k - c_k] = \sum_k |g(b_k) - g(a_k)|$. This completes the proof.

The role of strict monotonicity should be emphasized. Without it the image intervals may overlap and pile up, so that a small total length $\sum_k |g(b_k) - g(a_k)|$ no longer controls $\sum_k |f(g(b_k)) - f(g(a_k))|$ through the absolute continuity of f ; Problem 43 gives an explicit composition of two absolutely continuous functions (the inner one oscillating, hence not monotone) that is not absolutely continuous.

Problem 6.46. Verify the assertions made in the final remark of this section.

Solution. The remark reads: for a nondegenerate closed, bounded interval $[a, b]$, let $\mathcal{F}_{Lip}$, $\mathcal{F}_{AC}$, and $\mathcal{F}_{BV}$ denote the families of functions on $[a, b]$ that are Lipschitz, absolutely continuous, and of bounded variation, respectively. Then

$$\mathcal{F}_{Lip} \subset \mathcal{F}_{AC} \subset \mathcal{F}_{BV},$$

these inclusions, labelled (27) in the text, being strict; each of the three collections is closed with respect to the formation of linear combinations, a function in one of these collections has its total variation function in the same collection, and consequently, by (23), a function in one of these collections may be expressed as the difference of two increasing functions in the same collection. We verify these four assertions in turn.

Assertion 1: the inclusions hold and are strict.

Proposition 7 asserts that a Lipschitz function on $[a, b]$ is absolutely continuous on $[a, b]$, which is the inclusion $\mathcal{F}_{Lip} \subseteq \mathcal{F}_{AC}$. Theorem 8 asserts that an absolutely continuous function on $[a, b]$ is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation; this is the inclusion $\mathcal{F}_{AC} \subseteq \mathcal{F}_{BV}$.

To see the first inclusion is strict, define f on $[a, b]$ by $f(x) = \sqrt{x-a}$. Then f is not Lipschitz, since for $0 < t \leq b-a$,

$$\frac{|f(a+t) - f(a)|}{t} = \frac{\sqrt{t}}{t} = \frac{1}{\sqrt{t}} \rightarrow \infty \text{ as } t \rightarrow 0^+,$$

so no constant c can serve as a Lipschitz constant. On the other hand f is absolutely continuous. Indeed, f is continuous and increasing on $[a, b]$; given $\epsilon > 0$, choose $\eta \in (0, b-a)$ with $f(a+\eta) - f(a) = \sqrt{\eta} < \epsilon/2$, and observe that on $[a+\eta, b]$ the function f is differentiable with $0 < f'(x) = 1/(2\sqrt{x-a}) \leq M := 1/(2\sqrt{\eta})$, hence Lipschitz with constant M there by the Mean Value Theorem. Set $\delta = \epsilon/(2M)$ and let $\{(a_k, b_k)\}_{k=1}^n$ be a disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$. Split each (a_k, b_k) at the point $a + \eta$ if it contains it. Since f is increasing, the increment of f over (a_k, b_k) is nonnegative and equals the sum of its increments over the pieces; the pieces lying in $[a, a + \eta]$ are nonoverlapping, so their increments sum to at most $f(a + \eta) - f(a) < \epsilon/2$, while the pieces lying in $[a + \eta, b]$ have total length less than δ , so their increments sum to less than $M\delta = \epsilon/2$. Hence $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$ and f is absolutely continuous. (This is Problem 37(iii), transplanted to $[a, b]$.) Thus $f \in \mathcal{F}_{AC} \setminus \mathcal{F}_{Lip}$.

To see the second inclusion is strict, let φ be the Cantor-Lebesgue function on $[0, 1]$ and define ψ on $[a, b]$ by

$$\psi(x) = \varphi\left(\frac{x-a}{b-a}\right).$$

Then ψ is increasing on $[a, b]$, and an increasing real-valued function is of bounded variation: for any partition $P = \{x_0, \dots, x_k\}$ of $[a, b]$ we have $V(\psi, P) = \sum_{i=1}^k [\psi(x_i) - \psi(x_{i-1})] = \psi(b) - \psi(a)$, so $TV(\psi) = \psi(b) - \psi(a) < \infty$. But ψ is not absolutely continuous. For suppose it were; the affine map $L(t) = a + (b-a)t$ carries $[0, 1]$ onto $[a, b]$, carries disjoint open subintervals of $(0, 1)$ to disjoint open subintervals of (a, b) , and multiplies lengths by $b-a$. Hence, given $\epsilon > 0$, if δ responds to the ϵ challenge for ψ on $[a, b]$, then $\delta/(b-a)$ responds to the ϵ challenge for $\varphi = \psi \circ L$ on $[0, 1]$, so φ would be absolutely continuous on $[0, 1]$. This contradicts the Example on page 120, where it is shown that there is no response to the $\epsilon = 1$ challenge for φ . Thus $\psi \in \mathcal{F}_{BV} \setminus \mathcal{F}_{AC}$.

Assertion 2: each collection is closed under linear combinations.

Let α, β be real numbers and f, g functions on $[a, b]$.

If f and g are Lipschitz with constants c_f and c_g , then for all $u, v \in [a, b]$,

$$|(\alpha f + \beta g)(u) - (\alpha f + \beta g)(v)| \leq |\alpha| |f(u) - f(v)| + |\beta| |g(u) - g(v)| \leq (|\alpha|c_f + |\beta|c_g)|u - v|,$$

so $\alpha f + \beta g$ is Lipschitz. Hence $\mathcal{F}_{Lip}$ is closed under linear combinations.

If f and g are absolutely continuous, let $\epsilon > 0$ and choose δ_f responding to the $\epsilon/[2(|\alpha| + 1)]$ challenge for f and δ_g responding to the $\epsilon/[2(|\beta| + 1)]$ challenge for g ; set $\delta = \min\{\delta_f, \delta_g\}$. For a disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$,

$$\sum_{k=1}^n |(\alpha f + \beta g)(b_k) - (\alpha f + \beta g)(a_k)| \leq |\alpha| \sum_{k=1}^n |f(b_k) - f(a_k)| + |\beta| \sum_{k=1}^n |g(b_k) - g(a_k)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\mathcal{F}_{AC}$ is closed under linear combinations. (This is the assertion made on page 120 that linear combinations of absolutely continuous functions are absolutely continuous.)

If f and g are of bounded variation, then for every partition P of $[a, b]$, the same triangle-inequality estimate applied termwise gives

$$V(\alpha f + \beta g, P) \leq |\alpha|V(f, P) + |\beta|V(g, P) \leq |\alpha|TV(f) + |\beta|TV(g).$$

So the set of variations of $\alpha f + \beta g$ is bounded above and $TV(\alpha f + \beta g) \leq |\alpha|TV(f) + |\beta|TV(g) < \infty$. Hence $\mathcal{F}_{BV}$ is closed under linear combinations.

Assertion 3: the total variation function stays in the same collection.

For f of bounded variation on $[a, b]$ write $T_f(x) = TV(f_{[a, x]})$ for the total variation function of f . It is real-valued, since $TV(f_{[a, x]}) \leq TV(f) < \infty$, and it is increasing: by (21), $T_f(v) - T_f(u) = TV(f_{[u, v]}) \geq 0$ for $a \leq u < v \leq b$.

If $f \in \mathcal{F}_{BV}$, then T_f is real-valued and increasing, hence of bounded variation by the computation in Assertion 1 above; so $T_f \in \mathcal{F}_{BV}$.

If $f \in \mathcal{F}_{AC}$, then $f \in \mathcal{F}_{BV}$ by Theorem 8, so T_f is defined, and the last paragraph of the proof of Theorem 8 shows precisely that T_f is absolutely continuous: display (24) states that for each $\epsilon > 0$ there is a $\delta > 0$ such that for every disjoint collection $\{(c_k, d_k)\}_{k=1}^n$ of open subintervals of (a, b) ,

$$\sum_{k=1}^n |TV(f_{[a, d_k]}) - TV(f_{[a, c_k]})| < \epsilon \quad \text{if} \quad \sum_{k=1}^n [d_k - c_k] < \delta.$$

That is, $T_f \in \mathcal{F}_{AC}$.

If $f \in \mathcal{F}_{Lip}$ with Lipschitz constant c , then by the second Example of Section 6.3 (the one treating Lipschitz functions), applied to the restriction of f to a closed subinterval $[u, v]$ of $[a, b]$, which is Lipschitz there with the same constant c , we have $TV(f_{[u, v]}) \leq c \cdot [v - u]$; in particular $TV(f) \leq c(b - a) < \infty$, so $f \in \mathcal{F}_{BV}$ and T_f is defined. By (21), for $a \leq u < v \leq b$,

$$0 \leq T_f(v) - T_f(u) = TV(f_{[u, v]}) \leq c[v - u],$$

so $|T_f(v) - T_f(u)| \leq c|v - u|$ for all $u, v \in [a, b]$ and T_f is Lipschitz with the same constant c . Thus $T_f \in \mathcal{F}_{Lip}$.

Assertion 4: decomposition into increasing functions from the same collection.

Let $\mathcal{F}$ be any one of $\mathcal{F}_{Lip}$, $\mathcal{F}_{AC}$, $\mathcal{F}_{BV}$, and let $f \in \mathcal{F}$. By Assertion 1, f is of bounded variation, so Lemma 5 applies and (23) gives the explicit decomposition

$$f(x) = [f(x) + T_f(x)] - T_f(x) \quad \text{for all } x \in [a, b].$$

By Assertion 3, $T_f \in \mathcal{F}$; by Assertion 2 (closure under the linear combination $1 \cdot f + 1 \cdot T_f$), $f + T_f \in \mathcal{F}$. Both of these functions are increasing: T_f is increasing as noted above, and $f + T_f$ is increasing by (22), which states that $f(v) + TV(f_{[a,v]}) \geq f(u) + TV(f_{[a,u]})$ for $a \leq u < v \leq b$. Therefore f is the difference of two increasing functions belonging to $\mathcal{F}$, which is the final assertion of the remark. For $\mathcal{F} = \mathcal{F}_{AC}$ this recovers the first statement of Theorem 8.

Problem 6.47. Show that a function f is absolutely continuous on $[a, b]$ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) ,

$$\left| \sum_{k=1}^n [f(b_k) - f(a_k)] \right| < \epsilon \quad \text{if} \quad \sum_{k=1}^n [b_k - a_k] < \delta.$$

Solution. Call the displayed condition, in which the absolute value sits outside the sum, the weak criterion, and let the criterion for absolute continuity, in which the absolute values sit inside the sum, be the strong criterion. We must show the two are equivalent.

One implication is immediate. Suppose f is absolutely continuous on $[a, b]$, and let $\epsilon > 0$. Let $\delta > 0$ respond to the ϵ challenge regarding the criterion for the absolute continuity of f . If $\{(a_k, b_k)\}_{k=1}^n$ is a disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$, then, by the triangle inequality,

$$\left| \sum_{k=1}^n [f(b_k) - f(a_k)] \right| \leq \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

So the same δ responds to the ϵ challenge for the weak criterion.

Now suppose f satisfies the weak criterion; we show f is absolutely continuous. Let $\epsilon > 0$, and choose $\delta > 0$ to respond to the $\epsilon/2$ challenge regarding the weak criterion: for every finite disjoint collection $\{(c_j, d_j)\}_{j=1}^m$ of open intervals in (a, b) ,

$$\left| \sum_{j=1}^m [f(d_j) - f(c_j)] \right| < \epsilon/2 \quad \text{if} \quad \sum_{j=1}^m [d_j - c_j] < \delta.$$

We claim this δ responds to the ϵ challenge regarding the criterion for the absolute continuity of f . Indeed, let $\{(a_k, b_k)\}_{k=1}^n$ be a disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$. Sort the indices according to the sign of the increment of f : define

$$P = \{k : f(b_k) - f(a_k) \geq 0\}, \quad N = \{k : f(b_k) - f(a_k) < 0\},$$

so that $\{1, \dots, n\}$ is the disjoint union of P and N .

The subcollection $\{(a_k, b_k)\}_{k \in P}$ is itself a finite disjoint collection of open intervals in (a, b) , and

$$\sum_{k \in P} [b_k - a_k] \leq \sum_{k=1}^n [b_k - a_k] < \delta,$$

since all the terms are nonnegative. (Here and below, if the index set in question is empty, the corresponding sum is 0 and the estimate is trivially satisfied.) Hence, by the choice of δ for the weak criterion,

$$\sum_{k \in P} |f(b_k) - f(a_k)| = \sum_{k \in P} [f(b_k) - f(a_k)] = \left| \sum_{k \in P} [f(b_k) - f(a_k)] \right| < \epsilon/2,$$

the first equality because every term of the sum over P is nonnegative, and the second because a sum of nonnegative numbers equals its own absolute value.

The same reasoning applies to N : the subcollection $\{(a_k, b_k)\}_{k \in N}$ is a finite disjoint collection of open intervals in (a, b) whose lengths sum to less than δ , so by the same choice of δ ,

$$\sum_{k \in N} |f(b_k) - f(a_k)| = \sum_{k \in N} -[f(b_k) - f(a_k)] = \left| \sum_{k \in N} [f(b_k) - f(a_k)] \right| < \epsilon/2,$$

where now every term $f(b_k) - f(a_k)$ is negative, so the sum over N is negative and its absolute value is the sum of the absolute values of its terms.

Adding the two estimates,

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k \in P} |f(b_k) - f(a_k)| + \sum_{k \in N} |f(b_k) - f(a_k)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus δ responds to the ϵ challenge for absolute continuity, and f is absolutely continuous on $[a, b]$.

The point of the argument is that the two criteria differ only by the freedom to discard part of a disjoint collection: passing from a collection to a subcollection preserves both disjointness and the smallness of the total length, so the weak criterion may be applied separately to the intervals across which f increases and to those across which f decreases, and the two resulting estimates reassemble into the strong criterion.

Problem 6.48. The Cantor-Lebesgue function φ is continuous and increasing on $[0, 1]$. Conclude from Theorem 10 that φ is not absolutely continuous on $[0, 1]$. Compare this reasoning with that proposed in Problem 40.

Solution. Recall the construction of the Cantor set C : from $[0, 1]$ one deletes, at the n -th stage, 2^{n-1} open middle-third intervals, and C is what remains. The deleted intervals form a countable disjoint collection $\{I_j\}_{j=1}^{\infty}$ of open intervals with

$$m\left(\bigcup_{j=1}^{\infty} I_j\right) = m([0, 1] \setminus C) = 1 - m(C) = 1,$$

since $m(C) = 0$. The Cantor-Lebesgue function φ is, by its construction, constant on each I_j : on the j -th deleted interval it takes the constant dyadic value assigned at the stage at which that interval was removed.

Consequently φ is differentiable at every point of $\bigcup_j I_j$, with $\varphi' = 0$ there, because each such point has a neighbourhood (contained in the open interval I_j holding it) on which φ is constant. Since the complement of $\bigcup_j I_j$ in $[0, 1]$ is C , a set of measure zero, we conclude

$$\varphi'(x) = 0 \quad \text{for almost all } x \in (0, 1).$$

Now suppose, to argue by contradiction, that φ were absolutely continuous on $[0, 1]$. Theorem 10 asserts that if a function f is absolutely continuous on the closed, bounded interval $[a, b]$, then f is differentiable almost everywhere on (a, b) , its derivative f' is integrable over $[a, b]$, and

$$\int_a^b f' = f(b) - f(a).$$

Applying this to φ on $[0, 1]$, and using that φ' vanishes almost everywhere on $(0, 1)$ together with the fact that the integral of a function vanishing almost everywhere is zero,

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi' = \int_0^1 0 = 0.$$

But the Cantor-Lebesgue function satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$, so $\varphi(1) - \varphi(0) = 1$. This contradiction shows that φ is not absolutely continuous on $[0, 1]$.

Equivalently, in the language introduced just before these problems: φ is of bounded variation (being increasing) and its derivative vanishes almost everywhere, so φ is a singular function; and Theorem 10 tells us that an absolutely continuous function is singular if and only if it is constant. Since φ is not constant, it is not absolutely continuous.

Comparison with Problem 40. There one first shows (Problems 38 and 39) that an increasing absolutely continuous function on $[a, b]$ maps sets of measure zero to sets of measure zero. One then observes that $\psi(x) = x + \varphi(x)$ maps the Cantor set C , a set of measure zero, onto a set of measure 1 (page 52). If φ were absolutely continuous, then so would be ψ , being the sum of the absolutely continuous functions $x \mapsto x$ and φ ; and ψ is increasing, being the sum of an increasing and a strictly increasing function. So ψ would have to carry C to a set of measure zero, contradicting $m(\psi(C)) = 1$.

The two arguments turn on the same underlying fact, that φ accomplishes all of its increase on the null set C , but they extract the contradiction by different means. The Problem 40 argument is purely measure-theoretic: it uses only the definition of absolute continuity, through a covering estimate on outer measure, and never differentiates anything; on the other hand it needs the auxiliary function ψ and the nontrivial fact that $m(\psi(C)) = 1$. The present argument is shorter and more direct, requiring only the elementary observation that $\varphi' = 0$ off C , but it invokes the whole of the fundamental theorem of integral calculus for the Lebesgue integral, Theorem 10, which itself rests on Lebesgue's theorem on the differentiability of monotone functions, on Theorem 9, and on the Vitali Convergence Theorem. So the second argument buys brevity with much deeper machinery.

Problem 6.49. Let f be continuous on $[a, b]$ and differentiable almost everywhere on (a, b) . Show that

$$\int_a^b f' = f(b) - f(a)$$

if and only if

$$\int_a^b \left[\lim_{n \rightarrow \infty} \text{Diff}_{1/n} f \right] = \lim_{n \rightarrow \infty} \left[\int_a^b \text{Diff}_{1/n} f \right].$$

Solution. Throughout, f is extended to $[a, b + 1]$ by setting $f(x) = f(b)$ for $b < x \leq b + 1$, and for $0 < h \leq 1$,

$$\text{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h}, \quad \text{Av}_h f(x) = \frac{1}{h} \int_x^{x+h} f \quad \text{for } x \in [a, b],$$

as on page 113. Since f is continuous on $[a, b]$ and the extension is the constant $f(b)$ on $[b, b + 1]$, the extended function is continuous on $[a, b + 1]$. Hence for each natural number n the function $\text{Diff}_{1/n} f$ is continuous, and therefore measurable and bounded, on $[a, b]$; in particular it is integrable over $[a, b]$ and the right-hand integral below is defined for every n .

The proof consists of two observations, after which the two displayed assertions are seen to be one and the same.

Step 1: for every continuous f on $[a, b]$,

$$\lim_{n \rightarrow \infty} \left[\int_a^b \text{Diff}_{1/n} f \right] = f(b) - f(a).$$

Take $u = a$ and $v = b$ in the change of variables formula (14) to obtain the discrete formulation of the fundamental theorem of integral calculus,

$$\int_a^b \text{Diff}_h f = \text{Av}_h f(b) - \text{Av}_h f(a) \quad \text{for } 0 < h \leq 1.$$

Now f is identically equal to $f(b)$ on $(b, b + 1]$, so for $0 < h \leq 1$,

$$\text{Av}_h f(b) = \frac{1}{h} \int_b^{b+h} f = \frac{1}{h} \int_b^{b+h} f(b) dt = f(b),$$

the value of f at the single point b being irrelevant to the integral. On the other hand, by the continuity of f at a ,

$$|\text{Av}_h f(a) - f(a)| = \left| \frac{1}{h} \int_a^{a+h} [f(t) - f(a)] dt \right| \leq \sup_{a \leq t \leq a+h} |f(t) - f(a)| \longrightarrow 0 \text{ as } h \rightarrow 0^+.$$

Taking $h = 1/n$ and letting $n \rightarrow \infty$ gives

$$\lim_{n \rightarrow \infty} \left[\int_a^b \text{Diff}_{1/n} f \right] = f(b) - \lim_{n \rightarrow \infty} \text{Av}_{1/n} f(a) = f(b) - f(a),$$

which is exactly (29) in the proof of Theorem 10. Note that this uses only the continuity of f ; no differentiability and no absolute continuity is involved.

Step 2: $\lim_{n \rightarrow \infty} \text{Diff}_{1/n} f = f'$ almost everywhere on $[a, b]$.

Let $x \in (a, b)$ be a point at which f is differentiable; by hypothesis almost every point of (a, b) is such a point. Choose N with $1/N < b - x$. For $n \geq N$ we have $x + 1/n \in (a, b)$, so the extension of f beyond b plays no role, and

$$\text{Diff}_{1/n} f(x) = \frac{f(x + 1/n) - f(x)}{1/n} \longrightarrow f'(x) \text{ as } n \rightarrow \infty,$$

since $1/n \rightarrow 0^+$ and f is differentiable at x . Thus the sequence $\{\text{Diff}_{1/n} f\}$ converges pointwise almost everywhere on $[a, b]$, and its limit function is f' almost everywhere. (In particular f' , being the almost-everywhere pointwise limit of a sequence of continuous, hence measurable, functions, is measurable on $[a, b]$.)

Conclusion.

By Step 2 the two functions $\lim_{n \rightarrow \infty} \text{Diff}_{1/n} f$ and f' are defined and equal at almost every point of $[a, b]$. Two measurable functions that agree almost everywhere on $[a, b]$ have the same integral over $[a, b]$, in the strong sense that one is integrable if and only if the other is, and then their integrals are equal. Hence

$$\int_a^b \left[\lim_{n \rightarrow \infty} \text{Diff}_{1/n} f \right] = \int_a^b f',$$

each side being defined precisely when the other is. By Step 1,

$$\lim_{n \rightarrow \infty} \left[\int_a^b \text{Diff}_{1/n} f \right] = f(b) - f(a),$$

this limit always existing.

Substituting these two identities into the second displayed assertion turns it, term by term, into the first: the assertion

$$\int_a^b \left[\lim_{n \rightarrow \infty} \text{Diff}_{1/n} f \right] = \lim_{n \rightarrow \infty} \left[\int_a^b \text{Diff}_{1/n} f \right]$$

says that $\int_a^b f'$ equals $f(b) - f(a)$, and conversely. Therefore each of the two assertions holds if and only if the other does. (Both presuppose the same thing, namely that f' is integrable over $[a, b]$, equivalently that $\lim_n \text{Diff}_{1/n} f$ is; the Remark on page 114 shows that this can fail, for instance for $f(x) = x^2 \sin(1/x^2)$ on $[0, 1]$ with $f(0) = 0$.)

The content of the exercise is thus the following reformulation of the fundamental theorem of integral calculus for a continuous, almost everywhere differentiable f : the equality $\int_a^b f' = f(b) - f(a)$ holds if and only if one may pass the limit under the integral sign for the sequence of divided difference functions $\{\text{Diff}_{1/n} f\}$. This is precisely how Theorem 10 is proved for absolutely continuous f : Theorem 9 tells us that $\{\text{Diff}_h f\}_{0 < h \leq 1}$ is then uniformly integrable over $[a, b]$, and the Vitali Convergence Theorem licenses the interchange, giving (30). Conversely the Cantor-Lebesgue function φ shows that the interchange genuinely can fail: there $\lim_n \text{Diff}_{1/n} \varphi = \varphi' = 0$ almost everywhere, so the left-hand side is 0, while the right-hand side equals $\varphi(1) - \varphi(0) = 1$.

6.8 Problems 6.50–6.56

Problem 6.50. Let f be continuous on $[a, b]$ and differentiable almost everywhere on (a, b) . Show that if $\{\text{Diff}_{1/n} f\}$ is uniformly integrable over $[a, b]$, then

$$\int_a^b f' = f(b) - f(a).$$

Solution. Throughout we use the conventions of Section 6.2: the function f is extended to $[a, b+1]$ by setting $f \equiv f(b)$ on $(b, b+1]$, and for $0 < h \leq 1$,

$$\text{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h}, \quad \text{Av}_h f(x) = \frac{1}{h} \int_x^{x+h} f \quad \text{for all } x \in [a, b].$$

The extended function is continuous on $[a, b+1]$, since f is continuous on $[a, b]$ and the extension is constant, equal to $f(b)$, on $[b, b+1]$.

Step 1: the limit of the integrals of the divided differences. By the change of variables formula (14) of Section 6.2, taking $u = a$ and $v = b$,

$$\int_a^b \text{Diff}_h f = \text{Av}_h f(b) - \text{Av}_h f(a) \quad \text{for all } 0 < h \leq 1.$$

Fix $x \in \{a, b\}$. Since

$$|\text{Av}_h f(x) - f(x)| = \left| \frac{1}{h} \int_x^{x+h} [f(t) - f(x)] dt \right| \leq \sup_{x \leq t \leq x+h} |f(t) - f(x)|,$$

and f is continuous at x , we conclude that $\text{Av}_h f(x) \rightarrow f(x)$ as $h \rightarrow 0^+$. Therefore

$$\lim_{n \rightarrow \infty} \int_a^b \text{Diff}_{1/n} f = f(b) - f(a). \quad (*)$$

Step 2: passage to the limit under the integral sign. Each function $\text{Diff}_{1/n} f$ is continuous on $[a, b]$, since $x \mapsto f(x + 1/n)$ and $x \mapsto f(x)$ are continuous there; hence each $\text{Diff}_{1/n} f$ is measurable and bounded on $[a, b]$ and therefore integrable over $[a, b]$, a set of finite measure. Moreover, if $x \in (a, b)$ is a point at which f is differentiable, then

$$\lim_{n \rightarrow \infty} \text{Diff}_{1/n} f(x) = \lim_{n \rightarrow \infty} \frac{f(x + 1/n) - f(x)}{1/n} = f'(x),$$

so that $\{\text{Diff}_{1/n} f\}$ converges pointwise to f' almost everywhere on $[a, b]$; in particular f' , being the almost-everywhere pointwise limit of a sequence of measurable functions, is measurable.

By hypothesis the sequence $\{\text{Diff}_{1/n} f\}$ is uniformly integrable over $[a, b]$. Since $m([a, b]) < \infty$, the Vitali Convergence Theorem of Chapter 4 applies: f' is integrable over $[a, b]$ and

$$\lim_{n \rightarrow \infty} \int_a^b \text{Diff}_{1/n} f = \int_a^b \lim_{n \rightarrow \infty} \text{Diff}_{1/n} f = \int_a^b f'.$$

Step 3: conclusion. Comparing this with $(*)$, the two limits of the same sequence of numbers must agree, and therefore

$$\int_a^b f' = f(b) - f(a).$$

Problem 6.51. Let f be continuous on $[a, b]$ and differentiable almost everywhere on (a, b) . Suppose there is a nonnegative function g that is integrable over $[a, b]$ and

$$|\text{Diff}_{1/n} f| \leq g \quad \text{a.e. on } [a, b] \text{ for all } n.$$

Show that

$$\int_a^b f' = f(b) - f(a).$$

Solution. As in the preceding problem, f is extended to $[a, b+1]$ by $f \equiv f(b)$ on $(b, b+1]$; the extended function is continuous on $[a, b+1]$, and for $0 < h \leq 1$ the divided difference and average value functions on $[a, b]$ are

$$\text{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h}, \quad \text{Av}_h f(x) = \frac{1}{h} \int_x^{x+h} f.$$

Step 1: the limit of the integrals. By the change of variables formula (14) of Section 6.2 with $u = a$, $v = b$,

$$\int_a^b \text{Diff}_h f = \text{Av}_h f(b) - \text{Av}_h f(a) \quad \text{for } 0 < h \leq 1.$$

For $x \in \{a, b\}$,

$$|\text{Av}_h f(x) - f(x)| = \left| \frac{1}{h} \int_x^{x+h} [f(t) - f(x)] dt \right| \leq \sup_{x \leq t \leq x+h} |f(t) - f(x)| \rightarrow 0 \quad \text{as } h \rightarrow 0^+,$$

by the continuity of f at x . Hence

$$\lim_{n \rightarrow \infty} \int_a^b \text{Diff}_{1/n} f = f(b) - f(a). \quad (*)$$

Step 2: pointwise convergence. Each $\text{Diff}_{1/n} f$ is continuous, hence measurable, on $[a, b]$. If $x \in (a, b)$ is a point of differentiability of f , then $\text{Diff}_{1/n} f(x) \rightarrow f'(x)$. Since f is differentiable almost everywhere on (a, b) , the sequence $\{\text{Diff}_{1/n} f\}$ converges pointwise almost everywhere on $[a, b]$ to f' , and therefore f' is measurable.

Step 3: domination. By hypothesis there is a nonnegative integrable g on $[a, b]$ with $|\text{Diff}_{1/n} f| \leq g$ almost everywhere on $[a, b]$, for every n . Discarding the countable union of the exceptional sets, which is a set of measure zero, we may assume $|\text{Diff}_{1/n} f| \leq g$ on $[a, b] \sim E_0$ with $m(E_0) = 0$, for all n simultaneously. Passing to the limit in this inequality gives $|f'| \leq g$ almost everywhere on $[a, b]$, so f' is integrable over $[a, b]$ by the integral comparison test.

The Lebesgue Dominated Convergence Theorem of Chapter 4 now applies to the sequence $\{\text{Diff}_{1/n} f\}$, which converges pointwise almost everywhere on $[a, b]$ to f' and is dominated there by the integrable function g . It yields

$$\lim_{n \rightarrow \infty} \int_a^b \text{Diff}_{1/n} f = \int_a^b f'.$$

(Alternatively, a sequence of measurable functions on a set of finite measure that is dominated by a single integrable function is uniformly integrable, by the absolute continuity of integration, Proposition 23 of Chapter 4; so the conclusion also follows immediately from the preceding problem.)

Step 4: conclusion. Comparing the last display with $(*)$,

$$\int_a^b f' = f(b) - f(a).$$

Problem 6.52. Let f and g be absolutely continuous on $[a, b]$. Show that

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$

Solution. Step 1: the product $f \cdot g$ is absolutely continuous on $[a, b]$. An absolutely continuous function is continuous, so both f and g are continuous on the closed, bounded interval $[a, b]$ and hence bounded there: choose $M > 0$ with $|f| \leq M$ and $|g| \leq M$ on $[a, b]$. Let $\epsilon > 0$. Choose $\delta_1 > 0$ responding to the challenge $\epsilon/(2M)$ in the criterion of absolute continuity for f , and $\delta_2 > 0$ responding to the same challenge for g ; put $\delta = \min\{\delta_1, \delta_2\}$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$. Then, using the algebraic identity

$$f(b_k)g(b_k) - f(a_k)g(a_k) = f(b_k)[g(b_k) - g(a_k)] + g(a_k)[f(b_k) - f(a_k)],$$

we obtain

$$\sum_{k=1}^n |f(b_k)g(b_k) - f(a_k)g(a_k)| \leq M \sum_{k=1}^n |g(b_k) - g(a_k)| + M \sum_{k=1}^n |f(b_k) - f(a_k)| < M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} = \epsilon.$$

Thus $f \cdot g$ is absolutely continuous on $[a, b]$.

Step 2: the derivative of the product. By Theorem 10, each of f , g and $f \cdot g$ is differentiable almost everywhere on (a, b) , and each of f' , g' and $(f \cdot g)'$ is integrable over $[a, b]$. Let $x \in (a, b)$ be a point at which both f and g are differentiable; the set of such x has full measure in (a, b) , being the intersection of two sets of full measure. At such a point the usual product rule of elementary calculus is valid, since for $h \neq 0$ small,

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h},$$

and letting $h \rightarrow 0$, using the continuity of f at x , the right-hand side converges to $f(x)g'(x) + g(x)f'(x)$. Hence

$$(f \cdot g)' = f \cdot g' + f' \cdot g \quad \text{almost everywhere on } (a, b).$$

Step 3: integrability of each term. Since f and g are bounded by M and f' , g' are integrable over $[a, b]$, the comparison test gives that $f \cdot g'$ and $f' \cdot g$ are integrable over $[a, b]$, because $|f \cdot g'| \leq M|g'|$ and $|f' \cdot g| \leq M|f'|$ almost everywhere.

Step 4: conclusion. Apply Theorem 10 to the absolutely continuous function $f \cdot g$ and then use the linearity of integration, which is legitimate since both $f \cdot g'$ and $f' \cdot g$ are integrable:

$$f(b)g(b) - f(a)g(a) = \int_a^b (f \cdot g)' = \int_a^b [f \cdot g' + f' \cdot g] = \int_a^b f \cdot g' + \int_a^b f' \cdot g.$$

Rearranging,

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g,$$

which is the integration by parts formula for absolutely continuous functions.

Problem 6.53. Let the function f be absolutely continuous on $[a, b]$. Show that f is Lipschitz on $[a, b]$ if and only if there is a $c > 0$ for which $|f'| \leq c$ a.e. on $[a, b]$.

Solution. Since f is absolutely continuous on $[a, b]$, Theorem 10 tells us that f is differentiable almost everywhere on (a, b) , that f' is integrable over $[a, b]$, and that f is the indefinite integral of f' ; indeed, by Theorem 11 (or by applying Theorem 10 to the restriction of f to a closed subinterval, which is

again absolutely continuous),

$$f(v) - f(u) = \int_u^v f' \quad \text{for all } a \leq u \leq v \leq b. \quad (\dagger)$$

Suppose first that f is Lipschitz on $[a, b]$, so there is a $c > 0$ with

$$|f(u) - f(v)| \leq c|u - v| \quad \text{for all } u, v \in [a, b].$$

Let $x \in (a, b)$ be a point at which f is differentiable. For $h \neq 0$ small enough that $x + h \in [a, b]$,

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq \frac{c|h|}{|h|} = c.$$

Letting $h \rightarrow 0$ and using the continuity of the absolute value, we get $|f'(x)| \leq c$. Since f is differentiable at almost all $x \in (a, b)$, we conclude that $|f'| \leq c$ almost everywhere on $[a, b]$. (Observe that this half of the equivalence requires no absolute continuity assumption; it holds for any Lipschitz function, which in any case is absolutely continuous.)

Conversely, suppose there is a $c > 0$ with $|f'| \leq c$ almost everywhere on $[a, b]$. Let $u, v \in [a, b]$ with $u < v$. By $(\dagger)$ and the monotonicity of integration together with the triangle inequality for integrals,

$$|f(v) - f(u)| = \left| \int_u^v f' \right| \leq \int_u^v |f'| \leq \int_u^v c = c(v - u).$$

Here the middle inequality is legitimate because f' is integrable over $[u, v]$, and the last inequality holds because $|f'| \leq c$ almost everywhere on $[u, v]$ and altering an integrand on a set of measure zero does not change its integral. Since the resulting estimate is symmetric in u and v and trivial when $u = v$, we have

$$|f(u) - f(v)| \leq c|u - v| \quad \text{for all } u, v \in [a, b],$$

that is, f is Lipschitz on $[a, b]$ with Lipschitz constant c .

Problem 6.54. (i) Let f be a singular increasing function on $[a, b]$. Use the Vitali Covering Lemma to show that f has the following property: Given $\epsilon > 0$, $\delta > 0$, there is a finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) for which

$$\sum_{k=1}^n [b_k - a_k] < \delta \quad \text{and} \quad \sum_{k=1}^n [f(b_k) - f(a_k)] > f(b) - f(a) - \epsilon.$$

(ii) Let f be an increasing function on $[a, b]$ with the property described in part (i). Show that f is singular.

(iii) Let $\{f_n\}$ be a sequence of singular increasing functions on $[a, b]$ for which the series $\sum_{n=1}^{\infty} f_n(x)$ converges to a finite value for each $x \in [a, b]$. Define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for } x \in [a, b].$$

Show that f is also singular.

Solution. Recall that a function of bounded variation is called singular provided its derivative vanishes almost everywhere. An increasing function on $[a, b]$ is of bounded variation, with $TV(f) = f(b) - f(a)$. Part (i). Let $\epsilon > 0$ and $\delta > 0$ be given. Set

$$\eta = \frac{\epsilon}{b-a} > 0,$$

and let E be the set of points $x \in (a, b)$ at which f is differentiable with $f'(x) = 0$. Since f is singular,

$$m(E) = b - a.$$

Let $\mathcal{F}$ be the collection of all closed, bounded, nondegenerate intervals $[c, d] \subseteq (a, b)$ for which

$$f(d) - f(c) < \eta(d - c).$$

We claim $\mathcal{F}$ covers E in the sense of Vitali. Indeed, let $x \in E$ and let $\ell > 0$. Since $f'(x) = 0$, there is an $h_0 > 0$ with $[x - h_0, x + h_0] \subseteq (a, b)$, $2h_0 < \ell$, and

$$|f(x + t) - f(x)| \leq \frac{\eta}{2} |t| \quad \text{for all } |t| \leq h_0.$$

Then for $0 < h \leq h_0$, the interval $[x - h, x + h]$ contains x , has length $2h < \ell$, and

$$f(x + h) - f(x - h) = [f(x + h) - f(x)] + [f(x) - f(x - h)] \leq \frac{\eta}{2}h + \frac{\eta}{2}h = \eta h < \eta \cdot (2h).$$

So $[x - h, x + h] \in \mathcal{F}$. Thus each point of E is contained in intervals of $\mathcal{F}$ of arbitrarily small length, which is precisely the Vitali covering condition.

Since $m^*(E) = b - a < \infty$, the Vitali Covering Lemma provides a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left(E \sim \bigcup_{k=1}^n [c_k, d_k]\right) < \delta.$$

Relabel so that $c_1 < d_1 < c_2 < d_2 < \cdots < c_n < d_n$; since these are disjoint closed intervals contained in (a, b) we also have $a < c_1$ and $d_n < b$. Put $d_0 = a$ and $c_{n+1} = b$, and consider the $n + 1$ complementary open intervals

$$(a_j, b_j) := (d_{j-1}, c_j), \quad j = 1, \dots, n + 1.$$

Each is a nonempty open interval contained in (a, b) , and they are pairwise disjoint.

Their total length is small. Indeed, $(a, b) \sim \bigcup_{k=1}^n [c_k, d_k]$ differs from $E \sim \bigcup_{k=1}^n [c_k, d_k]$ by a subset of $(a, b) \sim E$, a set of measure zero; therefore

$$\sum_{j=1}^{n+1} [b_j - a_j] = (b - a) - \sum_{k=1}^n [d_k - c_k] = m\left((a, b) \sim \bigcup_{k=1}^n [c_k, d_k]\right) = m^*\left(E \sim \bigcup_{k=1}^n [c_k, d_k]\right) < \delta.$$

Their total f -increase is large. Since the intervals $[c_k, d_k]$ and the gaps (d_{j-1}, c_j) together exhaust $[a, b]$ and abut consecutively, the increments telescope:

$$\sum_{j=1}^{n+1} [f(b_j) - f(a_j)] = [f(b) - f(a)] - \sum_{k=1}^n [f(d_k) - f(c_k)].$$

Because each $[c_k, d_k]$ belongs to $\mathcal{F}$ and the $[c_k, d_k]$ are disjoint subintervals of $[a, b]$,

$$\sum_{k=1}^n [f(d_k) - f(c_k)] < \eta \sum_{k=1}^n [d_k - c_k] \leq \eta(b - a) = \epsilon$$

(if $n = 0$ the left-hand sum is $0 < \epsilon$ as well). Therefore

$$\sum_{j=1}^{n+1} [f(b_j) - f(a_j)] > f(b) - f(a) - \epsilon,$$

and the collection $\{(a_j, b_j)\}_{j=1}^{n+1}$ has both required properties.

Part (ii). Let f be increasing on $[a, b]$ with the stated property. By Lebesgue's Theorem f is differentiable almost everywhere on (a, b) , $f' \geq 0$ almost everywhere, and by Corollary 4 the function f' is integrable over $[a, b]$. We show $\int_a^b f' = 0$; since $f' \geq 0$ almost everywhere, Proposition 9 of Chapter 4 then forces $f' = 0$ almost everywhere, that is, f is singular.

Let $\epsilon > 0$. Since f' is integrable over $[a, b]$, by the absolute continuity of integration (Proposition 23 of Chapter 4) there is a $\delta > 0$ such that

$$\int_A f' < \epsilon \quad \text{whenever } A \subseteq [a, b] \text{ is measurable with } m(A) < \delta.$$

For this ϵ and δ , the assumed property gives a finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) with

$$\sum_{k=1}^n [b_k - a_k] < \delta \quad \text{and} \quad \sum_{k=1}^n [f(b_k) - f(a_k)] > f(b) - f(a) - \epsilon.$$

Let $U = \bigcup_{k=1}^n (a_k, b_k)$, so that $m(U) < \delta$ and hence $\int_U f' < \epsilon$.

The complement $[a, b] \sim U$ is a finite disjoint union of closed intervals $[\alpha_j, \beta_j]$, $j = 1, \dots, p$, whose endpoints are exactly a, b and the points a_k, b_k (a degenerate one, occurring when $b_k = a_{k+1}$, is a single point and contributes nothing to any of the sums below; discard such j). Consecutive intervals of the two families abut, so the increments again telescope:

$$\sum_{j=1}^p [f(\beta_j) - f(\alpha_j)] = [f(b) - f(a)] - \sum_{k=1}^n [f(b_k) - f(a_k)] < \epsilon.$$

Applying Corollary 4 to the restriction of the increasing function f to each $[\alpha_j, \beta_j]$, and then using the additivity of integration over domains,

$$\int_{[a,b] \sim U} f' = \sum_{j=1}^p \int_{\alpha_j}^{\beta_j} f' \leq \sum_{j=1}^p [f(\beta_j) - f(\alpha_j)] < \epsilon.$$

(The finitely many endpoints form a set of measure zero and contribute nothing.) Hence

$$0 \leq \int_a^b f' = \int_U f' + \int_{[a,b] \sim U} f' < 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, $\int_a^b f' = 0$, and f is singular.

Part (iii). Each f_n is increasing, so for $a \leq x < y \leq b$ we have $f(y) - f(x) = \sum_{n=1}^{\infty} [f_n(y) - f_n(x)] \geq 0$; here the series of nonnegative terms converges because both $\sum f_n(y)$ and $\sum f_n(x)$ converge. Thus f is a real-valued increasing function on $[a, b]$. We verify that f has the property of part (i); part (ii) will then show f is singular.

For $N \in \mathbb{N}$ set $s_N = \sum_{n=1}^N f_n$ and $r_N = f - s_N = \sum_{n>N} f_n$. Both are increasing on $[a, b]$, and s_N is singular, since a finite sum of functions differentiable at a point is differentiable there with derivative the sum of the derivatives, so $s'_N = \sum_{n=1}^N f'_n = 0$ almost everywhere on (a, b) .

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $s_N(b) - s_N(a) = \sum_{n=1}^N [f_n(b) - f_n(a)]$ converges to $f(b) - f(a)$ as $N \rightarrow \infty$, we may fix N with

$$r_N(b) - r_N(a) = [f(b) - f(a)] - [s_N(b) - s_N(a)] < \frac{\epsilon}{2}.$$

Apply part (i) to the singular increasing function s_N , with the challenges $\epsilon/2$ and δ : there is a finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) with

$$\sum_{k=1}^n [b_k - a_k] < \delta \quad \text{and} \quad \sum_{k=1}^n [s_N(b_k) - s_N(a_k)] > s_N(b) - s_N(a) - \frac{\epsilon}{2}.$$

Since r_N is increasing, $f(b_k) - f(a_k) \geq s_N(b_k) - s_N(a_k)$ for each k . Therefore

$$\sum_{k=1}^n [f(b_k) - f(a_k)] \geq \sum_{k=1}^n [s_N(b_k) - s_N(a_k)] > s_N(b) - s_N(a) - \frac{\epsilon}{2} = [f(b) - f(a)] - [r_N(b) - r_N(a)] - \frac{\epsilon}{2} > f(b) - f(a) - \epsilon.$$

Thus f possesses the property of part (i), and by part (ii) the function f is singular.

Problem 6.55. Let f be of bounded variation on $[a, b]$, and define $v(x) = TV(f_{[a,x]})$ for all $x \in [a, b]$.

(i) Show that $|f'| \leq v'$ a.e. on $[a, b]$, and infer from this that

$$\int_a^b |f'| \leq TV(f).$$

(ii) Show that the above is an equality if and only if f is absolutely continuous on $[a, b]$.

(iii) Compare parts (i) and (ii) with Corollaries 4 and 12, respectively.

Solution. Part (i). By (21) of Section 6.3, for $a \leq u < w \leq b$,

$$v(w) - v(u) = TV(f_{[a,w]}) - TV(f_{[a,u]}) = TV(f_{[u,w]}) \geq 0,$$

so v is a real-valued increasing function on $[a, b]$ with $v(a) = 0$ and $v(b) = TV(f)$. Moreover, taking the crudest partition $\{u, w\}$ of $[u, w]$,

$$|f(w) - f(u)| \leq TV(f_{[u,w]}) = v(w) - v(u) \quad \text{for all } a \leq u < w \leq b. \quad (*)$$

By Lebesgue's Theorem the increasing function v is differentiable almost everywhere on (a, b) , and by Corollary 6 the function f , being of bounded variation, is differentiable almost everywhere on (a, b) with f' integrable over $[a, b]$. Let $x \in (a, b)$ be a point at which both f and v are differentiable; almost every point of (a, b) is such a point. For small $h > 0$, $(*)$ with $u = x$, $w = x + h$ gives

$$\frac{|f(x+h) - f(x)|}{h} \leq \frac{v(x+h) - v(x)}{h}.$$

Letting $h \rightarrow 0^+$ yields $|f'(x)| \leq v'(x)$. Hence $|f'| \leq v'$ almost everywhere on $[a, b]$.

Now v is increasing on $[a, b]$, so Corollary 4 tells us v' is integrable over $[a, b]$ and

$$\int_a^b v' \leq v(b) - v(a) = TV(f).$$

Since $0 \leq |f'| \leq v'$ almost everywhere and $|f'|$ is integrable (as f' is), the monotonicity of integration gives

$$\int_a^b |f'| \leq \int_a^b v' \leq TV(f). \quad (**)$$

Part (ii). Suppose first that f is absolutely continuous on $[a, b]$. By Theorem 11, f is the indefinite integral of f' , so for any partition $P = \{x_0, x_1, \dots, x_k\}$ of $[a, b]$,

$$V(f, P) = \sum_{i=1}^k |f(x_i) - f(x_{i-1})| = \sum_{i=1}^k \left| \int_{x_{i-1}}^{x_i} f' \right| \leq \sum_{i=1}^k \int_{x_{i-1}}^{x_i} |f'| = \int_a^b |f'|.$$

Taking the supremum over all partitions, $TV(f) \leq \int_a^b |f'|$. Together with $(**)$ this gives $\int_a^b |f'| = TV(f)$.

Conversely, suppose $\int_a^b |f'| = TV(f)$. Fix $x \in (a, b)$. The restriction of f to $[a, x]$ is of bounded variation with total variation function $u \mapsto v(u)$ on $[a, x]$, and its derivative agrees with f' on (a, x) ; similarly for the restriction to $[x, b]$, whose total variation is $TV(f_{[x,b]}) = v(b) - v(x)$ by (20). Applying the inequality of part (i) to each restriction,

$$\int_a^x |f'| \leq TV(f_{[a,x]}) = v(x) \quad \text{and} \quad \int_x^b |f'| \leq TV(f_{[x,b]}) = v(b) - v(x).$$

Adding these and using additivity over domains together with the assumed equality,

$$v(b) = TV(f) = \int_a^b |f'| = \int_a^x |f'| + \int_x^b |f'| \leq v(x) + [v(b) - v(x)] = v(b).$$

Hence both inequalities in the preceding display are equalities; in particular

$$v(x) = \int_a^x |f'| = v(a) + \int_a^x |f'| \quad \text{for all } x \in [a, b]$$

(the cases $x = a$ and $x = b$ being immediate). Thus v is the indefinite integral over $[a, b]$ of the integrable function $|f'|$, and therefore, by Theorem 11, v is absolutely continuous on $[a, b]$.

Finally, (*) transfers absolute continuity from v to f . Let $\epsilon > 0$ and choose $\delta > 0$ responding to the ϵ challenge in the criterion of absolute continuity for v . If $\{(a_k, b_k)\}_{k=1}^n$ is a disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n [b_k - a_k] < \delta$, then by (*),

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \leq \sum_{k=1}^n [v(b_k) - v(a_k)] = \sum_{k=1}^n |v(b_k) - v(a_k)| < \epsilon.$$

So f is absolutely continuous on $[a, b]$.

Part (iii). Suppose f is increasing on $[a, b]$. Then f is of bounded variation with $TV(f) = f(b) - f(a)$, and $f' \geq 0$ wherever it exists, so $|f'| = f'$ almost everywhere. With these identifications, the inequality of part (i) reads

$$\int_a^b f' \leq f(b) - f(a),$$

which is exactly Corollary 4; and the equivalence of part (ii) reads: f is absolutely continuous on $[a, b]$ if and only if $\int_a^b f' = f(b) - f(a)$, which is exactly Corollary 12. (For decreasing f the same reduction holds, since then $TV(f) = f(a) - f(b)$ and $|f'| = -f'$.) Thus parts (i) and (ii) are precisely the extensions of Corollaries 4 and 12 from monotone functions to functions of bounded variation, the increment $f(b) - f(a)$ being replaced by the total variation $TV(f)$ and the derivative f' by $|f'|$.

Problem 6.56. Let g be strictly increasing and absolutely continuous on $[a, b]$.

(i) Show that for any open subset $\mathcal{O}$ of (a, b) ,

$$m(g(\mathcal{O})) = \int_{\mathcal{O}} g'(x) dx.$$

(ii) Show that for any G_δ subset E of (a, b) ,

$$m(g(E)) = \int_E g'(x) dx.$$

(iii) Show that for any subset E of $[a, b]$ that has measure 0, its image $g(E)$ also has measure 0, so that

$$m(g(E)) = 0 = \int_E g'(x) dx.$$

(iv) Show that for any measurable subset A of $[a, b]$,

$$m(g(A)) = \int_A g'(x) dx.$$

(v) Let $c = g(a)$ and $d = g(b)$. Show that for any simple function φ on $[c, d]$,

$$\int_c^d \varphi(y) dy = \int_a^b \varphi(g(x)) g'(x) dx.$$

(vi) Show that for any nonnegative integrable function f over $[c, d]$,

$$\int_c^d f(y) dy = \int_a^b f(g(x)) g'(x) dx.$$

(vii) Show that part (i) follows from (vi) in the case that f is the characteristic function of $\mathcal{O}$ and the composition is defined.

Solution. Throughout, $c = g(a)$ and $d = g(b)$. Since g is absolutely continuous, it is continuous, and being strictly increasing it is a one-to-one continuous map of $[a, b]$ onto $[c, d]$ (onto by the Intermediate Value Theorem). By Theorem 10, g is differentiable almost everywhere on (a, b) and g' is integrable over $[a, b]$; since g is increasing, $g' \geq 0$ almost everywhere. Note also that g carries an open interval $(u, w) \subseteq (a, b)$ onto the open interval $(g(u), g(w))$, and that g , being one-to-one, carries disjoint sets to disjoint sets and satisfies $g(S \sim T) = g(S) \sim g(T)$ whenever $T \subseteq S \subseteq [a, b]$.

Part (i). Let $\mathcal{O}$ be an open subset of (a, b) . Write $\mathcal{O} = \bigcup_k (a_k, b_k)$ as the countable disjoint union of its component open intervals, each contained in (a, b) . Then

$$g(\mathcal{O}) = \bigcup_k g((a_k, b_k)) = \bigcup_k (g(a_k), g(b_k)),$$

a countable disjoint union of open intervals; in particular $g(\mathcal{O})$ is open, hence measurable, and by the countable additivity of Lebesgue measure

$$m(g(\mathcal{O})) = \sum_k [g(b_k) - g(a_k)].$$

The restriction of g to $[a_k, b_k]$ is absolutely continuous, so Theorem 10 gives $g(b_k) - g(a_k) = \int_{a_k}^{b_k} g'$. Since $g' \geq 0$ almost everywhere, the partial sums of $\sum_k g' \chi_{(a_k, b_k)}$ increase pointwise almost everywhere to $g' \chi_{\mathcal{O}}$, so the Monotone Convergence Theorem of Chapter 4, together with the finite additivity of integration over disjoint domains, gives $\sum_k \int_{a_k}^{b_k} g' = \int_{\mathcal{O}} g'$. Hence

$$m(g(\mathcal{O})) = \sum_k \int_{a_k}^{b_k} g' = \int_{\mathcal{O}} g'.$$

Part (ii). Let E be a G_δ subset of (a, b) , say $E = \bigcap_{n=1}^\infty \mathcal{U}_n$ with each $\mathcal{U}_n$ open. Replacing $\mathcal{U}_n$ by $\mathcal{O}_n = (a, b) \cap \bigcap_{j=1}^n \mathcal{U}_j$, we may assume the $\mathcal{O}_n$ are open subsets of (a, b) , descending, with $E = \bigcap_n \mathcal{O}_n$. Because g is one-to-one,

$$g(E) = g\left(\bigcap_n \mathcal{O}_n\right) = \bigcap_n g(\mathcal{O}_n),$$

so $g(E)$ is a G_δ set, in particular measurable. The sets $g(\mathcal{O}_n)$ are descending and contained in $g((a, b)) \subseteq [c, d]$, which has finite measure; by the continuity of measure,

$$m(g(E)) = \lim_{n \rightarrow \infty} m(g(\mathcal{O}_n)) = \lim_{n \rightarrow \infty} \int_{\mathcal{O}_n} g',$$

the last equality by part (i). On the other hand, $\chi_{\mathcal{O}_n} \rightarrow \chi_E$ pointwise on $[a, b]$ (the $\mathcal{O}_n$ descend to E), and $0 \leq g' \chi_{\mathcal{O}_n} \leq g'$ almost everywhere with g' integrable; the Lebesgue Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{\mathcal{O}_n} g' = \lim_{n \rightarrow \infty} \int_a^b g' \chi_{\mathcal{O}_n} = \int_a^b g' \chi_E = \int_E g'.$$

Therefore $m(g(E)) = \int_E g'$.

Part (iii). Let $E \subseteq [a, b]$ with $m(E) = 0$, and let $\epsilon > 0$. Since g' is integrable over $[a, b]$, the absolute continuity of integration (Proposition 23 of Chapter 4) provides a $\delta > 0$ such that $\int_A g' < \epsilon$ whenever $A \subseteq [a, b]$ is measurable with $m(A) < \delta$. As $m(E \cap (a, b)) = 0$, by the outer regularity of Lebesgue measure there is an open set $\mathcal{O}$ with $E \cap (a, b) \subseteq \mathcal{O} \subseteq (a, b)$ and $m(\mathcal{O}) < \delta$. By part (i) and the monotonicity of outer measure,

$$m^*(g(E \cap (a, b))) \leq m(g(\mathcal{O})) = \int_{\mathcal{O}} g' < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $m^*(g(E \cap (a, b))) = 0$. Finally $g(E) \sim g(E \cap (a, b)) \subseteq \{g(a), g(b)\}$ is finite, so $m^*(g(E)) = 0$ and $g(E)$ is measurable with $m(g(E)) = 0$. Also $\int_E g' = 0$ since $m(E) = 0$. Thus $m(g(E)) = 0 = \int_E g'$.

Part (iv). Let A be a measurable subset of $[a, b]$ and put $A_0 = A \cap (a, b)$. Then $A \sim A_0 \subseteq \{a, b\}$ has measure zero and $g(A) \sim g(A_0) \subseteq \{c, d\}$; since a set of measure zero may be added or removed without affecting either side of the asserted identity, it suffices to treat A_0 .

By the approximation properties of measurable sets (Theorem 11 of Chapter 2) there is a G_δ set G' with $A_0 \subseteq G'$ and $m(G' \sim A_0) = 0$. Set $G = G' \cap (a, b)$; since (a, b) is open, G is again a G_δ set, and $A_0 \subseteq G \subseteq (a, b)$ with $m(G \sim A_0) = 0$. Because g is one-to-one,

$$g(A_0) = g(G) \sim g(G \sim A_0).$$

By part (ii), $g(G)$ is measurable with $m(g(G)) = \int_G g'$; by part (iii), $g(G \sim A_0)$ is measurable with measure zero. Hence $g(A_0)$ is measurable and, by the excision property,

$$m(g(A_0)) = m(g(G)) - m(g(G \sim A_0)) = m(g(G)) = \int_G g' = \int_{A_0} g' + \int_{G \sim A_0} g' = \int_{A_0} g',$$

the last step because $m(G \sim A_0) = 0$. Therefore $m(g(A)) = \int_A g'$.

Part (v). We first record the key lemma, which also settles the measurability question implicit in the statement. Note that a strictly increasing absolutely continuous g need not carry measurable sets back to measurable sets under g^{-1} (the set $\{g' = 0\}$ can have positive measure), so the composition $\varphi \circ g$ need not be measurable; what is always measurable, up to a set of measure zero, is the product $(\varphi \circ g) \cdot g'$.

Lemma. Let $E \subseteq [c, d]$ be measurable. Then $x \mapsto \chi_E(g(x)) g'(x)$ agrees almost everywhere on $[a, b]$ with a measurable function, and

$$\int_a^b \chi_E(g(x)) g'(x) dx = m(E).$$

Proof of the Lemma. By the approximation properties of measurable sets there are Borel sets B and N with

$$B \subseteq E \subseteq B \cup N, \quad m(E \sim B) = 0, \quad m(N) = 0, \quad N \subseteq [c, d]$$

(take B an F_σ set contained in E with $m(E \sim B) = 0$, and N a G_δ set of measure zero containing $E \sim B$). Since g is continuous, the preimages $g^{-1}(B)$ and $g^{-1}(N)$ are Borel subsets of $[a, b]$, hence measurable, and

$$\chi_B \circ g = \chi_{g^{-1}(B)}, \quad \chi_N \circ g = \chi_{g^{-1}(N)}$$

are measurable functions. Because g maps $[a, b]$ one-to-one onto $[c, d]$, we have $g(g^{-1}(S)) = S$ for every $S \subseteq [c, d]$. Applying part (iv) to the measurable sets $g^{-1}(B)$ and $g^{-1}(N)$,

$$\int_{g^{-1}(B)} g' = m(B) = m(E), \quad \int_{g^{-1}(N)} g' = m(N) = 0.$$

As $g' \geq 0$ almost everywhere, the second identity forces $g' = 0$ almost everywhere on $g^{-1}(N)$. Since $g^{-1}(B) \subseteq g^{-1}(E) \subseteq g^{-1}(B) \cup g^{-1}(N)$,

$$0 \leq \chi_{g^{-1}(E)} - \chi_{g^{-1}(B)} \leq \chi_{g^{-1}(N)} \quad \text{on } [a, b],$$

and multiplying by $g' \geq 0$ we conclude that

$$\chi_E(g(x)) g'(x) = \chi_{g^{-1}(E)}(x) g'(x) = \chi_{g^{-1}(B)}(x) g'(x) \quad \text{for almost all } x \in [a, b].$$

The right-hand function is measurable, and its integral over $[a, b]$ equals $\int_{g^{-1}(B)} g' = m(E)$. The Lemma is proved.

Now let φ be simple on $[c, d]$, written as $\varphi = \sum_{i=1}^p \alpha_i \chi_{E_i}$ with the α_i real and the E_i measurable subsets of $[c, d]$. Then

$$\varphi(g(x)) g'(x) = \sum_{i=1}^p \alpha_i \chi_{E_i}(g(x)) g'(x) \quad \text{for all } x \text{ at which } g'(x) \text{ exists,}$$

which by the Lemma agrees almost everywhere with a measurable function; moreover $|\varphi(g)g'| \leq \max_i |\alpha_i| \cdot p \cdot g'$ almost everywhere, so it is integrable over $[a, b]$. By the linearity of integration and the Lemma,

$$\int_a^b \varphi(g(x)) g'(x) dx = \sum_{i=1}^p \alpha_i \int_a^b \chi_{E_i}(g(x)) g'(x) dx = \sum_{i=1}^p \alpha_i m(E_i) = \int_c^d \varphi(y) dy.$$

Part (vi). Let f be nonnegative and integrable over $[c, d]$. By the Simple Approximation Theorem of Chapter 3 there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on $[c, d]$ that converges pointwise on $[c, d]$ to f . By part (v), for each n the function $(\varphi_n \circ g) \cdot g'$ agrees almost everywhere on $[a, b]$ with a measurable function ψ_n , and

$$\int_a^b \psi_n = \int_c^d \varphi_n.$$

Discard the union of the countably many exceptional sets of measure zero, together with the set where g' fails to exist or is negative; off this single set E_0 of measure zero we have, for every n , $\psi_n = (\varphi_n \circ g) g' \geq 0$ and $\psi_n \leq \psi_{n+1}$, and

$$\lim_{n \rightarrow \infty} \psi_n(x) = \lim_{n \rightarrow \infty} \varphi_n(g(x)) g'(x) = f(g(x)) g'(x).$$

Hence $(f \circ g) \cdot g'$ agrees almost everywhere on $[a, b]$ with the measurable function $\lim_n \psi_n$, and the Monotone Convergence Theorem, applied on $[a, b] \sim E_0$, gives

$$\int_a^b f(g(x)) g'(x) dx = \lim_{n \rightarrow \infty} \int_a^b \psi_n = \lim_{n \rightarrow \infty} \int_c^d \varphi_n = \int_c^d f(y) dy,$$

the final equality by the Monotone Convergence Theorem on $[c, d]$. In particular $(f \circ g) \cdot g'$ is integrable over $[a, b]$, since f is integrable over $[c, d]$.

Part (vii). The proviso that the composition be defined is what fixes the meaning of the substitution: the function f in (vi) lives on $[c, d]$, whereas $\mathcal{O} \subseteq (a, b)$, so the characteristic function to be substituted is the one whose composition with g is $\chi_{\mathcal{O}}$, namely $\chi_{g(\mathcal{O})}$.

Let $\mathcal{O}$ be an open subset of (a, b) . As shown in part (i) the set $g(\mathcal{O})$ is open and bounded, hence measurable with finite measure, so $f = \chi_{g(\mathcal{O})}$ is a nonnegative integrable function on $[c, d]$. Because g is one-to-one, for $x \in [a, b]$ we have $g(x) \in g(\mathcal{O})$ if and only if $x \in \mathcal{O}$; that is, the composition is defined and

$$f(g(x)) = \chi_{g(\mathcal{O})}(g(x)) = \chi_{\mathcal{O}}(x) \quad \text{for all } x \in [a, b].$$

Applying part (vi) to this f ,

$$m(g(\mathcal{O})) = \int_c^d \chi_{g(\mathcal{O})}(y) dy = \int_a^b \chi_{g(\mathcal{O})}(g(x)) g'(x) dx = \int_a^b \chi_{\mathcal{O}}(x) g'(x) dx = \int_{\mathcal{O}} g'(x) dx,$$

which is exactly the assertion of part (i). (Of course the argument for part (i) given above is needed as an ingredient in the proof of part (vi), so this is a consistency check rather than an independent derivation.)

6.9 Problems 6.57–6.63

Problem 6.57. Is the change of variables formula in the last part of the preceding problem true if we just assume g is increasing, not necessarily strictly?

[Preceding problem, part (vi): for g strictly increasing and absolutely continuous on $[a, b]$, $c = g(a)$,

$d = g(b)$, and f a nonnegative integrable function over $[c, d]$,

$$\int_c^d f(y) dy = \int_a^b f(g(x))g'(x) dx.$$

]

Solution. Yes. The formula persists verbatim: if g is increasing (not necessarily strictly) and absolutely continuous on $[a, b]$, if $c = g(a)$, $d = g(b)$, and if f is a nonnegative integrable function over $[c, d]$, then

$$\int_c^d f(y) dy = \int_a^b f(g(x))g'(x) dx.$$

What must change is the proof: the route taken in the preceding problem passes through the identity $m(g(\mathcal{O})) = \int_{\mathcal{O}} g'$, whose proof uses that g carries disjoint intervals to disjoint intervals, i.e. that g is one-to-one. We argue instead by pushing Lebesgue measure back through g .

Since g is increasing and continuous, $g([a, b]) = [c, d]$. Since g is absolutely continuous, g' exists a.e. on (a, b) , is nonnegative a.e., is integrable, and, by Theorem 10, $\int_u^v g' = g(v) - g(u)$ for all $a \leq u \leq v \leq b$. Step 1: a set function. For a Borel subset E of $[c, d]$ define

$$\nu(E) = \int_a^b \chi_E(g(x)) g'(x) dx.$$

This makes sense: g is continuous, hence Borel measurable, so $\chi_E \circ g = \chi_{g^{-1}(E)}$ is a measurable function, and the integrand is nonnegative and measurable. Moreover ν is a measure on the Borel subsets of $[c, d]$: $\nu(\emptyset) = 0$, and if $\{E_k\}$ is a countable disjoint collection of Borel sets then $\chi_{\cup_k E_k} \circ g = \sum_k \chi_{E_k} \circ g$ pointwise, so countable additivity of the integral of nonnegative measurable functions (a consequence of the Monotone Convergence Theorem) gives $\nu(\cup_k E_k) = \sum_k \nu(E_k)$. Finally $\nu([c, d]) = \int_a^b g' = g(b) - g(a) = d - c < \infty$, so ν is finite.

Step 2: ν agrees with Lebesgue measure on intervals. Let $c \leq \alpha \leq \beta \leq d$. Put

$$u = \inf\{x \in [a, b] : g(x) \geq \alpha\}, \quad v = \sup\{x \in [a, b] : g(x) \leq \beta\}.$$

Both sets are nonempty ($g(b) = d \geq \alpha$ and $g(a) = c \leq \beta$), so $u, v \in [a, b]$. By continuity $g(u) \geq \alpha$; if $u > a$ then $g(x) < \alpha$ for $x < u$ and continuity forces $g(u) = \alpha$, while if $u = a$ then $g(u) = c \leq \alpha \leq g(u)$. In either case $g(u) = \alpha$, and symmetrically $g(v) = \beta$; since g is increasing, $u \leq v$. Furthermore $g^{-1}([\alpha, \beta]) = [u, v]$: if $u \leq x \leq v$ then $\alpha = g(u) \leq g(x) \leq g(v) = \beta$, while if $g(x) \in [\alpha, \beta]$ then x belongs to both defining sets, so $x \geq u$ and $x \leq v$. Therefore

$$\nu([\alpha, \beta]) = \int_a^b \chi_{[u, v]}(x) g'(x) dx = \int_u^v g' = g(v) - g(u) = \beta - \alpha = m([\alpha, \beta]).$$

The same computation applied to the degenerate interval $[\alpha, \alpha]$ gives $\nu(\{\alpha\}) = 0$, so ν and m agree on every subinterval of $[c, d]$, open, closed, or half-open.

Step 3: $\nu = m$ on the Borel sets. An open subset $\mathcal{O}$ of (c, d) is the union of a countable disjoint collection of open intervals, so countable additivity of ν and of m together with Step 2 give $\nu(\mathcal{O}) = m(\mathcal{O})$. If $G \subseteq (c, d)$ is a G_δ set, write $G = \bigcap_k \mathcal{O}_k$ with $\mathcal{O}_k$ open and, replacing $\mathcal{O}_k$ by $\mathcal{O}_1 \cap \cdots \cap \mathcal{O}_k$, decreasing; since ν and m are finite measures, continuity from above gives

$$\nu(G) = \lim_k \nu(\mathcal{O}_k) = \lim_k m(\mathcal{O}_k) = m(G).$$

Now let E be any Borel subset of $[c, d]$. The singletons $\{c\}$ and $\{d\}$ are m -null, and by the degenerate case of Step 2 they are ν -null as well, so replacing E by $E \cap (c, d)$ changes neither $m(E)$ nor $\nu(E)$; assume therefore $E \subseteq (c, d)$. By the outer approximation of measurable sets, Theorem 11 of Chapter 2, there is a G_δ set $G_0 \supseteq E$ with $m(G_0 \setminus E) = 0$; put $G = G_0 \cap (c, d)$, which is again a G_δ set, satisfies

$E \subseteq G \subseteq (c, d)$, and has $m(G \setminus E) = 0$. Choose likewise a G_δ set H with $G \setminus E \subseteq H \subseteq (c, d)$ and $m(H) = 0$. By the preceding paragraph $\nu(H) = m(H) = 0$, so $\nu(G \setminus E) \leq \nu(H) = 0$ and

$$\nu(E) = \nu(G) - \nu(G \setminus E) = m(G) = m(E).$$

Thus $\nu = m$ on Borel subsets of $[c, d]$.

Step 4: null sets and Lebesgue measurable sets. Let $N \subseteq [c, d]$ have $m(N) = 0$ and choose a Borel set $B \supseteq N$ with $m(B) = 0$. By Step 3, $\int_a^b \chi_B(g(x))g'(x) dx = \nu(B) = 0$, and the integrand is nonnegative, so by Proposition 9 of Chapter 4 it vanishes almost everywhere; that is,

$$g'(x) = 0 \text{ for almost all } x \in g^{-1}(B) \supseteq g^{-1}(N). \quad (*)$$

Consequently, if $E \subseteq [c, d]$ is Lebesgue measurable and B_0 is a Borel set with $m(E \Delta B_0) = 0$, then $\chi_E(g(x))g'(x) = \chi_{B_0}(g(x))g'(x)$ for almost all $x \in [a, b]$, by $(*)$ applied to the null set $E \Delta B_0$. Since Lebesgue measure is complete, $x \mapsto \chi_E(g(x))g'(x)$ is measurable and

$$\int_a^b \chi_E(g(x))g'(x) dx = \nu(B_0) = m(B_0) = m(E). \quad (**)$$

In particular $(*)$ shows that altering f on a set of measure zero alters $(f \circ g) \cdot g'$ only on a set of measure zero, so the right-hand side of the change of variables formula does not depend on the representative chosen for f .

Step 5: simple functions, then general f . Let $\varphi = \sum_{k=1}^n a_k \chi_{E_k}$ be a nonnegative simple function on $[c, d]$, with the E_k measurable and $a_k \geq 0$. By linearity of the integral and $(**)$,

$$\int_a^b \varphi(g(x))g'(x) dx = \sum_{k=1}^n a_k \int_a^b \chi_{E_k}(g(x))g'(x) dx = \sum_{k=1}^n a_k m(E_k) = \int_c^d \varphi.$$

Finally let f be nonnegative and integrable over $[c, d]$. By the Simple Approximation Theorem there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on $[c, d]$ with $\varphi_n \rightarrow f$ pointwise. For almost all x in (a, b) , $g'(x)$ exists and is nonnegative, and then $\varphi_n(g(x))g'(x)$ increases to $f(g(x))g'(x)$. Two applications of the Monotone Convergence Theorem give

$$\int_a^b f(g(x))g'(x) dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(g(x))g'(x) dx = \lim_{n \rightarrow \infty} \int_c^d \varphi_n = \int_c^d f(y) dy.$$

Both sides are therefore equal, and in particular $(f \circ g) \cdot g'$ is integrable over $[a, b]$. This is the asserted formula, and strict monotonicity was nowhere used.

Remark. Geometrically, nothing goes wrong because a maximal interval on which a merely increasing absolutely continuous g is constant is collapsed by g to a single point, and by Theorem 10 the integral of g' over that interval is 0; such an interval therefore contributes nothing to either side. The only role strict monotonicity played in the preceding problem was to keep the images of disjoint intervals disjoint, and Steps 1 through 3 above replace that bookkeeping by the additivity of the measure ν .

Problem 6.58. Construct an absolutely continuous strictly increasing function f on $[0, 1]$ for which $f' = 0$ on a set of positive measure. (Hint: Let E be the relative complement in $[0, 1]$ of a generalized Cantor set of positive measure and f the indefinite integral of χ_E . See Problem 39 of Chapter 2 for the construction of such a Cantor set.)

Solution. The generalized Cantor set. Fix α with $0 < \alpha < 1$; we take $\alpha = 1/2$. Following Problem 39 of Chapter 2, construct a descending sequence $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$ of closed subsets of $[0, 1]$ as follows. Set $C_0 = [0, 1]$. Having obtained C_{n-1} as a disjoint union of 2^{n-1} closed intervals, delete from the interior of each of them the open middle interval of length $\alpha 3^{-n}$; the union of the 2^n remaining closed intervals is C_n . This is legitimate provided $\alpha 3^{-n}$ is smaller than the common length ℓ_{n-1} of the

intervals of C_{n-1} ; since

$$\ell_n = \frac{1}{2^n} \left(1 - \alpha \sum_{k=1}^n 2^{k-1} 3^{-k} \right) = \frac{1}{2^n} \left(1 - \alpha (1 - (2/3)^n) \right) \geq \frac{1 - \alpha}{2^n},$$

and $\alpha 3^{-n} < (1 - \alpha) 2^{-(n-1)}$ for $\alpha = 1/2$ and all $n \geq 1$, the construction proceeds. Put $C = \bigcap_{n \geq 0} C_n$. Then C is closed, being an intersection of closed sets. At stage n the total length deleted is $2^{n-1} \alpha 3^{-n}$, and the deleted open intervals are pairwise disjoint, so by countable additivity

$$m([0, 1] \setminus C) = \sum_{n=1}^{\infty} 2^{n-1} \alpha 3^{-n} = \frac{\alpha}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3} \right)^n = \alpha,$$

whence $m(C) = 1 - \alpha = 1/2 > 0$. Finally C contains no open interval: $C \subseteq C_n$, and C_n is a union of intervals of length $\ell_n \leq 2^{-n}$, so any interval contained in C has length at most 2^{-n} for every n , hence length 0. Thus C is a closed set with empty interior, i.e. C is nowhere dense, and $m(C) = 1/2$.

The function. Let $E = [0, 1] \setminus C$, an open subset of $[0, 1]$ with $m(E) = 1/2$, and define

$$f(x) = \int_0^x \chi_E \quad \text{for } x \in [0, 1].$$

f is absolutely continuous. The function χ_E is nonnegative and bounded, hence integrable over $[0, 1]$, and f is by definition the indefinite integral of χ_E over $[0, 1]$ (with $f(0) = 0$). By Theorem 11, f is absolutely continuous on $[0, 1]$.

f is strictly increasing. Let $0 \leq x < y \leq 1$. By additivity over domains of integration,

$$f(y) - f(x) = \int_x^y \chi_E = m(E \cap (x, y)).$$

Since C has empty interior, the interval (x, y) is not contained in C , so $E \cap (x, y) = (x, y) \setminus C$ is nonempty, and it is open in $\mathbb{R}$ because C is closed. A nonempty open subset of $\mathbb{R}$ contains an open interval and therefore has positive measure. Hence $f(y) - f(x) = m(E \cap (x, y)) > 0$, so f is strictly increasing on $[0, 1]$.

$f' = 0$ on a set of positive measure. By Theorem 14,

$$f'(x) = \frac{d}{dx} \int_0^x \chi_E = \chi_E(x) \quad \text{for almost all } x \in (0, 1).$$

Let Z be the set of measure zero off which this holds. Then for every $x \in C \cap (0, 1) \setminus Z$ we have $\chi_E(x) = 0$ and hence $f'(x) = 0$. Since $m(Z) = 0$ and $m(C) = 1/2$,

$$m(\{x \in [0, 1] : f'(x) = 0\}) \geq m(C \cap (0, 1) \setminus Z) = m(C) = \frac{1}{2} > 0.$$

Thus f is absolutely continuous and strictly increasing on $[0, 1]$, yet f' vanishes on a set of measure $1/2$.

Remark. This shows that the converse of the obvious implication fails badly: an absolutely continuous strictly increasing function need not have $f' > 0$ anywhere in particular, and indeed $f' = 0$ on a set of measure as close to 1 as we please, by taking α close to 0 in the construction. Note also that, by part (iv) of Problem 56 applied to this strictly increasing absolutely continuous f , $m(f(C)) = \int_C f' = \int_C \chi_E = 0$; so f is a homeomorphism of $[0, 1]$ onto $[0, 1/2]$ carrying a set of measure $1/2$ onto a set of measure zero.

Problem 6.59. For a nonnegative integrable function f over $[c, d]$, and a strictly increasing absolutely continuous function g on $[a, b]$ such that $g([a, b]) \subseteq [c, d]$, is it possible to justify the change of variables

formula

$$\int_{g(a)}^{g(b)} f(y) dy = \int_a^b f(g(x))g'(x) dx$$

by showing that

$$\frac{d}{dx} \left[\int_{g(a)}^{g(x)} f(s) ds - \int_a^x f(g(t))g'(t) dt \right] = 0 \quad \text{for almost all } x \in (a, b)?$$

Solution. No, not as it stands. The displayed assertion is a **true** statement, but establishing it is not easier than the theorem, and by itself it does not deliver the theorem. There are two independent difficulties, and repairing either of them requires exactly the machinery of Problem 56 that already proves the formula outright.

Write $F(y) = \int_{g(a)}^y f(s) ds$ for $y \in [g(a), g(b)]$, $\Phi(x) = \int_a^x f(g(t))g'(t) dt$, and $H = F \circ g - \Phi$ on $[a, b]$, which is the bracketed expression.

First difficulty: a vanishing derivative does not force constancy. Even granting $H' = 0$ almost everywhere on (a, b) , one may not conclude that H is constant. The Cantor-Lebesgue function is continuous, increasing, and has derivative zero almost everywhere, yet is not constant. To pass from $H' = 0$ a.e. to $H \equiv H(a) = 0$ one needs H to be absolutely continuous, for it is only then that Theorem 10 gives $H(x) - H(a) = \int_a^x H' = 0$.

Now $F \circ g$ is indeed absolutely continuous: given $\epsilon > 0$ choose $\delta > 0$ responding to the ϵ challenge for the absolutely continuous F , and then $\eta > 0$ responding to the δ challenge for g ; if $\{(x_k, y_k)\}_{k=1}^n$ is a disjoint collection of intervals in (a, b) with $\sum_k (y_k - x_k) < \eta$, then, g being strictly increasing, $\{(g(x_k), g(y_k))\}_{k=1}^n$ is a disjoint collection of intervals in $(g(a), g(b))$ of total length $\sum_k [g(y_k) - g(x_k)] < \delta$, so $\sum_k |F(g(y_k)) - F(g(x_k))| < \epsilon$.

But Φ is a legitimate real-valued absolutely continuous function only if $(f \circ g) \cdot g'$ is **integrable** over $[a, b]$; a priori this product is merely a nonnegative measurable function (measurable after replacing f by a Borel representative, which changes $(f \circ g)g'$ only on a set of measure zero), and its integral could conceivably be $+\infty$, in which case $\Phi \equiv +\infty$ on $(a, b]$ and the bracketed difference is not even defined. The integrability of $(f \circ g) \cdot g'$ is not a technicality one may waive: it is half of the conclusion of the theorem. So the proposed scheme presupposes a substantial part of what it is meant to prove.

Second difficulty: the chain rule is unavailable on a set that can have positive measure. Suppose we grant integrability, so that H is absolutely continuous and H' exists a.e. The natural computation is

$$H'(x) = \frac{d}{dx} F(g(x)) - f(g(x))g'(x) = F'(g(x))g'(x) - f(g(x))g'(x) = 0,$$

using Theorem 14 in the form $F' = f$ and the chain rule. But Theorem 14 gives $F'(y) = f(y)$ only for y outside some set $N \subseteq [g(a), g(b)]$ with $m(N) = 0$, and at points of N the function F may fail to be differentiable altogether. The computation is therefore valid only for $x \notin g^{-1}(N)$, and $g^{-1}(N)$ **can have positive measure**.

Indeed, let g be the function of the preceding problem: $C \subseteq [0, 1]$ a closed nowhere dense set with $m(C) = 1/2$, $E = [0, 1] \setminus C$, and $g(x) = \int_0^x \chi_E$, which is strictly increasing and absolutely continuous on $[0, 1]$ with $g([0, 1]) = [0, 1/2]$. Then $N := g(C)$ satisfies $m(N) = \int_C g' = \int_C \chi_E = 0$ by part (iv) of Problem 56, while $g^{-1}(N) = C$ has measure $1/2$ because g , being strictly increasing, is one-to-one. Now take

$$f = 1 + \chi_N \quad \text{on } [0, 1/2],$$

a nonnegative integrable function. Since $m(N) = 0$ we have $F(y) = \int_0^y f = y$, so $F'(y) = 1$ for every y , whereas $f(y) = 2$ for $y \in N$. Thus F is differentiable everywhere, yet the identity $F' = f$ that Theorem 14 supplies fails at every point of N , and consequently the step $F'(g(x)) = f(g(x))$ in the computation above is false for every x in the set C of measure $1/2$. The naive chain-rule verification of $H' = 0$ is therefore unavailable on a set of positive measure. (In this example $H' = 0$ a.e. is nevertheless true, since $g' = \chi_E = 0$ on C ; but that is a fact about this g , established by an argument the proposed scheme does not contain.)

What is actually needed to repair the argument. On $A := g^{-1}(N)$ one must show separately that both terms of H' vanish almost everywhere.

For the second term: $m(g(A)) \leq m(N) = 0$, and by part (iv) of Problem 56, $\int_A g' = m(g(A)) = 0$; since $g' \geq 0$ a.e., Proposition 9 of Chapter 4 gives $g' = 0$ a.e. on A , hence $f(g(x))g'(x) = 0$ for almost all $x \in A$.

For the first term one needs the corresponding statement for the increasing absolutely continuous function $\psi := F \circ g$, namely $\int_A \psi' = m(\psi(A))$. Granting this, $\psi(A) = F(g(A)) \subseteq F(N)$, and an increasing absolutely continuous function maps sets of measure zero onto sets of measure zero, so $m(\psi(A)) = 0$ and therefore, ψ' being nonnegative a.e., $\psi' = 0$ a.e. on A . (The mapping property is proved directly: given $\epsilon > 0$ let $\delta > 0$ respond to the ϵ challenge for the absolute continuity of F ; choose an open set $\mathcal{O}$ with $N \subseteq \mathcal{O} \subseteq (g(a), g(b))$ and $m(\mathcal{O}) < \delta$, and write $\mathcal{O}$ as the union of a countable disjoint collection $\{(\alpha_k, \beta_k)\}$ of open intervals. Since F is increasing, $F(\mathcal{O}) \subseteq \bigcup_k [F(\alpha_k), F(\beta_k)]$, and for each n the disjoint collection $\{(\alpha_k, \beta_k)\}_{k=1}^n$ has total length less than δ , so $\sum_{k=1}^n [F(\beta_k) - F(\alpha_k)] < \epsilon$; letting $n \rightarrow \infty$, $m^*(F(N)) \leq m^*(F(\mathcal{O})) \leq \epsilon$. As $\epsilon > 0$ was arbitrary, $m(F(N)) = 0$.) Combining, $H' = 0$ a.e. on A , and off A the chain-rule computation applies; so indeed $H' = 0$ a.e. on (a, b) , and, H being absolutely continuous with $H(a) = 0$, we get $H \equiv 0$, which is the change of variables formula. Conclusion. The scheme can be pushed through, but only after (i) knowing in advance that $(f \circ g)g'$ is integrable and (ii) invoking the identity $m(h(A)) = \int_A h'$ for increasing absolutely continuous h and measurable A , i.e. precisely parts (i)-(iv) of Problem 56. Since those parts already yield the change of variables formula directly, through parts (v) and (vi), the proposed route is not an independent justification: as an argument standing on its own it is invalid, and as a corrected argument it is a detour through the proof it was meant to replace. The honest answer to the question posed is therefore no.

Problem 6.60. Let f be absolutely continuous and singular on $[a, b]$. Show that f is constant. Also show that the Lebesgue decomposition of a function of bounded variation is unique if the singular function is required to vanish at $x = a$.

Solution. Recall the definitions in force. A function of bounded variation on $[a, b]$ is **singular** provided its derivative vanishes almost everywhere on $[a, b]$; and a **Lebesgue decomposition** of a function f of bounded variation on $[a, b]$ is an expression $f = g + h$ on $[a, b]$ in which g is absolutely continuous and h is singular. Note that every function of bounded variation on $[a, b]$ is, by Jordan's Theorem together with Lebesgue's Theorem, differentiable almost everywhere on (a, b) with integrable derivative (Corollary 6), so the notion of singularity is meaningful; and every absolutely continuous function on $[a, b]$ is of bounded variation, so both summands in a Lebesgue decomposition are of bounded variation.

An absolutely continuous singular function is constant. Let f be absolutely continuous and singular on $[a, b]$, and let $x \in [a, b]$. The restriction of f to $[a, x]$ is absolutely continuous, so Theorem 10, applied on the interval $[a, x]$, tells us that f' is integrable over $[a, x]$ and

$$\int_a^x f' = f(x) - f(a).$$

Since f is singular, $f' = 0$ almost everywhere on $[a, b]$, hence almost everywhere on $[a, x]$, and therefore $\int_a^x f' = 0$. Thus $f(x) = f(a)$ for every $x \in [a, b]$, i.e. f is constant. (Conversely a constant function is trivially absolutely continuous and singular, so the absolutely continuous singular functions are precisely the constants.)

Uniqueness of the Lebesgue decomposition. Let f be of bounded variation on $[a, b]$ and suppose

$$f = g_1 + h_1 = g_2 + h_2 \quad \text{on } [a, b],$$

where g_1, g_2 are absolutely continuous, h_1, h_2 are singular, and $h_1(a) = h_2(a) = 0$. Define

$$u = g_1 - g_2 = h_2 - h_1 \quad \text{on } [a, b].$$

First, u is absolutely continuous: $u = g_1 - g_2$ and a difference of absolutely continuous functions on

$[a, b]$ is absolutely continuous, since for a disjoint collection $\{(x_k, y_k)\}_{k=1}^n$ of intervals in (a, b) ,

$$\sum_{k=1}^n |u(y_k) - u(x_k)| \leq \sum_{k=1}^n |g_1(y_k) - g_1(x_k)| + \sum_{k=1}^n |g_2(y_k) - g_2(x_k)|,$$

so that if δ_1, δ_2 respond to the challenge $\epsilon/2$ for g_1, g_2 respectively, then $\delta = \min\{\delta_1, \delta_2\}$ responds to the challenge ϵ for u .

Second, u is singular: $u = h_2 - h_1$ is of bounded variation, being a difference of functions of bounded variation, and h_1, h_2 are differentiable almost everywhere with $h'_1 = h'_2 = 0$ almost everywhere; at every point at which both derivatives exist, $u' = h'_2 - h'_1$, so $u' = 0$ almost everywhere on $[a, b]$.

By the first part of the problem, u is constant on $[a, b]$. Its value at a is

$$u(a) = h_2(a) - h_1(a) = 0 - 0 = 0.$$

Hence $u \equiv 0$, that is $g_1 = g_2$ and $h_1 = h_2$ on $[a, b]$. The Lebesgue decomposition is therefore unique under the normalization that the singular part vanish at a .

Remark on existence with this normalization. The decomposition exhibited in the text, namely

$$g(x) = \int_a^x f', \quad h(x) = f(x) - \int_a^x f' \quad \text{for } x \in [a, b],$$

has g absolutely continuous by Theorem 11 and h singular by Theorem 14, and it satisfies $h(a) = f(a)$ rather than 0. Replacing the pair (g, h) by $(g + f(a), h - f(a))$ leaves g absolutely continuous and h singular and normalizes $h(a) = 0$; so a normalized Lebesgue decomposition exists, and by the above it is unique. Without the normalization the decomposition is unique only up to the addition of a constant to one summand and its subtraction from the other, which is precisely the indeterminacy the constant u measures.

Problem 6.61. Show that a real-valued function φ on (a, b) is convex if and only if for points $x_1, \dots, x_n$ in (a, b) and nonnegative numbers $\lambda_1, \dots, \lambda_n$ such that $\sum_{k=1}^n \lambda_k = 1$,

$$\varphi\left(\sum_{k=1}^n \lambda_k x_k\right) \leq \sum_{k=1}^n \lambda_k \varphi(x_k).$$

Use this to directly prove Jensen's Inequality for f a simple function.

Solution. Preliminary observation. If $x_1, \dots, x_n$ belong to (a, b) and $\lambda_1, \dots, \lambda_n$ are nonnegative with $\sum_k \lambda_k = 1$, then the point $x = \sum_k \lambda_k x_k$ again belongs to (a, b) . Indeed, with $\mu = \min_k x_k$ and $M = \max_k x_k$ we have $\mu = \sum_k \lambda_k \mu \leq \sum_k \lambda_k x_k \leq \sum_k \lambda_k M = M$, and $\mu, M \in (a, b)$, so $x \in [\mu, M] \subseteq (a, b)$. Thus the left-hand side of the asserted inequality is defined.

The condition is sufficient. Suppose the inequality holds for all such finite systems. Taking $n = 2$, $\lambda_1 = \lambda$ and $\lambda_2 = 1 - \lambda$ for $0 \leq \lambda \leq 1$ and $x_1, x_2 \in (a, b)$ gives

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda \varphi(x_1) + (1 - \lambda)\varphi(x_2),$$

which is exactly (38). Hence φ is convex.

The condition is necessary. Let φ be convex on (a, b) . We argue by induction on n . For $n = 1$ the inequality reads $\varphi(x_1) \leq \varphi(x_1)$, and for $n = 2$ it is (38).

Assume the inequality holds for every system of $n - 1$ points, where $n \geq 3$, and let $x_1, \dots, x_n \in (a, b)$ and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{k=1}^n \lambda_k = 1$ be given. If $\lambda_n = 1$ then $\lambda_k = 0$ for $k < n$ and the inequality is the identity $\varphi(x_n) \leq \varphi(x_n)$. So suppose $\lambda_n < 1$ and set

$$\mu = \sum_{k=1}^{n-1} \lambda_k = 1 - \lambda_n > 0, \quad y = \sum_{k=1}^{n-1} \frac{\lambda_k}{\mu} x_k.$$

The numbers λ_k/μ , $1 \leq k \leq n-1$, are nonnegative and sum to 1, so by the preliminary observation $y \in (a, b)$, and by the induction hypothesis

$$\varphi(y) \leq \sum_{k=1}^{n-1} \frac{\lambda_k}{\mu} \varphi(x_k).$$

Moreover $\sum_{k=1}^n \lambda_k x_k = \mu y + (1-\mu)x_n$, so (38), applied with $\lambda = \mu \in (0, 1]$ to the points y and x_n of (a, b) , gives

$$\varphi\left(\sum_{k=1}^n \lambda_k x_k\right) = \varphi(\mu y + (1-\mu)x_n) \leq \mu \varphi(y) + (1-\mu)\varphi(x_n) \leq \mu \sum_{k=1}^{n-1} \frac{\lambda_k}{\mu} \varphi(x_k) + \lambda_n \varphi(x_n) = \sum_{k=1}^n \lambda_k \varphi(x_k),$$

where the second inequality used $\mu > 0$ and the monotonicity of multiplication by the positive number μ . This completes the induction.

Jensen's Inequality for a simple function. Let φ be convex on $(-\infty, \infty)$ and let f be a simple function on $[0, 1]$; being simple it is measurable and bounded, hence integrable over $[0, 1]$, and $\varphi \circ f$ is simple, hence also integrable over $[0, 1]$. Write f in canonical form,

$$f = \sum_{k=1}^n c_k \chi_{E_k} \quad \text{on } [0, 1],$$

where $c_1, \dots, c_n$ are the distinct values taken by f and $E_k = f^{-1}(c_k)$, so that $\{E_k\}_{k=1}^n$ is a disjoint collection of measurable sets whose union is $[0, 1]$. Set $\lambda_k = m(E_k)$. Then each $\lambda_k \geq 0$ and, by the finite additivity of Lebesgue measure,

$$\sum_{k=1}^n \lambda_k = m\left(\bigcup_{k=1}^n E_k\right) = m([0, 1]) = 1.$$

By the definition of the integral of a simple function,

$$\int_0^1 f = \sum_{k=1}^n c_k m(E_k) = \sum_{k=1}^n \lambda_k c_k.$$

On the other hand $\varphi \circ f$ takes the value $\varphi(c_k)$ precisely on E_k , so $\varphi \circ f = \sum_{k=1}^n \varphi(c_k) \chi_{E_k}$ and

$$\int_0^1 (\varphi \circ f) = \sum_{k=1}^n \varphi(c_k) m(E_k) = \sum_{k=1}^n \lambda_k \varphi(c_k).$$

Applying the inequality just proved, with the points $c_1, \dots, c_n$ of $(-\infty, \infty)$ and the weights $\lambda_1, \dots, \lambda_n$, we conclude

$$\varphi\left(\int_0^1 f\right) = \varphi\left(\sum_{k=1}^n \lambda_k c_k\right) \leq \sum_{k=1}^n \lambda_k \varphi(c_k) = \int_0^1 (\varphi \circ f),$$

which is (42) for f simple.

Problem 6.62. Show that a continuous function on (a, b) is convex if and only if

$$\varphi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{\varphi(x_1) + \varphi(x_2)}{2} \quad \text{for all } x_1, x_2 \in (a, b).$$

Solution. Let φ be a continuous real-valued function on (a, b) . Call the displayed condition **midpoint convexity**.

Necessity. If φ is convex then (38) with $\lambda = 1/2$ is precisely midpoint convexity. (Continuity is not

needed for this direction.)

Sufficiency. Suppose φ is continuous and midpoint convex. Fix $x_1, x_2 \in (a, b)$; we must verify (38) for every $\lambda \in [0, 1]$. If $x_1 = x_2$ this is trivial, so assume $x_1 \neq x_2$. For $\lambda \in [0, 1]$ write

$$z(\lambda) = \lambda x_1 + (1 - \lambda)x_2,$$

a point of the closed interval with endpoints x_1, x_2 , which is contained in (a, b) ; the map $\lambda \mapsto z(\lambda)$ is continuous.

Step 1: dyadic λ . Let $D_n = \{k/2^n : k = 0, 1, \dots, 2^n\}$. We prove by induction on $n \geq 0$ that

$$\varphi(z(\lambda)) \leq \lambda\varphi(x_1) + (1 - \lambda)\varphi(x_2) \quad \text{for all } \lambda \in D_n. \quad (\dagger_n)$$

For $n = 0$ we have $\lambda \in \{0, 1\}$ and $(\dagger_0)$ is an identity. Assume $(\dagger_n)$ and let $\lambda = k/2^{n+1}$ with $0 \leq k \leq 2^{n+1}$. If k is even then $\lambda = (k/2)/2^n \in D_n$ and there is nothing to prove. If k is odd, then $1 \leq k \leq 2^{n+1} - 1$, and we set

$$\lambda' = \frac{k-1}{2^{n+1}}, \quad \lambda'' = \frac{k+1}{2^{n+1}}.$$

Both $k-1$ and $k+1$ are even and lie in $[0, 2^{n+1}]$, so $\lambda', \lambda'' \in D_n$, and $\lambda = \frac{1}{2}(\lambda' + \lambda'')$. Since z is an affine function of λ ,

$$z(\lambda) = \frac{z(\lambda') + z(\lambda'')}{2},$$

and $z(\lambda'), z(\lambda'') \in (a, b)$. Midpoint convexity and then $(\dagger_n)$ applied to λ' and to λ'' give

$$\begin{aligned} \varphi(z(\lambda)) &\leq \frac{\varphi(z(\lambda')) + \varphi(z(\lambda''))}{2} \\ &\leq \frac{1}{2} \left[\lambda' \varphi(x_1) + (1 - \lambda') \varphi(x_2) \right] + \frac{1}{2} \left[\lambda'' \varphi(x_1) + (1 - \lambda'') \varphi(x_2) \right] \\ &= \frac{\lambda' + \lambda''}{2} \varphi(x_1) + \left(1 - \frac{\lambda' + \lambda''}{2} \right) \varphi(x_2) \\ &= \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2). \end{aligned}$$

This establishes $(\dagger_{n+1})$ and completes the induction.

Step 2: arbitrary λ by continuity. Let $D = \bigcup_{n \geq 0} D_n$ be the set of dyadic rationals in $[0, 1]$; D is dense in $[0, 1]$, since the intervals $[k/2^n, (k+1)/2^n]$ have length $2^{-n} \rightarrow 0$. Given $\lambda \in [0, 1]$, choose a sequence $\{\lambda_j\}$ in D with $\lambda_j \rightarrow \lambda$. By Step 1,

$$\varphi(z(\lambda_j)) \leq \lambda_j \varphi(x_1) + (1 - \lambda_j) \varphi(x_2) \quad \text{for every } j.$$

Now $z(\lambda_j) \rightarrow z(\lambda)$, all these points lying in the closed interval with endpoints x_1, x_2 , a subset of (a, b) on which φ is continuous; hence $\varphi(z(\lambda_j)) \rightarrow \varphi(z(\lambda))$. The right-hand side converges to $\lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2)$. Passing to the limit in the inequality,

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) = \varphi(z(\lambda)) \leq \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2).$$

Since $x_1, x_2 \in (a, b)$ and $\lambda \in [0, 1]$ were arbitrary, φ satisfies (38) and is convex.

Remark. Continuity cannot be dropped. Regarding $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ and choosing a Hamel basis, one obtains additive functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, satisfying $\varphi(s+t) = \varphi(s) + \varphi(t)$, that are not linear; such a φ satisfies $\varphi((x_1+x_2)/2) = (\varphi(x_1) + \varphi(x_2))/2$ with equality, hence is midpoint convex, yet it is unbounded on every interval and therefore is not convex, since by Corollary 17 a convex function on an open interval is continuous.

Problem 6.63. A function on a general interval I is said to be convex provided it is continuous on I and (38) holds for all $x_1, x_2 \in I$. Is a convex function on a closed, bounded interval $[a, b]$ necessarily Lipschitz on $[a, b]$?

Solution. No. Corollary 17 asserts that a convex function on an open interval (a, b) is Lipschitz on each closed, bounded subinterval $[c, d]$ of (a, b) , and the proof there uses the finiteness of the one-sided derivatives $\varphi'(c^+)$ and $\varphi'(d^-)$ at **interior** points c and d . At an endpoint of $[a, b]$ there is no room to the left (or right) to run this argument, and the conclusion genuinely fails.

Counterexample. Define

$$\varphi(x) = -\sqrt{x} \quad \text{for } x \in [0, 1].$$

φ is convex on $[0, 1]$ in the sense stated. It is continuous on $[0, 1]$, being the composition of the continuous functions $x \mapsto \sqrt{x}$ and $t \mapsto -t$. On the open interval $(0, 1)$ it is twice differentiable with

$$\varphi'(x) = -\frac{1}{2}x^{-1/2}, \quad \varphi''(x) = \frac{1}{4}x^{-3/2} > 0,$$

so φ' is increasing on $(0, 1)$ and Proposition 15 shows that φ is convex on $(0, 1)$; that is, (38) holds for all $x_1, x_2 \in (0, 1)$ and all $\lambda \in [0, 1]$.

It remains to extend (38) to pairs involving the endpoints. Let $x_1, x_2 \in [0, 1]$ and $\lambda \in [0, 1]$. Choose sequences $\{x_1^{(j)}\}$ and $\{x_2^{(j)}\}$ in $(0, 1)$ with $x_1^{(j)} \rightarrow x_1$ and $x_2^{(j)} \rightarrow x_2$ (possible since $(0, 1)$ is dense in $[0, 1]$). For each j ,

$$\varphi(\lambda x_1^{(j)} + (1 - \lambda)x_2^{(j)}) \leq \lambda\varphi(x_1^{(j)}) + (1 - \lambda)\varphi(x_2^{(j)}).$$

Since $\lambda x_1^{(j)} + (1 - \lambda)x_2^{(j)} \rightarrow \lambda x_1 + (1 - \lambda)x_2$, all these points lying in $[0, 1]$, and since φ is continuous on $[0, 1]$, both sides converge; passing to the limit gives (38) for x_1, x_2 . Hence φ is convex on the closed, bounded interval $[0, 1]$.

φ is not Lipschitz on $[0, 1]$. For $0 < x \leq 1$,

$$\left| \frac{\varphi(x) - \varphi(0)}{x - 0} \right| = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}.$$

Given any $c > 0$, choose x with $0 < x < \min\{1, c^{-2}\}$; then $1/\sqrt{x} > c$, so $|\varphi(x) - \varphi(0)| > c|x - 0|$. Thus no constant c satisfies $|\varphi(u) - \varphi(v)| \leq c|u - v|$ for all $u, v \in [0, 1]$, and φ is not Lipschitz on $[0, 1]$.

What does survive. Two positive statements remain true, and they locate the failure precisely at the endpoints.

First, a convex function φ on $[a, b]$ is Lipschitz on every closed, bounded subinterval $[c, d]$ with $a < c \leq d < b$. Indeed the restriction of φ to (a, b) is convex there, so Corollary 17 applies. In the example above, $-\sqrt{x}$ is Lipschitz on $[c, 1]$ for each $c > 0$, with constant $1/(2\sqrt{c})$, but these constants blow up as $c \rightarrow 0^+$.

Second, a convex function on $[a, b]$ is bounded on $[a, b]$: it is continuous on a compact set. So the failure is not one of boundedness but purely one of the modulus of continuity near an endpoint.

Remark. The obstruction is exactly that a convex function on $[a, b]$ may have an infinite one-sided derivative at an endpoint: here $\varphi'(0^+) = \lim_{h \rightarrow 0^+} (-\sqrt{h})/h = -\infty$. Lemma 16 guarantees finite one-sided derivatives only at interior points, since its proof compares the slope at x with slopes of chords on both sides of x . Note that Lipschitzness is a strictly stronger demand than absolute continuity here: in this example $-\sqrt{x}$ is in fact absolutely continuous on $[0, 1]$, being the indefinite integral of the integrable function $-\frac{1}{2}x^{-1/2}$, and yet it is not Lipschitz. So the answer to the question is no even though the weaker conclusion of absolute continuity may still hold.

6.10 Problems 6.64–6.70

Problem 6.64. Let φ have a second derivative at each point in (a, b) . Show that φ is convex if and only if φ'' is nonnegative.

Solution. Throughout, (a, b) is an open interval and φ is real-valued on (a, b) with $\varphi''(x)$ existing (in particular φ' exists on all of (a, b) , since otherwise φ'' could not be formed) for every $x \in (a, b)$.

Suppose first that $\varphi'' \geq 0$ on (a, b) . This is exactly the hypothesis of the second assertion of Proposition 15: a function with a nonnegative second derivative on (a, b) is convex there. For completeness we recall why. Since $\varphi'' \geq 0$ on the interval (a, b) , the Mean Value Theorem applied to φ' shows that φ'

is increasing: for $u < v$ in (a, b) there is $c \in (u, v)$ with

$$\varphi'(v) - \varphi'(u) = \varphi''(c)(v - u) \geq 0.$$

Thus φ is differentiable on (a, b) with increasing derivative, and Proposition 15 tells us that φ is convex. Conversely, suppose φ is convex on (a, b) . Since φ is differentiable at every point of (a, b) by hypothesis, its one-sided derivatives coincide with φ' at each point, so the inequalities (40) of Lemma 16 read: for $u < v$ in (a, b) ,

$$\varphi'(u) = \varphi'(u^+) \leq \frac{\varphi(v) - \varphi(u)}{v - u} \leq \varphi'(v^-) = \varphi'(v).$$

Hence φ' is an increasing function on (a, b) . (This is also the content of Theorem 18, which asserts that the derivative of a convex function is increasing wherever it exists.)

Now fix $x \in (a, b)$. For all sufficiently small $h \neq 0$ the point $x + h$ lies in (a, b) , and since φ' is increasing the difference quotient

$$\frac{\varphi'(x + h) - \varphi'(x)}{h}$$

is nonnegative: if $h > 0$ then $x < x + h$ forces $\varphi'(x + h) - \varphi'(x) \geq 0$, while if $h < 0$ then $x + h < x$ forces $\varphi'(x + h) - \varphi'(x) \leq 0$, and in either case numerator and denominator have the same sign. By hypothesis the limit of this quotient as $h \rightarrow 0$ exists and equals $\varphi''(x)$. A limit of nonnegative numbers is nonnegative, so $\varphi''(x) \geq 0$.

Since $x \in (a, b)$ was arbitrary, $\varphi'' \geq 0$ on (a, b) . This completes both implications.

Problem 6.65. Suppose $a \geq 0$ and $b > 0$. Show that the function $\varphi(t) = (a + bt)^p$ is convex on $[0, \infty)$ for $1 \leq p < \infty$.

Solution. Since $[0, \infty)$ is not an open interval, convexity here is understood in the sense of Problem 63: φ is continuous on $[0, \infty)$ and the inequality (38),

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda\varphi(x_1) + (1 - \lambda)\varphi(x_2),$$

holds for all $x_1, x_2 \in [0, \infty)$ and all λ with $0 \leq \lambda \leq 1$.

Observe first that $a + bt > 0$ for every $t > 0$, since $a \geq 0$ and $b > 0$ give $a + bt \geq bt > 0$. Also $a + bt \geq 0$ for $t \geq 0$, so $(a + bt)^p$ is defined on all of $[0, \infty)$, and φ is continuous there, being the composition of the continuous function $t \mapsto a + bt$ with the function $s \mapsto s^p$, which is continuous on $[0, \infty)$ for $p \geq 1$. Convexity on the open interval $(0, \infty)$. For $t > 0$ we have $a + bt > 0$, so the chain rule gives

$$\varphi'(t) = pb(a + bt)^{p-1}, \quad \varphi''(t) = p(p-1)b^2(a + bt)^{p-2}.$$

(The exponents $p-1$ and $p-2$ may be negative, but this is harmless because the base $a + bt$ is strictly positive on $(0, \infty)$.) Since $p \geq 1$ we have $p(p-1) \geq 0$, and $b^2 > 0$, $(a + bt)^{p-2} > 0$; hence $\varphi''(t) \geq 0$ for all $t \in (0, \infty)$. By Proposition 15, φ is convex on the open interval $(0, \infty)$: for $x_1, x_2 \in (0, \infty)$ and $0 \leq \lambda \leq 1$,

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda\varphi(x_1) + (1 - \lambda)\varphi(x_2).$$

Call this the interior convexity inequality.

Extension to the endpoint 0. Let $x_1, x_2 \in [0, \infty)$ and $0 \leq \lambda \leq 1$ be arbitrary. If $\lambda = 0$ or $\lambda = 1$ the inequality is an identity, so assume $0 < \lambda < 1$. If both x_1 and x_2 are positive, the interior convexity inequality applies directly. If $x_1 = x_2 = 0$ the inequality is again an identity. There remains the case in which exactly one of the two points is 0; by symmetry (interchanging the roles of the points and replacing λ by $1 - \lambda$) we may assume $x_1 = 0$ and $x_2 > 0$.

For $0 < \varepsilon < x_2$ apply the interior convexity inequality to the points ε and x_2 of $(0, \infty)$:

$$\varphi(\lambda\varepsilon + (1 - \lambda)x_2) \leq \lambda\varphi(\varepsilon) + (1 - \lambda)\varphi(x_2).$$

Now let $\varepsilon \rightarrow 0^+$. Since φ is continuous on $[0, \infty)$ and $\lambda\varepsilon + (1 - \lambda)x_2 \rightarrow (1 - \lambda)x_2$, the left-hand side converges to $\varphi((1 - \lambda)x_2) = \varphi(\lambda \cdot 0 + (1 - \lambda)x_2)$, while the right-hand side converges to $\lambda\varphi(0) + (1 -$

$\lambda)\varphi(x_2)$. Passing to the limit in the inequality yields

$$\varphi(\lambda \cdot 0 + (1 - \lambda)x_2) \leq \lambda\varphi(0) + (1 - \lambda)\varphi(x_2),$$

which is the required inequality for the pair $(0, x_2)$.

Thus (38) holds for every pair of points of $[0, \infty)$ and every $\lambda \in [0, 1]$, and φ is continuous on $[0, \infty)$. Therefore $\varphi(t) = (a + bt)^p$ is convex on $[0, \infty)$ for $1 \leq p < \infty$.

Problem 6.66. For what functions φ is Jensen's Inequality always an equality?

Solution. Answer: precisely the affine functions, that is, the functions of the form $\varphi(t) = mt + c$ with m, c real constants.

Recall the statement of Jensen's Inequality: φ is convex on $(-\infty, \infty)$, f is integrable over $[0, 1]$, $\varphi \circ f$ is integrable over $[0, 1]$, and then

$$\varphi\left(\int_0^1 f(x) dx\right) \leq \int_0^1 (\varphi \circ f)(x) dx.$$

We ask for which convex φ equality holds for every such f .

Sufficiency. Suppose $\varphi(t) = mt + c$. Such a φ is convex, since it satisfies (38) with equality for every x_1, x_2 and λ . Let f be integrable over $[0, 1]$. Then $\varphi \circ f = mf + c$ is also integrable over $[0, 1]$, and by the linearity of the Lebesgue integral together with $\int_0^1 1 dx = 1$,

$$\int_0^1 (\varphi \circ f)(x) dx = m \int_0^1 f(x) dx + c = \varphi\left(\int_0^1 f(x) dx\right).$$

So Jensen's Inequality is an equality for every integrable f .

Necessity. Suppose now that φ is convex on $(-\infty, \infty)$ and that equality holds in Jensen's Inequality for every integrable f on $[0, 1]$ for which $\varphi \circ f$ is integrable. We show φ is affine.

Fix real numbers t_1, t_2 and λ with $0 \leq \lambda \leq 1$, and define the simple function

$$f = t_1 \cdot \chi_{[0, \lambda)} + t_2 \cdot \chi_{[\lambda, 1]}.$$

Then f is measurable and bounded on $[0, 1]$, hence integrable, and $\varphi \circ f = \varphi(t_1)\chi_{[0, \lambda)} + \varphi(t_2)\chi_{[\lambda, 1]}$ is likewise a bounded measurable, hence integrable, function. Since $m([0, \lambda)) = \lambda$ and $m([\lambda, 1]) = 1 - \lambda$, the additivity of the integral over the two pieces gives

$$\int_0^1 f = \lambda t_1 + (1 - \lambda)t_2, \quad \int_0^1 (\varphi \circ f) = \lambda\varphi(t_1) + (1 - \lambda)\varphi(t_2).$$

Equality in Jensen's Inequality for this f therefore reads

$$\varphi(\lambda t_1 + (1 - \lambda)t_2) = \lambda\varphi(t_1) + (1 - \lambda)\varphi(t_2)$$

for all real t_1, t_2 and all $\lambda \in [0, 1]$. Thus φ agrees on each interval with the chord over that interval. We now deduce that φ is affine. Put

$$m = \varphi(1) - \varphi(0), \quad c = \varphi(0), \quad \ell(t) = mt + c,$$

and let t be an arbitrary real number. There are three cases.

If $0 \leq t \leq 1$, write $t = (1 - t) \cdot 0 + t \cdot 1$ and apply the displayed equality with $t_1 = 0$, $t_2 = 1$, $\lambda = 1 - t$:

$$\varphi(t) = (1 - t)\varphi(0) + t\varphi(1) = \varphi(0) + t(\varphi(1) - \varphi(0)) = \ell(t).$$

If $t > 1$, then 1 lies between 0 and t : indeed $1 = \lambda \cdot 0 + (1 - \lambda)t$ with $1 - \lambda = 1/t \in (0, 1)$. The equality applied to the points 0 and t gives

$$\varphi(1) = \left(1 - \frac{1}{t}\right)\varphi(0) + \frac{1}{t}\varphi(t),$$

and solving for $\varphi(t)$,

$$\varphi(t) = t\varphi(1) - (t-1)\varphi(0) = \varphi(0) + t(\varphi(1) - \varphi(0)) = \ell(t).$$

If $t < 0$, then 0 lies between t and 1: with $\lambda = 1/(1-t) \in (0, 1)$ we have $1 - \lambda = -t/(1-t)$ and

$$\lambda t + (1 - \lambda) \cdot 1 = \frac{t}{1-t} - \frac{t}{1-t} = 0.$$

The equality applied to the points t and 1 gives

$$\varphi(0) = \frac{1}{1-t}\varphi(t) - \frac{t}{1-t}\varphi(1),$$

so that

$$\varphi(t) = (1-t)\varphi(0) + t\varphi(1) = \varphi(0) + t(\varphi(1) - \varphi(0)) = \ell(t).$$

In every case $\varphi(t) = mt + c$. Hence φ is affine, and the two implications together show that Jensen's Inequality is always an equality exactly for the affine functions.

Two remarks. First, the necessity argument used only simple functions f taking two values, so equality for all such f already forces φ to be affine. Second, this is consistent with the proof of Jensen's Inequality: there one integrates the supporting-line inequality $\varphi(t) \geq m(t - \alpha) + \varphi(\alpha)$, and equality in the integrated form forces $\varphi(f(x)) = m(f(x) - \alpha) + \varphi(\alpha)$ for almost all x ; demanding this for all f means φ coincides with a supporting line on all of $\mathbb{R}$, that is, φ is affine.

Problem 6.67. State and prove a version of Jensen's Inequality on a general closed, bounded interval $[a, b]$.

Solution. The interval $[0, 1]$ enters the statement of Jensen's Inequality through the single fact that it has measure 1: because of this, $\int_0^1 f$ is the average value of f , and dx restricted to $[0, 1]$ is a measure of total mass 1. On a general closed, bounded interval $[a, b]$ with $a < b$ one must therefore replace integrals by averages, dividing by $m([a, b]) = b - a$.

Jensen's Inequality on $[a, b]$. Let $a < b$, let φ be a convex function on $(-\infty, \infty)$, let f be integrable over $[a, b]$, and suppose $\varphi \circ f$ is also integrable over $[a, b]$. Then

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x) dx\right) \leq \frac{1}{b-a}\int_a^b (\varphi \circ f)(x) dx.$$

As in the case $[a, b] = [0, 1]$, the integrability assumption on $\varphi \circ f$ may be dropped when $\varphi \circ f$ is nonnegative, since then the right-hand side is defined in $[0, \infty]$ and the inequality is trivial if it equals $+\infty$; note also that $\varphi \circ f$ is measurable, because φ , being convex on $(-\infty, \infty)$, is continuous there (Corollary 17) and the composition of a continuous function with a measurable function is measurable by Proposition 7 of Chapter 3.

Proof. Set

$$\alpha = \frac{1}{b-a}\int_a^b f(x) dx.$$

Since f is integrable over $[a, b]$, the integral $\int_a^b f$ is a finite real number and hence $\alpha \in \mathbb{R}$.

By Lemma 16 the convex function φ has left-hand and right-hand derivatives at the point α and $\varphi'(\alpha^-) \leq \varphi'(\alpha^+)$. Choose a real number m with $\varphi'(\alpha^-) \leq m \leq \varphi'(\alpha^+)$. As observed in the text following Theorem 18, the line $y = m(t - \alpha) + \varphi(\alpha)$ is then a supporting line for the graph of φ at α , that is,

$$\varphi(t) \geq m(t - \alpha) + \varphi(\alpha) \quad \text{for all } t \in \mathbb{R}.$$

(Explicitly: for $t > \alpha$, Lemma 16 gives $(\varphi(t) - \varphi(\alpha))/(t - \alpha) \geq \varphi'(\alpha^+) \geq m$, and multiplying by $t - \alpha > 0$ yields the inequality; for $t < \alpha$, Lemma 16 gives $(\varphi(\alpha) - \varphi(t))/(\alpha - t) \leq \varphi'(\alpha^-) \leq m$, and multiplying by $\alpha - t > 0$ and rearranging yields it again; for $t = \alpha$ it is an identity.)

Since f is integrable over $[a, b]$, it is finite almost everywhere on $[a, b]$. Substituting $t = f(x)$ in the supporting-line inequality we obtain

$$\varphi(f(x)) \geq m(f(x) - \alpha) + \varphi(\alpha) \quad \text{for almost all } x \in [a, b].$$

Both sides are integrable over $[a, b]$: the left-hand side by hypothesis, the right-hand side because f is integrable and constants are integrable over the bounded interval $[a, b]$. Hence, by the monotonicity and linearity of the Lebesgue integral,

$$\begin{aligned} \int_a^b \varphi(f(x)) \, dx &\geq \int_a^b [m(f(x) - \alpha) + \varphi(\alpha)] \, dx \\ &= m \left[\int_a^b f(x) \, dx - \alpha(b - a) \right] + \varphi(\alpha)(b - a) \\ &= m[\alpha(b - a) - \alpha(b - a)] + \varphi(\alpha)(b - a) \\ &= \varphi(\alpha)(b - a), \end{aligned}$$

where we used $\int_a^b f = \alpha(b - a)$ by the definition of α . Dividing by $b - a > 0$ gives

$$\frac{1}{b - a} \int_a^b (\varphi \circ f)(x) \, dx \geq \varphi(\alpha) = \varphi \left(\frac{1}{b - a} \int_a^b f(x) \, dx \right),$$

which is the asserted inequality.

Two comments. First, taking $[a, b] = [0, 1]$ recovers the inequality (42) exactly, since then $b - a = 1$. Second, the normalization by $1/(b - a)$ cannot be dispensed with: if $\varphi(t) = t^2$, $f \equiv 1$ and $[a, b] = [0, 2]$, then $\varphi(\int_0^2 f) = 4$ while $\int_0^2 \varphi \circ f = 2$, so the unnormalized inequality fails. What makes the averaged form work is that $\frac{1}{b-a} dx$ is a measure of total mass 1 on $[a, b]$, which is precisely what the last line of the computation exploited.

Problem 6.68. Let f be integrable over $[0, 1]$. Show that

$$\exp \left[\int_0^1 f(x) \, dx \right] \leq \int_0^1 \exp(f(x)) \, dx.$$

Solution. Let $\varphi(t) = e^t$ for $t \in \mathbb{R}$. Then φ has the second derivative $\varphi''(t) = e^t > 0$ at every t , so by Proposition 15 (or by Problem 64) φ is convex on $(-\infty, \infty)$. Moreover φ is continuous, so by Proposition 7 of Chapter 3 the composition $\varphi \circ f = e^f$ is measurable on $[0, 1]$; being nonnegative, its integral $\int_0^1 e^f$ is defined as an element of $[0, \infty]$.

Set

$$\alpha = \int_0^1 f(x) \, dx,$$

a finite real number because f is integrable over $[0, 1]$.

If $\int_0^1 e^{f(x)} \, dx = +\infty$ the asserted inequality is trivially true, since its left-hand side e^α is finite. So assume $\int_0^1 e^f < \infty$; then e^f , being nonnegative, measurable and of finite integral, is integrable over $[0, 1]$, and Jensen's Inequality applies directly to φ and f to give

$$\exp \left[\int_0^1 f(x) \, dx \right] = \varphi \left(\int_0^1 f \right) \leq \int_0^1 (\varphi \circ f) = \int_0^1 \exp(f(x)) \, dx,$$

as required.

For completeness we give the underlying supporting-line argument explicitly, since it also shows that the integrability of e^f need not be assumed. Since $\varphi(t) = e^t$ is differentiable everywhere, its unique supporting line at α is the tangent line there, of slope $\varphi'(\alpha) = e^\alpha$; thus

$$e^t \geq e^\alpha(t - \alpha) + e^\alpha \quad \text{for all } t \in \mathbb{R}.$$

(This is the familiar inequality $e^s \geq 1 + s$ with $s = t - \alpha$, multiplied by $e^\alpha > 0$; it follows from the convexity of φ exactly as in the proof of Jensen's Inequality, using Lemma 16.)

Since f is integrable it is finite almost everywhere on $[0, 1]$, so substituting $t = f(x)$ gives

$$e^{f(x)} \geq e^\alpha (f(x) - \alpha) + e^\alpha \quad \text{for almost all } x \in [0, 1].$$

The right-hand side is integrable over $[0, 1]$, being $e^\alpha f$ plus a constant, and the left-hand side is nonnegative and measurable. By the monotonicity of the Lebesgue integral (valid for the integral of a nonnegative measurable function even when it takes the value $+\infty$), and using $\int_0^1 1 \, dx = 1$,

$$\int_0^1 e^{f(x)} \, dx \geq e^\alpha \left[\int_0^1 f(x) \, dx - \alpha \right] + e^\alpha = e^\alpha \cdot 0 + e^\alpha = e^\alpha,$$

because $\int_0^1 f = \alpha$. That is,

$$\int_0^1 \exp(f(x)) \, dx \geq \exp \left[\int_0^1 f(x) \, dx \right],$$

which is the desired inequality, with no integrability assumption on $\exp \circ f$ needed.

Problem 6.69. Let $\{\alpha_n\}$ be a sequence of nonnegative numbers whose sum is 1 and $\{\zeta_n\}$ a sequence of positive numbers. Show that

$$\prod_{n=1}^{\infty} \zeta_n^{\alpha_n} \leq \sum_{n=1}^{\infty} \alpha_n \zeta_n.$$

Solution. This is the weighted arithmetic-geometric mean inequality for countably many terms; we deduce it from Problem 68. The infinite product is understood as $\lim_{N \rightarrow \infty} \prod_{n=1}^N \zeta_n^{\alpha_n}$, and $\zeta^0 = 1$.

Write

$$S = \sum_{n=1}^{\infty} \alpha_n \zeta_n \in (0, \infty].$$

The sum is positive: since $\sum_n \alpha_n = 1$, some $\alpha_{n_0} > 0$, and $\zeta_{n_0} > 0$, so $S \geq \alpha_{n_0} \zeta_{n_0} > 0$. If $S = +\infty$ the asserted inequality is trivial, so assume $S < \infty$.

Step 1: the left-hand side is well defined. Applying the inequality $e^s \geq 1 + s$ (valid for all real s , since e^s lies above its tangent line at $s = 0$, as noted in the solution of Problem 68) with $s = \log \zeta$ gives $\zeta \geq 1 + \log \zeta$, that is, the elementary inequality

$$\log \zeta \leq \zeta - 1 \quad \text{for all } \zeta > 0.$$

Hence $(\log \zeta_n)^+ \leq (\zeta_n - 1)^+ \leq \zeta_n$ for every n , and therefore

$$\sum_{n=1}^{\infty} \alpha_n (\log \zeta_n)^+ \leq \sum_{n=1}^{\infty} \alpha_n \zeta_n = S < \infty.$$

Consequently the series $\sum_n \alpha_n \log \zeta_n$ has convergent positive part, so it is well defined as an element of $[-\infty, \infty)$, namely as the difference $\sum_n \alpha_n (\log \zeta_n)^+ - \sum_n \alpha_n (\log \zeta_n)^-$ of a finite number and a number in $[0, \infty]$; and its partial sums converge to this value in the extended sense (they tend to $-\infty$ exactly when $\sum_n \alpha_n (\log \zeta_n)^- = \infty$). Since $\prod_{n=1}^N \zeta_n^{\alpha_n} = \exp \left(\sum_{n=1}^N \alpha_n \log \zeta_n \right)$ and $\exp$ is continuous with $\exp(s) \rightarrow 0$ as $s \rightarrow -\infty$, the partial products converge in $[0, \infty)$ to

$$\prod_{n=1}^{\infty} \zeta_n^{\alpha_n} = \exp \left(\sum_{n=1}^{\infty} \alpha_n \log \zeta_n \right),$$

with the convention $\exp(-\infty) = 0$.

If $\sum_n \alpha_n \log \zeta_n = -\infty$, the left-hand side is $0 < S$ and we are done. So assume from now on that $\sum_n \alpha_n \log \zeta_n$ converges to a real number, and note that in view of the bound on the positive part this

means

$$\sum_{n=1}^{\infty} \alpha_n |\log \zeta_n| < \infty.$$

Step 2: construction of a function on $[0, 1]$. Put $A_0 = 0$ and $A_N = \sum_{n=1}^N \alpha_n$ for $N \geq 1$; the A_N increase to 1. Set

$$E_n = [A_{n-1}, A_n), \quad n = 1, 2, \dots$$

These are pairwise disjoint intervals with $m(E_n) = \alpha_n$ (empty when $\alpha_n = 0$) whose union is $[0, 1]$, and $[0, 1]$ differs from $[0, 1)$ by a single point, a set of measure zero. Define g on $[0, 1)$ by

$$g(x) = \zeta_n \quad \text{for } x \in E_n,$$

and define $g(1) = 1$, say. Then g is measurable and positive, and $f := \log g$ is measurable, with $f = \log \zeta_n$ on E_n .

Step 3: computing the two integrals. Both $|f|$ and g are nonnegative and measurable on $[0, 1]$, so by the countable additivity of integration over the disjoint sets E_n (a consequence of the Monotone Convergence Theorem of Chapter 4, applied to the increasing sequence of partial sums $\sum_{n=1}^N h \cdot \chi_{E_n}$ for a nonnegative measurable h),

$$\int_0^1 |f| = \sum_{n=1}^{\infty} \alpha_n |\log \zeta_n| < \infty, \quad \int_0^1 e^f = \int_0^1 g = \sum_{n=1}^{\infty} \alpha_n \zeta_n = S.$$

The first of these shows that f is integrable over $[0, 1]$, and the same countable additivity, now applied to f^+ and f^- separately and subtracted (both integrals being finite), gives

$$\int_0^1 f = \sum_{n=1}^{\infty} \alpha_n \log \zeta_n.$$

Step 4: applying Problem 68. Since f is integrable over $[0, 1]$, Problem 68 gives

$$\exp \left[\int_0^1 f(x) dx \right] \leq \int_0^1 \exp(f(x)) dx.$$

Substituting the two computations of Step 3,

$$\exp \left(\sum_{n=1}^{\infty} \alpha_n \log \zeta_n \right) \leq S = \sum_{n=1}^{\infty} \alpha_n \zeta_n.$$

By Step 1 the left-hand side is exactly $\prod_{n=1}^{\infty} \zeta_n^{\alpha_n}$. Therefore

$$\prod_{n=1}^{\infty} \zeta_n^{\alpha_n} \leq \sum_{n=1}^{\infty} \alpha_n \zeta_n,$$

as was to be shown.

Remark. One can also argue without measure theory, using only the inequality $\log \zeta \leq \zeta - 1$: applying it to $\zeta = \zeta_n/S$ gives $\log \zeta_n \leq \log S + \zeta_n/S - 1$, and multiplying by $\alpha_n \geq 0$ and summing over $n \leq N$ yields

$$\sum_{n=1}^N \alpha_n \log \zeta_n \leq A_N \log S + \frac{1}{S} \sum_{n=1}^N \alpha_n \zeta_n - A_N.$$

Letting $N \rightarrow \infty$, the right-hand side tends to $\log S + 1 - 1 = \log S$, so $\sum_n \alpha_n \log \zeta_n \leq \log S$, which is the same conclusion.

Problem 6.70. Let g be a positive measurable function on $[0, 1]$. Show that $\log \left(\int_0^1 g(x) dx \right) \geq \int_0^1 \log(g(x)) dx$ whenever each side is defined.

Solution. Since g is nonnegative and measurable on $[0, 1]$, the integral $\int_0^1 g$ is defined as an element of $[0, \infty]$; in fact it is strictly positive. Indeed, since $g > 0$ everywhere on $[0, 1]$, the measurable sets $E_k = \{x \in [0, 1] : g(x) > 1/k\}$ increase to $[0, 1]$, so by the continuity of measure $m(E_k) \rightarrow 1$, and hence $m(E_{k_0}) > 0$ for some k_0 ; then by the monotonicity of the integral $\int_0^1 g \geq \int_{E_{k_0}} g \geq m(E_{k_0})/k_0 > 0$. Thus $\int_0^1 g \in (0, \infty]$ and the left-hand side is defined in $(-\infty, \infty]$. The function $\log \circ g$ is measurable: $\log$ is continuous on $(0, \infty)$ and g is measurable and positive, so Proposition 7 of Chapter 3 applies. Its integral is defined provided not both $\int_0^1 (\log g)^+$ and $\int_0^1 (\log g)^-$ are infinite. We show that under these standing conventions the inequality

$$\log \left(\int_0^1 g(x) dx \right) \geq \int_0^1 \log(g(x)) dx$$

always holds; in fact we show that whenever $\int_0^1 g < \infty$ the right-hand side is automatically defined in $[-\infty, \infty)$, so that the proviso is needed only to make sense of the case $\int_0^1 g = \infty$.

Case 1: $\int_0^1 g = +\infty$. (This includes the case in which g takes the value $+\infty$ on a set of positive measure, if g is allowed to be extended real-valued.) Then the left-hand side is $\log(+\infty) = +\infty$, which dominates any value the right-hand side may take. The inequality holds.

Case 2: $A := \int_0^1 g(x) dx$ is finite, so $0 < A < \infty$ and g is integrable over $[0, 1]$; in particular g is finite almost everywhere. Put $f = \log g$, a measurable function on $[0, 1]$ which is finite almost everywhere, and note that $e^f = g$ almost everywhere.

We first check that $\int_0^1 f$ is defined. By the inequality $\log \zeta \leq \zeta - 1$ for $\zeta > 0$ (see the solution of Problem 69, where it was derived from $e^s \geq 1 + s$), we have pointwise almost everywhere

$$f^+ = (\log g)^+ \leq (g - 1)^+ \leq g,$$

and hence, by the monotonicity of the integral for nonnegative measurable functions,

$$\int_0^1 f^+ \leq \int_0^1 g = A < \infty.$$

Thus the positive part of f is always integrable, and $\int_0^1 f$ is defined as an element of $[-\infty, \infty)$.

If $\int_0^1 f = -\infty$, the asserted inequality reads $\log A \geq -\infty$, which is true since $A > 0$. So assume $\int_0^1 f$ is finite; then f is integrable over $[0, 1]$, and Problem 68 applies to f :

$$\exp \left[\int_0^1 f(x) dx \right] \leq \int_0^1 \exp(f(x)) dx.$$

Now $\exp(f(x)) = g(x)$ for almost all $x \in [0, 1]$, so the right-hand side equals $\int_0^1 g = A$. Therefore

$$\exp \left[\int_0^1 \log(g(x)) dx \right] \leq A.$$

Since $A > 0$ and $\log$ is an increasing function on $(0, \infty)$, taking logarithms in this inequality preserves it and gives

$$\int_0^1 \log(g(x)) dx \leq \log A = \log \left(\int_0^1 g(x) dx \right),$$

which is exactly the asserted inequality.

In both cases the inequality holds, and the proof is complete.

Remark. The inequality is the integral form of the statement that the geometric mean of g does not exceed its arithmetic mean: writing $\exp(\int_0^1 \log g)$ for the geometric mean of g over $[0, 1]$, what has been proved is

$$\exp\left(\int_0^1 \log g\right) \leq \int_0^1 g.$$

Problem 69 is the special case in which g is the countably-valued step function taking the value ζ_n on a set of measure α_n . Equality holds precisely when g is constant almost everywhere; indeed, equality forces equality almost everywhere in the supporting-line inequality $e^t \geq e^\alpha(t - \alpha) + e^\alpha$ used to prove Problem 68, and since e^t is strictly convex this happens only for $t = \alpha$, that is, only when $\log g$ is almost everywhere equal to its average.

6.11 Problem 6.71

Problem 6.71. (Nemytskii) Let φ be a continuous function on $\mathbb{R}$. Show that if there are constants for which (45) holds, then $\varphi \circ f$ is integrable over $[0, 1]$ whenever f is. Then show that if $\varphi \circ f$ is integrable over $[0, 1]$ whenever f is, then there are constants c_1 and c_2 for which (45) holds.

[The reference to (45) is a misnumbering. The growth condition intended is the one displayed in the discussion following Jensen's Inequality, where it carries the number (43), namely

$$|\varphi(x)| \leq c_1 + c_2|x| \quad \text{for all } x \in \mathbb{R}.$$

The text there observes that in the absence of this growth assumption $\varphi \circ f$ may fail to be integrable over $[0, 1]$, and refers the reader to this problem.]

Solution. Throughout, integrability over $[0, 1]$ carries its meaning from Chapter 4: the function is measurable on $[0, 1]$ and its absolute value has finite integral there. The condition in question is

$$|\varphi(x)| \leq c_1 + c_2|x| \quad \text{for all } x \in \mathbb{R}.$$

We call this displayed inequality the growth condition.

Sufficiency of the growth condition. Assume there are constants c_1, c_2 for which the growth condition holds, and let f be integrable over $[0, 1]$. Then f is measurable on $[0, 1]$ and, by Proposition 15 of Chapter 4, finite almost everywhere on $[0, 1]$.

A preliminary reduction is needed because φ is defined only on $\mathbb{R}$, so that $\varphi \circ f$ is a priori defined only where f is finite. Set $Z = \{x \in [0, 1] : |f(x)| = \infty\}$, so $m(Z) = 0$, and define $\tilde{f} = f$ on $[0, 1] \setminus Z$ and $\tilde{f} = 0$ on Z . Then $\tilde{f}$ is real-valued and measurable on $[0, 1]$ and agrees with f except on a set of measure zero, so $\int_0^1 |\tilde{f}| = \int_0^1 |f| < \infty$ and $\tilde{f}$ is integrable over $[0, 1]$; moreover $\varphi \circ \tilde{f}$ agrees with $\varphi \circ f$ at every point at which the latter is defined, hence almost everywhere on $[0, 1]$. Since functions that agree almost everywhere have the same integral, it suffices to prove that $\varphi \circ \tilde{f}$ is integrable over $[0, 1]$. We therefore assume from the outset that f is real-valued on $[0, 1]$.

Since φ is continuous on all of $\mathbb{R}$ and f is a real-valued measurable function on $[0, 1]$, Proposition 7 of Chapter 3 tells us that the composition $\varphi \circ f$ is measurable on $[0, 1]$.

Define $g = c_1 + c_2|f|$ on $[0, 1]$. By the growth condition,

$$|(\varphi \circ f)(x)| = |\varphi(f(x))| \leq c_1 + c_2|f(x)| = g(x) \quad \text{for all } x \in [0, 1],$$

so in particular $g \geq 0$ on $[0, 1]$, since the left-hand side is nonnegative. Moreover $|f|$ is integrable over $[0, 1]$ because f is, and constant functions are integrable over the set $[0, 1]$, which has finite measure; hence, by Theorem 17 of Chapter 4 (Linearity and Monotonicity of Integration), g is integrable over $[0, 1]$ and

$$\int_0^1 g = c_1 + c_2 \int_0^1 |f| < \infty,$$

so g is a nonnegative integrable function on $[0, 1]$ that dominates $|\varphi \circ f|$. By Proposition 16 of Chapter 4 (the Integral Comparison Test), $\varphi \circ f$ is integrable over $[0, 1]$. This is exactly the assertion made in the text's remark following Jensen's Inequality.

Necessity of the growth condition. Now assume that $\varphi \circ f$ is integrable over $[0, 1]$ whenever f is, and suppose, to get a contradiction, that there is no pair of constants c_1, c_2 for which the growth condition holds. Then for each natural number n the particular choice $c_1 = c_2 = n$ fails, that is, there is a point $x_n \in \mathbb{R}$ for which

$$|\varphi(x_n)| > n + n|x_n| = n(1 + |x_n|).$$

Using this sequence $\{x_n\}$ we manufacture an integrable f on $[0, 1]$ whose composition with φ fails to be integrable. For each index n set

$$\alpha_n = \frac{1}{2n^2(1 + |x_n|)} > 0.$$

Since $1 + |x_n| \geq 1$ we have $0 < \alpha_n \leq 1/(2n^2)$, and therefore

$$\sum_{n=1}^{\infty} \alpha_n \leq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12} < 1.$$

Put $s_0 = 0$ and $s_n = \sum_{k=1}^n \alpha_k$ for $n \geq 1$; then $\{s_n\}$ is strictly increasing and $s_n < 1$ for every n . Define the pairwise disjoint intervals

$$E_n = [s_{n-1}, s_n) \subseteq [0, 1], \quad m(E_n) = \alpha_n,$$

and let $E_0 = [0, 1] \setminus \bigcup_{n=1}^{\infty} E_n$. Define f on $[0, 1]$ by

$$f = \sum_{n=1}^{\infty} x_n \chi_{E_n},$$

that is, $f(t) = x_n$ for $t \in E_n$ and $f(t) = 0$ for $t \in E_0$. Each partial sum $\sum_{n=1}^N x_n \chi_{E_n}$ is a simple function, and since the E_n are disjoint these partial sums converge to f pointwise on $[0, 1]$; hence f is measurable on $[0, 1]$, and it is real-valued.

Claim: f is integrable over $[0, 1]$. The functions $h_N = \sum_{n=1}^N |x_n| \chi_{E_n}$ form an increasing sequence of nonnegative measurable functions converging pointwise on $[0, 1]$ to $|f|$ (at a point of E_m we have $h_N = |x_m| = |f|$ once $N \geq m$, and at a point of E_0 every h_N vanishes, as does f). Each h_N is a nonnegative simple function whose integral is, by definition, $\sum_{n=1}^N |x_n| m(E_n) = \sum_{n=1}^N |x_n| \alpha_n$. Hence by the Monotone Convergence Theorem,

$$\int_0^1 |f| = \lim_{N \rightarrow \infty} \int_0^1 h_N = \lim_{N \rightarrow \infty} \sum_{n=1}^N |x_n| \alpha_n = \sum_{n=1}^{\infty} \frac{|x_n|}{2n^2(1 + |x_n|)}.$$

Since $|x_n| \leq 1 + |x_n|$, each term is at most $1/(2n^2)$, and therefore

$$\int_0^1 |f| \leq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus f is integrable over $[0, 1]$.

Claim: $\varphi \circ f$ is not integrable over $[0, 1]$. The function $\varphi \circ f$ is measurable on $[0, 1]$, by Proposition 7 of Chapter 3 again, and it takes the value $\varphi(x_n)$ on E_n and the value $\varphi(0)$ on E_0 . Being nonnegative and measurable, $|\varphi \circ f|$ has a well-defined integral over any measurable subset of $[0, 1]$, with value in $[0, \infty]$. Fix a natural number N and set $A_N = \bigcup_{n=1}^N E_n$, a measurable subset of $[0, 1]$. Applying Theorem 11 of Chapter 4 (Additivity Over Domains of Integration) to the disjoint decomposition $[0, 1] = A_N \cup ([0, 1] \setminus A_N)$, and then to the disjoint decomposition $A_N = E_1 \cup \cdots \cup E_N$, and using that $|\varphi \circ f|$ is the constant $|\varphi(x_n)|$ on E_n so that $\int_{E_n} |\varphi \circ f| = |\varphi(x_n)| m(E_n) = \alpha_n |\varphi(x_n)|$, we obtain

$$\int_0^1 |\varphi \circ f| \geq \int_{A_N} |\varphi \circ f| = \sum_{n=1}^N \alpha_n |\varphi(x_n)|.$$

By the choice of x_n and of α_n ,

$$\alpha_n |\varphi(x_n)| > \frac{1}{2n^2(1+|x_n|)} \cdot n(1+|x_n|) = \frac{1}{2n},$$

and hence

$$\int_0^1 |\varphi \circ f| \geq \sum_{n=1}^N \frac{1}{2n} \quad \text{for every } N.$$

The harmonic series diverges, so letting $N \rightarrow \infty$ gives $\int_0^1 |\varphi \circ f| = \infty$. Therefore $\varphi \circ f$ is not integrable over $[0, 1]$.

The f just constructed is integrable over $[0, 1]$ while $\varphi \circ f$ is not, contradicting the standing hypothesis that $\varphi \circ f$ is integrable over $[0, 1]$ whenever f is. The supposition that no pair of constants satisfies the growth condition is therefore untenable, and there do exist constants c_1 and c_2 for which $|\varphi(x)| \leq c_1 + c_2|x|$ for all $x \in \mathbb{R}$.

It is worth noting in passing that the sequence $\{x_n\}$ produced above is necessarily unbounded: were it contained in some closed, bounded interval, the continuity of φ would make $|\varphi|$ bounded on that interval, whereas $|\varphi(x_n)| > n(1+|x_n|) \geq n$ for every n . This is only a comment on the shape of a counterexample; no use of it was made in the proof, which is valid for any sequence with the stated property.

Remark. Combining the two halves: for a continuous φ on $\mathbb{R}$, the Nemytskii operator $f \mapsto \varphi \circ f$ carries every function integrable over $[0, 1]$ to a function integrable over $[0, 1]$ if and only if φ obeys the linear growth bound $|\varphi(x)| \leq c_1 + c_2|x|$ on $\mathbb{R}$. In particular, in the absence of that growth assumption the function $\varphi \circ f$ appearing in Jensen's Inequality may fail to be integrable, which is the assertion the text makes when it refers the reader here. A concrete instance is $\varphi(x) = x^2$: it is continuous and convex but obeys no bound of the form $c_1 + c_2|x|$, and the construction above (take x_n with $x_n^2 > n(1+|x_n|)$, for instance $x_n = 2n$) produces an f integrable over $[0, 1]$ for which $\varphi \circ f$ is not.

7 The L^p Spaces: Completeness and Approximation

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7.1 Problems 7.1–7.7

Problem 7.1. For f in $C[a, b]$, define

$$\|f\|_1 = \int_a^b |f|.$$

Show that this is a norm on $C[a, b]$. Also show that there is no number $c \geq 0$ for which

$$\|f\|_{\max} \leq c\|f\|_1 \quad \text{for all } f \text{ in } C[a, b],$$

but there is a $c \geq 0$ for which

$$\|f\|_1 \leq c\|f\|_{\max} \quad \text{for all } f \text{ in } C[a, b].$$

Solution. Throughout, $[a, b]$ is a nondegenerate closed, bounded interval, so $a < b$.

The functional is well defined and finite. If f is continuous on $[a, b]$, then f is measurable and, attaining a maximum value, bounded; hence $|f|$ is a bounded measurable function on a set of finite measure and $\int_a^b |f|$ is a finite real number.

The Triangle Inequality. Let f, g belong to $C[a, b]$. Then $|f + g| \leq |f| + |g|$ pointwise on $[a, b]$, so by the monotonicity and linearity of integration,

$$\|f + g\|_1 = \int_a^b |f + g| \leq \int_a^b [|f| + |g|] = \int_a^b |f| + \int_a^b |g| = \|f\|_1 + \|g\|_1.$$

Positive Homogeneity. For a real number α , $|\alpha f| = |\alpha| |f|$ pointwise, so by the linearity of integration

$$\|\alpha f\|_1 = \int_a^b |\alpha| |f| = |\alpha| \int_a^b |f| = |\alpha| \|f\|_1.$$

Nonnegativity. Since $|f| \geq 0$, monotonicity of integration gives $\|f\|_1 \geq 0$. Clearly $\|0\|_1 = 0$. Conversely, suppose $\|f\|_1 = 0$ and $f \neq 0$, say $|f(x_0)| = \epsilon > 0$ for some x_0 in $[a, b]$. Since $|f|$ is continuous at x_0 , there is a $\delta > 0$ such that $|f(x)| > \epsilon/2$ for all x in $(x_0 - \delta, x_0 + \delta) \cap [a, b]$. This intersection contains a closed interval $[c, d] \subseteq [a, b]$ with $d - c > 0$ (take a one-sided interval if x_0 is an endpoint). By monotonicity of integration,

$$\|f\|_1 = \int_a^b |f| \geq \int_c^d |f| \geq \frac{\epsilon}{2}(d - c) > 0,$$

a contradiction. Hence $\|f\|_1 = 0$ if and only if $f = 0$. Note that here, in contrast with $L^1[a, b]$, the conclusion is that f vanishes at **every** point, not merely almost everywhere: continuity upgrades “a.e.” to “everywhere”. Thus $\|\cdot\|_1$ is a norm on $C[a, b]$.

There is no $c \geq 0$ with $\|f\|_{\max} \leq c\|f\|_1$ for all f . Choose a natural number N with $1/N < b - a$. For each natural number $n \geq N$ define f_n in $C[a, b]$ by

$$f_n(x) = \begin{cases} n(1 - n(x - a)) & \text{if } a \leq x \leq a + 1/n, \\ 0 & \text{if } a + 1/n < x \leq b. \end{cases}$$

Each f_n is continuous (the two formulas agree, with value 0, at $x = a + 1/n$), nonnegative, and takes its maximum value n at $x = a$; thus $\|f_n\|_{\max} = n$. On the other hand f_n vanishes off $[a, a + 1/n]$ and its graph there is a triangle of base $1/n$ and height n , so

$$\|f_n\|_1 = \int_a^{a+1/n} n(1 - n(x - a)) dx = \frac{1}{2} \cdot \frac{1}{n} \cdot n = \frac{1}{2}.$$

Hence $\|f_n\|_{\max} = n = 2n \cdot \|f_n\|_1$. If there were a $c \geq 0$ with $\|f\|_{\max} \leq c\|f\|_1$ for all f in $C[a, b]$, then $n \leq c/2$ for every $n \geq N$, which is impossible. Therefore no such c exists.

There is a $c \geq 0$ with $\|f\|_1 \leq c\|f\|_{\max}$ for all f . Take $c = b - a$. Indeed, for f in $C[a, b]$ we have $|f(x)| \leq \|f\|_{\max}$ for all x in $[a, b]$, so by the monotonicity of integration,

$$\|f\|_1 = \int_a^b |f| \leq \int_a^b \|f\|_{\max} = (b - a) \|f\|_{\max}.$$

Thus the maximum norm dominates the L^1 norm on $C[a, b]$ up to the constant $b - a$, but the two norms are not equivalent: convergence in $\|\cdot\|_1$ does not imply uniform convergence.

Problem 7.2. Let X be the family of all polynomials with real coefficients defined on $\mathbb{R}$. Show that this is a linear space. For a polynomial p , define $\|p\|$ to be the sum of the absolute values of the coefficients of p . Is this a norm?

Solution. X is a linear space. Let $\mathcal{F}$ denote the linear space of all real-valued functions on $\mathbb{R}$, with the usual pointwise addition and scalar multiplication. Then $X \subseteq \mathcal{F}$, and it suffices to show that X is a linear subspace of $\mathcal{F}$, that is, that X is closed under addition and scalar multiplication and contains the zero function.

The zero function is the polynomial all of whose coefficients are 0, so 0 belongs to X . Let p, q belong to X and let α, β be real numbers. Write

$$p(x) = \sum_{k=0}^n a_k x^k, \quad q(x) = \sum_{k=0}^n b_k x^k,$$

where we have padded the shorter list of coefficients with zeros so that both sums run to the same index n . Then for every real x ,

$$(\alpha p + \beta q)(x) = \alpha p(x) + \beta q(x) = \sum_{k=0}^n (\alpha a_k + \beta b_k) x^k,$$

which is again a polynomial with real coefficients. Hence $\alpha p + \beta q$ belongs to X , and X is a linear space.

Well-definedness of $\|\cdot\|$. The proposed functional is defined in terms of the coefficients, while the elements of X are **functions**; so we must first check that a polynomial function determines its coefficients. Suppose $\sum_{k=0}^n a_k x^k = \sum_{k=0}^n c_k x^k$ for every real x . Then $r(x) = \sum_{k=0}^n (a_k - c_k) x^k$ vanishes identically on $\mathbb{R}$. If some coefficient $a_k - c_k$ were nonzero, r would be a nonzero polynomial of some degree $d \leq n$ and hence would have at most d real roots, contradicting the fact that every real number is a root. Therefore $a_k = c_k$ for all k . Consequently each p in X has a unique coefficient sequence $(a_0, a_1, \dots)$, all but finitely many of whose terms vanish, and

$$\|p\| = \sum_{k \geq 0} |a_k|$$

is an unambiguously defined finite nonnegative real number.

Yes, $\|\cdot\|$ is a norm. Let p and q have coefficient sequences (a_k) and (b_k) , each with only finitely many nonzero terms, and let α be a real number.

The Triangle Inequality: by the computation above, the coefficient sequence of $p + q$ is $(a_k + b_k)$, so using the triangle inequality for real numbers termwise,

$$\|p + q\| = \sum_{k \geq 0} |a_k + b_k| \leq \sum_{k \geq 0} [|a_k| + |b_k|] = \sum_{k \geq 0} |a_k| + \sum_{k \geq 0} |b_k| = \|p\| + \|q\|.$$

(All sums have only finitely many nonzero terms, so these manipulations are finite sums.)

Positive Homogeneity: the coefficient sequence of αp is (αa_k) , so

$$\|\alpha p\| = \sum_{k \geq 0} |\alpha a_k| = |\alpha| \sum_{k \geq 0} |a_k| = |\alpha| \|p\|.$$

Nonnegativity: $\|p\| \geq 0$ as a sum of nonnegative numbers, and $\|p\| = 0$ if and only if $|a_k| = 0$ for every k , that is, if and only if every coefficient of p vanishes, which by the uniqueness of coefficients just established holds if and only if p is the zero function.

Hence $\|\cdot\|$ is a norm on X , and X with this norm is a normed linear space.

Two remarks. First, the uniqueness of coefficients is essential: had we taken X to be the polynomial functions on a finite set, or on an interval-free domain such as a single point, distinct coefficient lists could give the same function and $\|\cdot\|$ would not even be well defined. Second, this norm is genuinely different from, say, $\|p\|_{\max}$ on a bounded interval; identifying X with the space of finitely supported real sequences, $\|\cdot\|$ is exactly the restriction of the ℓ^1 norm.

Problem 7.3. For f in $L^1[a, b]$, define $\|f\| = \int_a^b x^2 |f(x)| dx$. Show that this is a norm on $L^1[a, b]$.

Solution. Recall that $L^1[a, b]$ is the linear space of (equivalence classes of) integrable functions on $[a, b]$, two functions being identified when they agree almost everywhere.

The functional is well defined and finite. Set $M = \max\{a^2, b^2\}$, so that $0 \leq x^2 \leq M$ for all x in $[a, b]$. If f belongs to $L^1[a, b]$ then $x \mapsto x^2 |f(x)|$ is a nonnegative measurable function on $[a, b]$, and by the monotonicity and linearity of integration

$$0 \leq \int_a^b x^2 |f(x)| dx \leq M \int_a^b |f| = M \|f\|_1 < \infty.$$

So $\|f\|$ is a finite nonnegative real number. Moreover, if $f = g$ a.e. on $[a, b]$ then $x^2 |f(x)| = x^2 |g(x)|$ a.e. on $[a, b]$, so the two have the same integral; hence $\|\cdot\|$ is constant on equivalence classes and is a genuine functional on $L^1[a, b]$.

The Triangle Inequality. Let f, g belong to $L^1[a, b]$. Since f and g are integrable, they are finite a.e. on $[a, b]$, and at each point where both are finite the triangle inequality for real numbers gives $|f(x) + g(x)| \leq |f(x)| + |g(x)|$. Multiplying by the nonnegative number x^2 ,

$$x^2 |f(x) + g(x)| \leq x^2 |f(x)| + x^2 |g(x)| \quad \text{for almost all } x \text{ in } [a, b].$$

Hence, by the monotonicity and linearity of integration,

$$\|f + g\| = \int_a^b x^2 |f + g| \leq \int_a^b x^2 |f| + \int_a^b x^2 |g| = \|f\| + \|g\|.$$

Positive Homogeneity. For a real number α we have $x^2 |\alpha f(x)| = |\alpha| x^2 |f(x)|$ for all x , so by the linearity of integration

$$\|\alpha f\| = \int_a^b |\alpha| x^2 |f| = |\alpha| \int_a^b x^2 |f| = |\alpha| \|f\|.$$

Nonnegativity. We have already observed $\|f\| \geq 0$, and plainly $\|0\| = 0$. Conversely suppose $\|f\| = 0$. The function $h(x) = x^2 |f(x)|$ is nonnegative and integrable over $[a, b]$ with $\int_a^b h = 0$; therefore, by Proposition 9 of Chapter 4, $h = 0$ a.e. on $[a, b]$. Let $Z = \{x \in [a, b] : h(x) \neq 0\}$, so $m(Z) = 0$. If x belongs to $[a, b] \sim (Z \cup \{0\})$, then $x \neq 0$, so $x^2 > 0$, and $x^2 |f(x)| = 0$ forces $f(x) = 0$. Since $m(Z \cup \{0\}) \leq m(Z) + m(\{0\}) = 0$, we conclude that $f = 0$ a.e. on $[a, b]$, that is, f is the zero element of $L^1[a, b]$.

Note that this last step is where the choice of weight matters: the weight x^2 is allowed to vanish, but only on the single point $x = 0$, a set of measure zero, so no nonzero element of $L^1[a, b]$ is annihilated. (The same argument works for any measurable weight $w \geq 0$ that is bounded on $[a, b]$ and satisfies $m(\{x : w(x) = 0\}) = 0$.)

Therefore $\|\cdot\|$ satisfies the Triangle Inequality, Positive Homogeneity, and Nonnegativity, and so is a norm on $L^1[a, b]$.

Problem 7.4. For f in $L^\infty[a, b]$, show that

$$\|f\|_\infty = \min \left\{ M \mid m\{x \in [a, b] \mid |f(x)| > M\} = 0 \right\}$$

and if, furthermore, f is continuous on $[a, b]$, that

$$\|f\|_\infty = \|f\|_{\max}.$$

Solution. Recall the definitions. A function f is essentially bounded on E provided there is some $M \geq 0$, called an essential upper bound for f , for which $|f(x)| \leq M$ for almost all x in E ; and $\|f\|_\infty$ is defined to be the infimum of the essential upper bounds for f . As usual $a < b$.

Put

$$S = \left\{ M \in \mathbb{R} \mid m\{x \in [a, b] : |f(x)| > M\} = 0 \right\}.$$

Each set $\{x \in [a, b] : |f(x)| > M\}$ is measurable because f is measurable, so the condition makes sense. Step 1: S is exactly the set of essential upper bounds for f . Indeed, the statement “ $|f(x)| \leq M$ for almost all x in $[a, b]$ ” says precisely that the set of x in $[a, b]$ at which the inequality fails, namely $\{x \in [a, b] : |f(x)| > M\}$, has measure zero. Moreover every member of S is automatically nonnegative: if $M < 0$ then $\{x \in [a, b] : |f(x)| > M\} = [a, b]$, which has measure $b - a > 0$, so M is not in S . Hence S is the set of essential upper bounds for f , and by definition

$$\|f\|_\infty = \inf S.$$

Since f belongs to $L^\infty[a, b]$, S is nonempty; and S is bounded below by 0, so this infimum is a finite nonnegative number.

Step 2: the infimum is attained, that is, $\|f\|_\infty$ itself belongs to S . Let n be a natural number. Since $\|f\|_\infty = \inf S$ and $\|f\|_\infty + 1/n > \|f\|_\infty$, there is an M_n in S with $M_n < \|f\|_\infty + 1/n$. Define

$$E_n = \{x \in [a, b] : |f(x)| > M_n\},$$

so that $m(E_n) = 0$ and $|f(x)| \leq M_n \leq \|f\|_\infty + 1/n$ for all x in $[a, b] \setminus E_n$. Set $E_\infty = \bigcup_{n=1}^\infty E_n$. By the countable subadditivity of Lebesgue measure, $m(E_\infty) = 0$. If x belongs to $[a, b] \setminus E_\infty$, then for every natural number n we have $|f(x)| \leq \|f\|_\infty + 1/n$, and letting $n \rightarrow \infty$ gives $|f(x)| \leq \|f\|_\infty$. Therefore

$$\{x \in [a, b] : |f(x)| > \|f\|_\infty\} \subseteq E_\infty,$$

and by the monotonicity of measure this set has measure zero. Hence $\|f\|_\infty$ belongs to S . Being a lower bound for S that lies in S , it is the minimum of S :

$$\|f\|_\infty = \min S = \min \left\{ M \mid m\{x \in [a, b] : |f(x)| > M\} = 0 \right\}.$$

Equivalently: the essential supremum of f is itself an essential upper bound for f , that is, the smallest essential upper bound.

Step 3: if f is continuous on $[a, b]$, then $\|f\|_\infty = \|f\|_{\max}$. Here $\|f\|_{\max} = \max_{x \in [a, b]} |f(x)|$, which exists since $|f|$ is continuous on the closed, bounded interval $[a, b]$ and therefore attains a maximum value.

First, $|f(x)| \leq \|f\|_{\max}$ for **every** x in $[a, b]$, so a fortiori for almost every x ; thus $\|f\|_{\max}$ belongs to S , and since $\|f\|_\infty = \min S$,

$$\|f\|_\infty \leq \|f\|_{\max}.$$

For the reverse inequality, suppose to the contrary that $\|f\|_\infty < \|f\|_{\max}$ and set $\epsilon = \|f\|_{\max} - \|f\|_\infty > 0$. Choose x_0 in $[a, b]$ with $|f(x_0)| = \|f\|_{\max}$. Since $|f|$ is continuous at x_0 , there is a $\delta > 0$ such that

$$|f(x)| > |f(x_0)| - \frac{\epsilon}{2} = \|f\|_\infty + \frac{\epsilon}{2} > \|f\|_\infty \quad \text{for all } x \text{ in } (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

Because $a < b$ and x_0 belongs to $[a, b]$, the set $(x_0 - \delta, x_0 + \delta) \cap [a, b]$ contains an interval of positive length and so has positive measure. But this set is contained in $\{x \in [a, b] : |f(x)| > \|f\|_\infty\}$, which by Step 2 has measure zero. This contradiction shows $\|f\|_{\max} \leq \|f\|_\infty$.

Combining the two inequalities, $\|f\|_\infty = \|f\|_{\max}$ for continuous f . (As the text observes, an element of $L^\infty[a, b]$ has at most one continuous representative, so this identifies the L^∞ norm with the maximum norm on the copy of $C[a, b]$ sitting inside $L^\infty[a, b]$.)

Problem 7.5. Show that ℓ^∞ and ℓ^1 are normed linear spaces.

Solution. Recall the definitions from the text: ℓ^∞ is the collection of bounded real sequences $a = (a_1, a_2, \dots)$ with

$$\|a\|_\infty = \sup_{1 \leq k < \infty} |a_k|,$$

and ℓ^1 is the collection of real sequences for which $\sum_{k=1}^\infty |a_k| < \infty$, with

$$\|a\|_1 = \sum_{k=1}^\infty |a_k|.$$

Both sit inside the linear space $\mathcal{S}$ of all real sequences, with coordinatewise addition and scalar multiplication, $(a+b)_k = a_k + b_k$ and $(\alpha a)_k = \alpha a_k$; so in each case it suffices to check that the collection is a linear subspace of $\mathcal{S}$ and that the proposed functional satisfies the three defining properties of a norm.

The space ℓ^∞ . Let a, b belong to ℓ^∞ and let α be a real number. For each index k ,

$$|a_k + b_k| \leq |a_k| + |b_k| \leq \|a\|_\infty + \|b\|_\infty,$$

so $a+b$ is a bounded sequence, and $\|a\|_\infty + \|b\|_\infty$ is an upper bound for the set $\{|a_k + b_k| : k \geq 1\}$. Since the supremum is the least upper bound,

$$\|a+b\|_\infty \leq \|a\|_\infty + \|b\|_\infty.$$

Also $|\alpha a_k| = |\alpha| |a_k| \leq |\alpha| \|a\|_\infty$ for each k , so αa is bounded. Thus ℓ^∞ is closed under linear combinations, and it contains the zero sequence, so it is a linear space.

For positive homogeneity, if $\alpha = 0$ then $\|\alpha a\|_\infty = 0 = |\alpha| \|a\|_\infty$. If $\alpha \neq 0$, then multiplication by the positive number $|\alpha|$ is an increasing bijection of $[0, \infty)$ onto itself, so

$$\|\alpha a\|_\infty = \sup_k |\alpha| |a_k| = |\alpha| \sup_k |a_k| = |\alpha| \|a\|_\infty.$$

(Concretely: $|\alpha| \|a\|_\infty$ is an upper bound for $\{|\alpha a_k|\}$; and if c is any upper bound for $\{|\alpha a_k|\}$, then $c/|\alpha|$ is an upper bound for $\{|a_k|\}$, so $c/|\alpha| \geq \|a\|_\infty$, that is, $c \geq |\alpha| \|a\|_\infty$.)

Finally $\|a\|_\infty \geq 0$ since it is a supremum of nonnegative numbers, and $\|a\|_\infty = 0$ if and only if $|a_k| \leq 0$ for every k , that is, if and only if $a_k = 0$ for every k , that is, if and only if a is the zero sequence. Hence ℓ^∞ is a normed linear space.

The space ℓ^1 . Let a, b belong to ℓ^1 and let α be a real number. For each natural number n , using the triangle inequality for real numbers termwise and then the fact that the full series have nonnegative terms,

$$\sum_{k=1}^n |a_k + b_k| \leq \sum_{k=1}^n |a_k| + \sum_{k=1}^n |b_k| \leq \|a\|_1 + \|b\|_1.$$

The partial sums of $\sum_{k=1}^\infty |a_k + b_k|$ form an increasing sequence bounded above by $\|a\|_1 + \|b\|_1$, hence converge to a finite limit that is at most this bound. Therefore $a+b$ belongs to ℓ^1 and

$$\|a+b\|_1 = \sum_{k=1}^\infty |a_k + b_k| \leq \|a\|_1 + \|b\|_1,$$

which is the Triangle Inequality. Likewise $\sum_{k=1}^n |\alpha a_k| = |\alpha| \sum_{k=1}^n |a_k| \leq |\alpha| \|a\|_1$ for every n , so αa belongs to ℓ^1 , and letting $n \rightarrow \infty$ in the identity $\sum_{k=1}^n |\alpha a_k| = |\alpha| \sum_{k=1}^n |a_k|$ gives

$$\|\alpha a\|_1 = |\alpha| \|a\|_1.$$

Since ℓ^1 contains the zero sequence and is closed under these two operations, it is a linear space.

For nonnegativity, $\|a\|_1$ is the limit of an increasing sequence of nonnegative partial sums, so $\|a\|_1 \geq 0$; and if $\|a\|_1 = 0$, then for each fixed k we have $|a_k| \leq \sum_{j=1}^\infty |a_j| = 0$, so $a_k = 0$. Thus $\|a\|_1 = 0$ if and only if a is the zero sequence. Hence ℓ^1 is a normed linear space.

Problem 7.6. Show that if Hölder's Inequality is true for normalized functions it is true in general.

Solution. Fix a measurable set E , a number p with $1 \leq p < \infty$, and let q be the conjugate of p . Hölder's Inequality (Theorem 1) asserts that if f belongs to $L^p(E)$ and g belongs to $L^q(E)$, then $f \cdot g$ is integrable over E and

$$\int_E |f \cdot g| \leq \|f\|_p \cdot \|g\|_q.$$

Recall that a function h in a normed linear space is called normalized, or a unit function, provided $\|h\| = 1$. So we assume the following.

Hypothesis. If f belongs to $L^p(E)$ with $\|f\|_p = 1$ and g belongs to $L^q(E)$ with $\|g\|_q = 1$, then $f \cdot g$ is integrable over E and $\int_E |f \cdot g| \leq 1$.

We must deduce the general inequality. Let f in $L^p(E)$ and g in $L^q(E)$ be arbitrary.

Two preliminary observations. First, $\|\cdot\|_p$ is positively homogeneous: for a real number α and h in $L^p(E)$, $|\alpha h|^p = |\alpha|^p |h|^p$, so by the linearity of integration

$$\|\alpha h\|_p = \left[\int_E |\alpha h|^p \right]^{1/p} = \left[|\alpha|^p \int_E |h|^p \right]^{1/p} = |\alpha| \left[\int_E |h|^p \right]^{1/p} = |\alpha| \|h\|_p,$$

and the same computation applies to $\|\cdot\|_q$ when $q < \infty$. (When the index is ∞ – which happens for q when $p = 1$ – positive homogeneity of $\|\cdot\|_\infty$ was verified in Section 7.1.) Second, for a finite index r , $\|h\|_r = 0$ if and only if $h = 0$ a.e. on E : this is Proposition 9 of Chapter 4 applied to the nonnegative integrable function $|h|^r$, whose integral vanishes exactly when $|h|^r = 0$ a.e. The same equivalence holds for the index ∞ : by the inequality (2) of Section 7.1, $|h| \leq \|h\|_\infty$ a.e. on E , so $\|h\|_\infty = 0$ forces $h = 0$ a.e.; and conversely if $h = 0$ a.e. then every $M \geq 0$ is an essential upper bound for h , so $\|h\|_\infty = 0$.

The degenerate cases. Suppose $\|f\|_p = 0$. Then $f = 0$ a.e. on E , hence $f \cdot g = 0$ a.e. on E ; so $f \cdot g$ is integrable over E with $\int_E |f \cdot g| = 0$, and the asserted inequality holds, indeed with equality, since $\|f\|_p \|g\|_q = 0$. The case $\|g\|_q = 0$ is identical. (Note $\|f\|_p$ and $\|g\|_q$ are finite, since f and g belong to $L^p(E)$ and $L^q(E)$ respectively.)

The general case. Assume now $\|f\|_p > 0$ and $\|g\|_q > 0$, and form the normalizations

$$f^\# = \frac{f}{\|f\|_p}, \quad g^\# = \frac{g}{\|g\|_q}.$$

These are scalar multiples of f and g , hence belong to $L^p(E)$ and $L^q(E)$ respectively, and by positive homogeneity

$$\|f^\#\|_p = \frac{1}{\|f\|_p} \|f\|_p = 1, \quad \|g^\#\|_q = \frac{1}{\|g\|_q} \|g\|_q = 1.$$

So $f^\#$ and $g^\#$ are normalized, and the Hypothesis applies to them: $f^\# \cdot g^\#$ is integrable over E and

$$\int_E |f^\# \cdot g^\#| \leq 1.$$

But pointwise on E ,

$$f \cdot g = (\|f\|_p f^\#) \cdot (\|g\|_q g^\#) = \|f\|_p \|g\|_q (f^\# \cdot g^\#),$$

a constant multiple of an integrable function. Hence $f \cdot g$ is integrable over E , and by the linearity of integration together with $|f \cdot g| = \|f\|_p \|g\|_q |f^\# \cdot g^\#|$,

$$\int_E |f \cdot g| = \|f\|_p \|g\|_q \int_E |f^\# \cdot g^\#| \leq \|f\|_p \cdot \|g\|_q.$$

This is exactly Hölder's Inequality for the arbitrary pair f, g . Therefore it suffices to prove the inequality for normalized functions, which is the reduction used in the text's proof of Theorem 1: one there assumes $\int_E |f|^p = 1$ and $\int_E |g|^q = 1$ and shows $\int_E |f \cdot g| \leq 1$.

Problem 7.7. Verify the assertions in the above two examples regarding the membership of the function f in $L^p(E)$.

[The two examples are: (i) for E of finite measure and $1 \leq p_1 < p_2 \leq \infty$, $L^{p_2}(E)$ is in general a proper subspace of $L^{p_1}(E)$ – for instance, let $E = (0, 1]$ and let f be defined by $f(x) = x^\alpha$ for $0 < x \leq 1$, where $-1/p_1 < \alpha < -1/p_2$; then f belongs to $L^{p_1}(E) \sim L^{p_2}(E)$. (ii) for E of infinite measure there are in general no inclusion relationships among the $L^p(E)$ spaces – for instance, for $E = (1, \infty)$ and f defined by

$$f(x) = \frac{x^{-1/2}}{1 + \ln x} \quad \text{for } x > 1,$$

f belongs to $L^p(E)$ if and only if $p = 2$.]

Solution. A computational lemma. For β a real number,

$$\int_{(0,1]} x^\beta dx = \begin{cases} \frac{1}{\beta+1} & \text{if } \beta > -1, \\ \infty & \text{if } \beta \leq -1, \end{cases} \quad \int_{(1,\infty)} x^\beta dx = \begin{cases} \frac{-1}{\beta+1} & \text{if } \beta < -1, \\ \infty & \text{if } \beta \geq -1. \end{cases}$$

To justify the first, let $g(x) = x^\beta$ on $(0, 1]$ and set $g_n = g \cdot \chi_{[1/n, 1]}$. Then $\{g_n\}$ is an increasing sequence of nonnegative measurable functions converging pointwise on $(0, 1]$ to g , so by the Monotone Convergence Theorem $\int_{(0,1]} g = \lim_n \int_{[1/n, 1]} g$. On $[1/n, 1]$ the function x^β is continuous, hence Riemann integrable, and by Theorem 3 of Chapter 4 its Riemann and Lebesgue integrals agree. For $\beta \neq -1$,

$$\int_{1/n}^1 x^\beta dx = \frac{1 - n^{-(\beta+1)}}{\beta+1},$$

which tends to $1/(\beta+1)$ if $\beta > -1$ (since then $n^{-(\beta+1)} \rightarrow 0$) and to ∞ if $\beta < -1$ (since then $n^{-(\beta+1)} \rightarrow \infty$ while $\beta+1 < 0$). For $\beta = -1$, $\int_{1/n}^1 x^{-1} dx = \ln n \rightarrow \infty$. The second formula is proved in the same way, with $g_n = g \cdot \chi_{(1, n]}$ and $\int_1^n x^\beta dx = (n^{\beta+1} - 1)/(\beta+1)$ for $\beta \neq -1$ and $\ln n$ for $\beta = -1$. Example (i). Let $E = (0, 1]$, let $1 \leq p_1 < p_2 \leq \infty$, and choose α with $-1/p_1 < \alpha < -1/p_2$, where $-1/p_2$ is read as 0 when $p_2 = \infty$. Such an α exists because $p_1 < p_2$ forces $-1/p_1 < -1/p_2$. Put $f(x) = x^\alpha$ for $0 < x \leq 1$; f is continuous and positive on E , hence measurable, and $\alpha < 0$.

f belongs to $L^{p_1}(E)$. Since $p_1 < \infty$, we must check $\int_E |f|^{p_1} < \infty$. Here $|f(x)|^{p_1} = x^{\alpha p_1}$, and $\alpha > -1/p_1$ gives $\alpha p_1 > -1$ (multiplying by the positive number p_1). By the lemma,

$$\int_{(0,1]} x^{\alpha p_1} dx = \frac{1}{\alpha p_1 + 1} < \infty.$$

f does not belong to $L^{p_2}(E)$. Suppose first $p_2 < \infty$. Then $\alpha < -1/p_2$ gives $\alpha p_2 < -1$, so by the lemma $\int_{(0,1]} |f|^{p_2} = \int_{(0,1]} x^{\alpha p_2} dx = \infty$, and f is not in $L^{p_2}(E)$. Now suppose $p_2 = \infty$, so that $\alpha < 0$. Let $M > 0$. Since $\alpha < 0$, $x^\alpha > M$ exactly when $x < M^{1/\alpha}$; hence

$$\{x \in (0, 1] : |f(x)| > M\} \supseteq (0, \min\{1, M^{1/\alpha}\}),$$

a set of positive measure. So no M is an essential upper bound for f , and f is not essentially bounded, that is, f is not in $L^\infty(E)$.

Thus f belongs to $L^{p_1}(E) \sim L^{p_2}(E)$, and the inclusion $L^{p_2}(E) \subseteq L^{p_1}(E)$ furnished by Corollary 3 is proper.

Example (ii). Let $E = (1, \infty)$ and $f(x) = x^{-1/2}/(1 + \ln x)$ for $x > 1$. On E we have $\ln x > 0$, so $1 + \ln x > 1$ and f is a well-defined, positive, continuous, hence measurable function, with

$$|f(x)|^p = \frac{x^{-p/2}}{(1 + \ln x)^p} \quad \text{for } 1 \leq p < \infty.$$

f belongs to $L^2(E)$. Here $|f(x)|^2 = 1/(x(1 + \ln x)^2)$. The function $F(x) = -1/(1 + \ln x)$ is continuously differentiable on $[1, n]$ with $F'(x) = 1/(x(1 + \ln x)^2)$, so by the Monotone Convergence Theorem (applied

to $|f|^2 \chi_{(1,n]}$ together with Theorem 3 of Chapter 4,

$$\int_{(1,\infty)} |f|^2 = \lim_{n \rightarrow \infty} \int_1^n \frac{dx}{x(1+\ln x)^2} = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{1+\ln n} \right] = 1 < \infty.$$

f does not belong to $L^p(E)$ for $1 \leq p < 2$. Put $\delta = 1 - p/2 > 0$. Substituting $t = \ln x$,

$$\frac{x^\delta}{(1+\ln x)^p} = \frac{e^{\delta t}}{(1+t)^p} \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

since an exponential with positive exponent grows faster than any power. Hence there is an $X > 1$ with $x^\delta \geq (1+\ln x)^p$ for all $x \geq X$, and for such x ,

$$|f(x)|^p = \frac{x^{-p/2}}{(1+\ln x)^p} \geq \frac{x^{-p/2}}{x^\delta} = x^{-p/2-1+p/2} = \frac{1}{x}.$$

By the monotonicity of integration and the lemma, $\int_{(1,\infty)} |f|^p \geq \int_{(X,\infty)} x^{-1} dx = \infty$. So f is not in $L^p(E)$.

A correction to the printed assertion. The example as printed claims that f belongs to $L^p(E)$ if and only if $p = 2$; the “only if” half fails for $p > 2$. Indeed, let $2 < p < \infty$. Since $1 + \ln x > 1$ on E , we have $(1 + \ln x)^p > 1$, so $|f(x)|^p \leq x^{-p/2}$, and $-p/2 < -1$, so by the lemma

$$\int_{(1,\infty)} |f|^p \leq \int_{(1,\infty)} x^{-p/2} dx = \frac{1}{p/2 - 1} < \infty.$$

Moreover f is bounded on E : for $x > 1$, $x^{-1/2} < 1$ and $1 + \ln x > 1$, so $0 < f(x) < 1$, and f belongs to $L^\infty(E)$ with $\|f\|_\infty \leq 1$. Hence the correct statement is

$$f \in L^p((1, \infty)) \iff 2 \leq p \leq \infty.$$

This still serves the purpose of the example, since it exhibits a function in $L^2(E) \sim L^1(E)$ and so shows that $L^{p_1}(E) \supseteq L^{p_2}(E)$ can fail for $p_1 < p_2$ when $m(E) = \infty$.

The repaired example. To get membership exactly at $p = 2$, take $E = (0, \infty)$ and

$$h(x) = \frac{x^{-1/2}}{1 + |\ln x|} \quad \text{for } x > 0,$$

which is continuous and positive on $(0, \infty)$, hence measurable. On $(1, \infty)$ we have $|\ln x| = \ln x$, so h agrees there with the f above and, by what we proved, $\int_{(1,\infty)} |h|^p < \infty$ if and only if $p \geq 2$ (and h is bounded on $(1, \infty)$). On $(0, 1)$ we have $|\ln x| = -\ln x$, so $|h|^2 = 1/(x(1 - \ln x)^2)$ has antiderivative $1/(1 - \ln x)$, and $\int_{1/n}^1 |h|^2 = 1 - 1/(1 + \ln n) \rightarrow 1$, so $\int_{(0,1)} |h|^2 < \infty$; while for $p > 2$, writing $\delta = p/2 - 1 > 0$ we have $x^{-\delta} \geq (1 - \ln x)^p$ for all sufficiently small $x > 0$ (again exponential beats power, via $t = -\ln x \rightarrow \infty$), whence $|h(x)|^p \geq x^{-p/2} x^\delta = 1/x$ there, and $\int_{(0,1)} x^{-1} dx = \infty$. Finally h is not in $L^\infty((0, \infty))$: for $0 < x < 1$, $h(x) = x^{-1/2}/(1 - \ln x)$ and, with $t = -\ln x$, this equals $e^{t/2}/(1+t) \rightarrow \infty$ as $t \rightarrow \infty$, so h exceeds any given M on a set of positive measure. Thus

$$h \in L^p((0, \infty)) \iff p = 2,$$

because failure at $p < 2$ comes from the tail at ∞ and failure at $p > 2$ from the singularity at 0. The same construction works at any exponent: for $1 \leq r < \infty$ set

$$h_r(x) = \frac{x^{-1/r}}{(1 + |\ln x|)^{2/r}} \quad \text{for } x > 0,$$

so that $|h_r(x)|^r = x^{-1}(1 + |\ln x|)^{-2}$, whose integral over $(0, \infty)$ equals $2 \int_0^\infty (1+t)^{-2} dt = 2$ after the substitutions $t = \ln x$ on $(1, \infty)$ and $t = -\ln x$ on $(0, 1)$; while for $p < r$ the integral of $|h_r|^p = x^{-p/r}(1 + |\ln x|)^{-2p/r}$ diverges at ∞ (compare with $1/x$ exactly as above, using $p/r < 1$), and for $p > r$ it diverges at 0; and h_r is unbounded near 0, so it is not in L^∞ . Thus h_r belongs to $L^p((0, \infty))$ precisely when $p = r$. Finally, the constant function 1 belongs to $L^\infty((0, \infty))$ but to no $L^p((0, \infty))$ with $p < \infty$. Hence for any $p_1 \neq p_2$ in $[1, \infty]$ there is a function in $L^{p_1}((0, \infty)) \sim L^{p_2}((0, \infty))$: on a set of infinite measure there are indeed no inclusion relationships among the L^p spaces.

7.2 Problems 7.8–7.14

Problem 7.8. Let f and g belong to $L^2(E)$. From the linearity of integration show that for any number λ ,

$$\lambda^2 \int_E f^2 + 2\lambda \int_E f \cdot g + \int_E g^2 = \int_E (\lambda f + g)^2 \geq 0.$$

From this and the quadratic formula directly derive the Cauchy-Schwarz Inequality.

Solution. Recall that f belongs to $L^2(E)$ means that f is measurable on E and f^2 is integrable over E .

Step 1: the product $f \cdot g$ is integrable. Since the nonnegative functions f^2 and g^2 are integrable over E , Proposition 13 of Chapter 4 tells us that f^2 and g^2 , and hence f and g , are finite a.e. on E . All the pointwise identities below are therefore valid a.e. on E , and that suffices, since by the excision property of integration a set of measure zero affects no integral. Moreover $f \cdot g$ is measurable, by Theorem 6 of Chapter 3. For a.e. x in E ,

$$|f(x)g(x)| \leq \frac{1}{2}(f^2(x) + g^2(x)),$$

since $0 \leq (|f(x)| - |g(x)|)^2$. The right-hand side is integrable over E , being one half of the sum of two integrable functions, so by the integral comparison test $f \cdot g$ is integrable over E . In particular all three numbers

$$A = \int_E f^2, \quad B = \int_E f \cdot g, \quad C = \int_E g^2$$

are finite, and $A \geq 0$, $C \geq 0$.

Step 2: the identity. Fix a real number λ . At a.e. point of E (namely wherever f and g are finite),

$$(\lambda f + g)^2 = \lambda^2 f^2 + 2\lambda f \cdot g + g^2.$$

Each of the three functions on the right is integrable over E , so their linear combination is integrable and, by the linearity of integration,

$$\int_E (\lambda f + g)^2 = \lambda^2 \int_E f^2 + 2\lambda \int_E f \cdot g + \int_E g^2 = A\lambda^2 + 2B\lambda + C.$$

Moreover $(\lambda f + g)^2 \geq 0$ on E , so by the monotonicity of integration $\int_E (\lambda f + g)^2 \geq 0$. Therefore

$$p(\lambda) := A\lambda^2 + 2B\lambda + C \geq 0 \quad \text{for every real } \lambda.$$

Step 3: the quadratic formula. Suppose first $A > 0$. Then p is a genuine quadratic with real coefficients whose roots are, by the quadratic formula,

$$\lambda_{\pm} = \frac{-2B \pm \sqrt{4B^2 - 4AC}}{2A}.$$

If $B^2 - AC > 0$ these are two distinct real numbers, and since $A > 0$ the parabola p is strictly negative at every λ strictly between them, contradicting $p \geq 0$. Hence

$$B^2 \leq AC, \quad \text{that is} \quad \left| \int_E f \cdot g \right| \leq \sqrt{\int_E f^2} \cdot \sqrt{\int_E g^2}. \quad (*)$$

If $A = 0$, then f^2 is a nonnegative function with zero integral, so by Proposition 9 of Chapter 4, $f^2 = 0$ a.e. on E ; thus $f = 0$ a.e. on E , hence $f \cdot g = 0$ a.e. on E and $B = 0$. Thus $(*)$ holds trivially in this case as well. The same argument covers $C = 0$. Therefore $(*)$ holds for all f, g in $L^2(E)$.

Step 4: the Cauchy-Schwarz Inequality. The functions $|f|$ and $|g|$ are measurable and $|f|^2 = f^2$, $|g|^2 = g^2$ are integrable, so $|f|$ and $|g|$ also belong to $L^2(E)$. Applying $(*)$ to the pair $|f|, |g|$ and noting $\int_E |f| \cdot |g| = \int_E |f \cdot g|$ because the integrand is nonnegative, we obtain

$$\int_E |f \cdot g| \leq \sqrt{\int_E f^2} \cdot \sqrt{\int_E g^2},$$

which is the Cauchy-Schwarz Inequality. (Step 1 already showed that $f \cdot g$ is integrable over E , which is the other assertion of that theorem.)

Problem 7.9. Show that in Young's Inequality there is equality if and only if $a = b = 1$.

Solution. Throughout, $1 < p < \infty$, $q = p/(p-1)$ is the conjugate of p , and a, b are positive numbers. Young's Inequality asserts

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

We first determine exactly when equality holds, then comment on the assertion as printed.

Sharpening the proof of Young's Inequality. Let $g(x) = \frac{1}{p}x^p + \frac{1}{q} - x$ for $x > 0$, the function used in the book's proof. Then

$$g'(x) = x^{p-1} - 1,$$

which is negative for $0 < x < 1$ and positive for $x > 1$, since $p-1 > 0$ so $x \mapsto x^{p-1}$ is strictly increasing on $(0, \infty)$ and takes the value 1 at $x = 1$. Hence g is strictly decreasing on $(0, 1]$ and strictly increasing on $[1, \infty)$, so g attains its minimum on $(0, \infty)$ at the single point $x = 1$. Since $g(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$, we conclude

$$x \leq \frac{1}{p}x^p + \frac{1}{q} \quad \text{for all } x > 0, \quad \text{with equality if and only if } x = 1. \quad (6)$$

Transfer to Young's Inequality. Because $p(q-1) = q$, we have $b^{p(q-1)} = b^q$. Dividing each side of Young's Inequality by the positive number b^q and putting

$$x_0 = \frac{a}{b^{q-1}} > 0$$

gives the equivalent statement

$$\frac{ab}{b^q} = \frac{a}{b^{q-1}} = x_0 \leq \frac{a^p}{p b^{p(q-1)}} + \frac{b^q}{q b^q} = \frac{x_0^p}{p} + \frac{1}{q},$$

which is exactly (6) evaluated at x_0 . Since dividing by the positive number b^q preserves equality as well as inequality, we conclude from (6):

$$ab = \frac{a^p}{p} + \frac{b^q}{q} \iff x_0 = 1 \iff a = b^{q-1} \iff a^p = b^{p(q-1)} = b^q.$$

Thus equality holds in Young's Inequality if and only if

$$a^p = b^q.$$

Relation to the printed assertion. If $a = b = 1$ then $a^p = 1 = b^q$, so equality does indeed hold; this is the easy half of the printed claim, and it is correct. The converse as printed is a misprint: the correct condition is $a^p = b^q$, which does not force $a = b = 1$. For example, when $p = q = 2$ Young's Inequality reads $ab \leq (a^2 + b^2)/2$, and there is equality for every pair with $a = b$, say $a = b = 3$. More generally, for any p and any $a > 0$ the number $b = a^{p/q}$ produces equality.

The printed statement becomes true if one asks for equality for all conjugate pairs at once. Suppose $a, b > 0$ and $a^p = b^q$ holds for every $p \in (1, \infty)$ with $q = p/(p-1)$. Letting $p \rightarrow \infty$, we have $q \rightarrow 1$, so $b^q \rightarrow b$, a finite positive number. If $a > 1$ then $a^p \rightarrow \infty$, and if $a < 1$ then $a^p \rightarrow 0$; neither limit can equal the positive finite number b . Hence $a = 1$, and then $b^q = a^p = 1$ forces $b = 1$. So $a = b = 1$ is precisely the condition for equality to hold in Young's Inequality simultaneously for all $1 < p < \infty$.

Problem 7.10. Show that in Holder's Inequality there is equality if and only if there are constants

α and β , not both zero, for which

$$\alpha|f|^p = \beta|g|^q \quad \text{a.e. on } E.$$

Solution. Here $1 < p < \infty$, q is the conjugate of p , f belongs to $L^p(E)$ and g belongs to $L^q(E)$; Holder's Inequality is

$$\int_E |f \cdot g| \leq \|f\|_p \|g\|_q. \quad (3)$$

(The exponents p and q appearing in the asserted relation presuppose $1 < p < \infty$, and that is the case we treat.)

A preliminary remark on Young's Inequality. Young's Inequality was stated for positive numbers, but it extends to all $a, b \geq 0$, together with its equality case from Problem 9: if $a, b \geq 0$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{with equality if and only if } a^p = b^q. \quad (7)$$

Indeed, for $a, b > 0$ this is Problem 9. If $a = 0$ and $b > 0$, then $ab = 0 < b^q/q$, so the inequality is strict, and correspondingly $a^p = 0 \neq b^q$; symmetrically if $b = 0 < a$. If $a = b = 0$ both sides vanish and $a^p = 0 = b^q$.

Sufficiency. Suppose $\alpha|f|^p = \beta|g|^q$ a.e. on E with α, β not both zero.

If $\alpha = 0$, then $\beta \neq 0$ and $\beta|g|^q = 0$ a.e., so $g = 0$ a.e. on E ; then $f \cdot g = 0$ a.e. and $\|g\|_q = 0$, so both sides of (3) are zero. Symmetrically if $\beta = 0$ then $f = 0$ a.e. and both sides of (3) are zero. If α and β are nonzero and of opposite sign, then $\alpha|f|^p = \beta|g|^q$ forces both sides to be simultaneously ≥ 0 and ≤ 0 , hence $|f|^p = 0$ and $|g|^q = 0$ a.e., and again both sides of (3) vanish.

So assume α, β are nonzero of the same sign; replacing the pair (α, β) by $(-\alpha, -\beta)$ if necessary, we may assume $\alpha > 0$ and $\beta > 0$. Put $c = \alpha/\beta > 0$, so that

$$|g|^q = c|f|^p \quad \text{a.e. on } E.$$

If $f = 0$ a.e. then $g = 0$ a.e. and (3) is an equality $0 = 0$. Otherwise $\|f\|_p > 0$. Since $\frac{1}{p} + \frac{1}{q} = 1$ gives $1 + \frac{p}{q} = p$, we have a.e. on E

$$|f \cdot g| = |f| \cdot (c|f|^p)^{1/q} = c^{1/q}|f|^{1+p/q} = c^{1/q}|f|^p,$$

so, integrating, $\int_E |f \cdot g| = c^{1/q}\|f\|_p^p$. On the other hand $\|g\|_q^q = \int_E |g|^q = c\|f\|_p^p$, whence

$$\|f\|_p \|g\|_q = \|f\|_p (c\|f\|_p^p)^{1/q} = c^{1/q}\|f\|_p^{1+p/q} = c^{1/q}\|f\|_p^p.$$

The two sides agree, so there is equality in (3).

Necessity. Suppose there is equality in (3). If $f = 0$ a.e. on E , take $\alpha = 1, \beta = 0$: then $\alpha|f|^p = 0 = \beta|g|^q$ a.e. Similarly if $g = 0$ a.e., take $\alpha = 0, \beta = 1$. So assume neither, and set

$$A = \|f\|_p > 0, \quad B = \|g\|_q > 0$$

(these are positive because, by Proposition 9 of Chapter 4, $\int_E |f|^p = 0$ would force $f = 0$ a.e.). Define the normalizations

$$F = \frac{|f|}{A}, \quad G = \frac{|g|}{B},$$

so that $\int_E F^p = 1$, $\int_E G^q = 1$, and, by the assumed equality in (3),

$$\int_E F \cdot G = \frac{1}{AB} \int_E |f \cdot g| = \frac{\|f\|_p \|g\|_q}{AB} = 1.$$

By (7) applied pointwise to the nonnegative numbers $F(x), G(x)$ (both are finite a.e. on E , since F^p and G^q are integrable),

$$h := \frac{F^p}{p} + \frac{G^q}{q} - F \cdot G \geq 0 \quad \text{a.e. on } E.$$

The function h is integrable, being a linear combination of integrable functions, and by the linearity of integration

$$\int_E h = \frac{1}{p} \int_E F^p + \frac{1}{q} \int_E G^q - \int_E F \cdot G = \frac{1}{p} + \frac{1}{q} - 1 = 0.$$

A nonnegative integrable function with zero integral vanishes a.e. (Proposition 9 of Chapter 4), so $h = 0$ a.e. on E . By the equality case in (7), this means

$$F^p = G^q \quad \text{a.e. on } E, \quad \text{that is} \quad \frac{|f|^p}{A^p} = \frac{|g|^q}{B^q} \quad \text{a.e. on } E.$$

Therefore

$$\alpha|f|^p = \beta|g|^q \quad \text{a.e. on } E, \quad \alpha = B^q > 0, \quad \beta = A^p > 0,$$

and in particular α, β are not both zero. This completes the proof.

Problem 7.11. For a point $x = (x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$, define T_x to be the step function on the interval $[1, n+1)$ that takes the value x_k on the interval $[k, k+1)$, for $1 \leq k \leq n$. For $p \geq 1$, define $\|x\|_p = \|T_x\|_p$, the norm of the function T_x in $L^p[1, n+1)$. Show that this defines a norm on $\mathbb{R}^n$. State and prove the Holder and Minkowski Inequalities for this norm.

Solution. Write $I = [1, n+1)$ and $I_k = [k, k+1)$ for $1 \leq k \leq n$, so that I is the disjoint union of the I_k and $m(I_k) = 1$ for each k . Thus

$$T_x = \sum_{k=1}^n x_k \chi_{I_k}$$

is a simple function on I ; it is measurable and bounded on a set of finite measure, so it belongs to $L^p(I)$ for every p with $1 \leq p \leq \infty$, and $\|x\|_p$ is a well-defined finite number.

An explicit formula. For $1 \leq p < \infty$, the function $|T_x|^p = \sum_{k=1}^n |x_k|^p \chi_{I_k}$ is simple, so by the additivity of integration over the disjoint sets I_k ,

$$\|x\|_p^p = \int_I |T_x|^p = \sum_{k=1}^n |x_k|^p m(I_k) = \sum_{k=1}^n |x_k|^p, \quad \text{so} \quad \|x\|_p = \left[\sum_{k=1}^n |x_k|^p \right]^{1/p}.$$

For $p = \infty$: $|T_x| \leq \max_{1 \leq k \leq n} |x_k|$ everywhere on I , and for each k the set where $|T_x| = |x_k|$ has measure $1 > 0$; hence the essential supremum of $|T_x|$ is $\max_{1 \leq k \leq n} |x_k|$ and $\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$.

The map T is linear and one-to-one. For x, y in $\mathbb{R}^n$ and λ in $\mathbb{R}$, evaluating on each I_k shows

$$T_{x+y} = T_x + T_y \quad \text{and} \quad T_{\lambda x} = \lambda T_x \quad \text{everywhere on } I.$$

Moreover, if $T_x = 0$ a.e. on I , then for each k the constant value x_k of T_x on I_k must be 0, since $m(I_k) = 1 > 0$; hence $x = 0$. So T is one-to-one even after identifying functions that agree a.e.

$\|\cdot\|_p$ is a norm on $\mathbb{R}^n$. Fix p with $1 \leq p \leq \infty$.

(i) Finiteness and nonnegativity are clear from the formulas above.

(ii) If $\|x\|_p = 0$, then $\|T_x\|_p = 0$, so $T_x = 0$ a.e. on I (for $1 \leq p < \infty$ this is Proposition 9 of Chapter 4 applied to the nonnegative function $|T_x|^p$; for $p = \infty$ it is immediate from the definition of the essential supremum). By the injectivity just proved, $x = 0$. Conversely $\|0\|_p = 0$.

(iii) Positive homogeneity: $\|\lambda x\|_p = \|T_{\lambda x}\|_p = \|\lambda T_x\|_p = |\lambda| \|T_x\|_p = |\lambda| \|x\|_p$, using the positive homogeneity of the L^p norm.

(iv) Triangle inequality: by linearity of T and Minkowski's Inequality in $L^p(I)$ (valid for $1 \leq p \leq \infty$),

$$\|x + y\|_p = \|T_{x+y}\|_p = \|T_x + T_y\|_p \leq \|T_x\|_p + \|T_y\|_p = \|x\|_p + \|y\|_p.$$

Hence $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

Holder's Inequality in $\mathbb{R}^n$. Let $1 \leq p \leq \infty$ and let q be the conjugate of p . Then for all x, y in $\mathbb{R}^n$,

$$\sum_{k=1}^n |x_k y_k| \leq \|x\|_p \|y\|_q,$$

that is, for $1 < p < \infty$,

$$\sum_{k=1}^n |x_k y_k| \leq \left[\sum_{k=1}^n |x_k|^p \right]^{1/p} \left[\sum_{k=1}^n |y_k|^q \right]^{1/q},$$

while for $p = 1$, $q = \infty$ it reads $\sum_{k=1}^n |x_k y_k| \leq (\sum_{k=1}^n |x_k|) \max_{1 \leq k \leq n} |y_k|$.

Proof. The pointwise product $T_x \cdot T_y$ is the step function on I taking the value $x_k y_k$ on I_k , that is, $T_x \cdot T_y = T_z$ where $z = (x_1 y_1, \dots, x_n y_n)$. Hence, as computed above with $p = 1$,

$$\int_I |T_x \cdot T_y| = \sum_{k=1}^n |x_k y_k|.$$

Since T_x belongs to $L^p(I)$ and T_y belongs to $L^q(I)$, Theorem 1 (Holder's Inequality) gives

$$\sum_{k=1}^n |x_k y_k| = \int_I |T_x \cdot T_y| \leq \|T_x\|_p \|T_y\|_q = \|x\|_p \|y\|_q.$$

Theorem 1 is stated for $1 \leq p < \infty$; the remaining case $p = \infty$, $q = 1$ is obtained by applying Theorem 1 to the pair T_y (in $L^1(I)$) and T_x (in $L^\infty(I)$), since $\int_I |T_y \cdot T_x| = \int_I |T_x \cdot T_y|$.

Minkowski's Inequality in $\mathbb{R}^n$. For $1 \leq p \leq \infty$ and all x, y in $\mathbb{R}^n$,

$$\left[\sum_{k=1}^n |x_k + y_k|^p \right]^{1/p} \leq \left[\sum_{k=1}^n |x_k|^p \right]^{1/p} + \left[\sum_{k=1}^n |y_k|^p \right]^{1/p} \quad (1 \leq p < \infty),$$

and $\max_k |x_k + y_k| \leq \max_k |x_k| + \max_k |y_k|$ for $p = \infty$.

Proof. This is exactly step (iv) above: $T_{x+y} = T_x + T_y$, so Minkowski's Inequality in $L^p(I)$ yields $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, which is the displayed inequality once the explicit formulas for $\|\cdot\|_p$ are inserted.

Problem 7.12. For $1 \leq p < \infty$ and a sequence $a = (a_1, a_2, \dots)$ in ℓ^p , define T_a to be the function on the interval $[1, \infty)$ that takes the value a_k on $[k, k+1)$, for $k = 1, 2, \dots$. Show that T_a belongs to $L^p[1, \infty)$ and that $\|a\|_p = \|T_a\|_p$. Use this to state and prove the Holder and Minkowski Inequalities in ℓ^p .

Solution. Write $I = [1, \infty)$ and $I_k = [k, k+1)$ for $k = 1, 2, \dots$, so that I is the disjoint union of the I_k and $m(I_k) = 1$ for every k . Recall that ℓ^p consists of those real sequences $a = (a_k)$ for which $\sum_{k=1}^\infty |a_k|^p < \infty$, normed by $\|a\|_p = [\sum_{k=1}^\infty |a_k|^p]^{1/p}$.

T_a is measurable. For any real number c ,

$$\{x \in I : T_a(x) > c\} = \bigcup \{I_k : a_k > c\},$$

a countable union of intervals and therefore measurable. Hence T_a is a measurable function on I . The same computation applies to $|T_a|^p$, which is the function taking the value $|a_k|^p$ on I_k .

T_a belongs to $L^p(I)$ and $\|T_a\|_p = \|a\|_p$. For each n set

$$\varphi_n = \sum_{k=1}^n |a_k|^p \chi_{I_k},$$

a nonnegative simple function; $\{\varphi_n\}$ is an increasing sequence which converges pointwise on I to $|T_a|^p$. By the Monotone Convergence Theorem and the additivity of integration over the disjoint sets $I_1, \dots, I_n$,

$$\int_I |T_a|^p = \lim_{n \rightarrow \infty} \int_I \varphi_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n |a_k|^p m(I_k) = \sum_{k=1}^\infty |a_k|^p < \infty.$$

Thus $|T_a|^p$ is integrable over I , so T_a belongs to $L^p(I)$, and taking p -th roots,

$$\|T_a\|_p = \left[\int_I |T_a|^p \right]^{1/p} = \left[\sum_{k=1}^{\infty} |a_k|^p \right]^{1/p} = \|a\|_p. \quad (8)$$

Observe also that the map $a \mapsto T_a$ is linear: evaluating on each I_k gives $T_{a+b} = T_a + T_b$ and $T_{\lambda a} = \lambda T_a$ on all of I . Note that (8) with $p = 1$ says: for any sequence (c_k) of nonnegative numbers, $\int_I T_c = \sum_{k=1}^{\infty} c_k$, a fact we use below (the computation above is valid for any nonnegative sequence, the value $+\infty$ being allowed on both sides).

Holder's Inequality in ℓ^p . Let $1 < p < \infty$ and let q be the conjugate of p . If $a = (a_k)$ belongs to ℓ^p and $b = (b_k)$ belongs to ℓ^q , then the sequence $(a_k b_k)$ belongs to ℓ^1 and

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \left[\sum_{k=1}^{\infty} |a_k|^p \right]^{1/p} \left[\sum_{k=1}^{\infty} |b_k|^q \right]^{1/q} = \|a\|_p \|b\|_q.$$

Proof. The pointwise product $T_a \cdot T_b$ takes the value $a_k b_k$ on I_k , that is, $T_a \cdot T_b = T_c$ where $c = (a_k b_k)$. By (8), T_a belongs to $L^p(I)$ and T_b belongs to $L^q(I)$, so Theorem 1 tells us that $T_a \cdot T_b$ is integrable over I and

$$\int_I |T_a \cdot T_b| \leq \|T_a\|_p \|T_b\|_q = \|a\|_p \|b\|_q.$$

By the remark following (8), applied to the nonnegative sequence $(|a_k b_k|)$,

$$\sum_{k=1}^{\infty} |a_k b_k| = \int_I |T_a \cdot T_b| \leq \|a\|_p \|b\|_q < \infty.$$

In particular the series converges, so $(a_k b_k)$ belongs to ℓ^1 , and the asserted inequality holds.

The case $p = 1$, $q = \infty$ is the same argument once one checks that T_b belongs to $L^\infty(I)$ with $\|T_b\|_\infty = \sup_k |b_k|$ when b belongs to ℓ^∞ : indeed $|T_b| \leq \sup_k |b_k|$ everywhere on I , so that number is an essential upper bound; and if M is an essential upper bound then, since $|T_b| = |b_k|$ on the set I_k of measure $1 > 0$, necessarily $|b_k| \leq M$ for every k , so $\sup_k |b_k| \leq M$. Holder's Inequality in $L^1(I)$ then gives $\sum_k |a_k b_k| = \int_I |T_a \cdot T_b| \leq \|T_a\|_1 \|T_b\|_\infty = \|a\|_1 \sup_k |b_k|$.

Minkowski's Inequality in ℓ^p . Let $1 \leq p < \infty$. If a and b belong to ℓ^p , then so does $a + b = (a_k + b_k)$, and

$$\left[\sum_{k=1}^{\infty} |a_k + b_k|^p \right]^{1/p} \leq \left[\sum_{k=1}^{\infty} |a_k|^p \right]^{1/p} + \left[\sum_{k=1}^{\infty} |b_k|^p \right]^{1/p},$$

that is, $\|a + b\|_p \leq \|a\|_p + \|b\|_p$.

Proof. By (8), T_a and T_b belong to $L^p(I)$, so by Minkowski's Inequality in $L^p(I)$ their sum $T_a + T_b = T_{a+b}$ belongs to $L^p(I)$ and

$$\|T_{a+b}\|_p \leq \|T_a\|_p + \|T_b\|_p.$$

Now $\int_I |T_{a+b}|^p = \sum_{k=1}^{\infty} |a_k + b_k|^p$ by the computation leading to (8) (valid for an arbitrary sequence, with $+\infty$ permitted), and the left-hand side is finite; hence $a + b$ belongs to ℓ^p and $\|a + b\|_p = \|T_{a+b}\|_p$. Substituting the three identities of (8) into the last display gives $\|a + b\|_p \leq \|a\|_p + \|b\|_p$.

Together with the evident positive homogeneity $\|\lambda a\|_p = |\lambda| \|a\|_p$ and the fact that $\|a\|_p = 0$ forces every $a_k = 0$, this shows that ℓ^p is a normed linear space for $1 \leq p < \infty$.

Problem 7.13. Show that if f is a bounded function on E that belongs to $L^{p_1}(E)$, then it belongs to $L^{p_2}(E)$ for any $p_2 > p_1$.

Solution. Let $1 \leq p_1 < \infty$, let f belong to $L^{p_1}(E)$, and suppose f is bounded on E : there is a number $M \geq 0$ for which

$$|f(x)| \leq M \quad \text{for all } x \text{ in } E.$$

(Only an essential bound is needed; the argument below is unchanged if $|f| \leq M$ merely a.e. on E ,

that is, if f belongs to $L^\infty(E)$ and $M = \|f\|_\infty$.) Let $p_2 > p_1$.

The case $p_2 = \infty$. Since f is measurable and $|f| \leq M$ on E , the number M is an essential upper bound for f , so f belongs to $L^\infty(E)$ with $\|f\|_\infty \leq M$.

The case $p_2 < \infty$. If $M = 0$ then $f \equiv 0$ on E and there is nothing to prove, so assume $M > 0$. The function f is measurable, hence so is $|f|^{p_2}$. Since $p_2 - p_1 > 0$ and $0 \leq |f(x)| \leq M$ for all x in E , monotonicity of $t \mapsto t^{p_2-p_1}$ on $[0, \infty)$ gives $|f(x)|^{p_2-p_1} \leq M^{p_2-p_1}$, and therefore

$$|f(x)|^{p_2} = |f(x)|^{p_1} \cdot |f(x)|^{p_2-p_1} \leq M^{p_2-p_1} |f(x)|^{p_1} \quad \text{for all } x \text{ in } E.$$

The function on the right is integrable over E , being a constant multiple of the integrable function $|f|^{p_1}$. By the integral comparison test, the nonnegative measurable function $|f|^{p_2}$ is integrable over E , and by the monotonicity and linearity of integration

$$\int_E |f|^{p_2} \leq M^{p_2-p_1} \int_E |f|^{p_1} < \infty.$$

Hence f belongs to $L^{p_2}(E)$; indeed, taking p_2 -th roots,

$$\|f\|_{p_2} \leq M^{1-p_1/p_2} \|f\|_{p_1}^{p_1/p_2}.$$

Remark. No assumption on $m(E)$ is needed here. This complements Corollary 3, which for $m(E) < \infty$ gives the opposite inclusion $L^{p_2}(E) \subseteq L^{p_1}(E)$ for $p_1 < p_2$; boundedness of f is what buys the inclusion in the direction of larger exponents, on a set of arbitrary measure.

Problem 7.14. Show that if $f(x) = \ln(1/x)$ for $x \in (0, 1]$, then f belongs to $L^p(0, 1]$ for all $1 \leq p < \infty$ but does not belong to $L^\infty(0, 1]$.

Solution. The function $f(x) = \ln(1/x)$ is continuous on $(0, 1]$, hence measurable, and $f \geq 0$ there since $1/x \geq 1$.

f does not belong to $L^\infty(0, 1]$. Let $M > 0$ be any number. Then

$$\{x \in (0, 1] : |f(x)| > M\} = \{x \in (0, 1] : \ln(1/x) > M\} = (0, e^{-M}),$$

a set of measure $e^{-M} > 0$. So no number M is an essential upper bound for f , and therefore f does not belong to $L^\infty(0, 1]$. (Equivalently, in the notation of Problem 4, the set of M for which $m\{x : |f(x)| > M\} = 0$ is empty, so $\|f\|_\infty = \infty$.)

A logarithmic estimate. We use the elementary inequality

$$\ln v \leq v - 1 \leq v \quad \text{for all } v > 0,$$

which follows from the fact that $v \mapsto v - 1 - \ln v$ has derivative $1 - 1/v$, negative on $(0, 1)$ and positive on $(1, \infty)$, hence attains its minimum value 0 at $v = 1$. Now fix p with $1 \leq p < \infty$ and put $\alpha = 1/(2p) > 0$. Applying the inequality with $v = u^\alpha$, where $u > 0$, gives $\alpha \ln u = \ln u^\alpha \leq u^\alpha$, that is,

$$\ln u \leq \frac{u^\alpha}{\alpha} = 2p u^{1/(2p)} \quad \text{for all } u > 0.$$

Taking $u = 1/x$ for x in $(0, 1]$ we obtain

$$0 \leq f(x) = \ln(1/x) \leq 2p x^{-1/(2p)},$$

and raising to the p -th power (both sides are nonnegative),

$$|f(x)|^p \leq (2p)^p x^{-1/2} \quad \text{for all } x \text{ in } (0, 1]. \quad (9)$$

The majorant is integrable. Let $g(x) = x^{-1/2}$ on $(0, 1]$ and let $g_n = g \cdot \chi_{[1/n, 1]}$ for $n \geq 2$. Each g_n is nonnegative and measurable, $\{g_n\}$ increases pointwise on $(0, 1]$ to g , and g_n vanishes off $[1/n, 1]$ and

is continuous, hence Riemann integrable, on $[1/n, 1]$. So by Theorem 3 of Chapter 4 together with the additivity of integration over domains,

$$\int_{(0,1]} g_n = \int_{[1/n,1]} x^{-1/2} = \int_{1/n}^1 x^{-1/2} dx = \left[2\sqrt{x} \right]_{1/n}^1 = 2 - \frac{2}{\sqrt{n}} \leq 2,$$

the last integral being a Riemann integral evaluated by the Fundamental Theorem of Calculus. By the Monotone Convergence Theorem,

$$\int_{(0,1]} x^{-1/2} dx = \lim_{n \rightarrow \infty} \left(2 - \frac{2}{\sqrt{n}} \right) = 2 < \infty,$$

so g is integrable over $(0, 1]$.

Conclusion. By (9), the nonnegative measurable function $|f|^p$ is dominated on $(0, 1]$ by the integrable function $(2p)^p g$. By the integral comparison test, $|f|^p$ is integrable over $(0, 1]$, and

$$\int_{(0,1]} |f|^p \leq (2p)^p \int_{(0,1]} x^{-1/2} dx = 2 (2p)^p < \infty.$$

Hence f belongs to $L^p(0, 1]$, with $\|f\|_p \leq 2^{1/p} (2p)$, for every p with $1 \leq p < \infty$, while f does not belong to $L^\infty(0, 1]$.

Remark. Since $m((0, 1]) = 1 < \infty$, Corollary 3 shows that the containments $L^{p_2}(0, 1] \subseteq L^{p_1}(0, 1]$ hold for $p_1 < p_2$; this f shows that the intersection of all the $L^p(0, 1]$ with $p < \infty$ is strictly larger than $L^\infty(0, 1]$.

7.3 Problems 7.15–7.21

Problem 7.15. Formulate and prove an extension of Holder's Inequality for the product of three functions.

Solution. Formulation. Let E be a measurable set and let p, q, r belong to $[1, \infty]$ satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

with the usual convention $1/\infty = 0$. If $f \in L^p(E)$, $g \in L^q(E)$ and $h \in L^r(E)$, then the product $f \cdot g \cdot h$ is integrable over E and

$$\int_E |f \cdot g \cdot h| \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$

Proof. We first dispose of the cases in which one of the exponents is infinite. Suppose $r = \infty$. Then $1/p + 1/q = 1$, so p and q are conjugate. Since $|h| \leq \|h\|_\infty$ a.e. on E (the essential supremum is itself an essential upper bound, by observation (2) of Section 7.1), monotonicity of integration and Holder's Inequality, Theorem 1 of Chapter 7, applied to f and g give

$$\int_E |f \cdot g \cdot h| \leq \|h\|_\infty \int_E |f \cdot g| \leq \|f\|_p \|g\|_q \|h\|_\infty,$$

and in particular $f \cdot g \cdot h$ is integrable. The cases $p = \infty$ and $q = \infty$ are identical after relabeling. Observe also that if two of the exponents were infinite the third would have to satisfy $1/p = 1$, that is $p = 1$, and the same argument applies. Finally, if one of the exponents equals 1, say $p = 1$, then $1/q + 1/r = 0$ forces $q = r = \infty$, a case already treated.

So assume $1 < p, q, r < \infty$. Define s by

$$\frac{1}{s} = \frac{1}{q} + \frac{1}{r} = 1 - \frac{1}{p}.$$

Then s is the conjugate of p , and since $1/s > 1/q$ and $1/s > 1/r$ we have $1 < s < q$ and $s < r$.

Step 1: $g \cdot h$ belongs to $L^s(E)$ and $\|g \cdot h\|_s \leq \|g\|_q \|h\|_r$. Put

$$\alpha = \frac{q}{s}, \quad \beta = \frac{r}{s}.$$

Both exceed 1, and

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{s}{q} + \frac{s}{r} = s \left(\frac{1}{q} + \frac{1}{r} \right) = s \cdot \frac{1}{s} = 1,$$

so α and β are conjugate. The function $|g|^s$ belongs to $L^\alpha(E)$, since $\int_E (|g|^s)^\alpha = \int_E |g|^q < \infty$, and likewise $|h|^s$ belongs to $L^\beta(E)$. By Holder's Inequality applied to this pair of functions and this pair of conjugate exponents, the product $|g|^s |h|^s$ is integrable over E and

$$\int_E |g|^s |h|^s \leq \left[\int_E |g|^{s\alpha} \right]^{1/\alpha} \cdot \left[\int_E |h|^{s\beta} \right]^{1/\beta} = \left[\int_E |g|^q \right]^{s/q} \cdot \left[\int_E |h|^r \right]^{s/r} = \|g\|_q^s \cdot \|h\|_r^s.$$

Since $|g \cdot h|^s = |g|^s |h|^s$, the product $g \cdot h$ is measurable with $|g \cdot h|^s$ integrable, so $g \cdot h \in L^s(E)$, and taking s -th roots in the displayed inequality yields $\|g \cdot h\|_s \leq \|g\|_q \|h\|_r$.

Step 2: apply Holder to f and $g \cdot h$. Since p and s are conjugate, $f \in L^p(E)$ and $g \cdot h \in L^s(E)$, Holder's Inequality tells us that $f \cdot (g \cdot h)$ is integrable over E and

$$\int_E |f \cdot g \cdot h| \leq \|f\|_p \cdot \|g \cdot h\|_s \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$

This is the asserted inequality, and the proof is complete. ■

Remark. An entirely parallel argument, by induction on n , gives the n -function version: if $1/p_1 + \cdots + 1/p_n = 1$ with each $p_k \in [1, \infty]$ and $f_k \in L^{p_k}(E)$, then $f_1 \cdots f_n$ is integrable over E and $\int_E |f_1 \cdots f_n| \leq \prod_{k=1}^n \|f_k\|_{p_k}$.

Alternative proof of the three-function case, all three exponents finite. Suppose now $1 < p, q, r < \infty$ (the cases in which some exponent is infinite were disposed of above, and there is no analogue of the term $|h|^r/r$ below when $r = \infty$). One may argue directly, as the book does for two functions in the proof of Theorem 1 of Chapter 7, by first normalizing. It suffices to prove the inequality when $\|f\|_p = \|g\|_q = \|h\|_r = 1$ (if one of the three norms vanishes then the corresponding function is zero a.e., the product is zero a.e., and the inequality is trivial; otherwise replace f, g, h by $f/\|f\|_p, g/\|g\|_q, h/\|h\|_r$ and use the positive homogeneity of both sides). Under this normalization we claim

$$|f| \cdot |g| \cdot |h| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q} + \frac{|h|^r}{r} \quad \text{a.e. on } E.$$

Indeed, since $|f|^p, |g|^q, |h|^r$ are integrable, the three functions are finite a.e. on E . At a point where all three values $a = |f|, b = |g|, c = |h|$ are positive and finite, the concavity of the logarithm together with the weights $1/p, 1/q, 1/r$, which are positive and sum to 1, gives

$$\ln \left(\frac{a^p}{p} + \frac{b^q}{q} + \frac{c^r}{r} \right) \geq \frac{1}{p} \ln a^p + \frac{1}{q} \ln b^q + \frac{1}{r} \ln c^r = \ln(abc),$$

and exponentiating yields the claim; if one of a, b, c is zero the claim is trivial. Integrating the claim over E and using the linearity of integration and the integral comparison test, $f \cdot g \cdot h$ is integrable over E and

$$\int_E |f \cdot g \cdot h| \leq \frac{1}{p} \int_E |f|^p + \frac{1}{q} \int_E |g|^q + \frac{1}{r} \int_E |h|^r = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 = \|f\|_p \|g\|_q \|h\|_r. \quad \blacksquare$$

Problem 7.16. Suppose that $\{f_n\}$ is bounded in $L^1[0, 1]$. Is $\{f_n\}$ uniformly integrable over $[0, 1]$?

Solution. No. Boundedness in L^1 does not imply uniform integrability. Corollary 2 of Chapter 7, which asserts that a family bounded in $L^p(E)$ is uniformly integrable, is stated for $1 < p < \infty$, and that

restriction is essential: the conclusion fails for $p = 1$.

Counterexample. For each natural number n define f_n on $[0, 1]$ by

$$f_n = n \cdot \chi_{[0, 1/n]}, \quad \text{that is,} \quad f_n(x) = \begin{cases} n, & 0 \leq x \leq 1/n, \\ 0, & 1/n < x \leq 1. \end{cases}$$

Each f_n is a nonnegative simple function, hence measurable, and

$$\|f_n\|_1 = \int_0^1 |f_n| = n \cdot m([0, 1/n]) = n \cdot \frac{1}{n} = 1.$$

Thus $\{f_n\}$ belongs to $L^1[0, 1]$ and is bounded there, with $\|f_n\|_1 = 1 \leq 1$ for all n .

We claim $\{f_n\}$ is not uniformly integrable over $[0, 1]$. Recall the definition: the family is uniformly integrable over $[0, 1]$ provided that for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_A |f_n| < \epsilon \quad \text{for all } n \text{ and all measurable } A \subseteq [0, 1] \text{ with } m(A) < \delta.$$

Take the challenge $\epsilon = 1/2$ and let $\delta > 0$ be arbitrary. Choose a natural number n with $1/n < \delta$ and set $A = [0, 1/n]$. Then A is a measurable subset of $[0, 1]$ with $m(A) = 1/n < \delta$, while

$$\int_A |f_n| = \int_0^{1/n} n = n \cdot \frac{1}{n} = 1 > \frac{1}{2} = \epsilon.$$

Hence no δ responds to the challenge $\epsilon = 1/2$, and the family fails to be uniformly integrable.

Why the argument for $p > 1$ breaks down at $p = 1$. In the proof of Corollary 2 one estimates, for measurable $A \subseteq E$,

$$\int_A |f| \leq \|f\|_p \cdot [m(A)]^{1/q},$$

where q is the conjugate of p . This is what forces the integral over a small set to be small uniformly. When $p = 1$ the conjugate exponent is $q = \infty$, the factor $[m(A)]^{1/q}$ degenerates to $[m(A)]^0 = 1$, and the estimate reduces to the vacuous $\int_A |f| \leq \|f\|_1$. The example above shows that no substitute estimate is available: the mass 1 carried by f_n can be concentrated on a set of arbitrarily small measure.

Remark. What is true is that a single integrable function is uniformly integrable in this sense, by Proposition 23 of Chapter 4, and more generally that any finite family of integrable functions is, by Proposition 24 of Chapter 4; the failure above is a failure of uniformity across an infinite family. Note also that $\{f_n\}$ converges to 0 pointwise on $(0, 1]$ while $\int_0^1 f_n = 1$ for all n , which is exactly the failure of passage to the limit under the integral sign that uniform integrability is designed to preclude.

Problem 7.17. For $1 < p < \infty$, suppose that $\{f_n\}$ is bounded in $L^p(\mathbb{R})$. Is $\{f_n\}$ tight?

Solution. No. Recall the definition from Section 5.1: a family $\mathcal{F}$ of measurable functions on E is tight over E provided that for each $\epsilon > 0$ there is a subset E_0 of E of finite measure for which

$$\int_{E \setminus E_0} |f| < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

Boundedness in $L^p(\mathbb{R})$ for $1 < p < \infty$ does not imply this. Thus while Corollary 2 of Chapter 7 upgrades L^p -boundedness to uniform integrability for $p > 1$, no analogous upgrade to tightness holds on a set of infinite measure. We give two counterexamples: the first shows the failure already for a single function, the second shows that the failure persists even for a sequence of integrable functions of constant L^1 -norm.

Counterexample 1 (a one-element family). Fix $1 < p < \infty$ and define

$$f(x) = \frac{1}{1 + |x|} \quad \text{for } x \in \mathbb{R},$$

and set $f_n = f$ for all n . This f is continuous, hence measurable, and

$$\int_{\mathbb{R}} |f|^p = 2 \int_0^\infty \frac{dx}{(1+x)^p} = \frac{2}{p-1} < \infty,$$

so $f \in L^p(\mathbb{R})$ and $\{f_n\}$ is bounded in $L^p(\mathbb{R})$. On the other hand

$$\int_{\mathbb{R}} |f| = 2 \int_0^\infty \frac{dx}{1+x} = \infty,$$

so f is not integrable over $\mathbb{R}$. Now let $E_0 \subseteq \mathbb{R}$ be measurable with finite measure. Since $0 \leq f \leq 1$ on $\mathbb{R}$, monotonicity gives $\int_{E_0} f \leq m(E_0) < \infty$. Because f is nonnegative and measurable, additivity of integration over domains holds in the extended sense:

$$\int_{\mathbb{R}} f = \int_{E_0} f + \int_{\mathbb{R} \sim E_0} f \quad \text{in } [0, \infty].$$

Were $\int_{\mathbb{R} \sim E_0} f$ finite, the right-hand side would be finite and f would be integrable over $\mathbb{R}$, which it is not. Hence $\int_{\mathbb{R} \sim E_0} f = \infty$. Hence no set of finite measure responds to the challenge $\epsilon = 1$, and $\{f_n\}$ is not tight over $\mathbb{R}$. (This is consistent with the observation recorded after the definition of tightness: a family that is both uniformly integrable and tight consists of integrable functions, and here f is not integrable.)

Counterexample 2 (integrable functions, escaping mass). One might object that the first example fails only because its single member is not integrable. Tightness can fail even for a sequence of integrable functions of constant norm. Define

$$f_n = \chi_{[n, n+1]} \quad \text{for } n = 1, 2, \dots$$

Each f_n is measurable, and for every r with $1 \leq r < \infty$,

$$\|f_n\|_r = \left[\int_{\mathbb{R}} \chi_{[n, n+1]} \right]^{1/r} = 1,$$

so $\{f_n\}$ is bounded in $L^p(\mathbb{R})$ (indeed in every $L^r(\mathbb{R})$, and each f_n is integrable with $\int_{\mathbb{R}} |f_n| = 1$).

We claim $\{f_n\}$ is not tight over $\mathbb{R}$. Let $E_0 \subseteq \mathbb{R}$ be measurable with $m(E_0) < \infty$. The sets $E_0 \cap [n, n+1]$, for $n = 1, 2, \dots$, are disjoint measurable subsets of E_0 , so by the countable additivity and monotonicity of Lebesgue measure,

$$\sum_{n=1}^{\infty} m(E_0 \cap [n, n+1]) = m(E_0 \cap [1, \infty)) \leq m(E_0) < \infty.$$

A convergent series has terms tending to 0, so $m(E_0 \cap [n, n+1]) \rightarrow 0$, and since $m(E_0 \cap [n, n+1]) = m(E_0 \cap [n, n+1])$ (the single point $n+1$ has measure zero), we may choose N with $m(E_0 \cap [N, N+1]) < 1/2$. For this N ,

$$\int_{\mathbb{R} \sim E_0} |f_N| = m([N, N+1] \sim E_0) = 1 - m(E_0 \cap [N, N+1]) > \frac{1}{2}.$$

Thus no set of finite measure responds to the challenge $\epsilon = 1/2$, and $\{f_n\}$ is not tight over $\mathbb{R}$.

Conclusion. The answer is no: for $1 < p < \infty$, boundedness in $L^p(\mathbb{R})$ implies uniform integrability over $\mathbb{R}$, by Corollary 2 of Chapter 7, but it does not imply tightness. Tightness controls the escape of mass to infinity, a phenomenon invisible to the L^p norm, which is invariant under translation.

Problem 7.18. Assume $m(E) < \infty$. For $f \in L^\infty(E)$, show that $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.

Solution. Throughout, write $M = \|f\|_\infty$, the essential supremum of f on E , that is, the infimum of the essential upper bounds for f . As established in Section 7.1 (observation (2) there), this infimum is attained: M is itself an essential upper bound, so that

$$|f| \leq M \quad \text{a.e. on } E. \quad (*)$$

The norms $\|f\|_p$ are finite. Since $m(E) < \infty$ and $f \in L^\infty(E)$, Corollary 3 of Chapter 7 (with $p_1 = p$ and $p_2 = \infty$) gives $L^\infty(E) \subseteq L^p(E)$ for every p with $1 \leq p < \infty$, so $\|f\|_p$ is a finite nonnegative number for each such p . Directly: by $(*)$ and the monotonicity of integration, $\int_E |f|^p \leq M^p m(E) < \infty$.

The degenerate case $m(E) = 0$. If $m(E) = 0$ then every nonnegative number is an essential upper bound for f , so $M = 0$; and $\int_E |f|^p = 0$ for every p , so $\|f\|_p = 0$. Thus the limit relation holds trivially. Assume henceforth that $m(E) > 0$, so $0 < m(E) < \infty$.

Step 1: $\limsup_{p \rightarrow \infty} \|f\|_p \leq M$. By $(*)$ and the monotonicity of integration, for each p with $1 \leq p < \infty$,

$$\int_E |f|^p \leq \int_E M^p = M^p \cdot m(E),$$

and taking p -th roots,

$$\|f\|_p \leq M \cdot [m(E)]^{1/p}.$$

Since $0 < m(E) < \infty$, we have $[m(E)]^{1/p} = \exp\left(\frac{1}{p} \ln m(E)\right) \rightarrow \exp(0) = 1$ as $p \rightarrow \infty$. Therefore

$$\limsup_{p \rightarrow \infty} \|f\|_p \leq M \cdot \lim_{p \rightarrow \infty} [m(E)]^{1/p} = M.$$

Step 2: $\liminf_{p \rightarrow \infty} \|f\|_p \geq M$. If $M = 0$ this is immediate, since $\|f\|_p \geq 0$ for all p . So assume $M > 0$ and let ϵ satisfy $0 < \epsilon < M$. Define

$$E_\epsilon = \{x \in E : |f(x)| > M - \epsilon\},$$

a measurable subset of E since f is measurable. We claim $m(E_\epsilon) > 0$. Indeed, if $m(E_\epsilon) = 0$, then $|f| \leq M - \epsilon$ a.e. on E , so $M - \epsilon$ would be an essential upper bound for f ; this contradicts the fact that M is the infimum of the essential upper bounds, since $M - \epsilon < M$. Hence $m(E_\epsilon) > 0$, and of course $m(E_\epsilon) \leq m(E) < \infty$.

Now, by the monotonicity of integration and the additivity of integration over domains, for each p with $1 \leq p < \infty$,

$$\int_E |f|^p \geq \int_{E_\epsilon} |f|^p \geq \int_{E_\epsilon} (M - \epsilon)^p = (M - \epsilon)^p \cdot m(E_\epsilon),$$

the middle inequality because $|f| > M - \epsilon > 0$ on E_ϵ and $t \mapsto t^p$ is increasing on $[0, \infty)$. Taking p -th roots,

$$\|f\|_p \geq (M - \epsilon) \cdot [m(E_\epsilon)]^{1/p}.$$

Since $0 < m(E_\epsilon) < \infty$, again $[m(E_\epsilon)]^{1/p} \rightarrow 1$ as $p \rightarrow \infty$, and therefore

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq (M - \epsilon) \cdot \lim_{p \rightarrow \infty} [m(E_\epsilon)]^{1/p} = M - \epsilon.$$

As $\epsilon \in (0, M)$ was arbitrary, letting $\epsilon \rightarrow 0^+$ gives $\liminf_{p \rightarrow \infty} \|f\|_p \geq M$.

Conclusion. Combining the two steps,

$$M \leq \liminf_{p \rightarrow \infty} \|f\|_p \leq \limsup_{p \rightarrow \infty} \|f\|_p \leq M,$$

so the limit exists and

$$\lim_{p \rightarrow \infty} \|f\|_p = M = \|f\|_\infty. \quad \blacksquare$$

Remark. The hypothesis $m(E) < \infty$ is needed for Step 1 (and to guarantee $f \in L^p(E)$ at all). For instance, with $E = \mathbb{R}$ and $f \equiv 1$ we have $\|f\|_\infty = 1$ while $\|f\|_p = \infty$ for every $p < \infty$. Step 2, by contrast, is valid on any measurable set $E \subseteq \mathbb{R}$: one still has $\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty$, where $\|f\|_p$ is read as ∞ when $|f|^p$ fails to be integrable. Indeed, if $m(E_\epsilon) = \infty$, then by the continuity of measure $m(E_\epsilon \cap [-n, n]) \rightarrow m(E_\epsilon)$, so for n large this intersection has finite positive measure and may be used in place of E_ϵ in the argument.

Problem 7.19. For $1 \leq p < \infty$, q the conjugate of p , and $f \in L^p(E)$, show that

$$\|f\|_p = \max_{g \in L^q(E), \|g\|_q \leq 1} \int_E f \cdot g.$$

Solution. Two things must be shown: that $\int_E f \cdot g \leq \|f\|_p$ for every $g \in L^q(E)$ with $\|g\|_q \leq 1$, so that $\|f\|_p$ is an upper bound for the set of such numbers; and that this upper bound is attained by some particular such g , so that the supremum is a maximum.

Observe first that the expression $\int_E f \cdot g$ makes sense: for $f \in L^p(E)$ and $g \in L^q(E)$ with p and q conjugate, Holder's Inequality, Theorem 1 of Chapter 7, asserts precisely that $f \cdot g$ is integrable over E , so $\int_E f \cdot g$ is a real number.

Step 1: $\|f\|_p$ is an upper bound. Let $g \in L^q(E)$ with $\|g\|_q \leq 1$. By the monotonicity of integration and Holder's Inequality,

$$\int_E f \cdot g \leq \int_E |f \cdot g| \leq \|f\|_p \cdot \|g\|_q \leq \|f\|_p \cdot 1 = \|f\|_p.$$

Hence

$$\sup \left\{ \int_E f \cdot g : g \in L^q(E), \|g\|_q \leq 1 \right\} \leq \|f\|_p.$$

Step 2: the bound is attained.

Case $f = 0$ in $L^p(E)$, that is, $f = 0$ a.e. on E . Then $\|f\|_p = 0$. Take $g = 0$, which belongs to $L^q(E)$ and has $\|g\|_q = 0 \leq 1$. Then $\int_E f \cdot g = 0 = \|f\|_p$, so the supremum is attained. (By Step 1 the supremum is ≤ 0 , and this g realizes the value 0.)

Case $f \neq 0$. Then $\|f\|_p > 0$, and in particular $m(E) > 0$. Consider the conjugate function of f introduced in Section 7.2,

$$f^* = \|f\|_p^{1-p} \cdot \operatorname{sgn}(f) \cdot |f|^{p-1},$$

where $\operatorname{sgn}(f)$ takes the value 1 where $f \geq 0$ and -1 where $f < 0$. As part of the statement of Holder's Inequality, Theorem 1 of Chapter 7, the book records that f^* belongs to $L^q(E)$ and that

$$\int_E f \cdot f^* = \|f\|_p \quad \text{and} \quad \|f^*\|_q = 1.$$

For completeness we recall the verification. Since $\operatorname{sgn}(f) \cdot f = |f|$ a.e. on E (f being finite a.e., as $|f|^p$ is integrable), we have

$$f \cdot f^* = \|f\|_p^{1-p} \cdot |f|^{p-1} \cdot \operatorname{sgn}(f) \cdot f = \|f\|_p^{1-p} \cdot |f|^p \quad \text{a.e. on } E,$$

so by the linearity of integration

$$\int_E f \cdot f^* = \|f\|_p^{1-p} \int_E |f|^p = \|f\|_p^{1-p} \cdot \|f\|_p^p = \|f\|_p.$$

As for the norm of f^* : when $p > 1$, q is finite and $q(p-1) = p$, so

$$\int_E |f^*|^q = \|f\|_p^{q(1-p)} \int_E |f|^{q(p-1)} = \|f\|_p^{-p} \int_E |f|^p = 1,$$

whence $f^* \in L^q(E)$ and $\|f^*\|_q = 1$; when $p = 1$ we have $\|f\|_p^{1-p} = 1$ and $|f|^{p-1} = 1$, so $f^* = \operatorname{sgn}(f)$, which belongs to $L^\infty(E)$ with $\|f^*\|_\infty = 1$: indeed $|f^*| = 1$ everywhere on E , so 1 is an essential upper bound for f^* , while no $c < 1$ is one, since $m(E) > 0$.

Thus $g = f^*$ is admissible in the competition, since $f^* \in L^q(E)$ and $\|f^*\|_q = 1 \leq 1$, and it achieves the value

$$\int_E f \cdot f^* = \|f\|_p.$$

Conclusion. By Step 1 no admissible g gives a value exceeding $\|f\|_p$, and by Step 2 some admissible g gives exactly $\|f\|_p$. Therefore the supremum is attained and

$$\|f\|_p = \max_{g \in L^q(E), \|g\|_q \leq 1} \int_E f \cdot g. \quad \blacksquare$$

Remark. The hypothesis $p < \infty$ matters. For $p = \infty$ the conjugate exponent is $q = 1$ and the analogous identity $\|f\|_\infty = \sup\{\int_E f \cdot g : \|g\|_1 \leq 1\}$ does hold when $m(E) > 0$, but the supremum need not be attained: for $E = [0, 1]$ and $f(x) = x$ one has $\|f\|_\infty = 1$, while $\int_0^1 x g(x) dx < 1$ for every $g \in L^1[0, 1]$ with $\|g\|_1 \leq 1$.

Problem 7.20. For $1 \leq p < \infty$, q the conjugate of p , and $f \in L^p(E)$, show that $f = 0$ if and only if

$$\int_E f \cdot g = 0 \quad \text{for all } g \in L^q(E).$$

Solution. Here, as always for the L^p spaces, the equation $f = 0$ means that f is the zero element of $L^p(E)$, that is, $f = 0$ a.e. on E . Note first that for every $g \in L^q(E)$ the product $f \cdot g$ is integrable over E , by Holder's Inequality, Theorem 1 of Chapter 7, so each of the integrals in question is a well-defined real number.

Necessity. Suppose $f = 0$ a.e. on E . Let $g \in L^q(E)$. Then $f \cdot g = 0$ a.e. on E , and a function that vanishes a.e. on E has $\int_E f \cdot g = 0$. Thus the stated condition holds.

Sufficiency. Suppose $\int_E f \cdot g = 0$ for all $g \in L^q(E)$. We argue by contradiction: assume $f \neq 0$ in $L^p(E)$, that is, $\|f\|_p > 0$. Consider the conjugate function of f from Section 7.2,

$$f^* = \|f\|_p^{1-p} \cdot \operatorname{sgn}(f) \cdot |f|^{p-1},$$

where $\operatorname{sgn}(f)$ equals 1 where $f \geq 0$ and -1 where $f < 0$. As recorded in the statement of Holder's Inequality, Theorem 1 of Chapter 7, f^* belongs to $L^q(E)$, with $\|f^*\|_q = 1$ and

$$\int_E f \cdot f^* = \|f\|_p.$$

Let us verify this last identity again. Since $|f|^p$ is integrable, f is finite a.e. on E , so $\operatorname{sgn}(f) \cdot f = |f|$ a.e. on E and therefore

$$f \cdot f^* = \|f\|_p^{1-p} \cdot |f|^{p-1} \cdot \operatorname{sgn}(f) \cdot f = \|f\|_p^{1-p} \cdot |f|^p \quad \text{a.e. on } E.$$

By the linearity of integration,

$$\int_E f \cdot f^* = \|f\|_p^{1-p} \int_E |f|^p = \|f\|_p^{1-p} \cdot \|f\|_p^p = \|f\|_p > 0.$$

That $f^* \in L^q(E)$ is immediate: for $p > 1$ the exponent q is finite and $q(p-1) = p$, so $\int_E |f^*|^q = \|f\|_p^{-p} \int_E |f|^p = 1$; for $p = 1$ we have $f^* = \operatorname{sgn}(f)$, a measurable function bounded in absolute value by 1, hence a member of $L^\infty(E)$.

So $g = f^*$ is a member of $L^q(E)$ with $\int_E f \cdot g = \|f\|_p > 0$, contradicting the hypothesis that this integral vanishes for every g in $L^q(E)$. Therefore $\|f\|_p = 0$. But $\int_E |f|^p = 0$ with $|f|^p$ nonnegative and measurable forces $|f|^p = 0$ a.e. on E , by Proposition 9 of Chapter 4; this is exactly the observation made in Section 7.1 that $\|f\|_p = 0$ if and only if f vanishes a.e. on E . We conclude $f = 0$ a.e. on E . $\blacksquare$

Alternative derivation of sufficiency. Granting the preceding problem, the implication is immediate: if $\int_E f \cdot g = 0$ for every $g \in L^q(E)$, then in particular this holds for every $g \in L^q(E)$ with $\|g\|_q \leq 1$, so

$$\|f\|_p = \max_{g \in L^q(E), \|g\|_q \leq 1} \int_E f \cdot g = 0,$$

and hence $f = 0$ a.e. on E .

Remark. The proof shows more: it is enough to test f against the single function $g = f^*$, or indeed to know only that $\sup\{\int_E f \cdot g : g \in L^q(E), \|g\|_q \leq 1\} = 0$. The hypothesis $p < \infty$ is used to know that the conjugate function f^* lies in $L^q(E)$; for $p = \infty$ and $q = 1$ the corresponding statement is still true when E is of σ -finite measure, but the witness must be built by exhaustion rather than taken directly as f^* , since $\text{sgn}(f) \cdot \chi_E$ need not be integrable.

Problem 7.21. For $1 \leq p \leq \infty$, find the values of the parameter λ for which

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^\lambda} \int_0^\epsilon f = 0 \quad \text{for all } f \in L^p[0, 1].$$

Solution. Answer. Let q be the conjugate of p . The property holds for all $f \in L^p[0, 1]$ precisely when

$$\lambda \leq \frac{1}{q} = 1 - \frac{1}{p} \quad \text{if } 1 \leq p < \infty, \quad \text{and} \quad \lambda < 1 \quad \text{if } p = \infty.$$

Thus for $p = 1$ the admissible λ are those with $\lambda \leq 0$; for $1 < p < \infty$ they are those with $\lambda \leq 1 - 1/p$; and for $p = \infty$ they are those with $\lambda < 1$, the endpoint $\lambda = 1$ now being excluded.

Throughout we take $0 < \epsilon \leq 1$, and we note that each $f \in L^p[0, 1]$ is integrable over $[0, 1]$: for $p = \infty$ this is because $m([0, 1]) < \infty$ and f is essentially bounded, and for $1 \leq p < \infty$ it is Corollary 3 of Chapter 7 with $p_1 = 1 \leq p_2 = p$. So $\int_0^\epsilon f$ is a well-defined real number for every ϵ .

Sufficiency, the case $1 \leq p < \infty$ and $\lambda \leq 1/q$. Let $f \in L^p[0, 1]$ and $0 < \epsilon \leq 1$. Both $u = f \cdot \chi_{[0, \epsilon]}$ and $v = \chi_{[0, \epsilon]}$ are measurable on $[0, 1]$, with $u \in L^p[0, 1]$ (since $|u| \leq |f|$) and $v \in L^q[0, 1]$. Since $|u \cdot v| = |f| \cdot \chi_{[0, \epsilon]}$, Holder's Inequality, Theorem 1 of Chapter 7, applied to u and v gives

$$\left| \int_0^\epsilon f \right| \leq \int_0^\epsilon |f| = \int_0^1 |u \cdot v| \leq \|u\|_p \cdot \|v\|_q = \|f \cdot \chi_{[0, \epsilon]}\|_p \cdot \epsilon^{1/q},$$

where $\|v\|_q = [\int_0^1 \chi_{[0, \epsilon]}^q]^{1/q} = \epsilon^{1/q}$ for $q < \infty$, while for $p = 1$, so that $q = \infty$, one reads $\|v\|_\infty = 1 = \epsilon^0 = \epsilon^{1/q}$. Consequently

$$\left| \frac{1}{\epsilon^\lambda} \int_0^\epsilon f \right| \leq \epsilon^{1/q - \lambda} \cdot \|f \cdot \chi_{[0, \epsilon]}\|_p.$$

Since $\lambda \leq 1/q$ and $0 < \epsilon \leq 1$, we have $\epsilon^{1/q - \lambda} \leq 1$. It remains to observe that

$$\|f \cdot \chi_{[0, \epsilon]}\|_p = \left[\int_0^\epsilon |f|^p \right]^{1/p} \longrightarrow 0 \quad \text{as } \epsilon \rightarrow 0^+.$$

This is Proposition 23 of Chapter 4 applied to the integrable function $|f|^p$: given $\eta > 0$ there is $\delta > 0$ such that $\int_A |f|^p < \eta^p$ whenever $A \subseteq [0, 1]$ is measurable with $m(A) < \delta$; taking $A = [0, \epsilon]$ with $\epsilon < \delta$ gives $\|f \cdot \chi_{[0, \epsilon]}\|_p < \eta$. Therefore the right-hand side of the displayed estimate tends to 0, and so does the left. This proves the assertion for all $\lambda \leq 1/q$.

Sufficiency, the case $p = \infty$ and $\lambda < 1$. Let $f \in L^\infty[0, 1]$. Since $|f| \leq \|f\|_\infty$ a.e. on $[0, 1]$, monotonicity of integration gives, for $0 < \epsilon \leq 1$,

$$\left| \frac{1}{\epsilon^\lambda} \int_0^\epsilon f \right| \leq \frac{1}{\epsilon^\lambda} \int_0^\epsilon |f| \leq \frac{1}{\epsilon^\lambda} \cdot \|f\|_\infty \cdot \epsilon = \|f\|_\infty \cdot \epsilon^{1-\lambda}.$$

Because $1 - \lambda > 0$, we have $\epsilon^{1-\lambda} \rightarrow 0$ as $\epsilon \rightarrow 0^+$, and the limit is 0.

Necessity. We show that for every remaining λ there is a single $f \in L^p[0, 1]$ for which the limit is not 0. The remaining values are: $\lambda > 1 - 1/p$ when $1 \leq p < \infty$, and $\lambda \geq 1$ when $p = \infty$. In every such case $\lambda > 0$; indeed for $p < \infty$ we have $\lambda > 1 - 1/p \geq 0$, and for $p = \infty$ we have $\lambda \geq 1$.

Define

$$f(x) = x^{\lambda-1} \text{ for } 0 < x \leq 1, \quad f(0) = 0.$$

This f is continuous on $(0, 1]$, hence measurable on $[0, 1]$.

We check $f \in L^p[0, 1]$ for the λ under consideration.

First suppose $1 \leq p < \infty$. Then $|f|^p = x^{(\lambda-1)p}$ on $(0, 1]$. Since $\lambda > 1 - 1/p$, we have $(\lambda - 1)p > -1$, so $(\lambda - 1)p + 1 > 0$. The functions $x^{(\lambda-1)p} \cdot \chi_{[1/n, 1]}$ are nonnegative and measurable and increase pointwise on $(0, 1]$ to $|f|^p$, and on each $[1/n, 1]$ the integrand is continuous, so its Lebesgue integral agrees with its Riemann integral, which the Fundamental Theorem of Calculus evaluates. Hence, by the Monotone Convergence Theorem,

$$\int_0^1 |f|^p = \lim_{n \rightarrow \infty} \int_{1/n}^1 x^{(\lambda-1)p} dx = \lim_{n \rightarrow \infty} \frac{1 - (1/n)^{(\lambda-1)p+1}}{(\lambda-1)p+1} = \frac{1}{(\lambda-1)p+1} < \infty,$$

the last equality because $(\lambda - 1)p + 1 > 0$. Hence $f \in L^p[0, 1]$. If instead $p = \infty$, then $\lambda \geq 1$, so $\lambda - 1 \geq 0$ and $0 \leq f(x) = x^{\lambda-1} \leq 1$ for $x \in (0, 1]$; thus f is bounded and $f \in L^\infty[0, 1]$.

Now compute, for $0 < \epsilon \leq 1$, again by the Monotone Convergence Theorem and the Fundamental Theorem of Calculus, using $\lambda > 0$,

$$\int_0^\epsilon f = \lim_{n \rightarrow \infty} \int_{1/n}^\epsilon x^{\lambda-1} dx = \lim_{n \rightarrow \infty} \frac{\epsilon^\lambda - (1/n)^\lambda}{\lambda} = \frac{\epsilon^\lambda}{\lambda},$$

where the sequence of integrals is taken over n with $1/n < \epsilon$. Therefore

$$\frac{1}{\epsilon^\lambda} \int_0^\epsilon f = \frac{1}{\lambda} \quad \text{for every } \epsilon \in (0, 1],$$

so that

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^\lambda} \int_0^\epsilon f = \frac{1}{\lambda} \neq 0.$$

Hence such a λ does not have the required property.

Conclusion. Combining sufficiency and necessity, the set of admissible parameters is exactly

$$\{\lambda : \lambda \leq 1/q\} = \{\lambda : \lambda \leq 1 - 1/p\} \quad \text{for } 1 \leq p < \infty, \quad \{\lambda : \lambda < 1\} \quad \text{for } p = \infty. \quad \blacksquare$$

Remark on the endpoint. The asymmetry at $p = \infty$ is worth noting. For $p < \infty$ the critical exponent $\lambda = 1/q$ is admissible because the quantity $\|f \cdot \chi_{[0, \epsilon]}\|_p$ carries an extra, though non-quantitative, decay to 0 as $\epsilon \rightarrow 0^+$; this is exactly the continuity of the integral expressed by Proposition 23 of Chapter 4. For $p = \infty$ the corresponding quantity is the essential supremum of f over $[0, \epsilon]$, which need not tend to 0 at all: for $f \equiv 1$ it is identically 1, and then $\epsilon^{-1} \int_0^\epsilon f = 1$ for all ϵ , so the critical exponent $\lambda = 1$ fails.

7.4 Problems 7.22–7.28

Problem 7.22. (Riesz) For $1 < p < \infty$, show that if the absolutely continuous function F on $[a, b]$ is the indefinite integral of an $L^p[a, b]$ function, then there is a constant $M > 0$ such that for any partition $\{x_0, \dots, x_n\}$ of $[a, b]$,

$$\sum_{k=1}^n \frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}} \leq M.$$

Solution. By hypothesis there is a function $f \in L^p[a, b]$ for which

$$F(x) = F(a) + \int_a^x f \quad \text{for all } x \in [a, b].$$

Let $q = p/(p-1)$ be the conjugate of p , so that $1 < q < \infty$ and $p/q = p-1$.

Set

$$M = \int_a^b |f|^p + 1 = \|f\|_p^p + 1,$$

which is a finite positive number since f belongs to $L^p[a, b]$. We claim this M works.

Let $\{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, so $a = x_0 < x_1 < \dots < x_n = b$. Fix an index k with $1 \leq k \leq n$. By the additivity of integration over domains,

$$F(x_k) - F(x_{k-1}) = \int_{x_{k-1}}^{x_k} f,$$

and therefore, by the monotonicity of integration,

$$|F(x_k) - F(x_{k-1})| \leq \int_{x_{k-1}}^{x_k} |f|.$$

Apply Hölder's Inequality (Theorem 1 of this chapter) on the interval $[x_{k-1}, x_k]$ to the pair of functions $|f|$, which belongs to $L^p[x_{k-1}, x_k]$, and the function identically equal to 1, which belongs to $L^q[x_{k-1}, x_k]$ because the interval has finite measure. We obtain

$$\int_{x_{k-1}}^{x_k} |f| \cdot 1 \leq \left[\int_{x_{k-1}}^{x_k} |f|^p \right]^{1/p} \left[\int_{x_{k-1}}^{x_k} 1^q \right]^{1/q} = \left[\int_{x_{k-1}}^{x_k} |f|^p \right]^{1/p} (x_k - x_{k-1})^{1/q}.$$

Combining the two displays and raising to the p -th power (legitimate, since $t \mapsto t^p$ is increasing on $[0, \infty)$),

$$|F(x_k) - F(x_{k-1})|^p \leq \left[\int_{x_{k-1}}^{x_k} |f|^p \right] (x_k - x_{k-1})^{p/q} = \left[\int_{x_{k-1}}^{x_k} |f|^p \right] (x_k - x_{k-1})^{p-1},$$

where we used $p/q = p(1 - 1/p) = p - 1$. Since $x_k - x_{k-1} = |x_k - x_{k-1}| > 0$, we may divide by $|x_k - x_{k-1}|^{p-1}$ to get

$$\frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}} \leq \int_{x_{k-1}}^{x_k} |f|^p.$$

Now sum over $k = 1, \dots, n$. Since the intervals $[x_{k-1}, x_k]$ have pairwise disjoint interiors and their union is $[a, b]$, the additivity of integration over domains gives

$$\sum_{k=1}^n \frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}} \leq \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f|^p = \int_a^b |f|^p < M.$$

Since the partition was arbitrary and M depends only on f , the proof is complete.

We remark on the mechanism: Hölder's Inequality was applied separately on each subinterval, and the exponent bookkeeping $p/q = p - 1$ is exactly what makes the factor $(x_k - x_{k-1})^{1/q}$ cancel against the denominator $|x_k - x_{k-1}|^{p-1}$, leaving behind on the k -th term only the local mass $\int_{x_{k-1}}^{x_k} |f|^p$. These local masses sum to $\int_a^b |f|^p$ over any partition whatsoever, which is why a single constant M serves for all partitions simultaneously. Note also that the hypothesis $p > 1$ is used only through the existence of the finite conjugate exponent q ; for $p = 1$ the displayed sum reduces to the total variation sums of F , which are bounded by $\int_a^b |f|$ by the same argument.

Problem 7.23. Provide an example of a Cauchy sequence of real numbers that is not rapidly Cauchy.

Solution. Here $X = \mathbb{R}$ normed by the absolute value. Recall the definition: a sequence $\{a_n\}$ in X is rapidly Cauchy provided there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \epsilon_k$ for which $|a_{k+1} - a_k| \leq \epsilon_k^2$ for all k .

Take

$$a_n = \frac{1}{n} \quad \text{for each natural number } n.$$

First, the sequence is Cauchy. Indeed $\{a_n\}$ converges to 0 in $\mathbb{R}$, and by Proposition 4 every convergent sequence in a normed linear space is Cauchy. (Directly: given $\epsilon > 0$ choose $N > 2/\epsilon$; then for $m, n \geq N$ we have $|a_n - a_m| \leq a_n + a_m \leq 2/N < \epsilon$.)

Second, the sequence is not rapidly Cauchy. Suppose, to get a contradiction, that it were. Then there would be a convergent series $\sum_{k=1}^{\infty} \epsilon_k$ of positive numbers with

$$|a_{k+1} - a_k| \leq \epsilon_k^2 \quad \text{for all } k.$$

But

$$|a_{k+1} - a_k| = \left| \frac{1}{k+1} - \frac{1}{k} \right| = \frac{1}{k(k+1)},$$

so that $\epsilon_k^2 \geq \frac{1}{k(k+1)}$ and hence, since $\epsilon_k > 0$,

$$\epsilon_k \geq \frac{1}{\sqrt{k(k+1)}} \geq \frac{1}{k+1} \quad \text{for all } k,$$

the last inequality because $k(k+1) \leq (k+1)^2$. Therefore, for each n ,

$$\sum_{k=1}^n \epsilon_k \geq \sum_{k=1}^n \frac{1}{k+1},$$

and the harmonic series diverges, so the partial sums on the right are unbounded. Thus $\sum_{k=1}^{\infty} \epsilon_k$ diverges, contradicting the choice of $\{\epsilon_k\}$. Hence $\{1/n\}$ is a Cauchy sequence of real numbers that is not rapidly Cauchy.

A remark on what is really going on. The computation isolates the general criterion. A sequence $\{a_n\}$ of real numbers is rapidly Cauchy if and only if $\sum_{k=1}^{\infty} \sqrt{|a_{k+1} - a_k|} < \infty$. For necessity, argue as above: $\epsilon_k \geq \sqrt{|a_{k+1} - a_k|}$. For sufficiency, put $\epsilon_k = \max\{\sqrt{|a_{k+1} - a_k|}, 2^{-k}\} > 0$; then $\epsilon_k^2 \geq |a_{k+1} - a_k|$ and $\sum_k \epsilon_k \leq \sum_k \sqrt{|a_{k+1} - a_k|} + \sum_k 2^{-k} < \infty$. So being rapidly Cauchy is a genuine strengthening of being Cauchy: it forces the increments to be summable at a square-root rate, which the increments $1/[k(k+1)]$ of $\{1/n\}$ fail to be, their square roots being comparable to the harmonic terms $1/k$. This also shows Proposition 5 is sharp: $\{1/n\}$ certainly has rapidly Cauchy subsequences, for instance $\{1/2^k\}$, whose increments are $2^{-(k+1)}$ with $\epsilon_k = 2^{-k/2}$ summable, but the full sequence is not rapidly Cauchy.

Problem 7.24. Let X be a normed linear space. Assume that $\{f_n\} \rightarrow f$ in X , $\{g_n\} \rightarrow g$ in X , and α and β are real numbers. Show that

$$\{\alpha f_n + \beta g_n\} \rightarrow \alpha f + \beta g \text{ in } X.$$

Solution. Since X is a linear space and f_n, f, g_n, g all belong to X , each $\alpha f_n + \beta g_n$ and also $\alpha f + \beta g$ belong to X . So the assertion to be proved is that

$$\lim_{n \rightarrow \infty} \|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\| = 0.$$

Fix a natural number n . Regrouping in the linear space X ,

$$(\alpha f_n + \beta g_n) - (\alpha f + \beta g) = \alpha[f_n - f] + \beta[g_n - g].$$

By the triangle inequality for the norm and then the positive homogeneity of the norm,

$$\|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\| \leq \|\alpha[f_n - f]\| + \|\beta[g_n - g]\| = |\alpha| \|f_n - f\| + |\beta| \|g_n - g\|.$$

Now let $\epsilon > 0$. Since $\{f_n\} \rightarrow f$ in X , there is a natural number N_1 such that

$$\|f_n - f\| < \frac{\epsilon}{2(|\alpha| + 1)} \quad \text{for all } n \geq N_1,$$

and since $\{g_n\} \rightarrow g$ in X , there is a natural number N_2 such that

$$\|g_n - g\| < \frac{\epsilon}{2(|\beta| + 1)} \quad \text{for all } n \geq N_2.$$

(The harmless +1 in the denominators covers the possibility that α or β vanishes.) Set $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,

$$\|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\| \leq |\alpha| \cdot \frac{\epsilon}{2(|\alpha| + 1)} + |\beta| \cdot \frac{\epsilon}{2(|\beta| + 1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where we used $|\alpha|/(|\alpha| + 1) < 1$ and $|\beta|/(|\beta| + 1) < 1$.

Since $\epsilon > 0$ was arbitrary,

$$\lim_{n \rightarrow \infty} \|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\| = 0,$$

that is, $\{\alpha f_n + \beta g_n\} \rightarrow \alpha f + \beta g$ in X .

Equivalently and more briefly: the right-hand side $|\alpha| \|f_n - f\| + |\beta| \|g_n - g\|$ of the displayed estimate is a sequence of nonnegative real numbers that converges to $|\alpha| \cdot 0 + |\beta| \cdot 0 = 0$, by the algebra of limits of sequences of real numbers; the squeeze principle then forces $\|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)\| \rightarrow 0$. Nothing beyond the three norm axioms was used, so the conclusion holds in every normed linear space; in particular it holds in $L^p(E)$ for $1 \leq p \leq \infty$, where it says that if $\{f_n\} \rightarrow f$ and $\{g_n\} \rightarrow g$ with respect to the $L^p(E)$ norm then $\{\alpha f_n + \beta g_n\} \rightarrow \alpha f + \beta g$ with respect to that norm. The special case $\alpha = \beta = 1$ says that addition is continuous with respect to the norm, and the case $\beta = 0$ says that scalar multiplication by a fixed scalar is continuous.

Problem 7.25. Assume that E has finite measure and $1 \leq p_1 < p_2 \leq \infty$. Show that if $\{f_n\} \rightarrow f$ in $L^{p_2}(E)$, then $\{f_n\} \rightarrow f$ in $L^{p_1}(E)$.

Solution. Since $m(E) < \infty$ and $1 \leq p_1 < p_2 \leq \infty$, Corollary 3 of this chapter tells us that

$$L^{p_2}(E) \subseteq L^{p_1}(E), \quad \|h\|_{p_1} \leq c \|h\|_{p_2} \quad \text{for all } h \in L^{p_2}(E),$$

where the constant c depends only on $m(E)$, p_1 and p_2 : namely

$$c = [m(E)]^{\frac{1}{p_1} - \frac{1}{p_2}} \quad \text{if } p_2 < \infty, \quad c = [m(E)]^{\frac{1}{p_1}} \quad \text{if } p_2 = \infty.$$

By hypothesis f and every f_n belong to $L^{p_2}(E)$, so by the inclusion just quoted f and every f_n belong to $L^{p_1}(E)$ as well. Thus the assertion that $\{f_n\} \rightarrow f$ in $L^{p_1}(E)$ makes sense, and what must be shown is that $\|f_n - f\|_{p_1} \rightarrow 0$.

Fix n . The difference $f_n - f$ belongs to $L^{p_2}(E)$, since $L^{p_2}(E)$ is a linear space. Applying the inequality of Corollary 3 to $h = f_n - f$,

$$\|f_n - f\|_{p_1} \leq c \|f_n - f\|_{p_2} \quad \text{for all } n.$$

By hypothesis $\lim_{n \rightarrow \infty} \|f_n - f\|_{p_2} = 0$, and c is a fixed finite constant, so the right-hand side tends to 0. Since $0 \leq \|f_n - f\|_{p_1} \leq c \|f_n - f\|_{p_2}$, the squeeze principle for sequences of real numbers gives

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{p_1} = 0,$$

that is, $\{f_n\} \rightarrow f$ in $L^{p_1}(E)$.

For completeness we recall the proof of the inequality of Corollary 3, since it is the whole content of the exercise. Suppose first $p_2 < \infty$. Put $p = p_2/p_1 > 1$ and let q be its conjugate, so $1/q = 1 - p_1/p_2$. For $h \in L^{p_2}(E)$ the function $|h|^{p_1}$ belongs to $L^p(E)$, because $\int_E (|h|^{p_1})^p = \int_E |h|^{p_2} < \infty$, while $g = \chi_E$ belongs to $L^q(E)$ because $m(E) < \infty$. Hölder's Inequality gives

$$\int_E |h|^{p_1} = \int_E |h|^{p_1} \cdot \chi_E \leq \left[\int_E |h|^{p_2} \right]^{1/p} \left[\int_E \chi_E \right]^{1/q} = \|h\|_{p_2}^{p_1} [m(E)]^{1-p_1/p_2}.$$

Raising to the power $1/p_1$ yields $\|h\|_{p_1} \leq [m(E)]^{\frac{1}{p_1} - \frac{1}{p_2}} \|h\|_{p_2}$. If instead $p_2 = \infty$, then $|h| \leq \|h\|_{\infty}$ a.e. on E , so

$$\int_E |h|^{p_1} \leq \|h\|_{\infty}^{p_1} m(E),$$

and taking $1/p_1$ powers gives $\|h\|_{p_1} \leq [m(E)]^{1/p_1} \|h\|_\infty$.

Note where finiteness of $m(E)$ is used: it is what makes χ_E belong to $L^q(E)$, respectively what makes the last integral finite. The conclusion genuinely fails without it. On $E = \mathbb{R}$, with $p_1 = 1$ and $p_2 = 2$, put $f = 0$ and $f_n = n^{-1}\chi_{[0, n]}$. Then

$$\|f_n\|_2 = \left[\int_0^n n^{-2} \right]^{1/2} = n^{-1/2} \rightarrow 0, \quad \|f_n\|_1 = \int_0^n n^{-1} = 1 \quad \text{for all } n,$$

so $\{f_n\} \rightarrow 0$ in $L^2(\mathbb{R})$ but $\{f_n\}$ does not converge to 0 in $L^1(\mathbb{R})$.

Problem 7.26. (The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f . For $1 \leq p < \infty$, suppose there is a function g in $L^p(E)$ such that for all n , $|f_n| \leq g$ a.e. on E . Prove that $\{f_n\} \rightarrow f$ in $L^p(E)$.

Solution. By possibly excising from E a set of measure zero we may assume that g is real-valued, that $\{f_n\} \rightarrow f$ pointwise on all of E , and that $|f_n| \leq g$ on all of E for every n : indeed, the union of the countably many exceptional sets, one for the pointwise convergence and one for each of the inequalities $|f_n| \leq g$, together with the set where $g = \infty$ (which has measure zero since $g \in L^p(E)$), is a set of measure zero, and excising a set of measure zero changes neither the hypotheses nor any of the integrals below.

First, every function in sight belongs to $L^p(E)$. Each f_n is measurable and $|f_n|^p \leq g^p$ on E , and g^p is integrable over E because $g \in L^p(E)$; by the monotonicity of integration for nonnegative measurable functions, $\int_E |f_n|^p \leq \int_E g^p < \infty$, so $f_n \in L^p(E)$. Moreover f is measurable, being the pointwise limit on E of the sequence of measurable functions $\{f_n\}$ (Proposition 9 of Chapter 3), and letting $n \rightarrow \infty$ in the inequality $|f_n(x)| \leq g(x)$ gives

$$|f(x)| \leq g(x) \quad \text{for all } x \in E.$$

Hence $\int_E |f|^p \leq \int_E g^p < \infty$ and $f \in L^p(E)$ too.

Now consider the sequence of nonnegative measurable functions $\{h_n\}$ defined by

$$h_n = |f_n - f|^p \quad \text{on } E.$$

Two observations.

- (i) $\{h_n\} \rightarrow 0$ pointwise on E . For each $x \in E$ we have $f_n(x) \rightarrow f(x)$ in $\mathbb{R}$, hence $|f_n(x) - f(x)| \rightarrow 0$, and since $t \mapsto t^p$ is continuous on $[0, \infty)$, $h_n(x) = |f_n(x) - f(x)|^p \rightarrow 0$.
- (ii) $\{h_n\}$ is dominated by a fixed integrable function. For each $x \in E$ and each n ,

$$|f_n(x) - f(x)| \leq |f_n(x)| + |f(x)| \leq g(x) + g(x) = 2g(x),$$

so that

$$0 \leq h_n(x) = |f_n(x) - f(x)|^p \leq 2^p [g(x)]^p.$$

The function $2^p g^p$ is integrable over E , again because g belongs to $L^p(E)$.

By the Lebesgue Dominated Convergence Theorem of Chapter 4, applied to the sequence $\{h_n\}$ of measurable functions on E , which converges pointwise on E to 0 and is dominated in absolute value by the integrable function $2^p g^p$,

$$\lim_{n \rightarrow \infty} \int_E |f_n - f|^p = \lim_{n \rightarrow \infty} \int_E h_n = \int_E 0 = 0.$$

Taking p -th roots, $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$, that is,

$$\{f_n\} \rightarrow f \text{ in } L^p(E).$$

Two remarks. First, no assumption is made about $m(E)$; the domination by a single L^p function supplies both the uniform integrability and the tightness of $\{|f_n - f|^p\}$ that would be needed to invoke the Vitali Convergence Theorem instead. Second, the statement fails for $p = \infty$ in this form: on $E = [0, 1]$ the sequence $f_n = \chi_{[0, 1/n]}$ is dominated by $g = \chi_{[0, 1]} \in L^\infty[0, 1]$ and converges pointwise a.e. to $f = 0$, but $\|f_n - f\|_\infty = 1$ for every n .

Problem 7.27. For E a measurable set and $1 \leq p < \infty$, assume $\{f_n\} \rightarrow f$ in $L^p(E)$. Show that there is a subsequence $\{f_{n_k}\}$ and a function $g \in L^p(E)$ for which $|f_{n_k}| \leq g$ a.e. on E for all k .

Solution. By possibly excising from E a set of measure zero we may assume that f and each f_n are real-valued on E ; this is legitimate since a function in $L^p(E)$ is finite a.e. on E and there are countably many functions in question.

Since $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$, we may choose inductively a strictly increasing sequence of natural numbers $\{n_k\}$ for which

$$\|f_{n_k} - f\|_p \leq \frac{1}{2^k} \quad \text{for all } k.$$

Indeed, having chosen $n_1 < \dots < n_{k-1}$, the convergence provides an index N with $\|f_n - f\|_p \leq 2^{-k}$ for all $n \geq N$, and we take $n_k = \max\{N, n_{k-1} + 1\}$.

Define, for each natural number N , the nonnegative measurable function

$$g_N = |f| + \sum_{k=1}^N |f_{n_k} - f| \quad \text{on } E,$$

and define

$$g(x) = \lim_{N \rightarrow \infty} g_N(x) = |f(x)| + \sum_{k=1}^{\infty} |f_{n_k}(x) - f(x)| \quad \text{for } x \in E,$$

the limit existing in $[0, \infty]$ because $\{g_N(x)\}$ is an increasing sequence of nonnegative real numbers. As a pointwise limit of measurable functions, g is measurable, by Proposition 9 of Chapter 3.

We claim $g \in L^p(E)$. Each g_N belongs to $L^p(E)$ and, by Minkowski's Inequality applied N times,

$$\|g_N\|_p \leq \|f\|_p + \sum_{k=1}^N \|f_{n_k} - f\|_p \leq \|f\|_p + \sum_{k=1}^N \frac{1}{2^k} \leq \|f\|_p + 1 =: C,$$

a bound independent of N . Now $\{g_N^p\}$ is an increasing sequence of nonnegative measurable functions on E converging pointwise on E to g^p (the map $t \mapsto t^p$ being increasing and continuous on $[0, \infty]$). By the Monotone Convergence Theorem of Chapter 4,

$$\int_E g^p = \lim_{N \rightarrow \infty} \int_E g_N^p = \lim_{N \rightarrow \infty} \|g_N\|_p^p \leq C^p < \infty.$$

Hence g^p is integrable over E . In particular g is finite a.e. on E ; redefining g to be 0 on the set of measure zero where it takes the value ∞ changes nothing in what follows, and with this convention g is a real-valued measurable function on E with $\int_E |g|^p < \infty$, so

$$g \in L^p(E).$$

Finally, the domination. Fix k and $x \in E$. Since $|f_{n_k}(x) - f(x)|$ is one of the terms of the series defining $g(x)$, and all terms are nonnegative,

$$|f_{n_k}(x)| \leq |f(x)| + |f_{n_k}(x) - f(x)| \leq |f(x)| + \sum_{j=1}^{\infty} |f_{n_j}(x) - f(x)| = g(x).$$

This holds for every $x \in E$ outside the set of measure zero on which g was redefined, and therefore

$$|f_{n_k}| \leq g \quad \text{a.e. on } E, \quad \text{for all } k,$$

as required.

Observe as a by-product that the same g dominates $|f|$, and that on $E \sim \{g = \infty\}$ the series $\sum_k |f_{n_k} - f|$ converges, so $\{f_{n_k}\} \rightarrow f$ pointwise a.e. on E . This recovers, along the way, the last assertion of the Riesz-Fischer Theorem: a sequence convergent in $L^p(E)$ has a subsequence convergent pointwise a.e. on E to the same limit.

Problem 7.28. Assume E has finite measure and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence of measurable functions that converges pointwise a.e. on E to f . For $1 \leq p < \infty$, show that $\{f_n\} \rightarrow f$ in $L^p(E)$ if there is a $\theta > 0$ such that $\{f_n\}$ belongs to and is bounded as a subset of $L^{p+\theta}(E)$.

Solution. Write $r = p + \theta$, so $1 \leq p < r < \infty$. By hypothesis every f_n belongs to $L^r(E)$ and there is a constant $M \geq 0$ with

$$\|f_n\|_r \leq M \quad \text{for all } n.$$

Since $m(E) < \infty$, Corollary 3 of this chapter gives $L^r(E) \subseteq L^p(E)$, so $\{f_n\}$ is in particular a sequence in $L^p(E)$.

Step 1: the limit f belongs to $L^p(E)$.

The function f is measurable, being the pointwise limit a.e. on E of measurable functions (Proposition 9 of Chapter 3). The sequence of nonnegative measurable functions $\{|f_n|^r\}$ converges pointwise a.e. on E to $|f|^r$, so Fatou's Lemma of Chapter 4 gives

$$\int_E |f|^r \leq \liminf_{n \rightarrow \infty} \int_E |f_n|^r = \liminf_{n \rightarrow \infty} \|f_n\|_r^r \leq M^r < \infty.$$

Hence $f \in L^r(E)$, and again by Corollary 3, $f \in L^p(E)$.

Step 2: $\{|f_n|^p\}$ is uniformly integrable over E .

Set

$$s = \frac{r}{p} = \frac{p + \theta}{p} > 1,$$

and consider the family $\mathcal{F} = \{|f_n|^p : n \in \mathbb{N}\}$ of functions on E . Each member belongs to $L^s(E)$, since

$$\int_E (|f_n|^p)^s = \int_E |f_n|^r \leq M^r < \infty,$$

and, taking s -th roots,

$$\| |f_n|^p \|_s = \left[\int_E |f_n|^r \right]^{p/r} = \|f_n\|_r^p \leq M^p \quad \text{for all } n.$$

So $\mathcal{F}$ is a family of functions in $L^s(E)$ with $1 < s < \infty$ that is bounded in $L^s(E)$ by the constant M^p . By Corollary 2 of this chapter, $\mathcal{F}$ is uniformly integrable over E .

For the reader who prefers the estimate in the open: let t be the conjugate of s , so $1/t = 1 - p/r = \theta/r$. If $A \subseteq E$ is measurable, Hölder's Inequality applied to $|f_n|^p \in L^s(A)$ and $\chi_A \in L^t(A)$ gives

$$\int_A |f_n|^p \leq \left[\int_A |f_n|^r \right]^{p/r} [m(A)]^{1/t} \leq M^p [m(A)]^{\theta/r},$$

so given $\epsilon > 0$ the choice $\delta = (\epsilon/(M^p + 1))^{r/\theta}$ responds to the ϵ challenge, uniformly in n .

Step 3: $\{|f_n|^p\}$ is tight over E .

Tightness over E requires, for each $\epsilon > 0$, a measurable subset $E_0 \subseteq E$ of finite measure with $\int_{E \sim E_0} |f_n|^p < \epsilon$ for all n . Since $m(E) < \infty$ by hypothesis, take $E_0 = E$; then $E \sim E_0 = \emptyset$ and the integrals are 0. So the family is tight over E , trivially.

Step 4: conclusion.

We have a sequence $\{f_n\}$ in $L^p(E)$ that converges pointwise a.e. on E to the function f , which belongs to $L^p(E)$ by Step 1, and by Steps 2 and 3 the sequence $\{|f_n|^p\}$ is uniformly integrable and tight over E . Theorem 8 of this chapter therefore yields

$$\{f_n\} \rightarrow f \text{ in } L^p(E).$$

Two remarks on the hypotheses. The finiteness of $m(E)$ cannot be dropped: on $E = \mathbb{R}$ with $p = 1$ and $\theta = 1$, the sequence $f_n = n^{-1} \chi_{[0,n]}$ satisfies $\|f_n\|_2 = n^{-1/2} \leq 1$ and converges pointwise to $f = 0$, yet $\|f_n - f\|_1 = 1$ for all n ; here tightness fails. Nor can the higher-integrability bound be dropped: on $E = [0, 1]$ the sequence $f_n = n \chi_{(0, 1/n]}$ converges pointwise on $[0, 1]$ to 0 and is bounded in no $L^{1+\theta}[0, 1]$, and indeed $\|f_n\|_1 = 1$ for all n , so $\{f_n\}$ does not converge to 0 in $L^1[0, 1]$; here uniform integrability fails. The role of the exponent gap $\theta > 0$ is exactly to buy, via Hölder's Inequality, the uniform smallness of $\int_A |f_n|^p$ on sets A of small measure.

7.5 Problems 7.29–7.35

Problem 7.29. Consider the linear space of polynomials on $[a, b]$ normed by $\|\cdot\|_{\max}$ norm. Is this normed linear space a Banach space?

Solution. No. The space is not complete, so it is not a Banach space.

Write $\mathcal{P}$ for the linear space of (restrictions to $[a, b]$ of) real polynomials, normed by

$$\|p\|_{\max} = \max_{x \in [a, b]} |p(x)|.$$

This is a norm: the maximum is attained since a polynomial is continuous on the compact interval $[a, b]$, positive homogeneity and the triangle inequality are inherited from the absolute value, and $\|p\|_{\max} = 0$ forces p to vanish at every point of $[a, b]$, hence at infinitely many points, so p is the zero polynomial. Thus $\mathcal{P}$ is a linear subspace of $C[a, b]$ carrying the maximum norm, and, as noted in this section, convergence with respect to $\|\cdot\|_{\max}$ is exactly uniform convergence on $[a, b]$.

We exhibit a Cauchy sequence in $\mathcal{P}$ that fails to converge in $\mathcal{P}$. For $x \in [a, b]$ put $t = t(x) = (x - a)/(b - a)$, so that t ranges over $[0, 1]$, and define, for each natural number n ,

$$p_n(x) = \sum_{k=0}^n \frac{1}{k!} \left(\frac{x-a}{b-a} \right)^k.$$

Each p_n is a polynomial in x , so p_n belongs to $\mathcal{P}$.

The sequence $\{p_n\}$ is Cauchy in $\mathcal{P}$. Indeed, for $m > n$ and $x \in [a, b]$,

$$|p_m(x) - p_n(x)| = \left| \sum_{k=n+1}^m \frac{t^k}{k!} \right| \leq \sum_{k=n+1}^m \frac{1}{k!} \leq \sum_{k=n+1}^{\infty} \frac{1}{k!},$$

since $0 \leq t \leq 1$. Hence

$$\|p_m - p_n\|_{\max} \leq \sum_{k=n+1}^{\infty} \frac{1}{k!} \quad \text{for all } m > n.$$

The series $\sum_{k=0}^{\infty} 1/k!$ converges (to e), so its tails tend to 0; given $\epsilon > 0$, choose N with $\sum_{k=N+1}^{\infty} 1/k! < \epsilon$ and then $\|p_m - p_n\|_{\max} < \epsilon$ for all $m, n \geq N$. Thus $\{p_n\}$ is Cauchy.

Suppose, to get a contradiction, that $\{p_n\}$ converged in $\mathcal{P}$ to some polynomial q . Convergence in the maximum norm implies uniform, hence pointwise, convergence on $[a, b]$. But for each fixed $x \in [a, b]$, $p_n(x)$ is the n -th partial sum of the exponential series evaluated at $t = (x - a)/(b - a)$, so

$$\lim_{n \rightarrow \infty} p_n(x) = e^{(x-a)/(b-a)}.$$

By uniqueness of pointwise limits,

$$q(x) = e^{(x-a)/(b-a)} \quad \text{for all } x \in [a, b].$$

Since the right-hand side is everywhere positive, q is not the zero polynomial; let d be its degree. Differentiate the last identity $d+1$ times on the open interval (a, b) . The left-hand side gives $q^{(d+1)} \equiv 0$, because differentiating a polynomial of degree d more than d times yields 0. The right-hand side gives

$$\frac{d^{d+1}}{dx^{d+1}} e^{(x-a)/(b-a)} = \frac{1}{(b-a)^{d+1}} e^{(x-a)/(b-a)} > 0 \quad \text{for all } x \in (a, b).$$

This is a contradiction. Hence no polynomial q is the limit of $\{p_n\}$ in $\mathcal{P}$.

Therefore $\mathcal{P}$ contains a Cauchy sequence with no limit in $\mathcal{P}$, so $\mathcal{P}$, normed by the maximum norm, is not complete and is not a Banach space.

A second proof, once one knows that $C[a, b]$ with the maximum norm is complete (Problem 31): the Weierstrass Approximation Theorem, quoted in Section 7.4, says $\mathcal{P}$ is dense in $C[a, b]$, while $\mathcal{P} \neq C[a, b]$, since for instance $x \mapsto e^x$ is continuous but is not a polynomial (by the derivative argument just given). A complete subspace of a normed linear space is closed in it, and a dense proper subset is not closed. Hence $\mathcal{P}$ is not complete.

Problem 7.30. Let $\{f_n\}$ be a sequence in $C[a, b]$ and $\sum_{k=1}^{\infty} a_k$ a convergent series of positive numbers such that

$$\|f_{k+1} - f_k\|_{\max} \leq a_k \quad \text{for all } k.$$

Prove that

$$|f_{n+k}(x) - f_k(x)| \leq \|f_{n+k} - f_k\|_{\max} \leq \sum_{j=n}^{\infty} a_j \quad \text{for all } k, n \text{ and all } x \in [a, b].$$

Conclude that there is a function $f \in C[a, b]$ such that $\{f_n\} \rightarrow f$ uniformly on $[a, b]$.

Solution. A remark on the printed display first. As it stands the subscripts do not match: with k fixed and n large the left side compares f_{n+k} with f_k , and telescoping only bounds that by $\sum_{j=k}^{\infty} a_j$, which need not be at most $\sum_{j=n}^{\infty} a_j$ when $n > k$. The intended inequality is the one recorded as (6) in this section, namely with the second index n rather than k :

$$|f_{n+k}(x) - f_n(x)| \leq \|f_{n+k} - f_n\|_{\max} \leq \sum_{j=n}^{\infty} a_j \quad \text{for all } n, k \text{ and all } x \in [a, b].$$

That is what we prove; the printed form follows from it whenever $k \geq n$, since then $\sum_{j=k}^{\infty} a_j \leq \sum_{j=n}^{\infty} a_j$. Step 1: the pointwise bound. By the very definition of the maximum norm on $C[a, b]$, for any g in $C[a, b]$ and any x in $[a, b]$,

$$|g(x)| \leq \max_{y \in [a, b]} |g(y)| = \|g\|_{\max}.$$

Applying this to $g = f_{n+k} - f_n$, which belongs to $C[a, b]$ because $C[a, b]$ is a linear space, gives

$$|f_{n+k}(x) - f_n(x)| \leq \|f_{n+k} - f_n\|_{\max} \quad \text{for all } x \in [a, b].$$

Step 2: the norm bound. Fix natural numbers n and k . Telescoping,

$$f_{n+k} - f_n = \sum_{j=n}^{n+k-1} [f_{j+1} - f_j],$$

so by the triangle inequality for the norm, applied $k-1$ times, and the hypothesis $\|f_{j+1} - f_j\|_{\max} \leq a_j$,

$$\|f_{n+k} - f_n\|_{\max} \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\|_{\max} \leq \sum_{j=n}^{n+k-1} a_j \leq \sum_{j=n}^{\infty} a_j,$$

the last inequality because the a_j are positive, so that a partial sum of a tail is at most the whole tail. Combining Steps 1 and 2 gives the asserted chain of inequalities.

Step 3: construction of the limit function. Since $\sum_{k=1}^{\infty} a_k$ converges, its tails

$$R_n = \sum_{j=n}^{\infty} a_j$$

are finite and $\lim_{n \rightarrow \infty} R_n = 0$.

Fix $x \in [a, b]$. By Steps 1 and 2, $|f_{n+k}(x) - f_n(x)| \leq R_n$ for all n and k ; since $R_n \rightarrow 0$, the sequence of real numbers $\{f_n(x)\}$ is Cauchy. The real numbers are complete, so this sequence converges. Define

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{for } x \in [a, b].$$

Step 4: the convergence is uniform. Fix n and $x \in [a, b]$ and let $k \rightarrow \infty$ in the inequality $|f_{n+k}(x) - f_n(x)| \leq R_n$. Since $f_{n+k}(x) \rightarrow f(x)$ and the absolute value is continuous,

$$|f(x) - f_n(x)| \leq R_n \quad \text{for all } x \in [a, b] \text{ and all } n.$$

The bound R_n does not depend on x and tends to 0: given $\epsilon > 0$, choose N with $R_n < \epsilon$ for $n \geq N$; then $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$ and all $x \in [a, b]$. Thus $\{f_n\} \rightarrow f$ uniformly on $[a, b]$.
 Step 5: f is continuous. This is the standard $\epsilon/3$ argument. Let $x_0 \in [a, b]$ and $\epsilon > 0$. Choose n with $\sup_{x \in [a, b]} |f(x) - f_n(x)| \leq R_n < \epsilon/3$. Since f_n is continuous at x_0 , there is a $\delta > 0$ such that $|f_n(x) - f_n(x_0)| < \epsilon/3$ whenever $x \in [a, b]$ and $|x - x_0| < \delta$. For such x ,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Hence f is continuous at x_0 ; as x_0 was arbitrary, f belongs to $C[a, b]$.

Therefore there is a function $f \in C[a, b]$ with $\{f_n\} \rightarrow f$ uniformly on $[a, b]$, that is, $\{f_n\} \rightarrow f$ in $C[a, b]$ normed by the maximum norm, and moreover $\|f - f_n\|_{\max} \leq \sum_{j=n}^{\infty} a_j$ for every n .

Problem 7.31. Use the preceding problem to show that $C[a, b]$, normed by the maximum norm, is a Banach space.

Solution. Recall first that $C[a, b]$, the linear space of continuous real-valued functions on the closed, bounded interval $[a, b]$, is normed by

$$\|f\|_{\max} = \max_{x \in [a, b]} |f(x)|,$$

the maximum being attained by the Extreme Value Theorem. We must show that every Cauchy sequence in $C[a, b]$ converges to a function in $C[a, b]$.

Let $\{f_n\}$ be a Cauchy sequence in $C[a, b]$.

Step 1: extraction of a subsequence with rapidly decreasing successive differences. We construct inductively a strictly increasing sequence of natural numbers $\{n_k\}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_{\max} \leq \left(\frac{1}{2}\right)^k \quad \text{for all } k.$$

Since $\{f_n\}$ is Cauchy, for each natural number k the Cauchy property applied with $\epsilon = (1/2)^k$ supplies a natural number M_k such that

$$\|f_n - f_m\|_{\max} \leq \left(\frac{1}{2}\right)^k \quad \text{for all } n, m \geq M_k.$$

Set $n_1 = M_1$ and, inductively, $n_{k+1} = \max\{M_{k+1}, n_k + 1\}$. Then $\{n_k\}$ is strictly increasing, and $n_k \geq M_k$ for every k (for $k = 1$ by definition, and for $k \geq 2$ because $n_k = \max\{M_k, n_{k-1} + 1\} \geq M_k$). Since both n_{k+1} and n_k are at least $n_k \geq M_k$, the displayed estimate with this k gives

$$\|f_{n_{k+1}} - f_{n_k}\|_{\max} \leq \left(\frac{1}{2}\right)^k.$$

Step 2: the subsequence converges in $C[a, b]$. Put $g_k = f_{n_k}$ and $a_k = (1/2)^k$. Then $\{g_k\}$ is a sequence in $C[a, b]$, the a_k are positive, the geometric series $\sum_{k=1}^{\infty} (1/2)^k$ converges (to 1), and by Step 1

$$\|g_{k+1} - g_k\|_{\max} \leq a_k \quad \text{for all } k.$$

These are exactly the hypotheses of the preceding problem. Hence there is a function f in $C[a, b]$ such that $\{g_k\} \rightarrow f$ uniformly on $[a, b]$. As observed at the start of this section, for a sequence and a function in $C[a, b]$, convergence in the maximum norm is the same as uniform convergence on $[a, b]$. Therefore the subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converges to f in $C[a, b]$.

Step 3: the whole sequence converges. The sequence $\{f_n\}$ is Cauchy in the normed linear space $C[a, b]$ and has a subsequence that converges in $C[a, b]$ to f . By Proposition 4 of this chapter, a Cauchy sequence in a normed linear space converges if it has a convergent subsequence. Hence $\{f_n\} \rightarrow f$ in $C[a, b]$.

Since every Cauchy sequence in $C[a, b]$ converges to a function in $C[a, b]$, the space $C[a, b]$ is complete, that is, $C[a, b]$ normed by the maximum norm is a Banach space.

Problem 7.32. Let $\{f_n\}$ be a sequence in $L^\infty(E)$ and $\sum_{k=1}^\infty a_k$ a convergent series of positive numbers such that

$$\|f_{k+1} - f_k\|_\infty \leq a_k \quad \text{for all } k.$$

Prove that there is a subset E_0 of E which has measure zero and

$$|f_{n+k}(x) - f_k(x)| \leq \|f_{n+k} - f_k\|_\infty \leq \sum_{j=n}^\infty a_j \quad \text{for all } k, n \text{ and all } x \in E \sim E_0.$$

Conclude that there is a function $f \in L^\infty(E)$ such that $\{f_n\} \rightarrow f$ uniformly on $E \sim E_0$.

Solution. As in the preceding problem, the subscripts in the printed display are misaligned: what is true, and what is needed, is the inequality with second index n ,

$$|f_{n+k}(x) - f_n(x)| \leq \|f_{n+k} - f_n\|_\infty \leq \sum_{j=n}^\infty a_j \quad \text{for all } n, k \text{ and all } x \in E \sim E_0,$$

which is the analogue for $L^\infty(E)$ of (6) of this section. This is what we prove. (The printed form follows for $k \geq n$, since then $\sum_{j=k}^\infty a_j \leq \sum_{j=n}^\infty a_j$.)

Step 1: choice of the exceptional set E_0 . Recall from Section 7.1 that for g in $L^\infty(E)$ the essential supremum $\|g\|_\infty$, defined as the infimum of the essential upper bounds of $|g|$, is itself an essential upper bound for $|g|$; indeed, if $M_i \downarrow \|g\|_\infty$ with each M_i an essential upper bound, then $\{x \in E \mid |g(x)| > \|g\|_\infty\} = \bigcup_i \{x \in E \mid |g(x)| > M_i\}$ is a countable union of sets of measure zero. Thus

$$|g(x)| \leq \|g\|_\infty \quad \text{for almost all } x \in E.$$

Consequently, for each natural number n , the set

$$A_n = \{x \in E \mid |f_n(x)| > \|f_n\|_\infty\}$$

has measure zero, and for all natural numbers n and k the set

$$B_{n,k} = \{x \in E \mid |f_{n+k}(x) - f_n(x)| > \|f_{n+k} - f_n\|_\infty\}$$

has measure zero, since $f_{n+k} - f_n$ belongs to $L^\infty(E)$ ($L^\infty(E)$ is a linear space). Define

$$E_0 = \left(\bigcup_{n=1}^\infty A_n \right) \cup \left(\bigcup_{n=1}^\infty \bigcup_{k=1}^\infty B_{n,k} \right).$$

This is a countable union of sets of measure zero, so by the countable subadditivity of measure, $m(E_0) = 0$. Note that on $E \sim E_0$ every f_n is finite, since there $|f_n(x)| \leq \|f_n\|_\infty < \infty$; hence all the differences below are honest differences of real numbers.

Step 2: the two inequalities. Fix n and k . Since $x \in E \sim E_0$ implies $x \notin B_{n,k}$, we have

$$|f_{n+k}(x) - f_n(x)| \leq \|f_{n+k} - f_n\|_\infty \quad \text{for all } x \in E \sim E_0.$$

For the second inequality, telescope and use the triangle inequality for $\|\cdot\|_\infty$, exactly as in (6) of this section:

$$f_{n+k} - f_n = \sum_{j=n}^{n+k-1} [f_{j+1} - f_j], \quad \|f_{n+k} - f_n\|_\infty \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\|_\infty \leq \sum_{j=n}^{n+k-1} a_j \leq \sum_{j=n}^\infty a_j,$$

the last step because the a_j are positive, so a partial sum of a tail is at most the whole tail. This establishes the asserted chain of inequalities on $E \sim E_0$.

Step 3: construction of f . Since $\sum_{k=1}^{\infty} a_k$ converges, its tails $R_n = \sum_{j=n}^{\infty} a_j$ are finite and $R_n \rightarrow 0$. Fix x in $E \sim E_0$. By Step 2, $|f_{n+k}(x) - f_n(x)| \leq R_n$ for all n, k , so $\{f_n(x)\}$ is a Cauchy sequence of real numbers, hence convergent since $\mathbb{R}$ is complete. Define

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \text{ for } x \in E \sim E_0, \quad f(x) = 0 \text{ for } x \in E_0.$$

Each f_n is measurable, hence so is $f_n \cdot \chi_{E \sim E_0}$, and f is the pointwise limit on E of the measurable functions $f_n \cdot \chi_{E \sim E_0}$; by Proposition 9 of Chapter 3, a pointwise limit of a sequence of measurable functions is measurable, so f is measurable on E .

Step 4: f belongs to $L^\infty(E)$ and the convergence is uniform off E_0 . Fix n and $x \in E \sim E_0$ and let $k \rightarrow \infty$ in $|f_{n+k}(x) - f_n(x)| \leq R_n$; by continuity of the absolute value,

$$|f(x) - f_n(x)| \leq R_n \text{ for all } x \in E \sim E_0 \text{ and all } n.$$

Taking $n = 1$ we get, for $x \in E \sim E_0$,

$$|f(x)| \leq |f_1(x)| + R_1 \leq \|f_1\|_\infty + \sum_{j=1}^{\infty} a_j < \infty,$$

so $|f|$ is bounded above by the constant $\|f_1\|_\infty + \sum_{j=1}^{\infty} a_j$ except on the set E_0 of measure zero. Thus f is measurable and essentially bounded, that is, f belongs to $L^\infty(E)$.

Finally, the bound R_n in the displayed inequality is independent of x and tends to 0: given $\epsilon > 0$ choose N with $R_n < \epsilon$ for $n \geq N$; then $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$ and all $x \in E \sim E_0$. Hence $\{f_n\} \rightarrow f$ uniformly on $E \sim E_0$. Since E_0 has measure zero, R_n is an essential upper bound for $|f - f_n|$, so $\|f - f_n\|_\infty \leq R_n \rightarrow 0$ and $\{f_n\} \rightarrow f$ in $L^\infty(E)$ as well.

Problem 7.33. Use the preceding problem to show that $L^\infty(E)$ is a Banach space.

Solution. Recall that $L^\infty(E)$ is the linear space of (equivalence classes, under equality almost everywhere, of) essentially bounded measurable functions on E , normed by the essential supremum

$$\|f\|_\infty = \text{ess sup}_E |f| = \inf\{M \geq 0 \mid |f| \leq M \text{ a.e. on } E\},$$

which was shown in Section 7.1 to be a norm on $L^\infty(E)$. We must show that every Cauchy sequence in $L^\infty(E)$ converges, with respect to $\|\cdot\|_\infty$, to a function in $L^\infty(E)$.

Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(E)$.

Step 1: extraction of a subsequence with rapidly decreasing successive differences. Exactly as in Problem 31, we construct inductively a strictly increasing sequence of natural numbers $\{n_k\}$ with

$$\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq \left(\frac{1}{2}\right)^k \text{ for all } k.$$

Indeed, since $\{f_n\}$ is Cauchy, for each natural number k there is a natural number M_k such that $\|f_n - f_m\|_\infty \leq (1/2)^k$ for all $n, m \geq M_k$. Set $n_1 = M_1$ and, inductively, $n_{k+1} = \max\{M_{k+1}, n_k + 1\}$. Then $\{n_k\}$ is strictly increasing and $n_{k+1} > n_k \geq M_k$, so $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq (1/2)^k$.

Step 2: the subsequence converges. Put $g_k = f_{n_k}$ and $a_k = (1/2)^k$. Then $\{g_k\}$ is a sequence in $L^\infty(E)$, the numbers a_k are positive, the geometric series $\sum_{k=1}^{\infty} (1/2)^k$ converges, and $\|g_{k+1} - g_k\|_\infty \leq a_k$ for all k . These are the hypotheses of the preceding problem. Hence there is a set $E_0 \subseteq E$ of measure zero and a function f in $L^\infty(E)$ such that $\{g_k\} \rightarrow f$ uniformly on $E \sim E_0$. As observed at the start of this section, for a sequence and a function in $L^\infty(E)$, convergence in $L^\infty(E)$ is equivalent to uniform convergence on the complement of a set of measure zero. Therefore the subsequence $\{f_{n_k}\}$ converges to f in $L^\infty(E)$. (Concretely, the estimate established in the preceding problem gives $\|f - g_k\|_\infty \leq \sum_{j=k}^{\infty} (1/2)^j = (1/2)^{k-1} \rightarrow 0$.)

Step 3: the whole sequence converges. The sequence $\{f_n\}$ is Cauchy in the normed linear space $L^\infty(E)$ and possesses a subsequence converging in $L^\infty(E)$ to f . By Proposition 4 of this chapter, $\{f_n\} \rightarrow f$ in $L^\infty(E)$.

Thus every Cauchy sequence in $L^\infty(E)$ converges to a function in $L^\infty(E)$: the space $L^\infty(E)$ is complete, hence a Banach space. This is the assertion that the text of Section 7.3 makes and defers to the present problem; note that it is not covered by the Riesz-Fischer Theorem or by Theorem 6, both of which are stated only for $1 \leq p < \infty$.

Problem 7.34. Prove that for $1 \leq p \leq \infty$, ℓ^p is a Banach space.

Solution. Here ℓ^p , for $1 \leq p < \infty$, is the linear space of real sequences $a = (a_1, a_2, \dots)$ for which $\sum_{k=1}^\infty |a_k|^p < \infty$, normed by

$$\|a\|_p = \left(\sum_{k=1}^\infty |a_k|^p \right)^{1/p},$$

and ℓ^∞ is the linear space of bounded real sequences, normed by $\|a\|_\infty = \sup_k |a_k|$. That these are linear spaces and that $\|\cdot\|_p$ is a norm on ℓ^p was established in Problem 12 (Minkowski's Inequality in ℓ^p); for $p = \infty$ the norm properties are immediate from those of the absolute value together with $\sup_k |a_k + b_k| \leq \sup_k |a_k| + \sup_k |b_k|$.

Throughout, write a sequence in ℓ^p as $\{a^{(n)}\}_{n=1}^\infty$, where $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, \dots)$, so that $a_k^{(n)}$ is the k -th term of the n -th sequence. The key observation, valid for every p with $1 \leq p \leq \infty$, is that for each fixed index k the coordinate functional is bounded by the norm:

$$|a_k| \leq \|a\|_p \quad \text{for all } a \in \ell^p \text{ and all } k.$$

For $p = \infty$ this is the definition of the supremum; for $p < \infty$ it holds because $|a_k|^p \leq \sum_{i=1}^\infty |a_i|^p$ and $t \mapsto t^{1/p}$ is increasing on $[0, \infty)$.

Case $1 \leq p < \infty$. Let $\{a^{(n)}\}$ be a Cauchy sequence in ℓ^p .

Step 1: the coordinatewise limit. Fix k . By the observation above applied to $a^{(n)} - a^{(m)}$,

$$|a_k^{(n)} - a_k^{(m)}| \leq \|a^{(n)} - a^{(m)}\|_p \quad \text{for all } n, m.$$

Since $\{a^{(n)}\}$ is Cauchy in ℓ^p , the right side is small for large n, m , so $\{a_k^{(n)}\}_n$ is a Cauchy sequence of real numbers. The real numbers are complete, so this sequence converges; define

$$a_k = \lim_{n \rightarrow \infty} a_k^{(n)}, \quad a = (a_1, a_2, \dots).$$

Step 2: $a^{(n)} - a$ lies in ℓ^p with small norm. Let $\epsilon > 0$. Choose a natural number N such that

$$\|a^{(n)} - a^{(m)}\|_p \leq \epsilon \quad \text{for all } n, m \geq N,$$

that is,

$$\sum_{k=1}^\infty |a_k^{(n)} - a_k^{(m)}|^p \leq \epsilon^p \quad \text{for all } n, m \geq N.$$

Fix $n \geq N$ and a natural number K . Discarding all but the first K terms of this series of nonnegative numbers gives

$$\sum_{k=1}^K |a_k^{(n)} - a_k^{(m)}|^p \leq \epsilon^p \quad \text{for all } m \geq N.$$

This is a finite sum, and by Step 1 each summand converges as $m \rightarrow \infty$ to $|a_k^{(n)} - a_k|^p$ (the maps $t \mapsto |t|$ and $t \mapsto t^p$ are continuous). Letting $m \rightarrow \infty$ therefore yields

$$\sum_{k=1}^K |a_k^{(n)} - a_k|^p \leq \epsilon^p.$$

The bound is independent of K ; letting $K \rightarrow \infty$, the increasing sequence of partial sums has a finite limit and

$$\sum_{k=1}^{\infty} |a_k^{(n)} - a_k|^p \leq \epsilon^p \quad \text{for all } n \geq N.$$

Hence $a^{(n)} - a$ belongs to ℓ^p and $\|a^{(n)} - a\|_p \leq \epsilon$ for all $n \geq N$.

Step 3: conclusion. Taking $n = N$ in Step 2 shows $a^{(N)} - a \in \ell^p$; since ℓ^p is a linear space and $a^{(N)} \in \ell^p$,

$$a = a^{(N)} - (a^{(N)} - a) \in \ell^p.$$

Moreover Step 2 says exactly that for every $\epsilon > 0$ there is an N with $\|a^{(n)} - a\|_p \leq \epsilon$ for $n \geq N$, that is, $\{a^{(n)}\} \rightarrow a$ in ℓ^p . Therefore ℓ^p is complete.

Case $p = \infty$. Let $\{a^{(n)}\}$ be Cauchy in ℓ^∞ . Since $|a_k^{(n)} - a_k^{(m)}| \leq \|a^{(n)} - a^{(m)}\|_\infty$ for every k , each coordinate sequence $\{a_k^{(n)}\}_n$ is Cauchy in $\mathbb{R}$; let $a_k = \lim_{n \rightarrow \infty} a_k^{(n)}$ and $a = (a_1, a_2, \dots)$. Given $\epsilon > 0$, choose N with $\|a^{(n)} - a^{(m)}\|_\infty \leq \epsilon$ for $n, m \geq N$, so that

$$|a_k^{(n)} - a_k^{(m)}| \leq \epsilon \quad \text{for all } n, m \geq N \text{ and all } k.$$

Fix $n \geq N$ and k , and let $m \rightarrow \infty$: by continuity of the absolute value, $|a_k^{(n)} - a_k| \leq \epsilon$. As k was arbitrary,

$$\sup_k |a_k^{(n)} - a_k| \leq \epsilon \quad \text{for all } n \geq N,$$

so $a^{(n)} - a$ is a bounded sequence, that is, $a^{(n)} - a \in \ell^\infty$, and $\|a^{(n)} - a\|_\infty \leq \epsilon$ for $n \geq N$. In particular $a = a^{(N)} - (a^{(N)} - a)$ is bounded, so $a \in \ell^\infty$, and $\{a^{(n)}\} \rightarrow a$ in ℓ^∞ . Therefore ℓ^∞ is complete.

In all cases $1 \leq p \leq \infty$ the normed linear space ℓ^p is complete, hence a Banach space.

An alternative argument for $1 \leq p < \infty$ uses Problem 12 and the Riesz-Fischer Theorem. For $a \in \ell^p$ let T_a be the function on $[1, \infty)$ taking the value a_k on $[k, k+1)$; then $a \mapsto T_a$ is linear and, by Problem 12, $\|a\|_p = \|T_a\|_p$, so it is a linear isometry of ℓ^p onto the subspace S of $L^p[1, \infty)$ consisting of those functions that are almost everywhere constant on each interval $[k, k+1)$. Since $L^p[1, \infty)$ is complete by the Riesz-Fischer Theorem, it suffices to check S is closed: if $g_n \in S$ and $\{g_n\} \rightarrow g$ in $L^p[1, \infty)$, then by the Riesz-Fischer Theorem a subsequence of $\{g_n\}$ converges to g pointwise almost everywhere, and on each $[k, k+1)$ the members of that subsequence are constants, so their limit g is almost everywhere equal on $[k, k+1)$ to a single number; hence $g \in S$. A linear space isometric to a closed subspace of a Banach space is itself complete.

Problem 7.35. Show that the space c of all convergent sequences of real numbers and the space c_0 of all sequences that converge to zero are Banach spaces with respect to the ℓ^∞ norm.

Solution. Write a real sequence as $a = (a_1, a_2, \dots)$ and recall $\|a\|_\infty = \sup_k |a_k|$.

Step 0: c and c_0 are linear subspaces of ℓ^∞ . A convergent sequence of real numbers is bounded, so $c_0 \subseteq c \subseteq \ell^\infty$. If a, b converge to L, M respectively and α, β are real, then $\alpha a + \beta b$ converges to $\alpha L + \beta M$ by the algebra of limits; so c is a linear subspace of ℓ^∞ , and taking $L = M = 0$ shows c_0 is a linear subspace of c . Each therefore inherits the norm $\|\cdot\|_\infty$ and is a normed linear space.

Step 1: a closed subspace of a Banach space is a Banach space. Let X be a Banach space and Y a linear subspace of X that is closed in X (that is, contains the limit of every sequence in Y that converges in X). Let $\{y_n\}$ be Cauchy in Y with respect to the norm of X . Then $\{y_n\}$ is Cauchy in X , so it converges in X to some $y \in X$; since Y is closed, $y \in Y$; and convergence in X of a sequence in Y to a point of Y is the same as convergence in Y , the norms being the same. Hence Y is complete.

By Problem 34, ℓ^∞ is a Banach space. So it suffices to prove that c and c_0 are closed subsets of ℓ^∞ .

Step 2: c is closed in ℓ^∞ . Let $\{a^{(n)}\}$ be a sequence in c that converges in ℓ^∞ to some $a \in \ell^\infty$; write $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, \dots)$ and let

$$L_n = \lim_{k \rightarrow \infty} a_k^{(n)},$$

which exists since $a^{(n)} \in c$. We claim $\{L_n\}$ is a Cauchy sequence of real numbers. Indeed, for all n, m and all k ,

$$|a_k^{(n)} - a_k^{(m)}| \leq \|a^{(n)} - a^{(m)}\|_\infty,$$

and letting $k \rightarrow \infty$ gives

$$|L_n - L_m| \leq \|a^{(n)} - a^{(m)}\|_\infty.$$

Since $\{a^{(n)}\}$ converges in ℓ^∞ , it is Cauchy there by Proposition 4, so the right side is small for large n, m ; hence $\{L_n\}$ is Cauchy in $\mathbb{R}$ and, $\mathbb{R}$ being complete, converges to some real number L .

We show $\lim_{k \rightarrow \infty} a_k = L$, so that $a \in c$. Let $\epsilon > 0$. Choose a natural number n so large that

$$\|a - a^{(n)}\|_\infty < \frac{\epsilon}{3} \quad \text{and} \quad |L_n - L| < \frac{\epsilon}{3},$$

possible since $\{a^{(n)}\} \rightarrow a$ in ℓ^∞ and $\{L_n\} \rightarrow L$ in $\mathbb{R}$. With this n fixed, choose K so that $|a_k^{(n)} - L_n| < \epsilon/3$ for all $k \geq K$. Then for $k \geq K$,

$$|a_k - L| \leq |a_k - a_k^{(n)}| + |a_k^{(n)} - L_n| + |L_n - L| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

where the first term was bounded using $|a_k - a_k^{(n)}| \leq \|a - a^{(n)}\|_\infty$. Hence $\{a_k\} \rightarrow L$, so $a \in c$ and c is closed in ℓ^∞ .

Step 3: c_0 is closed in ℓ^∞ . Let $\{a^{(n)}\}$ be a sequence in c_0 converging in ℓ^∞ to a . Let $\epsilon > 0$. Choose n with $\|a - a^{(n)}\|_\infty < \epsilon/2$, and then, since $\{a_k^{(n)}\}_k \rightarrow 0$, choose K with $|a_k^{(n)}| < \epsilon/2$ for all $k \geq K$. Then for $k \geq K$,

$$|a_k| \leq |a_k - a_k^{(n)}| + |a_k^{(n)}| \leq \|a - a^{(n)}\|_\infty + |a_k^{(n)}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{a_k\} \rightarrow 0$, so $a \in c_0$ and c_0 is closed in ℓ^∞ .

Conclusion. By Steps 2 and 3, c and c_0 are closed linear subspaces of the Banach space ℓ^∞ , so by Step 1 each is complete with respect to the ℓ^∞ norm. Therefore c and c_0 , normed by $\|\cdot\|_\infty$, are Banach spaces. (Since $c_0 \subseteq c \subseteq \ell^\infty$ and c_0 is closed in ℓ^∞ , it is in particular closed in c , so the completeness of c_0 also follows from that of c .)

7.6 Problems 7.36–7.42

Problem 7.36. Let $\mathcal{S}$ be a subset of a normed linear space X . Show that $\mathcal{S}$ is dense in X if and only if each $g \in X$ is the limit of a sequence in $\mathcal{S}$.

Solution. Recall the definition in force: for $\mathcal{F} \subseteq \mathcal{G}$ subsets of a normed linear space X with norm $\|\cdot\|$, we say $\mathcal{F}$ is dense in $\mathcal{G}$ provided that for each g in $\mathcal{G}$ and each $\epsilon > 0$ there is an f in $\mathcal{F}$ for which $\|f - g\| < \epsilon$. Recall also that a sequence $\{f_n\}$ in X converges to g in X , written $\lim_{n \rightarrow \infty} f_n = g$ in X , provided $\lim_{n \rightarrow \infty} \|f_n - g\| = 0$.

Suppose first that $\mathcal{S}$ is dense in X , and let g belong to X . For each natural number n , apply the definition of denseness with the choice $\epsilon = 1/n$: there is a function f_n in $\mathcal{S}$ for which

$$\|f_n - g\| < \frac{1}{n}.$$

This defines a sequence $\{f_n\}$ whose terms all lie in $\mathcal{S}$. Since $0 \leq \|f_n - g\| < 1/n$ for all n , the squeeze principle gives $\lim_{n \rightarrow \infty} \|f_n - g\| = 0$, that is, $\{f_n\} \rightarrow g$ in X . Thus g is the limit of a sequence in $\mathcal{S}$. As g was an arbitrary member of X , every g in X is such a limit.

Conversely, suppose each g in X is the limit of a sequence in $\mathcal{S}$. Let g belong to X and let $\epsilon > 0$. Choose a sequence $\{f_n\}$ in $\mathcal{S}$ with $\{f_n\} \rightarrow g$ in X , that is, with $\lim_{n \rightarrow \infty} \|f_n - g\| = 0$. By the definition of the limit of a sequence of real numbers, there is an index N for which $\|f_n - g\| < \epsilon$ whenever $n \geq N$. In particular $f := f_N$ is a function in $\mathcal{S}$ with $\|f - g\| < \epsilon$. Since g in X and $\epsilon > 0$ were arbitrary, $\mathcal{S}$ is dense in X .

Problem 7.37. Verify (13). [Assertion (13) of the text reads: for $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H} \subseteq X$,

if $\mathcal{F}$ is dense in $\mathcal{G}$ and $\mathcal{G}$ is dense in $\mathcal{H}$, then $\mathcal{F}$ is dense in $\mathcal{H}$.

Here X is a normed linear space with norm $\|\cdot\|$.]

Solution. Let X be a normed linear space with norm $\|\cdot\|$ and let $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H} \subseteq X$. Assume $\mathcal{F}$ is dense in $\mathcal{G}$ and $\mathcal{G}$ is dense in $\mathcal{H}$. Observe first that the inclusion $\mathcal{F} \subseteq \mathcal{H}$ required by the definition holds, since $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$.

Let h belong to $\mathcal{H}$ and let $\epsilon > 0$. Since $\mathcal{G}$ is dense in $\mathcal{H}$, applied to the function h in $\mathcal{H}$ and the positive number $\epsilon/2$, there is a function g in $\mathcal{G}$ for which

$$\|g - h\| < \frac{\epsilon}{2}.$$

Since $\mathcal{F}$ is dense in $\mathcal{G}$, applied now to this function g in $\mathcal{G}$ and the positive number $\epsilon/2$, there is a function f in $\mathcal{F}$ for which

$$\|f - g\| < \frac{\epsilon}{2}.$$

By the triangle inequality for the norm $\|\cdot\|$,

$$\|f - h\| = \|(f - g) + (g - h)\| \leq \|f - g\| + \|g - h\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus for the arbitrary h in $\mathcal{H}$ and $\epsilon > 0$ we have produced f in $\mathcal{F}$ with $\|f - h\| < \epsilon$. Therefore $\mathcal{F}$ is dense in $\mathcal{H}$, which is exactly the assertion (13).

Problem 7.38. Prove that the collection of polynomials with rational coefficients is countable.

Solution. Let $\mathcal{P}$ denote the collection of all polynomials with rational coefficients, and for each non-negative integer n let

$$\mathcal{P}_n = \{c_0 + c_1x + \cdots + c_nx^n : c_0, \dots, c_n \in \mathbb{Q}\}$$

be the collection of polynomials with rational coefficients of degree at most n . Every polynomial has a finite degree, so

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n.$$

We first show each $\mathcal{P}_n$ is countable. By Corollary 4 of Chapter 1, the set $\mathbb{Q}$ of rational numbers is countably infinite, so there is a one-to-one correspondence φ of $\mathbb{N}$ with $\mathbb{Q}$. Define

$$\Phi: \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n+1 \text{ times}} \longrightarrow \mathcal{P}_n, \quad \Phi(k_0, k_1, \dots, k_n) = \varphi(k_0) + \varphi(k_1)x + \cdots + \varphi(k_n)x^n.$$

Since φ maps $\mathbb{N}$ onto $\mathbb{Q}$, every choice of rational coefficients $c_0, \dots, c_n$ is realized, so Φ maps $\mathbb{N}^{n+1}$ onto $\mathcal{P}_n$. By Corollary 4 of Chapter 1 the Cartesian product $\mathbb{N}^{n+1}$ of $n+1$ copies of $\mathbb{N}$ is countably infinite, and $\mathcal{P}_n$ is nonempty. Therefore, by Theorem 5 of Chapter 1, which asserts that a nonempty set is countable if and only if it is the image of a function whose domain is a nonempty countable set, the set $\mathcal{P}_n$ is countable.

Finally, $\{\mathcal{P}_n\}_{n=0}^{\infty}$ is a countable collection of countable sets, so by Corollary 6 of Chapter 1 its union $\mathcal{P}$ is countable.

We remark that $\mathcal{P}$ is in fact countably infinite, since the constant polynomials c_0 with c_0 rational already form a subset of $\mathcal{P}$ equipotent to the infinite set $\mathbb{Q}$; hence $\mathcal{P}$ is not finite.

Problem 7.39. Let E be a measurable set, $1 \leq p < \infty$, q the conjugate of p , and $\mathcal{S}$ a dense subset of $L^q(E)$. Show that if $g \in L^p(E)$ and $\int_E f \cdot g = 0$ for all $f \in \mathcal{S}$, then $g = 0$.

Solution. Let g belong to $L^p(E)$ and suppose $\int_E f \cdot g = 0$ for every f in $\mathcal{S}$.

Step 1: the hypothesis propagates from $\mathcal{S}$ to all of $L^q(E)$. Let h be any function in $L^q(E)$. Since $\mathcal{S}$ is dense in $L^q(E)$, by Problem 36 there is a sequence $\{f_n\}$ in $\mathcal{S}$ with $\{f_n\} \rightarrow h$ in $L^q(E)$, that is, $\lim_{n \rightarrow \infty} \|f_n - h\|_q = 0$. For each n , the function $(f_n - h) \cdot g$ is integrable over E by Holder's Inequality (Theorem 1 of Chapter 7), since $f_n - h$ belongs to $L^q(E)$ and g belongs to $L^p(E)$; the same theorem gives

$$\left| \int_E f_n \cdot g - \int_E h \cdot g \right| = \left| \int_E (f_n - h) \cdot g \right| \leq \int_E |(f_n - h) \cdot g| \leq \|f_n - h\|_q \cdot \|g\|_p.$$

Here we used the linearity of integration and the fact that the absolute value of an integral does not exceed the integral of the absolute value. Since $\|g\|_p$ is a finite number and $\|f_n - h\|_q \rightarrow 0$, the right-hand side tends to 0. But $\int_E f_n \cdot g = 0$ for every n , because each f_n belongs to $\mathcal{S}$. Therefore

$$\int_E h \cdot g = \lim_{n \rightarrow \infty} \int_E f_n \cdot g = 0.$$

Thus $\int_E h \cdot g = 0$ for every h in $L^q(E)$.

Step 2: testing against the conjugate function of g . Suppose, to get a contradiction, that g is not the zero element of $L^p(E)$, that is, $\|g\|_p > 0$. Then, by Theorem 1 of Chapter 7, the conjugate function of g ,

$$g^* = \|g\|_p^{1-p} \cdot \operatorname{sgn}(g) \cdot |g|^{p-1},$$

belongs to $L^q(E)$ and satisfies (4) of that theorem, namely

$$\int_E g \cdot g^* = \|g\|_p \quad \text{and} \quad \|g^*\|_q = 1.$$

Applying Step 1 with $h = g^*$ gives $\int_E g^* \cdot g = 0$, so $\|g\|_p = 0$. This contradicts the assumption $\|g\|_p > 0$. Therefore $\|g\|_p = 0$. Since $\|g\|_p = [\int_E |g|^p]^{1/p} = 0$ with $|g|^p$ a nonnegative measurable function, Proposition 9 of Chapter 4 (a nonnegative measurable function with zero integral vanishes almost everywhere) gives $|g|^p = 0$, hence $g = 0$, almost everywhere on E . As elements of $L^p(E)$ are identified when they agree almost everywhere on E , we conclude $g = 0$ in $L^p(E)$.

We note the case $p = 1$, $q = \infty$ is included: then $g^* = \operatorname{sgn}(g)$, which is an essentially bounded measurable function on E , and Theorem 1 of Chapter 7 explicitly records that (4) holds for $p = 1$, $q = \infty$.

Problem 7.40. Verify the details in the proof of Theorem 11. [Theorem 11: Let E be a measurable set and $1 \leq p < \infty$. Then the normed linear space $L^p(E)$ is separable.]

Solution. Throughout, $1 \leq p < \infty$ is fixed. Recall the notation of the proof of Theorem 11: for a closed, bounded interval $[a, b]$, $\mathcal{S}[a, b]$ is the collection of step functions on $[a, b]$, and $\mathcal{S}'[a, b]$ is the subcollection of those step functions ψ that take rational values and for which there is a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ with ψ constant on (x_{k-1}, x_k) for $1 \leq k \leq n$ and x_k rational for $1 \leq k \leq n - 1$. There are four points left to the reader; we verify them in turn.

Detail 1: $\mathcal{S}'[a, b]$ is dense in $\mathcal{S}[a, b]$ with respect to the $L^p[a, b]$ norm.

Let ψ belong to $\mathcal{S}[a, b]$ and let $\epsilon > 0$. There is a partition $a = y_0 < y_1 < \dots < y_n = b$ of $[a, b]$ and real numbers $c_1, \dots, c_n$ with $\psi = c_k$ on (y_{k-1}, y_k) for $1 \leq k \leq n$. (The finitely many values $\psi(y_k)$ are irrelevant, since a change of a function on a set of measure zero changes neither its membership in $L^p[a, b]$ nor its L^p norm.)

First replace the values by rational ones. By the density of $\mathbb{Q}$ in $\mathbb{R}$, choose rational numbers $d_1, \dots, d_n$

with

$$|d_k - c_k| < \frac{\epsilon}{2(b-a)^{1/p} + 2} \quad \text{for } 1 \leq k \leq n,$$

and let φ be the step function equal to d_k on (y_{k-1}, y_k) and equal to 0 at the partition points. Then $|\varphi - \psi| \leq \epsilon/(2(b-a)^{1/p} + 2)$ except at finitely many points, so by the monotonicity of integration

$$\|\varphi - \psi\|_p \leq \frac{\epsilon}{2(b-a)^{1/p} + 2} \cdot (b-a)^{1/p} < \frac{\epsilon}{2}.$$

Now replace the interior partition points by rational ones. Let $M = \max_{1 \leq k \leq n} |d_k|$ and let $\delta = \frac{1}{2} \min_{1 \leq k \leq n} (y_k - y_{k-1}) > 0$. Choose a positive number

$$\eta < \min \left\{ \delta, \frac{1}{2n} \left[\frac{\epsilon}{2(2M+1)} \right]^p \right\},$$

and, again by the density of the rationals, choose rational numbers $x_1, \dots, x_{n-1}$ with $|x_k - y_k| < \eta$ for $1 \leq k \leq n-1$; set $x_0 = a$ and $x_n = b$. Since $\eta < \delta$, we still have $a = x_0 < x_1 < \dots < x_n = b$, so $P = \{x_0, \dots, x_n\}$ is a partition of $[a, b]$. Let φ' be the function equal to d_k on (x_{k-1}, x_k) for $1 \leq k \leq n$ and equal to 0 at the points of P . Then φ' takes rational values, is constant on each (x_{k-1}, x_k) , and its interior partition points $x_1, \dots, x_{n-1}$ are rational; hence φ' belongs to $\mathcal{S}'[a, b]$.

The functions φ and φ' agree except on the set

$$D = \{x_0, \dots, x_n, y_0, \dots, y_n\} \cup \bigcup_{k=1}^{n-1} [\min\{x_k, y_k\}, \max\{x_k, y_k\}],$$

whose measure is at most $(n-1) \cdot \eta \leq n\eta$, by the countable (here finite) subadditivity and monotonicity of Lebesgue measure. On D we have $|\varphi - \varphi'| \leq 2M$. Therefore

$$\|\varphi - \varphi'\|_p = \left[\int_D |\varphi - \varphi'|^p \right]^{1/p} \leq 2M \cdot [m(D)]^{1/p} \leq (2M+1) (n\eta)^{1/p} < \frac{\epsilon}{2},$$

by the choice of η . By Minkowski's Inequality,

$$\|\varphi' - \psi\|_p \leq \|\varphi' - \varphi\|_p + \|\varphi - \psi\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ψ in $\mathcal{S}[a, b]$ and $\epsilon > 0$ were arbitrary, $\mathcal{S}'[a, b]$ is dense in $\mathcal{S}[a, b]$ with respect to the L^p norm. Combined with Proposition 10, which tells us $\mathcal{S}[a, b]$ is dense in $L^p[a, b]$, assertion (13) applied to the inclusions $\mathcal{S}'[a, b] \subseteq \mathcal{S}[a, b] \subseteq L^p[a, b]$ shows that $\mathcal{S}'[a, b]$ is dense in $L^p[a, b]$.

Detail 2: $\mathcal{S}'[a, b]$ is a countable set.

We argue directly, without identifying functions that agree almost everywhere. Fix a natural number n and let $\mathcal{S}'_n[a, b]$ be the collection of those ψ in $\mathcal{S}'[a, b]$ admitting such a partition $P = \{x_0, \dots, x_n\}$ into exactly n subintervals. Such a ψ is completely determined by the rational interior partition points $x_1 < \dots < x_{n-1}$ in (a, b) , by the rational values $d_1, \dots, d_n$ that ψ takes on the successive open intervals (x_{k-1}, x_k) , and by the rational values $e_0, \dots, e_n$ that ψ takes at the points of P ; the endpoints $x_0 = a$ and $x_n = b$ are fixed. (A given ψ may admit more than one such partition; that is harmless, since all we need is a map onto $\mathcal{S}'_n[a, b]$.) Hence

$$(x_1, \dots, x_{n-1}, d_1, \dots, d_n, e_0, \dots, e_n) \mapsto \psi$$

is a mapping of a subset A_n of $\mathbb{Q}^{3n}$ onto $\mathcal{S}'_n[a, b]$. As in Problem 38, since $\mathbb{Q}$ is countably infinite by Corollary 4 of Chapter 1, $\mathbb{Q}^{3n}$ is equipotent to $\mathbb{N}^{3n}$, which is countable by that same corollary; the subset A_n is countable by Theorem 3 of Chapter 1; and therefore, by Theorem 5 of Chapter 1, $\mathcal{S}'_n[a, b]$ is countable. Since

$$\mathcal{S}'[a, b] = \bigcup_{n=1}^{\infty} \mathcal{S}'_n[a, b]$$

is the union of a countable collection of countable sets, Corollary 6 of Chapter 1 tells us $\mathcal{S}'[a, b]$ is countable.

Detail 3: $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is a countable subset of $L^p(\mathbb{R})$ that is dense in $L^p(\mathbb{R})$.

Recall $\mathcal{F}_n$ is the collection of functions on $\mathbb{R}$ that vanish outside $[-n, n]$ and whose restrictions to $[-n, n]$ belong to $\mathcal{S}'[-n, n]$. Each such function is bounded and vanishes outside a set of finite measure, so it belongs to $L^p(\mathbb{R})$. By Detail 2, $\mathcal{S}'[-n, n]$ is countable, and $f \mapsto f|_{[-n, n]}$ is a one-to-one correspondence of $\mathcal{F}_n$ with $\mathcal{S}'[-n, n]$; hence each $\mathcal{F}_n$ is countable, and by Corollary 6 of Chapter 1 so is $\mathcal{F}$.

To see that $\mathcal{F}$ is dense, let f belong to $L^p(\mathbb{R})$ and let $\epsilon > 0$. The sequence of nonnegative measurable functions $\{|f|^p \cdot \chi_{[-n, n]}\}$ is increasing and converges pointwise on $\mathbb{R}$ to $|f|^p$, so the Monotone Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{[-n, n]} |f|^p = \int_{\mathbb{R}} |f|^p < \infty.$$

Since $\int_{\mathbb{R}} |f|^p$ is finite, we may subtract and conclude there is a natural number n for which

$$\int_{\mathbb{R} \setminus [-n, n]} |f|^p = \int_{\mathbb{R}} |f|^p - \int_{[-n, n]} |f|^p < \left(\frac{\epsilon}{2}\right)^p.$$

Put $g = f \cdot \chi_{[-n, n]}$. Then $\|f - g\|_{L^p(\mathbb{R})} = \left[\int_{\mathbb{R} \setminus [-n, n]} |f|^p\right]^{1/p} < \epsilon/2$. The restriction of g to $[-n, n]$ belongs to $L^p[-n, n]$, so by Detail 1 there is a function ψ in $\mathcal{S}'[-n, n]$ with $\|g - \psi\|_{L^p[-n, n]} < \epsilon/2$. Extend ψ by 0 outside $[-n, n]$; the extension, still denoted ψ , belongs to $\mathcal{F}_n \subseteq \mathcal{F}$, and since g and ψ both vanish outside $[-n, n]$,

$$\|g - \psi\|_{L^p(\mathbb{R})} = \|g - \psi\|_{L^p[-n, n]} < \frac{\epsilon}{2}.$$

By Minkowski's Inequality, $\|f - \psi\|_{L^p(\mathbb{R})} \leq \|f - g\|_{L^p(\mathbb{R})} + \|g - \psi\|_{L^p(\mathbb{R})} < \epsilon$. Hence $\mathcal{F}$ is dense in $L^p(\mathbb{R})$, and $L^p(\mathbb{R})$ is separable.

Detail 4: passage from $\mathbb{R}$ to a general measurable set E .

Let E be measurable and set $\mathcal{F}_E = \{f|_E : f \in \mathcal{F}\}$. This is the image of the countable set $\mathcal{F}$ under the restriction map, hence countable by Theorem 5 of Chapter 1, and each $f|_E$ belongs to $L^p(E)$ because $\int_E |f|^p \leq \int_{\mathbb{R}} |f|^p < \infty$. Let h belong to $L^p(E)$ and let $\epsilon > 0$. Define $\hat{h}$ on $\mathbb{R}$ by $\hat{h} = h$ on E and $\hat{h} = 0$ on $\mathbb{R} \setminus E$. Then $\hat{h}$ is measurable on $\mathbb{R}$ and $\int_{\mathbb{R}} |\hat{h}|^p = \int_E |h|^p < \infty$, so $\hat{h}$ belongs to $L^p(\mathbb{R})$. By Detail 3 there is f in $\mathcal{F}$ with $\|\hat{h} - f\|_{L^p(\mathbb{R})} < \epsilon$. By the monotonicity of integration over the subset E of $\mathbb{R}$,

$$\|h - f|_E\|_{L^p(E)} = \left[\int_E |\hat{h} - f|^p\right]^{1/p} \leq \left[\int_{\mathbb{R}} |\hat{h} - f|^p\right]^{1/p} < \epsilon.$$

Thus $\mathcal{F}_E$ is a countable dense subset of $L^p(E)$, and therefore $L^p(E)$ is separable. This completes the verification of the details in the proof of Theorem 11.

Problem 7.41. Let E be a measurable set of finite measure and $1 \leq p_1 < p_2 < \infty$. Consider the linear space $L^{p_2}(E)$ normed by $\|\cdot\|_{p_1}$. Is this normed linear space a Banach space?

Solution. The answer is no, unless E has measure zero (in which case $L^{p_2}(E)$ consists of the single element 0 and is trivially complete). We assume $m(E) > 0$ and exhibit a sequence in $L^{p_2}(E)$ that is Cauchy with respect to $\|\cdot\|_{p_1}$ but converges in $\|\cdot\|_{p_1}$ to no member of $L^{p_2}(E)$.

First note the setting is legitimate: since $m(E) < \infty$ and $p_1 < p_2$, Corollary 3 of Chapter 7 gives $L^{p_2}(E) \subseteq L^{p_1}(E)$, so $\|\cdot\|_{p_1}$ is finite on $L^{p_2}(E)$; and it is a norm there, since it is a norm on $L^{p_1}(E)$ and $L^{p_2}(E)$ is a linear subspace of $L^{p_1}(E)$.

Step 1: a function in $L^{p_1}(E)$ that fails to belong to $L^{p_2}(E)$.

Define $\mu(t) = m(E \cap (-\infty, t])$ for $t \in \mathbb{R}$. Since $m(E) < \infty$, μ is real-valued, nondecreasing, and continuous: indeed, for $s < t$,

$$|\mu(t) - \mu(s)| = m(E \cap (s, t]) \leq t - s,$$

by the monotonicity of Lebesgue measure. Moreover $\lim_{t \rightarrow -\infty} \mu(t) = 0$ and $\lim_{t \rightarrow \infty} \mu(t) = m(E)$, by

the continuity of measure. Hence, by the Intermediate Value Theorem, μ takes every value in $(0, m(E))$. For each natural number k choose c_k with $\mu(c_k) = m(E) \cdot 2^{-k}$, and put $A_k = E \cap (-\infty, c_k]$, so that $m(A_k) = m(E) \cdot 2^{-k}$. Since μ is nondecreasing and $\mu(c_{k+1}) < \mu(c_k)$, we have $c_{k+1} \leq c_k$ and therefore $A_{k+1} \subseteq A_k$. Define

$$E_k = A_k \setminus A_{k+1}, \quad m_k := m(E_k) = m(A_k) - m(A_{k+1}) = m(E) \cdot 2^{-(k+1)} > 0,$$

using the excision property of measure, valid since $m(A_{k+1}) < \infty$. The sets $\{E_k\}_{k=1}^{\infty}$ are measurable, pairwise disjoint, and contained in E .

Now set, for each k ,

$$a_k = \left[\frac{1}{k^2 m_k} \right]^{1/p_1} > 0, \quad f = \sum_{k=1}^{\infty} a_k \chi_{E_k}.$$

The function f is nonnegative, real-valued and measurable on E : the sets E_k are disjoint, so f takes the value a_k on E_k and the value 0 on $E \setminus \bigcup_k E_k$, and it is the pointwise limit on E of the increasing sequence of simple functions $\sum_{k=1}^n a_k \chi_{E_k}$. For any $r > 0$ the sequence $\left\{ \sum_{k=1}^n a_k^r \chi_{E_k} \right\}$ increases pointwise on E to $|f|^r$, so the Monotone Convergence Theorem, together with the additivity of integration over the finite disjoint collection $\{E_k\}_{k=1}^n$ and the vanishing of f off $\bigcup_k E_k$, gives

$$\int_E |f|^r = \sum_{k=1}^{\infty} a_k^r m_k.$$

For $r = p_1$ we get

$$\int_E |f|^{p_1} = \sum_{k=1}^{\infty} \frac{1}{k^2 m_k} \cdot m_k = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

so f belongs to $L^{p_1}(E)$. For $r = p_2$, writing $\alpha = p_2/p_1 > 1$ and $m_k = m(E)2^{-(k+1)}$,

$$a_k^{p_2} m_k = [k^2 m_k]^{-\alpha} m_k = k^{-2\alpha} m_k^{1-\alpha} = k^{-2\alpha} \left[\frac{m(E)}{2^{k+1}} \right]^{1-\alpha} = C \cdot \frac{2^{k(\alpha-1)}}{k^{2\alpha}},$$

where $C = [m(E)/2]^{1-\alpha} > 0$. Since $\alpha-1 > 0$, the numerator grows geometrically while the denominator grows only polynomially, so $a_k^{p_2} m_k \rightarrow \infty$; in particular the series diverges and

$$\int_E |f|^{p_2} = \infty.$$

Thus f belongs to $L^{p_1}(E)$ but not to $L^{p_2}(E)$.

Step 2: a Cauchy sequence with no limit in the space.

Let $f_n = \sum_{k=1}^n a_k \chi_{E_k}$. Each f_n is a simple function on the finite measure set E , hence bounded and measurable, so f_n belongs to $L^{p_2}(E)$. Now $f - f_n = \sum_{k>n} a_k \chi_{E_k}$, which is a function of exactly the same form as f with the first n coefficients deleted; so the computation just performed, applied to it with $r = p_1$, gives

$$\|f - f_n\|_{p_1}^{p_1} = \int_E \left| \sum_{k>n} a_k \chi_{E_k} \right|^{p_1} = \sum_{k>n} \frac{1}{k^2} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

this being the tail of the convergent series $\sum 1/k^2$. Hence $\{f_n\} \rightarrow f$ in $L^{p_1}(E)$, and in particular, by Minkowski's Inequality,

$$\|f_n - f_m\|_{p_1} \leq \|f_n - f\|_{p_1} + \|f - f_m\|_{p_1} \rightarrow 0,$$

so $\{f_n\}$ is a Cauchy sequence in $L^{p_2}(E)$ with respect to the norm $\|\cdot\|_{p_1}$.

Suppose this sequence converged, with respect to $\|\cdot\|_{p_1}$, to some g in $L^{p_2}(E)$. Since $L^{p_2}(E) \subseteq L^{p_1}(E)$, both f and g lie in $L^{p_1}(E)$, and by Minkowski's Inequality

$$\|f - g\|_{p_1} \leq \|f - f_n\|_{p_1} + \|f_n - g\|_{p_1} \rightarrow 0,$$

so $\|f - g\|_{p_1} = 0$ and therefore, by Proposition 9 of Chapter 4, $f = g$ almost everywhere on E . But then $\int_E |f|^{p_2} = \int_E |g|^{p_2} < \infty$, contradicting Step 1.

Therefore $\{f_n\}$ is a Cauchy sequence in the normed linear space $(L^{p_2}(E), \|\cdot\|_{p_2})$ which does not converge in that space, and the space is not a Banach space.

The underlying reason is structural: Step 1 shows $L^{p_2}(E)$ is a proper subspace of $L^{p_1}(E)$, and the argument above shows it is a dense one (the simple functions are dense in $L^{p_1}(E)$ by Proposition 9 of Chapter 7 and, E having finite measure, they all belong to $L^{p_2}(E)$). By Problem 43, a subspace of a Banach space that is dense and complete in the inherited norm must be the whole space; since $L^{p_1}(E)$ is a Banach space by the Riesz-Fischer Theorem and $L^{p_2}(E)$ is a proper dense subspace of it, $L^{p_2}(E)$ cannot be complete in $\|\cdot\|_{p_1}$.

Problem 7.42. Exhibit a measurable set E for which $L^\infty(E)$ is separable. Show that $L^\infty(E)$ is not separable if the set E contains a nondegenerate interval.

Solution. Part 1: a measurable set E for which $L^\infty(E)$ is separable.

Take any measurable set of measure zero, for instance $E = \{0\}$, or $E = \mathbb{Q}$, or $E = \emptyset$. Let f be any measurable function on E . Since $m(E) = 0$, the inequality $|f| \leq 0$ holds almost everywhere on E (the exceptional set, being a subset of E , has measure zero). Thus 0 is an essential upper bound for $|f|$ on E , and therefore

$$\|f\|_\infty = \inf\{M \geq 0 : |f| \leq M \text{ a.e. on } E\} = 0.$$

Since the members of $L^\infty(E)$ are identified when they agree almost everywhere on E , every measurable f on E equals 0 as an element of $L^\infty(E)$; that is, $L^\infty(E) = \{0\}$. The one-element set $\{0\}$ is a countable subset of $L^\infty(E)$ that is dense in it, since for the only g in $L^\infty(E)$, namely $g = 0$, and any $\epsilon > 0$ we have $\|0 - g\|_\infty = 0 < \epsilon$. Hence $L^\infty(E)$ is separable.

Part 2: $L^\infty(E)$ is not separable when E contains a nondegenerate interval.

Suppose E contains a nondegenerate closed, bounded interval $[a, b]$, so $a < b$. (If E contains a nondegenerate interval of any kind, it contains such a closed, bounded one.) We argue by contradiction, following the pattern of the Example after Theorem 11.

Assume $L^\infty(E)$ is separable, and let $\{f_n\}_{n=1}^\infty$ be a countable subset of $L^\infty(E)$ that is dense in $L^\infty(E)$. For each x in $[a, b]$ define g_x on E to be the characteristic function of $[a, x]$, that is, $g_x = \chi_{[a, x]}$ restricted to E . Each g_x is measurable and bounded on E , so g_x belongs to $L^\infty(E)$. By the density of $\{f_n\}$, applied to g_x and $\epsilon = 1/2$, we may select a natural number $\eta(x)$ for which

$$\|g_x - f_{\eta(x)}\|_\infty < \frac{1}{2}.$$

We claim that for $a \leq x_1 < x_2 \leq b$,

$$\|g_{x_1} - g_{x_2}\|_\infty = 1.$$

Indeed, $g_{x_2} - g_{x_1}$ is the characteristic function of the set $(x_1, x_2]$, which is contained in E because $[a, b] \subseteq E$. Since $m((x_1, x_2]) = x_2 - x_1 > 0$, no number $M < 1$ can be an essential upper bound: the set where $|g_{x_2} - g_{x_1}| = 1 > M$ is $(x_1, x_2]$, of positive measure. On the other hand $|g_{x_2} - g_{x_1}| \leq 1$ everywhere on E . Hence the essential supremum equals 1, proving the claim.

Now suppose $\eta(x_1) = \eta(x_2) = k$ for some $a \leq x_1 < x_2 \leq b$. Then, by the triangle inequality for the norm $\|\cdot\|_\infty$,

$$1 = \|g_{x_1} - g_{x_2}\|_\infty \leq \|g_{x_1} - f_k\|_\infty + \|f_k - g_{x_2}\|_\infty < \frac{1}{2} + \frac{1}{2} = 1,$$

a contradiction. Therefore η is a one-to-one mapping of $[a, b]$ into $\mathbb{N}$.

Consequently $[a, b]$ is equipotent to a subset of $\mathbb{N}$, and so, by Theorem 3 of Chapter 1, $[a, b]$ is countable. But $[a, b]$ is a nondegenerate interval of real numbers and hence is uncountable, by Theorem 7 of Chapter 1. From this contradiction we conclude that no countable subset of $L^\infty(E)$ is dense in $L^\infty(E)$; that is, $L^\infty(E)$ is not separable.

Remark: containing an interval is not needed; $m(E) > 0$ suffices. If $m(E) > 0$, choose n with $m(E \cap [-n, n]) > 0$ and set $E_0 = E \cap [-n, n]$, so $0 < m(E_0) < \infty$. As in Problem 41, the function

$\mu(t) = m(E_0 \cap (-\infty, t])$ is continuous, nondecreasing, and takes every value in $(0, m(E_0))$; for each such t pick c_t with $\mu(c_t) = t$ and put $A_t = E_0 \cap (-\infty, c_t]$. For $s < t$ we have $A_s \subseteq A_t$ and $m(A_t \setminus A_s) = t - s > 0$, so exactly as above $\|\chi_{A_s} - \chi_{A_t}\|_\infty = 1$. Since $(0, m(E_0))$ is a nondegenerate interval, hence uncountable by Theorem 7 of Chapter 1, the same selection argument produces a one-to-one map of an uncountable set into $\mathbb{N}$, a contradiction. Thus $L^\infty(E)$ is separable if and only if $m(E) = 0$.

7.7 Problems 7.43–7.49

Problem 7.43. Suppose that X is a Banach space with norm $\|\cdot\|$. Let X_0 be a dense subspace of X . Assume that X_0 , when normed by the norm it inherits from X , is also a Banach space. Prove that $X = X_0$.

Solution. Since $X_0 \subseteq X$, we need only prove the reverse inclusion $X \subseteq X_0$.

Let x belong to X . By the definition of denseness in Section 7.4, for each natural number n there is a point x_n in X_0 with $\|x_n - x\| < 1/n$. (This is the implication recorded in Problem 36.) Thus there is a sequence $\{x_n\}$ in X_0 for which

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{in } X, \quad \text{that is,} \quad \lim_{n \rightarrow \infty} \|x_n - x\| = 0.$$

We claim $\{x_n\}$ is a Cauchy sequence in X . Indeed, let $\epsilon > 0$. Choose an index N for which $\|x_n - x\| < \epsilon/2$ for all $n \geq N$. Then for $n, m \geq N$, the triangle inequality gives

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Now every x_n belongs to X_0 , and the norm on X_0 is the restriction to X_0 of the norm on X . Therefore $\{x_n\}$ is a Cauchy sequence in the normed linear space X_0 . By assumption X_0 is a Banach space, that is, X_0 is complete. Hence there is a point x_0 in X_0 for which $\{x_n\} \rightarrow x_0$ with respect to the norm of X_0 . Again because the norm of X_0 is inherited from X , this means

$$\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0,$$

so that $\{x_n\} \rightarrow x_0$ in X as well.

Finally, limits with respect to a norm are unique: for every index n ,

$$\|x - x_0\| \leq \|x - x_n\| + \|x_n - x_0\|,$$

and the right-hand side tends to 0 as $n \rightarrow \infty$. Since $\|x - x_0\|$ is a nonnegative number that is no larger than a quantity tending to 0, we conclude $\|x - x_0\| = 0$ and therefore $x = x_0$.

Thus $x = x_0$ belongs to X_0 . Since x was an arbitrary point of X , we have $X \subseteq X_0$, and hence $X = X_0$. Observe the essential point: a complete subspace of a normed linear space is closed, and a dense closed subset of X is all of X .

Problem 7.44. For $1 \leq p < \infty$, show that the sequence space ℓ^p is separable. Show that the collection of sets of natural numbers is uncountable and conclude that ℓ^∞ is not separable.

Solution. We first prove the separability of ℓ^p for $1 \leq p < \infty$. For a natural number n , let

$$\mathcal{D}_n = \{(q_1, \dots, q_n, 0, 0, \dots) : q_1, \dots, q_n \text{ rational}\},$$

the set of sequences whose terms beyond the n -th vanish and whose first n terms are rational. Each such sequence has only finitely many nonzero terms, so it certainly belongs to ℓ^p . The map $(q_1, \dots, q_n) \mapsto (q_1, \dots, q_n, 0, 0, \dots)$ is a one-to-one correspondence of $\mathbb{Q}^n$ with $\mathcal{D}_n$, and a finite Cartesian product of countable sets is countable; hence each $\mathcal{D}_n$ is countable. Therefore

$$\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$$

is a countable union of countable sets and so is countable.

We claim $\mathcal{D}$ is dense in ℓ^p . Let $a = \{a_k\}_{k=1}^\infty$ belong to ℓ^p and let $\epsilon > 0$. Since the series $\sum_{k=1}^\infty |a_k|^p$ converges, its tails tend to 0; choose a natural number N for which

$$\sum_{k=N+1}^\infty |a_k|^p < (\epsilon/2)^p.$$

Let $a^{(N)} = (a_1, \dots, a_N, 0, 0, \dots)$. Then

$$\|a - a^{(N)}\|_p = \left(\sum_{k=N+1}^\infty |a_k|^p \right)^{1/p} < \epsilon/2.$$

By the density of the rational numbers in $\mathbb{R}$, for each $1 \leq k \leq N$ choose a rational q_k with

$$|a_k - q_k|^p < \frac{(\epsilon/2)^p}{N},$$

and set $q = (q_1, \dots, q_N, 0, 0, \dots)$, a member of $\mathcal{D}_N \subseteq \mathcal{D}$. Then

$$\|a^{(N)} - q\|_p = \left(\sum_{k=1}^N |a_k - q_k|^p \right)^{1/p} < \left(N \cdot \frac{(\epsilon/2)^p}{N} \right)^{1/p} = \epsilon/2.$$

By the Minkowski Inequality (that is, the triangle inequality for the ℓ^p norm),

$$\|a - q\|_p \leq \|a - a^{(N)}\|_p + \|a^{(N)} - q\|_p < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\mathcal{D}$ is a countable dense subset of ℓ^p , and ℓ^p is separable.

We next show that the collection of sets of natural numbers is uncountable. Let $\mathcal{P}(\mathbb{N})$ denote the collection of all subsets of $\mathbb{N}$. It is not finite, since the singletons $\{n\}$, $n \in \mathbb{N}$, are infinitely many distinct members. Suppose, to obtain a contradiction, that $\mathcal{P}(\mathbb{N})$ is countable. Then it may be enumerated as $\{A_n\}_{n=1}^\infty$. Define

$$A = \{n \in \mathbb{N} : n \notin A_n\}.$$

Then A is a subset of $\mathbb{N}$, so $A = A_m$ for some natural number m . Ask whether m belongs to A . If $m \in A$, then by the definition of A we have $m \notin A_m = A$, a contradiction. If $m \notin A = A_m$, then by the definition of A we have $m \in A$, again a contradiction. This contradiction shows that $\mathcal{P}(\mathbb{N})$ is uncountable.

Finally we show that ℓ^∞ is not separable. For $A \subseteq \mathbb{N}$, let χ_A denote the sequence whose k -th term is 1 if $k \in A$ and 0 if $k \notin A$. Each χ_A is a bounded sequence, so χ_A belongs to ℓ^∞ . If A and B are distinct subsets of $\mathbb{N}$, there is an index k belonging to one of them and not the other, and for that index the k -th terms of χ_A and χ_B differ by 1; since all terms of $\chi_A - \chi_B$ have absolute value at most 1,

$$\|\chi_A - \chi_B\|_\infty = 1 \quad \text{if } A \neq B.$$

Suppose, to obtain a contradiction, that ℓ^∞ is separable, and let $\{x_n\}_{n=1}^\infty$ be a countable dense subset of ℓ^∞ . For each subset A of $\mathbb{N}$, by density there is a natural number, which we call $\eta(A)$, for which

$$\|\chi_A - x_{\eta(A)}\|_\infty < 1/2.$$

We claim η is one-to-one. Indeed, suppose $A \neq B$ but $\eta(A) = \eta(B) = m$. Then, by the triangle inequality,

$$1 = \|\chi_A - \chi_B\|_\infty \leq \|\chi_A - x_m\|_\infty + \|x_m - \chi_B\|_\infty < 1/2 + 1/2 = 1,$$

which is a contradiction. Hence η is a one-to-one mapping of $\mathcal{P}(\mathbb{N})$ into $\mathbb{N}$, so $\mathcal{P}(\mathbb{N})$ is equipotent to a subset of $\mathbb{N}$ and therefore countable. This contradicts the uncountability of $\mathcal{P}(\mathbb{N})$ just established. Therefore ℓ^∞ is not separable.

This argument is the sequence-space analogue of the example in Section 7.4 showing that $L^\infty[a, b]$ is not separable.

Problem 7.45. Prove Theorem 12. [Theorem 12: Let E be a measurable set and $1 \leq p < \infty$. Then $C_c(E)$ is dense in $L^p(E)$. Here $C_c(E)$ denotes the linear space of continuous real-valued functions on E that vanish outside a bounded set.]

Solution. Let f belong to $L^p(E)$ and let $\epsilon > 0$. We must produce a function g in $C_c(E)$ for which $\|f - g\|_{L^p(E)} < \epsilon$.

Step 1: reduction to $E = \mathbb{R}$. Extend f to a function $\tilde{f}$ on all of $\mathbb{R}$ by setting $\tilde{f} = 0$ on $\mathbb{R} \setminus E$. Then $\tilde{f}$ is measurable and $\int_{\mathbb{R}} |\tilde{f}|^p = \int_E |f|^p < \infty$, so $\tilde{f}$ belongs to $L^p(\mathbb{R})$. If g is a continuous function on $\mathbb{R}$ that vanishes outside a bounded set, then its restriction to E is continuous on E and vanishes outside a bounded set, so this restriction belongs to $C_c(E)$; moreover, since $|f - g|^p \leq |\tilde{f} - g|^p$ on E and $E \subseteq \mathbb{R}$,

$$\|f - g\|_{L^p(E)} \leq \|\tilde{f} - g\|_{L^p(\mathbb{R})}.$$

So it suffices to approximate $\tilde{f}$, in the $L^p(\mathbb{R})$ norm, by a continuous function on $\mathbb{R}$ that vanishes outside a bounded set. Rename $\tilde{f}$ as f .

Step 2: truncation to a bounded interval. For each natural number n , $|f|^p \cdot \chi_{[-n,n]}$ is a nonnegative measurable function and these functions increase pointwise on $\mathbb{R}$ to $|f|^p$. By the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_{[-n,n]} |f|^p = \int_{\mathbb{R}} |f|^p,$$

and since this common value is finite we may choose a natural number n for which

$$\int_{\mathbb{R} \setminus [-n,n]} |f|^p = \int_{\mathbb{R}} |f|^p - \int_{[-n,n]} |f|^p < (\epsilon/3)^p.$$

Set $a = -n$, $b = n$, and $f_1 = f \cdot \chi_{[a,b]}$. Then f_1 belongs to $L^p(\mathbb{R})$ and

$$\|f - f_1\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R} \setminus [a,b]} |f|^p \right)^{1/p} < \epsilon/3.$$

Step 3: approximation by a step function. The restriction of f_1 to $[a,b]$ belongs to $L^p[a,b]$. By Proposition 10, the step functions on $[a,b]$ are dense in $L^p[a,b]$, so there is a step function ψ on $[a,b]$ for which

$$\|f_1 - \psi\|_{L^p[a,b]} < \epsilon/3.$$

Extend ψ to $\mathbb{R}$ by setting it equal to 0 outside $[a,b]$. Since f_1 also vanishes outside $[a,b]$, we then have $\|f_1 - \psi\|_{L^p(\mathbb{R})} < \epsilon/3$. By the definition of a step function there is a partition $a = x_0 < x_1 < \dots < x_m = b$ of $[a,b]$ and numbers $c_1, \dots, c_m$ for which $\psi = c_k$ on the open interval (x_{k-1}, x_k) for $1 \leq k \leq m$. The finitely many partition points form a set of measure zero, so

$$\psi = \sum_{k=1}^m c_k \cdot \chi_{(x_{k-1}, x_k)} \quad \text{a.e. on } \mathbb{R},$$

and hence ψ and this linear combination are the same member of $L^p(\mathbb{R})$.

Step 4: approximation of a characteristic function of an interval by a continuous function. Let (c, d) be a bounded open interval and let $\delta > 0$ with $2\delta < d - c$. Define h_δ on $\mathbb{R}$ by

$$h_\delta(x) = \begin{cases} 0, & x \leq c \text{ or } x \geq d, \\ 1, & c + \delta \leq x \leq d - \delta, \\ (x - c)/\delta, & c < x < c + \delta, \\ (d - x)/\delta, & d - \delta < x < d. \end{cases}$$

This function is piecewise linear and the definitions agree at the junction points, so h_δ is continuous on $\mathbb{R}$; it vanishes outside the bounded set $[c, d]$. Moreover $0 \leq h_\delta \leq 1$ and $h_\delta = \chi_{(c,d)}$ except on the

two intervals $(c, c + \delta)$ and $(d - \delta, d)$ together with the two points c, d . Therefore $|\chi_{(c,d)} - h_\delta| \leq 1$ and this difference vanishes outside a set of measure 2δ , so

$$\|\chi_{(c,d)} - h_\delta\|_{L^p(\mathbb{R})} \leq (2\delta)^{1/p},$$

which tends to 0 as $\delta \rightarrow 0^+$ since $p < \infty$. Hence $\chi_{(c,d)}$ may be approximated in $L^p(\mathbb{R})$, to within any prescribed positive tolerance, by a continuous function on $\mathbb{R}$ that vanishes outside a bounded set.

Step 5: conclusion. Apply Step 4 to each of the intervals (x_{k-1}, x_k) , $1 \leq k \leq m$: choose continuous functions g_k on $\mathbb{R}$, each vanishing outside a bounded set, for which

$$\|\chi_{(x_{k-1}, x_k)} - g_k\|_{L^p(\mathbb{R})} < \frac{\epsilon}{3m(1 + |c_k|)}.$$

Define $g = \sum_{k=1}^m c_k g_k$. As a finite linear combination of continuous functions vanishing outside bounded sets, g is continuous on $\mathbb{R}$ and vanishes outside a bounded set. By the Minkowski Inequality,

$$\|\psi - g\|_{L^p(\mathbb{R})} \leq \sum_{k=1}^m |c_k| \cdot \|\chi_{(x_{k-1}, x_k)} - g_k\|_{L^p(\mathbb{R})} < \sum_{k=1}^m \frac{|c_k| \epsilon}{3m(1 + |c_k|)} \leq \epsilon/3.$$

Therefore, by the Minkowski Inequality once more,

$$\|f - g\|_{L^p(\mathbb{R})} \leq \|f - f_1\|_{L^p(\mathbb{R})} + \|f_1 - \psi\|_{L^p(\mathbb{R})} + \|\psi - g\|_{L^p(\mathbb{R})} < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

By Step 1, the restriction of g to E belongs to $C_c(E)$ and satisfies $\|f - g\|_{L^p(E)} < \epsilon$. Since f in $L^p(E)$ and $\epsilon > 0$ were arbitrary, $C_c(E)$ is dense in $L^p(E)$.

Problem 7.46. Show that for $1 < p < \infty$ and any two numbers a and b ,

$$\left| \operatorname{sgn}(a) \cdot |a|^{1/p} - \operatorname{sgn}(b) \cdot |b|^{1/p} \right|^p \leq 2^p \cdot |a - b|.$$

Solution. Write $s = 1/p$, so that $0 < s < 1$, and define

$$\varphi(t) = \operatorname{sgn}(t) \cdot |t|^s \quad \text{for } t \in \mathbb{R}.$$

The asserted inequality is $|\varphi(a) - \varphi(b)|^p \leq 2^p |a - b|$, and since both sides are nonnegative and $t \mapsto t^{1/p}$ is increasing on $[0, \infty)$, it is equivalent to

$$|\varphi(a) - \varphi(b)| \leq 2 |a - b|^s.$$

Call this last inequality (*).

We first record a lemma.

Lemma (subadditivity of $t \mapsto t^s$ for $0 < s < 1$): for all $u, v \geq 0$,

$$(u + v)^s \leq u^s + v^s.$$

Indeed, if $u = v = 0$ this is trivial. Otherwise let $T = u + v > 0$. Since $0 \leq u/T \leq 1$ and $0 < s < 1$, we have $(u/T)^s \geq u/T$, because for $0 \leq x \leq 1$ raising to a power smaller than 1 increases the number: $x^s = x \cdot x^{s-1} \geq x$ as $x^{s-1} \geq 1$ when $0 < x \leq 1$ and $s - 1 < 0$ (and the inequality is clear when $x = 0$). Likewise $(v/T)^s \geq v/T$. Adding,

$$\frac{u^s + v^s}{T^s} \geq \frac{u}{T} + \frac{v}{T} = 1,$$

which is the assertion.

Now we prove (*). Note that φ is odd, $\varphi(-t) = -\varphi(t)$, so replacing the pair (a, b) by $(-a, -b)$ changes neither side of (*); and both sides of (*) are unchanged when a and b are interchanged. We use these two symmetries to reduce to two cases.

Case 1: a and b have the same sign, where we count 0 as having either sign. After replacing (a, b) by $(-a, -b)$ if necessary we may assume $a \geq 0$ and $b \geq 0$, and after interchanging a and b if necessary we

may assume $a \geq b \geq 0$. Then $\varphi(a) = a^s$ and $\varphi(b) = b^s$, and $|a - b| = a - b$. Apply the Lemma with $u = a - b \geq 0$ and $v = b \geq 0$:

$$a^s = (u + v)^s \leq (a - b)^s + b^s,$$

so that

$$|\varphi(a) - \varphi(b)| = a^s - b^s \leq (a - b)^s = |a - b|^s \leq 2|a - b|^s.$$

Case 2: a and b have strictly opposite signs. Interchanging a and b if necessary, we may assume $a > 0 > b$. Then

$$\varphi(a) - \varphi(b) = a^s + |b|^s, \quad |a - b| = a + |b|.$$

Since $t \mapsto t^s$ is increasing on $[0, \infty)$ and both a and $|b|$ are at most $a + |b|$,

$$a^s + |b|^s \leq (a + |b|)^s + (a + |b|)^s = 2(a + |b|)^s = 2|a - b|^s.$$

Hence $|\varphi(a) - \varphi(b)| \leq 2|a - b|^s$ in this case as well.

In both cases (*) holds. Raising both sides of (*) to the power $p > 0$, which preserves the inequality between nonnegative numbers, gives

$$\left| \operatorname{sgn}(a) \cdot |a|^{1/p} - \operatorname{sgn}(b) \cdot |b|^{1/p} \right|^p \leq 2^p \left(|a - b|^{1/p} \right)^p = 2^p \cdot |a - b|,$$

which is the desired inequality.

Problem 7.47. Show that for $1 < p < \infty$ and any two numbers a and b ,

$$|\operatorname{sgn}(a) \cdot |a|^p - \operatorname{sgn}(b) \cdot |b|^p| \leq p \cdot |a - b| (|a| + |b|)^{p-1}.$$

Solution. Define

$$\psi(t) = \operatorname{sgn}(t) \cdot |t|^p \quad \text{for } t \in \mathbb{R},$$

so that $\psi(t) = t^p$ for $t \geq 0$ and $\psi(t) = -(-t)^p$ for $t \leq 0$; in particular $\psi(0) = 0$ and ψ is odd.

We claim that ψ is differentiable on all of $\mathbb{R}$ with

$$\psi'(t) = p|t|^{p-1} \quad \text{for all } t \in \mathbb{R}.$$

For $t > 0$ we have $\psi = t^p$ in a neighborhood of t , so $\psi'(t) = p t^{p-1} = p|t|^{p-1}$. For $t < 0$ we have $\psi(t) = -(-t)^p$ in a neighborhood of t , so by the chain rule

$$\psi'(t) = -p(-t)^{p-1} \cdot (-1) = p(-t)^{p-1} = p|t|^{p-1}.$$

At $t = 0$, for $h \neq 0$,

$$\frac{\psi(h) - \psi(0)}{h} = \frac{\operatorname{sgn}(h)|h|^p}{h} = \frac{|h|^p}{|h|} = |h|^{p-1},$$

where we used $\operatorname{sgn}(h)/h = 1/|h|$. Since $p > 1$, the exponent $p - 1$ is positive, so $|h|^{p-1} \rightarrow 0$ as $h \rightarrow 0$. Hence $\psi'(0) = 0 = p \cdot |0|^{p-1}$, and the claim is proved.

Now let a and b be real numbers. If $a = b$ both sides of the asserted inequality equal 0 and there is nothing to prove, so assume $a \neq b$. The function ψ is differentiable, hence continuous, on the closed interval with endpoints a and b and differentiable on its interior. By the Mean Value Theorem there is a number c strictly between a and b for which

$$\psi(a) - \psi(b) = \psi'(c) (a - b) = p|c|^{p-1} (a - b).$$

Since c lies between a and b , we have $|c| \leq \max\{|a|, |b|\} \leq |a| + |b|$. As $p - 1 > 0$, the function $t \mapsto t^{p-1}$ is increasing on $[0, \infty)$, so $|c|^{p-1} \leq (|a| + |b|)^{p-1}$. Therefore

$$|\operatorname{sgn}(a)|a|^p - \operatorname{sgn}(b)|b|^p| = |\psi(a) - \psi(b)| = p|c|^{p-1} |a - b| \leq p \cdot |a - b| (|a| + |b|)^{p-1},$$

as was to be shown.

Problem 7.48. (Mazur) Let E be a measurable set and $1 < p < \infty$. For f in $L^1(E)$, define the function $\Phi(f)$ on E by

$$\Phi(f)(x) = \operatorname{sgn}(f(x)) |f(x)|^{1/p}.$$

Show that $\Phi(f)$ belongs to $L^p(E)$. Moreover, use Problem 46 to show that

$$\|\Phi(f) - \Phi(g)\|_p \leq 2^p \cdot \|f - g\|_1 \quad \text{for all } f, g \text{ in } L^1(E).$$

From this conclude that Φ is a continuous mapping of $L^1(E)$ into $L^p(E)$ in the sense that if $\{f_n\} \rightarrow f$ in $L^1(E)$, then $\{\Phi(f_n)\} \rightarrow \Phi(f)$ in $L^p(E)$. Then show that Φ is one-to-one and its image is $L^p(E)$. Find a formula for the inverse mapping. Use the preceding problem to conclude that the inverse mapping Φ^{-1} is a continuous mapping from $L^p(E)$ to $L^1(E)$.

Solution. Throughout, let $q = p/(p-1)$ be the conjugate of p , and define the two functions on $\mathbb{R}$

$$\varphi(t) = \operatorname{sgn}(t) |t|^{1/p}, \quad \psi(t) = \operatorname{sgn}(t) |t|^p,$$

so that $\Phi(f) = \varphi \circ f$.

Both φ and ψ are continuous on $\mathbb{R}$: each is continuous on $(0, \infty)$ and on $(-\infty, 0)$ by the continuity of powers, each is odd, and each tends to 0 as $t \rightarrow 0$. Moreover φ and ψ are strictly increasing on $\mathbb{R}$: on $[0, \infty)$ the functions $t \mapsto t^{1/p}$ and $t \mapsto t^p$ are strictly increasing and vanish only at 0, and both φ and ψ are odd, so each is negative on $(-\infty, 0)$, zero at 0, positive on $(0, \infty)$, and strictly increasing on each of the two half-lines. Finally, for all t ,

$$\psi(\varphi(t)) = \operatorname{sgn}(t) \left(|t|^{1/p}\right)^p = t \quad \text{and likewise} \quad \varphi(\psi(t)) = t,$$

using $\operatorname{sgn}(\varphi(t)) = \operatorname{sgn}(t)$ and $\operatorname{sgn}(\psi(t)) = \operatorname{sgn}(t)$. So φ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ with inverse ψ .

A preliminary remark on representatives. If f belongs to $L^1(E)$ then $|f|$ is a nonnegative integrable function, and if h belongs to $L^p(E)$ then $|h|^p$ is; so by Proposition 13 of Chapter 4 each such function is finite almost everywhere on E . The formulas below involve φ and ψ , which are defined only on $\mathbb{R}$, so throughout we work with a representative that is real-valued at every point of E ; altering a function on a set of measure zero changes neither the equivalence class it names nor any of the integrals computed below, so nothing depends on the choice.

Step 1: $\Phi(f)$ belongs to $L^p(E)$. Let f be in $L^1(E)$. Since f is measurable and φ is continuous on all of $\mathbb{R}$, the composition $\Phi(f) = \varphi \circ f$ is measurable. Furthermore, for all x in E ,

$$|\Phi(f)(x)|^p = \left(|f(x)|^{1/p}\right)^p = |f(x)|,$$

and $|f|$ is integrable over E . Hence $\int_E |\Phi(f)|^p = \int_E |f| < \infty$, so $\Phi(f)$ belongs to $L^p(E)$ and in fact

$$\|\Phi(f)\|_p = \|f\|_1^{1/p}.$$

Note also that if $f = g$ a.e. on E then $\Phi(f) = \Phi(g)$ a.e. on E , so Φ is well defined as a mapping of $L^1(E)$ into $L^p(E)$.

Step 2: the Lipschitz-type estimate. Let f, g belong to $L^1(E)$. By Problem 46, applied at each point x of E with $a = f(x)$ and $b = g(x)$,

$$|\Phi(f)(x) - \Phi(g)(x)|^p \leq 2^p |f(x) - g(x)| \quad \text{for all } x \in E.$$

Integrating this inequality between nonnegative measurable functions over E and using the monotonicity and linearity of integration,

$$\|\Phi(f) - \Phi(g)\|_p^p = \int_E |\Phi(f) - \Phi(g)|^p \leq 2^p \int_E |f - g| = 2^p \cdot \|f - g\|_1.$$

Equivalently, taking p -th roots,

$$\|\Phi(f) - \Phi(g)\|_p \leq 2 \cdot \|f - g\|_1^{1/p}.$$

A remark on the inequality as printed. The displayed assertion in the statement of the problem is

$$\|\Phi(f) - \Phi(g)\|_p \leq 2^p \cdot \|f - g\|_1,$$

with no exponent on the left-hand norm, and this is not correct as it stands. Indeed, take $E = [0, 1]$, $p = 2$, $g \equiv 0$, and $f \equiv c$ for a constant c with $0 < c < 1$. Then $\|f - g\|_1 = c$ while $\Phi(f) \equiv c^{1/2}$, so $\|\Phi(f) - \Phi(g)\|_2 = c^{1/2}$. The printed inequality would assert $c^{1/2} \leq 4c$, that is, $c \geq 1/16$; taking $c = 1/100$ gives $1/10 > 4/100$, a contradiction. The failure is intrinsic and not a matter of the constant: the left side is comparable to $\|f - g\|_1^{1/p}$, which dominates any fixed multiple of $\|f - g\|_1$ as $\|f - g\|_1 \rightarrow 0$, since $1/p < 1$.

The inequality that Problem 46 actually yields is the one displayed above for the p -th power of the norm, equivalently the $1/p$ -Holder estimate

$$\|\Phi(f) - \Phi(g)\|_p \leq 2 \cdot \|f - g\|_1^{1/p},$$

and this is what is used in what follows. It serves the purpose the problem intends, namely to force the continuity of Φ ; the printed form is a misprint for $\|\Phi(f) - \Phi(g)\|_p^p \leq 2^p \cdot \|f - g\|_1$.

Step 3: Φ is continuous. Suppose $\{f_n\} \rightarrow f$ in $L^1(E)$, that is, $\|f_n - f\|_1 \rightarrow 0$. By Step 2,

$$\|\Phi(f_n) - \Phi(f)\|_p \leq 2 \|f_n - f\|_1^{1/p} \rightarrow 0,$$

since $t \mapsto t^{1/p}$ is continuous at 0 with value 0. Thus $\{\Phi(f_n)\} \rightarrow \Phi(f)$ in $L^p(E)$, so Φ is continuous.

Step 4: Φ is one-to-one. Suppose f, g belong to $L^1(E)$ and $\Phi(f) = \Phi(g)$ in $L^p(E)$, that is, $\varphi(f(x)) = \varphi(g(x))$ for almost all x in E . Since φ is one-to-one on $\mathbb{R}$, it follows that $f(x) = g(x)$ for almost all x in E , that is, $f = g$ in $L^1(E)$.

Step 5: the image of Φ is all of $L^p(E)$, and the inverse formula. Let h belong to $L^p(E)$ and define

$$f = \psi \circ h, \quad \text{that is,} \quad f(x) = \operatorname{sgn}(h(x)) |h(x)|^p.$$

Since h is measurable and ψ is continuous on $\mathbb{R}$, f is measurable; and $|f| = |h|^p$ is integrable over E because h belongs to $L^p(E)$. Hence f belongs to $L^1(E)$, with $\|f\|_1 = \|h\|_p^p$. Moreover, since $\varphi \circ \psi$ is the identity on $\mathbb{R}$,

$$\Phi(f)(x) = \varphi(\psi(h(x))) = h(x) \quad \text{for all } x \in E.$$

So h lies in the image of Φ . Together with Step 4, Φ is a one-to-one mapping of $L^1(E)$ onto $L^p(E)$, and its inverse is given by

$$\Phi^{-1}(h)(x) = \operatorname{sgn}(h(x)) |h(x)|^{p-1} \quad \text{for } h \in L^p(E).$$

Step 6: Φ^{-1} is continuous. Let h, k belong to $L^p(E)$. By Problem 47, applied pointwise with $a = h(x)$ and $b = k(x)$,

$$|\Phi^{-1}(h)(x) - \Phi^{-1}(k)(x)| \leq p |h(x) - k(x)| (|h(x)| + |k(x)|)^{p-1} \quad \text{for all } x \in E.$$

The function $(|h| + |k|)^{p-1}$ belongs to $L^q(E)$, since

$$\int_E [(|h| + |k|)^{p-1}]^q = \int_E (|h| + |k|)^{(p-1) \cdot \frac{p}{p-1}} = \int_E (|h| + |k|)^p = \| |h| + |k| \|_p^p < \infty,$$

the finiteness following from the Minkowski Inequality, which gives $\| |h| + |k| \|_p \leq \|h\|_p + \|k\|_p < \infty$. Reading the same computation as an identity of norms,

$$\|(|h| + |k|)^{p-1}\|_q = \| |h| + |k| \|_p^{p/q} = \| |h| + |k| \|_p^{p-1} \leq (\|h\|_p + \|k\|_p)^{p-1},$$

where we used $p/q = p - 1$ and again the Minkowski Inequality. Now integrate the pointwise estimate over E and apply Holder's Inequality with the conjugate exponents p and q :

$$\begin{aligned} \|\Phi^{-1}(h) - \Phi^{-1}(k)\|_1 &= \int_E |\Phi^{-1}(h) - \Phi^{-1}(k)| \\ &\leq p \int_E |h - k| \cdot (|h| + |k|)^{p-1} \\ &\leq p \|h - k\|_p \cdot \|(|h| + |k|)^{p-1}\|_q \\ &\leq p (\|h\|_p + \|k\|_p)^{p-1} \|h - k\|_p. \end{aligned}$$

Suppose now that $\{h_n\} \rightarrow h$ in $L^p(E)$. Since $|\|h_n\|_p - \|h\|_p| \leq \|h_n - h\|_p \rightarrow 0$, the sequence of numbers $\{\|h_n\|_p\}$ is bounded, say $\|h_n\|_p \leq M$ for all n . Taking $k = h_n$ in the estimate just proved,

$$\|\Phi^{-1}(h_n) - \Phi^{-1}(h)\|_1 \leq p(M + \|h\|_p)^{p-1} \|h_n - h\|_p \rightarrow 0.$$

Therefore $\{\Phi^{-1}(h_n)\} \rightarrow \Phi^{-1}(h)$ in $L^1(E)$, so Φ^{-1} is a continuous mapping of $L^p(E)$ onto $L^1(E)$.

Problem 7.49. Use the preceding problem to show that the separability of $L^1(E)$ implies the separability of $L^p(E)$, for $1 < p < \infty$.

Solution. Assume $L^1(E)$ is separable and let $1 < p < \infty$. Let $\mathcal{D}$ be a countable dense subset of $L^1(E)$, enumerated as $\mathcal{D} = \{f_n\}_{n=1}^\infty$, with repetitions allowed in case $\mathcal{D}$ is finite. Let $\Phi: L^1(E) \rightarrow L^p(E)$ be the Mazur mapping of the preceding problem,

$$\Phi(f)(x) = \operatorname{sgn}(f(x)) |f(x)|^{1/p}.$$

Define

$$\Phi(\mathcal{D}) = \{\Phi(f_n) : n \in \mathbb{N}\}.$$

By the preceding problem each $\Phi(f_n)$ belongs to $L^p(E)$, and $\Phi(\mathcal{D})$ is the image of a countable set under a mapping, hence is countable.

We claim $\Phi(\mathcal{D})$ is dense in $L^p(E)$. Let h belong to $L^p(E)$ and let $\epsilon > 0$. By the preceding problem Φ maps $L^1(E)$ onto $L^p(E)$, so there is a function f in $L^1(E)$ with $\Phi(f) = h$; explicitly, $f = \Phi^{-1}(h)$ is given by $f(x) = \operatorname{sgn}(h(x)) |h(x)|^p$. Since $\mathcal{D}$ is dense in $L^1(E)$, there is an index n for which

$$\|f - f_n\|_1 < \left(\frac{\epsilon}{2}\right)^p.$$

By the estimate established in the preceding problem, namely

$$\|\Phi(u) - \Phi(v)\|_p^p \leq 2^p \|u - v\|_1, \quad \text{that is,} \quad \|\Phi(u) - \Phi(v)\|_p \leq 2 \|u - v\|_1^{1/p},$$

valid for all u, v in $L^1(E)$ (this is the pointwise inequality of Problem 46 integrated over E ; as noted in the solution to the preceding problem, this, and not the form printed there, is the correct estimate), we conclude

$$\|h - \Phi(f_n)\|_p = \|\Phi(f) - \Phi(f_n)\|_p \leq 2 \|f - f_n\|_1^{1/p} < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Thus every function in $L^p(E)$ can be approximated to within an arbitrary positive tolerance, in the $L^p(E)$ norm, by a member of the countable set $\Phi(\mathcal{D})$. Hence $\Phi(\mathcal{D})$ is a countable dense subset of $L^p(E)$, and $L^p(E)$ is separable.

The argument is simply that a continuous surjection carries a dense set onto a dense set: since Φ is continuous and onto, and $\mathcal{D}$ is dense in $L^1(E)$, the image $\Phi(\mathcal{D})$ is dense in $L^p(E)$; countability is preserved under images. This gives a second proof of the separability of $L^p(E)$ for $1 < p < \infty$, granted the case $p = 1$ of Theorem 11.

7.8 Problems 7.50–7.51

Problem 7.50. For $[a, b]$ a nondegenerate closed, bounded interval, show that there is no continuous mapping Φ from $L^1[a, b]$ onto $L^\infty[a, b]$.

Solution. Here, as in Problem 48, a mapping Φ between normed linear spaces is called continuous provided that whenever $\{f_n\} \rightarrow f$ in the domain, $\{\Phi(f_n)\} \rightarrow \Phi(f)$ in the target. (Since both $L^1[a, b]$ and $L^\infty[a, b]$ are metric spaces, this sequential continuity is the same as continuity in the sense of the norm topologies, so nothing is lost by using this formulation.)

The proof rests on two facts already established in this section: $L^1[a, b]$ is separable, while $L^\infty[a, b]$ is not.

Indeed, by Theorem 11, applied with $E = [a, b]$ and $p = 1$, the normed linear space $L^1[a, b]$ is separable; let

$$D = \{f_1, f_2, f_3, \dots\}$$

be a countable dense subset of $L^1[a, b]$. On the other hand, the Example following Theorem 11 shows that for a nondegenerate closed, bounded interval $[a, b]$ the normed linear space $L^\infty[a, b]$ is not separable. That Example is exactly where the nondegeneracy of $[a, b]$ enters: it considers the family $\{\chi_{[a, x]}\}_{a \leq x \leq b}$, for which

$$\|\chi_{[a, x_1]} - \chi_{[a, x_2]}\|_\infty = 1 \text{ whenever } a \leq x_1 < x_2 \leq b,$$

and it deduces that a countable dense subset $\{h_n\}_{n=1}^\infty$ of $L^\infty[a, b]$ would furnish, by assigning to each x in $[a, b]$ an index $\eta(x)$ with $\|\chi_{[a, x]} - h_{\eta(x)}\|_\infty < 1/2$, a one-to-one map η of $[a, b]$ into the natural numbers; this is impossible because $a < b$ forces $[a, b]$ to be uncountable.

Now argue by contradiction. Suppose Φ is a continuous mapping of $L^1[a, b]$ onto $L^\infty[a, b]$. Consider the image of the countable dense set D ,

$$\Phi(D) = \{\Phi(f_1), \Phi(f_2), \Phi(f_3), \dots\},$$

which is a countable subset of $L^\infty[a, b]$. We claim $\Phi(D)$ is dense in $L^\infty[a, b]$.

To see this, let g belong to $L^\infty[a, b]$. Since Φ maps $L^1[a, b]$ onto $L^\infty[a, b]$, there is a function f in $L^1[a, b]$ for which $\Phi(f) = g$. Since D is dense in $L^1[a, b]$, by Problem 36 there is a sequence $\{g_k\}_{k=1}^\infty$ of functions belonging to D for which $\{g_k\} \rightarrow f$ in $L^1[a, b]$. By the continuity of Φ ,

$$\{\Phi(g_k)\} \rightarrow \Phi(f) = g \text{ in } L^\infty[a, b].$$

Thus g is the limit, with respect to the $L^\infty[a, b]$ norm, of a sequence of functions belonging to $\Phi(D)$. Since g was an arbitrary function in $L^\infty[a, b]$, another application of Problem 36 tells us that $\Phi(D)$ is dense in $L^\infty[a, b]$.

Therefore $\Phi(D)$ is a countable dense subset of $L^\infty[a, b]$, that is, $L^\infty[a, b]$ is separable. This contradicts the Example cited above. We conclude that no such Φ exists: there is no continuous mapping of $L^1[a, b]$ onto $L^\infty[a, b]$.

Two remarks are worth recording. First, no linearity of Φ was used; the obstruction is purely topological, namely that separability is inherited by continuous images. Second, the same argument shows more generally that for any measurable set E and any $1 \leq p < \infty$ there is no continuous mapping of $L^p(E)$ onto $L^\infty(E)$ whenever E contains a nondegenerate interval, since $L^p(E)$ is separable by Theorem 11 while $L^\infty(E)$ is then not separable by Problem 42.

Problem 7.51. Use Lusin's Theorem to prove Theorem 12.

Solution. Theorem 12 asserts: for E a measurable set and $1 \leq p < \infty$, the linear space $C_c(E)$ of continuous real-valued functions on E that vanish outside a bounded set is dense in $L^p(E)$. Lusin's Theorem asserts: if u is a real-valued measurable function on E , then for each $\eta > 0$ there is a continuous function v on $\mathbb{R}$ and a closed set F contained in E for which $u = v$ on F and $m(E \setminus F) < \eta$. (The theorem is proved in Section 3.3 for $m(E) < \infty$, and Problem 29 of Chapter 3 extends it to sets of infinite measure, so we may use it for an arbitrary measurable set E .)

Let f belong to $L^p(E)$ and let $\epsilon > 0$. We must produce a function in $C_c(E)$ within ϵ of f in the $L^p(E)$ norm. We do so in three steps: first replace f by a bounded function vanishing outside a bounded set, then apply Lusin's Theorem to that function and repair the resulting continuous function so that it too is bounded and of bounded support, and finally estimate.

Step 1: reduction to a bounded function of bounded support. Since f belongs to $L^p(E)$, $|f|^p$ is integrable over E , and therefore, by Proposition 15 of Chapter 4 applied to $|f|^p$, $|f|^p$ is finite a.e. on E ; hence so is $|f|$. Let E_0 be the set of measure zero on which $|f| = \infty$. Redefine f to be 0 on E_0 . This alters f only on a set of measure zero, so it changes neither the equivalence class of f in $L^p(E)$ nor the value of $\|f - g\|_p$ for any g ; observe that we do not shrink the domain E , since the function we must finally produce is required to be continuous on all of E . We may therefore assume f is real-valued on

all of E . For each natural number n define the truncation

$$h_n(x) = \begin{cases} f(x) & \text{if } x \in E, |x| \leq n \text{ and } |f(x)| \leq n, \\ 0 & \text{otherwise on } E. \end{cases}$$

Each h_n is measurable, since it is f multiplied by the characteristic function of the measurable set $\{x \in E : |x| \leq n, |f(x)| \leq n\}$. Moreover $|h_n| \leq |f|$ on E for all n , and $\{h_n\} \rightarrow f$ pointwise on E : given $x \in E$, for all $n \geq \max\{|x|, |f(x)|\}$ we have $h_n(x) = f(x)$. Consequently

$$|h_n - f|^p \leq (|h_n| + |f|)^p \leq 2^p |f|^p \text{ on } E \text{ for all } n,$$

and $\{|h_n - f|^p\} \rightarrow 0$ pointwise on E . Since $|f|^p$ is integrable over E , the Lebesgue Dominated Convergence Theorem gives $\lim_{n \rightarrow \infty} \int_E |h_n - f|^p = 0$, that is, $\{h_n\} \rightarrow f$ in $L^p(E)$. Fix a natural number $n \geq 1$ for which

$$\|f - h_n\|_p < \epsilon/2,$$

and write $h = h_n$ and $M = n$. Thus h is a real-valued measurable function on E with $|h| \leq M$ on E and $h = 0$ on $\{x \in E : |x| > n\}$, and $M \geq 1$.

Step 2: Lusin's Theorem. Define

$$\eta = \frac{1}{2} \left(\frac{\epsilon}{4M} \right)^p > 0.$$

By Lusin's Theorem applied to the real-valued measurable function h on E (here E is an arbitrary measurable set, which is legitimate by the extension recorded above), there is a continuous function g_0 on $\mathbb{R}$ and a closed set F contained in E for which

$$h = g_0 \text{ on } F \text{ and } m(E \setminus F) < \eta.$$

The function g_0 need be neither bounded nor of bounded support, so we repair it by two continuous surgeries, each of which leaves the values on F unchanged.

First, truncate in the range. Define $\tau : \mathbb{R} \rightarrow \mathbb{R}$ by $\tau(t) = \max\{-M, \min\{t, M\}\}$. Then τ is continuous, $|\tau| \leq M$, and $\tau(t) = t$ whenever $|t| \leq M$. Set $g_1 = \tau \circ g_0$. Then g_1 is continuous on $\mathbb{R}$ with $|g_1| \leq M$, and for $x \in F$ we have $g_0(x) = h(x)$ with $|h(x)| \leq M$, so

$$g_1(x) = \tau(h(x)) = h(x) \text{ for all } x \in F.$$

Second, cut off in the domain. Define $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq n, \\ n+1-|x| & \text{if } n < |x| < n+1, \\ 0 & \text{if } |x| \geq n+1. \end{cases}$$

Then φ is continuous on $\mathbb{R}$, $0 \leq \varphi \leq 1$, and φ vanishes outside the bounded set $[-n-1, n+1]$. Define

$$g = g_1 \cdot \varphi \text{ on } \mathbb{R}.$$

Then g is continuous on $\mathbb{R}$, $|g| \leq M$, and g vanishes outside $[-n-1, n+1]$. Hence the restriction of g to E is a continuous real-valued function on E that vanishes outside a bounded set, that is, g belongs to $C_c(E)$. It is worth pausing on the fact that g also belongs to $L^p(E)$: a member of $C_c(E)$ need not be p -integrable, since continuity on E does not force boundedness on the bounded set outside of which the function vanishes, as the function $x \mapsto 1/x$ on $E = (0, 1)$ shows. Our g is however bounded, and so

$$\int_E |g|^p \leq M^p \cdot m(E \cap [-n-1, n+1]) \leq M^p(2n+2) < \infty.$$

Thus g belongs to $L^p(E)$, and the quantity $\|f - g\|_p$ estimated below is finite and the Minkowski Inequality is available.

We claim $g = h$ on F . Let $x \in F$. If $|x| \leq n$, then $\varphi(x) = 1$ and hence $g(x) = g_1(x) = h(x)$. If $|x| > n$, then, because x belongs to E and h vanishes on $\{x \in E : |x| > n\}$, we have $h(x) = 0$; therefore

$g_1(x) = h(x) = 0$ and so $g(x) = g_1(x)\varphi(x) = 0 = h(x)$. In either case $g(x) = h(x)$, and the claim is proved.

Step 3: the estimate. Since $h - g$ vanishes on F and $|h - g| \leq |h| + |g| \leq 2M$ on E ,

$$\int_E |h - g|^p = \int_{E \sim F} |h - g|^p \leq (2M)^p \cdot m(E \sim F) < (2M)^p \eta.$$

By the choice of η made in Step 2,

$$(2M)^p \eta = \frac{1}{2} \left(\frac{2M\epsilon}{4M} \right)^p = \frac{1}{2} \left(\frac{\epsilon}{2} \right)^p < \left(\frac{\epsilon}{2} \right)^p,$$

and therefore

$$\|h - g\|_p = \left(\int_E |h - g|^p \right)^{1/p} < \epsilon/2.$$

Finally, by the Minkowski Inequality (the triangle inequality for the $L^p(E)$ norm),

$$\|f - g\|_p \leq \|f - h\|_p + \|h - g\|_p < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since f in $L^p(E)$ and $\epsilon > 0$ were arbitrary, and g belongs to $C_c(E)$, we conclude that $C_c(E)$ is dense in $L^p(E)$. This is Theorem 12.

8 The L^p Spaces: Duality and Weak Convergence

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8.1 Problems 8.1–8.7

Problem 8.1. Verify (8), that is, verify that for a bounded linear functional T on a normed linear space X ,

$$\|T\|_* = \sup\{T(f) \mid f \in X, \|f\| \leq 1\}.$$

Solution. Recall the definition preceding (8): T is bounded provided there is an $M \geq 0$ for which the inequality (5),

$$|T(f)| \leq M \cdot \|f\| \quad \text{for all } f \in X,$$

holds, and $\|T\|_*$ is the infimum of the set $\mathcal{M}$ of all such M . Since T is bounded, $\mathcal{M}$ is a nonempty set of nonnegative numbers, so $\|T\|_* = \inf \mathcal{M}$ is a well-defined nonnegative number.

Write $B = \{f \in X \mid \|f\| \leq 1\}$ and

$$S = \sup\{T(f) \mid f \in B\}.$$

First observe that $S = \sup\{|T(f)| \mid f \in B\}$. Indeed, B is symmetric: if f belongs to B then so does $-f$, since $\|-f\| = |-1| \cdot \|f\| = \|f\| \leq 1$. By the linearity of T , $T(-f) = -T(f)$. Hence the set $\{T(f) \mid f \in B\}$ is symmetric about 0, and therefore its supremum equals the supremum of the absolute values of its members. In particular $S \geq T(0) = 0$.

We now show $S \leq \|T\|_*$ and $S \geq \|T\|_*$.

Let M belong to $\mathcal{M}$. For f in B we have, by (5), $|T(f)| \leq M \cdot \|f\| \leq M$. Taking the supremum over f in B gives $S \leq M$. Since M was an arbitrary member of $\mathcal{M}$, S is a lower bound for $\mathcal{M}$, and therefore

$$S \leq \inf \mathcal{M} = \|T\|_*.$$

Conversely, we claim S itself belongs to $\mathcal{M}$. Let f belong to X . If $f = 0$, then $T(f) = 0$ by linearity and (5) holds trivially with $M = S$ since $S \geq 0$. If $f \neq 0$, then $\|f\| > 0$ and the vector $h = f/\|f\|$ satisfies $\|h\| = 1$, so h belongs to B and hence $|T(h)| \leq S$. By the linearity of T , $T(h) = T(f)/\|f\|$, so

$$|T(f)| = \|f\| \cdot |T(h)| \leq S \cdot \|f\|.$$

Thus S belongs to $\mathcal{M}$, and therefore $\|T\|_* = \inf \mathcal{M} \leq S$.

Combining the two inequalities, $\|T\|_* = S$, which is (8).

Two remarks are worth recording, since they are used repeatedly in this section. First, the argument above shows that the infimum defining $\|T\|_*$ is attained: $\|T\|_* = S$ belongs to $\mathcal{M}$, so (5) holds with $M = \|T\|_*$, that is,

$$|T(f)| \leq \|T\|_* \cdot \|f\| \quad \text{for all } f \in X.$$

Second, by the linearity of T , applying this to $f - h$ yields (6):

$$|T(f) - T(h)| = |T(f - h)| \leq \|T\|_* \cdot \|f - h\| \quad \text{for all } f, h \in X.$$

Problem 8.2. Prove Proposition 1: Let X be a normed linear space. Then the collection of bounded linear functionals on X is a linear space on which $\|\cdot\|_*$ is a norm. This normed linear space is called the dual space of X and denoted by X^* .

Solution. Throughout we use the characterization (8) established in the preceding problem, together with the two consequences recorded there: the infimum defining $\|T\|_*$ is attained, and

$$\|T\|_* = \sup\{|T(f)| \mid f \in X, \|f\| \leq 1\}.$$

Let $\mathcal{L}$ denote the linear space of all linear functionals on X , with the pointwise linear operations

$$(\alpha \cdot T + \beta \cdot S)(f) = \alpha \cdot T(f) + \beta \cdot S(f),$$

and let X^* be the subset of $\mathcal{L}$ consisting of the bounded ones.

Step 1: X^* is a linear subspace of $\mathcal{L}$. The zero functional is bounded, so X^* is nonempty. Let T and S belong to X^* and let α, β be real numbers. The functional $\alpha \cdot T + \beta \cdot S$ is linear, since a pointwise linear combination of linear functionals is linear. Moreover, for f in X , by the triangle inequality in $\mathbb{R}$ and the boundedness of T and S ,

$$|(\alpha \cdot T + \beta \cdot S)(f)| \leq |\alpha| \cdot |T(f)| + |\beta| \cdot |S(f)| \leq (|\alpha| \cdot \|T\|_* + |\beta| \cdot \|S\|_*) \cdot \|f\|.$$

Hence $\alpha \cdot T + \beta \cdot S$ is bounded, and so belongs to X^* . Thus X^* is a linear space.

Step 2: $\|\cdot\|_*$ is a norm on X^* . We verify the three defining properties.

Nonnegativity and definiteness. By construction $\|T\|_*$ is the infimum of a nonempty set of nonnegative numbers, so $\|T\|_* \geq 0$. If $T = 0$, then $M = 0$ satisfies (5), so $\|T\|_* = 0$. Conversely, suppose $\|T\|_* = 0$. Since the infimum is attained, (5) holds with $M = 0$, that is, $|T(f)| \leq 0 \cdot \|f\| = 0$ for all f in X . Hence $T(f) = 0$ for all f , that is, $T = 0$.

Positive homogeneity. Let T belong to X^* and let α be a real number. Using the supremum formula,

$$\|\alpha \cdot T\|_* = \sup_{\|f\| \leq 1} |\alpha \cdot T(f)| = \sup_{\|f\| \leq 1} |\alpha| \cdot |T(f)| = |\alpha| \cdot \sup_{\|f\| \leq 1} |T(f)| = |\alpha| \cdot \|T\|_*,$$

where the third equality is the elementary fact that $\sup(c \cdot A) = c \cdot \sup A$ for a nonempty set A of reals bounded above and a constant $c \geq 0$.

The triangle inequality. Let T and S belong to X^* . For each f in X with $\|f\| \leq 1$,

$$|(T + S)(f)| \leq |T(f)| + |S(f)| \leq \|T\|_* + \|S\|_*.$$

Thus $\|T\|_* + \|S\|_*$ is an upper bound for the set whose supremum is $\|T + S\|_*$, and therefore

$$\|T + S\|_* \leq \|T\|_* + \|S\|_*.$$

Hence $\|\cdot\|_*$ is a norm on the linear space X^* , which completes the proof.

Problem 8.3. Let T be a linear functional on a normed linear space X . Show that T is bounded if and only if the continuity property (7) holds, that is, if and only if

$$\text{whenever } \{f_n\} \rightarrow f \text{ in } X, \text{ then } \{T(f_n)\} \rightarrow T(f).$$

Solution. Suppose first that T is bounded. Let $\{f_n\}$ be a sequence in X converging to f in X , that is, $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$. By (6),

$$|T(f_n) - T(f)| \leq \|T\|_* \cdot \|f_n - f\| \quad \text{for all } n.$$

The right-hand side converges to 0, so $\{T(f_n)\} \rightarrow T(f)$. Thus (7) holds. (This is exactly the inference the text makes from (6).)

Conversely, suppose T is not bounded; we show (7) fails. Since T is unbounded, for each natural number n the number $M = n$ fails to satisfy (5), so there is a vector f_n in X for which

$$|T(f_n)| > n \cdot \|f_n\|.$$

Observe that $f_n \neq 0$: if $f_n = 0$ then, by the linearity of T , $T(f_n) = 0$ and the displayed inequality would read $0 > 0$. Hence $\|f_n\| > 0$, and we may define

$$h_n = \frac{1}{n \cdot \|f_n\|} \cdot f_n.$$

Then

$$\|h_n\| = \frac{\|f_n\|}{n \cdot \|f_n\|} = \frac{1}{n},$$

so $\{h_n\} \rightarrow 0$ in X . On the other hand, by the linearity of T ,

$$|T(h_n)| = \frac{|T(f_n)|}{n \cdot \|f_n\|} > \frac{n \cdot \|f_n\|}{n \cdot \|f_n\|} = 1 \quad \text{for all } n.$$

Since $T(0) = 0$ by linearity, the sequence $\{T(h_n)\}$ does not converge to $T(0)$. Therefore the continuity property (7) fails for the convergent sequence $\{h_n\}$.

By contraposition, if (7) holds then T is bounded. Together with the first paragraph this establishes the equivalence.

Problem 8.4. A functional T on a normed linear space X is said to be Lipschitz provided there is a $c \geq 0$ such that

$$|T(g) - T(h)| \leq c\|g - h\| \quad \text{for all } g, h \in X.$$

The infimum of such c 's is called the Lipschitz constant for T . Show that a linear functional is bounded if and only if it is Lipschitz, in which case its Lipschitz constant is $\|T\|_*$.

Solution. Let T be a linear functional on X . Denote by $\mathcal{C}$ the set of numbers $c \geq 0$ for which the Lipschitz inequality

$$|T(g) - T(h)| \leq c \cdot \|g - h\| \quad \text{for all } g, h \in X$$

holds, and by $\mathcal{M}$ the set of numbers $M \geq 0$ for which (5) holds, that is, $|T(f)| \leq M \cdot \|f\|$ for all f in X . By definition, T is Lipschitz exactly when $\mathcal{C} \neq \emptyset$, in which case its Lipschitz constant is $c_0 = \inf \mathcal{C}$; and T is bounded exactly when $\mathcal{M} \neq \emptyset$, in which case $\|T\|_* = \inf \mathcal{M}$.

We claim that in fact $\mathcal{C} = \mathcal{M}$; the assertion of the problem follows at once from this, since then one set is empty if and only if the other is, and the two infima coincide.

$\mathcal{M} \subseteq \mathcal{C}$. Let M belong to $\mathcal{M}$ and let g, h belong to X . By the linearity of T , $T(g) - T(h) = T(g - h)$, and therefore

$$|T(g) - T(h)| = |T(g - h)| \leq M \cdot \|g - h\|.$$

Thus M belongs to $\mathcal{C}$.

$\mathcal{C} \subseteq \mathcal{M}$. Let c belong to $\mathcal{C}$ and let f belong to X . By the linearity of T , $T(0) = 0$. Taking $g = f$ and $h = 0$ in the Lipschitz inequality gives

$$|T(f)| = |T(f) - T(0)| \leq c \cdot \|f - 0\| = c \cdot \|f\|.$$

Thus c belongs to $\mathcal{M}$.

Hence $\mathcal{C} = \mathcal{M}$. Consequently T is bounded if and only if T is Lipschitz, and in that case

$$c_0 = \inf \mathcal{C} = \inf \mathcal{M} = \|T\|_*,$$

that is, the Lipschitz constant of T equals $\|T\|_*$.

We note in passing that, as observed in Problem 1, the infimum is attained: $\|T\|_*$ itself belongs to $\mathcal{M} = \mathcal{C}$, so the Lipschitz inequality actually holds with $c = \|T\|_*$. This is precisely inequality (6) of the text.

Problem 8.5. Let E be a measurable set and $1 \leq p < \infty$. Show that the functions in $L^p(E)$ that vanish outside a bounded set are dense in $L^p(E)$. Show that this is false for $L^\infty(\mathbb{R})$.

Solution. Let $\mathcal{F}$ denote the collection of functions in $L^p(E)$ that vanish outside a bounded set.

First, $\mathcal{F}$ is a subset of $L^p(E)$, and we must show that for each f in $L^p(E)$ and each $\epsilon > 0$ there is a member of $\mathcal{F}$ within ϵ of f in the $L^p(E)$ norm.

So let f belong to $L^p(E)$. For each natural number n define

$$f_n = f \cdot \chi_{E \cap [-n, n]} \quad \text{on } E,$$

that is, $f_n = f$ on $E \cap [-n, n]$ and $f_n = 0$ on $E \setminus [-n, n]$. Each f_n is measurable, $|f_n| \leq |f|$ on E , so

f_n belongs to $L^p(E)$, and f_n vanishes outside the bounded set $[-n, n]$. Thus each f_n belongs to $\mathcal{F}$. Consider the sequence of nonnegative measurable functions

$$h_n = |f - f_n|^p = |f|^p \cdot \chi_{E \setminus [-n, n]} \quad \text{on } E.$$

For each x in E there is a natural number N with $|x| \leq N$, and then $h_n(x) = 0$ for all $n \geq N$; hence $\{h_n\} \rightarrow 0$ pointwise on E . Moreover $0 \leq h_n \leq |f|^p$ on E for all n , and $|f|^p$ is integrable over E since f belongs to $L^p(E)$ and $p < \infty$. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_E |f - f_n|^p = \lim_{n \rightarrow \infty} \int_E h_n = 0,$$

that is, $\lim_{n \rightarrow \infty} \|f - f_n\|_p = 0$. Therefore f is a limit in $L^p(E)$ of members of $\mathcal{F}$, and since f was arbitrary, $\mathcal{F}$ is dense in $L^p(E)$.

Note where the hypothesis $p < \infty$ entered: it is needed both to write $\|f - f_n\|_p^p = \int_E |f - f_n|^p$ and to have the integrable dominating function $|f|^p$.

Now we show the corresponding statement is false for $L^\infty(\mathbb{R})$. Let f be the constant function $f \equiv 1$ on $\mathbb{R}$; then f belongs to $L^\infty(\mathbb{R})$ with $\|f\|_\infty = 1$.

Let g belong to $L^\infty(\mathbb{R})$ and vanish outside a bounded set. Since the members of $L^\infty(\mathbb{R})$ are identified when they agree almost everywhere, we read this hypothesis in the only way that is invariant under that identification: there is a bounded set B for which $g = 0$ almost everywhere on $\mathbb{R} \setminus B$. Choose a natural number n with $B \subseteq [-n, n]$. Then $g = 0$ almost everywhere on $\mathbb{R} \setminus [-n, n]$, and hence

$$|f - g| = |1 - 0| = 1 \quad \text{almost everywhere on } \mathbb{R} \setminus [-n, n].$$

Let c be any number with $0 \leq c < 1$. The set $\{x \in \mathbb{R} \mid |f(x) - g(x)| > c\}$ contains $\mathbb{R} \setminus [-n, n]$ except for a set of measure zero, and $\mathbb{R} \setminus [-n, n]$ has infinite measure; hence that set has positive measure. Therefore c is not an essential upper bound for $|f - g|$. As this holds for every $c < 1$,

$$\|f - g\|_\infty \geq 1.$$

Thus every function in $L^\infty(\mathbb{R})$ that vanishes outside a bounded set is at distance at least 1 from f . Consequently no such function lies within $1/2$ of f , and the functions in $L^\infty(\mathbb{R})$ that vanish outside a bounded set are not dense in $L^\infty(\mathbb{R})$.

Problem 8.6. Establish the Riesz Representation Theorem in the case $p = 1$ by first showing, in the notation of the proof of the theorem, that the function Φ is Lipschitz and therefore it is absolutely continuous. Then follow the $p > 1$ proof.

Solution. We prove Theorem 5 for $p = 1$: if $[a, b]$ is a closed, bounded interval and T is a bounded linear functional on $L^1[a, b]$, then there is a function g in $L^\infty[a, b]$ for which

$$T(f) = \int_a^b g \cdot f \quad \text{for all } f \in L^1[a, b].$$

Here $q = \infty$ is the conjugate of $p = 1$. At the end we indicate how the passage from $[a, b]$ to a general measurable set E , carried out in the text, goes through verbatim for $p = 1$.

Step 1: Φ is Lipschitz, hence absolutely continuous. Following the notation of the proof of Theorem 5, define

$$\Phi(x) = T(\chi_{[a, x]}) \quad \text{for } x \in [a, b],$$

which makes sense because $\chi_{[a, x]}$ belongs to $L^1[a, b]$. As in the text, if $a \leq c \leq d \leq b$, then $\chi_{[c, d]} = \chi_{[a, d]} - \chi_{[a, c]}$, so by the linearity of T ,

$$\Phi(d) - \Phi(c) = T(\chi_{[a, d]}) - T(\chi_{[a, c]}) = T(\chi_{[c, d]}).$$

Since T is bounded, by (6) applied with $h = 0$, or directly by (5) with $M = \|T\|_*$,

$$|\Phi(d) - \Phi(c)| = |T(\chi_{[c, d]})| \leq \|T\|_* \cdot \|\chi_{[c, d]}\|_1 = \|T\|_* \cdot (d - c).$$

As this holds for all $a \leq c \leq d \leq b$, we conclude

$$|\Phi(u) - \Phi(v)| \leq \|T\|_* \cdot |u - v| \quad \text{for all } u, v \in [a, b],$$

that is, Φ is Lipschitz on $[a, b]$ with Lipschitz constant at most $\|T\|_*$.

A Lipschitz function on $[a, b]$ is absolutely continuous. Indeed, let $\epsilon > 0$ and set $\delta = \epsilon/(\|T\|_* + 1) > 0$. If $\{(a_k, b_k)\}_{k=1}^n$ is a finite disjoint collection of intervals in (a, b) with $\sum_{k=1}^n (b_k - a_k) < \delta$, then

$$\sum_{k=1}^n |\Phi(b_k) - \Phi(a_k)| \leq \|T\|_* \cdot \sum_{k=1}^n (b_k - a_k) < \|T\|_* \cdot \frac{\epsilon}{\|T\|_* + 1} < \epsilon.$$

Thus Φ is absolutely continuous on $[a, b]$. (This is the only place where the argument differs from the case $p > 1$: there one had the weaker estimate $\sum_k |\Phi(b_k) - \Phi(a_k)| \leq \|T\|_* \cdot [\sum_k (b_k - a_k)]^{1/p}$, which for $p = 1$ degenerates to exactly the Lipschitz estimate above.)

Step 2: recovery of g . By Theorem 10 of Chapter 6, an absolutely continuous function on $[a, b]$ is differentiable almost everywhere on $[a, b]$, its derivative is integrable over $[a, b]$, and it is the indefinite integral of its derivative. Hence $g = \Phi'$ is integrable over $[a, b]$ and

$$\Phi(x) - \Phi(a) = \int_a^x g \quad \text{for all } x \in [a, b].$$

Since $\chi_{[a,a]}$ is the zero function, $\Phi(a) = T(0) = 0$, so $\Phi(x) = \int_a^x g$ on $[a, b]$. Therefore, for $a \leq c \leq d \leq b$,

$$T(\chi_{[c,d]}) = \Phi(d) - \Phi(c) = \int_c^d g = \int_a^b g \cdot \chi_{[c,d]}.$$

Both T and the functional $f \mapsto \int_a^b g \cdot f$ are linear on the linear space of step functions on $[a, b]$ (the latter is properly defined there since g is integrable and step functions are bounded), and every step function is a finite linear combination of functions of the form $\chi_{[c,d]}$, modulo a set of measure zero, on which T is unchanged because T respects equality almost everywhere. Hence

$$T(f) = \int_a^b g \cdot f \quad \text{for all step functions } f \text{ on } [a, b].$$

Step 3: from step functions to simple functions. Let f be a simple function on $[a, b]$. By Proposition 10 of the preceding chapter and its proof, there is a sequence $\{\varphi_n\}$ of step functions on $[a, b]$ that converges to f in $L^1[a, b]$ and is uniformly pointwise bounded on $[a, b]$, say $|\varphi_n| \leq C$ on $[a, b]$ for all n . By the Riesz-Fischer Theorem of the preceding chapter, a subsequence $\{\varphi_{n_k}\}$ converges to f pointwise almost everywhere on $[a, b]$. Since T is bounded, the continuity property (7) gives

$$\lim_{k \rightarrow \infty} T(\varphi_{n_k}) = T(f).$$

On the other hand, $|g \cdot \varphi_{n_k}| \leq C \cdot |g|$ on $[a, b]$, where $C \cdot |g|$ is integrable, and $\{g \cdot \varphi_{n_k}\} \rightarrow g \cdot f$ pointwise almost everywhere. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{k \rightarrow \infty} \int_a^b g \cdot \varphi_{n_k} = \int_a^b g \cdot f.$$

Since $T(\varphi_{n_k}) = \int_a^b g \cdot \varphi_{n_k}$ for every k by Step 2, we conclude

$$T(f) = \int_a^b g \cdot f \quad \text{for all simple functions } f \text{ on } [a, b].$$

Step 4: g belongs to $L^\infty[a, b]$. Since T is bounded, for every simple function f in $L^1[a, b]$,

$$\left| \int_a^b g \cdot f \right| = |T(f)| \leq \|T\|_* \cdot \|f\|_1.$$

Thus g is integrable over $[a, b]$ and satisfies hypothesis (9) of Lemma 4 with $p = 1$ and $M = \|T\|_*$. Lemma 4, in the case $p = 1$, asserts precisely that g then belongs to $L^\infty[a, b]$ with $\|g\|_\infty \leq \|T\|_*$; recall its proof in that case: if $\|T\|_*$ were not an essential upper bound for $|g|$, then by the continuity of measure the set $E_\epsilon = \{x \in [a, b] \mid |g(x)| > \|T\|_* + \epsilon\}$ would have positive measure for some $\epsilon > 0$, and taking $f = \operatorname{sgn}(g) \cdot \chi_A$ for a measurable subset A of E_ϵ of finite positive measure gives $\int_a^b g \cdot f \geq (\|T\|_* + \epsilon) \cdot m(A) > \|T\|_* \cdot \|f\|_1$, a contradiction.

Step 5: conclusion. Since g belongs to $L^\infty[a, b] = L^q[a, b]$, Proposition 2 tells us that the functional $\mathcal{R}_g: f \mapsto \int_a^b g \cdot f$ is a bounded linear functional on $L^1[a, b]$ with $\|\mathcal{R}_g\|_* = \|g\|_\infty$. By Step 3, the two bounded linear functionals T and $\mathcal{R}_g$ agree on the simple functions, which by Proposition 9 of the preceding chapter form a dense subspace of $L^1[a, b]$. By Proposition 3 they agree on all of $L^1[a, b]$:

$$T(f) = \int_a^b g \cdot f \quad \text{for all } f \in L^1[a, b].$$

This proves Theorem 5 for $p = 1$.

Finally, the extension to a general measurable set E proceeds exactly as in the proof of the Riesz Representation Theorem for the dual of $L^p(E)$, with one adjustment. Uniqueness and the identity $\mathcal{R}_{g_1} - \mathcal{R}_{g_2} = \mathcal{R}_{g_1 - g_2}$ are unchanged. For $E = \mathbb{R}$, restricting T to $L^1[-n, n]$ produces, by the case just proved, functions g_n in $L^\infty[-n, n]$ with $\|g_n\|_\infty \leq \|T\|_*$ which are consistent with one another almost everywhere, and hence a measurable function g on $\mathbb{R}$ agreeing with each g_n almost everywhere on $[-n, n]$. In place of the Fatou's Lemma step (which was used to pass from $\int_{-n}^n |g|^q \leq (\|T\|_*)^q$ to $g \in L^q(\mathbb{R})$), one argues directly: for each n , $|g| \leq \|T\|_*$ almost everywhere on $[-n, n]$, so the set where $|g| > \|T\|_*$ is the countable union over n of sets of measure zero and hence has measure zero. Thus g belongs to $L^\infty(\mathbb{R})$ with $\|g\|_\infty \leq \|T\|_*$. Since $\mathcal{R}_g$ and T are bounded and agree on the functions in $L^1(\mathbb{R})$ vanishing outside a bounded set, which by the preceding problem form a dense subspace of $L^1(\mathbb{R})$, Proposition 3 gives $\mathcal{R}_g = T$. The passage from $\mathbb{R}$ to an arbitrary measurable set E , by extending functions in $L^1(E)$ by zero, is identical to the text's.

Problem 8.7. State and prove a Riesz Representation Theorem for the bounded linear functionals on ℓ^p , $1 \leq p < \infty$.

Solution. Recall the notation. For $1 \leq p < \infty$, ℓ^p is the linear space of real sequences $x = \{x_k\}_{k=1}^\infty$ for which $\sum_{k=1}^\infty |x_k|^p < \infty$, normed by $\|x\|_p = [\sum_{k=1}^\infty |x_k|^p]^{1/p}$; and ℓ^∞ is the linear space of bounded real sequences $y = \{y_k\}$, normed by $\|y\|_\infty = \sup_{k \geq 1} |y_k|$. That these are normed linear spaces is the content of Minkowski's Inequality. For each natural number k , let e_k denote the sequence whose k th term is 1 and whose other terms are 0.

Theorem (Riesz Representation for the dual of ℓ^p). Let $1 \leq p < \infty$ and let q be the conjugate of p . For each $y = \{y_k\}$ in ℓ^q , define the functional $\mathcal{R}_y$ on ℓ^p by

$$\mathcal{R}_y(x) = \sum_{k=1}^{\infty} y_k \cdot x_k \quad \text{for all } x = \{x_k\} \in \ell^p.$$

Then $\mathcal{R}_y$ is a properly defined bounded linear functional on ℓ^p with $\|\mathcal{R}_y\|_* = \|y\|_q$, and for each bounded linear functional T on ℓ^p there is a unique y in ℓ^q for which

$$\mathcal{R}_y = T, \quad \text{and} \quad \|T\|_* = \|y\|_q.$$

Thus $y \mapsto \mathcal{R}_y$ is a linear isometry of ℓ^q onto $(\ell^p)^*$.

Proof.

Step 1: Hölder's Inequality for sequences. Let y belong to ℓ^q and x to ℓ^p . We claim

$$\sum_{k=1}^{\infty} |y_k \cdot x_k| \leq \|y\|_q \cdot \|x\|_p.$$

If $p = 1$ and $q = \infty$ this is immediate, since $|y_k| \leq \|y\|_\infty$ for every k . Suppose $1 < p < \infty$. If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x = 0$ or $y = 0$ and the inequality is trivial. Otherwise, by homogeneity we may assume $\|x\|_p = \|y\|_q = 1$. Young's Inequality asserts that $\alpha \cdot \beta \leq \alpha^p/p + \beta^q/q$ for nonnegative α, β . Applying it termwise with $\alpha = |x_k|$, $\beta = |y_k|$ and summing,

$$\sum_{k=1}^{\infty} |y_k \cdot x_k| \leq \frac{1}{p} \sum_{k=1}^{\infty} |x_k|^p + \frac{1}{q} \sum_{k=1}^{\infty} |y_k|^q = \frac{1}{p} + \frac{1}{q} = 1 = \|y\|_q \cdot \|x\|_p.$$

Step 2: $\mathcal{R}_y$ is a bounded linear functional with $\|\mathcal{R}_y\|_* = \|y\|_q$. By Step 1 the series defining $\mathcal{R}_y(x)$ converges absolutely, so $\mathcal{R}_y$ is properly defined, and

$$|\mathcal{R}_y(x)| \leq \sum_{k=1}^{\infty} |y_k \cdot x_k| \leq \|y\|_q \cdot \|x\|_p.$$

Linearity follows from the linearity of termwise operations together with the fact that absolutely convergent series may be added and scaled termwise. Hence $\mathcal{R}_y$ is bounded and $\|\mathcal{R}_y\|_* \leq \|y\|_q$.

For the reverse inequality, suppose first $1 < p < \infty$ and $y \neq 0$. Define $x^* = \{x_k^*\}$ by

$$x_k^* = \operatorname{sgn}(y_k) \cdot |y_k|^{q-1}.$$

Since q is the conjugate of p we have $p(q-1) = q$, so $|x_k^*|^p = |y_k|^q$; hence x^* belongs to ℓ^p with $\|x^*\|_p = \|y\|_q^{q/p}$, which is positive and finite. Moreover

$$\mathcal{R}_y(x^*) = \sum_{k=1}^{\infty} |y_k|^q = \|y\|_q^q.$$

Therefore, by (8),

$$\|\mathcal{R}_y\|_* \geq \frac{\mathcal{R}_y(x^*)}{\|x^*\|_p} = \|y\|_q^{q-q/p} = \|y\|_q,$$

since $q - q/p = q(1 - 1/p) = q \cdot (1/q) = 1$. If $y = 0$ the equality $\|\mathcal{R}_y\|_* = \|y\|_q$ is trivial.

Now suppose $p = 1$, $q = \infty$. For each k , the sequence $\operatorname{sgn}(y_k) \cdot e_k$ has ℓ^1 norm at most 1 and $\mathcal{R}_y(\operatorname{sgn}(y_k) \cdot e_k) = |y_k|$. By (8), $\|\mathcal{R}_y\|_* \geq |y_k|$ for every k , and taking the supremum over k gives $\|\mathcal{R}_y\|_* \geq \|y\|_\infty$.

In both cases $\|\mathcal{R}_y\|_* = \|y\|_q$.

Step 3: uniqueness. The map $y \mapsto \mathcal{R}_y$ is linear, since $\mathcal{R}_{y_1} - \mathcal{R}_{y_2} = \mathcal{R}_{y_1 - y_2}$ by the linearity of absolutely convergent summation. Hence if $\mathcal{R}_{y_1} = \mathcal{R}_{y_2}$, then $\mathcal{R}_{y_1 - y_2} = 0$, so by Step 2, $\|y_1 - y_2\|_q = 0$ and therefore $y_1 = y_2$.

Step 4: the sequences of finite support are dense in ℓ^p . Let X_0 denote the linear span of $\{e_k\}_{k=1}^{\infty}$, that is, the set of sequences with only finitely many nonzero terms. Let x belong to ℓ^p and, for each n , let $x^{[n]} = \sum_{k=1}^n x_k \cdot e_k$, which belongs to X_0 . Then

$$\|x - x^{[n]}\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p,$$

which is the tail of the convergent series $\sum_{k=1}^{\infty} |x_k|^p$ and therefore tends to 0 as $n \rightarrow \infty$. Hence $\{x^{[n]}\} \rightarrow x$ in ℓ^p , and X_0 is dense in ℓ^p . (It is exactly here that $p < \infty$ is needed; in ℓ^∞ the constant sequence $\{1, 1, 1, \dots\}$ is at distance 1 from X_0 .)

Step 5: existence. Let T be a bounded linear functional on ℓ^p . Define

$$y_k = T(e_k) \quad \text{for } k = 1, 2, \dots$$

We claim $y = \{y_k\}$ belongs to ℓ^q with $\|y\|_q \leq \|T\|_*$.

Suppose first $1 < p < \infty$. Fix a natural number n and define $x^{(n)}$ in X_0 by

$$x^{(n)} = \sum_{k=1}^n \operatorname{sgn}(y_k) \cdot |y_k|^{q-1} \cdot e_k.$$

By the linearity of T ,

$$T(x^{(n)}) = \sum_{k=1}^n \operatorname{sgn}(y_k) \cdot |y_k|^{q-1} \cdot y_k = \sum_{k=1}^n |y_k|^q,$$

while, using $p(q-1) = q$,

$$\|x^{(n)}\|_p = \left[\sum_{k=1}^n |y_k|^{p(q-1)} \right]^{1/p} = \left[\sum_{k=1}^n |y_k|^q \right]^{1/p}.$$

Set $A_n = \sum_{k=1}^n |y_k|^q$. Since T is bounded, $A_n = T(x^{(n)}) \leq \|T\|_* \cdot A_n^{1/p}$. If $A_n = 0$ then certainly $A_n^{1/q} \leq \|T\|_*$; if $A_n > 0$, then A_n is a finite positive number and dividing by $A_n^{1/p}$ gives

$$A_n^{1-1/p} = A_n^{1/q} \leq \|T\|_*.$$

Thus $[\sum_{k=1}^n |y_k|^q]^{1/q} \leq \|T\|_*$ for every n ; letting $n \rightarrow \infty$ shows y belongs to ℓ^q with $\|y\|_q \leq \|T\|_*$. If $p = 1$ and $q = \infty$, then for each k , since $\|e_k\|_1 = 1$,

$$|y_k| = |T(e_k)| \leq \|T\|_* \cdot \|e_k\|_1 = \|T\|_*,$$

so y belongs to ℓ^∞ with $\|y\|_\infty \leq \|T\|_*$.

Step 6: conclusion. By Step 5, y belongs to ℓ^q , so by Step 2, $\mathcal{R}_y$ is a bounded linear functional on ℓ^p . For each k ,

$$\mathcal{R}_y(e_k) = y_k = T(e_k),$$

so by linearity T and $\mathcal{R}_y$ agree on the linear span X_0 of $\{e_k\}$. By Step 4, X_0 is a dense subspace of ℓ^p , and both functionals are bounded, so by Proposition 3 they agree on all of ℓ^p . That is, $T = \mathcal{R}_y$. Its uniqueness is Step 3, and by Step 2,

$$\|T\|_* = \|\mathcal{R}_y\|_* = \|y\|_q.$$

This completes the proof.

Remark. Steps 2 and 6 together say that $y \mapsto \mathcal{R}_y$ is a norm-preserving linear bijection of ℓ^q onto $(\ell^p)^*$; in this sense $(\ell^p)^* = \ell^q$ for $1 \leq p < \infty$. As with L^∞ , the result fails at $p = \infty$: Step 4 breaks down, and there are bounded linear functionals on ℓ^∞ that are not of the form $\mathcal{R}_y$ for any y in ℓ^1 .

8.2 Problems 8.8–8.14

Problem 8.8. Let c be the linear space of real sequences that converge to a real number and c_0 the subspace of c comprising sequences that converge to 0. Norm each of these linear spaces with the ℓ^∞ norm. Determine the dual space of c and of c_0 .

Solution. Throughout, for $x = (x_1, x_2, \dots)$ in c we write $\|x\|_\infty = \sup_k |x_k|$, we let e_k denote the sequence whose k -th component is 1 and whose other components are 0, and we let $\mathbf{1} = (1, 1, 1, \dots)$. Both e_k and $\mathbf{1}$ belong to c , and each e_k belongs to c_0 . Define $\operatorname{sgn}(t) = 1$ if $t \geq 0$ and $\operatorname{sgn}(t) = -1$ if $t < 0$, so that $t \cdot \operatorname{sgn}(t) = |t|$ for every real t .

We show that c_0^* and c^* are each isometrically isomorphic to ℓ^1 , but by different pairings: a functional on c_0 is integration against an ℓ^1 sequence, while a functional on c is such a pairing together with a multiple of the limit functional.

Part 1: the dual of c_0 .

For $a = (a_1, a_2, \dots)$ in ℓ^1 define

$$T_a(x) = \sum_{k=1}^{\infty} a_k x_k \quad \text{for all } x \in c_0.$$

The series converges absolutely since $\sum_k |a_k x_k| \leq \|x\|_\infty \sum_k |a_k|$, and this same estimate shows T_a is a

bounded linear functional on c_0 with

$$\|T_a\|_* \leq \|a\|_1 = \sum_{k=1}^{\infty} |a_k|.$$

Conversely, let T belong to c_0^* and set $a_k = T(e_k)$ for each index k . Fix a natural number n and let x be the sequence with $x_k = \text{sgn}(a_k)$ for $1 \leq k \leq n$ and $x_k = 0$ for $k > n$. Then x belongs to c_0 and $\|x\|_{\infty} \leq 1$, while by linearity

$$\sum_{k=1}^n |a_k| = \sum_{k=1}^n a_k x_k = T\left(\sum_{k=1}^n x_k e_k\right) = T(x) \leq \|T\|_* \|x\|_{\infty} \leq \|T\|_*.$$

Since n is arbitrary, a belongs to ℓ^1 and $\|a\|_1 \leq \|T\|_*$.

Moreover $T = T_a$. Indeed, if x belongs to c_0 and $s_n = \sum_{k=1}^n x_k e_k$, then

$$\|x - s_n\|_{\infty} = \sup_{k > n} |x_k| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

because $x_k \rightarrow 0$. Since T is bounded, it is continuous, so

$$T(x) = \lim_{n \rightarrow \infty} T(s_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k x_k = T_a(x).$$

Combining the two inequalities gives $\|T_a\|_* = \|a\|_1$. Therefore $a \mapsto T_a$ is a linear bijection of ℓ^1 onto c_0^* that preserves norms: $c_0^* = \ell^1$ isometrically, the pairing being $\langle a, x \rangle = \sum_k a_k x_k$. The map is one-to-one because it is norm preserving, and onto by the preceding paragraph.

Part 2: the dual of c .

For a sequence $a = (a_0, a_1, a_2, \dots)$ indexed from 0 with $\|a\|_1 = \sum_{k \geq 0} |a_k| < \infty$, define

$$S_a(x) = a_0 \cdot \lim_{k \rightarrow \infty} x_k + \sum_{k=1}^{\infty} a_k x_k \quad \text{for all } x \in c.$$

Since $|x_k| \leq \|x\|_{\infty}$ for every k and also $|\lim_k x_k| \leq \|x\|_{\infty}$, the series converges absolutely and

$$|S_a(x)| \leq \left(|a_0| + \sum_{k \geq 1} |a_k|\right) \|x\|_{\infty},$$

so S_a is a bounded linear functional on c with $\|S_a\|_* \leq \|a\|_1$.

Conversely let T belong to c^* . Its restriction to the subspace c_0 is a bounded linear functional on c_0 of norm at most $\|T\|_*$, so by Part 1 there is a sequence $(a_1, a_2, \dots)$ in ℓ^1 , namely $a_k = T(e_k)$, with

$$T(y) = \sum_{k=1}^{\infty} a_k y_k \quad \text{for all } y \in c_0.$$

Put $b = T(\mathbf{1})$ and $a_0 = b - \sum_{k \geq 1} a_k$, which is a well-defined real number since $\sum_k |a_k| < \infty$. Let x belong to c and let $\ell = \lim_k x_k$. Then $y = x - \ell \mathbf{1}$ belongs to c_0 , so by linearity

$$T(x) = T(y) + \ell T(\mathbf{1}) = \sum_{k=1}^{\infty} a_k (x_k - \ell) + \ell b = \sum_{k=1}^{\infty} a_k x_k + \ell \left(b - \sum_{k \geq 1} a_k\right),$$

where splitting the series is legitimate because both $\sum_k a_k x_k$ and $\sum_k a_k \ell$ converge absolutely. Hence $T = S_a$ for $a = (a_0, a_1, a_2, \dots)$, which belongs to ℓ^1 .

It remains to check $\|S_a\|_* = \|a\|_1$; only the inequality $\|S_a\|_* \geq \|a\|_1$ is left. Let $\epsilon > 0$ and choose a natural number N with $\sum_{k > N} |a_k| < \epsilon$. Define x by

$$x_k = \text{sgn}(a_k) \text{ for } 1 \leq k \leq N, \quad x_k = \text{sgn}(a_0) \text{ for } k > N.$$

Then x is eventually constant, so it belongs to c with $\lim_k x_k = \operatorname{sgn}(a_0)$, and $\|x\|_\infty \leq 1$. Therefore

$$S_a(x) = |a_0| + \sum_{k=1}^N |a_k| + \sum_{k>N} a_k \operatorname{sgn}(a_0) \geq |a_0| + \sum_{k=1}^N |a_k| - \sum_{k>N} |a_k| > |a_0| + \sum_{k \geq 1} |a_k| - 2\epsilon,$$

using $\sum_{k=1}^N |a_k| > \sum_{k \geq 1} |a_k| - \epsilon$. Thus $\|S_a\|_* \geq \|a\|_1 - 2\epsilon$ for every $\epsilon > 0$, and so $\|S_a\|_* = \|a\|_1$.

Conclusion. Both dual spaces are isometrically isomorphic to ℓ^1 . Explicitly, every bounded linear functional on c_0 has the form $x \mapsto \sum_{k \geq 1} a_k x_k$ for a unique $a \in \ell^1$, with $\|T\|_* = \|a\|_1$; and every bounded linear functional on c has the form

$$x \mapsto a_0 \lim_{k \rightarrow \infty} x_k + \sum_{k=1}^{\infty} a_k x_k$$

for a unique summable $(a_0, a_1, \dots)$, with $\|T\|_* = |a_0| + \sum_{k \geq 1} |a_k|$. In each case uniqueness of the representing sequence follows from the norm identity just established: the correspondence is linear and norm preserving, hence one-to-one. Note finally that although c_0 is a subspace of c and both duals are isometrically isomorphic to ℓ^1 , this isometry is not implemented by restriction of functionals: the restriction map $c^* \rightarrow c_0^*$ discards a_0 , so it is onto but not one-to-one.

Problem 8.9. Let $[a, b]$ be a closed, bounded interval and $C[a, b]$ be normed by the maximum norm. Let x_0 belong to $[a, b]$. Define the linear functional T on $C[a, b]$ by $T(f) = f(x_0)$. Show that T is bounded and is given by Riemann-Stieltjes integration against a function of bounded variation.

Solution. Boundedness. T is clearly linear. For f in $C[a, b]$,

$$|T(f)| = |f(x_0)| \leq \max_{x \in [a, b]} |f(x)| = \|f\|_{\max},$$

so T is bounded and $\|T\|_* \leq 1$. Taking $f \equiv 1$, which has $\|f\|_{\max} = 1$ and $T(f) = 1$, we get $\|T\|_* = 1$. The representing function. Define g on $[a, b]$ as follows.

If $x_0 > a$, let g be the characteristic function of $[x_0, b]$, that is, $g = 0$ on $[a, x_0)$ and $g = 1$ on $[x_0, b]$.

If $x_0 = a$, let g be the characteristic function of $(a, b]$, that is, $g(a) = 0$ and $g = 1$ on $(a, b]$.

In either case g is a nondecreasing function taking only the values 0 and 1, with exactly one jump, of size 1, located at x_0 . For any partition $a = t_0 < t_1 < \dots < t_n = b$ the sum $\sum_{i=1}^n |g(t_i) - g(t_{i-1})|$ telescopes in absolute value to $g(b) - g(a) = 1$ because g is nondecreasing. Hence g is of bounded variation with $TV(g) = 1$.

Claim: for every f in $C[a, b]$ the Riemann-Stieltjes integral $\int_a^b f dg$ exists and equals $f(x_0)$.

Recall that $\int_a^b f dg$ is the limit, as the mesh of the partition tends to 0, of the Riemann-Stieltjes sums

$$S(f, g, P, \xi) = \sum_{i=1}^n f(\xi_i) [g(t_i) - g(t_{i-1})],$$

taken over partitions $P : a = t_0 < \dots < t_n = b$ with tags $\xi_i \in [t_{i-1}, t_i]$.

Since g takes only the values 0 and 1 and increases from 0 to 1 exactly once, at most one difference $g(t_i) - g(t_{i-1})$ is nonzero, and when nonzero it equals 1. Precisely: let i be the unique index with $g(t_{i-1}) = 0$ and $g(t_i) = 1$. Then

$$S(f, g, P, \xi) = f(\xi_i), \quad \xi_i \in [t_{i-1}, t_i].$$

We show x_0 lies in $[t_{i-1}, t_i]$ as well, so that $|\xi_i - x_0| \leq t_i - t_{i-1} \leq \operatorname{mesh}(P)$.

Consider first the case $x_0 > a$, where $g = \chi_{[x_0, b]}$. Then $g(t_{i-1}) = 0$ forces $t_{i-1} < x_0$, and $g(t_i) = 1$ forces $t_i \geq x_0$; hence $t_{i-1} < x_0 \leq t_i$. In the case $x_0 = a$, where $g = \chi_{(a, b]}$, we have $g(t_0) = g(a) = 0$ and $g(t_1) = 1$, so $i = 1$ and $x_0 = a = t_0$ lies in $[t_0, t_1]$.

In both cases, given $\epsilon > 0$, uniform continuity of f on the compact interval $[a, b]$ provides $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ whenever $|s - t| < \delta$ with s, t in $[a, b]$. If $\operatorname{mesh}(P) < \delta$ then $|\xi_i - x_0| < \delta$, whence

$$|S(f, g, P, \xi) - f(x_0)| = |f(\xi_i) - f(x_0)| < \epsilon$$

for every choice of tags. Therefore the Riemann-Stieltjes sums converge to $f(x_0)$ as the mesh tends to 0, so $\int_a^b f dg$ exists and

$$T(f) = f(x_0) = \int_a^b f dg \quad \text{for all } f \in C[a, b].$$

Thus T is a bounded linear functional on $C[a, b]$, of norm 1, given by Riemann-Stieltjes integration against the function g of bounded variation, and indeed $\|T\|_* = 1 = TV(g)$. This exhibits the point-evaluation functional as an instance of the second example of this section, and is consistent with the Riesz theorem, quoted in the Remark following the Riesz Representation Theorem for the dual of $L^p(E)$ and proved in Chapter 21, which asserts that every bounded linear functional on $C[a, b]$ arises this way.

Problem 8.10. Let f belong to $C[a, b]$. Show that there is a function g that is of bounded variation on $[a, b]$ for which

$$\int_a^b f dg = \|f\|_{\max} \quad \text{and} \quad TV(g) = 1.$$

Solution. A word on the printed statement. The second requirement is printed as $TV(f) = 1$, which cannot be what is meant: f is an arbitrary member of $C[a, b]$ and continuous functions need not be of bounded variation at all, let alone of total variation 1. The intended requirement is $TV(g) = 1$, the function g being the one whose existence is asserted, and that is what we prove: for each f in $C[a, b]$ there is a function g of bounded variation on $[a, b]$ with $\int_a^b f dg = \|f\|_{\max}$ and $TV(g) = 1$. Since f is continuous on the compact interval $[a, b]$, the Extreme Value Theorem applied to the continuous function $|f|$ provides a point x_0 in $[a, b]$ at which $|f|$ attains its maximum value:

$$|f(x_0)| = \max_{x \in [a, b]} |f(x)| = \|f\|_{\max}.$$

Let $\sigma = 1$ if $f(x_0) \geq 0$ and $\sigma = -1$ if $f(x_0) < 0$, so that $\sigma f(x_0) = |f(x_0)| = \|f\|_{\max}$ and $|\sigma| = 1$. Let g_0 be the jump function attached to the point x_0 that was constructed in the preceding problem: if $x_0 > a$, let g_0 be the characteristic function of $[x_0, b]$; if $x_0 = a$, let g_0 be the characteristic function of $(a, b]$. As shown there, g_0 is of bounded variation on $[a, b]$ with $TV(g_0) = 1$, the Riemann-Stieltjes integral $\int_a^b h dg_0$ exists for every h in $C[a, b]$, and

$$\int_a^b h dg_0 = h(x_0) \quad \text{for all } h \in C[a, b].$$

(Briefly: g_0 is nondecreasing with a single jump, of size 1, at x_0 , so its variation over every partition is $g_0(b) - g_0(a) = 1$; and for every partition $a = t_0 < \dots < t_n = b$ exactly one difference $g_0(t_i) - g_0(t_{i-1})$ is nonzero, that one equals 1 and its interval $[t_{i-1}, t_i]$ contains x_0 , so each Riemann-Stieltjes sum for the partition P equals $h(\xi_i)$ with $|\xi_i - x_0| \leq \text{mesh}(P)$, and uniform continuity of h on $[a, b]$ forces these sums to converge to $h(x_0)$ as the mesh tends to 0.)

Define $g = \sigma \cdot g_0$.

First, g is of bounded variation and $TV(g) = 1$. Indeed, for every partition $a = t_0 < t_1 < \dots < t_n = b$,

$$\sum_{i=1}^n |g(t_i) - g(t_{i-1})| = |\sigma| \sum_{i=1}^n |g_0(t_i) - g_0(t_{i-1})| = \sum_{i=1}^n |g_0(t_i) - g_0(t_{i-1})|,$$

so the two functions have exactly the same variation over every partition; taking the supremum over all partitions gives $TV(g) = TV(g_0) = 1$.

Second, since every Riemann-Stieltjes sum for the pair (f, g) is σ times the corresponding sum for (f, g_0) , namely

$$\sum_{i=1}^n f(\xi_i) [g(t_i) - g(t_{i-1})] = \sigma \sum_{i=1}^n f(\xi_i) [g_0(t_i) - g_0(t_{i-1})],$$

the integral $\int_a^b f dg$ exists and equals $\sigma \int_a^b f dg_0$. Therefore

$$\int_a^b f dg = \sigma f(x_0) = |f(x_0)| = \|f\|_{\max}.$$

This proves the assertion. (Note that the conclusion holds also when $f \equiv 0$: then $\|f\|_{\max} = 0$ and $\int_a^b f dg = 0$, while g still has total variation 1.)

Remark. This is the $C[a, b]$ analogue of formula (27) of this chapter. Write $T(h) = \int_a^b h dg = \sigma h(x_0)$ for h in $C[a, b]$. By inequality (4) of Section 8.1, $|T(h)| \leq TV(g) \cdot \|h\|_{\max} = \|h\|_{\max}$ for every h , while $T(h) = 1$ for the constant function $h \equiv \sigma$, which has maximum norm 1; hence $\|T\|_* = 1$. Since also $T(f) = \|f\|_{\max}$, the function g produces, concretely, the norm-one functional on $C[a, b]$ attaining its norm at f that the Hahn-Banach Theorem, proved in Section 2 of Chapter 14, supplies abstractly.

Problem 8.11. Let $[a, b]$ be a closed, bounded interval and $C[a, b]$ be normed by the maximum norm. Let T be a bounded linear functional on $C[a, b]$. For $x \in [a, b]$, let g_x be the member of $C[a, b]$ that is linear on $[a, x]$ and on $[x, b]$ with $g_x(a) = 0$, $g_x(x) = x - a$ and $g_x(b) = x - a$. Define $\Phi(x) = T(g_x)$ for $x \in [a, b]$. Show that Φ is Lipschitz on $[a, b]$.

Solution. First we identify g_x explicitly. On $[a, x]$ the function g_x is linear with $g_x(a) = 0$ and $g_x(x) = x - a$, so its slope is 1 and $g_x(t) = t - a$ there. On $[x, b]$ it is linear with $g_x(x) = x - a$ and $g_x(b) = x - a$, so it is constant, $g_x(t) = x - a$ there. (When $x = a$ or $x = b$ one of the two pieces degenerates, and the formula still holds.) In short,

$$g_x(t) = \min\{t, x\} - a \quad \text{for all } t \in [a, b].$$

In particular g_x is continuous on $[a, b]$, so g_x does belong to $C[a, b]$ and $\Phi(x) = T(g_x)$ is well defined. Estimate of $\|g_x - g_y\|_{\max}$. Let x, y belong to $[a, b]$ and assume without loss of generality that $y \leq x$. Then, for t in $[a, b]$,

$$g_x(t) - g_y(t) = \min\{t, x\} - \min\{t, y\} = \begin{cases} 0, & a \leq t \leq y, \\ t - y, & y \leq t \leq x, \\ x - y, & x \leq t \leq b. \end{cases}$$

Indeed: if $t \leq y \leq x$ both minima equal t ; if $y \leq t \leq x$ then $\min\{t, x\} = t$ and $\min\{t, y\} = y$; if $t \geq x \geq y$ then the minima are x and y . In every case $0 \leq g_x(t) - g_y(t) \leq x - y$, and therefore

$$\|g_x - g_y\|_{\max} = \max_{t \in [a, b]} |g_x(t) - g_y(t)| \leq x - y = |x - y|.$$

(In fact equality holds, the value $x - y$ being attained at $t = x$, but only the inequality is needed.)

Conclusion. Since T is linear and bounded, for all x, y in $[a, b]$,

$$|\Phi(x) - \Phi(y)| = |T(g_x) - T(g_y)| = |T(g_x - g_y)| \leq \|T\|_* \cdot \|g_x - g_y\|_{\max} \leq \|T\|_* \cdot |x - y|.$$

Thus Φ is Lipschitz on $[a, b]$ with Lipschitz constant at most $\|T\|_*$, which is finite because T is bounded.

Remark. Two comments. First, Problem 4 of this section identifies boundedness of a linear functional with its being Lipschitz, with Lipschitz constant $\|T\|_*$; what is proved above is the different (and easier) assertion that the numerical function $x \mapsto T(g_x)$ is Lipschitz on $[a, b]$, and it uses only the boundedness of T together with the estimate $\|g_x - g_y\|_{\max} \leq |x - y|$. Second, Φ here plays the role of the function Φ built in the proof of the Riesz Representation Theorem for the dual of L^p , where $\Phi(x) = T(\chi_{[a, x]})$; those characteristic functions are not continuous, so they are unavailable in $C[a, b]$, and the piecewise linear functions g_x replace them. Being Lipschitz on a closed, bounded interval, Φ is absolutely continuous there, by Chapter 6, and this is the first step toward the theorem of Riesz, quoted in the Remark at the end of Section 8.1 and proved in Chapter 21, that every bounded linear functional on $C[a, b]$ is given by Stieltjes integration against a function of bounded variation.

Problem 8.12. Show that the sequence defined in the first example of this section does not converge strongly to $f \equiv 0$ in $L^p[0, 1]$ for all $1 \leq p < \infty$.

Solution. The sequence in question is the sequence of Radamacher functions: for $I = [0, 1]$ and a natural number n , f_n is the step function on I defined by

$$f_n(x) = (-1)^k \quad \text{for } \frac{k}{2^n} \leq x < \frac{k+1}{2^n}, \quad 0 \leq k \leq 2^n - 1,$$

with, say, $f_n(1) = 1$ so that f_n is defined on all of I . (The book prints the index restriction as $0 \leq k < 2^n - 1$. That must be a misprint for $0 \leq k \leq 2^n - 1$: only the latter has the intervals exhaust $[0, 1)$, and only the latter supports the assertions made in the example itself, that $\|f_n\|_p \leq 1$ and that $|f_n - f_m|$ takes the value 2 on a set of measure $1/2$ when $n \neq m$. With that reading, the definition determines f_n everywhere on $[0, 1)$ and leaves only the single point $x = 1$ free.)

Fix $1 \leq p < \infty$ and a natural number n . The half-open intervals $[k/2^n, (k+1)/2^n)$, for $0 \leq k \leq 2^n - 1$, are disjoint and their union is $[0, 1)$, which differs from I by the single point 1, a set of measure zero. On each of these intervals $|f_n| = |(-1)^k| = 1$. Hence $|f_n| = 1$ almost everywhere on I , and therefore $|f_n|^p = 1$ almost everywhere on I . Consequently

$$\|f_n - f\|_p^p = \|f_n\|_p^p = \int_0^1 |f_n|^p = \int_0^1 1 = 1,$$

so $\|f_n - f\|_p = 1$ for every n , where $f \equiv 0$.

Since the sequence of real numbers $\{\|f_n - f\|_p\}$ is constantly equal to 1, it does not converge to 0. By the definition of strong convergence in a normed linear space, $\{f_n\}$ does not converge strongly to $f \equiv 0$ in $L^p[0, 1]$. The same computation shows that no subsequence of $\{f_n\}$ converges strongly to $f \equiv 0$, since every subsequence also has $\|f_{n_k} - f\|_p = 1$ for all k .

Remark. This is exactly the point of the example: for $1 < p < \infty$ the Radamacher sequence does converge weakly to $f \equiv 0$ in $L^p(I)$. Indeed $\{f_n\}$ is bounded in $L^p(I)$, and for x in $[0, 1]$ the integral $\int_0^x f_n$ has absolute value at most 2^{-n} , because on each pair of consecutive intervals of length 2^{-n} the function f_n integrates to 0, leaving at most one incompletely traversed interval of length 2^{-n} on which $|f_n| = 1$. Thus $\lim_{n \rightarrow \infty} \int_0^x f_n = 0 = \int_0^x f$ for all x in $[0, 1]$, and Theorem 11 tells us that $\{f_n\} \rightharpoonup f$ in $L^p(I)$. So this sequence witnesses the failure of the converse of the implication “strong convergence implies weak convergence.”

Problem 8.13. Fix real numbers α and β . For each natural number n , consider the step function f_n defined on $I = [0, 1]$ by

$$f_n(x) = (1 - (-1)^k)\alpha/2 + (1 + (-1)^k)\beta/2 \quad \text{for } k/2^n \leq x < (k+1)/2^n, \quad 0 \leq k < 2^n - 1.$$

For $1 < p < \infty$, show that $\{f_n\}$ converges weakly in $L^p(I)$ to the constant function that takes the value $(\alpha + \beta)/2$. For $\alpha \neq \beta$, show that no subsequence of $\{f_n\}$ converges strongly in $L^p(I)$.

Solution. Throughout, the index k is understood to run over $0 \leq k \leq 2^n - 1$, so that the intervals $[k/2^n, (k+1)/2^n)$ exhaust $[0, 1)$; the printed upper restriction is a misprint. The value of f_n at the single point 1 is irrelevant, since it affects nothing in $L^p(I)$.

Reduction to the Radamacher functions. If k is even then $(-1)^k = 1$ and the defining formula gives $f_n = 0 \cdot \alpha + \beta = \beta$; if k is odd then $(-1)^k = -1$ and the formula gives $f_n = \alpha + 0 \cdot \beta = \alpha$. Let r_n be the n -th Radamacher function of the first example of this section, so $r_n = (-1)^k$ on $[k/2^n, (k+1)/2^n)$. Then, on each such interval,

$$\frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2} r_n = \begin{cases} \beta, & k \text{ even,} \\ \alpha, & k \text{ odd,} \end{cases}$$

so that, almost everywhere on I ,

$$f_n = \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2} r_n.$$

Let f denote the constant function on I with value $(\alpha + \beta)/2$.

Boundedness in $L^p(I)$. Set $M = \max\{|\alpha|, |\beta|\}$. Then $|f_n| \leq M$ almost everywhere on I , so $\|f_n\|_p \leq M \cdot m(I)^{1/p} = M$ for all n : the sequence $\{f_n\}$ is bounded in $L^p(I)$. Also f belongs to $L^p(I)$.

The key estimate. We claim that

$$\left| \int_0^x r_n \right| \leq \frac{1}{2^n} \quad \text{for all } x \in [0, 1] \text{ and all } n.$$

For $x = 1$ this is clear: the 2^n intervals contribute $(-1)^k 2^{-n}$ each, and since 2^n is even these cancel in pairs, so $\int_0^1 r_n = 0$. Let $0 \leq x < 1$ and let m be the integer part of $2^n x$, so $0 \leq m \leq 2^n - 1$ and $m/2^n \leq x < (m+1)/2^n$. Split

$$\int_0^x r_n = \int_0^{m/2^n} r_n + \int_{m/2^n}^x r_n = \frac{1}{2^n} \sum_{k=0}^{m-1} (-1)^k + (-1)^m \left(x - \frac{m}{2^n} \right).$$

The first term is 0 if m is even and $1/2^n$ if m is odd, since consecutive terms of the alternating sum cancel. Write $\delta = x - m/2^n$, so $0 \leq \delta < 1/2^n$. If m is even the total is δ , which lies in $[0, 2^{-n}]$; if m is odd the total is $2^{-n} - \delta$, which lies in $(0, 2^{-n}]$. In both cases the absolute value is at most 2^{-n} , proving the claim.

Weak convergence for $1 < p < \infty$. By the claim and linearity of integration, for each x in $[0, 1]$,

$$\int_0^x f_n = \frac{\alpha + \beta}{2} x + \frac{\beta - \alpha}{2} \int_0^x r_n, \quad \left| \int_0^x f_n - \int_0^x f \right| \leq \frac{|\beta - \alpha|}{2} \cdot \frac{1}{2^n}.$$

Hence $\lim_{n \rightarrow \infty} \int_0^x f_n = \int_0^x f$ for every x in $[0, 1]$ (indeed uniformly in x). Since $\{f_n\}$ is a bounded sequence in $L^p[0, 1]$, f belongs to $L^p[0, 1]$, and $1 < p < \infty$, Theorem 11 tells us that this is necessary and sufficient for weak convergence. Therefore $\{f_n\}$ converges weakly in $L^p(I)$ to the constant function $f \equiv (\alpha + \beta)/2$.

No strongly convergent subsequence when $\alpha \neq \beta$. First compute the norms. Among the indices $0 \leq k \leq 2^n - 1$ there are 2^{n-1} even ones and 2^{n-1} odd ones, and each corresponding interval has measure 2^{-n} . Hence $f_n = \beta$ on a set of measure $1/2$ and $f_n = \alpha$ on a complementary set of measure $1/2$, so for every n

$$\|f_n\|_p^p = \frac{|\alpha|^p + |\beta|^p}{2}, \quad \text{while} \quad \|f\|_p^p = \left| \frac{\alpha + \beta}{2} \right|^p.$$

We claim that for $p > 1$ and $\alpha \neq \beta$,

$$\left| \frac{\alpha + \beta}{2} \right|^p < \frac{|\alpha|^p + |\beta|^p}{2}.$$

Suppose first that $|\alpha| \neq |\beta|$. The function $\varphi(s) = s^p$ on $[0, \infty)$ has $\varphi''(s) = p(p-1)s^{p-2} > 0$ for $s > 0$, so φ' is strictly increasing on $(0, \infty)$ and φ is strictly convex on $[0, \infty)$. Applied to the distinct points $|\alpha|$ and $|\beta|$ this gives

$$\left(\frac{|\alpha| + |\beta|}{2} \right)^p < \frac{|\alpha|^p + |\beta|^p}{2}.$$

Since $|(\alpha + \beta)/2| \leq (|\alpha| + |\beta|)/2$ and φ is nondecreasing on $[0, \infty)$, the claim follows in this case. Suppose next that $|\alpha| = |\beta|$ but $\alpha \neq \beta$. Then $\alpha = -\beta$ and $\alpha \neq 0$, so the left side is 0 while the right side is $|\alpha|^p > 0$; the claim holds here too.

Consequently

$$\|f\|_p < \left(\frac{|\alpha|^p + |\beta|^p}{2} \right)^{1/p} = \liminf_{n \rightarrow \infty} \|f_n\|_p,$$

the sequence $\{\|f_n\|_p\}$ being constant. Corollary 13 states that, for $1 < p < \infty$ and $\{f_n\}$ converging weakly to f in $L^p(E)$, a subsequence of $\{f_n\}$ converges strongly to f if and only if $\|f\|_p = \liminf_{n \rightarrow \infty} \|f_n\|_p$. Since that equality fails, no subsequence of $\{f_n\}$ converges strongly to f .

Finally, no subsequence converges strongly to anything else either. Suppose a subsequence $\{f_{n_j}\}$ converged strongly in $L^p(I)$ to some g . Strong convergence implies weak convergence, so $\{f_{n_j}\} \rightharpoonup g$. But every subsequence of a weakly convergent sequence converges weakly to the same limit, so $\{f_{n_j}\} \rightharpoonup f$ as well. Since weak sequential limits in $L^p(E)$ are unique, $g = f$, contradicting the preceding paragraph. Therefore, for $\alpha \neq \beta$, no subsequence of $\{f_n\}$ converges strongly in $L^p(I)$.

Problem 8.14. Let h be a continuous function defined on all of $\mathbb{R}$ that is periodic of period T and $\int_0^T h = 0$. Let $[a, b]$ be a closed, bounded interval and for each natural number n , define the function f_n on $[a, b]$ by $f_n(x) = h(nx)$. Define $f \equiv 0$ on $[a, b]$. Show that for $1 \leq p < \infty$, $\{f_n\}$ converges weakly to f in $L^p[a, b]$.

Solution. Fix $1 \leq p < \infty$ and let q be the conjugate of p . By Proposition 6, since $f \equiv 0$ it suffices to prove

$$\lim_{n \rightarrow \infty} \int_a^b g \cdot f_n = 0 \quad \text{for every } g \in L^q[a, b].$$

We prove the stronger statement that this holds for every g in $L^1[a, b]$; since $[a, b]$ has finite measure, $L^q[a, b]$ is contained in $L^1[a, b]$ for every q with $1 < q \leq \infty$ (immediately for $q = \infty$, and by Hölder's Inequality for $1 < q < \infty$, taking the conjugate exponent against the function identically 1, which is integrable over $[a, b]$), and for $p = 1$ we have $q = \infty$, while for $p > 1$ we have $1 < q < \infty$. So the stronger statement covers all $1 \leq p < \infty$ at once.

Step 1: h is bounded, and each f_n belongs to $L^p[a, b]$. The function h is continuous on the compact interval $[0, T]$, hence attains a finite maximum there; put $K = \max_{[0, T]} |h|$. Every real number u may be written $u = t + jT$ with $t \in [0, T)$ and j an integer, and $h(u) = h(t)$ by periodicity, so $|h| \leq K$ on all of $\mathbb{R}$. Consequently each $f_n(x) = h(nx)$ is continuous on $[a, b]$ with $|f_n| \leq K$, so f_n belongs to $L^p[a, b]$ and $\|f_n\|_p \leq K(b - a)^{1/p}$. In particular $\{f_n\}$ is bounded in $L^p[a, b]$.

Step 2: the antiderivative of h is bounded. Define

$$H(u) = \int_0^u h \quad \text{for } u \in \mathbb{R}.$$

H is continuous, being an indefinite integral of a continuous function. We claim H is periodic of period T . For any real u ,

$$H(u + T) - H(u) = \int_u^{u+T} h.$$

Substituting $t = s - T$ and using $h(t + T) = h(t)$,

$$\int_T^{u+T} h(s) ds = \int_0^u h(t + T) dt = \int_0^u h,$$

and therefore

$$\int_u^{u+T} h = \int_u^T h + \int_T^{u+T} h = \int_u^T h + \int_0^u h = \int_0^T h = 0.$$

(The manipulations are valid for every real u , the integrals being ordinary Riemann integrals of a continuous function, with the usual orientation convention.) Hence $H(u + T) = H(u)$ for all u . Being continuous and periodic, H is bounded: with $M = \max_{[0, T]} |H|$, the same reduction as in Step 1 gives $|H| \leq M$ on all of $\mathbb{R}$.

Step 3: the integral of f_n over any subinterval tends to 0, uniformly. Let $a \leq c \leq d \leq b$ and let n be a natural number. The change of variable $u = nx$ gives

$$\int_c^d f_n = \int_c^d h(nx) dx = \frac{1}{n} \int_{nc}^{nd} h(u) du = \frac{H(nd) - H(nc)}{n},$$

and hence

$$\left| \int_c^d f_n \right| \leq \frac{2M}{n} \quad \text{for all } a \leq c \leq d \leq b.$$

Step 4: the conclusion for step functions. Let φ be a step function on $[a, b]$: there is a partition $a = x_0 < x_1 < \cdots < x_N = b$ and real numbers $c_1, \dots, c_N$ with $\varphi = c_i$ on the open interval (x_{i-1}, x_i) for each i . The finitely many partition points form a set of measure zero, so

$$\int_a^b \varphi \cdot f_n = \sum_{i=1}^N c_i \int_{x_{i-1}}^{x_i} f_n, \quad \left| \int_a^b \varphi \cdot f_n \right| \leq \frac{2M}{n} \sum_{i=1}^N |c_i|,$$

by Step 3. Since $\sum_{i=1}^N |c_i|$ does not depend on n , we conclude

$$\lim_{n \rightarrow \infty} \int_a^b \varphi \cdot f_n = 0 \quad \text{for every step function } \varphi \text{ on } [a, b].$$

Step 5: the conclusion for every g in $L^1[a, b]$. Let g belong to $L^1[a, b]$ and let $\epsilon > 0$. By Proposition 10 of the preceding chapter, the step functions are dense in $L^1[a, b]$, so there is a step function φ on $[a, b]$ with

$$\|g - \varphi\|_1 < \frac{\epsilon}{2(K+1)}.$$

For every n , since $|f_n| \leq K$ on $[a, b]$,

$$\left| \int_a^b (g - \varphi) \cdot f_n \right| \leq K \int_a^b |g - \varphi| = K \|g - \varphi\|_1 < \frac{\epsilon}{2}.$$

By Step 4 there is a natural number N such that $\left| \int_a^b \varphi \cdot f_n \right| < \epsilon/2$ for all $n \geq N$. Hence for $n \geq N$,

$$\left| \int_a^b g \cdot f_n \right| \leq \left| \int_a^b (g - \varphi) \cdot f_n \right| + \left| \int_a^b \varphi \cdot f_n \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\lim_{n \rightarrow \infty} \int_a^b g \cdot f_n = 0$.

Conclusion. Let $1 \leq p < \infty$ and let q be the conjugate of p . Every g in $L^q[a, b]$ belongs to $L^1[a, b]$, so by Step 5,

$$\lim_{n \rightarrow \infty} \int_a^b g \cdot f_n = 0 = \int_a^b g \cdot f \quad \text{for all } g \in L^q[a, b].$$

By Proposition 6, $\{f_n\}$ converges weakly to $f \equiv 0$ in $L^p[a, b]$ for every $1 \leq p < \infty$.

Remark. The Riemann-Lebesgue Lemma, given as an example in this section, is the special case $h(u) = \sin u$, $T = 2\pi$, $[a, b] = [-\pi, \pi]$. Note also that unless $h \equiv 0$ the convergence is not strong for any p : since $|h|^p$ is continuous, periodic of period T and not identically zero, the same argument applied to $|h|^p - \gamma$, where $\gamma = (1/T) \int_0^T |h|^p > 0$, shows $\|f_n\|_p^p = \int_a^b |h(nx)|^p dx \rightarrow \gamma(b-a) > 0$, whereas $\|f\|_p = 0$.

8.3 Problems 8.15–8.21

Problem 8.15. Let $1 < p < \infty$ and f_0 belong to $L^p(\mathbb{R})$. For each natural number n , define $f_n(x) = f_0(x - n)$ for all x . Define $f \equiv 0$ on $\mathbb{R}$. Show that $\{f_n\} \rightharpoonup f$ in $L^p(\mathbb{R})$. Is this true for $p = 1$?

Solution. Throughout, q denotes the conjugate of p ; since $p > 1$ we have $q < \infty$.

Step 1: the sequence is bounded in $L^p(\mathbb{R})$. Lebesgue measure is translation invariant, so for each n ,

$$\int_{\mathbb{R}} |f_n|^p = \int_{\mathbb{R}} |f_0(x - n)|^p dx = \int_{\mathbb{R}} |f_0|^p,$$

that is, $\|f_n\|_p = \|f_0\|_p$ for all n . Hence $\{f_n\}$ is a bounded sequence in $L^p(\mathbb{R})$.

Step 2: verification of the criterion of Theorem 10. Since $\{f_n\}$ is bounded in $L^p(\mathbb{R})$, $f \equiv 0$ belongs to $L^p(\mathbb{R})$, and $p > 1$, Theorem 10 tells us that $\{f_n\} \rightharpoonup f$ in $L^p(\mathbb{R})$ if and only if

$$\lim_{n \rightarrow \infty} \int_A f_n = \int_A f = 0 \quad \text{for every measurable set } A \text{ of finite measure.}$$

So let $A \subseteq \mathbb{R}$ be measurable with $m(A) < \infty$. By the translation invariance of Lebesgue measure,

$$\int_A f_n = \int_A f_0(x-n) dx = \int_{A-n} f_0,$$

where $A-n = \{x-n : x \in A\}$.

Let $\epsilon > 0$. Since $A \cap [-M, M] \uparrow A$ as $M \rightarrow \infty$ and $m(A) < \infty$, the continuity of measure provides an $M > 0$ for which

$$m(A \setminus [-M, M]) < \epsilon.$$

Write $A_M = A \cap [-M, M]$ and $B_M = A \setminus [-M, M]$. Then

$$\left| \int_A f_n \right| \leq \int_{A_M-n} |f_0| + \int_{B_M-n} |f_0|.$$

We estimate the second term with Hölder's Inequality, applied to $|f_0|$ and the characteristic function of $B_M - n$; since $m(B_M - n) = m(B_M) < \epsilon$ and $q < \infty$,

$$\int_{B_M-n} |f_0| \leq \|f_0\|_p \cdot [m(B_M - n)]^{1/q} \leq \|f_0\|_p \cdot \epsilon^{1/q}.$$

This is the only place where $p > 1$ is used, and it is used decisively: for $p = 1$ the factor $[m(B_M - n)]^{1/q}$ is replaced by 1 and the estimate is worthless.

For the first term, observe that $A_M - n \subseteq [-M-n, M-n]$, so again by Hölder's Inequality,

$$\int_{A_M-n} |f_0| \leq \left[\int_{-M-n}^{M-n} |f_0|^p \right]^{1/p} \cdot (2M)^{1/q}.$$

Now $|f_0|^p$ is integrable over $\mathbb{R}$, so the set function $A \mapsto \int_A |f_0|^p$ is a finite measure on the Lebesgue measurable sets. The sets $(-\infty, M-n]$ decrease, as $n \rightarrow \infty$, to the empty set, and so by the continuity of this measure,

$$\lim_{n \rightarrow \infty} \int_{-M-n}^{M-n} |f_0|^p = 0, \quad \text{hence} \quad \lim_{n \rightarrow \infty} \int_{-M-n}^{M-n} |f_0| = 0.$$

Therefore $\lim_{n \rightarrow \infty} \int_{A_M-n} |f_0| = 0$.

Combining the two estimates,

$$\limsup_{n \rightarrow \infty} \left| \int_A f_n \right| \leq \|f_0\|_p \cdot \epsilon^{1/q}.$$

As $\epsilon > 0$ was arbitrary and $q < \infty$, we conclude $\lim_{n \rightarrow \infty} \int_A f_n = 0$. By Theorem 10, $\{f_n\} \rightarrow f \equiv 0$ in $L^p(\mathbb{R})$.

Step 3: the case $p = 1$. The assertion fails. Take $f_0 = \chi_{[0,1]}$, which belongs to $L^1(\mathbb{R})$. Then $f_n = \chi_{[n,n+1]}$. The function $g \equiv 1$ belongs to $L^\infty(\mathbb{R})$, the conjugate space of $L^1(\mathbb{R})$, and

$$\int_{\mathbb{R}} g \cdot f_n = 1 \text{ for all } n, \quad \text{while} \quad \int_{\mathbb{R}} g \cdot f = 0.$$

By Proposition 6, $\{f_n\}$ does not converge weakly to $f \equiv 0$ in $L^1(\mathbb{R})$. (More generally, for any $f_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} f_0 \neq 0$ the same test function $g \equiv 1$ shows the failure, since $\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f_0$ for every n .) This is exactly the example presented in the text just before Theorem 12, where the verification of the criterion of Theorem 10 is left as “an exercise in the use of continuity of measure” – which is what Step 2 above carries out.

Problem 8.16. Let E be a measurable set, $\{f_n\}$ a sequence in $L^2(E)$ and f belong to $L^2(E)$.

Suppose

$$\lim_{n \rightarrow \infty} \int_E f_n \cdot f = \lim_{n \rightarrow \infty} \int_E f_n^2 = \int_E f^2.$$

Show that $\{f_n\}$ converges strongly to f in $L^2(E)$.

Solution. Since 2 is its own conjugate, Hölder's Inequality tells us that the product $f_n \cdot f$ of two functions in $L^2(E)$ is integrable over E , so each of the integrals in the hypothesis is a well-defined real number. In particular $\int_E f^2 = \|f\|_2^2$ is finite.

Fix an index n . The function $f_n - f$ belongs to the linear space $L^2(E)$, and

$$(f_n - f)^2 = f_n^2 - 2f_n \cdot f + f^2 \quad \text{pointwise on } E.$$

Each of the three functions on the right is integrable over E : f_n^2 and f^2 because f_n and f belong to $L^2(E)$, and $f_n \cdot f$ by Hölder's Inequality as just observed. Therefore, by the linearity of integration,

$$\|f_n - f\|_2^2 = \int_E (f_n - f)^2 = \int_E f_n^2 - 2 \int_E f_n \cdot f + \int_E f^2.$$

Now pass to the limit. By hypothesis

$$\lim_{n \rightarrow \infty} \int_E f_n^2 = \int_E f^2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_E f_n \cdot f = \int_E f^2,$$

and these are limits of real sequences, so by the linearity of convergence for sequences of real numbers,

$$\lim_{n \rightarrow \infty} \|f_n - f\|_2^2 = \int_E f^2 - 2 \int_E f^2 + \int_E f^2 = 0.$$

Hence $\lim_{n \rightarrow \infty} \|f_n - f\|_2 = 0$, that is, $\{f_n\}$ converges strongly to f in $L^2(E)$.

We remark that this computation is precisely the argument used in the text to prove the Radon-Riesz Theorem in the case $p = 2$: there the first hypothesis, $\lim_{n \rightarrow \infty} \int_E f_n \cdot f = \int_E f^2$, is obtained from the weak convergence $\{f_n\} \rightharpoonup f$ in $L^2(E)$ by taking f itself as the test function in Proposition 6 (legitimate because f belongs to $L^2(E) = L^q(E)$), and the second hypothesis is the convergence of the norms.

Problem 8.17. Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. Consider the following four properties: (i) $\{f_n\}$ converges pointwise to f almost everywhere on E , (ii) $\{f_n\} \rightharpoonup f$ in $L^p(E)$, (iii) $\{\|f_n\|_p\}$ converges to $\|f\|_p$, and (iv) $\{f_n\} \rightarrow f$ in $L^p(E)$. If $\{f_n\}$ possesses two of these properties, does a subsequence possess all four properties?

Solution. A remark on the statement: the text as printed reads “(iii) $\{\|f_n\|_p\}$ converges to $\{\|f_n\|_p\}$ ”, which is a misprint; the property intended, and the only one that makes the question non-trivial, is that $\{\|f_n\|_p\}$ converges to $\|f\|_p$. We read (iii) that way throughout. Likewise (ii) is weak convergence and (iv) is strong (norm) convergence.

The answer is yes for five of the six pairs, and no for the pair (i), (ii).

We first record three implications that will be used repeatedly. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$ with $1 < p < \infty$ and let $f \in L^p(E)$.

(A) Property (i) implies property (ii). This is exactly Theorem 12: a bounded sequence in $L^p(E)$, $1 < p < \infty$, that converges pointwise a.e. on E to f converges weakly to f in $L^p(E)$.

(B) Properties (ii) and (iii) together imply property (iv). This is the Radon-Riesz Theorem: if $\{f_n\} \rightharpoonup f$ in $L^p(E)$ with $1 < p < \infty$, then $\{f_n\} \rightarrow f$ in $L^p(E)$ if and only if $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$.

(C) Property (iv) implies properties (ii) and (iii), and implies that some subsequence has property (i). Indeed, strong convergence implies weak convergence in any normed linear space, since $|T(f_n) - T(f)| \leq \|T\|_* \cdot \|f_n - f\|$ for every $T \in X^*$; convergence of the norms follows from $|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p$; and the Riesz-Fischer Theorem of the preceding chapter states that if $\{f_n\} \rightarrow f$ in $L^p(E)$, then a

subsequence of $\{f_n\}$ converges pointwise a.e. on E to f .

Now consider the six pairs.

The pairs (i) and (iv), (ii) and (iv), (iii) and (iv). Each contains property (iv). By (C), the whole sequence then has properties (ii), (iii), (iv), and some subsequence has in addition property (i). Since properties (ii), (iii), (iv) are inherited by every subsequence (a subsequence of a strongly convergent sequence converges strongly to the same limit, and (ii), (iii) then follow from (C)), that subsequence possesses all four properties. So the answer is yes.

The pair (i) and (iii). By (A), property (i) and the boundedness of $\{f_n\}$ give property (ii). Property (ii) together with the assumed property (iii) gives property (iv) by (B). Thus the whole sequence possesses all four properties, and the answer is yes.

The pair (ii) and (iii). By (B) the whole sequence has property (iv), hence also (ii) and (iii); and by (C) a subsequence in addition has property (i). That subsequence possesses all four properties. The answer is yes.

The pair (i) and (ii). Here the answer is no. Note first that by (A) this pair carries no more information than property (i) alone. Let $E = [0, 1]$, let $f \equiv 0$, and for each natural number n define

$$f_n = n^{1/p} \cdot \chi_{(0, 1/n)}.$$

Each f_n belongs to $L^p[0, 1]$ and

$$\|f_n\|_p = \left[\int_0^{1/n} n \right]^{1/p} = 1 \text{ for all } n,$$

so $\{f_n\}$ is a bounded sequence in $L^p[0, 1]$. If $0 < x \leq 1$, then $f_n(x) = 0$ once $n > 1/x$, so $\{f_n\}$ converges to $f \equiv 0$ pointwise on $(0, 1]$ and therefore almost everywhere on $[0, 1]$: property (i) holds. By Theorem 12, $\{f_n\} \rightharpoonup f$ in $L^p[0, 1]$: property (ii) holds.

But $\|f\|_p = 0$ while $\|f_{n_k}\|_p = 1$ for every subsequence $\{f_{n_k}\}$, so no subsequence has property (iii); and since strong convergence would force convergence of the norms by (C), no subsequence has property (iv) either. Hence no subsequence possesses all four properties.

In summary: if $\{f_n\}$ possesses two of the four properties, then a subsequence possesses all four, except in the single case where the two properties are (i) and (ii).

Problem 8.18. Let X be a normed linear space and $\{f_n\} \rightharpoonup f$ in X . Suppose $\{\|f_n\|\}$ is unbounded. Show that, by possibly taking a subsequence and relabeling, we may suppose $\|f_n\| \geq \alpha_n = n \cdot 3^n$ for all n . Then show that, by possibly taking a further subsequence and relabeling, we may suppose $\{\|f_n\|/\alpha_n\} \rightarrow \alpha \in [1, \infty]$. Define $g_n = \alpha_n/\|f_n\| \cdot f_n$ for each n . Show that $\{g_n\}$ converges weakly and $\|g_n\| = n \cdot 3^n$ for all n .

Solution. Two elementary observations will be used. First, a subsequence of a weakly convergent sequence converges weakly to the same limit: if $\{f_n\} \rightharpoonup f$ in X and $\{f_{n_k}\}$ is a subsequence, then for each $T \in X^*$ the sequence $\{T(f_{n_k})\}$ is a subsequence of the convergent sequence of real numbers $\{T(f_n)\}$ and hence converges to the same limit $T(f)$. Second, since $\{\|f_n\|\}$ is unbounded, for every $M > 0$ the set $\{n : \|f_n\| > M\}$ is infinite: were it finite for some M , then $\|f_n\| \leq M'$ for all n , where M' is the maximum of M and the finitely many values $\|f_n\|$ with $\|f_n\| > M$, contradicting unboundedness. Step 1: arranging $\|f_n\| \geq \alpha_n = n \cdot 3^n$.

We select indices inductively. Using the second observation with $M = 1 \cdot 3^1$, choose n_1 with $\|f_{n_1}\| \geq 1 \cdot 3^1$. Having chosen $n_1 < n_2 < \cdots < n_{k-1}$, use the second observation with $M = k \cdot 3^k$: infinitely many indices n satisfy $\|f_n\| > k \cdot 3^k$, so one of them exceeds n_{k-1} ; call it n_k . This produces a strictly increasing sequence of indices $\{n_k\}$ with

$$\|f_{n_k}\| \geq k \cdot 3^k = \alpha_k \text{ for all } k.$$

By the first observation, $\{f_{n_k}\} \rightharpoonup f$ in X . Relabeling this subsequence as $\{f_n\}$, we may and do suppose

$$\|f_n\| \geq \alpha_n = n \cdot 3^n \text{ for all } n.$$

In particular $\|f_n\| \geq 3 > 0$ for every n , so no f_n is the zero vector and the quotients below are defined.

Step 2: arranging $\{\|f_n\|/\alpha_n\} \rightarrow \alpha \in [1, \infty]$.

Set $\beta_n = \|f_n\|/\alpha_n$. By Step 1, $\beta_n \geq 1$ for all n . If $\{\beta_n\}$ already converges in the extended sense to some $\alpha \in [1, \infty]$, there is nothing to do.

In general it does not, and here one must be careful about what “relabeling” does. If we retain the indices $n_1 < n_2 < \dots$ and relabel f_{n_k} as the k th term, the ratio attached to the k th term of the new sequence is not $\beta_{n_k} = \|f_{n_k}\|/\alpha_{n_k}$ but rather $\|f_{n_k}\|/\alpha_k$; since $n_k \geq k$ and $\{\alpha_n\}$ is increasing, $\alpha_k \leq \alpha_{n_k}$, so relabeling can only increase the ratio. Thus one cannot simply apply the Bolzano-Weierstrass Theorem to $\{\beta_n\}$ and read off the conclusion: a finite limit for $\{\beta_{n_k}\}$ need not survive the reindexing. Nor can the finiteness of α be arranged in general. If, for instance, $\|f_n\| = 3^{n^2}$ for all n , then for any indices $n_1 < n_2 < \dots$ one has $n_k \geq k$ and hence

$$\frac{\|f_{n_k}\|}{\alpha_k} = \frac{3^{n_k^2}}{k \cdot 3^k} \geq \frac{3^{k^2}}{k \cdot 3^k} \rightarrow \infty,$$

so every relabeled subsequence has ratios tending to ∞ . This is why the value $\alpha = \infty$ must be allowed in the assertion.

What is always available is the following selection, and it suffices. By Step 1, $\|f_n\| \geq n \cdot 3^n$, so $\lim_{n \rightarrow \infty} \|f_n\| = \infty$. Hence, for each natural number k , the inequality $\|f_n\| \geq k \cdot \alpha_k$ holds for all sufficiently large n , since $k \cdot \alpha_k$ is a fixed real number. Choose inductively $n_1 < n_2 < \dots$ with

$$\|f_{n_k}\| \geq k \cdot \alpha_k \text{ for all } k.$$

By the first observation, $\{f_{n_k}\} \rightharpoonup f$ in X . Relabeling f_{n_k} as the k th term, the relabeled sequence satisfies $\|f_k\| \geq k \cdot \alpha_k \geq \alpha_k$, so the normalization of Step 1 persists, and

$$\frac{\|f_k\|}{\alpha_k} \geq k \rightarrow \infty,$$

so the relabeled ratios converge to $\alpha = \infty \in [1, \infty]$. Thus, after this further passage to a subsequence and relabeling, we may suppose both

$$\|f_n\| \geq \alpha_n \text{ for all } n \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\|f_n\|}{\alpha_n} = \alpha \in [1, \infty].$$

Step 3: the sequence $\{g_n\}$.

Define $c_n = \alpha_n/\|f_n\|$ and $g_n = c_n \cdot f_n$. Since $\|f_n\| > 0$, c_n is a well-defined positive real number, and by the homogeneity of the norm,

$$\|g_n\| = c_n \|f_n\| = \frac{\alpha_n}{\|f_n\|} \cdot \|f_n\| = \alpha_n = n \cdot 3^n \text{ for all } n.$$

It remains to show that $\{g_n\}$ converges weakly in X . By Step 2, c_n is the reciprocal of $\|f_n\|/\alpha_n$, and this last quantity converges to $\alpha \in [1, \infty]$. Hence

$$\lim_{n \rightarrow \infty} c_n = c, \quad \text{where } c = 1/\alpha \in (0, 1] \text{ if } \alpha < \infty \text{ and } c = 0 \text{ if } \alpha = \infty.$$

(If $\alpha = \infty$, then $\beta_n \rightarrow \infty$ and so $c_n = 1/\beta_n \rightarrow 0$; if $\alpha \in [1, \infty)$, then $\alpha \geq 1 > 0$ and $c_n \rightarrow 1/\alpha$ by the quotient rule for limits of real sequences.)

Now let T belong to X^* . By the linearity of T ,

$$T(g_n) = T(c_n f_n) = c_n \cdot T(f_n).$$

Since $\{T(f_n)\}$ converges to the real number $T(f)$ and $\{c_n\}$ converges to the real number c , the product rule for limits of real sequences gives

$$\lim_{n \rightarrow \infty} T(g_n) = c \cdot T(f) = T(c \cdot f),$$

the last equality again by linearity of T . As $T \in X^*$ was arbitrary, $\{g_n\} \rightharpoonup c \cdot f$ in X , where $c \cdot f$ is the vector f/α if $\alpha < \infty$ and is the zero vector if $\alpha = \infty$. Thus $\{g_n\}$ converges weakly in X and $\|g_n\| = n \cdot 3^n$ for all n .

This is precisely the justification of the phrase “without loss of generality (see Problem 18), by possibly taking scalar multiples of a subsequence” in the proof of Theorem 7: to derive a contradiction from the assumption that a weakly convergent sequence in $L^p(E)$ has unbounded norms, one may replace the sequence by $\{g_n\}$, which is a sequence of scalar multiples of a subsequence, still converges weakly, and satisfies the normalization (18), namely $\|g_n\| = n \cdot 3^n$ for all n .

Problem 8.19. For $1 \leq p < \infty$, let $\{\zeta_n\}$ be a bounded sequence in ℓ^p and ζ belong to ℓ^p . Show that $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^p if and only if it converges componentwise, that is, for each index k ,

$$\lim_{n \rightarrow \infty} \zeta_n^k = \zeta^k \quad \text{where } \zeta_n = \langle \zeta_n^1, \zeta_n^2, \dots \rangle \text{ and } \zeta = \langle \zeta^1, \zeta^2, \dots \rangle.$$

Solution. Throughout, q denotes the conjugate of p . By Problem 7 of this chapter (the Riesz Representation Theorem for ℓ^p), for $1 \leq p < \infty$ every bounded linear functional T on ℓ^p is of the form

$$T(\xi) = \sum_{k=1}^{\infty} \eta^k \xi^k \text{ for all } \xi \in \ell^p,$$

for a unique $\eta = (\eta^1, \eta^2, \dots)$ in ℓ^q , and $\|T\|_* = \|\eta\|_q$. Thus $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^p if and only if

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \eta^k \zeta_n^k = \sum_{k=1}^{\infty} \eta^k \zeta^k \text{ for every } \eta \in \ell^q.$$

We refer to this displayed condition as the pairing criterion.

Weak convergence implies componentwise convergence, for every p with $1 \leq p < \infty$. Fix an index k and define $T_k(\xi) = \xi^k$ for $\xi \in \ell^p$. Then T_k is linear, and

$$|T_k(\xi)| = |\xi^k| \leq \left[\sum_{j=1}^{\infty} |\xi^j|^p \right]^{1/p} = \|\xi\|_p,$$

so T_k is a bounded linear functional on ℓ^p . (No representation theorem is needed for this direction: T_k is manifestly bounded.) If $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^p , then by the definition of weak convergence $\lim_{n \rightarrow \infty} T_k(\zeta_n) = T_k(\zeta)$, that is, $\lim_{n \rightarrow \infty} \zeta_n^k = \zeta^k$. Since k was arbitrary, $\{\zeta_n\}$ converges componentwise to ζ .

Componentwise convergence implies weak convergence, when $1 < p < \infty$. Assume now $p > 1$, so that $q < \infty$. Since $\{\zeta_n\}$ is bounded in ℓ^p and $\zeta \in \ell^p$, the number

$$M = \sup_n \|\zeta_n - \zeta\|_p \leq \sup_n \|\zeta_n\|_p + \|\zeta\|_p$$

is finite. Let $\eta \in \ell^q$ and let $\epsilon > 0$. Since $\sum_{k=1}^{\infty} |\eta^k|^q < \infty$, the tail of this convergent series is small: choose a natural number N for which

$$\left[\sum_{k=N+1}^{\infty} |\eta^k|^q \right]^{1/q} < \epsilon.$$

For each n , by the linearity of the (absolutely convergent) series and the triangle inequality,

$$\left| \sum_{k=1}^{\infty} \eta^k \zeta_n^k - \sum_{k=1}^{\infty} \eta^k \zeta^k \right| \leq \sum_{k=1}^N |\eta^k| |\zeta_n^k - \zeta^k| + \left| \sum_{k=N+1}^{\infty} \eta^k (\zeta_n^k - \zeta^k) \right|.$$

Hölder's Inequality for series – the counting-measure form of Theorem 1 of the preceding chapter, proved by the identical argument from Young's Inequality – applied to the sequences $\langle 0, \dots, 0, \eta^{N+1}, \eta^{N+2}, \dots \rangle$ in ℓ^q and $\zeta_n - \zeta$ in ℓ^p , bounds the second term:

$$\left| \sum_{k=N+1}^{\infty} \eta^k (\zeta_n^k - \zeta^k) \right| \leq \left[\sum_{k=N+1}^{\infty} |\eta^k|^q \right]^{1/q} \cdot \|\zeta_n - \zeta\|_p \leq \epsilon M.$$

The first term is a sum of the fixed finite number N of terms, each of which tends to 0 as $n \rightarrow \infty$ by componentwise convergence; hence it tends to 0. Consequently

$$\limsup_{n \rightarrow \infty} \left| \sum_{k=1}^{\infty} \eta^k \zeta_n^k - \sum_{k=1}^{\infty} \eta^k \zeta^k \right| \leq \epsilon M.$$

Since $\epsilon > 0$ was arbitrary and M is a fixed finite number, the limit is 0. Thus the pairing criterion holds and $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^p .

(This second half is the ℓ^p analogue of Proposition 9, proved above directly so that no transfer of that proposition from $L^p(E)$ to the counting measure is needed: the family $\mathcal{F} = \{e_k\}_{k \geq 1}$, where e_k has k th component 1 and all other components 0, has as its linear span the finitely supported sequences, and this span is dense in ℓ^q when $q < \infty$. Componentwise convergence is exactly the statement that $\lim_n \langle \zeta_n, e_k \rangle = \langle \zeta, e_k \rangle$ for each k .)

A remark on the case $p = 1$. The implication just proved genuinely requires $p > 1$, and the statement of the problem is not correct at $p = 1$: only the first half survives. For $p = 1$ the conjugate exponent is $q = \infty$, the tail estimate above is unavailable, and the linear span of $\{e_k\}$ is not dense in ℓ^∞ . Indeed, let $\zeta_n = e_n$ and $\zeta = 0$. Then $\|e_n\|_1 = 1$, so $\{\zeta_n\}$ is bounded in ℓ^1 , and for each fixed k we have $e_n^k = 0$ as soon as $n > k$, so $\{\zeta_n\}$ converges componentwise to $\zeta = 0$. But $\eta = (1, 1, 1, \dots)$ belongs to ℓ^∞ and

$$\sum_{k=1}^{\infty} \eta^k \zeta_n^k = 1 \text{ for all } n, \quad \text{while} \quad \sum_{k=1}^{\infty} \eta^k \zeta^k = 0,$$

so $\{\zeta_n\}$ does not converge weakly to ζ in ℓ^1 . This is the assertion of Problem 21.

Problem 8.20. Let $1 \leq p_1 < p_2 < \infty$, $\{f_n\}$ be a sequence in $L^{p_2}[0, 1]$ and f belong to $L^{p_2}[0, 1]$. What is the relationship between $\{f_n\} \rightharpoonup f$ in $L^{p_2}[0, 1]$ and $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0, 1]$?

Solution. The answer: weak convergence in $L^{p_2}[0, 1]$ implies weak convergence in $L^{p_1}[0, 1]$; the converse implication is false in general, but it does hold under the additional hypothesis that $\{f_n\}$ is bounded in $L^{p_2}[0, 1]$.

Let q_1 and q_2 be the conjugates of p_1 and p_2 . Since $1 \leq p_1 < p_2 < \infty$ we have $1 < q_2 < q_1 \leq \infty$. Since $[0, 1]$ has finite measure, Corollary 3 of the preceding chapter tells us that

$$L^{p_2}[0, 1] \subseteq L^{p_1}[0, 1] \quad \text{and} \quad L^{q_1}[0, 1] \subseteq L^{q_2}[0, 1],$$

with $\|h\|_{p_1} \leq \|h\|_{p_2}$ and $\|h\|_{q_2} \leq \|h\|_{q_1}$, since $m([0, 1]) = 1$ makes the constant c in that corollary equal to 1. (For $q_1 = \infty$ the second inclusion is the case $p_2 = \infty$ of that corollary.) In particular, the functions f_n and f all belong to $L^{p_1}[0, 1]$, so the second mode of convergence is at least meaningful.

Weak convergence in L^{p_2} implies weak convergence in L^{p_1} . Suppose $\{f_n\} \rightharpoonup f$ in $L^{p_2}[0, 1]$. By Proposition 6 this means

$$\lim_{n \rightarrow \infty} \int_0^1 g \cdot f_n = \int_0^1 g \cdot f \text{ for all } g \in L^{q_2}[0, 1].$$

Let g belong to $L^{q_1}[0, 1]$. By the inclusion above, g belongs to $L^{q_2}[0, 1]$, so the displayed limit holds for this g . Since $g \in L^{q_1}[0, 1]$ was arbitrary, Proposition 6, applied now with the exponent p_1 , gives $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0, 1]$.

The converse fails in general. Let $\theta = \frac{1}{2}(1/p_1 + 1/p_2)$, so that $1/p_2 < \theta < 1/p_1$, and for each natural number n define

$$f_n = n^\theta \cdot \chi_{(0,1/n)} \text{ on } [0, 1].$$

Each f_n is a bounded measurable function on $[0, 1]$ and so belongs to $L^{p_2}[0, 1]$. For any r with $1 \leq r < \infty$,

$$\|f_n\|_r = \left[n^{\theta r} \cdot \frac{1}{n} \right]^{1/r} = n^{\theta - 1/r}.$$

Take $f \equiv 0$. Since $\theta - 1/p_1 < 0$ we get $\|f_n - f\|_{p_1} = n^{\theta - 1/p_1} \rightarrow 0$, so $\{f_n\}$ converges strongly, and hence weakly, to $f \equiv 0$ in $L^{p_1}[0, 1]$. On the other hand $\theta - 1/p_2 > 0$, so

$$\|f_n\|_{p_2} = n^{\theta - 1/p_2} \rightarrow \infty,$$

and $\{f_n\}$ is unbounded in $L^{p_2}[0, 1]$. By Theorem 7, a sequence that converges weakly in $L^{p_2}[0, 1]$ is bounded in $L^{p_2}[0, 1]$. Therefore $\{f_n\}$ does not converge weakly in $L^{p_2}[0, 1]$ to f , nor indeed to any function whatsoever. So weak convergence in $L^{p_1}[0, 1]$ does not imply weak convergence in $L^{p_2}[0, 1]$. The converse holds if $\{f_n\}$ is bounded in $L^{p_2}[0, 1]$. Suppose $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0, 1]$ and $\{f_n\}$ is bounded in $L^{p_2}[0, 1]$. Let A be a measurable subset of $[0, 1]$. Since $[0, 1]$ has finite measure, χ_A belongs to $L^r[0, 1]$ for every r with $1 \leq r \leq \infty$, and in particular $\chi_A \in L^{q_1}[0, 1]$. Hence, by Proposition 6 applied with the exponent p_1 ,

$$\lim_{n \rightarrow \infty} \int_A f_n = \lim_{n \rightarrow \infty} \int_0^1 \chi_A \cdot f_n = \int_0^1 \chi_A \cdot f = \int_A f.$$

Since $\{f_n\}$ is bounded in $L^{p_2}[0, 1]$, f belongs to $L^{p_2}[0, 1]$, and $1 < p_2 < \infty$, Theorem 10 tells us that this last property is equivalent to $\{f_n\} \rightharpoonup f$ in $L^{p_2}[0, 1]$.

Summarizing: weak convergence in the larger exponent is the stronger statement; it implies weak convergence in every smaller exponent, and the two are equivalent exactly when one supplements weak convergence in $L^{p_1}[0, 1]$ with boundedness in $L^{p_2}[0, 1]$, which by Theorem 7 is in any case a necessary condition for weak convergence in $L^{p_2}[0, 1]$.

Problem 8.21. For $1 \leq p < \infty$ and each index n , let $e_n \in \ell^p$ have n th component 1 and other components vanish. Show that if $p > 1$, then $\{e_n\}$ converges weakly to 0 in ℓ^p , but no subsequence converges strongly to 0. Show that $\{e_n\}$ does not converge weakly in ℓ^1 .

Solution. For every p with $1 \leq p < \infty$ we have

$$\|e_n\|_p = \left[\sum_{k=1}^{\infty} |e_n^k|^p \right]^{1/p} = 1 \text{ for all } n,$$

so $\{e_n\}$ is a sequence of unit vectors, and in particular is bounded in ℓ^p .

Weak convergence to 0 when $p > 1$. Let q be the conjugate of p ; since $p > 1$, $q < \infty$. By Problem 7 of this chapter (the Riesz Representation Theorem for ℓ^p), every bounded linear functional T on ℓ^p has the form

$$T(\xi) = \sum_{k=1}^{\infty} \eta^k \xi^k \text{ for all } \xi \in \ell^p,$$

for some $\eta \in \ell^q$. For such a T ,

$$T(e_n) = \sum_{k=1}^{\infty} \eta^k e_n^k = \eta^n.$$

Since $\eta \in \ell^q$ with $q < \infty$, the series $\sum_{k=1}^{\infty} |\eta^k|^q$ converges, and therefore its terms tend to 0; hence $|\eta^n|^q \rightarrow 0$ and so $\eta^n \rightarrow 0$. Thus

$$\lim_{n \rightarrow \infty} T(e_n) = 0 = T(0) \text{ for every } T \in (\ell^p)^*,$$

that is, $\{e_n\} \rightharpoonup 0$ in ℓ^p . (Alternatively: $\{e_n\}$ is bounded in ℓ^p and converges componentwise to 0, since for each fixed k we have $e_n^k = 0$ whenever $n \neq k$; now apply Problem 19.) No subsequence converges strongly to 0. For any subsequence $\{e_{n_k}\}$ we have

$$\|e_{n_k} - 0\|_p = \|e_{n_k}\|_p = 1 \text{ for all } k,$$

so $\{\|e_{n_k} - 0\|_p\}$ does not converge to 0. Hence no subsequence of $\{e_n\}$ converges strongly to 0 in ℓ^p . (This is consistent with the Radon-Riesz Theorem: $\{e_n\} \rightharpoonup 0$ while $\lim_n \|e_n\|_p = 1 \neq 0 = \|0\|_p$, so the norms do not converge to the norm of the weak limit, and strong convergence must fail. Equivalently, by Corollary 13, a subsequence converges strongly to the weak limit 0 if and only if $\|0\|_p = \liminf \|e_n\|_p$, and here the left side is 0 while the right side is 1.)

Failure of weak convergence in ℓ^1 . Suppose, to get a contradiction, that $\{e_n\} \rightharpoonup \zeta$ in ℓ^1 for some $\zeta = (\zeta^1, \zeta^2, \dots)$ in ℓ^1 .

First we identify ζ . Fix an index k and define $T_k(\xi) = \xi^k$ for $\xi \in \ell^1$. This is a linear functional, and $|T_k(\xi)| = |\xi^k| \leq \sum_{j=1}^{\infty} |\xi^j| = \|\xi\|_1$, so T_k is bounded on ℓ^1 . Weak convergence gives

$$\zeta^k = T_k(\zeta) = \lim_{n \rightarrow \infty} T_k(e_n) = \lim_{n \rightarrow \infty} e_n^k = 0,$$

because $e_n^k = 0$ for all $n \neq k$. Since k was arbitrary, $\zeta = 0$.

Next we contradict this. Integration against the bounded sequence $\eta = \langle 1, 1, 1, \dots \rangle$ gives the functional

$$T(\xi) = \sum_{k=1}^{\infty} \xi^k \text{ for } \xi \in \ell^1.$$

The series converges absolutely for each $\xi \in \ell^1$, T is linear, and $|T(\xi)| \leq \sum_{k=1}^{\infty} |\xi^k| = \|\xi\|_1$, so T is a bounded linear functional on ℓ^1 . But

$$T(e_n) = 1 \text{ for all } n, \quad \text{while} \quad T(\zeta) = T(0) = 0,$$

so $\{T(e_n)\}$ does not converge to $T(\zeta)$. This contradicts the weak convergence of $\{e_n\}$ to ζ .

Therefore $\{e_n\}$ does not converge weakly in ℓ^1 . Notice that the argument needed no representation theorem for $(\ell^1)^*$: the two explicit bounded functionals T_k and T suffice. Notice also the contrast with the case $p > 1$: there $\{e_n\}$ is a bounded sequence with no strongly convergent subsequence, yet it does converge weakly, whereas in ℓ^1 even weak convergence fails.

8.4 Problems 8.22–8.28

Problem 8.22. State and prove the Radon-Riesz Theorem in ℓ^2 .

Solution. Statement. Let $\{\zeta_n\}$ be a sequence in ℓ^2 and let ζ belong to ℓ^2 . Suppose $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^2 . Then $\{\zeta_n\} \rightarrow \zeta$ strongly in ℓ^2 if and only if

$$\lim_{n \rightarrow \infty} \|\zeta_n\|_2 = \|\zeta\|_2.$$

Here, for $\eta = (\eta^1, \eta^2, \dots)$ in ℓ^2 we write $\|\eta\|_2 = (\sum_{k=1}^{\infty} |\eta^k|^2)^{1/2}$, and for ξ, η in ℓ^2 we write $\langle \xi, \eta \rangle = \sum_{k=1}^{\infty} \xi^k \eta^k$; this series converges absolutely by the Cauchy-Schwarz (Hoelder) Inequality, and $|\langle \xi, \eta \rangle| \leq \|\xi\|_2 \|\eta\|_2$.

Preliminary: a description of weak convergence in ℓ^2 . By the Riesz Representation Theorem for the dual of ℓ^p , $1 \leq p < \infty$ (Problem 7 of this chapter), applied with $p = 2$, whose conjugate is $q = 2$: a functional T belongs to $(\ell^2)^*$ if and only if there is a unique η in ℓ^2 for which

$$T(\xi) = \sum_{k=1}^{\infty} \eta^k \xi^k = \langle \xi, \eta \rangle \quad \text{for all } \xi \in \ell^2,$$

and moreover $\|T\|_* = \|\eta\|_2$. Therefore, by the definition of weak convergence in a normed linear space, $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^2 if and only if

$$\lim_{n \rightarrow \infty} \langle \zeta_n, \eta \rangle = \langle \zeta, \eta \rangle \quad \text{for all } \eta \in \ell^2. \quad (*)$$

Proof of the theorem. First suppose $\{\zeta_n\} \rightarrow \zeta$ strongly, that is, $\lim_{n \rightarrow \infty} \|\zeta_n - \zeta\|_2 = 0$. In any linear space normed by $\|\cdot\|$ the triangle inequality gives

$$|\|\xi\| - \|\eta\|| \leq \|\xi - \eta\| \quad \text{for all } \xi, \eta,$$

and hence $|\|\zeta_n\|_2 - \|\zeta\|_2| \leq \|\zeta_n - \zeta\|_2 \rightarrow 0$. Thus $\lim_{n \rightarrow \infty} \|\zeta_n\|_2 = \|\zeta\|_2$. Observe that this half uses neither the weak convergence hypothesis nor any special property of ℓ^2 .

Conversely, assume $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^2 and $\lim_{n \rightarrow \infty} \|\zeta_n\|_2 = \|\zeta\|_2$. Since each $\zeta_n - \zeta$ belongs to ℓ^2 , we may expand the square of its norm: for each n ,

$$\begin{aligned} \|\zeta_n - \zeta\|_2^2 &= \sum_{k=1}^{\infty} (\zeta_n^k - \zeta^k)^2 \\ &= \sum_{k=1}^{\infty} \left[(\zeta_n^k)^2 - 2\zeta_n^k \zeta^k + (\zeta^k)^2 \right] \\ &= \|\zeta_n\|_2^2 - 2\langle \zeta_n, \zeta \rangle + \|\zeta\|_2^2. \end{aligned}$$

The regrouping of the three series is legitimate because each of the three separately converges absolutely: $\sum_k (\zeta_n^k)^2$ and $\sum_k (\zeta^k)^2$ converge since $\zeta_n, \zeta \in \ell^2$, and $\sum_k |\zeta_n^k \zeta^k| \leq \|\zeta_n\|_2 \|\zeta\|_2 < \infty$ by the Cauchy-Schwarz Inequality.

Now ζ itself belongs to ℓ^2 , so we may take $\eta = \zeta$ in $(*)$ to conclude

$$\lim_{n \rightarrow \infty} \langle \zeta_n, \zeta \rangle = \langle \zeta, \zeta \rangle = \|\zeta\|_2^2.$$

Together with the assumed convergence of the norms, $\lim_{n \rightarrow \infty} \|\zeta_n\|_2^2 = \|\zeta\|_2^2$, we infer that

$$\lim_{n \rightarrow \infty} \|\zeta_n - \zeta\|_2^2 = \|\zeta\|_2^2 - 2\|\zeta\|_2^2 + \|\zeta\|_2^2 = 0.$$

Therefore $\{\zeta_n\} \rightarrow \zeta$ strongly in ℓ^2 . This completes the proof.

Remark. This is exactly the argument the text gives for the Radon-Riesz Theorem in $L^2(E)$, with integration replaced by summation, that is, with Lebesgue measure replaced by counting measure on the natural numbers; the identity $(a - b)^2 = a^2 - 2ab + b^2$ is what makes the case $p = 2$ so simple.

A companion corollary. Just as Corollary 13 follows from the Radon-Riesz Theorem in $L^p(E)$, we get: if $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^2 , then a subsequence of $\{\zeta_n\}$ converges strongly to ζ in ℓ^2 if and only if $\|\zeta\|_2 = \liminf \|\zeta_n\|_2$. Indeed, if $\|\zeta\|_2 = \liminf \|\zeta_n\|_2$, choose a subsequence $\{\zeta_{n_k}\}$ with $\lim_{k \rightarrow \infty} \|\zeta_{n_k}\|_2 = \|\zeta\|_2$; this subsequence still converges weakly to ζ , so by the theorem just proved it converges strongly to ζ . Conversely, if some subsequence converges strongly to ζ , then along it the norms converge to $\|\zeta\|_2$, so $\liminf \|\zeta_n\|_2 \leq \|\zeta\|_2$; the reverse inequality $\|\zeta\|_2 \leq \liminf \|\zeta_n\|_2$ holds for any weakly convergent sequence. Indeed, this is trivial if $\zeta = 0$; and if $\zeta \neq 0$, then for each n the Cauchy-Schwarz Inequality gives $\langle \zeta_n, \zeta \rangle \leq \|\zeta_n\|_2 \|\zeta\|_2$, so passing to the limit inferior (which preserves termwise inequalities) yields $\|\zeta\|_2^2 = \lim_{n \rightarrow \infty} \langle \zeta_n, \zeta \rangle \leq \liminf \|\zeta_n\|_2 \cdot \|\zeta\|_2$, and division by $\|\zeta\|_2 > 0$ gives the claim.

Problem 8.23. Let $[a, b]$ be a closed, bounded interval. Suppose $\{f_n\} \rightharpoonup f$ in $C[a, b]$. Show that $\{f_n\}$ converges pointwise on $[a, b]$ to f .

Solution. Throughout, $C[a, b]$ is normed by the maximum norm,

$$\|g\|_{\max} = \max_{x \in [a, b]} |g(x)|,$$

which is defined and finite for every g in $C[a, b]$ by the Extreme Value Theorem.

Fix a point x_0 in $[a, b]$ and define the evaluation functional T_{x_0} on $C[a, b]$ by

$$T_{x_0}(g) = g(x_0) \quad \text{for all } g \in C[a, b].$$

Step 1: T_{x_0} is a bounded linear functional on $C[a, b]$. Linearity is immediate from the definition of the linear structure on $C[a, b]$: for g, h in $C[a, b]$ and real α, β , the function $\alpha g + \beta h$ is evaluated pointwise, so

$$T_{x_0}(\alpha g + \beta h) = (\alpha g + \beta h)(x_0) = \alpha g(x_0) + \beta h(x_0) = \alpha T_{x_0}(g) + \beta T_{x_0}(h).$$

For boundedness, observe that for every g in $C[a, b]$,

$$|T_{x_0}(g)| = |g(x_0)| \leq \max_{x \in [a, b]} |g(x)| = \|g\|_{\max},$$

since x_0 is one of the points over which the maximum is taken. Hence T_{x_0} is bounded and $\|T_{x_0}\|_* \leq 1$. In fact $\|T_{x_0}\|_* = 1$: the constant function $g \equiv 1$ belongs to $C[a, b]$, has $\|g\|_{\max} = 1$, and $T_{x_0}(g) = 1$. (This is the content of Problem 9 of this chapter.) Therefore T_{x_0} belongs to the dual space $(C[a, b])^*$. *Step 2: apply the definition of weak convergence.* By hypothesis $\{f_n\} \rightharpoonup f$ in $C[a, b]$, which by definition means

$$\lim_{n \rightarrow \infty} T(f_n) = T(f) \quad \text{for every } T \in (C[a, b])^*.$$

By Step 1 the functional T_{x_0} is a legitimate choice of T , and therefore

$$\lim_{n \rightarrow \infty} f_n(x_0) = \lim_{n \rightarrow \infty} T_{x_0}(f_n) = T_{x_0}(f) = f(x_0).$$

Since x_0 was an arbitrary point of $[a, b]$, the sequence $\{f_n\}$ converges to f pointwise on all of $[a, b]$.

Remark. The converse implication is false: pointwise convergence in $C[a, b]$ does not imply weak convergence. On $[0, 1]$, for each natural number n let f_n be the continuous function that vanishes outside the interval $(0, 1/n)$, is linear on $[0, 1/(2n)]$ and on $[1/(2n), 1/n]$, and takes the value n at $x = 1/(2n)$. For each x in $(0, 1]$ we have $f_n(x) = 0$ once $n > 1/x$, and $f_n(0) = 0$ for every n ; hence $\{f_n\}$ converges pointwise on $[0, 1]$ to $f \equiv 0$. But $T(g) = \int_0^1 g$ defines a bounded linear functional on $C[0, 1]$, since $|T(g)| \leq \|g\|_{\max}$, and $T(f_n)$ is the area of a triangle of base $1/n$ and height n , namely

$$T(f_n) = \frac{1}{2} \cdot \frac{1}{n} \cdot n = \frac{1}{2} \quad \text{for all } n,$$

which does not converge to $T(f) = 0$. So $\{f_n\}$ does not converge weakly to f in $C[0, 1]$.

Problem 8.24. Let $[a, b]$ be a closed, bounded interval. Suppose $\{f_n\} \rightharpoonup f$ in $L^\infty[a, b]$. Show that

$$\lim_{n \rightarrow \infty} \int_a^x f_n = \int_a^x f \quad \text{for all } x \in [a, b].$$

Solution. First note that all of the integrals in question are defined and finite. Indeed, $[a, b]$ has finite measure, so if h belongs to $L^\infty[a, b]$ then h is measurable and $|h| \leq \|h\|_\infty$ almost everywhere on $[a, b]$, whence

$$\int_a^b |h| \leq \|h\|_\infty \cdot (b - a) < \infty,$$

so h belongs to $L^1[a, b]$ and $\int_a^x h$ is a finite real number for each x in $[a, b]$.

Step 1: integration against an L^1 function is a bounded linear functional on L^∞ . Let g belong to $L^1[a, b]$ and define T_g on $L^\infty[a, b]$ by

$$T_g(h) = \int_a^b g \cdot h \quad \text{for all } h \in L^\infty[a, b].$$

This is well defined: $g \cdot h$ is measurable, and $|g \cdot h| \leq \|h\|_\infty \cdot |g|$ almost everywhere on $[a, b]$, so by the

monotonicity of integration $g \cdot h$ is integrable over $[a, b]$ and

$$|T_g(h)| \leq \int_a^b |g \cdot h| \leq \|g\|_1 \cdot \|h\|_\infty.$$

Linearity of T_g follows from the linearity of integration. Hence T_g is a bounded linear functional on $L^\infty[a, b]$ with $\|T_g\|_* \leq \|g\|_1$; that is, T_g belongs to $(L^\infty[a, b])^*$. (This is precisely the observation made in Section 8.1 preceding the problems: each g in $L^1[a, b]$ induces such a functional, although not every bounded linear functional on $L^\infty[a, b]$ arises this way. Note that this direction is Hölder's Inequality with $p = \infty$, $q = 1$, and it is all we need; we do not need any representation theorem for the dual of L^∞ .)

Step 2: choose the right g . Fix x in $[a, b]$ and let

$$g = \chi_{[a, x]},$$

the characteristic function of the interval $[a, x]$ (interpreted as $\chi_{\{a\}}$, hence as the zero function in L^1 , when $x = a$). Then g is measurable and

$$\int_a^b |g| = \int_a^b \chi_{[a, x]} = x - a < \infty,$$

so g belongs to $L^1[a, b]$. Moreover, for every h in $L^\infty[a, b]$,

$$T_g(h) = \int_a^b \chi_{[a, x]} \cdot h = \int_a^x h.$$

Step 3: apply weak convergence. By hypothesis $\{f_n\} \rightharpoonup f$ in $L^\infty[a, b]$, which by the definition of weak convergence in a normed linear space means that

$$\lim_{n \rightarrow \infty} T(f_n) = T(f) \quad \text{for every } T \in (L^\infty[a, b])^*.$$

By Steps 1 and 2, $T_g = T_{\chi_{[a, x]}}$ is such a functional. Therefore

$$\lim_{n \rightarrow \infty} \int_a^x f_n = \lim_{n \rightarrow \infty} T_g(f_n) = T_g(f) = \int_a^x f.$$

Since x was an arbitrary point of $[a, b]$, this holds for all x in $[a, b]$, as required.

Remark. The same argument, verbatim, shows more: $\lim_{n \rightarrow \infty} \int_A f_n = \int_A f$ for every measurable subset A of $[a, b]$, since χ_A belongs to $L^1[a, b]$. By linearity it then follows that $\lim_{n \rightarrow \infty} \int_a^b \varphi \cdot f_n = \int_a^b \varphi \cdot f$ for every simple function φ on $[a, b]$. One would like to pass from the simple functions to all of $L^1[a, b]$ by density, as in Proposition 9 of this chapter; that passage requires in addition that $\{\|f_n\|_\infty\}$ be bounded, and we do not assert that here, since the boundedness of a weakly convergent sequence in a general normed linear space is the Uniform Boundedness Principle, which is not available at this point in the book. What we have proved above uses only the single functional $T_{\chi_{[a, x]}}$ and needs no such input.

Problem 8.25. Let X be a normed linear space. Suppose that for each $f \in X$ there is bounded linear functional $T \in X^*$ for which $T(f) = \|f\|$ and $\|T\|_* = 1$.

- (i) Prove that if $\{f_n\}$ converges weakly in X to both f_1 and f_2 , then $f_1 = f_2$.
- (ii) Prove that if $\{f_n\} \rightharpoonup f$ in X , then $\|f\| \leq \liminf \|f_n\|$.

Solution. Recall the definition: $\{f_n\} \rightharpoonup f$ in X means that $\lim_{n \rightarrow \infty} T(f_n) = T(f)$ for every T in X^* . Recall also that for T in X^* ,

$$|T(h)| \leq \|T\|_* \cdot \|h\| \quad \text{for all } h \in X. \quad (\dagger)$$

(By the Hahn-Banach Theorem, proved in Section 2 of Chapter 14, the hypothesis imposed on X holds in every normed linear space; for $X = L^p(E)$ with $1 \leq p < \infty$ it is realized concretely by the conjugate

function, as the Remark at the end of Section 8.2 records in display (27). If $X = \{0\}$ both assertions are vacuous, so we assume $X \neq \{0\}$, which is what makes the normalization $\|T\|_* = 1$ possible.)

(i) *Uniqueness of the weak limit.* Suppose $\{f_n\} \rightharpoonup f_1$ and $\{f_n\} \rightharpoonup f_2$. Consider the vector

$$h = f_1 - f_2,$$

which belongs to X since X is a linear space. By hypothesis there is a bounded linear functional T in X^* for which

$$T(h) = \|h\| \quad \text{and} \quad \|T\|_* = 1.$$

Apply the definition of weak convergence twice to this single functional T . Since $\{f_n\} \rightharpoonup f_1$,

$$\lim_{n \rightarrow \infty} T(f_n) = T(f_1);$$

since $\{f_n\} \rightharpoonup f_2$,

$$\lim_{n \rightarrow \infty} T(f_n) = T(f_2).$$

A sequence of real numbers has at most one limit, so $T(f_1) = T(f_2)$. By the linearity of T ,

$$\|h\| = T(h) = T(f_1 - f_2) = T(f_1) - T(f_2) = 0.$$

A norm vanishes only at the zero vector, so $h = 0$, that is, $f_1 = f_2$. Thus a sequence in X can converge weakly to at most one member of X .

(ii) *Weak lower semicontinuity of the norm.* Suppose $\{f_n\} \rightharpoonup f$ in X . By hypothesis there is T in X^* with

$$T(f) = \|f\| \quad \text{and} \quad \|T\|_* = 1.$$

For each index n we infer from (†) that

$$T(f_n) \leq |T(f_n)| \leq \|T\|_* \cdot \|f_n\| = \|f_n\|.$$

Since $\{f_n\} \rightharpoonup f$, the sequence of real numbers $\{T(f_n)\}$ converges to $T(f)$; a convergent sequence of real numbers has limit inferior equal to its limit, so

$$\|f\| = T(f) = \lim_{n \rightarrow \infty} T(f_n) = \liminf_{n \rightarrow \infty} T(f_n).$$

Finally, the limit inferior is monotone with respect to the termwise inequality $T(f_n) \leq \|f_n\|$, and therefore

$$\|f\| = \liminf_{n \rightarrow \infty} T(f_n) \leq \liminf_{n \rightarrow \infty} \|f_n\|.$$

(If $\{\|f_n\|\}$ is unbounded the right-hand side may be $+\infty$, in which case the inequality is trivially true.) This is the desired inequality.

Remark. These are the abstract versions of two facts proved concretely for $L^p(E)$ in Section 8.2: the uniqueness of weak sequential limits in $L^p(E)$, established just after Proposition 6 using the conjugate function of $f_1 - f_2$, and the right-hand inequality of (17) in Theorem 7. Comparing the two arguments makes plain that the only property of the conjugate function used there is the pair of identities (27), $T(f) = \|f\|_p$ and $\|T\|_* = 1$.

Problem 8.26. (Uniform Boundedness Principle) Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p . Suppose $\{f_n\}$ is a sequence in $L^p(E)$ such that for each $g \in L^q(E)$, the sequence $\{\int_E g \cdot f_n\}$ is bounded. Show that $\{f_n\}$ is bounded in $L^p(E)$.

Solution. Recall from Theorem 1 of Chapter 7 that for f in $L^p(E)$ with $f \neq 0$ the **conjugate function**

$$f^* = \|f\|_p^{1-p} \operatorname{sgn}(f) |f|^{p-1}$$

belongs to $L^q(E)$ and satisfies

$$\int_E f^* \cdot f = \|f\|_p \quad \text{and} \quad \|f^*\|_q = 1. \quad (1)$$

We also use Hoelder's Inequality: $|\int_E h \cdot f| \leq \|h\|_q \cdot \|f\|_p$ for $h \in L^q(E)$, $f \in L^p(E)$.

The proof is by contradiction, and is the gliding-hump construction used in the proof of Theorem 7; there weak convergence entered only through the fact that a convergent sequence of real numbers is bounded, and that is exactly what is assumed here.

Step 1: normalization. Assume, contrary to what is to be proved, that $\{\|f_n\|_p\}$ is unbounded. Observe first that then, for every $M \geq 0$ and every index N , there is an index $n > N$ with $\|f_n\|_p \geq M$: otherwise the terms beyond N would all be bounded by M , and, there being only finitely many terms with index at most N , the whole sequence of norms would be bounded. Hence we may choose inductively indices $n_1 < n_2 < \dots$ with $\|f_{n_k}\|_p \geq k \cdot 3^k$ for each k . For each k set

$$c_k = \frac{k \cdot 3^k}{\|f_{n_k}\|_p} \in (0, 1] \quad \text{and} \quad h_k = c_k \cdot f_{n_k}.$$

Then h_k belongs to $L^p(E)$ and $\|h_k\|_p = k \cdot 3^k$. Moreover the hypothesis passes to $\{h_k\}$: if g belongs to $L^q(E)$ and M is a bound for $\{|\int_E g \cdot f_n|\}$, then since $0 < c_k \leq 1$,

$$\left| \int_E g \cdot h_k \right| = c_k \left| \int_E g \cdot f_{n_k} \right| \leq M \quad \text{for all } k.$$

So it suffices to derive a contradiction from the existence of a sequence $\{h_n\}$ in $L^p(E)$ with

$$\|h_n\|_p = n \cdot 3^n \quad \text{for all } n, \quad (2)$$

for which $\{\int_E g \cdot h_n\}$ is bounded for every g in $L^q(E)$. (This normalization is Problem 18 of this chapter, carried out here directly.) Note each $h_n \neq 0$, so each conjugate function $(h_n)^*$ is defined.

Step 2: choice of signs. We select inductively real numbers $\{\epsilon_k\}$ with $\epsilon_k = \pm 1/3^k$. Put $\epsilon_1 = 1/3$. Suppose $\epsilon_1, \dots, \epsilon_{n-1}$ have been chosen, and set

$$A_n = \int_E \left[\sum_{k=1}^{n-1} \epsilon_k (h_k)^* \right] \cdot h_n$$

(with $A_1 = 0$). Define

$$\epsilon_n = \frac{1}{3^n} \quad \text{if } A_n \geq 0, \quad \epsilon_n = -\frac{1}{3^n} \quad \text{if } A_n < 0.$$

Each integral above is finite, since $\sum_{k=1}^{n-1} \epsilon_k (h_k)^*$ belongs to $L^q(E)$ and Hoelder's Inequality applies. With this choice, using (1) and (2),

$$\int_E \left[\sum_{k=1}^n \epsilon_k (h_k)^* \right] \cdot h_n = A_n + \epsilon_n \int_E (h_n)^* \cdot h_n = A_n + \epsilon_n \|h_n\|_p.$$

If $A_n \geq 0$ then $\epsilon_n > 0$ and the right-hand side is at least $\|h_n\|_p/3^n$; if $A_n < 0$ then $\epsilon_n < 0$ and the right-hand side is at most $-\|h_n\|_p/3^n$. In either case the two terms have the same sign, and therefore

$$\left| \int_E \left[\sum_{k=1}^n \epsilon_k (h_k)^* \right] \cdot h_n \right| \geq \frac{1}{3^n} \|h_n\|_p = \frac{n \cdot 3^n}{3^n} = n \quad \text{for all } n. \quad (3)$$

Also, by (1), $\|\epsilon_k (h_k)^*\|_q = |\epsilon_k| \cdot \|(h_k)^*\|_q = 1/3^k$ for every k .

Step 3: construction of g . Since $\sum_{k=1}^{\infty} \|\epsilon_k (h_k)^*\|_q = \sum_{k=1}^{\infty} 1/3^k = 1/2 < \infty$, the partial sums of the series $\sum_{k=1}^{\infty} \epsilon_k (h_k)^*$ form a Cauchy sequence in $L^q(E)$: for $m < n$,

$$\left\| \sum_{k=m+1}^n \epsilon_k (h_k)^* \right\|_q \leq \sum_{k=m+1}^n \frac{1}{3^k} \leq \frac{1}{2 \cdot 3^m}.$$

The space $L^q(E)$ is complete: for $1 \leq q < \infty$ this is the Riesz-Fischer Theorem of Chapter 7, while for $q = \infty$ (the case $p = 1$) it is Problem 33 of Chapter 7. Hence there is a function g in $L^q(E)$ with

$$g = \sum_{k=1}^{\infty} \epsilon_k(h_k)^*,$$

the series converging in $L^q(E)$, and for each n ,

$$\left\| g - \sum_{k=1}^n \epsilon_k(h_k)^* \right\|_q = \left\| \sum_{k=n+1}^{\infty} \epsilon_k(h_k)^* \right\|_q \leq \sum_{k=n+1}^{\infty} \frac{1}{3^k} = \frac{1}{2 \cdot 3^n}. \quad (4)$$

Step 4: the contradiction. Fix a natural number n . Write $g = \sum_{k=1}^n \epsilon_k(h_k)^* + r_n$, where $r_n = g - \sum_{k=1}^n \epsilon_k(h_k)^*$. By the triangle inequality, (3), Hoelder's Inequality, (4) and (2),

$$\begin{aligned} \left| \int_E g \cdot h_n \right| &\geq \left| \int_E \left[\sum_{k=1}^n \epsilon_k(h_k)^* \right] \cdot h_n \right| - \left| \int_E r_n \cdot h_n \right| \\ &\geq n - \|r_n\|_q \cdot \|h_n\|_p \\ &\geq n - \frac{1}{2 \cdot 3^n} \cdot n \cdot 3^n \\ &= n - \frac{n}{2} = \frac{n}{2}. \end{aligned}$$

Thus $\{\int_E g \cdot h_n\}$ is unbounded. But g belongs to $L^q(E)$, so by the hypothesis inherited in Step 1 this sequence must be bounded. This contradiction shows the assumption of Step 1 is false.

Therefore $\{\|f_n\|_p\}$ is bounded, that is, $\{f_n\}$ is a bounded sequence in $L^p(E)$.

Remark. Theorem 7 is the special case of this result in which $\{f_n\} \rightarrow f$ in $L^p(E)$: then for each g in $L^q(E)$ the sequence $\{\int_E g \cdot f_n\}$ converges, by Proposition 6, and a convergent sequence of real numbers is bounded. The name of this problem records that it is the L^p instance of the general Uniform Boundedness Principle for a family of bounded linear functionals on a Banach space, whose usual proof runs through the Baire Category Theorem; here the concrete structure of $L^q(E)$ supplies the sliding function g by hand.

Problem 8.27. Let $[a, b]$ be a nondegenerate closed, bounded interval. In the Banach space $C[a, b]$, normed by the maximum norm, find a bounded sequence that fails to have any strongly convergent subsequence.

Solution. Since $[a, b]$ is nondegenerate, $a < b$. Write $\ell = b - a > 0$. For each natural number n define

$$\alpha_n = a + \frac{\ell}{2^{n+1}}, \quad \beta_n = a + \frac{\ell}{2^n}, \quad m_n = \frac{\alpha_n + \beta_n}{2},$$

so that

$$a < \alpha_n < m_n < \beta_n \leq b \quad \text{and} \quad \beta_n - \alpha_n = \frac{\ell}{2^{n+1}} > 0.$$

The open intervals $(\alpha_n, \beta_n) = (a + \ell/2^{n+1}, a + \ell/2^n)$, $n = 1, 2, \dots$, are pairwise disjoint subintervals of $(a, b]$, since the sequence $\{a + \ell/2^n\}$ is strictly decreasing.

Define the “tent” functions

$$f_n(x) = \max \left\{ 0, 1 - \frac{2|x - m_n|}{\beta_n - \alpha_n} \right\} \quad \text{for } x \in [a, b].$$

The sequence lies in $C[a, b]$ and is bounded. Each f_n is continuous on $[a, b]$, being the maximum of the two continuous functions $x \mapsto 0$ and $x \mapsto 1 - 2|x - m_n|/(\beta_n - \alpha_n)$. Directly from the definition, $0 \leq f_n \leq 1$ on $[a, b]$, while $f_n(m_n) = 1$. Hence

$$\|f_n\|_{\max} = 1 \quad \text{for all } n,$$

so $\{f_n\}$ is a bounded sequence in $C[a, b]$.

Observe also that f_n vanishes outside (α_n, β_n) : if $|x - m_n| \geq (\beta_n - \alpha_n)/2$, that is, if $x \notin (\alpha_n, \beta_n)$, then $1 - 2|x - m_n|/(\beta_n - \alpha_n) \leq 0$ and so $f_n(x) = 0$.

No subsequence converges strongly. Let m and n be distinct natural numbers. Since the intervals (α_n, β_n) and (α_m, β_m) are disjoint and m_n belongs to (α_n, β_n) , the point m_n lies outside (α_m, β_m) , so $f_m(m_n) = 0$. Therefore

$$\|f_n - f_m\|_{\max} \geq |f_n(m_n) - f_m(m_n)| = |1 - 0| = 1.$$

Now suppose some subsequence $\{f_{n_k}\}$ converged strongly in $C[a, b]$, say to f . Then $\{f_{n_k}\}$ would be a Cauchy sequence in the maximum norm, since for all j, k ,

$$\|f_{n_j} - f_{n_k}\|_{\max} \leq \|f_{n_j} - f\|_{\max} + \|f - f_{n_k}\|_{\max} \rightarrow 0.$$

In particular there would be an index K with $\|f_{n_j} - f_{n_k}\|_{\max} < 1$ for all $j, k \geq K$. But the indices n_K and n_{K+1} are distinct, so by the estimate above $\|f_{n_K} - f_{n_{K+1}}\|_{\max} \geq 1$. This contradiction shows that no subsequence of $\{f_n\}$ converges strongly in $C[a, b]$.

Thus $\{f_n\}$ is a bounded sequence in $C[a, b]$ with no strongly convergent subsequence; the closed unit ball of $C[a, b]$ is not sequentially compact. This illustrates the failure, in the infinite dimensional space $C[a, b]$, of the Bolzano-Weierstrass property enjoyed by each Euclidean space $\mathbb{R}^n$, and it is the analogue for $C[a, b]$ of the Rademacher function example given at the start of Section 8.2 for $L^p[0, 1]$.

Remark. Other choices are available, for instance $f_n(x) = \sin\left(\frac{n\pi(x-a)}{b-a}\right)$, but for those one must still do work to bound $\|f_n - f_m\|_{\max}$ away from 0. The disjointly supported tents make the separation $\|f_n - f_m\|_{\max} \geq 1$ immediate, with no computation.

Problem 8.28. For $1 \leq p \leq \infty$, find a bounded sequence in ℓ^p that fails to have any strongly convergent subsequence.

Solution. For each natural number n let e_n be the sequence whose n th component is 1 and whose other components are 0, that is,

$$e_n = (\underbrace{0, \dots, 0}_{n-1}, 1, 0, 0, \dots), \quad e_n^k = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

We claim that $\{e_n\}$ is a bounded sequence in ℓ^p , for every p with $1 \leq p \leq \infty$, that possesses no strongly convergent subsequence.

Membership and boundedness. If $1 \leq p < \infty$, then

$$\|e_n\|_p = \left(\sum_{k=1}^{\infty} |e_n^k|^p \right)^{1/p} = (1)^{1/p} = 1,$$

since exactly one term of the series is nonzero and it equals 1. If $p = \infty$, then

$$\|e_n\|_{\infty} = \sup_k |e_n^k| = 1.$$

In every case e_n belongs to ℓ^p and $\|e_n\|_p = 1$, so $\{e_n\}$ is a bounded sequence in ℓ^p .

Separation of the terms. Let $m \neq n$. The sequence $e_n - e_m$ has n th component 1, m th component -1 , and all other components 0. Hence, for $1 \leq p < \infty$,

$$\|e_n - e_m\|_p = (|1|^p + |-1|^p)^{1/p} = 2^{1/p} \geq 1,$$

and for $p = \infty$,

$$\|e_n - e_m\|_{\infty} = \max\{|1|, |-1|\} = 1.$$

So in every case

$$\|e_n - e_m\|_p \geq 1 \quad \text{whenever } m \neq n. \quad (*)$$

No subsequence converges strongly. Suppose, to the contrary, that some subsequence $\{e_{n_k}\}$ converged strongly in ℓ^p to some ζ in ℓ^p . A strongly convergent sequence in a normed linear space is Cauchy: given $\epsilon = 1/2$ there would be an index K such that $\|e_{n_k} - \zeta\|_p < 1/4$ for all $k \geq K$, and then for all $j, k \geq K$,

$$\|e_{n_j} - e_{n_k}\|_p \leq \|e_{n_j} - \zeta\|_p + \|\zeta - e_{n_k}\|_p < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Taking $j = K$ and $k = K + 1$, the indices n_K and n_{K+1} are distinct because the subsequence indices are strictly increasing, so $(*)$ gives $\|e_{n_K} - e_{n_{K+1}}\|_p \geq 1 > 1/2$. This contradiction shows that no subsequence of $\{e_n\}$ converges strongly in ℓ^p .

Therefore, for every p with $1 \leq p \leq \infty$, the sequence $\{e_n\}$ is bounded in ℓ^p but has no strongly convergent subsequence.

Remark. The contrast with weak convergence is instructive, and is the content of Problem 21 of this chapter: for $1 < p < \infty$ the same sequence $\{e_n\}$ does converge weakly to 0 in ℓ^p . Indeed, by the Riesz Representation Theorem for the dual of ℓ^p (Problem 7 of this chapter) every T in $(\ell^p)^*$ is of the form $T(\xi) = \sum_k \eta^k \xi^k$ for a single η in ℓ^q , and $T(e_n) = \eta^n \rightarrow 0$ because $\sum_k |\eta^k|^q < \infty$ forces the terms of a convergent series to tend to 0. In ℓ^1 , by contrast, $\{e_n\}$ does not converge weakly to 0: the dual of ℓ^1 contains the functional $T(\xi) = \sum_k \xi^k$ induced by the bounded sequence $\eta \equiv 1$ in ℓ^∞ , and $T(e_n) = 1$ for every n . So strong and weak convergence genuinely differ here, and no weakly convergent subsequence can be upgraded to a strongly convergent one.

8.5 Problems 8.29–8.35

Problem 8.29. Let E be a measurable set that contains a nondegenerate interval. Show that there is a bounded sequence in $L^1(E)$ that fails to have a weakly convergent subsequence. Exhibit a measurable set E for which every bounded sequence in $L^1(E)$ has a weakly convergent subsequence.

Solution. Throughout we use the definition of weak convergence in $L^1(E)$: a sequence $\{f_n\}$ in $L^1(E)$ converges weakly to f in $L^1(E)$ provided

$$\lim_{n \rightarrow \infty} \int_E f_n \cdot g = \int_E f \cdot g \quad \text{for all } g \in L^\infty(E),$$

since $L^\infty(E)$ is the conjugate space of $p = 1$. We will repeatedly use the following remark: if $h \in L^1(E)$, $A \subseteq E$ is measurable, and we set $g = \chi_A \cdot \text{sgn}(h)$, where $\text{sgn}(h)(x)$ equals 1 if $h(x) > 0$, equals -1 if $h(x) < 0$, and equals 0 otherwise, then g is measurable and $|g| \leq 1$, so $g \in L^\infty(E)$, and $\int_E h \cdot g = \int_A |h|$. Part 1: a bounded sequence with no weakly convergent subsequence.

Let $[a, b] \subseteq E$ with $a < b$. This is the construction of the Example of this section, transplanted to $[a, b]$. For each natural number n put

$$I_n = \left[a, a + \frac{b-a}{n} \right], \quad f_n = \frac{n}{b-a} \cdot \chi_{I_n} \quad \text{on } E,$$

that is, f_n vanishes on $E \setminus I_n$. Each f_n is measurable and $\|f_n\|_1 = \int_E |f_n| = \frac{n}{b-a} \cdot m(I_n) = 1$, so $\{f_n\}$ is a bounded sequence in $L^1(E)$.

Suppose, to get a contradiction, that a subsequence $\{f_{n_k}\}$ converges weakly in $L^1(E)$ to some $f \in L^1(E)$.

First, $f = 0$ almost everywhere on $A = E \setminus [a, b]$. Indeed, take $g = \chi_A \cdot \text{sgn}(f) \in L^\infty(E)$. Every f_{n_k} vanishes on A , so $\int_E f_{n_k} \cdot g = 0$ for every k , and therefore

$$\int_A |f| = \int_E f \cdot g = \lim_{k \rightarrow \infty} \int_E f_{n_k} \cdot g = 0.$$

By Proposition 9 of Chapter 4, a nonnegative integrable function with zero integral vanishes almost everywhere, so $f = 0$ a.e. on A .

Second, $f = 0$ almost everywhere on $[a, b]$. Fix a natural number $j \geq 2$ and set $J_j = [a + (b - a)/j, b]$ and $g_j = \chi_{J_j} \cdot \text{sgn}(f) \in L^\infty(E)$. If $n_k > j$ then $a + (b - a)/n_k < a + (b - a)/j$, so $I_{n_k} \cap J_j = \emptyset$ and hence $\int_E f_{n_k} \cdot g_j = 0$. Letting $k \rightarrow \infty$ gives $\int_{J_j} |f| = \int_E f \cdot g_j = 0$, so $f = 0$ a.e. on J_j . Since $(a, b] = \bigcup_{j \geq 2} J_j$ is a countable union and a countable union of sets of measure zero has measure zero, $f = 0$ a.e. on $(a, b]$, hence a.e. on $[a, b]$.

Therefore $f = 0$ almost everywhere on E , and so $\int_E f = 0$. On the other hand $\chi_E \in L^\infty(E)$, so weak convergence applied to $g = \chi_E$ gives

$$0 = \int_E f = \lim_{k \rightarrow \infty} \int_E f_{n_k} = \lim_{k \rightarrow \infty} 1 = 1,$$

a contradiction. Hence no subsequence of $\{f_n\}$ converges weakly in $L^1(E)$.

Part 2: a set E for which every bounded sequence in $L^1(E)$ has a weakly convergent subsequence.

Take E to be any set of measure zero, for instance $E = \emptyset$ or $E = \{0\}$. Then every measurable function on E vanishes almost everywhere on E , so $L^1(E)$ consists of the single element 0 and $\int_E f \cdot g = 0$ for all $f \in L^1(E)$, $g \in L^\infty(E)$. Consequently every sequence in $L^1(E)$ converges weakly to 0; in particular every bounded sequence has a weakly convergent subsequence.

This is in fact the only kind of example, and it is worth recording why. Suppose $m(E) > 0$. We produce a bounded sequence in $L^1(E)$ with no weakly convergent subsequence, which shows that Part 1 has nothing to do with E containing an interval. Choose a natural number N with $m(E \cap [-N, N]) > 0$ and set $E_0 = E \cap [-N, N]$, so $0 < m(E_0) < \infty$. Define $\varphi(t) = m(E_0 \cap [-N, t])$ for $t \in [-N, N]$. Then φ is nondecreasing and satisfies $|\varphi(t) - \varphi(s)| \leq |t - s|$ by the monotonicity and countable additivity of measure, so φ is continuous; moreover $\varphi(-N) = 0$ and $\varphi(N) = m(E_0) > 0$. Let $t_* = \sup\{t \in [-N, N] : \varphi(t) = 0\}$. By continuity $\varphi(t_*) = 0$, so $t_* < N$, and $\varphi(t) > 0$ for every $t > t_*$. For each natural number n let $\delta_n = \min\{1/n, (N - t_*)/2\} > 0$ and

$$B_n = E_0 \cap (t_*, t_* + \delta_n], \quad m(B_n) = \varphi(t_* + \delta_n) - \varphi(t_*) = \varphi(t_* + \delta_n) > 0.$$

The sequence $\{\delta_n\}$ is nonincreasing, so $\{B_n\}$ is a descending sequence of measurable subsets of E of positive finite measure, and $\bigcap_n B_n \subseteq \bigcap_n (t_*, t_* + \delta_n) = \emptyset$ because $\delta_n \rightarrow 0$. Define $f_n = \chi_{B_n}/m(B_n)$ on E ; then $\|f_n\|_1 = 1$.

Suppose a subsequence $\{f_{n_k}\}$ converged weakly to $f \in L^1(E)$. Fix a natural number m . For every k with $n_k \geq m$ we have $B_{n_k} \subseteq B_m$, so f_{n_k} vanishes on $E \setminus B_m$; testing against $g = \chi_{E \setminus B_m} \cdot \text{sgn}(f) \in L^\infty(E)$ gives $\int_{E \setminus B_m} |f| = 0$ and hence $f = 0$ a.e. on $E \setminus B_m$. Since $\bigcap_m B_m = \emptyset$ we have $E = \bigcup_{m=1}^\infty (E \setminus B_m)$, and a countable union of sets of measure zero has measure zero, so $f = 0$ a.e. on E . Testing against χ_E then yields $0 = \int_E f = \lim_k \int_E f_{n_k} = 1$, a contradiction. Thus for every E with $m(E) > 0$ there is a bounded sequence in $L^1(E)$ with no weakly convergent subsequence, and the sets E asked for in the second part of the problem are exactly the sets of measure zero.

Problem 8.30. Let X be a normed linear space, $\{T_n\}$ be a sequence in X^* , and T belong to X^* . Show that $\{T_n\} \rightarrow T$ with respect to the $\|\cdot\|_*$ norm if and only if

$$\lim_{n \rightarrow \infty} T_n(f) = T(f) \quad \text{uniformly on } \{f \in X \mid \|f\| \leq 1\}.$$

Solution. Write $B = \{f \in X \mid \|f\| \leq 1\}$ for the closed unit ball of X . Recall that for a bounded linear functional S on X , the norm $\|S\|_*$ is defined as the infimum of the numbers $M \geq 0$ for which

$$|S(f)| \leq M \cdot \|f\| \quad \text{for all } f \in X,$$

and that this inequality holds for $M = \|S\|_*$ itself.

Lemma. For every $S \in X^*$,

$$\|S\|_* = \sup\{|S(f)| : f \in B\}.$$

Indeed, let s denote the supremum on the right; it is finite because $|S(f)| \leq \|S\|_* \cdot \|f\| \leq \|S\|_*$ for $f \in B$, and this same inequality shows $s \leq \|S\|_*$. Conversely, $|S(f)| \leq s \cdot \|f\|$ for every $f \in X$: this

is clear for $f = 0$ since $S(0) = 0$ by linearity, and if $f \neq 0$ then $f/\|f\|$ belongs to B , so that by the homogeneity of S ,

$$\frac{|S(f)|}{\|f\|} = \left| S\left(\frac{f}{\|f\|}\right) \right| \leq s.$$

Thus s is one of the numbers M occurring in the definition of $\|S\|_*$, and therefore $\|S\|_* \leq s$. Hence $\|S\|_* = s$, proving the lemma.

Since X^* is a linear space, $T_n - T$ belongs to X^* for every n , and the lemma applied to $S = T_n - T$ gives

$$\|T_n - T\|_* = \sup_{f \in B} |T_n(f) - T(f)|.$$

Call this identity the norm formula.

Suppose first that $\{T_n\} \rightarrow T$ with respect to $\|\cdot\|_*$. Let $\epsilon > 0$. Choose an index N for which $\|T_n - T\|_* < \epsilon$ whenever $n \geq N$. Then for every $n \geq N$ and every $f \in B$,

$$|T_n(f) - T(f)| \leq \|T_n - T\|_* \cdot \|f\| \leq \|T_n - T\|_* < \epsilon.$$

The index N was chosen independently of $f \in B$, so $\{T_n(f)\} \rightarrow T(f)$ uniformly on B .

Conversely, suppose $\{T_n(f)\} \rightarrow T(f)$ uniformly on B . Let $\epsilon > 0$. Choose an index N for which

$$|T_n(f) - T(f)| < \epsilon/2 \quad \text{for all } n \geq N \text{ and all } f \in B.$$

Taking the supremum over $f \in B$ and using the norm formula, for $n \geq N$ we get $\|T_n - T\|_* \leq \epsilon/2 < \epsilon$. Since $\epsilon > 0$ was arbitrary, $\lim_{n \rightarrow \infty} \|T_n - T\|_* = 0$, that is, $\{T_n\} \rightarrow T$ in the $\|\cdot\|_*$ norm.

Thus convergence in the dual norm is precisely uniform convergence of the values on the closed unit ball of X .

Problem 8.31. Is the sequence defined in the last example of this section uniformly integrable?

Solution. The sequence in question is the one from the Example preceding the definition of weak sequential compactness: for $I = [0, 1]$ and each natural number n ,

$$I_n = [0, 1/n], \quad f_n = n \cdot \chi_{I_n}.$$

The answer is no: $\{f_n\}$ is not uniformly integrable over $[0, 1]$.

Recall the definition from Chapter 4: a family $\mathcal{F}$ of measurable functions on E is uniformly integrable over E provided for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $f \in \mathcal{F}$, if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \epsilon$.

Take the challenge $\epsilon = 1/2$ and let $\delta > 0$ be arbitrary. Choose a natural number n with $1/n < \delta$ and set $A = I_n = [0, 1/n]$. Then A is a measurable subset of $[0, 1]$ with $m(A) = 1/n < \delta$, while

$$\int_A |f_n| = \int_0^{1/n} n \, dx = n \cdot \frac{1}{n} = 1 > \frac{1}{2}.$$

Thus no $\delta > 0$ responds to the $\epsilon = 1/2$ challenge, and $\{f_n\}$ fails to be uniformly integrable. The point is that the whole of the mass of f_n , which has total integral 1 for every n , sits on a set of measure $1/n$; uniform integrability forbids exactly this concentration of mass on sets of small measure.

Two independent confirmations are worth recording.

First, $\{f_n\} \rightarrow 0$ pointwise on $(0, 1]$, hence a.e. on $[0, 1]$: given $x \in (0, 1]$, we have $f_n(x) = 0$ as soon as $n > 1/x$. But

$$\lim_{n \rightarrow \infty} \int_0^1 f_n = 1 \neq 0 = \int_0^1 0.$$

Since $[0, 1]$ has finite measure, were $\{f_n\}$ uniformly integrable the Vitali Convergence Theorem of Chapter 4 would force these two limits to agree. Hence $\{f_n\}$ is not uniformly integrable.

Second, the answer is exactly what is required for consistency with the Remark that closes this section: by the theorem of Dunford and Pettis quoted there, a bounded sequence in $L^1(E)$ that is uniformly

integrable possesses a weakly convergent subsequence. The Example shows that this bounded sequence in $L^1[0, 1]$ has no weakly convergent subsequence, so it cannot be uniformly integrable.

Problem 8.32. For $p = 1$, at what point does the proof of Theorem 14 fail?

Solution. Set $p = 1$, so that the conjugate exponent is $q = \infty$ and the space X appearing in the proof is $X = L^\infty(E)$. Run through the proof of Theorem 14 with these values, taking E to be a set of positive measure (for a null set there is nothing to prove, and indeed by Problem 29 the theorem is true for $p = 1$ exactly on the null sets).

The conclusions of the opening steps survive, though one citation must be repaired. Given a bounded sequence $\{f_n\}$ in $L^1(E)$, the functionals

$$T_n(g) = \int_E f_n \cdot g \quad \text{for } g \in X = L^\infty(E)$$

are still well defined, linear, and bounded, with $\|T_n\|_* = \|f_n\|_1$. The proof obtains this from Proposition 2 with p and q interchanged, and strictly speaking that invocation is already illegitimate: Proposition 2 assumes that the exponent of the space on which the functional is defined is finite, and here that exponent is $q = \infty$. The conclusion is nevertheless true, by a direct argument that does not use Proposition 2. Hölder's Inequality gives $|\int_E f_n \cdot g| \leq \|f_n\|_1 \cdot \|g\|_\infty$, so T_n is bounded with $\|T_n\|_* \leq \|f_n\|_1$; and $g = \text{sgn}(f_n)$ is measurable with $\|g\|_\infty \leq 1$ and $T_n(g) = \int_E |f_n| = \|f_n\|_1$, so $\|T_n\|_* = \|f_n\|_1$. Hence $\{T_n\}$ is still a bounded sequence in X^* . This first gap is therefore only apparent; it is a defect of citation, not of substance.

The first substantive failure occurs at the very next sentence, the invocation of separability. The proof asserts that $X = L^q(E)$ is separable, citing Theorem 11 of Chapter 7 (printed in the proof as Theorem 11 of Chapter 6) and explicitly restricting to $1 < q < \infty$ at that step. That theorem guarantees the separability of $L^q(E)$ only for $1 \leq q < \infty$, and for $q = \infty$ the hypothesis is simply false. This is recorded in the Example of Section 7.4, which shows that $L^\infty[a, b]$ is not separable; we recall the argument. If $[a, b] \subseteq E$ is a nondegenerate interval, then

$$\|\chi_{[a, x_1]} - \chi_{[a, x_2]}\|_\infty = 1 \quad \text{whenever } a \leq x_1 < x_2 \leq b,$$

so $\{\chi_{[a, x]}\}_{x \in (a, b]}$ is an uncountable family of functions in $L^\infty(E)$ any two of which are at distance 1; a countable set cannot be dense, since the balls of radius $1/2$ about the members of such a family are disjoint and each would have to contain a point of the dense set. Hence $L^\infty(E)$ is not separable and Helley's Theorem, whose hypothesis is precisely the separability of X , does not apply. The failure is genuine at the level of Helley's proof as well: that proof selects, by a diagonal argument, a subsequence for which $\{T_{n_k}(f_j)\}$ converges at each member of a countable dense set $\{f_j\}$, and then uses the density of $\{f_j\}$ together with the uniform bound to conclude that $\{T_{n_k}(f)\}$ is Cauchy for every $f \in X$. With no countable dense set available, the diagonal argument controls the subsequence at only countably many elements of $L^\infty(E)$ and yields no information at the remaining ones.

There is a second, independent failure further on. Even if one were handed a subsequence $\{T_{n_k}\}$ and a $T \in X^*$ with $\lim_k T_{n_k}(g) = T(g)$ for all $g \in L^\infty(E)$, the proof next invokes the Riesz Representation Theorem with p and q interchanged in order to produce $f \in L^1(E)$ with

$$T(g) = \int_E f \cdot g \quad \text{for all } g \in L^\infty(E).$$

This is a representation theorem for the dual of $L^\infty(E)$, and the Riesz Representation Theorem of Section 8.1 is established only for the dual of $L^r(E)$ with $1 \leq r < \infty$; it is not available for $r = \infty$. It is moreover false, as the Remark closing Section 8.1 already warns: there are bounded linear functionals on $L^\infty[a, b]$ that are not given by integration against a function in $L^1[a, b]$, the dual of L^∞ being described instead by the theorem of Kantorovitch proved in Section 3 of Chapter 19. Here is the argument. Granting the Hahn-Banach Theorem of Chapter 14, extend the functional $g \mapsto g(0)$, defined and of norm 1 on the subspace $C[0, 1]$ of $L^\infty[0, 1]$, to a bounded linear functional Λ on all of $L^\infty[0, 1]$. If

$\Lambda(g) = \int_0^1 f \cdot g$ for some $f \in L^1[0, 1]$, choose continuous g_n with $0 \leq g_n \leq 1$, $g_n(0) = 1$, and $g_n = 0$ off $[0, 1/n]$; then $\Lambda(g_n) = 1$ for all n , while $|\int_0^1 f g_n| \leq \int_0^{1/n} |f| \rightarrow 0$ by the continuity of integration. This contradiction shows that no such f exists.

Finally, neither of these two failures can be repaired, because the conclusion itself is false for $p = 1$: the Example of this section exhibits a bounded sequence in $L^1[0, 1]$ with no weakly convergent subsequence. In summary: the proof breaks down at the appeal to the separability of $X = L^q(E)$, and again at the appeal to the Riesz Representation Theorem for the dual of $L^q(E)$, both of which require $q < \infty$, that is, $p > 1$.

Problem 8.33. Show that in ℓ^p , $1 \leq p < \infty$, every bounded sequence in ℓ^p has a weakly convergent subsequence.

Solution. We prove the assertion for $1 < p < \infty$, which is the range in which it is true, and then show by an explicit counterexample that the case $p = 1$ must be excluded from the statement.

Weak convergence in a normed linear space X means, by the definition of Section 8.2, that $\lim_n T(x^{(n)}) = T(x)$ for every $T \in X^*$. So before anything else we must know what the bounded linear functionals on ℓ^p are. We prove this from scratch, since the Riesz Representation Theorem of Section 8.1 is stated for $L^p(E)$ with E a measurable set of real numbers and is not directly applicable to the sequence spaces.

Duality Lemma. Let $1 \leq p < \infty$ and let q be the conjugate of p . For each $g = \{g_j\} \in \ell^q$ the formula

$$T_g(x) = \sum_{j=1}^{\infty} x_j \cdot g_j \quad \text{for } x = \{x_j\} \in \ell^p$$

defines a bounded linear functional on ℓ^p with $\|T_g\|_* = \|g\|_q$, and every $T \in (\ell^p)^*$ is of this form.

Proof. Hölder's Inequality for series gives $\sum_j |x_j g_j| \leq \|x\|_p \cdot \|g\|_q$, so the series defining $T_g(x)$ converges absolutely, T_g is properly defined, it is linear by the linearity of summation, and $\|T_g\|_* \leq \|g\|_q$.

Conversely, let $T \in (\ell^p)^*$ and put $g_j = T(e_j)$, where e_j is the sequence with j -th term 1 and all other terms 0. We first show $g = \{g_j\} \in \ell^q$ with $\|g\|_q \leq \|T\|_*$. Fix a natural number N and set $S_N = \sum_{j \leq N} |g_j|^q$; we may assume $S_N > 0$. Suppose first $p > 1$, so $1 < q < \infty$. Define x by $x_j = |g_j|^{q-1} \operatorname{sgn}(g_j)$ for $j \leq N$ and $x_j = 0$ for $j > N$. Then x has finite support, so $x = \sum_{j \leq N} x_j e_j$ and, by the linearity of T , $T(x) = \sum_{j \leq N} x_j g_j = S_N$. Since $(q-1)p = q$, we have $\|x\|_p = S_N^{1/p}$, and therefore

$$S_N = T(x) \leq \|T\|_* \cdot \|x\|_p = \|T\|_* \cdot S_N^{1/p},$$

whence $S_N^{1/q} = S_N^{1-1/p} \leq \|T\|_*$. If instead $p = 1$, so $q = \infty$, then for each j the choice $x = \operatorname{sgn}(g_j) e_j$ has $\|x\|_1 = 1$ and $T(x) = |g_j|$, so $|g_j| \leq \|T\|_*$. In either case $\|g\|_q \leq \|T\|_*$, so $g \in \ell^q$ and T_g is defined. Finally, T and T_g are bounded linear functionals on ℓ^p that agree on every finitely supported sequence, by the linearity of both. The finitely supported sequences are dense in ℓ^p : for $x \in \ell^p$, the truncation y_N agreeing with x in the first N places and vanishing thereafter satisfies $\|x - y_N\|_p^p = \sum_{j > N} |x_j|^p$, which tends to 0 as $N \rightarrow \infty$ because $p < \infty$ and $\sum_j |x_j|^p$ converges. Two bounded linear functionals that agree on a dense subset of X are equal, since each is continuous. Hence $T = T_g$, and taking $x = g^* = \|g\|_q^{1-q} \operatorname{sgn}(g) |g|^{q-1}$ in the case $q < \infty$, respectively $x = \operatorname{sgn}(g_j) e_j$ with $|g_j|$ near $\|g\|_\infty$ in the case $q = \infty$, gives the reverse inequality $\|T_g\|_* \geq \|g\|_q$. This proves the lemma.

Consequently, if q is the conjugate of p , then $\{x^{(n)}\} \rightarrow x$ weakly in ℓ^p if and only if

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} x_j^{(n)} \cdot g_j = \sum_{j=1}^{\infty} x_j \cdot g_j \quad \text{for every } g = \{g_j\} \in \ell^q.$$

The case $1 < p < \infty$. Let $\{x^{(n)}\}$ be a sequence in ℓ^p with $\|x^{(n)}\|_p \leq M$ for all n , where $M \geq 0$.

Step 1: a coordinatewise convergent subsequence. For every index j and every n ,

$$|x_j^{(n)}| \leq \left(\sum_{i=1}^{\infty} |x_i^{(n)}|^p \right)^{1/p} = \|x^{(n)}\|_p \leq M,$$

so each coordinate sequence is a bounded sequence of real numbers. We now run the diagonal argument used in the proof of Helley's Theorem. Since $\{x_1^{(n)}\}_n$ is bounded, the Bolzano-Weierstrass Theorem provides a strictly increasing sequence of natural numbers $\{s(1, n)\}$ and a number a_1 with $\lim_n x_1^{(s(1, n))} = a_1$. Since $\{x_2^{(s(1, n))}\}_n$ is bounded, there is a subsequence $\{s(2, n)\}$ of $\{s(1, n)\}$ and a number a_2 with $\lim_n x_2^{(s(2, n))} = a_2$. Continuing inductively we obtain, for each j , a strictly increasing sequence $\{s(j, n)\}_n$ that is a subsequence of $\{s(j-1, n)\}_n$, and a number a_j , with $\lim_n x_j^{(s(j, n))} = a_j$. Set $n_k = s(k, k)$. Then $\{n_k\}$ is strictly increasing, and for each fixed j the tail $\{n_k\}_{k \geq j}$ is a subsequence of $\{s(j, n)\}_n$; therefore

$$\lim_{k \rightarrow \infty} x_j^{(n_k)} = a_j \quad \text{for every index } j.$$

Step 2: the limit belongs to ℓ^p . Fix a natural number N . Since the sum is finite, we may pass to the limit termwise:

$$\sum_{j=1}^N |a_j|^p = \lim_{k \rightarrow \infty} \sum_{j=1}^N |x_j^{(n_k)}|^p \leq \limsup_{k \rightarrow \infty} \|x^{(n_k)}\|_p^p \leq M^p.$$

The partial sums of the nonnegative series $\sum_j |a_j|^p$ are thus bounded by M^p , so $a = \{a_j\}$ belongs to ℓ^p and $\|a\|_p \leq M$. By Minkowski's Inequality, $\|x^{(n_k)} - a\|_p \leq 2M$ for every k .

Step 3: $\{x^{(n_k)}\}$ converges weakly to a . Let $g \in \ell^q$ and let $\epsilon > 0$. Since $\sum_j |g_j|^q < \infty$, its tails tend to zero, so we may choose N with

$$\left(\sum_{j>N} |g_j|^q \right)^{1/q} < \frac{\epsilon}{2(2M+1)}.$$

By Hölder's Inequality applied to the tail,

$$\left| \sum_{j>N} (x_j^{(n_k)} - a_j) g_j \right| \leq \|x^{(n_k)} - a\|_p \cdot \left(\sum_{j>N} |g_j|^q \right)^{1/q} \leq 2M \cdot \frac{\epsilon}{2(2M+1)} < \frac{\epsilon}{2}$$

for every k . On the other hand $\sum_{j \leq N} (x_j^{(n_k)} - a_j) g_j$ is a sum of N terms, each of which tends to 0 as $k \rightarrow \infty$ by Step 1; choose K so that this finite sum has absolute value less than $\epsilon/2$ for all $k \geq K$. Then for $k \geq K$,

$$\left| \sum_{j=1}^{\infty} x_j^{(n_k)} g_j - \sum_{j=1}^{\infty} a_j g_j \right| \leq \left| \sum_{j \leq N} (x_j^{(n_k)} - a_j) g_j \right| + \left| \sum_{j>N} (x_j^{(n_k)} - a_j) g_j \right| < \epsilon.$$

Since $\epsilon > 0$ and $g \in \ell^q$ were arbitrary, $\{x^{(n_k)}\}$ converges weakly in ℓ^p to $a \in \ell^p$. This proves the assertion for $1 < p < \infty$.

The case $p = 1$. Here the argument above breaks down at Step 3: the conjugate exponent is $q = \infty$, and for $g \in \ell^\infty$ the quantity $\sup_{j>N} |g_j|$ need not be small, so the tail cannot be controlled. Steps 1 and 2 survive verbatim and produce a subsequence converging coordinatewise to some $a \in \ell^1$, but this subsequence need not converge weakly, and no other subsequence need do so either. Indeed, the assertion is false for $p = 1$.

To see this, let e_n denote the sequence whose n -th term is 1 and whose other terms are 0. Then $\|e_n\|_1 = 1$, so $\{e_n\}$ is a bounded sequence in ℓ^1 . Let $\{e_{n_k}\}$ be any subsequence. Define $g = \{g_j\}$ by

$$g_j = \begin{cases} (-1)^k & \text{if } j = n_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

This is well defined because $\{n_k\}$ is strictly increasing, and $\|g\|_\infty = 1$, so $g \in \ell^\infty$, the conjugate space of ℓ^1 . But

$$\sum_{j=1}^{\infty} (e_{n_k})_j \cdot g_j = g_{n_k} = (-1)^k,$$

and this sequence of real numbers does not converge. Hence $\{e_{n_k}\}$ does not converge weakly in ℓ^1 to any element of ℓ^1 . Since the subsequence was arbitrary, the bounded sequence $\{e_n\}$ in ℓ^1 has no weakly convergent subsequence, and the hypothesis $1 \leq p < \infty$ in the statement of the problem must be replaced by $1 < p < \infty$. This is exactly parallel to Theorem 14 and to the Example of this section, where the same restriction on p appears for the same reason: the conjugate space must be separable, and ℓ^∞ is not.

Problem 8.34. Let $\{f_n\}$ be a sequence of functions on $[0, 1]$, each of which is of bounded variation and for which $\{TV(f_n)\}$ is bounded. Show that there is a subsequence $\{f_{n_k}\}$ with the property that for each continuous function g on $[0, 1]$, the sequence of integrals $\left\{\int_0^1 g(x) df_{n_k}(x)\right\}$ is Cauchy.

Solution. (The printed statement carries a stray dx after the Stieltjes differential $df_{n_k}(x)$; the integral intended is the Riemann-Stieltjes integral $\int_0^1 g(x) df_{n_k}(x)$, and that is what we treat.)

Let $M \geq 0$ be such that $TV(f_n) \leq M$ for all n ; such an M exists because $\{TV(f_n)\}$ is assumed to be a bounded sequence of numbers, and each $TV(f_n)$ is finite since f_n is of bounded variation.

Take $X = C[0, 1]$, the linear space of continuous real-valued functions on $[0, 1]$ normed by the maximum norm $\|g\|_{\max} = \max_{0 \leq x \leq 1} |g(x)|$.

Step 1: the functionals. For each natural number n define

$$T_n(g) = \int_0^1 g(x) df_n(x) \quad \text{for } g \in C[0, 1],$$

the integral being taken in the sense of Riemann-Stieltjes. By the second example of Section 8.1, since f_n is of bounded variation on $[0, 1]$ this integral is properly defined for every continuous g , the functional T_n is linear, and

$$|T_n(g)| \leq TV(f_n) \cdot \|g\|_{\max} \leq M \cdot \|g\|_{\max} \quad \text{for all } g \in C[0, 1].$$

The estimate is immediate from the definition of the Riemann-Stieltjes integral as a limit of sums: for any partition $0 = x_0 < x_1 < \cdots < x_m = 1$ and any choice of points $\xi_i \in [x_{i-1}, x_i]$,

$$\left| \sum_{i=1}^m g(\xi_i) [f_n(x_i) - f_n(x_{i-1})] \right| \leq \|g\|_{\max} \sum_{i=1}^m |f_n(x_i) - f_n(x_{i-1})| \leq \|g\|_{\max} \cdot TV(f_n).$$

Hence each T_n belongs to X^* and $\{T_n\}$ satisfies the boundedness hypothesis (28) of Helley's Theorem with the single constant M .

Step 2: $C[0, 1]$ is separable. Let $\mathcal{P}$ be the collection of polynomials with rational coefficients, restricted to $[0, 1]$. Since for each degree d the set of $(d+1)$ -tuples of rationals is countable and a countable union of countable sets is countable, $\mathcal{P}$ is countable. Let $g \in C[0, 1]$ and $\epsilon > 0$. By the Weierstrass Approximation Theorem there is a polynomial $p(x) = \sum_{j=0}^d c_j x^j$ with $\|g - p\|_{\max} < \epsilon/2$. Choose rationals r_j with $|c_j - r_j| < \epsilon/(2(d+1))$ and set $\tilde{p}(x) = \sum_{j=0}^d r_j x^j$, a member of $\mathcal{P}$. Since $0 \leq x^j \leq 1$ on $[0, 1]$,

$$\|p - \tilde{p}\|_{\max} \leq \sum_{j=0}^d |c_j - r_j| < \frac{\epsilon}{2},$$

and therefore $\|g - \tilde{p}\|_{\max} < \epsilon$. Thus $\mathcal{P}$ is a countable dense subset of $C[0, 1]$, so $C[0, 1]$ is a separable normed linear space.

Step 3: apply Helley's Theorem. The space $X = C[0, 1]$ is a separable normed linear space and $\{T_n\}$ is a sequence in X^* with $|T_n(g)| \leq M \cdot \|g\|_{\max}$ for all $g \in X$ and all n . By Helley's Theorem there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and a $T \in X^*$ for which

$$\lim_{k \rightarrow \infty} T_{n_k}(g) = T(g) \quad \text{for all } g \in C[0, 1].$$

Step 4: conclusion. Fix a continuous function g on $[0, 1]$. By Step 3 the sequence of real numbers

$$\left\{ \int_0^1 g(x) df_{n_k}(x) \right\}_{k=1}^{\infty} = \{T_{n_k}(g)\}_{k=1}^{\infty}$$

converges, and every convergent sequence of real numbers is Cauchy. Hence this sequence of integrals is Cauchy, and the single subsequence $\{f_{n_k}\}$ furnished by Helley's Theorem works simultaneously for every continuous g , as required.

We remark that more is true. The limit functional T is a bounded linear functional on $C[0, 1]$, and according to the Remark closing Section 8.1, the theorem of Riesz proved in Chapter 21 tells us that every bounded linear functional on $C[a, b]$ is given by Lebesgue-Stieltjes integration against a function of bounded variation. So there is in fact a function f of bounded variation on $[0, 1]$ with $\lim_k \int_0^1 g(x) df_{n_k}(x) = \int_0^1 g(x) df(x)$ for every continuous g on $[0, 1]$; this is the classical Helley selection principle. The exercise as posed asks only for the Cauchy property, which does not need this identification of the limit.

Problem 8.35. Let X be a normed linear space and $\{T_n\}$ a sequence in X^* for which there is an $M \geq 0$ such that $\|T_n\|_* \leq M$ for all n . Let $\mathcal{S}$ be a dense subset of X such that $\{T_n(g)\}$ is Cauchy for all $g \in \mathcal{S}$.

- (i) Show that $\{T_n(g)\}$ is Cauchy for all $g \in X$.
- (ii) Define $T(g) = \lim_{n \rightarrow \infty} T_n(g)$ for all $g \in X$. Show that T is linear. Then show that T is bounded.

Solution. This is the final step of the proof of Helley's Theorem, isolated and proved in general.

Throughout we use two standing facts. First, since $\|T_n\|_*$ is the norm of the bounded linear functional T_n , the defining inequality holds with $M_n = \|T_n\|_*$, so

$$|T_n(h)| \leq \|T_n\|_* \cdot \|h\| \leq M \cdot \|h\| \quad \text{for all } h \in X \text{ and all } n.$$

Second, $\mathcal{S}$ being dense in X means that for each $g \in X$ and each $\eta > 0$ there is an $h \in \mathcal{S}$ with $\|g - h\| < \eta$.

Part (i). Let $g \in X$ and let $\epsilon > 0$. Choose $h \in \mathcal{S}$ with

$$\|g - h\| < \frac{\epsilon}{3(M+1)},$$

which is possible by the density of $\mathcal{S}$; note $M+1 > 0$, so the right-hand side is a legitimate positive number even if $M = 0$. Since $\{T_n(h)\}$ is Cauchy, there is an index N for which

$$|T_n(h) - T_m(h)| < \frac{\epsilon}{3} \quad \text{for all } n, m \geq N.$$

Let $n, m \geq N$. Using the linearity of T_n and T_m and the estimates above,

$$\begin{aligned} |T_n(g) - T_m(g)| &\leq |T_n(g) - T_n(h)| + |T_n(h) - T_m(h)| + |T_m(h) - T_m(g)| \\ &= |T_n(g - h)| + |T_n(h) - T_m(h)| + |T_m(h - g)| \\ &\leq M \cdot \|g - h\| + \frac{\epsilon}{3} + M \cdot \|g - h\| \\ &< \frac{2M}{3(M+1)} \cdot \epsilon + \frac{\epsilon}{3} \\ &\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

where in the last line we used $M/(M+1) \leq 1$. Since $\epsilon > 0$ was arbitrary, $\{T_n(g)\}$ is a Cauchy sequence of real numbers, for every $g \in X$.

Part (ii). The real numbers are complete, so by part (i) the limit

$$T(g) = \lim_{n \rightarrow \infty} T_n(g)$$

exists as a real number for every $g \in X$; thus T is a properly defined real-valued function on X . T is linear. Let $g, h \in X$ and let α, β be real numbers. Each T_n is linear, so for every n ,

$$T_n(\alpha g + \beta h) = \alpha T_n(g) + \beta T_n(h).$$

The sequences $\{T_n(g)\}$ and $\{T_n(h)\}$ converge to $T(g)$ and $T(h)$ respectively, so by the linearity of limits of convergent sequences of real numbers, the right-hand side converges to $\alpha T(g) + \beta T(h)$. The left-hand side converges to $T(\alpha g + \beta h)$, again by part (i) applied to the vector $\alpha g + \beta h \in X$. Limits of convergent real sequences are unique, and therefore

$$T(\alpha g + \beta h) = \alpha T(g) + \beta T(h).$$

Hence T is linear.

T is bounded. Let $g \in X$. For every n we have $|T_n(g)| \leq M \cdot \|g\|$. Since $\{T_n(g)\} \rightarrow T(g)$ and the absolute value function is continuous, $\{|T_n(g)|\} \rightarrow |T(g)|$, and a limit of a convergent sequence of real numbers each of which is at most $M \cdot \|g\|$ is itself at most $M \cdot \|g\|$. Therefore

$$|T(g)| = \lim_{n \rightarrow \infty} |T_n(g)| \leq M \cdot \|g\| \quad \text{for all } g \in X.$$

Thus T is a bounded linear functional on X , that is, T belongs to X^* , and moreover $\|T\|_* \leq M$ since $\|T\|_*$ is the infimum of the constants for which such an inequality holds.

8.6 Problems 8.36–8.42

Problem 8.36. Show that the conclusion of Helley's Theorem is not true for $X = L^\infty[0, 1]$.

Solution. Helley's Theorem asserts that if X is a **separable** normed linear space and $\{T_n\}$ is a sequence in X^* for which $|T_n(f)| \leq M \cdot \|f\|$ for all f in X and all n , then some subsequence $\{T_{n_k}\}$ converges pointwise on X to some T in X^* . We exhibit a uniformly bounded sequence of bounded linear functionals on $X = L^\infty[0, 1]$ no subsequence of which converges pointwise on all of X . (Separability is exactly what fails here: $L^\infty[0, 1]$ is not separable.)

For each natural number n define $T_n: L^\infty[0, 1] \rightarrow \mathbb{R}$ by

$$T_n(f) = n \int_0^{1/n} f.$$

Each T_n is linear by the linearity of integration, and since $|f| \leq \|f\|_\infty$ a.e. on $[0, 1]$,

$$|T_n(f)| \leq n \int_0^{1/n} |f| \leq n \cdot \frac{1}{n} \cdot \|f\|_\infty = \|f\|_\infty.$$

Thus each T_n belongs to X^* and the sequence $\{T_n\}$ satisfies the boundedness hypothesis (28) of Helley's Theorem with $M = 1$.

We claim no subsequence of $\{T_n\}$ converges pointwise on X . Let $\{T_{n_k}\}$ be any subsequence. Since $n_1 < n_2 < \dots$ is strictly increasing, we may select indices $k_1 < k_2 < \dots$ inductively so that, writing $m_j = n_{k_j}$,

$$m_{j+1} \geq j \cdot m_j \quad \text{for all } j \geq 1;$$

this is possible because $n_k \rightarrow \infty$. In particular $m_j/m_{j+1} \leq 1/j$ for all $j \geq 1$.

Define f on $[0, 1]$ by

$$f = \sum_{j=1}^{\infty} (-1)^j \chi_{A_j}, \quad A_j = \left(\frac{1}{m_{j+1}}, \frac{1}{m_j} \right],$$

and $f = 0$ on $(1/m_1, 1]$ and at 0. The intervals A_j are pairwise disjoint, so f is a well-defined measurable function with $|f| \leq 1$; hence $f \in L^\infty[0, 1]$ and $\|f\|_\infty \leq 1$.

Fix j . Since $(0, 1/m_j] = A_j \cup (0, 1/m_{j+1}]$ and this union is disjoint,

$$T_{m_j}(f) = m_j \int_{A_j} f + m_j \int_0^{1/m_{j+1}} f = (-1)^j m_j \left(\frac{1}{m_j} - \frac{1}{m_{j+1}} \right) + m_j \int_0^{1/m_{j+1}} f.$$

The first term equals $(-1)^j (1 - m_j/m_{j+1})$, and the second is bounded in absolute value by $m_j \cdot (1/m_{j+1}) \leq 1/j$. Since $m_j/m_{j+1} \leq 1/j$ as well, we obtain

$$|T_{m_j}(f) - (-1)^j| \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

Therefore $T_{m_{2i}}(f) \rightarrow 1$ and $T_{m_{2i+1}}(f) \rightarrow -1$ as $i \rightarrow \infty$, so the sequence of real numbers $\{T_{m_j}(f)\}$ does not converge.

But $\{T_{m_j}\}$ is a subsequence of $\{T_{n_k}\}$. If $\{T_{n_k}(f)\}$ converged, every one of its subsequences would converge to the same limit, which is false. Hence $\{T_{n_k}(f)\}$ fails to converge, so $\{T_{n_k}\}$ does not converge pointwise on X .

Since the subsequence $\{T_{n_k}\}$ was arbitrary, no subsequence of $\{T_n\}$ converges pointwise on $L^\infty[0, 1]$, and the conclusion of Helly's Theorem fails for $X = L^\infty[0, 1]$.

Problem 8.37. Let E have finite measure and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. If one of the following properties holds, determine, for each of the other properties, if a subsequence has that other property. The cases $p = 1$ and $p > 1$ should be considered.

- (i) $\{f_n\} \rightarrow f$ in $L^p(E)$.
- (ii) $\{f_n\} \rightharpoonup f$ in $L^p(E)$.
- (iii) $\{f_n\} \rightarrow f$ pointwise a.e. on E .
- (iv) $\{f_n\} \rightarrow f$ in measure.

Solution. Throughout, $m(E) < \infty$, q is the conjugate of p , and $M \geq 0$ is a bound with $\|f_n\|_p \leq M$ for all n . We treat the twelve implications; the answers are collected at the end.

Two counterexamples. Take $E = [0, 1]$ and $f \equiv 0$.

The moving spike. Let $g_n = n^{1/p} \chi_{(0, 1/n)}$. Then $\|g_n\|_p = 1$ for all n , so $\{g_n\}$ is bounded in $L^p[0, 1]$, and $g_n(x) \rightarrow 0$ for every $x \in (0, 1]$, so $\{g_n\} \rightarrow 0$ pointwise a.e.; by Proposition 3 of Chapter 5 it also converges to 0 in measure. No subsequence converges to 0 in L^p , since $\|g_{n_k}\|_p = 1$ for every k . When $p = 1$, $g_n = n \chi_{(0, 1/n)}$ and $\int_0^1 g_n \cdot 1 = 1$ for all n , while $\int_0^1 0 \cdot 1 = 0$; as the constant function 1 lies in $L^\infty[0, 1] = L^q[0, 1]$, Proposition 6 shows that no subsequence of $\{g_n\}$ converges weakly to 0 in $L^1[0, 1]$.

The Rademacher functions. Let $r_n(x) = 1$ if $[2^n x]$ is even and $r_n(x) = -1$ otherwise; equivalently r_n is +1 on the even dyadic intervals $[k2^{-n}, (k+1)2^{-n})$ and -1 on the odd ones. Then $|r_n| = 1$, so $\|r_n\|_p = 1$ and $\{r_n\}$ is bounded in $L^p[0, 1]$.

We claim $\{r_n\} \rightharpoonup 0$ in $L^p[0, 1]$ for every $p \in [1, \infty)$. By Proposition 6 it suffices to show that $\int_0^1 r_n h \rightarrow 0$ for every $h \in L^q[0, 1]$. First observe that

$$0 \leq \int_0^x r_n \leq 2^{-n} \quad \text{for all } x \in [0, 1],$$

because on each interval $[2j \cdot 2^{-n}, (2j+2) \cdot 2^{-n}]$ the function r_n equals +1 on the first half and -1 on the second, so $t \mapsto \int_0^t r_n$ rises from 0 to 2^{-n} and falls back to 0. Consequently $\left| \int_c^d r_n \right| \leq 2^{-n}$ for every subinterval $[c, d]$ of $[0, 1]$, and therefore $\int_0^1 r_n \psi \rightarrow 0$ for every step function ψ on $[0, 1]$, such a ψ being a finite linear combination of characteristic functions of intervals.

Now let $h \in L^q[0, 1]$ and $\varepsilon > 0$. Since $[0, 1]$ has finite measure, h belongs to $L^1[0, 1]$, and by Proposition 10 of Chapter 7 the step functions are dense in $L^1[0, 1]$; choose a step function ψ with $\|h - \psi\|_1 < \varepsilon$. Since $|r_n| \leq 1$,

$$\left| \int_0^1 r_n h \right| \leq \int_0^1 |h - \psi| + \left| \int_0^1 r_n \psi \right| < \varepsilon + \left| \int_0^1 r_n \psi \right|.$$

Letting $n \rightarrow \infty$ gives $\limsup_{n \rightarrow \infty} \left| \int_0^1 r_n h \right| \leq \varepsilon$, and as $\varepsilon > 0$ was arbitrary, $\int_0^1 r_n h \rightarrow 0$. Hence $\{r_n\} \rightarrow 0$ in $L^p[0, 1]$ for every $p \in [1, \infty)$.

On the other hand no subsequence of $\{r_n\}$ has any of the properties (i), (iii), (iv) with limit 0. Indeed $\|r_{n_k}\|_p = 1$ for all k , so no subsequence converges to 0 in L^p . If some subsequence converged to 0 pointwise a.e., then, since $|r_{n_k}| \leq 1$ and the constant 1 is integrable over $[0, 1]$, the Dominated Convergence Theorem applied to $|r_{n_k}|^p$ would give $\|r_{n_k}\|_p \rightarrow 0$, a contradiction. If some subsequence converged to 0 in measure, then by Theorem 4 of Chapter 5 (Riesz) a further subsequence would converge to 0 pointwise a.e., which we just excluded.

(i) implies (ii), (iii) up to a subsequence, and (iv), for all $p \in [1, \infty)$. Suppose $\{f_n\} \rightarrow f$ in $L^p(E)$. For $h \in L^q(E)$, Hölder's Inequality gives $|\int_E (f_n - f)h| \leq \|f_n - f\|_p \|h\|_q \rightarrow 0$, so by Proposition 6, $\{f_n\} \rightarrow f$: (ii) holds for the whole sequence. By the Riesz–Fischer Theorem there is a subsequence of $\{f_n\}$ that converges to f pointwise a.e. on E , so (iii) holds for a subsequence. (Not for the whole sequence: the functions $\chi_{[0,1/2]}, \chi_{[1/2,1]}, \chi_{[0,1/4]}, \chi_{[1/4,1/2]}, \dots$ converge to 0 in $L^p[0, 1]$ but at no point.) Finally, for $\eta > 0$, Chebychev's Inequality applied to $|f_n - f|^p$ gives

$$m\{x \in E : |f_n(x) - f(x)| \geq \eta\} \leq \frac{1}{\eta^p} \int_E |f_n - f|^p \rightarrow 0,$$

so $\{f_n\} \rightarrow f$ in measure: (iv) holds for the whole sequence.

(iii) implies (iv), for all p . This is Proposition 3 of Chapter 5, which requires only that $m(E) < \infty$.

(iv) implies (iii) up to a subsequence, for all p . This is Theorem 4 of Chapter 5 (Riesz).

(iii) implies (ii), and (iv) implies (ii), when $p > 1$; both fail when $p = 1$. Assume $p > 1$, so $1 \leq q < \infty$, and assume (iii). Let $h \in L^q(E)$ and $\varepsilon > 0$. Since $|h|^q$ is integrable over E , by the continuity of integration there is a $\delta > 0$ such that $\int_A |h|^q < \varepsilon^q$ whenever $A \subseteq E$ is measurable with $m(A) < \delta$. Since $m(E) < \infty$ and $\{f_n\} \rightarrow f$ pointwise a.e. on E , Egoroff's Theorem provides a measurable $F \subseteq E$ with $m(E \setminus F) < \delta$ on which $\{f_n\} \rightarrow f$ uniformly. By Hölder's Inequality, $\int_F |h| \leq \|h\|_q m(E)^{1/p} < \infty$, so

$$\left| \int_F (f_n - f)h \right| \leq \left(\sup_F |f_n - f| \right) \int_F |h| \rightarrow 0.$$

Also, by Hölder's Inequality and Minkowski's Inequality,

$$\left| \int_{E \setminus F} (f_n - f)h \right| \leq \|f_n - f\|_p \left(\int_{E \setminus F} |h|^q \right)^{1/q} \leq (M + \|f\|_p) \varepsilon.$$

Hence $\limsup_n \left| \int_E (f_n - f)h \right| \leq (M + \|f\|_p) \varepsilon$ for every $\varepsilon > 0$, so $\int_E f_n h \rightarrow \int_E f h$. By Proposition 6, $\{f_n\} \rightarrow f$ in $L^p(E)$: the whole sequence converges weakly.

If instead (iv) holds and $p > 1$, apply the preceding to subsequences: every subsequence of $\{f_n\}$ still converges to f in measure, hence by Theorem 4 of Chapter 5 has a further subsequence converging to f pointwise a.e., which by the paragraph above converges weakly to f . Thus for each $h \in L^q(E)$ every subsequence of the real sequence $\{\int_E f_n h\}$ has a further subsequence converging to $\int_E f h$, so the whole sequence $\{\int_E f_n h\}$ converges to $\int_E f h$, and $\{f_n\} \rightarrow f$.

For $p = 1$ both implications fail, even for subsequences: the moving spike $g_n = n\chi_{(0,1/n)}$ is bounded in $L^1[0, 1]$, converges to 0 pointwise a.e. and in measure, yet no subsequence converges weakly to 0 in $L^1[0, 1]$.

Nothing implies (i) except (i), for any p . The moving spike $\{g_n\}$ satisfies (ii) when $p > 1$ (by the paragraph above), and satisfies (iii) and (iv) for every $p \in [1, \infty)$, but no subsequence converges to 0 in L^p . For $p = 1$ the Rademacher sequence $\{r_n\}$ satisfies (ii) with no subsequence satisfying (i). So (ii), (iii), (iv) each fail to yield (i) along a subsequence.

(ii) implies neither (iii) nor (iv) along a subsequence, for any $p \in [1, \infty)$. The Rademacher sequence $\{r_n\}$ is bounded in $L^p[0, 1]$ and converges weakly to 0, but as shown above no subsequence converges to 0 pointwise a.e. or in measure (or in L^p).

Summary. Write “yes” if the second property holds for a suitable subsequence and “no” otherwise.

- (i) $\Rightarrow$ (ii): yes (whole sequence), all p . (i) $\Rightarrow$ (iii): yes (subsequence), all p . (i) $\Rightarrow$ (iv): yes (whole sequence), all p .

- (ii) $\Rightarrow$ (i): no, all p . (ii) $\Rightarrow$ (iii): no, all p . (ii) $\Rightarrow$ (iv): no, all p .
- (iii) $\Rightarrow$ (i): no, all p . (iii) $\Rightarrow$ (ii): yes (whole sequence) if $p > 1$; no if $p = 1$. (iii) $\Rightarrow$ (iv): yes (whole sequence), all p .
- (iv) $\Rightarrow$ (i): no, all p . (iv) $\Rightarrow$ (ii): yes (whole sequence) if $p > 1$; no if $p = 1$. (iv) $\Rightarrow$ (iii): yes (subsequence), all p .

The boundedness of $\{f_n\}$ in $L^p(E)$ is used only in the implications (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii); by Theorem 7 it is automatic under (i) and (ii).

Problem 8.38. For $1 < p < \infty$ and each index n , let $e_n \in \ell^p$ have n th component 1 and other components 0. Show that $\{e_n\}$ converges weakly to 0 in ℓ^p , but no subsequence converges strongly to 0. Find a subsequence whose arithmetic means converge strongly to 0 in ℓ^p .

Solution. Let q be the conjugate of p , so $1 < q < \infty$ and $(q-1)p = q$.

Weak convergence. By definition, $\{e_n\} \rightarrow 0$ in ℓ^p means $T(e_n) \rightarrow 0$ for every bounded linear functional T on ℓ^p . So let T belong to $(\ell^p)^*$ and put $y_n = T(e_n)$. We show $\sum_{n=1}^{\infty} |y_n|^q < \infty$, which forces $y_n \rightarrow 0$. Fix a natural number N and define

$$x = \sum_{n=1}^N \operatorname{sgn}(y_n) |y_n|^{q-1} e_n,$$

a vector with only finitely many nonzero components, so $x \in \ell^p$. Since $(q-1)p = q$,

$$\|x\|_p = \left(\sum_{n=1}^N |y_n|^{(q-1)p} \right)^{1/p} = \left(\sum_{n=1}^N |y_n|^q \right)^{1/p}.$$

By linearity, $T(x) = \sum_{n=1}^N \operatorname{sgn}(y_n) |y_n|^{q-1} y_n = \sum_{n=1}^N |y_n|^q$. Hence, writing $S_N = \sum_{n=1}^N |y_n|^q$,

$$S_N = T(x) \leq \|T\|_* \cdot \|x\|_p = \|T\|_* \cdot S_N^{1/p}.$$

If $S_N > 0$, dividing by $S_N^{1/p}$ gives $S_N^{1-1/p} = S_N^{1/q} \leq \|T\|_*$; this also holds trivially if $S_N = 0$. Thus $S_N \leq \|T\|_*^q$ for every N , so $\sum_{n=1}^{\infty} |y_n|^q \leq \|T\|_*^q < \infty$. The terms of a convergent series tend to 0, so $T(e_n) = y_n \rightarrow 0 = T(0)$. Since $T \in (\ell^p)^*$ was arbitrary, $\{e_n\} \rightarrow 0$ in ℓ^p .

No subsequence converges strongly. For every n , $\|e_n\|_p = 1$. Hence for any subsequence $\{e_{n_k}\}$ we have $\|e_{n_k} - 0\|_p = 1$ for all k , so $\{e_{n_k}\}$ does not converge strongly to 0. (This is the promised illustration that weak convergence is strictly weaker than strong convergence: the sequence $\{e_n\}$ lies on the unit sphere of ℓ^p and converges weakly to a point of norm 0.)

Arithmetic means. In fact the arithmetic means of the full sequence $\{e_n\}$ converge strongly to 0, so we may take the subsequence to be $\{e_n\}$ itself; more generally, this works for every subsequence. Let $\{e_{n_k}\}$ be any subsequence and set

$$\sigma_k = \frac{e_{n_1} + e_{n_2} + \cdots + e_{n_k}}{k}.$$

Since $n_1 < n_2 < \cdots$, the vectors $e_{n_1}, \dots, e_{n_k}$ have disjoint supports, so σ_k has exactly k nonzero components, each equal to $1/k$. Therefore

$$\|\sigma_k\|_p = \left(k \cdot \frac{1}{k^p} \right)^{1/p} = (k^{1-p})^{1/p} = k^{(1-p)/p} = \frac{1}{k^{1-1/p}}.$$

Because $p > 1$ we have $1 - 1/p > 0$, so $\|\sigma_k\|_p \rightarrow 0$ as $k \rightarrow \infty$; that is, the arithmetic means converge strongly to 0 in ℓ^p .

This is exactly the conclusion of the Banach–Saks Theorem in this example. Note that the computation fails for $p = 1$: there $\|\sigma_k\|_1 = 1$ for all k , and indeed $\{e_n\}$ does not converge weakly to 0 in ℓ^1 , since the bounded linear functional $x \mapsto \sum_n x_n$ (represented by the element $(1, 1, 1, \dots)$ of ℓ^∞) takes the value 1 at every e_n .

Problem 8.39. Show that if a sequence of real numbers $\{a_n\}$ converges to a , then the sequence of arithmetic means also converges to a .

Solution. For each natural number n define the n th arithmetic mean

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

We must show $\sigma_n \rightarrow a$.

Replacing a_n by $a_n - a$ replaces σ_n by $\sigma_n - a$, since

$$\frac{(a_1 - a) + \cdots + (a_n - a)}{n} = \sigma_n - a.$$

Hence there is no loss of generality in assuming $a = 0$, and we must show $\sigma_n \rightarrow 0$.

Since $\{a_n\}$ converges, it is bounded: choose $M \geq 0$ with $|a_n| \leq M$ for all n .

Let $\varepsilon > 0$. Since $a_n \rightarrow 0$, there is a natural number N such that

$$|a_n| < \frac{\varepsilon}{2} \quad \text{for all } n > N.$$

For $n > N$ split the sum at N :

$$|\sigma_n| = \left| \frac{1}{n} \sum_{k=1}^n a_k \right| \leq \frac{1}{n} \sum_{k=1}^N |a_k| + \frac{1}{n} \sum_{k=N+1}^n |a_k| \leq \frac{NM}{n} + \frac{n-N}{n} \cdot \frac{\varepsilon}{2} \leq \frac{NM}{n} + \frac{\varepsilon}{2}.$$

Here N and M are fixed, so $NM/n \rightarrow 0$ as $n \rightarrow \infty$. Choose a natural number $N' > N$ with $NM/n < \varepsilon/2$ for all $n > N'$, that is, $N' \geq 2NM/\varepsilon$. Then for all $n > N'$,

$$|\sigma_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\sigma_n \rightarrow 0$. In the general case, $\sigma_n \rightarrow a$.

The converse is false: for $a_n = (-1)^n$ the arithmetic means converge to 0 while $\{a_n\}$ diverges. So the passage to arithmetic means is a genuine strengthening of the summability of a sequence, which is precisely why it is useful in the Banach–Saks Theorem.

Problem 8.40. State and prove the Banach–Saks Theorem in ℓ^2 .

Solution. Statement. Let $\{x_n\}$ be a sequence in ℓ^2 that converges weakly in ℓ^2 to x . Then there is a subsequence $\{x_{n_k}\}$ for which the sequence of arithmetic means converges strongly to x in ℓ^2 , that is,

$$\lim_{k \rightarrow \infty} \left\| \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k} - x \right\|_2 = 0.$$

For $u = \{u_n\}$ and $v = \{v_n\}$ in ℓ^2 write $\langle u, v \rangle = \sum_{n=1}^{\infty} u_n v_n$; by the Cauchy–Schwarz (Hölder) Inequality this series converges absolutely and $|\langle u, v \rangle| \leq \|u\|_2 \|v\|_2$, and $\langle u, u \rangle = \|u\|_2^2$.

A preliminary lemma: every $T \in (\ell^2)^*$ has the form $T(u) = \langle u, y \rangle$ for some $y \in \ell^2$. Let $e_n \in \ell^2$ have n th component 1 and the others 0, and set $y_n = T(e_n)$. Exactly as in Problem 38 (with $p = q = 2$), for each N the vector $w = \sum_{n=1}^N y_n e_n$ satisfies $T(w) = \sum_{n=1}^N y_n^2$ and $\|w\|_2 = \left(\sum_{n=1}^N y_n^2 \right)^{1/2}$, whence

$\left(\sum_{n=1}^N y_n^2 \right)^{1/2} \leq \|T\|_*$. Letting $N \rightarrow \infty$ shows $y = \{y_n\} \in \ell^2$. The functionals T and $u \mapsto \langle u, y \rangle$ are both bounded and linear on ℓ^2 and agree on each e_n , hence on the linear span of $\{e_n\}$, which is the set of vectors with finitely many nonzero components. That span is dense in ℓ^2 , since for $u \in \ell^2$ the truncations $\sum_{n \leq N} u_n e_n$ satisfy $\|u - \sum_{n \leq N} u_n e_n\|_2^2 = \sum_{n > N} u_n^2 \rightarrow 0$. Two continuous functions agreeing on a dense subset agree everywhere, so $T(u) = \langle u, y \rangle$ for all $u \in \ell^2$.

Consequently, $\{x_n\} \rightharpoonup x$ in ℓ^2 if and only if $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ for every $y \in \ell^2$.

Proof of the theorem. Replacing each x_n by $x_n - x$ replaces the arithmetic means by their translates by $-x$, so we may assume $x = 0$; thus $\langle x_n, y \rangle \rightarrow 0$ for every $y \in \ell^2$, and we must produce a subsequence whose arithmetic means converge strongly to 0.

Boundedness of $\{x_n\}$. The book's proof invokes Theorem 7 at this point; here is the ℓ^2 analogue, proved in full. For each n define $T_n(u) = \langle u, x_n \rangle$ for $u \in \ell^2$. By the Cauchy-Schwarz Inequality $|T_n(u)| \leq \|u\|_2 \|x_n\|_2$, while $T_n(x_n) = \|x_n\|_2^2$; hence $T_n \in (\ell^2)^*$ and $\|T_n\|_* = \|x_n\|_2$. By the preliminary lemma, weak convergence of $\{x_n\}$ to 0 says exactly that $T_n(u) = \langle u, x_n \rangle \rightarrow 0$ for each $u \in \ell^2$; in particular $\sup_n |T_n(u)| < \infty$ for each $u \in \ell^2$.

Suppose, to get a contradiction, that $\{\|T_n\|_*\}$ is unbounded. We use the following elementary observation: for a bounded linear functional T on a normed linear space, any point z , and any $r > 0$,

$$\sup \{|T(w)| : \|w - z\| \leq r\} \geq r \|T\|_*,$$

since for $\|u\| \leq r$ we have $\max \{|T(z+u)|, |T(z-u)|\} \geq \frac{1}{2} (|T(z+u)| + |T(z-u)|) \geq |T(u)|$, and $\sup_{\|u\| \leq r} |T(u)| = r \|T\|_*$.

Choose indices $k_1 < k_2 < \dots$ with $\|T_{k_j}\|_* \geq 4^j$. Put $z_0 = 0$ and choose inductively, using the observation with $r = 3^{-j}$ and the fact that $2/3 < 1$, points $z_j \in \ell^2$ with

$$\|z_j - z_{j-1}\|_2 \leq 3^{-j} \quad \text{and} \quad |T_{k_j}(z_j)| \geq \frac{2}{3} \cdot 3^{-j} \|T_{k_j}\|_*.$$

Write $d_j = z_j - z_{j-1}$, so $\sum_j \|d_j\|_2 \leq \sum_j 3^{-j} < \infty$. For each coordinate i we have $|d_j(i)| \leq \|d_j\|_2$, so the series $z(i) = \sum_{j \geq 1} d_j(i)$ converges absolutely; and for every N and every m , Minkowski's Inequality for

finite sums gives $\left(\sum_{i \leq N} \left| \sum_{j \leq m} d_j(i) \right|^2 \right)^{1/2} \leq \sum_{j \geq 1} \|d_j\|_2$, so letting $m \rightarrow \infty$ and then $N \rightarrow \infty$ shows $z = \{z(i)\} \in \ell^2$. The same estimate applied to the tail $\sum_{j > m} d_j$ gives $\|z - z_m\|_2 \leq \sum_{j > m} 3^{-j} = \frac{1}{2} \cdot 3^{-m}$. Therefore, for every j ,

$$|T_{k_j}(z)| \geq |T_{k_j}(z_j)| - \|T_{k_j}\|_* \|z - z_j\|_2 \geq \left(\frac{2}{3} - \frac{1}{2} \right) 3^{-j} \|T_{k_j}\|_* \geq \frac{1}{6} \left(\frac{4}{3} \right)^j,$$

which is unbounded in j . This contradicts $\sup_n |T_n(z)| < \infty$. Hence $\sup_n \|x_n\|_2 = \sup_n \|T_n\|_* < \infty$, and we may choose $M \geq 0$ with

$$\|x_n\|_2^2 \leq M \quad \text{for all } n.$$

(The argument just given is the Uniform Boundedness Principle; Problem 26 of this chapter is its L^p counterpart.)

We choose the subsequence inductively so that the partial sums

$$s_k = x_{n_1} + x_{n_2} + \dots + x_{n_k}$$

satisfy

$$\|s_k\|_2^2 \leq (2 + M)k \quad \text{for all } k. \quad (*)$$

Set $n_1 = 1$. Then $\|s_1\|_2^2 = \|x_1\|_2^2 \leq M \leq (2 + M) \cdot 1$, so $(*)$ holds for $k = 1$.

Suppose $n_1 < n_2 < \dots < n_k$ have been chosen with $(*)$ holding for that k . The vector s_k belongs to ℓ^2 , so by weak convergence $\langle s_k, x_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. Hence we may choose a natural number $n_{k+1} > n_k$ with

$$|\langle s_k, x_{n_{k+1}} \rangle| \leq 1.$$

Since $s_{k+1} = s_k + x_{n_{k+1}}$, expanding the inner product gives

$$\|s_{k+1}\|_2^2 = \|s_k\|_2^2 + 2\langle s_k, x_{n_{k+1}} \rangle + \|x_{n_{k+1}}\|_2^2 \leq (2 + M)k + 2 + M = (2 + M)(k + 1),$$

so $(*)$ holds for $k + 1$. This completes the induction and produces a subsequence $\{x_{n_k}\}$ satisfying $(*)$ for every k .

Finally, for each k ,

$$\left\| \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k} \right\|_2^2 = \frac{\|s_k\|_2^2}{k^2} \leq \frac{(2+M)k}{k^2} = \frac{2+M}{k},$$

which tends to 0 as $k \rightarrow \infty$. Therefore the arithmetic means of $\{x_{n_k}\}$ converge strongly to 0 in ℓ^2 , and in the general case to x . ■

Remark. Combining this with Problem 33 (every bounded sequence in ℓ^p , $1 \leq p < \infty$, has a weakly convergent subsequence) yields: every bounded sequence in ℓ^2 has a subsequence whose arithmetic means converge strongly in ℓ^2 . Problem 38 exhibits the theorem in action: the sequence $\{e_n\}$ converges weakly but not strongly to 0, yet its arithmetic means converge strongly to 0.

Problem 8.41. Let E be a measurable set and $1 \leq p < \infty$. Let T be a continuous linear functional on $L^p[a, b]$ and $K = \{f \in L^p(E) \mid \|f\|_p \leq 1\}$. Find a function $f_0 \in K$ for which

$$T(f_0) \geq T(f) \quad \text{for all } f \text{ in } K.$$

Solution. Here T is a continuous, that is bounded, linear functional on $L^p(E)$ and K is the closed unit ball of $L^p(E)$. We give the answer for $1 < p < \infty$, first by an appeal to Theorem 17 and then by an explicit formula, and we then show that for $p = 1$ a maximizer need not exist, so that the hypothesis must be read as $1 < p < \infty$.

Existence by Theorem 17 ($1 < p < \infty$). As observed in the second Example of this section, $K = \{f \in L^p(E) \mid \|f\|_p \leq 1\}$ is a closed, bounded, convex subset of $L^p(E)$: it is bounded by definition; it is convex by Minkowski's Inequality, since for $f, g \in K$ and $\lambda \in [0, 1]$, $\|\lambda f + (1-\lambda)g\|_p \leq \lambda\|f\|_p + (1-\lambda)\|g\|_p \leq 1$; and it is closed because $\|\|f_n\|_p - \|f\|_p\| \leq \|f_n - f\|_p$, so the norms of a strongly convergent sequence converge to the norm of the limit. The functional $-T$ is linear, hence convex (equality holds in the convexity inequality), and it is continuous since T is bounded: $|-T(f_n) - (-T)(f)| \leq \|T\|_* \|f_n - f\|_p$. Therefore Theorem 17 applies to $C = K$ and the continuous convex functional $-T$: there is $f_0 \in K$ with $-T(f_0) \leq -T(f)$ for all $f \in K$, that is,

$$T(f_0) \geq T(f) \quad \text{for all } f \in K.$$

An explicit maximizer ($1 < p < \infty$). Let q be the conjugate of p , so $1 < q < \infty$. By the Riesz Representation Theorem for the Dual of $L^p(E)$ there is a function $g \in L^q(E)$ with

$$T(f) = \int_E f \cdot g \quad \text{for all } f \in L^p(E), \quad \|T\|_* = \|g\|_q.$$

If $g = 0$ a.e. on E , then T is identically 0 and $f_0 = 0$ works. Otherwise $\|g\|_q > 0$, and we set

$$f_0 = \frac{\operatorname{sgn}(g) |g|^{q-1}}{\|g\|_q^{q-1}},$$

where $\operatorname{sgn}(g)$ is 1 where $g > 0$, -1 where $g < 0$, and 0 where $g = 0$. This is a measurable function, and since $(q-1)p = q$,

$$\int_E |f_0|^p = \frac{1}{\|g\|_q^{(q-1)p}} \int_E |g|^{(q-1)p} = \frac{1}{\|g\|_q^q} \int_E |g|^q = 1.$$

So $f_0 \in L^p(E)$ with $\|f_0\|_p = 1$, hence $f_0 \in K$. Moreover $f_0 \cdot g = |g|^q / \|g\|_q^{q-1}$, so

$$T(f_0) = \frac{1}{\|g\|_q^{q-1}} \int_E |g|^q = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q = \|T\|_*.$$

On the other hand, for any $f \in K$, $T(f) \leq |T(f)| \leq \|T\|_* \|f\|_p \leq \|T\|_*$. Therefore

$$T(f_0) = \|T\|_* \geq T(f) \quad \text{for all } f \in K,$$

which is the required conclusion, and it identifies the maximum value of T on K as $\|T\|_*$.

The case $p = 1$. Here $q = \infty$, and the maximum need not be attained. Take $E = [0, 1]$, $p = 1$, and $T(f) = \int_0^1 xf(x) dx$, integration against $g(x) = x \in L^\infty[0, 1]$; T is a bounded linear functional with $\|T\|_* = \|g\|_\infty = 1$. The value 1 is not attained on K : if $f \in L^1[0, 1]$ with $\|f\|_1 \leq 1$ and f is not zero a.e., then, since $1 - x > 0$ for all $x \in [0, 1]$ and $|f| > 0$ on a set of positive measure, $\int_0^1 (1 - x)|f(x)| dx > 0$, so

$$T(f) \leq \int_0^1 x|f(x)| dx = \int_0^1 |f| - \int_0^1 (1 - x)|f| < \int_0^1 |f| \leq 1,$$

while $T(0) = 0 < 1$. Thus $T(f) < 1$ for every $f \in K$. Yet the supremum is 1: for $f_n = n\chi_{[1-1/n, 1]}$ we have $\|f_n\|_1 = 1$ and

$$T(f_n) = n \int_{1-1/n}^1 x dx = \frac{n}{2} \left(1 - \left(1 - \frac{1}{n} \right)^2 \right) = 1 - \frac{1}{2n} \rightarrow 1.$$

So no $f_0 \in K$ maximizes T . This is consistent with Theorem 17, whose hypothesis is $1 < p < \infty$: it is exactly the weak sequential compactness of K furnished by Theorem 14, which fails in L^1 , that is lost. More precisely, when $p = 1$ and T is given by integration against $g \in L^\infty(E)$ with g not zero a.e., a maximizer exists if and only if the set $A = \{x \in E : |g(x)| = \|g\|_\infty\}$ has positive measure. If $m(A) > 0$, choose a measurable $B \subseteq A$ with $0 < m(B) < \infty$ and put $f_0 = \text{sgn}(g)\chi_B/m(B)$; then $\|f_0\|_1 = 1$ and $T(f_0) = \|g\|_\infty = \|T\|_*$. If $m(A) = 0$, then $\|g\|_\infty - |g| > 0$ a.e. on E , so for $f \in K$ that is not zero a.e. the set on which $|f| > 0$ has positive measure and hence $\int_E (\|g\|_\infty - |g|)|f| > 0$; therefore

$$T(f) \leq \int_E |g||f| = \|g\|_\infty \int_E |f| - \int_E (\|g\|_\infty - |g|)|f| < \|g\|_\infty \int_E |f| \leq \|T\|_*,$$

while $T(0) = 0 < \|T\|_*$. On the other hand $\sup_{f \in K} T(f) = \|T\|_*$: given $\varepsilon > 0$, the set $\{x \in E : |g(x)| > \|g\|_\infty - \varepsilon\}$ has positive measure, so it contains a measurable subset B with $0 < m(B) < \infty$, and $f = \text{sgn}(g)\chi_B/m(B)$ belongs to K with $T(f) = \frac{1}{m(B)} \int_B |g| \geq \|g\|_\infty - \varepsilon$. So the supremum is not attained and no maximizer exists.

Problem 8.42. (Nemytskii) Let E be a measurable set and p_1, p_2 belong to $[1, \infty)$. Suppose φ is a continuous real-valued function defined on $\mathbb{R}$ for which there are constants c_1 and c_2 such that $|\varphi(s)| \leq c_1 + c_2|s|^{p_1/p_2}$ for all real numbers s . Let $\{f_n\}$ be a sequence in $L^{p_1}(E)$. Show that

$$\text{if } \{f_n\} \rightarrow f \text{ in } L^{p_1}(E), \text{ then } \{\varphi \circ f_n\} \rightarrow \varphi \circ f \text{ in } L^{p_2}.$$

Solution. We may assume $c_1 \geq 0$ and $c_2 \geq 0$, since replacing c_1 and c_2 by their absolute values only weakens the growth bound. We also assume, as in Corollary 18 of this section, that E has finite measure, or else that $c_1 = 0$; the argument below is the same in the two cases.

Some such hypothesis is genuinely needed. For $f \equiv 0$ the growth bound gives only $|\varphi \circ f| \leq c_1$, and the constant function c_1 does not belong to $L^{p_2}(E)$ when $m(E) = \infty$ and $c_1 > 0$, so the conclusion need not even be meaningful. Worse, the asserted convergence can fail outright on a set of infinite measure. Take $E = \mathbb{R}$ and $p_1 = p_2 = 1$, and let $\varphi(s) = |s|^{1/2}$ for $|s| \leq 1$ and $\varphi(s) = 1$ for $|s| > 1$; then φ is continuous on $\mathbb{R}$ and $|\varphi(s)| \leq 1$ for all s , so the growth bound holds with $c_1 = 1$ and $c_2 = 0$. Put $f_n = n^{-2}\chi_{[0, n]}$ and $f \equiv 0$. Then $\|f_n - f\|_1 = n^{-2} \cdot n = 1/n \rightarrow 0$, so $\{f_n\} \rightarrow f$ in $L^1(\mathbb{R})$, while $\varphi \circ f_n = n^{-1}\chi_{[0, n]}$ and $\varphi \circ f \equiv 0$, so

$$\|\varphi \circ f_n - \varphi \circ f\|_1 = n^{-1} \cdot n = 1 \quad \text{for all } n.$$

We use repeatedly the elementary inequality: for $r \geq 1$ and $a, b \geq 0$,

$$(a + b)^r \leq (2 \max\{a, b\})^r = 2^r \max\{a^r, b^r\} \leq 2^r (a^r + b^r). \quad (1)$$

Step 1: $\varphi \circ h$ belongs to $L^{p_2}(E)$ for every $h \in L^{p_1}(E)$. Since φ is continuous and h is measurable and

finite a.e. on E , the composition $\varphi \circ h$ is measurable on E . By the growth bound and (1) with $r = p_2$,

$$|\varphi \circ h|^{p_2} \leq (c_1 + c_2|h|^{p_1/p_2})^{p_2} \leq 2^{p_2} (c_1^{p_2} + c_2^{p_2}|h|^{p_1}) \quad \text{a.e. on } E.$$

The right-hand side is integrable over E : $|h|^{p_1}$ is integrable since $h \in L^{p_1}(E)$, and the constant $c_1^{p_2}$ is integrable over E by our standing assumption. By the integral comparison test, $|\varphi \circ h|^{p_2}$ is integrable over E , that is, $\varphi \circ h \in L^{p_2}(E)$. In particular $\varphi \circ f$ and every $\varphi \circ f_n$ lie in $L^{p_2}(E)$, so the assertion is meaningful.

Step 2: reduction to a subsequence argument. Suppose, to get a contradiction, that $\{f_n\} \rightarrow f$ in $L^{p_1}(E)$ but $\{\varphi \circ f_n\}$ fails to converge to $\varphi \circ f$ in $L^{p_2}(E)$. Then there is an $\varepsilon > 0$ and a subsequence $\{f_{n_j}\}$ for which

$$\|\varphi \circ f_{n_j} - \varphi \circ f\|_{p_2} \geq \varepsilon \quad \text{for all } j. \quad (2)$$

Of course $\{f_{n_j}\} \rightarrow f$ in $L^{p_1}(E)$ as well. We will produce a further subsequence along which $\|\varphi \circ f_{n_j} - \varphi \circ f\|_{p_2} \rightarrow 0$, contradicting (2).

Step 3: a dominated, pointwise convergent further subsequence. Since $\{f_{n_j}\} \rightarrow f$ in $L^{p_1}(E)$, we may choose a subsequence, which we relabel $\{h_k\}$, that is rapidly Cauchy, that is,

$$\|h_{k+1} - h_k\|_{p_1} \leq 2^{-k} \quad \text{for all } k.$$

Define

$$g = |h_1| + \sum_{k=1}^{\infty} |h_{k+1} - h_k|,$$

the sum of a series of nonnegative measurable functions, so g is a nonnegative measurable function on E (possibly infinite at some points). Let $g_m = |h_1| + \sum_{k=1}^m |h_{k+1} - h_k|$. By Minkowski's Inequality,

$$\|g_m\|_{p_1} \leq \|h_1\|_{p_1} + \sum_{k=1}^m \|h_{k+1} - h_k\|_{p_1} \leq \|h_1\|_{p_1} + \sum_{k=1}^{\infty} 2^{-k} = \|h_1\|_{p_1} + 1.$$

Since $\{g_m^{p_1}\}$ is an increasing sequence of nonnegative measurable functions converging pointwise to g^{p_1} , the Monotone Convergence Theorem gives

$$\int_E g^{p_1} = \lim_{m \rightarrow \infty} \int_E g_m^{p_1} \leq (\|h_1\|_{p_1} + 1)^{p_1} < \infty.$$

Thus $g \in L^{p_1}(E)$; in particular $g < \infty$ a.e. on E . Consequently the series $h_1 + \sum_{k \geq 1} (h_{k+1} - h_k)$ converges absolutely at almost every point of E , and its m th partial sum is h_{m+1} . Hence $\{h_k\}$ converges pointwise a.e. on E to some measurable function h , and the triangle inequality gives

$$|h_k| \leq g_{k-1} \leq g \quad \text{a.e. on } E \text{ for all } k, \quad |h| \leq g \quad \text{a.e. on } E.$$

Moreover $\{h_k\} \rightarrow f$ in $L^{p_1}(E)$, being a subsequence of $\{f_{n_j}\}$. By the Riesz–Fischer Theorem some subsequence of $\{h_k\}$ converges to f pointwise a.e. on E ; since the whole sequence $\{h_k\}$ converges pointwise a.e. to h , we conclude $h = f$ a.e. on E . So, after modifying f on a set of measure zero (which changes neither $\|\cdot\|_{p_2}$ nor anything else),

$$\{h_k\} \rightarrow f \text{ pointwise a.e. on } E, \quad |h_k| \leq g \text{ and } |f| \leq g \text{ a.e. on } E.$$

Step 4: passing to the limit. Since φ is continuous on $\mathbb{R}$ and $h_k(x) \rightarrow f(x)$ for almost all $x \in E$,

$$\varphi(h_k(x)) \rightarrow \varphi(f(x)) \quad \text{for almost all } x \in E,$$

so $\{|\varphi \circ h_k - \varphi \circ f|^{p_2}\} \rightarrow 0$ pointwise a.e. on E . We dominate this sequence. By the growth bound and the estimates $|h_k| \leq g$, $|f| \leq g$ a.e.,

$$|\varphi \circ h_k| + |\varphi \circ f| \leq 2c_1 + 2c_2 g^{p_1/p_2} \quad \text{a.e. on } E \text{ for all } k,$$

and therefore, by (1) with $r = p_2$,

$$|\varphi \circ h_k - \varphi \circ f|^{p_2} \leq \left(2c_1 + 2c_2 g^{p_1/p_2}\right)^{p_2} \leq 2^{p_2} ((2c_1)^{p_2} + (2c_2)^{p_2} g^{p_1}) =: G \quad \text{a.e. on } E.$$

Exactly as in Step 1, G is integrable over E , since g^{p_1} is integrable and the constant $(2c_1)^{p_2}$ is integrable over E by our standing assumption. Thus the Lebesgue Dominated Convergence Theorem applies and yields

$$\lim_{k \rightarrow \infty} \int_E |\varphi \circ h_k - \varphi \circ f|^{p_2} = 0,$$

that is, $\|\varphi \circ h_k - \varphi \circ f\|_{p_2} \rightarrow 0$.

But $\{h_k\}$ is a subsequence of $\{f_{n_j}\}$, so this contradicts (2). Therefore no such ε and subsequence exist, and

$$\{\varphi \circ f_n\} \rightarrow \varphi \circ f \quad \text{in } L^{p_2}(E).$$

Remark. This is the assertion the book leaves as an exercise in the Example preceding Lemma 16, where E has finite measure, $1 < p < \infty$, φ is continuous and convex with $|\varphi(s)| \leq a + b|s|^p$, and $T(f) = \int_E \varphi \circ f$. Taking $p_1 = p$ and $p_2 = 1$ above, the growth bound reads $|\varphi(s)| \leq c_1 + c_2|s|^p$, and the conclusion is that $\{f_n\} \rightarrow f$ in $L^p(E)$ implies $\{\varphi \circ f_n\} \rightarrow \varphi \circ f$ in $L^1(E)$; hence $|T(f_n) - T(f)| \leq \int_E |\varphi \circ f_n - \varphi \circ f| \rightarrow 0$, so T is properly defined (Step 1) and continuous on $L^p(E)$. The map $f \mapsto \varphi \circ f$ is the Nemytskii operator associated with φ ; the content of the problem is that it carries $L^{p_1}(E)$ continuously into $L^{p_2}(E)$ under the stated growth condition.

8.7 Problems 8.43–8.45

Problem 8.43. (Beppo Levi) Let E be a measurable set, $1 \leq p < \infty$, and C a closed bounded convex subset of $L^p(E)$. Show that for any function $f_0 \in L^p(E)$, there is a function g_0 in C for which

$$\|g_0 - f_0\|_p \leq \|g - f_0\|_p \quad \text{for all } g \in C.$$

Solution. Assume, as is implicit in the assertion, that C is nonempty.

The case $1 < p < \infty$. Fix $f_0 \in L^p(E)$ and define the real-valued functional T on C by

$$T(g) = \|g - f_0\|_p \quad \text{for all } g \in C.$$

This is properly defined, since $g - f_0$ belongs to $L^p(E)$ whenever g does.

T is convex. Indeed, let g and h belong to C and $\lambda \in [0, 1]$. Then $\lambda g + (1 - \lambda)h$ belongs to C since C is convex, and

$$\lambda g + (1 - \lambda)h - f_0 = \lambda(g - f_0) + (1 - \lambda)(h - f_0),$$

so that, by the Minkowski Inequality,

$$T(\lambda g + (1 - \lambda)h) \leq \lambda\|g - f_0\|_p + (1 - \lambda)\|h - f_0\|_p = \lambda T(g) + (1 - \lambda)T(h).$$

T is continuous on C . Indeed, by the triangle inequality for the norm $\|\cdot\|_p$,

$$|T(g) - T(h)| = \left| \|g - f_0\|_p - \|h - f_0\|_p \right| \leq \|g - h\|_p \quad \text{for all } g, h \in C,$$

so if $\{g_n\}$ is a sequence in C converging strongly in $L^p(E)$ to $g \in C$, then $\{T(g_n)\} \rightarrow T(g)$.

Thus C is a closed, bounded, convex subset of $L^p(E)$ and T is a continuous convex functional on C . Since $1 < p < \infty$, Theorem 17 tells us that T takes on a minimum value on C : there is a function $g_0 \in C$ for which $T(g_0) \leq T(g)$ for all $g \in C$, that is,

$$\|g_0 - f_0\|_p \leq \|g - f_0\|_p \quad \text{for all } g \in C.$$

The case $p = 1$. For $p = 1$ the assertion is false, and the above proof genuinely breaks down: Theorem 17 rests on Theorem 14 (weak sequential compactness of bounded sequences), which fails in L^1 . Here

is a counterexample. Take $E = [0, 1]$, $f_0 \equiv 0$, and

$$C = \left\{ f \in L^1[0, 1] \mid \int_0^1 xf(x) dx = 1 \text{ and } \|f\|_1 \leq 3 \right\}.$$

C is nonempty: since $\int_{1/2}^1 x dx = 3/8$, the function $f = (8/3)\chi_{[1/2, 1]}$ belongs to C , as $\|f\|_1 = 4/3 \leq 3$. C is convex, being the intersection of the ball $\{\|f\|_1 \leq 3\}$ (convex by Minkowski's Inequality) with the set on which the linear functional $f \mapsto \int_0^1 xf$ takes the value 1.

C is closed: if $\{f_n\}$ is a sequence in C converging in $L^1[0, 1]$ to f , then

$$\left| \int_0^1 xf_n - \int_0^1 xf \right| \leq \int_0^1 |f_n - f| = \|f_n - f\|_1 \rightarrow 0,$$

so $\int_0^1 xf = 1$; and $\|f\|_1 = \lim \|f_n\|_1 \leq 3$. Of course C is bounded.

Now compute the distance from $f_0 = 0$ to C . For $f \in C$,

$$1 = \int_0^1 xf(x) dx \leq \int_0^1 x|f(x)| dx \leq \int_0^1 |f| = \|f\|_1,$$

so $\|f - f_0\|_1 \geq 1$ for all $f \in C$. On the other hand, for each natural number $n \geq 1$ set

$$a_n = \left(\int_{1-1/n}^1 x dx \right)^{-1} = \left(\frac{1}{n} - \frac{1}{2n^2} \right)^{-1}, \quad f_n = a_n \chi_{[1-1/n, 1]}.$$

Then $\int_0^1 xf_n = 1$ and

$$\|f_n\|_1 = \frac{a_n}{n} = \frac{1}{1 - 1/(2n)} \leq 2 \leq 3,$$

so $f_n \in C$, while $\|f_n - f_0\|_1 = 1/(1 - 1/(2n)) \rightarrow 1$. Hence

$$\inf_{f \in C} \|f - f_0\|_1 = 1.$$

This infimum is not attained. Suppose $f \in C$ had $\|f\|_1 = 1$. Then

$$0 = \|f\|_1 - \int_0^1 xf(x) dx = \int_0^1 (|f(x)| - xf(x)) dx \geq \int_0^1 (1-x)|f(x)| dx \geq 0,$$

since $|f| - xf \geq |f| - x|f| = (1-x)|f| \geq 0$ on $[0, 1]$. Therefore $(1-x)|f(x)| = 0$ for almost all $x \in [0, 1]$, and hence $f = 0$ almost everywhere on $[0, 1]$, so $f = 0$ in $L^1[0, 1]$. This contradicts $\int_0^1 xf = 1$. Thus no $g_0 \in C$ satisfies $\|g_0 - f_0\|_1 \leq \|g - f_0\|_1$ for all $g \in C$.

So the conclusion holds exactly for $1 < p < \infty$; the hypothesis in the statement should read $1 < p < \infty$, which is the range for which Theorem 17 is available.

Problem 8.44. (Banach-Saks) For a natural number n , define the function f_n on $[0, 1]$ by setting

$$f_n(x) = 1 \text{ for } k/2^n + 1/2^{2n+1} \leq x < (k+1)/2^n \text{ and } 0 \leq k \leq 2^n - 1,$$

and $f_n(x) = 1 - 2^{n+1}$ elsewhere on $[0, 1]$. Define $f \equiv 0$ on $[0, 1]$.

(i) Show that

$$\left| \int_0^x f_n \right| \leq 1/2^n \text{ for all } x \in [0, 1] \text{ and all } n,$$

and therefore

$$\lim_{n \rightarrow \infty} \int_0^x f_n = \int_0^x f \text{ for all } x \in [0, 1].$$

(ii) Define E to be the subset of $[0, 1]$ on which $f_n = 1$ for all n . Show that

$$\int_E f_n = m(E) > 0 \quad \text{for all } n.$$

- (iii) Show that $\|f_n\|_1 \leq 2$ for all n . Infer from part (ii) that $\{f_n\}$ is a bounded sequence in $L^1[0, 1]$ that does not converge weakly in $L^1[0, 1]$ to f . Does this and part (i) contradict Theorem 11?
- (iv) For $1 < p < \infty$, infer from part (ii) that $\{f_n\}$ is a sequence in $L^p[0, 1]$ that does not converge weakly in $L^p[0, 1]$ to f . Does this and part (i) contradict Theorem 11?

Solution. Throughout, fix a natural number n and write

$$I_k = [k/2^n, (k+1)/2^n), \quad J_k = [k/2^n, k/2^n + 1/2^{2n+1}) \subseteq I_k, \quad 0 \leq k \leq 2^n - 1.$$

The intervals I_k partition $[0, 1)$, each has length $1/2^n$, and each J_k has length $1/2^{2n+1}$. By definition $f_n = 1 - 2^{n+1}$ on J_k and $f_n = 1$ on $I_k \setminus J_k$. The single point $x = 1$ lies in no I_k , so it falls under the *elsewhere* clause and $f_n(1) = 1 - 2^{n+1}$. Accordingly put

$$B_n = \{1\} \cup \bigcup_{k=0}^{2^n-1} J_k, \quad A_n = [0, 1] \setminus B_n.$$

The J_k are disjoint and $\{1\}$ has measure zero, so

$$m(B_n) = 2^n \cdot \frac{1}{2^{2n+1}} = \frac{1}{2^{n+1}}, \quad m(A_n) = 1 - \frac{1}{2^{n+1}},$$

and $f_n = 1$ on A_n , $f_n = 1 - 2^{n+1}$ on B_n . Each f_n is a simple function, hence bounded and measurable, so it belongs to $L^p[0, 1]$ for every $1 \leq p \leq \infty$.

The key computation is that f_n has mean zero on each I_k :

$$\int_{I_k} f_n = 1 \cdot \left(\frac{1}{2^n} - \frac{1}{2^{2n+1}} \right) + (1 - 2^{n+1}) \cdot \frac{1}{2^{2n+1}} = \frac{1}{2^n} - \frac{2^{n+1}}{2^{2n+1}} = \frac{1}{2^n} - \frac{1}{2^n} = 0.$$

Call this identity (*). Note that $m(I_k) = 1/2^n$ and $m(J_k) = 1/2^{2n+1}$ were used, together with $f_n = 1$ on $I_k \setminus J_k$ and $f_n = 1 - 2^{n+1}$ on J_k .

(i) Let $x \in [0, 1]$. If $x = 1$, then by (*), $\int_0^1 f_n = \sum_{k=0}^{2^n-1} \int_{I_k} f_n = 0$, and the bound holds. Otherwise let k be the unique index with $x \in I_k$, and write $x = k/2^n + t$ with $0 \leq t < 1/2^n$. By (*) the integrals over $I_0, \dots, I_{k-1}$ all vanish, so

$$\int_0^x f_n = \int_{k/2^n}^{k/2^n+t} f_n =: \varphi(t).$$

If $0 \leq t \leq 1/2^{2n+1}$, the interval of integration lies in J_k , where $f_n = 1 - 2^{n+1}$, so

$$\varphi(t) = (1 - 2^{n+1})t,$$

which decreases from $\varphi(0) = 0$ to $\varphi(1/2^{2n+1}) = 1/2^{2n+1} - 1/2^n$. If $1/2^{2n+1} \leq t < 1/2^n$, then

$$\varphi(t) = \left(\frac{1}{2^{2n+1}} - \frac{1}{2^n} \right) + \left(t - \frac{1}{2^{2n+1}} \right),$$

since $f_n = 1$ on $I_k \setminus J_k$; this increases with t and its supremum, at $t = 1/2^n$, is 0. Hence φ takes values between $1/2^{2n+1} - 1/2^n$ and 0, so

$$\left| \int_0^x f_n \right| = |\varphi(t)| \leq \frac{1}{2^n} - \frac{1}{2^{2n+1}} \leq \frac{1}{2^n}.$$

Since $f \equiv 0$, $\int_0^x f = 0$ for every x , and since $1/2^n \rightarrow 0$,

$$\lim_{n \rightarrow \infty} \int_0^x f_n = 0 = \int_0^x f \quad \text{for all } x \in [0, 1].$$

(ii) Since $1 - 2^{n+1} \neq 1$, the set on which $f_n = 1$ is exactly A_n . Therefore

$$E = \bigcap_{n=1}^{\infty} A_n = [0, 1] \setminus \bigcup_{n=1}^{\infty} B_n,$$

which is measurable. By the countable subadditivity and monotonicity of Lebesgue measure,

$$m\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} m(B_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2},$$

and hence

$$m(E) \geq 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

Since $f_n \equiv 1$ on E ,

$$\int_E f_n = m(E) \geq \frac{1}{2} > 0 \text{ for all } n.$$

(iii) We compute the L^1 norm:

$$\|f_n\|_1 = \int_{A_n} |f_n| + \int_{B_n} |f_n| = \left(1 - \frac{1}{2^{n+1}}\right) + (2^{n+1} - 1) \cdot \frac{1}{2^{n+1}} = 2 - \frac{1}{2^n} \leq 2.$$

So $\{f_n\}$ is a bounded sequence in $L^1[0, 1]$.

The conjugate of $p = 1$ is $q = \infty$, and χ_E belongs to $L^\infty[0, 1]$. If $\{f_n\}$ converged weakly to $f \equiv 0$ in $L^1[0, 1]$, then, testing against χ_E ,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n \cdot \chi_E = \int_0^1 f \cdot \chi_E = 0.$$

But by part (ii), $\int_0^1 f_n \cdot \chi_E = \int_E f_n = m(E) \geq 1/2$ for every n , so this limit is $m(E) > 0$, a contradiction. Hence $\{f_n\}$ does not converge weakly in $L^1[0, 1]$ to f .

This does not contradict Theorem 11, because Theorem 11 is stated only for $1 < p < \infty$. What parts (i) and (iii) do show is that the theorem genuinely fails at $p = 1$: here $\{f_n\}$ is a bounded sequence in $L^1[0, 1]$, $f \in L^1[0, 1]$, and $\lim_n \int_0^x f_n = \int_0^x f$ for every $x \in [0, 1]$, and yet $\{f_n\}$ does not converge weakly to f in $L^1[0, 1]$. The reason is the one recorded in the text after Theorem 11: the proof of Theorem 11 runs through Proposition 9 and requires the linear span of the characteristic functions of intervals, that is, the step functions, to be dense in $L^q[0, 1]$; for $q = \infty$ the step functions are not dense in $L^\infty[0, 1]$. This is exactly the remark the text makes immediately after Theorem 11, and the function χ_E above is a witness to it. Indeed, we claim that $\|\chi_E - \varphi\|_\infty \geq 1/2$ for every step function φ on $[0, 1]$. First observe that if I is an interval with $m(I \cap E) > 0$, then $m(I \setminus E) > 0$: choose n with $1/2^n < m(I)$, so that I contains some I_k in its entirety, hence contains J_k , and $J_k \subseteq B_n \subseteq [0, 1] \setminus E$ has positive measure. Now let φ be a step function, determined by a finite partition of $[0, 1]$ into intervals on each of which φ is constant. Since $m(E) \geq 1/2 > 0$, at least one partition interval I has $m(I \cap E) > 0$; by the observation, $m(I \setminus E) > 0$ as well. If c is the value of φ on I , then $\chi_E - \varphi$ equals $1 - c$ on a subset of I of positive measure and equals $-c$ on a subset of I of positive measure, so

$$\|\chi_E - \varphi\|_\infty \geq \max\{|1 - c|, |c|\} \geq \frac{1}{2},$$

which proves the claim.

(iv) Fix $1 < p < \infty$ and let $q = p/(p - 1)$ be its conjugate. Each f_n is bounded and measurable on a set of finite measure, so $\{f_n\}$ is a sequence in $L^p[0, 1]$, and $f \equiv 0$ belongs to $L^p[0, 1]$. Also χ_E belongs to $L^q[0, 1]$, since $m(E) < \infty$. Exactly as in part (iii), if $\{f_n\}$ converged weakly to f in $L^p[0, 1]$, then

$$0 = \int_0^1 f \cdot \chi_E = \lim_{n \rightarrow \infty} \int_0^1 f_n \cdot \chi_E = \lim_{n \rightarrow \infty} m(E) = m(E) \geq \frac{1}{2},$$

a contradiction. So $\{f_n\}$ does not converge weakly in $L^p[0, 1]$ to f .

Again there is no contradiction with Theorem 11, but now for a different reason: Theorem 11 assumes that $\{f_n\}$ is a *bounded* sequence in $L^p[0, 1]$, and for $p > 1$ this sequence is not bounded. Indeed,

$$\|f_n\|_p^p = \int_{A_n} 1 + \int_{B_n} (2^{n+1} - 1)^p = \left(1 - \frac{1}{2^{n+1}}\right) + \frac{(2^{n+1} - 1)^p}{2^{n+1}} \geq \frac{(2^{n+1} - 1)^p}{2^{n+1}}.$$

Since $2^{n+1} - 1 \geq 2^n$, we get $\|f_n\|_p^p \geq 2^{np}/2^{n+1} = 2^{n(p-1)-1}$, and hence

$$\|f_n\|_p \geq 2^{(n(p-1)-1)/p} \rightarrow \infty \text{ as } n \rightarrow \infty,$$

because $p > 1$. So the hypothesis of Theorem 11 fails, and the theorem says nothing about this sequence.

Problem 8.45. Find a sequence $\{g_n\}$ in $L^2[0, 1]$ that has no Cauchy subsequence. Use this subsequence and the denseness of the continuous functions in $L^2[0, 1]$ to find a sequence of continuous functions on $[0, 1]$ for which no subsequence converges in $L^2[0, 1]$ to a function in $L^2[0, 1]$.

Solution. *A sequence with no Cauchy subsequence.* For each natural number n let

$$J_n = (2^{-n}, 2^{-n+1}] \subseteq [0, 1], \quad g_n = 2^{n/2} \chi_{J_n}.$$

Each g_n is a bounded measurable function, so it belongs to $L^2[0, 1]$, and since $m(J_n) = 2^{-n}$,

$$\|g_n\|_2^2 = \int_0^1 2^n \chi_{J_n} = 2^n \cdot 2^{-n} = 1.$$

The intervals J_n are pairwise disjoint, so for $n \neq m$ the functions g_n and g_m have disjoint supports and therefore $g_n \cdot g_m = 0$ on $[0, 1]$. Consequently, for $n \neq m$,

$$\|g_n - g_m\|_2^2 = \int_0^1 (g_n^2 - 2g_n g_m + g_m^2) = \|g_n\|_2^2 + \|g_m\|_2^2 = 2,$$

that is, $\|g_n - g_m\|_2 = \sqrt{2}$ whenever $n \neq m$.

Now let $\{g_{n_k}\}$ be any subsequence. Since the indices n_k are distinct, $\|g_{n_j} - g_{n_k}\|_2 = \sqrt{2}$ for all $j \neq k$. Taking $\epsilon = 1$ in the definition of a Cauchy sequence, there is no index K beyond which all pairwise distances are less than 1. Hence no subsequence of $\{g_n\}$ is Cauchy in $L^2[0, 1]$; in particular, since a convergent sequence is Cauchy, no subsequence of $\{g_n\}$ converges in $L^2[0, 1]$.

Passing to continuous functions. By Theorem 12 of Chapter 7, $C_c(E)$ is dense in $L^p(E)$ for $1 \leq p < \infty$; taking $E = [0, 1]$ and $p = 2$, and noting that $[0, 1]$ is itself bounded so that $C_c[0, 1]$ is the whole space of continuous functions on $[0, 1]$, the continuous functions on $[0, 1]$ are dense in $L^2[0, 1]$. Hence for each natural number n we may choose a continuous function h_n on $[0, 1]$ for which

$$\|h_n - g_n\|_2 < \frac{1}{n}.$$

We claim that no subsequence of $\{h_n\}$ converges in $L^2[0, 1]$ to a function in $L^2[0, 1]$. Suppose, to the contrary, that $\{h_{n_k}\}$ converges in $L^2[0, 1]$ to some $h \in L^2[0, 1]$. A convergent sequence in a normed linear space is Cauchy (Proposition 4 of Chapter 7), so $\{h_{n_k}\}$ is Cauchy in $L^2[0, 1]$. But then, for all indices j and k , by the triangle inequality,

$$\|g_{n_j} - g_{n_k}\|_2 \leq \|g_{n_j} - h_{n_j}\|_2 + \|h_{n_j} - h_{n_k}\|_2 + \|h_{n_k} - g_{n_k}\|_2 < \frac{1}{n_j} + \|h_{n_j} - h_{n_k}\|_2 + \frac{1}{n_k}.$$

Given $\epsilon > 0$, choose K_1 so that $\|h_{n_j} - h_{n_k}\|_2 < \epsilon/3$ for $j, k \geq K_1$, and choose K_2 so that $1/n_k < \epsilon/3$ for $k \geq K_2$; this is possible since $n_k \geq k$. Then for $j, k \geq \max\{K_1, K_2\}$ we have $\|g_{n_j} - g_{n_k}\|_2 < \epsilon$. Thus $\{g_{n_k}\}$ would be a Cauchy subsequence of $\{g_n\}$, contradicting what was proved above. Therefore no subsequence of the sequence $\{h_n\}$ of continuous functions converges in $L^2[0, 1]$ to a function in $L^2[0, 1]$.

Remark. Observe that $\|h_n\|_2 \leq \|g_n\|_2 + \|h_n - g_n\|_2 < 1 + 1/n \leq 2$ for all n , so $\{h_n\}$ is a bounded sequence of continuous functions; replacing h_n by $h_n/2$ (which changes nothing, since a convergent subsequence of $\{h_n/2\}$ would give one of $\{h_n\}$) produces continuous functions with $\int_0^1 |h_n|^2 \leq 1$ for all n and with no subsequence converging in $L^2[0, 1]$. Thus the Bolzano-Weierstrass Theorem does not extend to bounded sets in $L^2[0, 1]$, even to bounded sets of continuous functions. Note also that, by the Riesz-Fischer Theorem, $L^2[0, 1]$ is complete, so for a sequence in $L^2[0, 1]$ convergence to a function in $L^2[0, 1]$ is equivalent to being Cauchy; this is why the absence of Cauchy subsequences is exactly the absence of convergent subsequences.

9 Metric Spaces: General Properties

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9.1 Problems 9.1–9.7

Problem 9.1. Show that two metrics ρ and τ on the same set X are equivalent if and only if there is a $c > 0$ such that for all $u, v \in X$,

$$\frac{1}{c} \tau(u, v) \leq \rho(u, v) \leq c \tau(u, v).$$

Solution. Recall the definition: ρ and τ are equivalent provided there are positive numbers c_1 and c_2 such that

$$c_1 \cdot \tau(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \cdot \tau(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Suppose first that ρ and τ are equivalent, and let $c_1, c_2 > 0$ be as in the definition. Define

$$c = \max \left\{ \frac{1}{c_1}, c_2 \right\}.$$

Then $c > 0$, since both $1/c_1$ and c_2 are positive. Moreover $c \geq c_2$, so for all $u, v \in X$,

$$\rho(u, v) \leq c_2 \tau(u, v) \leq c \tau(u, v),$$

where we used $\tau(u, v) \geq 0$. Also $c \geq 1/c_1$, and hence $1/c \leq c_1$; therefore, again using $\tau(u, v) \geq 0$,

$$\frac{1}{c} \tau(u, v) \leq c_1 \tau(u, v) \leq \rho(u, v).$$

Combining the two displays, $\frac{1}{c} \tau(u, v) \leq \rho(u, v) \leq c \tau(u, v)$ for all $u, v \in X$, which is the asserted inequality.

Conversely, suppose there is a $c > 0$ with

$$\frac{1}{c} \tau(u, v) \leq \rho(u, v) \leq c \tau(u, v) \quad \text{for all } u, v \in X.$$

Set $c_1 = 1/c$ and $c_2 = c$. Both are positive, and the displayed inequality is exactly

$$c_1 \cdot \tau(u, v) \leq \rho(u, v) \leq c_2 \cdot \tau(u, v) \quad \text{for all } u, v \in X,$$

so ρ and τ are equivalent.

It is worth recording the symmetry that this reformulation makes transparent: the condition $\frac{1}{c} \tau \leq \rho \leq c \tau$ is unchanged if the roles of ρ and τ are interchanged, since it is equivalent to $\frac{1}{c} \rho \leq \tau \leq c \rho$. Indeed, from $\rho \leq c \tau$ we get $\frac{1}{c} \rho \leq \tau$, and from $\frac{1}{c} \tau \leq \rho$ we get $\tau \leq c \rho$. Hence equivalence of metrics is a symmetric relation. It is also reflexive (take $c = 1$) and transitive: if $\frac{1}{c} \tau \leq \rho \leq c \tau$ and $\frac{1}{c'} \sigma \leq \tau \leq c' \sigma$, then

$$\frac{1}{cc'} \sigma \leq \frac{1}{c} \tau \leq \rho \leq c \tau \leq cc' \sigma,$$

so ρ and σ are equivalent with constant cc' . Thus equivalence of metrics is an equivalence relation on the set of metrics on X .

Problem 9.2. Show that the following define equivalent metrics on $\mathbb{R}^n$:

$$\rho^*(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|;$$

$$\rho^+(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}.$$

Solution. Write $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ for points of $\mathbb{R}^n$.

Both are metrics. Each arises from a norm on the linear space $\mathbb{R}^n$ by the recipe $\rho(x, y) = \|x - y\|$ of

formula (1) of Section 9.1. Define

$$\|u\|_* = |u_1| + \cdots + |u_n|, \quad \|u\|_+ = \max\{|u_1|, \dots, |u_n|\}.$$

Both are nonnegative real-valued. Each vanishes exactly at $u = 0$: $\|u\|_* = 0$ forces every $|u_i| = 0$ since the $|u_i|$ are nonnegative and sum to 0, and $\|u\|_+ = 0$ forces $|u_i| \leq 0$, hence $u_i = 0$, for each i . Homogeneity is immediate from $|\alpha u_i| = |\alpha| |u_i|$: summing gives $\|\alpha u\|_* = |\alpha| \|u\|_*$, and taking maxima gives $\|\alpha u\|_+ = |\alpha| \|u\|_+$ (for $\alpha = 0$ both sides are 0). For the triangle inequality, the inequality $|u_i + v_i| \leq |u_i| + |v_i|$ for real numbers gives, upon summation over i , that $\|u + v\|_* \leq \|u\|_* + \|v\|_*$; and for each i ,

$$|u_i + v_i| \leq |u_i| + |v_i| \leq \|u\|_+ + \|v\|_+,$$

so taking the maximum over i on the left gives $\|u + v\|_+ \leq \|u\|_+ + \|v\|_+$. Thus $\|\cdot\|_*$ and $\|\cdot\|_+$ are norms, and, as observed in Section 9.1, the induced functions $\rho^*(x, y) = \|x - y\|_*$ and $\rho^+(x, y) = \|x - y\|_+$ are metrics: properties (i)-(iii) of a metric are immediate from the corresponding properties of the norm together with $\| - u \| = \| - 1 \| \|u\| = \|u\|$, and the triangle inequality (iv) for the metric is, as noted in the text, equivalent to the triangle inequality for the norm, via $x - y = (x - z) + (z - y)$.

Equivalence. Fix $x, y \in \mathbb{R}^n$ and put $t_i = |x_i - y_i| \geq 0$ for $1 \leq i \leq n$, so that

$$\rho^*(x, y) = t_1 + \cdots + t_n, \quad \rho^+(x, y) = \max\{t_1, \dots, t_n\}.$$

Choose an index k with $t_k = \max\{t_1, \dots, t_n\}$. Since every $t_i \geq 0$,

$$\rho^+(x, y) = t_k \leq t_1 + \cdots + t_n = \rho^*(x, y).$$

On the other hand, each $t_i \leq t_k$, so

$$\rho^*(x, y) = t_1 + \cdots + t_n \leq n t_k = n \cdot \rho^+(x, y).$$

Therefore

$$1 \cdot \rho^+(x, y) \leq \rho^*(x, y) \leq n \cdot \rho^+(x, y) \quad \text{for all } x, y \in \mathbb{R}^n,$$

which is precisely the definition of equivalence with $c_1 = 1$ and $c_2 = n$. (In the formulation of the preceding problem, one may take $c = n$, since $n \geq 1$ gives $\frac{1}{n}\rho^+ \leq \rho^+ \leq \rho^* \leq n\rho^+$.)

Problem 9.3. Find a metric on $\mathbb{R}^n$ that fails to be equivalent to either of those defined in the preceding problem.

Solution. Take ρ to be the discrete metric on $X = \mathbb{R}^n$, described in Section 9.1:

$$\rho(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

ρ is a metric. Clearly $\rho(x, y) \geq 0$, $\rho(x, y) = 0$ if and only if $x = y$, and $\rho(x, y) = \rho(y, x)$, since the defining condition $x = y$ is symmetric in x and y . For the triangle inequality, let $x, y, z \in \mathbb{R}^n$. If $x = y$ then $\rho(x, y) = 0 \leq \rho(x, z) + \rho(z, y)$ because ρ is nonnegative. If $x \neq y$, then z cannot equal both x and y ; hence at least one of $\rho(x, z)$, $\rho(z, y)$ equals 1, and therefore

$$\rho(x, y) = 1 \leq \rho(x, z) + \rho(z, y).$$

Next, ρ is not equivalent to ρ^* . Suppose it were. Then there would be a $c_1 > 0$ with

$$c_1 \cdot \rho^*(x, y) \leq \rho(x, y) \leq 1 \quad \text{for all } x, y \in \mathbb{R}^n,$$

the final inequality because ρ takes only the values 0 and 1. Thus $\rho^*(x, y) \leq 1/c_1$ for all x, y ; that is, ρ^* is a bounded function on $\mathbb{R}^n \times \mathbb{R}^n$. But ρ^* is unbounded: taking $x = 0$ and $y_t = (t, 0, \dots, 0)$ for $t > 0$ gives

$$\rho^*(0, y_t) = |t - 0| + 0 + \cdots + 0 = t,$$

which exceeds $1/c_1$ once $t > 1/c_1$. This contradiction shows ρ and ρ^* are not equivalent.

ρ is not equivalent to ρ^+ . The same argument applies verbatim, since $\rho^+(0, y_t) = \max\{t, 0, \dots, 0\} = t$ is likewise unbounded in t . Alternatively, invoke the preceding problem: ρ^* and ρ^+ are equivalent, and, as shown in Problem 1, equivalence of metrics is a transitive relation; so if ρ were equivalent to ρ^+ it would be equivalent to ρ^* , which we have just excluded.

The essential mechanism is that ρ is a bounded metric while ρ^* and ρ^+ are unbounded, and the inequality $c_1\rho^* \leq \rho$ required by equivalence transfers boundedness. Any bounded metric on $\mathbb{R}^n$ would do equally well; for instance $\sigma(x, y) = \min\{1, \rho^*(x, y)\}$ is a bounded metric, hence not equivalent to either ρ^* or ρ^+ . That σ is a metric is checked as follows: it is nonnegative, vanishes exactly when $\rho^*(x, y) = 0$, that is, when $x = y$, and is symmetric; for the triangle inequality, fix x, y, z and consider the two cases. If $\sigma(x, z) = 1$ or $\sigma(z, y) = 1$, then $\sigma(x, z) + \sigma(z, y) \geq 1 \geq \sigma(x, y)$. Otherwise $\sigma(x, z) = \rho^*(x, z)$ and $\sigma(z, y) = \rho^*(z, y)$, so

$$\sigma(x, y) \leq \rho^*(x, y) \leq \rho^*(x, z) + \rho^*(z, y) = \sigma(x, z) + \sigma(z, y).$$

Problem 9.4. For a closed, bounded interval $[a, b]$, consider the set $X = C[a, b]$ of continuous real-valued functions on $[a, b]$. Show that the metric induced by the maximum norm and that induced by the $L^1[a, b]$ norm are not equivalent.

Solution. Here $[a, b]$ is a nondegenerate interval, so $a < b$. Denote the two metrics on $X = C[a, b]$ by

$$\rho_{\max}(f, g) = \|f - g\|_{\max} = \max\{|f(x) - g(x)| : x \in [a, b]\}, \quad \rho_1(f, g) = \|f - g\|_1 = \int_a^b |f - g|.$$

Both are metrics on X : the maximum norm is a norm on $C[a, b]$ as recalled in Section 9.1, and the $L^1[a, b]$ norm restricted to $C[a, b]$ is a norm as well, since a continuous function on $[a, b]$ with $\int_a^b |h| = 0$ vanishes identically. (Indeed, if h is continuous and $h(x_0) \neq 0$ for some $x_0 \in [a, b]$, then by continuity $|h| > |h(x_0)|/2$ on a nondegenerate subinterval $I \subseteq [a, b]$ containing x_0 , whence by the monotonicity of integration $\int_a^b |h| \geq \frac{1}{2}|h(x_0)| \cdot \ell(I) > 0$.) The metrics they induce are then given by formula (1) of Section 9.1.

One of the two comparisons required for equivalence does hold: since $|f(x) - g(x)| \leq \|f - g\|_{\max}$ for every $x \in [a, b]$, monotonicity of the integral gives

$$\rho_1(f, g) = \int_a^b |f - g| \leq (b - a) \|f - g\|_{\max} = (b - a) \rho_{\max}(f, g).$$

We show that no inequality in the opposite direction is possible; that is, there is no $c > 0$ with $\rho_{\max}(f, g) \leq c\rho_1(f, g)$ for all $f, g \in X$. This suffices: by the preceding problems, if $\rho_{\max}$ and ρ_1 were equivalent then, in either order of the two metrics in the definition, there would be a $c > 0$ with $\frac{1}{c}\rho_1 \leq \rho_{\max} \leq c\rho_1$ throughout $X \times X$, and in particular $\rho_{\max} \leq c\rho_1$.

Choose a natural number N with $1/N < b - a$, and for each natural number $n \geq N$ define $f_n : [a, b] \rightarrow \mathbb{R}$ by

$$f_n(x) = \max\{0, 1 - n(x - a)\}.$$

Each f_n is continuous, being the maximum of two continuous functions, so $f_n \in X$. Explicitly, $f_n(x) = 1 - n(x - a)$ for $a \leq x \leq a + 1/n$ and $f_n(x) = 0$ for $a + 1/n \leq x \leq b$; note $a + 1/n \leq a + 1/N < b$, so this describes f_n on all of $[a, b]$.

Let $g \equiv 0$, the zero function, which belongs to X . Since $f_n(a) = 1$ and $0 \leq f_n \leq 1$ everywhere,

$$\rho_{\max}(f_n, g) = \max\{|f_n(x)| : x \in [a, b]\} = 1.$$

On the other hand, f_n vanishes on $[a + 1/n, b]$ and the region under its graph over $[a, a + 1/n]$ is a triangle of base $1/n$ and height 1, so

$$\rho_1(f_n, g) = \int_a^b |f_n| = \int_a^{a+1/n} (1 - n(x - a)) dx = \left[(x - a) - \frac{n}{2}(x - a)^2 \right]_a^{a+1/n} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

Now suppose, for the sake of contradiction, that the two metrics were equivalent. Then there would be a $c > 0$ with $\rho_{\max}(f, g) \leq c \rho_1(f, g)$ for all $f, g \in X$. Applying this to $f = f_n$ and $g \equiv 0$ gives

$$1 \leq c \cdot \frac{1}{2n} \quad \text{for every } n \geq N,$$

that is, $n \leq c/2$ for every $n \geq N$. This is false for n large, by the Archimedean property of $\mathbb{R}$. This contradiction shows that no such c exists, and hence the metric induced by the maximum norm and the metric induced by the $L^1[a, b]$ norm are not equivalent.

Informally: convergence in the L^1 metric does not control the size of a function at a single point, and the tall thin spikes f_n have maximum distance 1 from 0 while their L^1 distance from 0 tends to 0.

Problem 9.5. The Nikodym Metric. Let E be a Lebesgue measurable set of real numbers of finite measure, X the set of Lebesgue measurable subsets of E , and m Lebesgue measure. For $A, B \in X$, define $\rho(A, B) = m(A \Delta B)$, where $A \Delta B = [A \sim B] \cup [B \sim A]$, the symmetric difference of A and B . Show that this is a pseudometric on X . Define two measurable sets to be equivalent provided their symmetric difference has measure zero. Show that ρ induces a metric on the collection of equivalence classes. Finally, show that for $A, B \in X$,

$$\rho(A, B) = \int_E |\chi_A - \chi_B|,$$

where χ_A and χ_B are the characteristic functions of A and B , respectively.

Solution. Throughout, A, B, C denote members of X , that is, Lebesgue measurable subsets of E ; recall $m(E) < \infty$.

ρ is well defined and real-valued. Since the Lebesgue measurable sets form a σ -algebra, $A \sim B = A \cap B^c$ and $B \sim A$ are measurable, hence so is their union $A \Delta B$. Moreover $A \Delta B \subseteq E$, so by the monotonicity of measure $\rho(A, B) = m(A \Delta B) \leq m(E) < \infty$. Thus $\rho: X \times X \rightarrow \mathbb{R}$.

Property (i). $\rho(A, B) = m(A \Delta B) \geq 0$, since measure is nonnegative.

Property (ii) in the pseudometric form. If $A = B$ then $A \sim B = B \sim A = \emptyset$, so $A \Delta B = \emptyset$ and $\rho(A, A) = m(\emptyset) = 0$. The converse implication fails, which is why ρ is only a pseudometric: for $E = [0, 1]$, $A = [0, 1]$ and $B = (0, 1]$ we have $A \Delta B = \{0\}$, so $\rho(A, B) = 0$ although $A \neq B$.

Property (iii). The definition $A \Delta B = [A \sim B] \cup [B \sim A]$ is unchanged when A and B are interchanged, since union is commutative. Hence $A \Delta B = B \Delta A$ and $\rho(A, B) = \rho(B, A)$.

Property (iv), the triangle inequality. We claim that for all $A, B, C \in X$,

$$A \Delta B \subseteq [A \Delta C] \cup [C \Delta B].$$

Indeed, let $x \in A \Delta B$. Two cases. If $x \in A \sim B$, then $x \in A$ and $x \notin B$. Should $x \in C$, then $x \in C \sim B \subseteq C \Delta B$; should $x \notin C$, then $x \in A \sim C \subseteq A \Delta C$. Either way x lies in the right-hand side. If instead $x \in B \sim A$, the same argument with the roles of A and B interchanged, using $A \Delta C = C \Delta A$ and $C \Delta B = B \Delta C$, again places x in the right-hand side. This proves the claim.

Now by the monotonicity and the finite subadditivity of Lebesgue measure,

$$\rho(A, B) = m(A \Delta B) \leq m([A \Delta C] \cup [C \Delta B]) \leq m(A \Delta C) + m(C \Delta B) = \rho(A, C) + \rho(C, B).$$

Hence ρ satisfies (i), (iii), (iv) and $\rho(A, A) = 0$ for all A : it is a pseudometric on X .

The induced metric on equivalence classes. Define $A \cong B$ provided $m(A \Delta B) = 0$, i.e. provided $\rho(A, B) = 0$. This is exactly the relation described in Section 9.1 for a general pseudometric, and it is an equivalence relation. Reflexivity: $\rho(A, A) = 0$. Symmetry: ρ is symmetric. Transitivity: if $\rho(A, B) = 0$ and $\rho(B, C) = 0$, then by the triangle inequality and nonnegativity, $0 \leq \rho(A, C) \leq \rho(A, B) + \rho(B, C) = 0$, so $\rho(A, C) = 0$.

Let $X/\cong$ denote the collection of equivalence classes and, for classes $[A]$ and $[B]$, define

$$\tilde{\rho}([A], [B]) = \rho(A, B).$$

This is well defined, that is, independent of the representatives chosen. Suppose $A \cong A'$ and $B \cong B'$. Two applications of the triangle inequality give

$$\rho(A, B) \leq \rho(A, A') + \rho(A', B') + \rho(B', B) = 0 + \rho(A', B') + 0 = \rho(A', B'),$$

where we used $\rho(B', B) = \rho(B, B') = 0$; interchanging the roles of the primed and unprimed sets gives the reverse inequality, so $\rho(A, B) = \rho(A', B')$.

Finally, $\tilde{\rho}$ is a metric on $X/\cong$. Nonnegativity and symmetry are inherited from ρ , as is the triangle inequality:

$$\tilde{\rho}([A], [B]) = \rho(A, B) \leq \rho(A, C) + \rho(C, B) = \tilde{\rho}([A], [C]) + \tilde{\rho}([C], [B]).$$

And $\tilde{\rho}([A], [B]) = 0$ means $\rho(A, B) = 0$, which is precisely $A \cong B$, that is, $[A] = [B]$; conversely $[A] = [B]$ gives $\tilde{\rho}([A], [B]) = \rho(A, A) = 0$. So $\tilde{\rho}$ vanishes exactly on the diagonal, and $(X/\cong, \tilde{\rho})$ is a metric space.

The integral formula. Fix $A, B \in X$ and let $x \in E$. The number $\chi_A(x) - \chi_B(x)$ takes only the values $-1, 0, 1$, and it is nonzero precisely when exactly one of the statements $x \in A$, $x \in B$ holds. Explicitly: if $x \in A \cap B$ or $x \in E \sim [A \cup B]$, then $\chi_A(x) = \chi_B(x)$ and the difference is 0; if $x \in A \sim B$ the difference is 1; if $x \in B \sim A$ it is -1 . Since $A \Delta B$ is exactly the set of points lying in exactly one of A and B , we conclude

$$|\chi_A(x) - \chi_B(x)| = \chi_{A \Delta B}(x) \quad \text{for all } x \in E.$$

The function $\chi_{A \Delta B}$ is a nonnegative measurable simple function on E , bounded by 1, and E has finite measure, so it is integrable over E and its integral is the measure of the set on which it takes the value 1, namely

$$\int_E |\chi_A - \chi_B| = \int_E \chi_{A \Delta B} = m(A \Delta B) = \rho(A, B),$$

using $A \Delta B \subseteq E$. This is the asserted identity.

Problem 9.6. Show that for $a, b, c \geq 0$,

$$\text{if } a \leq b + c, \text{ then } \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

Solution. Define $\varphi: [0, \infty) \rightarrow \mathbb{R}$ by

$$\varphi(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}.$$

This is well defined since $1+t \geq 1 > 0$ for $t \geq 0$.

The function φ is increasing on $[0, \infty)$. Indeed, if $0 \leq s \leq t$, then

$$\varphi(t) - \varphi(s) = \frac{1}{1+s} - \frac{1}{1+t} = \frac{(1+t) - (1+s)}{(1+s)(1+t)} = \frac{t-s}{(1+s)(1+t)} \geq 0,$$

because the numerator is nonnegative and the denominator is positive.

Now let $a, b, c \geq 0$ with $a \leq b + c$. Since $b + c \geq 0$ as well, monotonicity of φ gives

$$\frac{a}{1+a} = \varphi(a) \leq \varphi(b+c) = \frac{b+c}{1+b+c} = \frac{b}{1+b+c} + \frac{c}{1+b+c}.$$

It remains to compare each of the last two terms with the corresponding term in the conclusion. Since $c \geq 0$, we have $1+b+c \geq 1+b > 0$, so

$$\frac{b}{1+b+c} \leq \frac{b}{1+b},$$

using $b \geq 0$ (a nonnegative numerator over a larger positive denominator gives a smaller quotient). Symmetrically, since $b \geq 0$ we have $1+b+c \geq 1+c > 0$ and $c \geq 0$, so

$$\frac{c}{1+b+c} \leq \frac{c}{1+c}.$$

Adding the last two inequalities and combining with the preceding display,

$$\frac{a}{1+a} \leq \frac{b}{1+b+c} + \frac{c}{1+b+c} \leq \frac{b}{1+b} + \frac{c}{1+c},$$

which is the asserted inequality.

Problem 9.7. Let E be a Lebesgue measurable set of real numbers that has finite measure and X the set of Lebesgue measurable real-valued functions on E . For $f, g \in X$, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Use the preceding problem to show that this is a pseudometric on X . Define two measurable functions to be equivalent provided they are equal a.e. on E . Show that ρ induces a metric on the collection of equivalence classes.

Solution. Throughout let $\varphi(t) = t/(1+t)$ for $t \geq 0$, the function studied in the preceding problem; it is increasing, $\varphi(0) = 0$, and $0 \leq \varphi(t) < 1$ for all $t \geq 0$, since $t < 1+t$.

ρ is well defined and real-valued. Let $f, g \in X$. Then $f - g$ is measurable, hence so is $|f - g|$. Extend φ to a continuous real-valued function on all of $\mathbb{R}$ by setting $\varphi(t) = t/(1+|t|)$; by Proposition 7 of Chapter 3, the composition

$$h_{f,g} = \varphi \circ |f - g| = \frac{|f - g|}{1 + |f - g|}$$

is measurable on E . It satisfies $0 \leq h_{f,g} < 1$ on E , so it is a bounded measurable function on the set E of finite measure, and hence, by Theorem 4 of Chapter 4, it is integrable over E . Thus $\rho(f, g) = \int_E h_{f,g}$ is a well-defined real number, and by the monotonicity of integration

$$0 \leq \rho(f, g) \leq m(E) < \infty.$$

Properties (i) and (iii). Nonnegativity was just noted. Symmetry is immediate, since $|f - g| = |g - f|$ pointwise on E , so $h_{f,g} = h_{g,f}$ and therefore $\rho(f, g) = \rho(g, f)$.

ρ vanishes on the diagonal. If $f = g$ on E , then $h_{f,g} \equiv \varphi(0) = 0$ and $\rho(f, f) = 0$. Note that ρ is only a pseudometric: if f and g differ on a set of measure zero but are not identical, then, as shown below, $\rho(f, g) = 0$ while $f \neq g$.

Property (iv), the triangle inequality. Let $f, g, k \in X$ and let $x \in E$. Put

$$a = |f(x) - k(x)|, \quad b = |f(x) - g(x)|, \quad c = |g(x) - k(x)|.$$

These are nonnegative real numbers, and the triangle inequality for absolute values gives $a \leq b + c$. Hence by the preceding problem,

$$\frac{|f(x) - k(x)|}{1 + |f(x) - k(x)|} \leq \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} + \frac{|g(x) - k(x)|}{1 + |g(x) - k(x)|},$$

that is, $h_{f,k} \leq h_{f,g} + h_{g,k}$ pointwise on E . All three functions are integrable over E , so by the monotonicity and linearity of integration,

$$\rho(f, k) = \int_E h_{f,k} \leq \int_E (h_{f,g} + h_{g,k}) = \int_E h_{f,g} + \int_E h_{g,k} = \rho(f, g) + \rho(g, k).$$

Therefore ρ is a pseudometric on X .

Identification of the null pairs. We claim that for $f, g \in X$,

$$\rho(f, g) = 0 \quad \text{if and only if} \quad f = g \text{ a.e. on } E.$$

Indeed, $h_{f,g}$ is a nonnegative measurable function on E , so by Proposition 9 of Chapter 4, $\int_E h_{f,g} = 0$ if and only if $h_{f,g} = 0$ a.e. on E . (Proposition 9 concerns the integral of a nonnegative measurable

function as defined in Section 4.3; for our $h_{f,g}$, which is bounded, measurable, and vanishes outside the set E of finite measure, that supremum agrees with the integral of Section 4.2 used above. Indeed $h_{f,g}$ is itself one of the competitors h in the supremum (8), so the supremum is at least $\int_E h_{f,g}$, while every competitor $h \leq h_{f,g}$ has $\int_E h \leq \int_E h_{f,g}$ by the monotonicity of Theorem 5 of Chapter 4.) Now for a point $x \in E$, $h_{f,g}(x) = \varphi(|f(x) - g(x)|)$ vanishes if and only if $|f(x) - g(x)| = 0$, since $\varphi(t) = 0$ with $t \geq 0$ forces $t = 0$ (as $1 + t > 0$). So the set on which $h_{f,g}$ vanishes is exactly the set on which $f = g$, and $h_{f,g} = 0$ a.e. on E if and only if $f = g$ a.e. on E . This proves the claim, and in particular shows ρ genuinely is not a metric on X (take $E = [0, 1]$, $f \equiv 0$ and $g = \chi_{\{0\}}$).

The induced metric on equivalence classes. Define $f \cong g$ provided $f = g$ a.e. on E . This is an equivalence relation on X : reflexivity and symmetry are clear, and if $f = g$ a.e. and $g = k$ a.e., then the set where $f \neq k$ is contained in the union of the two sets of measure zero where $f \neq g$ and where $g \neq k$, hence has measure zero by the countable subadditivity and monotonicity of Lebesgue measure. By the claim just proved, $f \cong g$ if and only if $\rho(f, g) = 0$, so this is precisely the equivalence relation associated with the pseudometric ρ as described in Section 9.1.

For equivalence classes $[f], [g] \in X/\cong$, define

$$\tilde{\rho}([f], [g]) = \rho(f, g).$$

This does not depend on the representatives. Suppose $f \cong f'$ and $g \cong g'$; then $\rho(f, f') = \rho(g, g') = 0$, and two applications of the triangle inequality give

$$\rho(f, g) \leq \rho(f, f') + \rho(f', g') + \rho(g', g) = \rho(f', g'),$$

using symmetry for the last term; interchanging the primed and unprimed functions gives the reverse inequality, so $\rho(f, g) = \rho(f', g')$.

Finally, $\tilde{\rho}$ is a metric on $X/\cong$. It is nonnegative and symmetric, and satisfies the triangle inequality, all inherited directly from the corresponding properties of ρ :

$$\tilde{\rho}([f], [k]) = \rho(f, k) \leq \rho(f, g) + \rho(g, k) = \tilde{\rho}([f], [g]) + \tilde{\rho}([g], [k]).$$

Moreover $\tilde{\rho}([f], [g]) = 0$ if and only if $\rho(f, g) = 0$, which by the claim holds if and only if $f = g$ a.e. on E , that is, if and only if $[f] = [g]$. Hence $\tilde{\rho}$ separates distinct equivalence classes, and $(X/\cong, \tilde{\rho})$ is a metric space.

9.2 Problems 9.8–9.14

Problem 9.8. For $0 < p < 1$, show that

$$(a + b)^p \leq a^p + b^p \text{ for all } a, b \geq 0.$$

Solution. We first record an elementary inequality: if $0 < p < 1$ and $0 \leq t \leq 1$, then

$$t \leq t^p.$$

Indeed, this is clear for $t = 0$, while for $0 < t \leq 1$ we have $t^p/t = t^{p-1} = t^{-(1-p)} \geq 1$, since $0 < t \leq 1$ and $1 - p > 0$ force $t^{-(1-p)} \geq 1$; multiplying through by $t > 0$ gives $t^p \geq t$.

Now let $a, b \geq 0$. If $a = b = 0$ the asserted inequality reads $0 \leq 0$ and there is nothing to prove. So assume $a + b > 0$ and set

$$t = \frac{a}{a + b}, \quad 1 - t = \frac{b}{a + b},$$

so that both t and $1 - t$ lie in $[0, 1]$. By the inequality of the preceding paragraph applied to t and to $1 - t$,

$$1 = t + (1 - t) \leq t^p + (1 - t)^p = \frac{a^p}{(a + b)^p} + \frac{b^p}{(a + b)^p} = \frac{a^p + b^p}{(a + b)^p},$$

where we used that $(a/(a + b))^p = a^p/(a + b)^p$ and similarly for b . Multiplying both sides of this

inequality by the positive number $(a+b)^p$ yields

$$(a+b)^p \leq a^p + b^p.$$

This holds for all $a, b \geq 0$, as required.

We remark, for later use, that the function $t \mapsto t^p$ is increasing on $[0, \infty)$: if $0 \leq s \leq t$ then $s^p \leq t^p$. Indeed, if $s = 0$ this reads $0 \leq t^p$, which holds; and if $0 < s \leq t$, then $t/s \geq 1$, so $(t/s)^p \geq 1$ because $u \mapsto u^p = e^{p \log u}$ is increasing in u for $p > 0$ and equals 1 at $u = 1$, and multiplying $(t/s)^p \geq 1$ by $s^p > 0$ gives $t^p \geq s^p$. Combined with the inequality just proved, this gives the frequently used consequence that for $a, b, c \geq 0$,

$$\text{if } a \leq b + c, \text{ then } a^p \leq (b + c)^p \leq b^p + c^p.$$

Problem 9.9. For E a Lebesgue measurable set of real numbers, $0 < p < 1$, and g and h Lebesgue measurable functions on E that have integrable p th powers, define

$$\rho_p(h, g) = \int_E |g - h|^p.$$

Use the preceding problem to show that this is a pseudometric on the collection of Lebesgue measurable functions on E that have integrable p th powers. Define two such functions to be equivalent provided they are equal a.e. on E . Show that $\rho_p(\cdot, \cdot)$ induces a metric on the collection of equivalence classes.

Solution. Let X denote the collection of Lebesgue measurable real-valued functions f on E for which $|f|^p$ is integrable over E .

The formula is well defined and finite. Let g, h belong to X . The function $g - h$ is measurable, being a difference of measurable real-valued functions. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\varphi(t) = |t|^p$; this is continuous on all of $\mathbb{R}$, since $p > 0$. Hence $|g - h|^p = \varphi \circ (g - h)$ is measurable, by Proposition 7 of Chapter 3, which asserts that the composition of a continuous real-valued function on $\mathbb{R}$ with a measurable real-valued function is measurable. Moreover, pointwise on E we have $|g - h| \leq |g| + |h|$, so by the monotonicity of $t \mapsto t^p$ and the preceding problem,

$$|g - h|^p \leq (|g| + |h|)^p \leq |g|^p + |h|^p.$$

Since $|g|^p$ and $|h|^p$ are integrable over E , so is their sum, and therefore by the monotonicity of integration for nonnegative measurable functions, $|g - h|^p$ is integrable over E and

$$0 \leq \rho_p(h, g) = \int_E |g - h|^p \leq \int_E |g|^p + \int_E |h|^p < \infty.$$

Thus ρ_p is a finite, nonnegative real-valued function on $X \times X$.

Symmetry. Since $|g - h| = |h - g|$ pointwise on E , we have $\rho_p(h, g) = \rho_p(g, h)$.

Vanishing on the diagonal. For $h \in X$, $|h - h|^p \equiv 0$ on E , so $\rho_p(h, h) = 0$.

The triangle inequality. Let f, g, h belong to X . At each point of E , the triangle inequality for the absolute value gives

$$|g - h| \leq |g - f| + |f - h|.$$

Since $t \mapsto t^p$ is increasing on $[0, \infty)$, raising both sides to the power p and then invoking the preceding problem with $a = |g - f|$ and $b = |f - h|$ gives, pointwise on E ,

$$|g - h|^p \leq (|g - f| + |f - h|)^p \leq |g - f|^p + |f - h|^p.$$

All three functions here are nonnegative and integrable over E , so by the monotonicity and linearity of integration,

$$\rho_p(h, g) = \int_E |g - h|^p \leq \int_E |g - f|^p + \int_E |f - h|^p = \rho_p(f, g) + \rho_p(h, f).$$

Thus ρ_p satisfies all the requirements of a metric except possibly the requirement that $\rho_p(h, g) = 0$ only if $h = g$; that is, ρ_p is a pseudometric on X .

Identification of the null pairs. We claim that for $g, h \in X$,

$$\rho_p(h, g) = 0 \text{ if and only if } g = h \text{ a.e. on } E.$$

The function $|g-h|^p$ is nonnegative and integrable over E . By Proposition 9 of Chapter 4, a nonnegative integrable function has integral zero over E if and only if it vanishes a.e. on E . Now $|g(x) - h(x)|^p = 0$ if and only if $|g(x) - h(x)| = 0$, that is, $g(x) = h(x)$, since $t^p = 0$ only for $t = 0$. Hence $\rho_p(h, g) = 0$ if and only if $g = h$ a.e. on E , which proves the claim. In particular ρ_p is genuinely only a pseudometric and not a metric: any two distinct functions in X that agree a.e. (for instance the zero function and the characteristic function of a single point, when E contains a point) are at ρ_p -distance 0.

The induced metric. Define $h \cong g$ provided $h = g$ a.e. on E . This is an equivalence relation on X : it is reflexive and symmetric trivially, and if $h = g$ off a set of measure zero Z_1 and $g = k$ off a set of measure zero Z_2 , then $h = k$ off $Z_1 \cup Z_2$, which has measure zero since the union of two sets of measure zero has measure zero. By the claim above, $h \cong g$ if and only if $\rho_p(h, g) = 0$, so this is exactly the equivalence relation that the pseudometric ρ_p determines, as described in the discussion of pseudometrics in Section 9.1.

Let $X/\cong$ denote the collection of equivalence classes and, for classes $[h]$ and $[g]$, define

$$\tilde{\rho}_p([h], [g]) = \rho_p(h, g).$$

This is well defined: suppose $h' \cong h$ and $g' \cong g$, say $h' = h$ off a set Z_1 of measure zero and $g' = g$ off a set Z_2 of measure zero, and put $Z = Z_1 \cup Z_2$, a set of measure zero. Then $|g' - h'|^p = |g - h|^p$ at every point of $E \sim Z$. Both $|g' - h'|^p$ and $|g - h|^p$ are integrable over E , by the first paragraph applied to the pairs g', h' and g, h , so Proposition 15 of Chapter 4 applies to each and gives

$$\rho_p(h', g') = \int_E |g' - h'|^p = \int_{E \sim Z} |g' - h'|^p = \int_{E \sim Z} |g - h|^p = \int_E |g - h|^p = \rho_p(h, g).$$

Finally, $\tilde{\rho}_p$ is a metric on $X/\cong$. It is nonnegative, real-valued and symmetric, and satisfies the triangle inequality, because ρ_p has all these properties and $\tilde{\rho}_p$ is computed by choosing representatives. And if $\tilde{\rho}_p([h], [g]) = 0$, then $\rho_p(h, g) = 0$, so $h = g$ a.e. on E , that is, $h \cong g$ and therefore $[h] = [g]$. Conversely $[h] = [g]$ gives $\tilde{\rho}_p([h], [g]) = \rho_p(h, h) = 0$. Hence ρ_p induces a metric on the collection of equivalence classes.

Problem 9.10. Let $\{(X_n, \rho_n)\}_{n=1}^\infty$ be a countable collection of metric spaces. Use Problem 6 to show that ρ_* defines a metric on the Cartesian product $\prod_{n=1}^\infty X_n$, where for points $x = \{x_n\}$ and $y = \{y_n\}$ in $\prod_{n=1}^\infty X_n$,

$$\rho_*(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} \cdot \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

Solution. Throughout, write $X = \prod_{n=1}^\infty X_n$ and, for a point $x = \{x_n\}$ and $y = \{y_n\}$ of X ,

$$d_n(x, y) = \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

Recall from Problem 6 that for $a, b, c \geq 0$,

$$\text{if } a \leq b + c, \text{ then } \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

(For completeness: the function $\varphi(t) = t/(1+t) = 1 - 1/(1+t)$ is increasing on $[0, \infty)$, so $a \leq b + c$ gives $\varphi(a) \leq \varphi(b+c) = \frac{b}{1+b+c} + \frac{c}{1+b+c} \leq \frac{b}{1+b} + \frac{c}{1+c}$.)

The series converges, and ρ_* is a finite nonnegative function. Since $\rho_n(x_n, y_n) \geq 0$, we have $0 \leq d_n(x, y) < 1$ for every n , so the n th term of the defining series lies in $[0, 2^{-n}]$. The series is therefore

a series of nonnegative terms dominated termwise by the convergent geometric series $\sum 2^{-n}$; by the comparison test it converges, and

$$0 \leq \rho_*(x, y) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus ρ_* is a well-defined nonnegative real-valued function on $X \times X$.

$\rho_*(x, y) = 0$ if and only if $x = y$. All terms of the series are nonnegative, so the sum vanishes if and only if every term vanishes, that is, if and only if $d_n(x, y) = 0$ for all n . Since $\varphi(t) = t/(1+t)$ vanishes only at $t = 0$, this happens if and only if $\rho_n(x_n, y_n) = 0$ for every n , and since each ρ_n is a metric on X_n this happens if and only if $x_n = y_n$ for every n , that is, if and only if $x = y$ as points of the Cartesian product.

Symmetry. Each ρ_n is symmetric, so $d_n(x, y) = d_n(y, x)$ for every n , and hence $\rho_*(x, y) = \rho_*(y, x)$.

The triangle inequality. Let $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$ belong to X . Fix n and set

$$a = \rho_n(x_n, z_n), \quad b = \rho_n(x_n, y_n), \quad c = \rho_n(y_n, z_n),$$

all nonnegative. The triangle inequality for the metric ρ_n gives $a \leq b + c$, so Problem 6 yields

$$d_n(x, z) = \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c} = d_n(x, y) + d_n(y, z).$$

Multiply by $2^{-n} > 0$ and sum over n . Each of the three series

$$\sum_{n=1}^{\infty} 2^{-n} d_n(x, z), \quad \sum_{n=1}^{\infty} 2^{-n} d_n(x, y), \quad \sum_{n=1}^{\infty} 2^{-n} d_n(y, z)$$

converges, as shown above, so the termwise inequality passes to the limits of the partial sums and we obtain

$$\rho_*(x, z) \leq \rho_*(x, y) + \rho_*(y, z).$$

All the requirements of a metric are met, so ρ_* is a metric on $\prod_{n=1}^{\infty} X_n$. This is the countable analogue of the product metric τ defined in Section 9.1 for a product of two metric spaces.

Problem 9.11. Let (X, ρ) be a metric space and A any set for which there is a one-to-one mapping f of A onto the set X . Show that there is a unique metric on A for which f is an isometry of metric spaces. (This is the sense in which an isometry amounts merely to a relabeling of the points in a space.)

Solution. Recall the definition from Section 9.1: a mapping f from a metric space (A, σ) to a metric space (X, ρ) is an isometry provided it maps A onto X and, for all $a_1, a_2 \in A$,

$$\rho(f(a_1), f(a_2)) = \sigma(a_1, a_2).$$

Uniqueness. Suppose σ is a metric on A for which $f: (A, \sigma) \rightarrow (X, \rho)$ is an isometry. Then by the very definition of isometry, for all a_1, a_2 in A ,

$$\sigma(a_1, a_2) = \rho(f(a_1), f(a_2)).$$

The right-hand side is determined by f and ρ alone. Hence there is at most one such metric, namely the one given by this formula; if σ and σ' both make f an isometry, then $\sigma(a_1, a_2) = \rho(f(a_1), f(a_2)) = \sigma'(a_1, a_2)$ for all $a_1, a_2 \in A$, so $\sigma = \sigma'$.

Existence. Define $\sigma: A \times A \rightarrow \mathbb{R}$ by

$$\sigma(a_1, a_2) = \rho(f(a_1), f(a_2)) \text{ for all } a_1, a_2 \in A.$$

We verify that σ is a metric on A .

- (i) Nonnegativity and finiteness: $\sigma(a_1, a_2) = \rho(f(a_1), f(a_2)) \geq 0$ and is a real number, since ρ is a nonnegative real-valued function on $X \times X$.
- (ii) $\sigma(a_1, a_2) = 0$ if and only if $a_1 = a_2$: if $a_1 = a_2$ then $f(a_1) = f(a_2)$ and so $\sigma(a_1, a_2) = \rho(f(a_1), f(a_1)) = 0$. Conversely, if $\sigma(a_1, a_2) = 0$, then $\rho(f(a_1), f(a_2)) = 0$, and since ρ is a metric on X this forces $f(a_1) = f(a_2)$; because f is one-to-one, $a_1 = a_2$. (This is the only place where the injectivity of f is used, and it is essential: without it σ would only be a pseudometric.)
- (iii) Symmetry: $\sigma(a_1, a_2) = \rho(f(a_1), f(a_2)) = \rho(f(a_2), f(a_1)) = \sigma(a_2, a_1)$, by the symmetry of ρ .
- (iv) The triangle inequality: for a_1, a_2, a_3 in A , the triangle inequality for ρ applied to the three points $f(a_1), f(a_2), f(a_3)$ of X gives

$$\sigma(a_1, a_3) = \rho(f(a_1), f(a_3)) \leq \rho(f(a_1), f(a_2)) + \rho(f(a_2), f(a_3)) = \sigma(a_1, a_2) + \sigma(a_2, a_3).$$

Thus (A, σ) is a metric space. Moreover f maps A onto X by hypothesis and satisfies $\rho(f(a_1), f(a_2)) = \sigma(a_1, a_2)$ by the definition of σ ; hence f is an isometry of (A, σ) onto (X, ρ) . Together with the uniqueness established above, this shows that σ is the one and only metric on A that makes f an isometry.

Problem 9.12. Show that the triangle inequality for Euclidean space $\mathbb{R}^n$ follows from the triangle inequality for $L^2[0, 1]$.

Solution. Fix the natural number n and partition $[0, 1]$ into the n disjoint measurable subintervals

$$I_k = \left[\frac{k-1}{n}, \frac{k}{n} \right), \quad k = 1, \dots, n-1, \quad I_n = \left[\frac{n-1}{n}, 1 \right],$$

each of measure $1/n$. Define a mapping Φ from $\mathbb{R}^n$ into the space of real-valued functions on $[0, 1]$ by

$$\Phi(x) = \sqrt{n} \sum_{k=1}^n x_k \chi_{I_k} \text{ for } x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

where χ_{I_k} is the characteristic function of I_k . Each $\Phi(x)$ is a simple, hence measurable, function on $[0, 1]$; it is bounded and $[0, 1]$ has finite measure, so $|\Phi(x)|^2$ is integrable over $[0, 1]$ and $\Phi(x)$ belongs to $L^2[0, 1]$.

Φ is linear. Since the I_k are disjoint, for $x, y \in \mathbb{R}^n$ and real α we have, pointwise on $[0, 1]$,

$$\Phi(x + y) = \sqrt{n} \sum_{k=1}^n (x_k + y_k) \chi_{I_k} = \Phi(x) + \Phi(y), \quad \Phi(\alpha x) = \alpha \Phi(x).$$

Φ preserves norms. Because the I_k are disjoint, $\Phi(x)^2 = n \sum_{k=1}^n x_k^2 \chi_{I_k}$ pointwise on $[0, 1]$. Hence, by the linearity of integration and $m(I_k) = 1/n$,

$$\|\Phi(x)\|_2^2 = \int_0^1 |\Phi(x)|^2 = n \sum_{k=1}^n x_k^2 m(I_k) = n \sum_{k=1}^n x_k^2 \cdot \frac{1}{n} = \sum_{k=1}^n x_k^2.$$

Taking square roots,

$$\|\Phi(x)\|_2 = [x_1^2 + \dots + x_n^2]^{1/2} = \|x\|,$$

the Euclidean norm of x .

The triangle inequality for the Euclidean norm. Let $x, y \in \mathbb{R}^n$. Using the two displayed properties of Φ and the triangle inequality for the $L^2[0, 1]$ norm (the Minkowski Inequality for $p = 2$),

$$\|x + y\| = \|\Phi(x + y)\|_2 = \|\Phi(x) + \Phi(y)\|_2 \leq \|\Phi(x)\|_2 + \|\Phi(y)\|_2 = \|x\| + \|y\|.$$

The triangle inequality for the Euclidean metric. As observed in Section 9.1, the triangle inequality for a norm is equivalent to the triangle inequality for the metric it induces. Explicitly, let $x, y, z \in \mathbb{R}^n$ and apply the inequality just proved to the vectors $x - y$ and $y - z$:

$$\rho(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = \rho(x, y) + \rho(y, z).$$

This is the triangle inequality for Euclidean space $\mathbb{R}^n$, deduced from the triangle inequality for $L^2[0, 1]$.

Problem 9.13. In a metric space X , is it possible for $r > 0$ and two distinct points u and v in X to have $B(u, r) = B(v, r)$? Is this possible in Euclidean space $\mathbb{R}^n$? Is it possible in a normed linear space?

Solution. The answers are: yes for a general metric space; no in $\mathbb{R}^n$; and no in any normed linear space. A general metric space: it is possible. Let X be any set containing at least two points, equipped with the discrete metric ρ , so that $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$. Take $r = 2$. For every point $x \in X$ and every $x' \in X$ we have $\rho(x', x) \leq 1 < 2$, so

$$B(x, 2) = \{x' \in X \mid \rho(x', x) < 2\} = X.$$

Hence if u and v are any two distinct points of X , then $B(u, 2) = X = B(v, 2)$. So in a general metric space two distinct points can certainly determine the same open ball of a given radius. (Concretely: $X = \{u, v\}$ with the discrete metric.)

Euclidean space $\mathbb{R}^n$: it is not possible. This is a special case of the normed linear space argument below, since the Euclidean metric on $\mathbb{R}^n$ is the metric induced by the Euclidean norm. We give the argument in that generality.

A normed linear space: it is not possible. Let X be a normed linear space with norm $\|\cdot\|$ and induced metric $\rho(x, y) = \|x - y\|$, let $r > 0$, and let u, v be distinct points of X . Set

$$d = \|u - v\| > 0 \quad \text{and} \quad e = \frac{1}{d}(u - v),$$

so that, by the homogeneity of the norm, $\|e\| = \frac{1}{d}\|u - v\| = 1$.

Choose a real number s with

$$\max\{0, r - d\} < s < r.$$

Such an s exists because $r - d < r$ and $0 < r$, so the interval $(\max\{0, r - d\}, r)$ is nonempty. Define the point

$$w = u + se \in X.$$

First, w belongs to $B(u, r)$: indeed $\rho(w, u) = \|w - u\| = \|se\| = s\|e\| = s < r$.

Second, w does not belong to $B(v, r)$: since $u - v = de$, we have

$$w - v = (u - v) + se = de + se = (d + s)e,$$

so, again by homogeneity of the norm,

$$\rho(w, v) = \|w - v\| = (d + s)\|e\| = d + s.$$

By the choice of s we have $s > r - d$, hence $d + s > r$, and therefore $\rho(w, v) > r$, so $w \notin B(v, r)$.

Thus $w \in B(u, r) \setminus B(v, r)$, and in particular $B(u, r) \neq B(v, r)$. Since u and v were arbitrary distinct points and $r > 0$ arbitrary, in a normed linear space distinct points always determine distinct open balls of any given radius. Applying this to $\mathbb{R}^n$ with the Euclidean norm answers the second question in the negative as well.

What makes the discrete example possible, and what the linear structure rules out, is the existence of points at distance exactly prescribed along the segment from v through u : in a normed linear space one can always move from u directly away from v by any amount less than r , and thereby escape $B(v, r)$ while remaining in $B(u, r)$.

Problem 9.14. Let (X, ρ) be a metric space in which $\{u_n\} \rightarrow u$ and $\{v_n\} \rightarrow v$. Show that $\{\rho(u_n, v_n)\} \rightarrow \rho(u, v)$.

Solution. Recall that $\{u_n\} \rightarrow u$ in (X, ρ) means that $\{\rho(u_n, u)\} \rightarrow 0$ as a sequence of real numbers, that is, for each $\epsilon > 0$ there is an index N such that $\rho(u_n, u) < \epsilon$ for all $n \geq N$.

A quadrilateral inequality. We claim that for all points a, b, c, d in X ,

$$|\rho(a, b) - \rho(c, d)| \leq \rho(a, c) + \rho(b, d).$$

To see this, apply the triangle inequality twice:

$$\rho(a, b) \leq \rho(a, c) + \rho(c, b) \leq \rho(a, c) + \rho(c, d) + \rho(d, b),$$

so, using the symmetry of ρ ,

$$\rho(a, b) - \rho(c, d) \leq \rho(a, c) + \rho(b, d).$$

Interchanging the roles of the pair (a, b) with the pair (c, d) in this computation gives

$$\rho(c, d) - \rho(a, b) \leq \rho(c, a) + \rho(d, b) = \rho(a, c) + \rho(b, d).$$

Since the absolute value of a real number is the larger of it and its negative, the two displayed inequalities together give the claim.

Conclusion. Apply the claim with $a = u_n$, $b = v_n$, $c = u$, $d = v$:

$$|\rho(u_n, v_n) - \rho(u, v)| \leq \rho(u_n, u) + \rho(v_n, v) \text{ for every index } n.$$

Let $\epsilon > 0$. Since $\{u_n\} \rightarrow u$, there is an index N_1 such that $\rho(u_n, u) < \epsilon/2$ for all $n \geq N_1$; since $\{v_n\} \rightarrow v$, there is an index N_2 such that $\rho(v_n, v) < \epsilon/2$ for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,

$$|\rho(u_n, v_n) - \rho(u, v)| \leq \rho(u_n, u) + \rho(v_n, v) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the sequence of real numbers $\{\rho(u_n, v_n)\}$ converges to $\rho(u, v)$.

Equivalently, this says that the metric $\rho: X \times X \rightarrow \mathbb{R}$ is a continuous mapping when $X \times X$ is given the product metric of Section 9.1, since convergence in that product metric is exactly coordinatewise convergence.

9.3 Problems 9.15–9.21

Problem 9.15. Let X be a metric space, x belong to X and $r > 0$.

- (i) Show that $\overline{B}(x, r)$ is closed and contains $B(x, r)$.
- (ii) Show that in a normed linear space X the closed ball $\overline{B}(x, r)$ is the closure of the open ball $B(x, r)$, but this is not so in a general metric space.

Solution. Throughout, ρ denotes the metric on X , and

$$B(x, r) = \{x' \in X \mid \rho(x', x) < r\}, \quad \overline{B}(x, r) = \{x' \in X \mid \rho(x', x) \leq r\}.$$

- (i) The inclusion $B(x, r) \subseteq \overline{B}(x, r)$ is immediate, since $\rho(x', x) < r$ implies $\rho(x', x) \leq r$.

To show $\overline{B}(x, r)$ is closed we show, by Proposition 4, that its complement $X \sim \overline{B}(x, r)$ is open. Let y belong to $X \sim \overline{B}(x, r)$, so that $\rho(y, x) > r$. Define

$$\delta = \rho(y, x) - r > 0.$$

We claim $B(y, \delta) \subseteq X \sim \overline{B}(x, r)$. Indeed, let z belong to $B(y, \delta)$. By the triangle inequality for ρ ,

$$\rho(y, x) \leq \rho(y, z) + \rho(z, x) < \delta + \rho(z, x),$$

so that

$$\rho(z, x) > \rho(y, x) - \delta = r.$$

Hence z does not belong to $\overline{B}(x, r)$. Thus every point of $X \sim \overline{B}(x, r)$ has an open ball about it that is contained in $X \sim \overline{B}(x, r)$; that is, $X \sim \overline{B}(x, r)$ is open. By Proposition 4, $\overline{B}(x, r)$ is closed.

- (ii) Now let X be a normed linear space, with norm $\|\cdot\|$ and induced metric $\rho(u, v) = \|u - v\|$. Denote by $\overline{B}(x, r)$ the closure of the open ball.

First, $\overline{B}(x, r)$ is a closed set containing $B(x, r)$ by part (i), so Proposition 3 (the closure of a set is the smallest closed set containing it) gives

$$\overline{B(x, r)} \subseteq \overline{B}(x, r).$$

For the reverse inclusion, let y satisfy $\|y - x\| \leq r$. For each natural number n define

$$y_n = x + \left(1 - \frac{1}{n}\right)(y - x),$$

which is a point of X because X is a linear space. By positive homogeneity of the norm,

$$\|y_n - x\| = \left(1 - \frac{1}{n}\right)\|y - x\| \leq \left(1 - \frac{1}{n}\right)r < r,$$

since $r > 0$. Thus y_n belongs to $B(x, r)$ for every n , so $\{y_n\}$ is a sequence in $B(x, r)$. Moreover

$$\|y_n - y\| = \frac{1}{n}\|y - x\| \leq \frac{r}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

so $\{y_n\} \rightarrow y$. By Proposition 6, y is a point of closure of $B(x, r)$, that is, y belongs to $\overline{B}(x, r)$. Therefore $\overline{B}(x, r) \subseteq \overline{B(x, r)}$, and the two sets are equal.

Finally we exhibit a metric space in which the equality fails. Let X be any set containing at least two points, equipped with the discrete metric

$$\rho(u, v) = \begin{cases} 0 & \text{if } u = v, \\ 1 & \text{if } u \neq v. \end{cases}$$

Fix $x \in X$ and take $r = 1$. Then

$$B(x, 1) = \{x' \in X \mid \rho(x', x) < 1\} = \{x\}, \quad \overline{B}(x, 1) = \{x' \in X \mid \rho(x', x) \leq 1\} = X.$$

In the discrete metric every subset of X is open, since for each point u of a subset S we have $B(u, 1/2) = \{u\} \subseteq S$. In particular $X \sim \{x\}$ is open, so by Proposition 4 the set $\{x\} = B(x, 1)$ is closed, and hence equals its own closure:

$$\overline{B(x, 1)} = \{x\}.$$

Since X has at least two points, $\{x\} \neq X = \overline{B}(x, 1)$. So in this metric space the closure of the open ball is a proper subset of the closed ball of the same radius.

Problem 9.16. Prove Proposition 2.

[Proposition 2 Let X be a subspace of the metric space Y and E a subset of X . Then E is open in X if and only if $E = X \cap \mathcal{O}$, where $\mathcal{O}$ is open in Y .]

Solution. Let ρ be the metric on Y ; by definition of metric subspace, the metric on X is the restriction of ρ to $X \times X$. For a point x of X and $r > 0$ write

$$B_Y(x, r) = \{y \in Y \mid \rho(y, x) < r\}, \quad B_X(x, r) = \{x' \in X \mid \rho(x', x) < r\}$$

for the open balls in Y and in X respectively. Directly from these definitions,

$$B_X(x, r) = X \cap B_Y(x, r) \quad \text{for all } x \in X, r > 0. \quad (*)$$

First suppose $E = X \cap \mathcal{O}$ with $\mathcal{O}$ open in Y . Let x belong to E . Then x belongs to $\mathcal{O}$, and since $\mathcal{O}$ is open in Y there is an $r > 0$ for which $B_Y(x, r) \subseteq \mathcal{O}$. Therefore, by (*),

$$B_X(x, r) = X \cap B_Y(x, r) \subseteq X \cap \mathcal{O} = E.$$

Thus each point of E is the center of an open ball of X contained in E , that is, E is open in X .

Conversely, suppose E is open in X . If $E = \emptyset$, take $\mathcal{O} = \emptyset$, which is open in Y by Proposition 1, and then $E = X \cap \mathcal{O}$. Otherwise, for each $x \in E$ choose, by the openness of E in X , a number $r_x > 0$ for which

$$B_X(x, r_x) \subseteq E.$$

Define

$$\mathcal{O} = \bigcup_{x \in E} B_Y(x, r_x).$$

Each $B_Y(x, r_x)$ is an open subset of Y (an open ball is open, as verified in the text), so $\mathcal{O}$ is open in Y by Proposition 1, which asserts that the union of any collection of open sets is open. Using (*) and the distributivity of intersection over union,

$$X \cap \mathcal{O} = \bigcup_{x \in E} (X \cap B_Y(x, r_x)) = \bigcup_{x \in E} B_X(x, r_x).$$

Every set in this last union is contained in E , so $X \cap \mathcal{O} \subseteq E$. On the other hand, each $x \in E$ belongs to $B_X(x, r_x)$, so $E \subseteq X \cap \mathcal{O}$. Therefore $E = X \cap \mathcal{O}$ with $\mathcal{O}$ open in Y .

Problem 9.17. Prove Proposition 7.

[Proposition 7 Let ρ and σ be equivalent metrics on a nonempty set X . Then a subset of X is open in the metric space (X, ρ) if and only if it is open in the metric space (X, σ) .]

Solution. By the definition of equivalent metrics, there are positive numbers c_1 and c_2 such that

$$c_1 \cdot \sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \cdot \sigma(x_1, x_2) \quad \text{for all } x_1, x_2 \in X. \quad (*)$$

For $x \in X$ and $r > 0$ write

$$B_\rho(x, r) = \{x' \in X \mid \rho(x', x) < r\}, \quad B_\sigma(x, r) = \{x' \in X \mid \sigma(x', x) < r\}.$$

We first record two inclusions between these balls.

Claim 1: $B_\sigma(x, r/c_2) \subseteq B_\rho(x, r)$ for every $x \in X$ and $r > 0$. Indeed, if $\sigma(x', x) < r/c_2$, then the right-hand inequality in (*) gives

$$\rho(x', x) \leq c_2 \cdot \sigma(x', x) < c_2 \cdot \frac{r}{c_2} = r,$$

so x' belongs to $B_\rho(x, r)$.

Claim 2: $B_\rho(x, c_1 r) \subseteq B_\sigma(x, r)$ for every $x \in X$ and $r > 0$. Indeed, if $\rho(x', x) < c_1 r$, then the left-hand inequality in (*) gives

$$c_1 \cdot \sigma(x', x) \leq \rho(x', x) < c_1 r,$$

and dividing by $c_1 > 0$ yields $\sigma(x', x) < r$, so x' belongs to $B_\sigma(x, r)$.

Now let $\mathcal{O}$ be a subset of X that is open in (X, ρ) . Let x belong to $\mathcal{O}$. There is an $r > 0$ for which $B_\rho(x, r) \subseteq \mathcal{O}$. Set $r' = r/c_2 > 0$. By Claim 1,

$$B_\sigma(x, r') \subseteq B_\rho(x, r) \subseteq \mathcal{O}.$$

Thus every point of $\mathcal{O}$ is the center of an open ball with respect to σ that is contained in $\mathcal{O}$; that is, $\mathcal{O}$ is open in (X, σ) .

Conversely, let $\mathcal{O}$ be open in (X, σ) and let x belong to $\mathcal{O}$. There is an $r > 0$ for which $B_\sigma(x, r) \subseteq \mathcal{O}$. Set $r'' = c_1 r > 0$. By Claim 2,

$$B_\rho(x, r'') \subseteq B_\sigma(x, r) \subseteq \mathcal{O},$$

so $\mathcal{O}$ is open in (X, ρ) .

Hence the open subsets of (X, ρ) and of (X, σ) are exactly the same. Consequently, by Proposition 4 the closed subsets agree as well. Convergence also agrees: if $\{x_n\} \rightarrow x$ in (X, ρ) , then $\sigma(x_n, x) \leq \rho(x_n, x)/c_1 \rightarrow 0$ by the left-hand inequality in (*), so $\{x_n\} \rightarrow x$ in (X, σ) ; and if $\{x_n\} \rightarrow x$ in (X, σ) , then $\rho(x_n, x) \leq c_2 \cdot \sigma(x_n, x) \rightarrow 0$ by the right-hand inequality in (*).

Problem 9.18. Let X be a subspace of the metric space Y and A a subset of X . Show that A is closed in X if and only if $A = X \cap F$, where F is closed in Y .

Solution. Since $A \subseteq X \subseteq Y$, we will repeatedly use the elementary set identity

$$X \sim (X \sim A) = A, \quad (*)$$

valid because $A \subseteq X$, and the identity

$$X \cap (Y \sim S) = X \sim S \quad \text{for any } S \subseteq Y, \quad (**)$$

valid because $X \subseteq Y$.

Suppose first that A is closed in X . By Proposition 4 applied in the metric space X , the set $X \sim A$ is open in X . By Proposition 2 there is a set $\mathcal{O}$, open in Y , with

$$X \sim A = X \cap \mathcal{O}.$$

Put $F = Y \sim \mathcal{O}$. By Proposition 4 applied in the metric space Y , the set F is closed in Y . Using $(**)$ and then $(*)$,

$$X \cap F = X \cap (Y \sim \mathcal{O}) = X \sim \mathcal{O} = X \sim (X \cap \mathcal{O}) = X \sim (X \sim A) = A.$$

Here the third equality holds because for a point of X , belonging to $\mathcal{O}$ is the same as belonging to $X \cap \mathcal{O}$. Thus $A = X \cap F$ with F closed in Y .

Conversely, suppose $A = X \cap F$ with F closed in Y . Put $\mathcal{O} = Y \sim F$, which is open in Y by Proposition 4. Then, using $(**)$,

$$X \cap \mathcal{O} = X \cap (Y \sim F) = X \sim F = X \sim (X \cap F) = X \sim A,$$

where again the third equality holds because a point of X lies in F precisely when it lies in $X \cap F$. Hence $X \sim A$ is the intersection of X with a set open in Y , so by Proposition 2 the set $X \sim A$ is open in X . By Proposition 4 applied in the metric space X , its complement in X , namely $X \sim (X \sim A) = A$, is closed in X .

Problem 9.19. Let X be a subspace of the metric space Y .

(i) If $\mathcal{O}$ is an open subset of the metric subspace X , is $\mathcal{O}$ an open subset of Y ? What if X is an open subset of Y ?

(ii) If F is a closed subset of the metric subspace X , is F a closed subset of Y ? What if X is a closed subset of Y ?

Solution. (i) In general the answer is no. Take $Y = \mathbb{R}$ with the Euclidean metric and $X = [0, 1]$ as a metric subspace. The set $\mathcal{O} = [0, 1] = X$ is open in the metric space X , since X is the whole space and Proposition 1 tells us the whole space is open. But $\mathcal{O}$ is not open in $\mathbb{R}$: no open ball $(-\epsilon, \epsilon)$ centered at the point $0 \in \mathcal{O}$ is contained in $[0, 1]$, since such a ball contains the negative number $-\epsilon/2$.

However, if X is an open subset of Y , then the answer is yes. Indeed, let $\mathcal{O}$ be open in the metric subspace X . By Proposition 2 there is a set $\mathcal{U}$, open in Y , with $\mathcal{O} = X \cap \mathcal{U}$. Since X is open in Y and $\mathcal{U}$ is open in Y , Proposition 1 tells us that the intersection of two open subsets of Y is open in Y . Hence $\mathcal{O} = X \cap \mathcal{U}$ is open in Y .

(ii) In general the answer is no. Take $Y = \mathbb{R}$ with the Euclidean metric and $X = (0, 1)$ as a metric subspace. The set $F = (0, 1) = X$ is closed in the metric space X , since by Proposition 5 the whole space is closed. But F is not closed in $\mathbb{R}$: the point 0 is a point of closure of $(0, 1)$ in $\mathbb{R}$, because every neighborhood of 0 in $\mathbb{R}$ contains an interval $(-\epsilon, \epsilon)$ and hence contains points of $(0, 1)$; yet 0 does not belong to F . (Equivalently, by Proposition 6, the sequence $\{1/n\}_{n \geq 2}$ lies in F and converges in $\mathbb{R}$ to $0 \notin F$.)

However, if X is a closed subset of Y , then the answer is yes. Indeed, let F be closed in the metric

subspace X . By the preceding problem there is a set G , closed in Y , with $F = X \cap G$. Since X is closed in Y and G is closed in Y , Proposition 5 tells us that the intersection of any collection of closed subsets of Y is closed in Y . Hence $F = X \cap G$ is closed in Y .

Problem 9.20. For a subset E of a metric space X , a point $x \in X$ is called an interior point of E provided there is an open ball centered at x that is contained in E : the collection of interior points of E is called the interior of E and denoted by $\text{int } E$. Show that $\text{int } E$ is always open and E is open if and only if $E = \text{int } E$.

Solution. Let ρ denote the metric on X , and for $x \in X$ and $r > 0$ let $B(x, r) = \{x' \in X \mid \rho(x', x) < r\}$ be the open ball centered at x of radius r .

We first observe that $\text{int } E \subseteq E$. Indeed, if x is an interior point of E , there is an $r > 0$ with $B(x, r) \subseteq E$; since $\rho(x, x) = 0 < r$, the point x belongs to $B(x, r)$ and hence to E .

Next we show $\text{int } E$ is open. Let x belong to $\text{int } E$ and choose $r > 0$ with $B(x, r) \subseteq E$. We claim

$$B(x, r) \subseteq \text{int } E.$$

Let x' belong to $B(x, r)$. As verified in the text, an open ball is an open set; explicitly, with $r' = r - \rho(x', x) > 0$ we have $B(x', r') \subseteq B(x, r)$. Therefore

$$B(x', r') \subseteq B(x, r) \subseteq E,$$

so x' is an interior point of E , that is, x' belongs to $\text{int } E$. This proves the claim. Thus every point of $\text{int } E$ is the center of an open ball contained in $\text{int } E$, and hence $\text{int } E$ is open. (Note that if $\text{int } E$ is empty it is open by Proposition 1, and the argument above is vacuous in that case.)

Finally we prove the stated characterization of openness.

Suppose E is open. Let x belong to E . By the definition of an open set there is an open ball centered at x contained in E , so x belongs to $\text{int } E$. Hence $E \subseteq \text{int } E$. Combined with the inclusion $\text{int } E \subseteq E$ established above, $E = \text{int } E$.

Conversely, suppose $E = \text{int } E$. Since $\text{int } E$ is open, so is E . (Directly: if $x \in E = \text{int } E$, then by the definition of interior point there is an open ball centered at x contained in E ; as x was arbitrary, E is open.)

Problem 9.21. For a subset E of a metric space X , a point $x \in X$ is called an exterior point of E provided there is an open ball centered at x that is contained in $X \sim E$: the collection of exterior points of E is called the exterior of E and denoted by $\text{ext } E$. Show that $\text{ext } E$ is always open. Show that E is closed if and only if $X \sim E = \text{ext } E$.

Solution. Let ρ denote the metric on X and let $B(x, r) = \{x' \in X \mid \rho(x', x) < r\}$ denote the open ball centered at x of radius r .

Compare the definition of exterior point with the definition of interior point in the preceding problem: a point $x \in X$ is an exterior point of E if and only if there is an open ball centered at x contained in $X \sim E$, that is, if and only if x is an interior point of the set $X \sim E$. Therefore

$$\text{ext } E = \text{int}(X \sim E). \quad (*)$$

Openness of $\text{ext } E$. By the preceding problem, the interior of any subset of X is open; applying this to the subset $X \sim E$ and using (*), we conclude that $\text{ext } E$ is open. For completeness, here is the argument written out directly. Let x belong to $\text{ext } E$ and choose $r > 0$ with $B(x, r) \subseteq X \sim E$. If x' belongs to $B(x, r)$, then setting $r' = r - \rho(x', x) > 0$ we have, as verified in the text, $B(x', r') \subseteq B(x, r) \subseteq X \sim E$, so x' belongs to $\text{ext } E$. Hence $B(x, r) \subseteq \text{ext } E$, and $\text{ext } E$ is open.

The characterization of closedness. By the preceding problem applied to the set $X \sim E$,

$$X \sim E \text{ is open} \iff X \sim E = \text{int}(X \sim E).$$

By (*), the right-hand condition is exactly $X \sim E = \text{ext } E$. By Proposition 4, a subset of a metric space X is open if and only if its complement in X is closed; since $X \sim (X \sim E) = E$, this says that $X \sim E$ is open if and only if E is closed. Chaining the two equivalences,

$$E \text{ is closed} \iff X \sim E \text{ is open} \iff X \sim E = \text{ext } E,$$

which is the assertion to be proved.

Two remarks. First, the inclusion $\text{ext } E \subseteq X \sim E$ always holds, since by the preceding problem $\text{int}(X \sim E) \subseteq X \sim E$; so the condition $X \sim E = \text{ext } E$ carries content only in the direction $X \sim E \subseteq \text{ext } E$. Second, it is worth recording the direct form of the result: E is closed exactly when every point not in E has an open ball about it missing E entirely, which is precisely the statement that no point of $X \sim E$ is a point of closure of E , that is, that E contains all of its points of closure.

9.4 Problems 9.22–9.28

Problem 9.22. For a subset E of a metric space X , a point $x \in X$ is called a boundary point of E provided every open ball centered at x contains points in E and points in $X \sim E$: the collection of boundary points of E is called the boundary of E and denoted by $\text{bd } E$. Show (i) that $\text{bd } E$ is always closed, (ii) that E is open if and only if $E \cap \text{bd } E = \emptyset$, and (iii) that E is closed if and only if $\text{bd } E \subseteq E$.

Solution. Throughout, recall that a point $x \in X$ is a point of closure of a set A provided every neighborhood of x contains a point of A ; since every neighborhood of x contains an open ball centered at x , and every open ball centered at x is itself a neighborhood of x , this is equivalent to the statement that every open ball centered at x contains a point of A .

Preliminary identity. Directly from the definitions,

$$\text{bd } E = \overline{E} \cap \overline{X \sim E}.$$

Indeed, $x \in \text{bd } E$ means precisely that every open ball centered at x meets E and meets $X \sim E$, i.e. that x is a point of closure of E and a point of closure of $X \sim E$.

Note also the symmetry

$$\text{bd}(X \sim E) = \overline{X \sim E} \cap \overline{X \sim (X \sim E)} = \overline{X \sim E} \cap \overline{E} = \text{bd } E,$$

since $X \sim (X \sim E) = E$. So a set and its complement have the same boundary.

(i) $\text{bd } E$ is closed. By Proposition 3, the closure of any subset of X is closed; hence $\overline{E}$ and $\overline{X \sim E}$ are closed. By Proposition 5, the intersection of any collection of closed subsets of X is closed. Therefore $\text{bd } E = \overline{E} \cap \overline{X \sim E}$ is closed.

(ii) E is open if and only if $E \cap \text{bd } E = \emptyset$.

Suppose first that E is open, and let $x \in E$. By the definition of an open set there is an $r > 0$ with $B(x, r) \subseteq E$. Then $B(x, r)$ contains no point of $X \sim E$, so x fails the defining condition for a boundary point: $x \notin \text{bd } E$. As $x \in E$ was arbitrary, $E \cap \text{bd } E = \emptyset$.

Conversely, suppose $E \cap \text{bd } E = \emptyset$, and let $x \in E$. Then $x \notin \text{bd } E$, so there is some $r > 0$ such that the ball $B(x, r)$ fails to contain a point of E or fails to contain a point of $X \sim E$. But x itself belongs to $B(x, r) \cap E$ (as $\rho(x, x) = 0 < r$), so $B(x, r)$ does contain a point of E . Hence $B(x, r)$ contains no point of $X \sim E$, that is, $B(x, r) \subseteq E$. Since each point of E is the center of an open ball contained in E , the set E is open.

(iii) E is closed if and only if $\text{bd } E \subseteq E$.

By Proposition 4, E is closed if and only if its complement $X \sim E$ is open. By part (ii) applied to the set $X \sim E$, the set $X \sim E$ is open if and only if

$$(X \sim E) \cap \text{bd}(X \sim E) = \emptyset.$$

By the symmetry noted above, $\text{bd}(X \sim E) = \text{bd } E$, so this condition reads $(X \sim E) \cap \text{bd } E = \emptyset$, which says exactly that no boundary point of E lies outside E , i.e. $\text{bd } E \subseteq E$. Combining the two equivalences: E is closed if and only if $\text{bd } E \subseteq E$.

Problem 9.23. Let A and B be subsets of a metric space X . Show that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. Also, show that $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$ and $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$.

Solution. Recall that $x \in \overline{A}$ means that every neighborhood of x contains a point of A ; equivalently, every open ball $B(x, r)$, $r > 0$, contains a point of A (every such ball is a neighborhood of x , and every neighborhood of x contains such a ball).

Monotonicity. Assume $A \subseteq B$ and let $x \in \overline{A}$. Given $r > 0$, the ball $B(x, r)$ contains a point of A , and that point lies in B since $A \subseteq B$. Hence every ball centered at x contains a point of B , i.e. $x \in \overline{B}$. Thus $\overline{A} \subseteq \overline{B}$. (This is the assertion recorded in the proof of Proposition 3.)

The union. We prove both inclusions.

Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, monotonicity gives $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$, so

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

For the reverse inclusion, let $x \notin \overline{A \cup B}$. Since $x \notin \overline{A}$, there is $r_1 > 0$ with $B(x, r_1) \cap A = \emptyset$; since $x \notin \overline{B}$, there is $r_2 > 0$ with $B(x, r_2) \cap B = \emptyset$. Put $r = \min\{r_1, r_2\} > 0$. Then $B(x, r) \subseteq B(x, r_1)$ and $B(x, r) \subseteq B(x, r_2)$, so

$$B(x, r) \cap (A \cup B) = (B(x, r) \cap A) \cup (B(x, r) \cap B) = \emptyset,$$

whence $x \notin \overline{A \cup B}$. Taking contrapositives, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, and therefore

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

(Alternatively: by Proposition 3 each of $\overline{A}$, $\overline{B}$ is closed, so by Proposition 5 their union is closed; it contains $A \cup B$, and $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, again by Proposition 3.)

The intersection. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, monotonicity yields $\overline{A \cap B} \subseteq \overline{A}$ and $\overline{A \cap B} \subseteq \overline{B}$, hence

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

This inclusion may be strict, so no equality holds in general. Take $X = \mathbb{R}$ with the usual metric, $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. Then $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$, while $\overline{A} = \overline{B} = \mathbb{R}$ because every open interval contains both a rational and an irrational number; thus $\overline{A} \cap \overline{B} = \mathbb{R}$.

Problem 9.24. Show that for a subset E of a metric space X , the closure of E is the intersection of all closed subsets of X that contain E .

Solution. Let $\mathcal{F}$ denote the collection of all closed subsets of X that contain E , and set

$$C = \bigcap_{F \in \mathcal{F}} F.$$

The collection $\mathcal{F}$ is nonempty: by Proposition 5 the whole space X is closed, and clearly $E \subseteq X$, so $X \in \mathcal{F}$. Hence the intersection defining C is taken over a nonempty family and is a well-defined subset of X . We show $\overline{E} = C$ by proving each inclusion.

First, $\overline{E} \subseteq C$. Let $F \in \mathcal{F}$, so F is closed and $E \subseteq F$. By the monotonicity of closure (established in the proof of Proposition 3: if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$), $\overline{E} \subseteq \overline{F}$. Since F is closed, F contains all of its points of closure, that is $\overline{F} = F$. Hence $\overline{E} \subseteq F$. As F was an arbitrary member of $\mathcal{F}$, we conclude $\overline{E} \subseteq \bigcap_{F \in \mathcal{F}} F = C$. (This is exactly the final assertion of Proposition 3: $\overline{E}$ is the smallest closed set containing E .)

Second, $C \subseteq \overline{E}$. By Proposition 3, the set $\overline{E}$ is closed, and $E \subseteq \overline{E}$ always holds. Therefore $\overline{E}$ is itself one of the sets in the family $\mathcal{F}$. An intersection of a family of sets is contained in each member of the family, so

$$C = \bigcap_{F \in \mathcal{F}} F \subseteq \overline{E}.$$

The two inclusions give $\overline{E} = C$, i.e. the closure of E is precisely the intersection of all closed subsets of X containing E .

Two remarks. Since by Proposition 5 the intersection of any collection of closed subsets of X is closed, this description makes it evident again that $\overline{E}$ is closed. And it exhibits $\overline{E}$ as the smallest closed superset of E : it is closed, contains E , and is contained in every closed set containing E .

Problem 9.25. Exhibit a continuous mapping that is not uniformly continuous and a uniformly continuous mapping that is not Lipschitz.

Solution. In both examples the metric spaces are subsets of $\mathbb{R}$ with the usual metric $\rho(u, v) = |u - v|$. A continuous mapping that is not uniformly continuous. Let $X = \mathbb{R}$, $Y = \mathbb{R}$ and $f(x) = x^2$. f is continuous: fix $x \in \mathbb{R}$ and let $\epsilon > 0$. Put $\delta = \min\{1, \epsilon/(2|x| + 1)\}$. If $|x' - x| < \delta$ then $|x' + x| \leq |x' - x| + 2|x| < \delta + 2|x|$, so

$$|f(x') - f(x)| = |x' - x| \cdot |x' + x| < \delta \cdot (2|x| + 1) \leq \epsilon.$$

Thus the ϵ - δ criterion holds at every point of $\mathbb{R}$, and so by the ϵ - δ Criterion for Continuity the mapping f is continuous on $\mathbb{R}$. (Note that δ here depends on x as well as on ϵ ; the next paragraph shows that this dependence cannot be removed.)

f is not uniformly continuous: take the challenge $\epsilon_0 = 1$ and let $\delta > 0$ be arbitrary. Choose a natural number n with $n\delta > 1$, and set $u = n$, $v = n + \delta/2$. Then $|u - v| = \delta/2 < \delta$, while

$$|f(u) - f(v)| = |u - v| |u + v| = \frac{\delta}{2} \left(2n + \frac{\delta}{2} \right) = n\delta + \frac{\delta^2}{4} > 1 = \epsilon_0.$$

So no $\delta > 0$ responds to the challenge $\epsilon_0 = 1$, and f fails to be uniformly continuous.

A uniformly continuous mapping that is not Lipschitz. Let $X = [0, 1]$, $Y = \mathbb{R}$ and $g(x) = \sqrt{x}$.

First we establish the inequality

$$|\sqrt{u} - \sqrt{v}| \leq \sqrt{|u - v|} \quad \text{for all } u, v \geq 0.$$

By symmetry we may assume $u \geq v \geq 0$. Since all terms are nonnegative,

$$(\sqrt{v} + \sqrt{u - v})^2 = v + 2\sqrt{v(u - v)} + (u - v) \geq u,$$

and taking square roots (the square root is increasing on $[0, \infty)$) gives $\sqrt{v} + \sqrt{u - v} \geq \sqrt{u}$, that is $\sqrt{u} - \sqrt{v} \leq \sqrt{u - v} = \sqrt{|u - v|}$, as claimed.

g is uniformly continuous: given $\epsilon > 0$, set $\delta = \epsilon^2 > 0$. If $u, v \in [0, 1]$ satisfy $|u - v| < \delta$, then

$$|g(u) - g(v)| = |\sqrt{u} - \sqrt{v}| \leq \sqrt{|u - v|} < \sqrt{\delta} = \epsilon.$$

Since δ depends only on ϵ and not on the points, g is uniformly continuous on $[0, 1]$.

g is not Lipschitz: suppose there were a $c \geq 0$ with $|g(u) - g(v)| \leq c|u - v|$ for all $u, v \in [0, 1]$. Take $v = 0$ and $u = 1/n^2$ for a natural number n . Then

$$\frac{1}{n} = \left| \sqrt{1/n^2} - \sqrt{0} \right| \leq c \cdot \frac{1}{n^2},$$

so $n \leq c$ for every natural number n , which is impossible. Hence no such c exists.

These two examples show that the implications recorded in the text, Lipschitz $\Rightarrow$ uniformly continuous $\Rightarrow$ continuous, are both strict.

Problem 9.26. Show that every mapping from a metric space (X, ρ) to a metric space (Y, σ) is continuous if ρ is the discrete metric.

Solution. Recall the discrete metric on a nonempty set X : $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$. The key observation is that for every $x \in X$ the open ball of radius 1 centered at x reduces to the single point x :

$$B(x, 1) = \{x' \in X \mid \rho(x', x) < 1\} = \{x\},$$

because $\rho(x', x)$ takes only the values 0 and 1, and $\rho(x', x) < 1$ forces $\rho(x', x) = 0$, i.e. $x' = x$ by property (ii) of a metric.

Now let f be any mapping from X to Y , and let $x \in X$. We verify the ϵ - δ criterion for continuity at x . Let $\epsilon > 0$ be given and take $\delta = 1$. If $x' \in X$ satisfies $\rho(x, x') < \delta = 1$, then by the observation above $x' = x$, so $f(x') = f(x)$ and therefore

$$\sigma(f(x), f(x')) = \sigma(f(x), f(x)) = 0 < \epsilon.$$

Equivalently, $f(B(x, 1)) = \{f(x)\} \subseteq B(f(x), \epsilon)$. By the ϵ - δ Criterion for Continuity, f is continuous at x . Since $x \in X$ was arbitrary, f is continuous.

Two alternative arguments. Using the definition of continuity by sequences: if $\{x_n\} \rightarrow x$ in (X, ρ) , then choosing $\epsilon = 1$ in the definition of convergence gives an index N with $\rho(x_n, x) < 1$, hence $x_n = x$, for all $n \geq N$; consequently $f(x_n) = f(x)$ for $n \geq N$, so $\sigma(f(x_n), f(x)) = 0$ for $n \geq N$ and $\{f(x_n)\} \rightarrow f(x)$.

Using Proposition 8: every subset A of X is open in the discrete metric, since for each $x \in A$ we have $B(x, 1) = \{x\} \subseteq A$. Hence for every open subset $\mathcal{O}$ of Y the set $f^{-1}(\mathcal{O})$, being a subset of X , is open in X ; by Proposition 8, f is continuous.

Note that the argument shows more: taking $\delta = 1$ independently of the point x , every such mapping is in fact uniformly continuous.

Problem 9.27. Suppose there is a continuous, one-to-one mapping from a metric space (X, ρ) to a metric space (Y, σ) , where σ is the discrete metric. Show that every subset of X is open.

Solution. Let $f: X \rightarrow Y$ be continuous and one-to-one, where σ is the discrete metric on Y , so that $\sigma(y, y') = 0$ if $y = y'$ and $\sigma(y, y') = 1$ if $y \neq y'$.

Step 1: every singleton subset of X is open. Fix $x \in X$. Apply the ϵ - δ Criterion for Continuity at x with the challenge $\epsilon = 1$: there is a $\delta > 0$ for which

$$f(B(x, \delta)) \subseteq B(f(x), 1).$$

As in the preceding problem, in the discrete metric $B(f(x), 1) = \{f(x)\}$, since σ takes only the values 0 and 1. Hence $f(x') = f(x)$ for every $x' \in B(x, \delta)$. Because f is one-to-one, this forces $x' = x$. Therefore

$$B(x, \delta) = \{x\},$$

and in particular the ball $B(x, \delta)$, which is centered at x , is contained in $\{x\}$. So $\{x\}$ is open.

Step 2: every subset of X is open. Let $A \subseteq X$. If $A = \emptyset$, it is open by Proposition 1. Otherwise

$$A = \bigcup_{x \in A} \{x\},$$

a union of a collection of open subsets of X by Step 1, and hence open by Proposition 1.

Therefore every subset of X is open. (Equivalently, by Proposition 4, every subset of X is also closed, since the complement of any subset is a subset and hence open.)

Alternative phrasing of Step 1 using Proposition 8: for each $x \in X$ the singleton $\{f(x)\} = B(f(x), 1)$ is an open subset of Y , being an open ball. Since f is continuous, Proposition 8 gives that $f^{-1}(\{f(x)\})$ is open in X ; and since f is one-to-one, $f^{-1}(\{f(x)\}) = \{x\}$. So $\{x\}$ is open.

Note where injectivity is needed: without it the conclusion fails. For instance, if $X = \mathbb{R}$ with the usual metric, Y any set with the discrete metric, and f constant, then f is continuous but the subsets of X are certainly not all open.

Problem 9.28. For a metric space (X, ρ) , show that the metric $\rho: X \times X \rightarrow \mathbb{R}$ is continuous, where $X \times X$ has the product metric.

Solution. The product metric τ on $X \times X$ is defined, as in Section 9.1, by

$$\tau((x_1, x_2), (y_1, y_2)) = \left\{ [\rho(x_1, y_1)]^2 + [\rho(x_2, y_2)]^2 \right\}^{1/2},$$

and $\mathbb{R}$ carries its usual metric $|s - t|$. We show that $\rho: (X \times X, \tau) \rightarrow \mathbb{R}$ is Lipschitz, with Lipschitz constant $c = \sqrt{2}$; since a Lipschitz mapping is uniformly continuous, and a uniformly continuous mapping is continuous, this proves the assertion.

Step 1: a quadrilateral inequality. For all $x_1, x_2, y_1, y_2 \in X$,

$$|\rho(x_1, x_2) - \rho(y_1, y_2)| \leq \rho(x_1, y_1) + \rho(x_2, y_2).$$

Indeed, two applications of the triangle inequality for ρ give

$$\rho(x_1, x_2) \leq \rho(x_1, y_1) + \rho(y_1, x_2) \leq \rho(x_1, y_1) + \rho(y_1, y_2) + \rho(y_2, x_2),$$

and since $\rho(y_2, x_2) = \rho(x_2, y_2)$ by the symmetry of a metric, this rearranges to

$$\rho(x_1, x_2) - \rho(y_1, y_2) \leq \rho(x_1, y_1) + \rho(x_2, y_2).$$

Interchanging the roles of the pair (x_1, x_2) and the pair (y_1, y_2) gives the same bound for $\rho(y_1, y_2) - \rho(x_1, x_2)$, and the two together give the stated inequality on absolute values.

Step 2: comparison with the product metric. For nonnegative reals a, b we have $2ab \leq a^2 + b^2$, hence

$$(a + b)^2 = a^2 + 2ab + b^2 \leq 2(a^2 + b^2), \quad \text{so} \quad a + b \leq \sqrt{2} (a^2 + b^2)^{1/2}.$$

Applying this with $a = \rho(x_1, y_1)$ and $b = \rho(x_2, y_2)$ and combining with Step 1,

$$|\rho(x_1, x_2) - \rho(y_1, y_2)| \leq \sqrt{2} \left\{ [\rho(x_1, y_1)]^2 + [\rho(x_2, y_2)]^2 \right\}^{1/2} = \sqrt{2} \tau((x_1, x_2), (y_1, y_2)).$$

Step 3: conclusion. To keep the two roles of the symbol ρ apart, write $F: X \times X \rightarrow \mathbb{R}$ for the mapping under consideration, so that $F(x_1, x_2) = \rho(x_1, x_2)$. With $u = (x_1, x_2)$ and $v = (y_1, y_2)$ denoting points of $X \times X$, Step 2 says

$$|F(u) - F(v)| \leq \sqrt{2} \cdot \tau(u, v) \quad \text{for all } u, v \in X \times X,$$

which is precisely the Lipschitz condition for F with constant $c = \sqrt{2} > 0$. As observed in the text, a Lipschitz mapping is uniformly continuous: given $\epsilon > 0$, the choice $\delta = \epsilon/\sqrt{2}$ responds to the challenge, since $\tau(u, v) < \delta$ implies $|F(u) - F(v)| \leq \sqrt{2} \tau(u, v) < \epsilon$. And a uniformly continuous mapping is continuous, as the text infers from the ϵ - δ Criterion for Continuity at a point. Hence the metric ρ , viewed as the mapping F from $X \times X$ with the product metric to $\mathbb{R}$ with its usual metric, is continuous.

Remark. Step 1 alone already yields the assertion of Problem 14: if $\{u_n\} \rightarrow u$ and $\{v_n\} \rightarrow v$ in X , then

$$|\rho(u_n, v_n) - \rho(u, v)| \leq \rho(u_n, u) + \rho(v_n, v) \rightarrow 0,$$

so $\{\rho(u_n, v_n)\} \rightarrow \rho(u, v)$ in $\mathbb{R}$. Conversely, the elementary bounds

$$\max\{a, b\} \leq (a^2 + b^2)^{1/2} \leq \sqrt{2} \max\{a, b\} \quad (a, b \geq 0)$$

show that a sequence converges in $(X \times X, \tau)$ if and only if both of its coordinate sequences converge in (X, ρ) ; so the continuity of ρ on $X \times X$ established above and the statement of Problem 14 are two formulations of the same fact.

9.5 Problems 9.29–9.35

Problem 9.29. Let z be a point in the metric space (X, ρ) . Define the function $f: X \rightarrow \mathbb{R}$ by $f(x) = \rho(x, z)$. Show that f is uniformly continuous.

Solution. We show that f is Lipschitz with Lipschitz constant $c = 1$, and then invoke the Example following the definition of uniform continuity, which observes that a Lipschitz mapping is uniformly continuous.

Let $u, v \in X$. By the triangle inequality for the metric ρ ,

$$\rho(u, z) \leq \rho(u, v) + \rho(v, z),$$

so that

$$f(u) - f(v) = \rho(u, z) - \rho(v, z) \leq \rho(u, v).$$

Interchanging the roles of u and v in this argument gives

$$f(v) - f(u) = \rho(v, z) - \rho(u, z) \leq \rho(v, u) = \rho(u, v),$$

where we used the symmetry of ρ . Since $|f(u) - f(v)|$ equals either $f(u) - f(v)$ or $f(v) - f(u)$, the two displayed inequalities together give

$$|f(u) - f(v)| \leq \rho(u, v) \quad \text{for all } u, v \in X.$$

The metric on $\mathbb{R}$ is $\sigma(s, t) = |s - t|$, so the above says precisely that

$$\sigma(f(u), f(v)) \leq 1 \cdot \rho(u, v) \quad \text{for all } u, v \in X,$$

that is, f is Lipschitz with $c = 1$.

To exhibit the uniform continuity explicitly: let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then $\delta > 0$, and if $u, v \in X$ satisfy $\rho(u, v) < \delta$, we have

$$|f(u) - f(v)| \leq \rho(u, v) < \delta = \epsilon.$$

Since the choice of δ depends only on ϵ and not on the points u, v , the function f is uniformly continuous on X .

Problem 9.30. Show that the composition of uniformly continuous mappings between metric spaces, when defined, is uniformly continuous.

Solution. Let (X, ρ) , (Y, σ) and (Z, τ) be metric spaces, and let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be uniformly continuous. We must show that the composition $g \circ f: X \rightarrow Z$, which is defined since the range of f lies in the domain of g , is uniformly continuous.

Let $\epsilon > 0$ be given.

Since $g: Y \rightarrow Z$ is uniformly continuous, there is an $\eta > 0$ such that

$$\text{for all } y_1, y_2 \in Y, \text{ if } \sigma(y_1, y_2) < \eta, \text{ then } \tau(g(y_1), g(y_2)) < \epsilon.$$

Observe that η depends only on ϵ .

Now apply the uniform continuity of $f: X \rightarrow Y$ to the positive number η : there is a $\delta > 0$ such that

$$\text{for all } u, v \in X, \text{ if } \rho(u, v) < \delta, \text{ then } \sigma(f(u), f(v)) < \eta.$$

Observe that δ depends only on η , and hence, through η , only on ϵ . In particular δ is independent of the points of X at which it will be used.

We claim this δ responds to the challenge ϵ for the mapping $g \circ f$. Indeed, let $u, v \in X$ satisfy $\rho(u, v) < \delta$. By the choice of δ ,

$$\sigma(f(u), f(v)) < \eta.$$

Since $f(u)$ and $f(v)$ are points of Y , the choice of η applied to the pair $y_1 = f(u)$, $y_2 = f(v)$ yields

$$\tau(g(f(u)), g(f(v))) < \epsilon, \quad \text{that is,} \quad \tau((g \circ f)(u), (g \circ f)(v)) < \epsilon.$$

Thus for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $u, v \in X$, if $\rho(u, v) < \delta$ then $\tau((g \circ f)(u), (g \circ f)(v)) < \epsilon$. By definition, $g \circ f$ is uniformly continuous.

Finally, an arbitrary finite composition of uniformly continuous mappings is uniformly continuous, by induction on the number of mappings composed, the case of two mappings being what we have just proved.

Problem 9.31. Show that a continuous mapping between metric spaces remains continuous if an equivalent metric is imposed on the domain and an equivalent metric is imposed on the range.

Solution. Recall the definition: two metrics ρ and ρ' on a set X are equivalent provided there are positive numbers c_1 and c_2 such that

$$c_1 \rho(x_1, x_2) \leq \rho'(x_1, x_2) \leq c_2 \rho(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Let $f: X \rightarrow Y$ be continuous as a mapping from (X, ρ) to (Y, σ) . Let ρ' be a metric on X equivalent to ρ and let σ' be a metric on Y equivalent to σ . We must show f is continuous as a mapping from (X, ρ') to (Y, σ') .

Argument by open sets. Let $\mathcal{O}$ be an open subset of the metric space (Y, σ') . Since σ' and σ are equivalent metrics on Y , Proposition 7 tells us that a subset of Y is open in (Y, σ') if and only if it is open in (Y, σ) . Hence $\mathcal{O}$ is open in (Y, σ) . Since $f: (X, \rho) \rightarrow (Y, \sigma)$ is continuous, Proposition 8 tells us that $f^{-1}(\mathcal{O})$ is an open subset of (X, ρ) . Since ρ and ρ' are equivalent metrics on X , Proposition 7, applied now on X , tells us that $f^{-1}(\mathcal{O})$ is also open in (X, ρ') .

Thus the inverse image under f of every open subset of (Y, σ') is an open subset of (X, ρ') . By Proposition 8 again, now in the direction that this criterion implies continuity, the mapping $f: (X, \rho') \rightarrow (Y, \sigma')$ is continuous.

A direct ϵ - δ argument. It is instructive to verify this without Propositions 7 and 8. Choose positive constants with

$$c_1 \rho(x_1, x_2) \leq \rho'(x_1, x_2) \leq c_2 \rho(x_1, x_2), \quad d_1 \sigma(y_1, y_2) \leq \sigma'(y_1, y_2) \leq d_2 \sigma(y_1, y_2)$$

for all $x_1, x_2 \in X$ and all $y_1, y_2 \in Y$.

Fix $x \in X$ and let $\epsilon > 0$. Since $f: (X, \rho) \rightarrow (Y, \sigma)$ is continuous at x , the ϵ - δ criterion for continuity provides a $\delta > 0$ such that

$$\text{if } \rho(x, x') < \delta, \text{ then } \sigma(f(x), f(x')) < \epsilon/d_2.$$

Set $\delta' = c_1 \delta > 0$. Suppose $x' \in X$ satisfies $\rho'(x, x') < \delta'$. From $c_1 \rho(x, x') \leq \rho'(x, x') < c_1 \delta$ we get $\rho(x, x') < \delta$, whence $\sigma(f(x), f(x')) < \epsilon/d_2$ and therefore

$$\sigma'(f(x), f(x')) \leq d_2 \sigma(f(x), f(x')) < d_2 \cdot \frac{\epsilon}{d_2} = \epsilon.$$

So the ϵ - δ criterion holds for $f: (X, \rho') \rightarrow (Y, \sigma')$ at the arbitrary point x , and f is continuous with respect to the new pair of metrics.

Note that only the inequalities $c_1 \rho \leq \rho'$ and $\sigma' \leq d_2 \sigma$ were used; the remaining two inequalities give the converse implication, so continuity with respect to one pair of metrics is in fact equivalent to continuity with respect to the other.

Problem 9.32. For a nonempty subset E of the metric space (X, ρ) and a point $x \in X$, define the

distance from x to E , $\text{dist}(x, E)$, as follows:

$$\text{dist}(x, E) = \inf\{\rho(x, y) \mid y \in E\}.$$

- (i) Show that the distance function $f: X \rightarrow \mathbb{R}$ defined by $f(x) = \text{dist}(x, E)$, for $x \in X$, is continuous.
- (ii) Show that $\{x \in X \mid \text{dist}(x, E) = 0\} = \overline{E}$.

Solution. First note that f is well defined and real-valued: since E is nonempty, the set $\{\rho(x, y) \mid y \in E\}$ is a nonempty set of real numbers bounded below by 0, so its infimum exists and satisfies $0 \leq f(x) < \infty$.

(i) We prove the Lipschitz estimate

$$|f(u) - f(v)| \leq \rho(u, v) \quad \text{for all } u, v \in X,$$

which, as noted in the Example following the definition of uniform continuity, implies that f is uniformly continuous and in particular continuous.

Let $u, v \in X$ and let $y \in E$ be arbitrary. By the triangle inequality,

$$\rho(u, y) \leq \rho(u, v) + \rho(v, y).$$

Since $f(u) = \text{dist}(u, E)$ is a lower bound for $\{\rho(u, y) \mid y \in E\}$, we have $f(u) \leq \rho(u, y)$, and therefore

$$f(u) \leq \rho(u, v) + \rho(v, y) \quad \text{for every } y \in E.$$

Thus $f(u) - \rho(u, v)$ is a lower bound for the set $\{\rho(v, y) \mid y \in E\}$. Since $\text{dist}(v, E)$ is the greatest lower bound of that set,

$$f(u) - \rho(u, v) \leq \text{dist}(v, E) = f(v), \quad \text{that is,} \quad f(u) - f(v) \leq \rho(u, v).$$

Interchanging u and v and using the symmetry of ρ gives $f(v) - f(u) \leq \rho(u, v)$. Combining the two inequalities yields $|f(u) - f(v)| \leq \rho(u, v)$, as claimed.

Explicitly, given $\epsilon > 0$, the choice $\delta = \epsilon$ has the property that $\rho(u, v) < \delta$ implies $|f(u) - f(v)| < \epsilon$; so f is uniformly continuous on X , hence continuous.

(ii) Recall that $\overline{E}$, the closure of E , is the set of points of closure of E , and that a point $x \in X$ is a point of closure of E provided every open ball $B(x, r)$ centered at x contains a point of E .

Suppose first that $f(x) = \text{dist}(x, E) = 0$. Let $r > 0$. Since 0 is the greatest lower bound of $\{\rho(x, y) \mid y \in E\}$ and $r > 0$, the number r is not a lower bound of this set; hence there is a point $y \in E$ with $\rho(x, y) < r$, that is, $y \in B(x, r) \cap E$. As $r > 0$ was arbitrary, every open ball centered at x contains a point of E , so x is a point of closure of E and thus $x \in \overline{E}$.

Conversely, suppose $x \in \overline{E}$. Let $r > 0$. Then $B(x, r)$ contains a point y of E , so $\rho(x, y) < r$ and hence

$$0 \leq \text{dist}(x, E) \leq \rho(x, y) < r.$$

Since this holds for every $r > 0$, we must have $\text{dist}(x, E) = 0$; for if $\text{dist}(x, E) = c > 0$, taking $r = c$ would give the contradiction $c < c$.

Therefore $\{x \in X \mid \text{dist}(x, E) = 0\} = \overline{E}$.

In particular, since a set is closed if and only if it equals its closure, E is closed if and only if $E = \{x \in X \mid \text{dist}(x, E) = 0\}$.

Problem 9.33. Show that a subset E of a metric space X is open if and only if there is a continuous real-valued function f on X for which $E = \{x \in X \mid f(x) > 0\}$.

Solution. Let (X, ρ) be a metric space and $E \subseteq X$.

Sufficiency. Suppose there is a continuous function $f: X \rightarrow \mathbb{R}$ with $E = \{x \in X \mid f(x) > 0\}$. The set $\mathcal{O} = (0, \infty)$ is an open subset of $\mathbb{R}$: indeed, if $t > 0$ then the ball $B(t, t) = (0, 2t)$ is contained in

$(0, \infty)$. Since f is continuous, Proposition 8 tells us that $f^{-1}(\mathcal{O})$ is an open subset of X . But

$$f^{-1}((0, \infty)) = \{x \in X \mid f(x) > 0\} = E,$$

so E is open.

Necessity. Suppose E is open. We construct such an f .

Case 1: $E = X$. Take $f \equiv 1$, the constant function. A constant function is continuous (it maps every sequence to a constant sequence), and $\{x \in X \mid f(x) > 0\} = X = E$.

Case 2: $E \neq X$. Then the complement $F = X \sim E$ is a nonempty subset of X , and it is closed: by Proposition 4, a subset of a metric space is open if and only if its complement is closed, and the complement of F in X is the open set E . Define

$$f(x) = \text{dist}(x, F) = \inf\{\rho(x, y) \mid y \in F\} \quad \text{for } x \in X.$$

This is well defined and real-valued because F is nonempty, and by part (i) of the preceding problem f is continuous (indeed Lipschitz, with $|f(u) - f(v)| \leq \rho(u, v)$).

We claim $E = \{x \in X \mid f(x) > 0\}$. Since $\rho \geq 0$, each set $\{\rho(x, y) \mid y \in F\}$ is bounded below by 0, so $f \geq 0$ on all of X ; consequently $\{x \in X \mid f(x) > 0\} = X \sim \{x \in X \mid f(x) = 0\}$, and it suffices to show that

$$\{x \in X \mid f(x) = 0\} = F.$$

By part (ii) of the preceding problem, $\{x \in X \mid \text{dist}(x, F) = 0\} = \overline{F}$. Since F is closed, $\overline{F} = F$. Hence $\{x \mid f(x) = 0\} = F$, and therefore

$$\{x \in X \mid f(x) > 0\} = X \sim \{x \in X \mid f(x) = 0\} = X \sim F = E.$$

Thus in either case there is a continuous real-valued f on X with $E = \{x \in X \mid f(x) > 0\}$, and the proof is complete.

Remark. No separate treatment of $E = \emptyset$ is needed. If $X \neq \emptyset$ then $E = \emptyset$ falls under Case 2, where $F = X$ and $f \equiv 0$, since $\rho(x, x) = 0$ for each $x \in X$; if $X = \emptyset$ then $E = X$ and Case 1 applies. In any event one may simply take $f \equiv -1$, for which $\{x \in X \mid f(x) > 0\} = \emptyset = E$.

Problem 9.34. Show that a subset E of a metric space X is closed if and only if there is a continuous real-valued function f on X for which $E = f^{-1}(0)$.

Solution. Let (X, ρ) be a metric space and $E \subseteq X$. Here $f^{-1}(0)$ denotes $f^{-1}(\{0\}) = \{x \in X \mid f(x) = 0\}$. Sufficiency. Suppose $f: X \rightarrow \mathbb{R}$ is continuous and $E = f^{-1}(0)$. The set $\mathbb{R} \sim \{0\}$ is open in $\mathbb{R}$: if $t \neq 0$, then the ball $B(t, |t|)$ consists of points s with $|s - t| < |t|$, and no such s can be 0. Since f is continuous, Proposition 8 tells us that $f^{-1}(\mathbb{R} \sim \{0\})$ is open in X . Taking inverse images commutes with complements: a point $x \in X$ lies in $f^{-1}(\mathbb{R} \sim \{0\})$ if and only if $f(x) \neq 0$, that is, if and only if $x \notin f^{-1}(\{0\})$. Hence

$$f^{-1}(\mathbb{R} \sim \{0\}) = X \sim f^{-1}(\{0\}) = X \sim E.$$

Thus $X \sim E$ is open, and therefore, by Proposition 4, E is closed.

Necessity. Suppose E is closed.

Case 1: $E = \emptyset$. Take $f \equiv 1$. This constant function is continuous and $f^{-1}(0) = \emptyset = E$.

Case 2: $E \neq \emptyset$. Define

$$f(x) = \text{dist}(x, E) = \inf\{\rho(x, y) \mid y \in E\} \quad \text{for } x \in X.$$

Since E is nonempty this is a well-defined real-valued function, and by part (i) of Problem 32 it is continuous (indeed Lipschitz, satisfying $|f(u) - f(v)| \leq \rho(u, v)$).

By part (ii) of Problem 32,

$$f^{-1}(0) = \{x \in X \mid \text{dist}(x, E) = 0\} = \overline{E}.$$

Since E is closed, $E = \overline{E}$, and therefore $f^{-1}(0) = E$.

In both cases we have produced a continuous real-valued f on X with $E = f^{-1}(0)$, which completes the proof.

Remark. The verification in the sufficiency direction shows more generally that the inverse image of a closed set under a continuous mapping is closed; this is the content of the next problem.

Problem 9.35. Let X and Y be metric spaces. Prove that $f: X \rightarrow Y$ is continuous if and only if $f^{-1}(C)$ is closed in X whenever C is closed in Y .

Solution. The proof rests on Proposition 8, which characterizes continuity by the openness of inverse images of open sets, together with the set-theoretic identity

$$f^{-1}(Y \sim A) = X \sim f^{-1}(A) \quad \text{for every } A \subseteq Y,$$

which we refer to as the complementation identity. To verify it: a point $x \in X$ belongs to $f^{-1}(Y \sim A)$ if and only if $f(x) \in Y \sim A$, that is, if and only if $f(x) \notin A$; and this holds if and only if $x \notin f^{-1}(A)$, that is, if and only if $x \in X \sim f^{-1}(A)$. (Note that f is defined on all of X , so every $x \in X$ has an image $f(x) \in Y$, and the equivalence is valid.)

Necessity. Suppose f is continuous, and let C be a closed subset of Y . Then $\mathcal{O} = Y \sim C$ is open in Y . By Proposition 8, $f^{-1}(\mathcal{O})$ is open in X . By the complementation identity with $A = C$,

$$f^{-1}(\mathcal{O}) = f^{-1}(Y \sim C) = X \sim f^{-1}(C).$$

Hence $X \sim f^{-1}(C)$ is open in X , and therefore, by Proposition 4, $f^{-1}(C)$ is closed in X .

Sufficiency. Suppose $f^{-1}(C)$ is closed in X whenever C is closed in Y . Let $\mathcal{O}$ be an open subset of Y . Then $C = Y \sim \mathcal{O}$ is closed in Y , so by hypothesis $f^{-1}(C)$ is closed in X . By the complementation identity with $A = \mathcal{O}$,

$$X \sim f^{-1}(\mathcal{O}) = f^{-1}(Y \sim \mathcal{O}) = f^{-1}(C),$$

which is closed in X . Therefore $f^{-1}(\mathcal{O})$, being the complement in X of the closed set $f^{-1}(C)$, is open in X by Proposition 4.

Thus the inverse image under f of every open subset of Y is open in X , and Proposition 8 gives that $f: X \rightarrow Y$ is continuous.

Combining the two directions: f is continuous if and only if $f^{-1}(C)$ is closed in X for every closed subset C of Y .

9.6 Problems 9.36–9.42

Problem 9.36. Let $X = C[a, b]$. Define the function $\psi: X \rightarrow \mathbb{R}$ by

$$\psi(f) = \int_a^b f(x) dx \quad \text{for each } f \text{ in } X.$$

Show that ψ is Lipschitz on the metric space X , where X has the metric induced by the maximum norm.

Solution. Recall that the metric on $X = C[a, b]$ induced by the maximum norm is

$$\rho(f, g) = \|f - g\|_{\max} = \max_{x \in [a, b]} |f(x) - g(x)|,$$

the maximum being attained because $|f - g|$ is continuous on the closed, bounded interval $[a, b]$ (by the Extreme Value Theorem). On $\mathbb{R}$ we use the usual metric $|s - t|$. To say that ψ is Lipschitz is to say that there is a constant $c \geq 0$ for which

$$|\psi(f) - \psi(g)| \leq c \cdot \rho(f, g) \quad \text{for all } f, g \in X.$$

We claim $c = b - a$ works.

First observe that ψ is well defined: each f in $C[a, b]$ is continuous, hence Lebesgue measurable and bounded on $[a, b]$, and $[a, b]$ has finite measure, so f is integrable over $[a, b]$ and $\int_a^b f$ is a real number. Now let f and g belong to X . By the linearity of integration,

$$\psi(f) - \psi(g) = \int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b [f(x) - g(x)] dx.$$

Since $|\int_a^b h| \leq \int_a^b |h|$ for an integrable h , we have

$$|\psi(f) - \psi(g)| \leq \int_a^b |f(x) - g(x)| dx.$$

For every x in $[a, b]$ we have the pointwise bound $|f(x) - g(x)| \leq \|f - g\|_{\max}$, and the integral is monotone, so

$$\int_a^b |f(x) - g(x)| dx \leq \int_a^b \|f - g\|_{\max} dx = (b - a) \|f - g\|_{\max}.$$

Combining the two displays,

$$|\psi(f) - \psi(g)| \leq (b - a) \|f - g\|_{\max} = (b - a) \rho(f, g).$$

Thus ψ is Lipschitz with Lipschitz constant $b - a$. In particular ψ is uniformly continuous on X : given $\epsilon > 0$, take $\delta = \epsilon/(b - a + 1)$; if $\rho(f, g) < \delta$ then $|\psi(f) - \psi(g)| \leq (b - a)\rho(f, g) < \epsilon$.

It is worth noting that the constant $b - a$ is sharp: taking $g = 0$ and $f \equiv 1$ gives $|\psi(f) - \psi(g)| = b - a$ while $\rho(f, g) = 1$, so no smaller constant can serve.

We remark finally that the first display in the chain above already gives $|\psi(f) - \psi(g)| \leq \int_a^b |f - g| = \|f - g\|_1$, so ψ is also Lipschitz, with constant 1, for the metric that X inherits from the $L^1[a, b]$ norm. The estimate asked for here is the one against the maximum norm, which is the metric used for $C[a, b]$ throughout this chapter.

Problem 9.37. In a metric space X , show (i) that a convergent sequence is Cauchy and (ii) that a Cauchy sequence is bounded.

Solution. Let (X, ρ) be a metric space and $\{x_n\}$ a sequence in X . Recall the definitions in force: $\{x_n\}$ converges to $x \in X$ provided $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$; $\{x_n\}$ is Cauchy provided for each $\epsilon > 0$ there is an index N for which $\rho(x_n, x_m) < \epsilon$ whenever $n, m \geq N$; and a nonempty subset E of X is bounded provided $\text{diam } E = \sup\{\rho(x, y) \mid x, y \in E\}$ is finite. A sequence is called bounded provided the set of its terms is a bounded subset of X .

Part (i): a convergent sequence is Cauchy.

Suppose $\{x_n\} \rightarrow x$ in X . Let $\epsilon > 0$. Since $\rho(x_n, x) \rightarrow 0$, there is an index N such that

$$\rho(x_n, x) < \epsilon/2 \quad \text{for all } n \geq N.$$

Let $n, m \geq N$. By the triangle inequality and the symmetry of ρ ,

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\{x_n\}$ is Cauchy.

Part (ii): a Cauchy sequence is bounded.

Suppose $\{x_n\}$ is Cauchy. Apply the Cauchy criterion with $\epsilon = 1$: there is an index N for which

$$\rho(x_n, x_m) < 1 \quad \text{whenever } n, m \geq N.$$

In particular, taking $m = N$,

$$\rho(x_n, x_N) < 1 \quad \text{for all } n \geq N.$$

The remaining terms $x_1, \dots, x_{N-1}$ are finite in number, so we may define

$$M = \max\{1, \rho(x_1, x_N), \rho(x_2, x_N), \dots, \rho(x_{N-1}, x_N)\},$$

a finite real number (the maximum of a finite set of reals exists). Then

$$\rho(x_n, x_N) \leq M \quad \text{for every index } n,$$

since for $n < N$ this holds by the definition of M , and for $n \geq N$ we have $\rho(x_n, x_N) < 1 \leq M$.

Now let $E = \{x_n \mid n \in \mathbb{N}\}$ be the set of terms of the sequence. For any two indices n and m , the triangle inequality gives

$$\rho(x_n, x_m) \leq \rho(x_n, x_N) + \rho(x_N, x_m) \leq M + M = 2M.$$

Hence $\text{diam } E \leq 2M < \infty$, that is, E is a bounded subset of X . Equivalently, E is contained in the closed ball of radius M about x_N . Therefore $\{x_n\}$ is bounded.

Combining (i) and (ii): every convergent sequence in a metric space is bounded, since it is Cauchy by (i) and hence bounded by (ii).

Problem 9.38. In a metric space X , show that a Cauchy sequence converges if and only if it has a convergent subsequence.

Solution. Let (X, ρ) be a metric space and let $\{x_n\}$ be a Cauchy sequence in X .

Suppose first that $\{x_n\}$ converges, say to $x \in X$. Then $\{x_n\}$ is itself a subsequence of $\{x_n\}$ (take $n_k = k$), and it converges. So $\{x_n\}$ has a convergent subsequence. (Note that this implication does not use the Cauchy hypothesis; indeed every subsequence of a convergent sequence converges to the same limit, since if $\{n_k\}$ is strictly increasing then $n_k \geq k$ for every k , so $\rho(x_{n_k}, x) \rightarrow 0$ whenever $\rho(x_n, x) \rightarrow 0$.)

Conversely, suppose $\{x_n\}$ is Cauchy and possesses a subsequence $\{x_{n_k}\}$ that converges to some point $x \in X$. We claim that the whole sequence $\{x_n\}$ converges to x .

Let $\epsilon > 0$. Since $\{x_n\}$ is Cauchy, there is an index N for which

$$\rho(x_n, x_m) < \epsilon/2 \quad \text{whenever } n, m \geq N.$$

Since $\{x_{n_k}\}$ converges to x , there is an index K_1 for which

$$\rho(x_{n_k}, x) < \epsilon/2 \quad \text{whenever } k \geq K_1.$$

Because $\{n_k\}$ is a strictly increasing sequence of natural numbers, $n_k \geq k$ for every k ; hence choosing any index k_0 with $k_0 \geq \max\{K_1, N\}$ we have both $n_{k_0} \geq k_0 \geq N$ and $\rho(x_{n_{k_0}}, x) < \epsilon/2$.

Now let $n \geq N$. Since $n_{k_0} \geq N$ as well, the Cauchy estimate applies to the pair (n, n_{k_0}) , and the triangle inequality gives

$$\rho(x_n, x) \leq \rho(x_n, x_{n_{k_0}}) + \rho(x_{n_{k_0}}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus for each $\epsilon > 0$ there is an index N with $\rho(x_n, x) < \epsilon$ for all $n \geq N$; that is, $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$, so $\{x_n\}$ converges to x .

This completes the proof of both implications. The hypothesis that $\{x_n\}$ be Cauchy is essential for the converse: in $\mathbb{R}$ the sequence $x_n = (-1)^n$ has convergent subsequences but does not converge. This is the fact used in the proof of Proposition 10 (completeness of $C[a, b]$) and in the proof of the Cantor Intersection Theorem, where a Cauchy sequence is shown to converge by exhibiting a convergent subsequence.

Problem 9.39. Suppose that $\{x_n\}$ is a sequence in a complete metric space (X, ρ) and for each index n , $\rho(x_n, x_{n+1}) < 1/2^n$. Show that $\{x_n\}$ converges. Does $\{x_n\}$ converge if for each index n , $\rho(x_n, x_{n+1}) < 1/n$?

Solution. First part. Assume $\rho(x_n, x_{n+1}) < 1/2^n$ for every index n . We show $\{x_n\}$ is Cauchy; completeness of X then delivers convergence.

Let n and m be indices with $m > n$. Repeated application of the triangle inequality along the chain $x_n, x_{n+1}, \dots, x_m$ gives

$$\rho(x_n, x_m) \leq \sum_{j=n}^{m-1} \rho(x_j, x_{j+1}) < \sum_{j=n}^{m-1} \frac{1}{2^j} < \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}},$$

the last equality being the sum of a geometric series with ratio $1/2$. By symmetry of ρ the same bound holds when $m < n$ with n replaced by $\min\{n, m\}$; and it is trivially true when $m = n$. Hence

$$\rho(x_n, x_m) \leq \frac{1}{2^{\min\{n, m\}-1}} \quad \text{for all } n, m.$$

Now let $\epsilon > 0$. Since $1/2^{N-1} \rightarrow 0$ as $N \rightarrow \infty$, choose N with $1/2^{N-1} < \epsilon$. If $n, m \geq N$ then $\min\{n, m\} \geq N$, so

$$\rho(x_n, x_m) \leq \frac{1}{2^{\min\{n, m\}-1}} \leq \frac{1}{2^{N-1}} < \epsilon.$$

Therefore $\{x_n\}$ is a Cauchy sequence in X . Since (X, ρ) is complete, $\{x_n\}$ converges to a point of X . The essential feature used is that $\sum_{n=1}^{\infty} 1/2^n$ converges; the identical argument shows more generally that if $\rho(x_n, x_{n+1}) \leq a_n$ for all n , where $\sum a_n < \infty$, then $\{x_n\}$ is Cauchy, since the tails $\sum_{j \geq n} a_j$ tend to 0. This is exactly the device used in the proof of Proposition 10 for $C[a, b]$.

Second part. The answer is no: the condition $\rho(x_n, x_{n+1}) < 1/n$ does not force convergence, even in a complete space.

For a counterexample take $X = \mathbb{R}$ with the usual metric $\rho(s, t) = |s - t|$, which is complete (this is the completeness axiom for the real numbers, as noted in the text). Define

$$x_n = \sum_{k=1}^n \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}.$$

Then for each index n ,

$$\rho(x_n, x_{n+1}) = |x_{n+1} - x_n| = \frac{1}{n+2} < \frac{1}{n},$$

so the hypothesis of the second question is met. But $\{x_n\}$ does not converge: the harmonic series diverges, so $x_n \rightarrow \infty$. Explicitly, grouping terms,

$$x_{2^m-1} \geq \sum_{k=2}^{2^m} \frac{1}{k} \geq \sum_{i=1}^{m-1} \sum_{k=2^i+1}^{2^{i+1}} \frac{1}{k} \geq \sum_{i=1}^{m-1} 2^i \cdot \frac{1}{2^{i+1}} = \frac{m-1}{2},$$

which is unbounded in m . An unbounded sequence cannot converge, since by Problem 37 a convergent sequence is Cauchy and hence bounded. Therefore $\{x_n\}$ diverges.

The point of the contrast is that $\sum_n 1/2^n$ converges while $\sum_n 1/n$ does not; the hypothesis $\rho(x_n, x_{n+1}) < 1/n$ controls consecutive terms but permits the accumulated displacement $\sum_{j=n}^{m-1} 1/j$ between x_n and x_m to remain large no matter how big n is.

Problem 9.40. Provide an example of a descending countable collection of closed, nonempty sets of real numbers whose intersection is empty. Does this contradict the Cantor Intersection Theorem?

Solution. For each index n define

$$F_n = [n, \infty) = \{x \in \mathbb{R} \mid x \geq n\}.$$

Each F_n is nonempty (it contains n). Each F_n is closed in $\mathbb{R}$: its complement $(-\infty, n)$ is open, since for $x < n$ the open interval of radius $n - x$ about x is contained in $(-\infty, n)$. The collection is descending, that is $F_{n+1} \subseteq F_n$ for every n , because $x \geq n+1$ implies $x \geq n$.

The intersection is empty:

$$\bigcap_{n=1}^{\infty} F_n = \emptyset.$$

Indeed, if x belonged to every F_n , then $x \geq n$ for every natural number n , contradicting the Archimedean property of $\mathbb{R}$ (for each real x there is a natural number n with $n > x$).

Thus $\{F_n\}_{n=1}^{\infty}$ is a descending countable collection of nonempty closed subsets of $\mathbb{R}$ with empty intersection.

This does not contradict the Cantor Intersection Theorem. That theorem asserts that a metric space X is complete if and only if whenever $\{F_n\}_{n=1}^{\infty}$ is a contracting sequence of nonempty closed subsets of X , there is a point $x \in X$ for which $\bigcap_{n=1}^{\infty} F_n = \{x\}$. By definition, a descending sequence $\{F_n\}$ of nonempty subsets of X is contracting provided

$$\lim_{n \rightarrow \infty} \text{diam}(F_n) = 0,$$

where $\text{diam } E = \sup\{|x - y| \mid x, y \in E\}$. Our sequence fails this hypothesis completely: for every n ,

$$\text{diam}(F_n) = \sup\{|x - y| \mid x, y \geq n\} = \infty,$$

since F_n contains points n and $n+k$ for every natural number k . So $\{F_n\}$ is not a contracting sequence, and the theorem says nothing about it. In particular no conclusion about $\bigcap_n F_n$ is available from the theorem, and the completeness of $\mathbb{R}$ is not impugned.

A final remark explains why unboundedness was unavoidable here, so that no example of the required kind could have had contracting, or even merely bounded, sets. Suppose $\{F_n\}$ is a descending sequence of nonempty closed subsets of $\mathbb{R}$ and some F_N is bounded; we show $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. For each $n \geq N$ select a point $y_n \in F_n$. Since $\{F_n\}$ is descending, every y_n with $n \geq N$ belongs to F_N , so $\{y_n\}_{n \geq N}$ is a bounded sequence of real numbers. By the Bolzano-Weierstrass Theorem it has a subsequence $\{y_{n_k}\}$ converging to some real number y . Fix an index m . For all k with $n_k \geq \max\{m, N\}$ we have $y_{n_k} \in F_{n_k} \subseteq F_m$, since $\{F_n\}$ is descending; thus y is the limit of a sequence in the closed set F_m , and hence $y \in F_m$ by Proposition 6. As m was arbitrary, $y \in \bigcap_{n=1}^{\infty} F_n$, and the intersection is not empty. Consequently, in any example of the required kind, every one of the sets must be unbounded, that is, must have infinite diameter, exactly as in the example above.

Problem 9.41. Let ρ and σ be equivalent metrics on a nonempty set X . Show that (X, ρ) is complete if and only if (X, σ) is complete.

Solution. Recall the definition: two metrics ρ and σ on a set X are equivalent provided there are positive numbers c_1 and c_2 such that

$$c_1 \cdot \sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \cdot \sigma(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Call this pair of inequalities (*).

Observe first that this relation is symmetric in ρ and σ . Dividing (*) through by the positive constants gives, for all $x_1, x_2 \in X$,

$$\frac{1}{c_2} \rho(x_1, x_2) \leq \sigma(x_1, x_2) \leq \frac{1}{c_1} \rho(x_1, x_2),$$

which is (*) with the roles of ρ and σ interchanged and with the constants $1/c_2$ and $1/c_1$, both positive. Consequently it suffices to prove one implication; the other follows by exchanging the names of the two metrics.

We prove two auxiliary facts.

Claim 1: a sequence $\{x_n\}$ in X is Cauchy in (X, ρ) if and only if it is Cauchy in (X, σ) .

Suppose $\{x_n\}$ is σ -Cauchy and let $\epsilon > 0$. Since $c_2 > 0$, there is an index N such that $\sigma(x_n, x_m) < \epsilon/c_2$ whenever $n, m \geq N$. Then by the right-hand inequality of (*), for $n, m \geq N$,

$$\rho(x_n, x_m) \leq c_2 \cdot \sigma(x_n, x_m) < c_2 \cdot \frac{\epsilon}{c_2} = \epsilon.$$

So $\{x_n\}$ is ρ -Cauchy. The converse follows from the symmetry just noted.

Claim 2: a sequence $\{x_n\}$ converges to a point x in (X, ρ) if and only if it converges to the same point x in (X, σ) .

If $\rho(x_n, x) \rightarrow 0$, then by the left-hand inequality of $(*)$,

$$0 \leq \sigma(x_n, x) \leq \frac{1}{c_1} \rho(x_n, x) \rightarrow 0,$$

so $\sigma(x_n, x) \rightarrow 0$. Again the converse follows by symmetry. (This is also immediate from Proposition 7, which asserts that equivalent metrics on X determine the same open sets, hence the same closed sets and the same convergent sequences with the same limits.)

Now suppose (X, ρ) is complete, and let $\{x_n\}$ be a Cauchy sequence in (X, σ) . By Claim 1, $\{x_n\}$ is Cauchy in (X, ρ) . By the completeness of (X, ρ) , there is a point $x \in X$ with $\rho(x_n, x) \rightarrow 0$. By Claim 2, $\sigma(x_n, x) \rightarrow 0$, so $\{x_n\}$ converges in (X, σ) to the point x of X . Since $\{x_n\}$ was an arbitrary σ -Cauchy sequence, (X, σ) is complete.

By the symmetry of the equivalence relation established at the outset, the same argument with ρ and σ interchanged shows that if (X, σ) is complete then so is (X, ρ) . Therefore (X, ρ) is complete if and only if (X, σ) is complete.

It is worth stressing that equivalence of metrics in the sense of $(*)$ is strictly stronger than the requirement that the two metrics induce the same topology, and it is the two-sided linear comparison that makes Claim 1 work. Completeness is not a topological invariant: on $X = (0, 1)$ the usual metric $\rho(s, t) = |s - t|$ and the metric $\sigma(s, t) = |\tan(\pi s - \pi/2) - \tan(\pi t - \pi/2)|$ induce exactly the same open subsets of $(0, 1)$, yet (X, ρ) is not complete (the sequence $x_n = 1/n$ is ρ -Cauchy with no limit in X) while (X, σ) is complete, being isometric to $\mathbb{R}$. These two metrics are therefore not equivalent in the sense of $(*)$. What the present problem shows is that the stronger relation $(*)$ does preserve completeness, because it preserves the class of Cauchy sequences and not merely the class of convergent ones.

Problem 9.42. Prove that the product of two complete metric spaces is complete.

Solution. Let (X_1, ρ_1) and (X_2, ρ_2) be complete metric spaces, and let τ be the product metric on the Cartesian product $X_1 \times X_2$, defined for (x_1, x_2) and (y_1, y_2) in $X_1 \times X_2$ by

$$\tau((x_1, x_2), (y_1, y_2)) = \left\{ [\rho_1(x_1, y_1)]^2 + [\rho_2(x_2, y_2)]^2 \right\}^{1/2}.$$

We must show that $(X_1 \times X_2, \tau)$ is complete.

We first record the elementary comparison estimates that drive the proof. For nonnegative reals a and b we have $a^2 \leq a^2 + b^2$ and $b^2 \leq a^2 + b^2$, and also $a^2 + b^2 \leq (a + b)^2$. Taking square roots (the square root is increasing on $[0, \infty)$),

$$\max\{a, b\} \leq \sqrt{a^2 + b^2} \leq a + b.$$

Applying this with $a = \rho_1(x_1, y_1)$ and $b = \rho_2(x_2, y_2)$ gives, for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $X_1 \times X_2$,

$$\rho_1(x_1, y_1) \leq \tau(x, y), \quad \rho_2(x_2, y_2) \leq \tau(x, y),$$

and

$$\tau(x, y) \leq \rho_1(x_1, y_1) + \rho_2(x_2, y_2).$$

Now let $\{z_n\}$ be a Cauchy sequence in $(X_1 \times X_2, \tau)$, and write $z_n = (x_n, y_n)$ with $x_n \in X_1$ and $y_n \in X_2$. The coordinate sequences are Cauchy. Indeed, let $\epsilon > 0$. There is an index N such that $\tau(z_n, z_m) < \epsilon$ whenever $n, m \geq N$. For such n, m the first pair of estimates gives

$$\rho_1(x_n, x_m) \leq \tau(z_n, z_m) < \epsilon \quad \text{and} \quad \rho_2(y_n, y_m) \leq \tau(z_n, z_m) < \epsilon.$$

Hence $\{x_n\}$ is Cauchy in (X_1, ρ_1) and $\{y_n\}$ is Cauchy in (X_2, ρ_2) .

Since (X_1, ρ_1) is complete, there is a point $x \in X_1$ with $\lim_{n \rightarrow \infty} \rho_1(x_n, x) = 0$. Since (X_2, ρ_2) is complete, there is a point $y \in X_2$ with $\lim_{n \rightarrow \infty} \rho_2(y_n, y) = 0$. Set $z = (x, y)$, a point of $X_1 \times X_2$.

We claim $\{z_n\}$ converges to z in the product metric. Let $\epsilon > 0$. Choose N_1 with $\rho_1(x_n, x) < \epsilon/2$ for $n \geq N_1$, and N_2 with $\rho_2(y_n, y) < \epsilon/2$ for $n \geq N_2$. Put $N = \max\{N_1, N_2\}$. Then for $n \geq N$, by the second estimate above,

$$\tau(z_n, z) \leq \rho_1(x_n, x) + \rho_2(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $\lim_{n \rightarrow \infty} \tau(z_n, z) = 0$, that is, $\{z_n\}$ converges in $X_1 \times X_2$ to the point z of $X_1 \times X_2$.

Since every Cauchy sequence in $(X_1 \times X_2, \tau)$ converges to a point of $X_1 \times X_2$, the product is complete.

Two remarks. First, the argument in fact establishes the sharper statement that a sequence in $X_1 \times X_2$ is τ -Cauchy (respectively, τ -convergent) if and only if both of its coordinate sequences are Cauchy (respectively, convergent) in their factors; the forward direction is the display above, and the reverse direction follows from $\tau(z, w) \leq \rho_1 + \rho_2$ exactly as in the last paragraph. Second, the same proof applies verbatim to any of the standard product metrics

$$\tau_1 = \rho_1 + \rho_2, \quad \tau_\infty = \max\{\rho_1, \rho_2\},$$

since the inequalities $\tau_\infty \leq \tau \leq \tau_1 \leq 2\tau_\infty$ show that all three are equivalent metrics on $X_1 \times X_2$ in the sense of the definition of Section 9.1; by Problem 41 completeness for one is completeness for all. Finally, by induction, the product of finitely many complete metric spaces is complete; the countable case is treated in Problem 45.

9.7 Problems 9.43–9.49

Problem 9.43. For a mapping f of the metric space (X, ρ) to the metric space (Y, σ) , show that f is uniformly continuous if and only if for any two sequences $\{u_n\}$ and $\{v_n\}$ in X ,

$$\text{if } \lim_{n \rightarrow \infty} \rho(u_n, v_n) = 0, \text{ then } \lim_{n \rightarrow \infty} \sigma(f(u_n), f(v_n)) = 0.$$

Solution. Recall that f is uniformly continuous provided that for each $\epsilon > 0$ there is a $\delta > 0$ such that for all u, v in X , if $\rho(u, v) < \delta$ then $\sigma(f(u), f(v)) < \epsilon$.

Necessity. Assume f is uniformly continuous, and let $\{u_n\}, \{v_n\}$ be sequences in X with $\lim_{n \rightarrow \infty} \rho(u_n, v_n) = 0$. Let $\epsilon > 0$. Choose $\delta > 0$ responding to this ϵ in the definition of uniform continuity. Since $\rho(u_n, v_n) \rightarrow 0$, there is an index N such that $\rho(u_n, v_n) < \delta$ for all $n \geq N$. Then, by the choice of δ ,

$$\sigma(f(u_n), f(v_n)) < \epsilon \quad \text{for all } n \geq N.$$

Since $\epsilon > 0$ was arbitrary, $\lim_{n \rightarrow \infty} \sigma(f(u_n), f(v_n)) = 0$.

Sufficiency. We argue by contraposition: assume f is not uniformly continuous and produce two sequences that violate the stated implication. Failure of uniform continuity means: there is an $\epsilon_0 > 0$ such that for every $\delta > 0$ there exist points u, v in X with

$$\rho(u, v) < \delta \quad \text{while} \quad \sigma(f(u), f(v)) \geq \epsilon_0.$$

For each index n apply this with $\delta = 1/n$ to select points u_n, v_n in X for which

$$\rho(u_n, v_n) < \frac{1}{n} \quad \text{and} \quad \sigma(f(u_n), f(v_n)) \geq \epsilon_0.$$

Then $\{u_n\}$ and $\{v_n\}$ are sequences in X with $\lim_{n \rightarrow \infty} \rho(u_n, v_n) = 0$, since $0 \leq \rho(u_n, v_n) < 1/n$ for every n . On the other hand $\sigma(f(u_n), f(v_n)) \geq \epsilon_0 > 0$ for every n , so the sequence of real numbers $\{\sigma(f(u_n), f(v_n))\}$ certainly does not converge to 0. Thus the sequential criterion fails.

Equivalently: if the sequential criterion holds for every pair of sequences, then f must be uniformly continuous. This completes the proof.

Problem 9.44. Use the outline below to prove the following extension property for uniformly continuous mappings: Let X and Y be metric spaces, with Y complete, and f a uniformly continuous mapping from a subset E of X to Y . Then f has a unique uniformly continuous extension to a mapping $\bar{f}$ of $\bar{E}$ to Y .

- (i) Show that f maps Cauchy sequences in E to Cauchy sequences in Y .
- (ii) For $x \in \bar{E}$, choose a sequence $\{x_n\}$ in E that converges to x and define $\bar{f}(x)$ to be the limit of $\{f(x_n)\}$. Use Problem 43 to show that $\bar{f}(x)$ is properly defined.
- (iii) Show that $\bar{f}$ is uniformly continuous on $\bar{E}$.
- (iv) Show that the above extension is unique since any two such extensions are continuous mappings on $\bar{E}$ that take the same values on the dense subset E of $\bar{E}$.

Solution. Write ρ for the metric on X and σ for the metric on Y . We record once the elementary inequality that will be used repeatedly: for points a, b, c, e of a metric space with metric d ,

$$|d(a, b) - d(c, e)| \leq d(a, c) + d(b, e),$$

which follows from two applications of the triangle inequality. In particular, if $a_n \rightarrow a$ and $b_n \rightarrow b$ then $d(a_n, b_n) \rightarrow d(a, b)$; the metric is continuous in its two variables jointly.

(i) f carries Cauchy sequences to Cauchy sequences. Let $\{x_n\}$ be a Cauchy sequence in E and let $\epsilon > 0$. By the uniform continuity of f there is a $\delta > 0$ such that $\sigma(f(u), f(v)) < \epsilon$ whenever $u, v \in E$ and $\rho(u, v) < \delta$. Since $\{x_n\}$ is Cauchy there is an index N with $\rho(x_n, x_m) < \delta$ for all $n, m \geq N$. Hence $\sigma(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geq N$. Therefore $\{f(x_n)\}$ is a Cauchy sequence in Y .

(ii) The definition of $\bar{f}$ is proper. Let $x \in \bar{E}$. By Proposition 6 of this chapter, a point of closure of E is the limit of a sequence in E , so there is a sequence $\{x_n\}$ in E with $x_n \rightarrow x$. A convergent sequence is Cauchy (Problem 37), so $\{x_n\}$ is Cauchy in E , and by (i) $\{f(x_n)\}$ is a Cauchy sequence in Y . Since Y is complete, $\{f(x_n)\}$ converges to a point of Y ; call it y .

We must check y does not depend on the chosen sequence. Let $\{x_n\}$ and $\{x'_n\}$ be two sequences in E converging to x , with $f(x_n) \rightarrow y$ and $f(x'_n) \rightarrow y'$. Then

$$0 \leq \rho(x_n, x'_n) \leq \rho(x_n, x) + \rho(x, x'_n) \rightarrow 0,$$

so $\lim_{n \rightarrow \infty} \rho(x_n, x'_n) = 0$. Since f is uniformly continuous on E , Problem 43 (applied to f regarded as a mapping of the metric subspace E into Y) gives

$$\lim_{n \rightarrow \infty} \sigma(f(x_n), f(x'_n)) = 0.$$

On the other hand, by the continuity of the metric noted above, $\sigma(f(x_n), f(x'_n)) \rightarrow \sigma(y, y')$. Hence $\sigma(y, y') = 0$, that is, $y = y'$. Therefore

$$\bar{f}(x) := \lim_{n \rightarrow \infty} f(x_n) \quad (\{x_n\} \subseteq E, x_n \rightarrow x)$$

is a well-defined mapping of $\bar{E}$ into Y . Moreover, if $x \in E$ we may take the constant sequence $x_n = x$ for all n , and then $\bar{f}(x) = f(x)$. Thus $\bar{f}$ is an extension of f .

(iii) $\bar{f}$ is uniformly continuous on $\bar{E}$. Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$u, v \in E, \rho(u, v) < \delta \implies \sigma(f(u), f(v)) < \epsilon/2.$$

We claim that if $x, x' \in \bar{E}$ and $\rho(x, x') < \delta/2$, then $\sigma(\bar{f}(x), \bar{f}(x')) \leq \epsilon/2 < \epsilon$. Indeed, choose sequences $\{x_n\}, \{x'_n\}$ in E with $x_n \rightarrow x$ and $x'_n \rightarrow x'$; by (ii), $f(x_n) \rightarrow \bar{f}(x)$ and $f(x'_n) \rightarrow \bar{f}(x')$. By the continuity of ρ , $\rho(x_n, x'_n) \rightarrow \rho(x, x') < \delta/2$, so there is an index N with $\rho(x_n, x'_n) < \delta$ for $n \geq N$, and therefore

$$\sigma(f(x_n), f(x'_n)) < \epsilon/2 \quad \text{for all } n \geq N.$$

Letting $n \rightarrow \infty$ and using the continuity of σ ,

$$\sigma(\bar{f}(x), \bar{f}(x')) = \lim_{n \rightarrow \infty} \sigma(f(x_n), f(x'_n)) \leq \epsilon/2 < \epsilon.$$

Since $\epsilon > 0$ was arbitrary and $\delta/2$ depends only on ϵ , $\bar{f}$ is uniformly continuous on $\bar{E}$.

(iv) Uniqueness. Suppose g and h are continuous mappings of $\bar{E}$ into Y that agree with f on E ; in particular this covers any two uniformly continuous extensions of f . Let $x \in \bar{E}$ and choose a sequence $\{x_n\}$ in E with $x_n \rightarrow x$; such a sequence exists by Proposition 6 of this chapter, that is, precisely because E is dense in $\bar{E}$. By Proposition 8 of this chapter, continuity of a mapping between metric spaces is equivalent to sequential continuity, so

$$g(x) = \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} h(x_n) = h(x),$$

the limits being taken in Y and the middle equalities holding because $x_n \in E$. Since $x \in \bar{E}$ was arbitrary, $g = h$. Hence the uniformly continuous extension $\bar{f}$ constructed in (ii)-(iii) is the only one.

Problem 9.45. Consider the countable collection of metric spaces $\{(X_n, \rho_n)\}_{n=1}^{\infty}$. For the Cartesian product of these sets $Z = \prod_{n=1}^{\infty} X_n$, define σ on $Z \times Z$ by setting, for $x = \{x_n\}$, $y = \{y_n\}$,

$$\sigma(x, y) = \sum_{n=1}^{\infty} 2^{-n} \rho_n^*(x_n, y_n) \quad \text{where each } \rho_n^* = \rho_n / (1 + \rho_n).$$

- (i) Show that σ is a metric.
- (ii) Show that (Z, σ) is complete if and only if each (X_n, ρ_n) is complete.

Solution. Throughout, each X_n is assumed nonempty. (This is the only reading under which the assertion in (ii) is correct: if some X_j were empty, then $Z = \emptyset$ would be complete no matter how the remaining factors behave.)

A scalar lemma. Let $\varphi(t) = t/(1+t)$ for $t \geq 0$. Then φ is strictly increasing, since $\varphi'(t) = (1+t)^{-2} > 0$; $0 \leq \varphi(t) < 1$; $\varphi(t) = 0$ only for $t = 0$; and φ is subadditive: for $s, t \geq 0$,

$$\varphi(s+t) = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = \varphi(s) + \varphi(t).$$

Its inverse on $[0, 1)$ is $\psi(u) = u/(1-u)$.

Step 1: each $\rho_n^* = \varphi \circ \rho_n$ is a metric on X_n , and it has the same Cauchy and convergent sequences as ρ_n . Nonnegativity and symmetry are inherited from ρ_n ; $\rho_n^*(u, v) = 0$ iff $\rho_n(u, v) = 0$ iff $u = v$. For the triangle inequality, using monotonicity and then subadditivity of φ ,

$$\rho_n^*(u, w) = \varphi(\rho_n(u, w)) \leq \varphi(\rho_n(u, v) + \rho_n(v, w)) \leq \rho_n^*(u, v) + \rho_n^*(v, w).$$

Furthermore $\rho_n^* \leq \rho_n$, while for $\epsilon > 0$ the implication $\rho_n^*(u, v) < \varphi(\epsilon) \Rightarrow \rho_n(u, v) < \epsilon$ holds because φ is strictly increasing and $\varphi(\epsilon) > 0$. These two comparisons give, for any $\epsilon > 0$ and any $u, v \in X_n$,

$$\rho_n(u, v) < \epsilon \implies \rho_n^*(u, v) < \epsilon, \quad \rho_n^*(u, v) < \varphi(\epsilon) \implies \rho_n(u, v) < \epsilon,$$

Hence a sequence in X_n is ρ_n -Cauchy if and only if it is ρ_n^* -Cauchy, and it ρ_n -converges to a point u if and only if it ρ_n^* -converges to u . (We do not claim that ρ_n and ρ_n^* are equivalent metrics in the sense defined in Section 9.1: that definition demands two-sided bounds by positive constant multiples of each other, and such bounds fail as soon as ρ_n is unbounded, since $\rho_n^* < 1$ always. The one-sided comparisons just proved are all we shall use.)

Step 2: (i) σ is a metric on Z . Since $0 \leq 2^{-n} \rho_n^*(x_n, y_n) \leq 2^{-n}$, the defining series converges and $0 \leq \sigma(x, y) \leq 1$; so σ is a well-defined finite, nonnegative, symmetric function. If $\sigma(x, y) = 0$ then every term of a convergent series of nonnegative terms vanishes, so $\rho_n^*(x_n, y_n) = 0$, i.e. $x_n = y_n$, for every n ; thus $x = y$. Conversely $\sigma(x, x) = 0$. For the triangle inequality, apply the triangle inequality for ρ_n^* in each coordinate and sum the (convergent, nonnegative) series:

$$\sigma(x, z) = \sum_{n=1}^{\infty} 2^{-n} \rho_n^*(x_n, z_n) \leq \sum_{n=1}^{\infty} 2^{-n} [\rho_n^*(x_n, y_n) + \rho_n^*(y_n, z_n)] = \sigma(x, y) + \sigma(y, z).$$

Step 3: σ -Cauchyness and σ -convergence are coordinatewise. Fix m . From the definition, since all terms are nonnegative,

$$2^{-m} \rho_m^*(x_m, y_m) \leq \sigma(x, y) \quad \text{for all } x, y \in Z. \quad (*)$$

Conversely, given $\epsilon > 0$ choose N with $\sum_{n>N} 2^{-n} < \epsilon/2$; then for all $x, y \in Z$,

$$\sigma(x, y) \leq \sum_{n=1}^N 2^{-n} \rho_n^*(x_n, y_n) + \frac{\epsilon}{2}. \quad (**)$$

Let $\{x^{(k)}\}$ be a sequence in Z , $x^{(k)} = \{x_n^{(k)}\}_{n \geq 1}$.

If $\{x^{(k)}\}$ is σ -Cauchy, then by $(*)$ each coordinate sequence $\{x_n^{(k)}\}_{k \geq 1}$ is ρ_n^* -Cauchy, hence ρ_n -Cauchy by Step 1. Conversely, if every coordinate sequence is Cauchy, let $\epsilon > 0$ and pick N as in $(**)$; choose K so large that $\rho_n^*(x_n^{(k)}, x_n^{(j)}) < \epsilon/2$ for all $k, j \geq K$ and all $n = 1, \dots, N$ (finitely many conditions). Then for $k, j \geq K$, by $(**)$,

$$\sigma(x^{(k)}, x^{(j)}) \leq \frac{\epsilon}{2} \sum_{n=1}^N 2^{-n} + \frac{\epsilon}{2} < \epsilon.$$

So $\{x^{(k)}\}$ is σ -Cauchy. The identical argument, with $x^{(j)}$ replaced by a fixed $x \in Z$, shows that $x^{(k)} \rightarrow x$ in (Z, σ) if and only if $x_n^{(k)} \rightarrow x_n$ in (X_n, ρ_n) for every n .

Step 4: (ii) the equivalence. Suppose each (X_n, ρ_n) is complete, and let $\{x^{(k)}\}$ be a Cauchy sequence in (Z, σ) . By Step 3 each coordinate sequence $\{x_n^{(k)}\}_{k \geq 1}$ is Cauchy in (X_n, ρ_n) , so it converges to some $x_n \in X_n$. Put $x = \{x_n\} \in Z$. By Step 3 again, $x^{(k)} \rightarrow x$ in (Z, σ) . Hence (Z, σ) is complete.

Conversely, suppose (Z, σ) is complete and fix an index m ; we show (X_m, ρ_m) is complete. For each $n \neq m$ select a point $a_n \in X_n$. Let $\{u^{(k)}\}$ be a Cauchy sequence in (X_m, ρ_m) , and define $x^{(k)} \in Z$ by

$$x_m^{(k)} = u^{(k)}, \quad x_n^{(k)} = a_n \quad (n \neq m).$$

Every coordinate sequence of $\{x^{(k)}\}$ is Cauchy — the m -th by hypothesis, the others because they are constant — so by Step 3 $\{x^{(k)}\}$ is Cauchy in (Z, σ) , and by completeness it converges to some $x \in Z$. By Step 3 the m -th coordinates converge: $u^{(k)} \rightarrow x_m$ in (X_m, ρ_m) . Thus every Cauchy sequence in X_m converges in X_m , i.e. (X_m, ρ_m) is complete. Since m was arbitrary, every factor is complete.

Problem 9.46. For each index n , define $f_n(x) = \alpha x^n + \beta \cos(x/n)$ for $0 \leq x \leq 1$. For what values of the parameters α and β is the sequence $\{f_n\}$ a Cauchy sequence in the metric space $C[0, 1]$?

Solution. Here $C[0, 1]$ carries the uniform metric

$$\rho(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|,$$

so a sequence is Cauchy in $C[0, 1]$ precisely when it is uniformly Cauchy on $[0, 1]$.

Answer: $\{f_n\}$ is Cauchy in $C[0, 1]$ if and only if $\alpha = 0$; the parameter β may be any real number.

Write $f_n = g_n + h_n$ where

$$g_n(x) = \alpha x^n, \quad h_n(x) = \beta \cos(x/n), \quad 0 \leq x \leq 1.$$

Step 1: $\{h_n\}$ is always uniformly Cauchy. Using $1 - \cos t = 2 \sin^2(t/2)$ and $|\sin s| \leq |s|$, for $0 \leq x \leq 1$,

$$|\cos(x/n) - 1| = 2 \sin^2\left(\frac{x}{2n}\right) \leq 2 \left(\frac{x}{2n}\right)^2 = \frac{x^2}{2n^2} \leq \frac{1}{2n^2}.$$

Hence, denoting by c the constant function with value β ,

$$\rho(h_n, c) = \max_{0 \leq x \leq 1} |\beta| |\cos(x/n) - 1| \leq \frac{|\beta|}{2n^2} \longrightarrow 0.$$

So $h_n \rightarrow c$ in $C[0, 1]$; in particular $\{h_n\}$ is a Cauchy sequence in $C[0, 1]$ (a convergent sequence is Cauchy, Problem 37), for every value of β .

Step 2: $\{f_n\}$ is Cauchy if and only if $\{g_n\}$ is Cauchy. Since $g_n = f_n - h_n$ and $f_n = g_n + h_n$, and since for any functions u_n, v_n one has

$$\rho(u_n + v_n, u_m + v_m) \leq \rho(u_n, u_m) + \rho(v_n, v_m),$$

the sum of two Cauchy sequences in $C[0, 1]$ is Cauchy. By Step 1 the sequence $\{h_n\}$, and likewise $\{-h_n\}$, is Cauchy; therefore $\{f_n\} = \{g_n + h_n\}$ is Cauchy if $\{g_n\}$ is, and $\{g_n\} = \{f_n + (-h_n)\}$ is Cauchy if $\{f_n\}$ is.

Step 3: $\{g_n\}$ is Cauchy if and only if $\alpha = 0$. If $\alpha = 0$ then $g_n \equiv 0$ for every n and the sequence is trivially Cauchy.

Suppose $\alpha \neq 0$. For each index n put $x_n = 2^{-1/n}$, so that $0 < x_n \leq 1$ and

$$x_n^n = \frac{1}{2}, \quad x_n^{2n} = \frac{1}{4}.$$

Then

$$\rho(g_n, g_{2n}) = \max_{0 \leq x \leq 1} |\alpha| |x^n - x^{2n}| \geq |\alpha| |x_n^n - x_n^{2n}| = \frac{|\alpha|}{4} > 0.$$

Thus with $\epsilon = |\alpha|/4$ there is, for every index N , a pair of indices $n, m \geq N$ (namely n and $m = 2n$ with $n \geq N$) for which $\rho(g_n, g_m) \geq \epsilon$. Hence $\{g_n\}$ is not Cauchy.

Combining Steps 2 and 3: $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$ exactly when $\alpha = 0$, and then, by Step 1, f_n converges uniformly on $[0, 1]$ to the constant function β . For $\alpha \neq 0$ the sequence fails to be Cauchy for every β . (This is consistent with the completeness of $C[0, 1]$: for $\alpha \neq 0$ the pointwise limit of f_n is β on $[0, 1)$ and $\alpha + \beta$ at $x = 1$, a discontinuous function, so no uniform limit in $C[0, 1]$ can exist.)

Problem 9.47. Let $\mathcal{D}$ be the subspace of $C[0, 1]$ consisting of the continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ that are differentiable on $(0, 1)$. Is $\mathcal{D}$ complete?

Solution. No: $\mathcal{D}$ is not complete.

Recall that $C[0, 1]$ carries the metric induced by the maximum norm, $\rho(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$, that this metric space is complete by Proposition 10 of this chapter, and that by Proposition 11 of this chapter a subset of a complete metric space is a complete metric subspace if and only if it is a closed subset. So it suffices to exhibit a sequence in $\mathcal{D}$ that is Cauchy in $\mathcal{D}$ but whose limit in $C[0, 1]$ fails to be differentiable on all of $(0, 1)$.

The example. Define

$$f(x) = \left|x - \frac{1}{2}\right|, \quad f_n(x) = \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{1}{n^2}} \quad (0 \leq x \leq 1, n \geq 1).$$

Each f_n belongs to $\mathcal{D}$. The function $t \mapsto \sqrt{t^2 + n^{-2}}$ is the composition of the polynomial $t \mapsto t^2 + n^{-2}$, which takes values in $[n^{-2}, \infty) \subseteq (0, \infty)$, with the square root, which is differentiable on $(0, \infty)$. Hence f_n is continuous on $[0, 1]$ and differentiable at every point of $(0, 1)$ (indeed at every real number), with

$$f'_n(x) = \frac{x - \frac{1}{2}}{\sqrt{(x - \frac{1}{2})^2 + n^{-2}}}.$$

$f_n \rightarrow f$ uniformly on $[0, 1]$. Put $t = x - \frac{1}{2}$ and $a = 1/n > 0$. Rationalizing,

$$0 \leq \sqrt{t^2 + a^2} - |t| = \frac{a^2}{\sqrt{t^2 + a^2} + |t|} \leq \frac{a^2}{a} = a.$$

Hence

$$\rho(f_n, f) = \max_{0 \leq x \leq 1} |f_n(x) - f(x)| \leq \frac{1}{n} \longrightarrow 0.$$

So $f_n \rightarrow f$ in the metric space $C[0, 1]$; in particular $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$, and since all its terms lie in $\mathcal{D}$, it is a Cauchy sequence in the metric subspace $\mathcal{D}$.
 $f \notin \mathcal{D}$. The point $\frac{1}{2}$ lies in $(0, 1)$, and for $h \neq 0$,

$$\frac{f(\frac{1}{2} + h) - f(\frac{1}{2})}{h} = \frac{|h|}{h} = \begin{cases} 1, & h > 0, \\ -1, & h < 0, \end{cases}$$

so the difference quotient has no limit as $h \rightarrow 0$: f is not differentiable at $\frac{1}{2}$. Thus f is continuous on $[0, 1]$ but $f \notin \mathcal{D}$.

Conclusion. Suppose, for the sake of contradiction, that $\{f_n\}$ converged in the metric space $\mathcal{D}$ to some $g \in \mathcal{D}$. Then $\rho(f_n, g) \rightarrow 0$, so $f_n \rightarrow g$ in $C[0, 1]$ as well. Limits in a metric space are unique, and $f_n \rightarrow f$ in $C[0, 1]$; hence $g = f$. But $f \notin \mathcal{D}$, a contradiction. Therefore $\{f_n\}$ is a Cauchy sequence in $\mathcal{D}$ with no limit in $\mathcal{D}$, and $\mathcal{D}$ is not complete.

(Equivalently: f is a point of $C[0, 1]$ that is a limit of points of $\mathcal{D}$ but does not belong to $\mathcal{D}$, so $\mathcal{D}$ is not a closed subset of the complete space $C[0, 1]$. In fact $\mathcal{D}$ is very far from closed: it contains all polynomials, so by the Weierstrass Approximation Theorem $\mathcal{D}$ is dense in $C[0, 1]$ while being a proper subset of it.)

Problem 9.48. Define $\mathcal{L}$ to be the subspace of $C[0, 1]$ consisting of the functions $f : [0, 1] \rightarrow \mathbb{R}$ that are Lipschitz. Is $\mathcal{L}$ complete?

Solution. No: $\mathcal{L}$ is not complete.

Recall that $f : [0, 1] \rightarrow \mathbb{R}$ is Lipschitz provided there is a constant $c \geq 0$ with $|f(u) - f(v)| \leq c|u - v|$ for all $u, v \in [0, 1]$; such an f is (uniformly) continuous, so $\mathcal{L} \subseteq C[0, 1]$. As in the preceding problem, $C[0, 1]$ with $\rho(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$ is complete by Proposition 10 of this chapter, and by Proposition 11 of this chapter a metric subspace of a complete metric space is complete if and only if it is a closed subset. We exhibit a Cauchy sequence in $\mathcal{L}$ whose limit in $C[0, 1]$ is not Lipschitz.

The example. Define

$$f(x) = \sqrt{x}, \quad f_n(x) = \sqrt{x + \frac{1}{n}} \quad (0 \leq x \leq 1, n \geq 1).$$

Each f_n belongs to $\mathcal{L}$. For $u, v \in [0, 1]$, rationalizing,

$$|f_n(u) - f_n(v)| = \frac{|u - v|}{\sqrt{u + \frac{1}{n}} + \sqrt{v + \frac{1}{n}}} \leq \frac{|u - v|}{2/\sqrt{n}} = \frac{\sqrt{n}}{2} |u - v|,$$

since $u + \frac{1}{n} \geq \frac{1}{n}$ and $v + \frac{1}{n} \geq \frac{1}{n}$. So f_n is Lipschitz with constant $\sqrt{n}/2$, and f_n is continuous on $[0, 1]$; thus $f_n \in \mathcal{L}$.

$f_n \rightarrow f$ uniformly on $[0, 1]$. For $0 \leq x \leq 1$,

$$0 \leq \sqrt{x + \frac{1}{n}} - \sqrt{x} = \frac{1/n}{\sqrt{x + \frac{1}{n}} + \sqrt{x}} \leq \frac{1/n}{\sqrt{1/n}} = \frac{1}{\sqrt{n}},$$

so $\rho(f_n, f) \leq 1/\sqrt{n} \rightarrow 0$. Hence $f_n \rightarrow f$ in $C[0, 1]$, and therefore $\{f_n\}$ is a Cauchy sequence in $C[0, 1]$; since every f_n lies in $\mathcal{L}$, it is a Cauchy sequence in the metric subspace $\mathcal{L}$.

$f \notin \mathcal{L}$. Suppose $\sqrt{\cdot}$ were Lipschitz on $[0, 1]$ with constant c . Taking $v = 0$ and $u = x \in (0, 1]$ gives $\sqrt{x} \leq cx$, that is, $c \geq 1/\sqrt{x}$ for every $x \in (0, 1]$. Letting $x \rightarrow 0^+$ makes the right-hand side unbounded, which is impossible for a finite c . So $f(x) = \sqrt{x}$ is continuous on $[0, 1]$ but not Lipschitz.

Conclusion. If $\{f_n\}$ had a limit g in the metric space $\mathcal{L}$, then $f_n \rightarrow g$ in $C[0, 1]$; by uniqueness of limits in $C[0, 1]$ and $f_n \rightarrow f$, we would get $g = f$, contradicting $f \notin \mathcal{L}$. Hence $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}$ that fails to converge in $\mathcal{L}$, and $\mathcal{L}$ is not complete.

(As before, this says exactly that $\mathcal{L}$ is not a closed subset of $C[0, 1]$. Since $\mathcal{L}$ contains all polynomials, the Weierstrass Approximation Theorem shows $\mathcal{L}$ is in fact a dense proper subspace of $C[0, 1]$.)

Problem 9.49. For a metric space (X, ρ) , complete the following outline of a proof of Theorem 13:

- (i) If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X , show that $\{\rho(x_n, y_n)\}$ is a Cauchy sequence of real numbers and therefore converges.
- (ii) Define X' to be the set of Cauchy sequences in X . For two Cauchy sequences in X , $\{x_n\}$ and $\{y_n\}$, define $\rho'(\{x_n\}, \{y_n\}) = \lim \rho(x_n, y_n)$. Show that this defines a pseudometric ρ' on X' .
- (iii) Define two members of X' , that is, two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X , to be equivalent, provided $\rho'(\{x_n\}, \{y_n\}) = 0$. Show that this is an equivalence relation in X' and denote by $\hat{X}$ the set of equivalence classes. Define the distance $\hat{\rho}$ between two equivalence classes to be the ρ' distance between representatives of the classes. Show that $\hat{\rho}$ is properly defined and is a metric on $\hat{X}$.
- (iv) Show that the metric space $(\hat{X}, \hat{\rho})$ is complete. [Hint: If $\{x_n\}$ is a Cauchy sequence from X , we may assume (by taking subsequences) that $\rho(x_n, x_{n+1}) < 2^{-n}$ for all n . If $\{\{x_{n,m}\}_{n=1}^{\infty}\}_{m=1}^{\infty}$ is a sequence of such Cauchy sequences that represents a Cauchy sequence in $\hat{X}$, then the sequence $\{x_{n,n}\}_{n=1}^{\infty}$ is a Cauchy sequence from X that represents the limit of the Cauchy sequences from $\hat{X}$.]
- (v) Define the mapping h from X to $\hat{X}$ by defining, for $x \in X$, $h(x)$ to be the equivalence class of the constant sequence all of whose terms are x . Show that $h(X)$ is dense in $\hat{X}$ and that $\hat{\rho}(h(u), h(v)) = \rho(u, v)$ for all $u, v \in X$.
- (vi) Define the set $\tilde{X}$ to be the disjoint union of X and $\hat{X} \sim h(X)$. For $u, v \in \tilde{X}$, define $\tilde{\rho}(u, v)$ as follows: $\tilde{\rho}(u, v) = \rho(u, v)$ if $u, v \in X$; $\tilde{\rho}(u, v) = \hat{\rho}(u, v)$ for $u, v \in \hat{X} \sim h(X)$; and $\tilde{\rho}(u, v) = \hat{\rho}(h(u), v)$ for $u \in X, v \in \hat{X} \sim h(X)$. From the preceding two parts conclude that the metric space $(\tilde{X}, \tilde{\rho})$ is a complete metric space containing (X, ρ) as a dense subspace.

Solution. We use repeatedly the quadrilateral inequality, valid in any metric space and obtained from two applications of the triangle inequality:

$$|\rho(a, b) - \rho(c, d)| \leq \rho(a, c) + \rho(b, d). \quad (\text{Q})$$

Part (i). Let $\{x_n\}, \{y_n\}$ be Cauchy in X and let $\epsilon > 0$. Choose N so large that $\rho(x_n, x_m) < \epsilon/2$ and $\rho(y_n, y_m) < \epsilon/2$ for all $n, m \geq N$. By (Q), for $n, m \geq N$,

$$|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(x_n, x_m) + \rho(y_n, y_m) < \epsilon.$$

Thus $\{\rho(x_n, y_n)\}$ is a Cauchy sequence of real numbers, and since $\mathbb{R}$ is complete it converges.

Part (ii). By (i) the formula $\rho'(\{x_n\}, \{y_n\}) = \lim_{n \rightarrow \infty} \rho(x_n, y_n)$ assigns a real number to every ordered pair of elements of X' , and this number is nonnegative as a limit of nonnegative numbers. Symmetry is clear from the symmetry of ρ , and $\rho'(\{x_n\}, \{x_n\}) = \lim \rho(x_n, x_n) = 0$. For the triangle inequality, if $\{x_n\}, \{y_n\}, \{z_n\} \in X'$ then $\rho(x_n, z_n) \leq \rho(x_n, y_n) + \rho(y_n, z_n)$ for every n ; letting $n \rightarrow \infty$ and using that all three limits exist,

$$\rho'(\{x_n\}, \{z_n\}) \leq \rho'(\{x_n\}, \{y_n\}) + \rho'(\{y_n\}, \{z_n\}).$$

Hence ρ' is a pseudometric. It is not a metric: distinct Cauchy sequences with the same limit (for instance $\{1/n\}$ and $\{-1/n\}$ in $\mathbb{R}$) are at ρ' -distance 0.

Part (iii). Define $s \sim t$ for $s, t \in X'$ to mean $\rho'(s, t) = 0$. Reflexivity holds since $\rho'(s, s) = 0$; symmetry holds since ρ' is symmetric; and if $\rho'(s, t) = 0$ and $\rho'(t, w) = 0$ then $0 \leq \rho'(s, w) \leq \rho'(s, t) + \rho'(t, w) = 0$, giving transitivity. Let $\hat{X} = X'/\sim$ and write $[s]$ for the class of s .

Set $\hat{\rho}([s], [t]) = \rho'(s, t)$. This is properly defined: if $s \sim s'$ and $t \sim t'$, then by the pseudometric triangle inequality used twice,

$$|\rho'(s, t) - \rho'(s', t')| \leq \rho'(s, s') + \rho'(t, t') = 0,$$

so $\rho'(s, t) = \rho'(s', t')$. (The displayed inequality is the analogue of (Q) for a pseudometric, proved the same way.) Nonnegativity, symmetry, and the triangle inequality for $\hat{\rho}$ are inherited from ρ' . Finally $\hat{\rho}([s], [t]) = 0$ means $\rho'(s, t) = 0$, that is $s \sim t$, that is $[s] = [t]$. So $\hat{\rho}$ is a metric on $\hat{X}$.

Part (iv). Call a Cauchy sequence $\{x_n\}$ in X rapid provided $\rho(x_n, x_{n+1}) < 2^{-n}$ for all n .

Every class in $\widehat{X}$ has a rapid representative. Indeed, let $\{x_n\}$ be Cauchy. Choose indices $n_1 < n_2 < \dots$ with $n_k \geq k$ and $\rho(x_p, x_q) < 2^{-k}$ for all $p, q \geq n_k$. Then $\{x_{n_k}\}_k$ is rapid, and

$$\rho'(\{x_n\}, \{x_{n_k}\}) = \lim_{k \rightarrow \infty} \rho(x_k, x_{n_k}) = 0$$

because $k, n_k \geq k$ and $\{x_n\}$ is Cauchy. So $\{x_{n_k}\} \sim \{x_n\}$.

If $\{x_n\}$ is rapid then for $m \geq n$,

$$\rho(x_n, x_m) \leq \sum_{j=n}^{m-1} \rho(x_j, x_{j+1}) < \sum_{j=n}^{\infty} 2^{-j} = 2^{1-n}. \quad (1)$$

Consequently, if $s = \{x_n\}$ and $t = \{y_n\}$ are both rapid, then by (Q) and (1), for $m \geq n$,

$$|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(x_n, x_m) + \rho(y_n, y_m) < 2^{2-n};$$

letting $m \rightarrow \infty$ gives the key estimate

$$|\rho(x_n, y_n) - \widehat{\rho}([s], [t])| \leq 2^{2-n} \quad \text{for all } n. \quad (2)$$

Now let $\{\xi_m\}$ be a Cauchy sequence in $(\widehat{X}, \widehat{\rho})$, and for each m choose a rapid representative $\{x_{n,m}\}_{n=1}^{\infty}$ of ξ_m . Put $z_n = x_{n,n}$.

The diagonal sequence $\{z_n\}$ is Cauchy in X . Fix n, m and let $k \geq \max\{n, m\}$. Using the triangle inequality, then (1) twice and (2) once,

$$\rho(z_n, z_m) \leq \rho(x_{n,n}, x_{k,n}) + \rho(x_{k,n}, x_{k,m}) + \rho(x_{k,m}, x_{m,m}) < 2^{1-n} + [\widehat{\rho}(\xi_n, \xi_m) + 2^{2-k}] + 2^{1-m}.$$

Since this holds for every $k \geq \max\{n, m\}$, letting $k \rightarrow \infty$ yields

$$\rho(z_n, z_m) \leq 2^{1-n} + 2^{1-m} + \widehat{\rho}(\xi_n, \xi_m). \quad (3)$$

Given $\epsilon > 0$, choose N with $\widehat{\rho}(\xi_n, \xi_m) < \epsilon/2$ for $n, m \geq N$ and with $2^{1-N} < \epsilon/4$; then $\rho(z_n, z_m) < \epsilon$ for $n, m \geq N$. So $\{z_n\} \in X'$; let $\zeta = [\{z_n\}] \in \widehat{X}$.

$\{\xi_m\}$ converges to ζ . By definition,

$$\widehat{\rho}(\xi_m, \zeta) = \rho'(\{x_{n,m}\}_n, \{z_n\}_n) = \lim_{n \rightarrow \infty} \rho(x_{n,m}, x_{n,n}).$$

Let $\epsilon > 0$ and choose N with $\widehat{\rho}(\xi_m, \xi_n) < \epsilon/2$ for all $n, m \geq N$ and with $2^{2-N} < \epsilon/2$. For $m \geq N$ and $n \geq N$, the sequences $\{x_{\cdot, m}\}$ and $\{x_{\cdot, n}\}$ are rapid, so (2) gives

$$\rho(x_{n,m}, x_{n,n}) \leq \widehat{\rho}(\xi_m, \xi_n) + 2^{2-n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Letting $n \rightarrow \infty$ with $m \geq N$ fixed gives $\widehat{\rho}(\xi_m, \zeta) \leq \epsilon$. Hence $\widehat{\rho}(\xi_m, \zeta) \rightarrow 0$, that is $\xi_m \rightarrow \zeta$ in $\widehat{X}$. Therefore $(\widehat{X}, \widehat{\rho})$ is complete.

Part (v). For $x \in X$ the constant sequence $c_x = (x, x, x, \dots)$ is Cauchy, so $h(x) = [c_x]$ is a well-defined element of $\widehat{X}$. For $u, v \in X$,

$$\widehat{\rho}(h(u), h(v)) = \rho'(c_u, c_v) = \lim_{n \rightarrow \infty} \rho(u, v) = \rho(u, v),$$

so h is an isometry of X into $\widehat{X}$; in particular h is one-to-one.

To see $h(X)$ is dense in $\widehat{X}$, let $\xi \in \widehat{X}$, choose a representative Cauchy sequence $\{x_n\}$, and let $r > 0$. Choose N with $\rho(x_n, x_m) < r/2$ for all $n, m \geq N$. Then

$$\widehat{\rho}(h(x_N), \xi) = \rho'(c_{x_N}, \{x_n\}) = \lim_{m \rightarrow \infty} \rho(x_N, x_m) \leq \frac{r}{2} < r,$$

the limit existing by (i) and being at most $r/2$ because $\rho(x_N, x_m) < r/2$ for every $m \geq N$. So the open ball of radius r about ξ contains the point $h(x_N)$ of $h(X)$. Since $\xi \in \widehat{X}$ and $r > 0$ were arbitrary, every nonempty open subset of $\widehat{X}$ contains a point of $h(X)$; that is, $h(X)$ is dense in $\widehat{X}$.

Part (vi). Let $\tilde{X}$ be the disjoint union of X and $\hat{X} \sim h(X)$, with $\tilde{\rho}$ defined as in the statement (and symmetrically in the mixed case, $\tilde{\rho}(v, u) = \hat{\rho}(h(u), v)$ for $u \in X, v \in \hat{X} \sim h(X)$). Define

$$H : \tilde{X} \rightarrow \hat{X}, \quad H(u) = h(u) \quad (u \in X), \quad H(v) = v \quad (v \in \hat{X} \sim h(X)).$$

Since h maps X one-to-one onto $h(X)$ and the identity maps $\hat{X} \sim h(X)$ onto itself, and since $\hat{X}$ is the disjoint union of $h(X)$ and $\hat{X} \sim h(X)$, the map H is a one-to-one mapping of $\tilde{X}$ onto $\hat{X}$.

Moreover $\tilde{\rho}(u, v) = \hat{\rho}(H(u), H(v))$ for all $u, v \in \tilde{X}$. Indeed, there are three cases, and each is exactly one clause of the definition of $\tilde{\rho}$: if $u, v \in X$ then $\hat{\rho}(H(u), H(v)) = \hat{\rho}(h(u), h(v)) = \rho(u, v)$ by part (v); if $u, v \in \hat{X} \sim h(X)$ then $\hat{\rho}(H(u), H(v)) = \hat{\rho}(u, v)$; and if $u \in X, v \in \hat{X} \sim h(X)$ then $\hat{\rho}(H(u), H(v)) = \hat{\rho}(h(u), v)$.

Consequently $\tilde{\rho}$ is a metric on $\tilde{X}$: nonnegativity, symmetry and the triangle inequality transfer immediately from $\hat{\rho}$ through the bijection H , and $\tilde{\rho}(u, v) = 0$ forces $H(u) = H(v)$, hence $u = v$ since H is one-to-one. Thus H is an isometry of $(\tilde{X}, \tilde{\rho})$ onto $(\hat{X}, \hat{\rho})$.

An isometry onto carries Cauchy sequences to Cauchy sequences and convergent sequences to convergent sequences, in both directions; therefore $(\tilde{X}, \tilde{\rho})$ is complete because $(\hat{X}, \hat{\rho})$ is complete by part (iv). Explicitly, if $\{u_k\}$ is Cauchy in $\tilde{X}$ then $\{H(u_k)\}$ is Cauchy in $\hat{X}$, so $H(u_k) \rightarrow \xi$ for some $\xi \in \hat{X}$; writing $u = H^{-1}(\xi)$ we get $\tilde{\rho}(u_k, u) = \hat{\rho}(H(u_k), \xi) \rightarrow 0$.

By the first clause of the definition, $\tilde{\rho}(u, v) = \rho(u, v)$ for all $u, v \in X$, so (X, ρ) is a metric subspace of $(\tilde{X}, \tilde{\rho})$. Finally X is dense in $\tilde{X}$: let $u \in \tilde{X}$ and $\epsilon > 0$. Since $h(X) = H(X)$ is dense in $\hat{X}$ by part (v), there is $x \in X$ with $\hat{\rho}(h(x), H(u)) < \epsilon$, and then $\tilde{\rho}(x, u) = \hat{\rho}(H(x), H(u)) < \epsilon$. So every open ball in $\tilde{X}$ meets X .

Therefore $(\tilde{X}, \tilde{\rho})$ is a complete metric space, X is a dense subset of $\tilde{X}$, and $\rho(u, v) = \tilde{\rho}(u, v)$ for all $u, v \in X$. This is precisely the assertion of Theorem 13.

9.8 Problems 9.50–9.56

Problem 9.50. Show that any two completions of a metric space X are isometric by way of an isometry that is the identity mapping on X .

Solution. Let (X, ρ) be a metric space and let $(\tilde{X}_1, \tilde{\rho}_1)$ and $(\tilde{X}_2, \tilde{\rho}_2)$ be two completions of (X, ρ) . By the definition following Theorem 13, this means: each $\tilde{X}_i$ is a complete metric space, X is a dense subset of $\tilde{X}_i$, and

$$\tilde{\rho}_i(u, v) = \rho(u, v) \quad \text{for all } u, v \in X, \quad i = 1, 2.$$

We construct an isometry f of $\tilde{X}_1$ onto $\tilde{X}_2$ that fixes each point of X .

We use twice the following elementary consequence of the triangle inequality, valid in any metric space (Y, σ) : if $y_n \rightarrow y$ and $z_n \rightarrow z$ in Y , then

$$|\sigma(y_n, z_n) - \sigma(y, z)| \leq \sigma(y_n, y) + \sigma(z_n, z),$$

so that $\lim_{n \rightarrow \infty} \sigma(y_n, z_n) = \sigma(y, z)$. Indeed, $\sigma(y_n, z_n) \leq \sigma(y_n, y) + \sigma(y, z) + \sigma(z, z_n)$ and symmetrically $\sigma(y, z) \leq \sigma(y, y_n) + \sigma(y_n, z_n) + \sigma(z_n, z)$.

Definition of f . Let $u \in \tilde{X}_1$. Since X is dense in $\tilde{X}_1$, there is a sequence $\{u_n\}$ in X with $u_n \rightarrow u$ in $\tilde{X}_1$. A convergent sequence is Cauchy (Problem 37), so $\{u_n\}$ is Cauchy with respect to $\tilde{\rho}_1$. But the terms u_n lie in X , and on X the three metrics $\tilde{\rho}_1, \rho, \tilde{\rho}_2$ agree. Hence $\{u_n\}$ is a Cauchy sequence in the metric space $\tilde{X}_2$, and since $\tilde{X}_2$ is complete it converges to a point of $\tilde{X}_2$. Define

$$f(u) = \lim_{n \rightarrow \infty} u_n \quad (\text{limit taken in } \tilde{X}_2).$$

This is properly defined. Suppose $\{u_n\}$ and $\{u'_n\}$ are two sequences in X that converge to u in $\tilde{X}_1$, with $\tilde{X}_2$ -limits w and w' respectively. Then

$$\rho(u_n, u'_n) = \tilde{\rho}_1(u_n, u'_n) \leq \tilde{\rho}_1(u_n, u) + \tilde{\rho}_1(u, u'_n) \rightarrow 0,$$

while by the observation above, applied in $\tilde{X}_2$,

$$\tilde{\rho}_2(w, w') = \lim_{n \rightarrow \infty} \tilde{\rho}_2(u_n, u'_n) = \lim_{n \rightarrow \infty} \rho(u_n, u'_n) = 0.$$

Therefore $w = w'$.

f is the identity on X . If $u \in X$, we may take $u_n = u$ for all n , since this sequence lies in X and converges to u in $\tilde{X}_1$. Hence $f(u) = u$.

f preserves distances. Let $u, v \in \tilde{X}_1$ and choose sequences $\{u_n\}, \{v_n\}$ in X with $u_n \rightarrow u$ and $v_n \rightarrow v$ in $\tilde{X}_1$. By the definition of f , $u_n \rightarrow f(u)$ and $v_n \rightarrow f(v)$ in $\tilde{X}_2$. Applying the observation above in $\tilde{X}_2$ and then in $\tilde{X}_1$, and using that the metrics agree on X ,

$$\tilde{\rho}_2(f(u), f(v)) = \lim_{n \rightarrow \infty} \tilde{\rho}_2(u_n, v_n) = \lim_{n \rightarrow \infty} \rho(u_n, v_n) = \lim_{n \rightarrow \infty} \tilde{\rho}_1(u_n, v_n) = \tilde{\rho}_1(u, v).$$

In particular f is one-to-one: if $f(u) = f(v)$ then $\tilde{\rho}_1(u, v) = \tilde{\rho}_2(f(u), f(v)) = 0$, so $u = v$.

f maps $\tilde{X}_1$ onto $\tilde{X}_2$. Let $w \in \tilde{X}_2$. Since X is dense in $\tilde{X}_2$, there is a sequence $\{w_n\}$ in X with $w_n \rightarrow w$ in $\tilde{X}_2$. As above, $\{w_n\}$ is Cauchy with respect to $\tilde{\rho}_2$, hence, the terms lying in X , Cauchy with respect to ρ and therefore with respect to $\tilde{\rho}_1$. Since $\tilde{X}_1$ is complete, $\{w_n\}$ converges in $\tilde{X}_1$ to some point u . The sequence $\{w_n\}$ then lies in X and converges to u in $\tilde{X}_1$, so by the definition of f ,

$$f(u) = \lim_{n \rightarrow \infty} w_n = w \quad (\text{limit in } \tilde{X}_2),$$

the last equality because limits in the metric space $\tilde{X}_2$ are unique. Hence f is onto.

Thus f maps $\tilde{X}_1$ onto $\tilde{X}_2$ and satisfies $\tilde{\rho}_2(f(u), f(v)) = \tilde{\rho}_1(u, v)$ for all $u, v \in \tilde{X}_1$; that is, f is an isometry, and it is the identity mapping on X . Any two completions of X are therefore isometric by way of an isometry that is the identity on X .

Finally, such an isometry is unique. If g is any continuous mapping of $\tilde{X}_1$ into $\tilde{X}_2$ with $g = f$ on X , then for $u \in \tilde{X}_1$ and $\{u_n\}$ a sequence in the dense set X converging to u , continuity gives $g(u) = \lim_n g(u_n) = \lim_n f(u_n) = f(u)$. Since an isometry is certainly continuous, f is the only isometry of $\tilde{X}_1$ onto $\tilde{X}_2$ that is the identity on X .

Problem 9.51. Consider the metric space $\mathbb{Q}$ consisting of the rational numbers with the metric induced by the absolute value. Which subspaces of $\mathbb{Q}$ are complete and which are compact?

Solution. Throughout, E denotes a subset of $\mathbb{Q}$, regarded as a metric space with the metric $\rho(p, q) = |p - q|$, and $\overline{E}$ denotes the closure of E in $\mathbb{R}$. The answer is:

E is a complete metric space if and only if E is a closed subset of $\mathbb{R}$; and E is a compact metric space if and only if E is a closed and bounded subset of $\mathbb{R}$, that is, if and only if E is a compact subset of $\mathbb{R}$.

Completeness. Suppose first that E is closed as a subset of $\mathbb{R}$. Let $\{q_n\}$ be a Cauchy sequence in the metric space E . Since the metric on E is the restriction of the metric on $\mathbb{R}$, $\{q_n\}$ is a Cauchy sequence of real numbers, and $\mathbb{R}$ is complete, so $q_n \rightarrow x$ for some $x \in \mathbb{R}$. Every ball about x in $\mathbb{R}$ contains points of E , namely tails of $\{q_n\}$, so $x \in \overline{E} = E$. Since the metric on E is the restriction of that on $\mathbb{R}$, $\{q_n\}$ converges to x in the metric space E . Hence E is complete.

Conversely, suppose the metric space E is complete and let $x \in \overline{E}$. By the characterization of closure points by sequences (Proposition 6 of this chapter, applied in $\mathbb{R}$), there is a sequence $\{q_n\}$ in E with $q_n \rightarrow x$ in $\mathbb{R}$. A convergent sequence is Cauchy (Problem 37), so $\{q_n\}$ is Cauchy in $\mathbb{R}$ and hence in E . By completeness of E , $q_n \rightarrow q$ for some $q \in E$; this convergence also holds in $\mathbb{R}$. Limits in $\mathbb{R}$ are unique, so $x = q \in E$. Thus $\overline{E} = E$, that is, E is closed in $\mathbb{R}$.

Compactness. Total boundedness of E is an intrinsic property of the metric space E and does not refer to the ambient space. By Proposition 15, a subset of Euclidean space $\mathbb{R}^n$ is bounded if and only if it is totally bounded; taking $n = 1$, the subset E of $\mathbb{R}$ is totally bounded if and only if it is bounded. By the Characterization of Compactness Theorem (Theorem 16), E is compact if and only if E is complete and totally bounded, hence, by the previous paragraph, if and only if E is a closed and bounded subset of $\mathbb{R}$. By the Heine–Borel Theorem this is exactly the statement that E is a compact subset of $\mathbb{R}$.

(Consistently, a subset of $\mathbb{Q}$ that is closed and bounded in $\mathbb{R}$ contains all of its real points of closure, and these therefore are rational.)

Which sets these are. The description above is intrinsic once one notes what it forbids. Claim: no nonempty subset of $\mathbb{Q}$ that is open in $\mathbb{Q}$ is complete. Indeed, let $U \subseteq \mathbb{Q}$ be open in $\mathbb{Q}$ and nonempty, and let $q \in U$. There is an $r > 0$ with $\{p \in \mathbb{Q} : |p - q| < r\} \subseteq U$. Choose an irrational number ξ with $|\xi - q| < r/2$, which is possible since the irrationals are dense in $\mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, choose rationals p_n with $|p_n - \xi| < \min\{1/n, r/2\}$. Then $|p_n - q| < r$, so $\{p_n\}$ is a sequence in U ; it converges in $\mathbb{R}$ to ξ and hence is Cauchy. If it converged to a point $p \in U \subseteq \mathbb{Q}$, then by uniqueness of limits in $\mathbb{R}$ we would get $p = \xi$, contradicting the irrationality of ξ . So U is not complete.

Consequently every complete subspace E of $\mathbb{Q}$ has empty interior in $\mathbb{Q}$. For suppose E contained a nonempty subset U that is open in $\mathbb{Q}$. By the first part of the solution E is closed in $\mathbb{R}$. The sequence $\{p_n\}$ constructed in the preceding paragraph from U lies in $U \subseteq E$ and converges in $\mathbb{R}$ to the irrational number ξ , so $\xi \in \overline{E} = E \subseteq \mathbb{Q}$, a contradiction. Being closed in $\mathbb{R}$, such an E is also closed in $\mathbb{Q}$, so its closure in $\mathbb{Q}$ is E itself; having empty interior in $\mathbb{Q}$, each complete subspace of $\mathbb{Q}$ is therefore nowhere dense in $\mathbb{Q}$. In particular $\mathbb{Q}$ itself is neither complete nor compact, since $\mathbb{Q}$ is not a closed subset of $\mathbb{R}$, and no ball in $\mathbb{Q}$ is complete or compact, a ball being nonempty and open in $\mathbb{Q}$.

Indeed no point of $\mathbb{Q}$ has a compact neighborhood. Suppose $K \subseteq \mathbb{Q}$ were a neighborhood of $q \in \mathbb{Q}$ that is compact as a metric subspace of $\mathbb{Q}$, say $\{p \in \mathbb{Q} : |p - q| < r\} \subseteq K$ with $r > 0$. By Theorem 16 a compact metric space is complete, so by the first part of this solution K is a closed subset of $\mathbb{R}$. Choose an irrational ξ with $|\xi - q| < r/2$ and rationals p_n with $|p_n - \xi| < \min\{1/n, r/2\}$, as in the preceding paragraph. Then $|p_n - q| < r$, so each p_n lies in K , and $p_n \rightarrow \xi$ in $\mathbb{R}$, so ξ is a point of closure of K in $\mathbb{R}$ and hence $\xi \in K \subseteq \mathbb{Q}$. This contradicts the irrationality of ξ . Thus $\mathbb{Q}$ is nowhere locally compact.

Examples. Every finite subset of $\mathbb{Q}$ is compact. The set $\mathbb{Z}$ is closed and unbounded in $\mathbb{R}$, hence a complete but noncompact subspace of $\mathbb{Q}$. The set $E = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is a closed and bounded subset of $\mathbb{R}$ consisting of rationals, hence an infinite compact subspace of $\mathbb{Q}$. The set $E = \mathbb{Q} \cap [0, 1]$ is bounded but not closed in $\mathbb{R}$, hence totally bounded but neither complete nor compact.

Problem 9.52. Let $B = B(x, r)$ be an open ball in Euclidean space $\mathbb{R}^n$. Show that B fails to be compact by (i) showing B is not sequentially compact, (ii) finding an open cover of B without any finite subcover, and (iii) showing B is not closed.

Solution. Here $r > 0$ and $B = B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$, regarded as a metric subspace of $\mathbb{R}^n$ with the Euclidean metric. Fix the unit vector $e = (1, 0, \dots, 0)$, so $\|e\| = 1$, and set

$$z = x + re, \quad y_k = x + \left(1 - \frac{1}{k}\right) re \quad \text{for } k \in \mathbb{N}.$$

Then $\|y_k - x\| = (1 - 1/k)r < r$, so each y_k belongs to B , while $\|z - x\| = r$, so z does not belong to B . Moreover

$$\|y_k - z\| = \frac{r}{k} \rightarrow 0,$$

so $\{y_k\}$ converges to z in $\mathbb{R}^n$.

(i) B is not sequentially compact. Consider the sequence $\{y_k\}$ in B just defined. Let $\{y_{k_j}\}$ be any subsequence. Since $\{y_k\}$ converges to z in $\mathbb{R}^n$, so does every subsequence. Suppose $\{y_{k_j}\}$ converged, in the metric space B , to a point $w \in B$. The metric on B is the restriction of the Euclidean metric, so $\{y_{k_j}\}$ would also converge to w in $\mathbb{R}^n$. Limits in a metric space are unique, so $w = z$; but $z \notin B$, a contradiction. Hence no subsequence of $\{y_k\}$ converges to a point of B , and B is not sequentially compact. By the Characterization of Compactness Theorem (Theorem 16), B is not compact.

(ii) An open cover of B with no finite subcover. For each natural number k put

$$\mathcal{O}_k = B\left(x, \left(1 - \frac{1}{k}\right)r\right),$$

the open ball in $\mathbb{R}^n$ of radius $(1 - 1/k)r$ about x . Each $\mathcal{O}_k$ is an open subset of $\mathbb{R}^n$ contained in B , hence $\mathcal{O}_k = \mathcal{O}_k \cap B$ is an open subset of the metric subspace B (Proposition 2). The collection $\{\mathcal{O}_k\}_{k=1}^\infty$

covers B : if $y \in B$ then $\|y - x\| < r$, so, choosing by the Archimedean property a natural number k with $1/k < 1 - \|y - x\|/r$, we get $\|y - x\| < (1 - 1/k)r$ and thus $y \in \mathcal{O}_k$. The collection is ascending, $\mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \dots$, so any finite subcollection $\mathcal{O}_{k_1}, \dots, \mathcal{O}_{k_m}$ has union $\mathcal{O}_K$, where $K = \max\{k_1, \dots, k_m\}$. But the point

$$u = x + \left(1 - \frac{1}{2K}\right)re$$

satisfies $\|u - x\| = (1 - 1/(2K))r < r$, so $u \in B$, while $\|u - x\| = (1 - 1/(2K))r > (1 - 1/K)r$, so $u \notin \mathcal{O}_K$. Hence no finite subcollection covers B , and B is not compact.

(iii) B is not closed, and a compact subspace must be closed. The point $z = x + re$ is the limit of the sequence $\{y_k\}$ of points of B , so by Proposition 6 it is a point of closure of B in $\mathbb{R}^n$; since $z \notin B$, the set B is not a closed subset of $\mathbb{R}^n$.

To conclude noncompactness, we check that a compact subspace K of a metric space X is a closed subset of X . Let w be a point of closure of K in X . By Proposition 6 there is a sequence $\{w_k\}$ in K with $w_k \rightarrow w$ in X . By Theorem 16, K is sequentially compact, so some subsequence $\{w_{k_j}\}$ converges to a point $w' \in K$; this convergence holds in X as well, since the metric on K is the restriction of that on X . But $\{w_{k_j}\}$, being a subsequence of $\{w_k\}$, also converges to w in X . By uniqueness of limits, $w = w' \in K$. Hence K is closed in X .

Since B is not a closed subset of $\mathbb{R}^n$, the metric subspace B is not compact.

Problem 9.53. When is a nonempty set X with the discrete metric a compact metric space?

Solution. Recall the discrete metric on a nonempty set X : $\rho(u, v) = 0$ if $u = v$, and $\rho(u, v) = 1$ if $u \neq v$. The answer is: (X, ρ) is compact if and only if X is a finite set.

Suppose first that X is finite, say $X = \{x_1, \dots, x_m\}$. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X . For each index i with $1 \leq i \leq m$ choose $\lambda_i \in \Lambda$ with $x_i \in \mathcal{O}_{\lambda_i}$. Then $\{\mathcal{O}_{\lambda_i}\}_{i=1}^m$ is a finite subcollection of the cover whose union contains every point of X , hence a finite subcover. Therefore X is compact. (This argument uses nothing about the metric: a finite metric space is always compact.)

Now suppose X is infinite; we show X is not compact. For each $u \in X$ we have, by the definition of ρ ,

$$B(u, 1/2) = \{v \in X : \rho(u, v) < 1/2\} = \{u\},$$

so every singleton, and hence every subset of X , is open. Thus $\{\{u\}\}_{u \in X}$ is an open cover of X . A finite subcollection $\{u_1\}, \dots, \{u_m\}$ has union the finite set $\{u_1, \dots, u_m\}$, which is a proper subset of the infinite set X and therefore fails to cover X . So this open cover admits no finite subcover, and X is not compact.

The same conclusion follows from the Characterization of Compactness Theorem (Theorem 16), and this route explains where the failure lies. The discrete metric space X is always complete: if $\{x_n\}$ is Cauchy, then choosing N so that $\rho(x_n, x_m) < 1/2$ for $n, m \geq N$ forces $x_n = x_m$ for all $n, m \geq N$, so $\{x_n\}$ is eventually equal to the constant x_N and hence converges to $x_N \in X$. So by Theorem 16, X is compact if and only if it is totally bounded. Taking $\epsilon = 1/2$ in the definition of total boundedness, X is covered by finitely many balls of radius $1/2$, each of which is a singleton by the computation above; hence X is finite. Conversely a finite metric space is trivially totally bounded, being the union of the finitely many balls $B(x, \epsilon)$, $x \in X$, for any $\epsilon > 0$.

Thus a nonempty set with the discrete metric is a compact metric space precisely when it is finite.

Problem 9.54. Let ρ and σ be equivalent metrics on a nonempty set X . Show that the metric space (X, ρ) is compact if and only if the metric space (X, σ) is compact.

Solution. By the definition of equivalence of metrics there are positive numbers c_1 and c_2 for which

$$c_1 \cdot \sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \cdot \sigma(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

The key point is that the two metrics determine the same open sets; this is Proposition 7, whose short

proof we include for completeness. Write $B_\rho(x, \epsilon)$ and $B_\sigma(x, \epsilon)$ for the balls with respect to ρ and σ . Let $x \in X$ and $\epsilon > 0$. If $\sigma(x, y) < \epsilon/c_2$, then $\rho(x, y) \leq c_2 \cdot \sigma(x, y) < \epsilon$; hence

$$B_\sigma(x, \epsilon/c_2) \subseteq B_\rho(x, \epsilon).$$

If $\rho(x, y) < c_1\epsilon$, then $c_1 \cdot \sigma(x, y) \leq \rho(x, y) < c_1\epsilon$, so $\sigma(x, y) < \epsilon$; hence

$$B_\rho(x, c_1\epsilon) \subseteq B_\sigma(x, \epsilon).$$

Now let $\mathcal{O}$ be open in (X, ρ) and let $x \in \mathcal{O}$. There is an $\epsilon > 0$ with $B_\rho(x, \epsilon) \subseteq \mathcal{O}$, and then $B_\sigma(x, \epsilon/c_2) \subseteq B_\rho(x, \epsilon) \subseteq \mathcal{O}$. Thus $\mathcal{O}$ is open in (X, σ) . Interchanging the roles of the two metrics, using the second inclusion in place of the first, shows that every set open in (X, σ) is open in (X, ρ) . Hence the metric spaces (X, ρ) and (X, σ) have exactly the same open sets.

Suppose now that (X, ρ) is compact, and let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be a cover of X by sets open in (X, σ) . By the preceding paragraph each $\mathcal{O}_\lambda$ is open in (X, ρ) , so $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is an open cover of the compact space (X, ρ) and therefore has a finite subcollection whose union is X . That same finite subcollection is a finite subcover of the original cover. Hence (X, σ) is compact. By symmetry, if (X, σ) is compact then so is (X, ρ) .

Therefore (X, ρ) is compact if and only if (X, σ) is compact.

Alternatively, one may argue through Theorem 16. The displayed inequalities show that $\rho(x_n, x) \rightarrow 0$ if and only if $\sigma(x_n, x) \rightarrow 0$, so the two metrics have the same convergent sequences with the same limits; hence a sequence has a subsequence converging in (X, ρ) if and only if it has one converging in (X, σ) . So (X, ρ) is sequentially compact if and only if (X, σ) is, and by the Characterization of Compactness Theorem the same equivalence holds for compactness.

Problem 9.55. Show that the Cartesian product of two compact metric spaces also is compact.

Solution. Let (X_1, ρ_1) and (X_2, ρ_2) be compact metric spaces and let τ be the product metric on the Cartesian product $X_1 \times X_2$, defined in Section 9.1 by

$$\tau((x_1, x_2), (y_1, y_2)) = \{[\rho_1(x_1, y_1)]^2 + [\rho_2(x_2, y_2)]^2\}^{1/2}.$$

We first record how convergence in $(X_1 \times X_2, \tau)$ relates to convergence in the factors. From the definition of τ we have, for all points (x_1, x_2) and (y_1, y_2) of the product,

$$\max\{\rho_1(x_1, y_1), \rho_2(x_2, y_2)\} \leq \tau((x_1, x_2), (y_1, y_2)) \leq \rho_1(x_1, y_1) + \rho_2(x_2, y_2),$$

the left inequality because each summand under the square root is dominated by the sum, and the right inequality because $a^2 + b^2 \leq (a + b)^2$ for $a, b \geq 0$. Consequently a sequence $\{(x_n, y_n)\}$ in $X_1 \times X_2$ converges to (x, y) with respect to τ if and only if $\{x_n\}$ converges to x in (X_1, ρ_1) and $\{y_n\}$ converges to y in (X_2, ρ_2) .

We show that $X_1 \times X_2$ is sequentially compact; by the Characterization of Compactness Theorem (Theorem 16) this is equivalent to compactness.

Let $\{(x_n, y_n)\}_{n=1}^\infty$ be a sequence in $X_1 \times X_2$. Since X_1 is compact, it is sequentially compact by Theorem 16. Applying this to the sequence $\{x_n\}$ in X_1 , there is a strictly increasing sequence of indices $\{n_k\}$ and a point $x \in X_1$ for which

$$\lim_{k \rightarrow \infty} \rho_1(x_{n_k}, x) = 0.$$

Now consider the sequence $\{y_{n_k}\}_{k=1}^\infty$ in X_2 . Since X_2 is compact, it too is sequentially compact by Theorem 16, so there is a strictly increasing sequence of indices $\{k_j\}$ and a point $y \in X_2$ for which

$$\lim_{j \rightarrow \infty} \rho_2(y_{n_{k_j}}, y) = 0.$$

Set $m_j = n_{k_j}$. Then $\{m_j\}$ is strictly increasing, so $\{(x_{m_j}, y_{m_j})\}$ is a subsequence of $\{(x_n, y_n)\}$. Moreover $\{x_{m_j}\}$ is a subsequence of the convergent sequence $\{x_{n_k}\}$ and hence also converges to x , while $\{y_{m_j}\}$

converges to y by construction. By the equivalence recorded above,

$$\lim_{j \rightarrow \infty} \tau((x_{m_j}, y_{m_j}), (x, y)) = 0,$$

that is, the subsequence $\{(x_{m_j}, y_{m_j})\}$ converges in $X_1 \times X_2$ to the point (x, y) of $X_1 \times X_2$.

Thus every sequence in $X_1 \times X_2$ has a subsequence converging to a point of $X_1 \times X_2$, so $X_1 \times X_2$ is sequentially compact. By Theorem 16, $(X_1 \times X_2, \tau)$ is compact.

Problem 9.56. Show that the Cartesian product of two totally bounded metric spaces also is totally bounded.

Solution. Let (X_1, ρ_1) and (X_2, ρ_2) be totally bounded metric spaces and let τ be the product metric on $X_1 \times X_2$,

$$\tau((x_1, x_2), (y_1, y_2)) = \{[\rho_1(x_1, y_1)]^2 + [\rho_2(x_2, y_2)]^2\}^{1/2}.$$

We must show that for each $\epsilon > 0$ the space $X_1 \times X_2$ can be covered by a finite number of open balls, with respect to τ , of radius ϵ .

Let $\epsilon > 0$ and put $\delta = \epsilon/\sqrt{2} > 0$. Since X_1 is totally bounded, there are points $u_1, \dots, u_m$ in X_1 with

$$X_1 = \bigcup_{i=1}^m B_{\rho_1}(u_i, \delta),$$

and since X_2 is totally bounded, there are points $v_1, \dots, v_n$ in X_2 with

$$X_2 = \bigcup_{j=1}^n B_{\rho_2}(v_j, \delta).$$

(If either factor is empty the product is empty and there is nothing to prove, so we may assume both are nonempty and hence that $m, n \geq 1$.)

Consider the finite collection of $m \cdot n$ open balls in $(X_1 \times X_2, \tau)$,

$$\{B_\tau((u_i, v_j), \epsilon) : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

We claim these cover $X_1 \times X_2$. Let $(x_1, x_2) \in X_1 \times X_2$. By the first display there is an index i with $\rho_1(x_1, u_i) < \delta$, and by the second there is an index j with $\rho_2(x_2, v_j) < \delta$. Therefore

$$\tau((x_1, x_2), (u_i, v_j)) = \{[\rho_1(x_1, u_i)]^2 + [\rho_2(x_2, v_j)]^2\}^{1/2} < \{\delta^2 + \delta^2\}^{1/2} = \delta\sqrt{2} = \epsilon,$$

so $(x_1, x_2) \in B_\tau((u_i, v_j), \epsilon)$.

Hence for each $\epsilon > 0$ the space $X_1 \times X_2$ is covered by a finite number of open balls of radius ϵ ; that is, $(X_1 \times X_2, \tau)$ is totally bounded.

Note in passing that this gives a second proof of the preceding problem. If X_1 and X_2 are compact, then by the Characterization of Compactness Theorem (Theorem 16) each is complete and totally bounded. The product of two complete metric spaces is complete (Problem 42), and by what we have just shown the product is totally bounded; so by Theorem 16 again the product is compact.

9.9 Problems 9.57–9.63

Problem 9.57. For E contained in a metric space X , show that the subspace E is totally bounded if and only if for each $\epsilon > 0$, E can be covered by a finite number of open balls (open in X) of radius ϵ which have centers belonging to X .

Solution. Throughout, ρ denotes the metric on X , and for $x \in X$ and $r > 0$ we write

$$B(x, r) = \{y \in X \mid \rho(x, y) < r\}$$

for the open ball in X . The metric subspace E carries the restricted metric, so the open ball in E of radius r about a point $y \in E$ is

$$B_E(y, r) = \{z \in E \mid \rho(y, z) < r\} = B(y, r) \cap E.$$

By the definition of total boundedness, the subspace E is totally bounded provided that for each $\epsilon > 0$ there are finitely many points $y_1, \dots, y_n$ belonging to E itself, with $E = \bigcup_{k=1}^n B_E(y_k, \epsilon)$. What is to be shown is that the a priori weaker requirement, in which the centers are merely required to lie in the ambient space X , is equivalent to this.

Assume first that the subspace E is totally bounded, and let $\epsilon > 0$. Choose $y_1, \dots, y_n$ in E with $E = \bigcup_{k=1}^n B_E(y_k, \epsilon)$. Since $B_E(y_k, \epsilon) \subseteq B(y_k, \epsilon)$ we have

$$E \subseteq \bigcup_{k=1}^n B(y_k, \epsilon),$$

a cover of E by finitely many open balls of X of radius ϵ whose centers y_k belong to E and hence to X . This is the asserted covering property.

Now assume conversely that for each $\epsilon > 0$ the set E can be covered by finitely many open balls of X of radius ϵ with centers in X . Let $\epsilon > 0$ be given and apply the hypothesis with $\epsilon/2$ in place of ϵ : there are points $x_1, \dots, x_n$ in X for which

$$E \subseteq \bigcup_{k=1}^n B(x_k, \epsilon/2).$$

Discard those indices k for which $B(x_k, \epsilon/2) \cap E = \emptyset$; the remaining balls still cover E . Relabel so that $B(x_k, \epsilon/2) \cap E \neq \emptyset$ for $1 \leq k \leq m$, and for each such k select a point

$$y_k \in B(x_k, \epsilon/2) \cap E.$$

Thus each y_k belongs to E . We claim that

$$B(x_k, \epsilon/2) \subseteq B(y_k, \epsilon) \quad \text{for } 1 \leq k \leq m.$$

Indeed, if $z \in B(x_k, \epsilon/2)$, then by the triangle inequality

$$\rho(z, y_k) \leq \rho(z, x_k) + \rho(x_k, y_k) < \epsilon/2 + \epsilon/2 = \epsilon,$$

since $\rho(x_k, y_k) < \epsilon/2$ by the choice of y_k . This proves the claim. Consequently

$$E \subseteq \bigcup_{k=1}^m B(x_k, \epsilon/2) \subseteq \bigcup_{k=1}^m B(y_k, \epsilon),$$

and therefore, intersecting with E ,

$$E = E \cap \bigcup_{k=1}^m B(y_k, \epsilon) = \bigcup_{k=1}^m (B(y_k, \epsilon) \cap E) = \bigcup_{k=1}^m B_E(y_k, \epsilon).$$

Since the centers $y_1, \dots, y_m$ lie in E , this exhibits a cover of the metric space E by finitely many open balls of E of radius ϵ . As $\epsilon > 0$ was arbitrary, the subspace E is totally bounded.

In the terminology introduced in the text, the collection $\{B(x_k, \epsilon)\}_{k=1}^n$ appearing in the statement is called a finite ϵ -net for E ; what has been proved is precisely the assertion left as an exercise there, namely that the metric subspace E is totally bounded if and only if for each $\epsilon > 0$ there is a finite ϵ -net for E , so that it is not necessary to require the centers of the balls of the net to belong to E .

Problem 9.58. Let E be a subset of the compact metric space X . Show the metric subspace E is compact if and only if E is a closed subset of X .

Solution. Denote the metric on X by ρ ; the subspace E carries the restriction of ρ . Observe at the outset that a sequence $\{x_n\}$ of points of E converges to a point $x \in E$ with respect to the subspace metric if and only if it converges to x as a sequence in X , since in both cases convergence means $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$. We use throughout the Characterization of Compactness for a Metric Space, Theorem 16 of Chapter 9, which tells us that a metric space is compact if and only if it is sequentially compact, and Proposition 6 of Chapter 9, which tells us that a subset E of X is closed if and only if whenever a sequence in E converges to a point $x \in X$, the limit x belongs to E .

Assume first that E is a closed subset of X . Let $\{x_n\}$ be a sequence in the metric space E . Regard it as a sequence in X . Since X is compact, it is sequentially compact by Theorem 16 of Chapter 9, and hence there is a subsequence $\{x_{n_k}\}$ that converges to some point $x \in X$. Each x_{n_k} belongs to E and E is closed, so by Proposition 6 of Chapter 9 the limit x belongs to E . By the observation of the preceding paragraph, $\{x_{n_k}\}$ converges to x in the metric space E . Thus every sequence in E has a subsequence converging to a point of E , that is, E is sequentially compact, and therefore, again by Theorem 16 of Chapter 9, the metric subspace E is compact.

Assume conversely that the metric subspace E is compact. To show that E is a closed subset of X , by Proposition 6 of Chapter 9 it suffices to show that if $\{x_n\}$ is a sequence in E converging to a point $x \in X$, then $x \in E$. So let such a sequence be given. Since E is compact, it is sequentially compact by Theorem 16 of Chapter 9, so some subsequence $\{x_{n_k}\}$ converges, in the metric space E , to a point $y \in E$. But then $\{x_{n_k}\}$ converges to y as a sequence in X as well. On the other hand, $\{x_{n_k}\}$ is a subsequence of a sequence in X converging to x , so it converges to x in X . Since limits of convergent sequences in a metric space are unique (if $\{z_n\}$ converges to both u and v , then $0 \leq \rho(u, v) \leq \rho(u, z_n) + \rho(z_n, v) \rightarrow 0$, so $\rho(u, v) = 0$ and $u = v$), we conclude $x = y \in E$. Hence E is closed in X .

Note that the compactness of X was used only in the first implication; the second implication shows quite generally that a compact subspace of any metric space is a closed subset of that space.

Problem 9.59. (Frechet Intersection Theorem). Let $\{F_n\}_{n=1}^\infty$ be a descending countable collection of nonempty closed subsets of a compact metric space X . Show that $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

Solution. That the collection is descending means $F_{n+1} \subseteq F_n$ for every natural number n , and hence, by induction, $F_m \subseteq F_n$ whenever $m \geq n$.

Consider the collection $\mathcal{F} = \{F_n\}_{n=1}^\infty$ of closed subsets of X . We claim $\mathcal{F}$ has the finite intersection property, that is, every finite subcollection of $\mathcal{F}$ has nonempty intersection. Indeed, let $F_{n_1}, \dots, F_{n_k}$ be a finite subcollection and put $N = \max\{n_1, \dots, n_k\}$. Since the collection is descending, $F_N \subseteq F_{n_j}$ for each j with $1 \leq j \leq k$, and therefore

$$\emptyset \neq F_N \subseteq \bigcap_{j=1}^k F_{n_j},$$

where $F_N \neq \emptyset$ by hypothesis. Thus $\mathcal{F}$ has the finite intersection property.

Since X is compact, Proposition 14 of Chapter 9 asserts that every collection of closed subsets of X possessing the finite intersection property has nonempty intersection. Applying this to $\mathcal{F}$ we conclude

$$\bigcap_{n=1}^\infty F_n \neq \emptyset.$$

For completeness we also give a direct argument using sequential compactness, which avoids appeal to Proposition 14. For each natural number n choose, using the nonemptiness of F_n , a point $x_n \in F_n$. Since X is compact it is sequentially compact by Theorem 16 of Chapter 9, so there is a subsequence $\{x_{n_k}\}$ converging to a point $x_0 \in X$. Fix a natural number N . For all indices k with $n_k \geq N$ — and all but finitely many k have this property, since $\{n_k\}$ is strictly increasing — we have $x_{n_k} \in F_{n_k} \subseteq F_N$.

Thus the tail of the convergent sequence $\{x_{n_k}\}$ lies in F_N , and this tail also converges to x_0 . Since F_N is closed, Proposition 6 of Chapter 9 tells us that $x_0 \in F_N$. As N was an arbitrary natural number,

$$x_0 \in \bigcap_{n=1}^{\infty} F_n,$$

so the intersection is nonempty.

Observe that no hypothesis on the diameters of the F_n is needed here, in contrast with the Cantor Intersection Theorem for complete metric spaces, where one assumes $\lim_{n \rightarrow \infty} \text{diam } F_n = 0$; compactness of X is what replaces that assumption.

Problem 9.60. For a subset E of a metric space X , show that E is totally bounded if and only if its closure $\overline{E}$ is totally bounded.

Solution. Let ρ be the metric on X and $B(x, r)$ the open ball in X of radius r about x . Throughout we use the criterion established in the preceding problem (Problem 57 of Chapter 9): a subset A of X is totally bounded if and only if for each $\epsilon > 0$ there are finitely many points $x_1, \dots, x_n$ of X for which $A \subseteq \bigcup_{k=1}^n B(x_k, \epsilon)$.

Suppose first that $\overline{E}$ is totally bounded. Since $E \subseteq \overline{E}$, any finite collection of balls of radius ϵ covering $\overline{E}$ also covers E . Hence, by the criterion, E is totally bounded. (More generally, this argument shows that any subset of a totally bounded set is totally bounded.)

Suppose conversely that E is totally bounded, and let $\epsilon > 0$ be given. Apply the criterion to E with $\epsilon/2$: there are points $x_1, \dots, x_n$ in X for which

$$E \subseteq \bigcup_{k=1}^n B(x_k, \epsilon/2).$$

We claim that

$$\overline{E} \subseteq \bigcup_{k=1}^n B(x_k, \epsilon).$$

Indeed, let $y \in \overline{E}$. Since y belongs to the closure of E , the ball $B(y, \epsilon/2)$ meets E ; choose $z \in E$ with $\rho(y, z) < \epsilon/2$. Since $z \in E$, there is an index k with $z \in B(x_k, \epsilon/2)$, that is, $\rho(z, x_k) < \epsilon/2$. By the triangle inequality,

$$\rho(y, x_k) \leq \rho(y, z) + \rho(z, x_k) < \epsilon/2 + \epsilon/2 = \epsilon,$$

so $y \in B(x_k, \epsilon)$. This proves the claim.

Thus for each $\epsilon > 0$ the set $\overline{E}$ is covered by finitely many open balls of X of radius ϵ with centers in X , and therefore, by the criterion of Problem 57 of Chapter 9, $\overline{E}$ is totally bounded.

Problem 9.61. For a subset E of a complete metric space X , show that E is totally bounded if and only if its closure $\overline{E}$ is compact.

Solution. Suppose first that E is totally bounded. By the preceding problem (Problem 60 of Chapter 9), the closure $\overline{E}$ is then totally bounded as a subspace of X . Moreover $\overline{E}$ is a closed subset of X and X is complete, so by Proposition 11 of Chapter 9 the metric subspace $\overline{E}$ is complete. Thus the metric space $\overline{E}$ is both complete and totally bounded, and therefore, by the Characterization of Compactness for a Metric Space (Theorem 16 of Chapter 9, whose implication (i) $\Rightarrow$ (ii) is Proposition 17 of Chapter 9), $\overline{E}$ is compact.

Suppose conversely that $\overline{E}$ is compact. By Theorem 16 of Chapter 9, applied to the metric space $\overline{E}$, the space $\overline{E}$ is complete and totally bounded; in particular the subspace $\overline{E}$ of X is totally bounded. Since $E \subseteq \overline{E}$, and since by Problem 57 of Chapter 9 total boundedness of a subset A of X is equivalent to the existence, for each $\epsilon > 0$, of a finite cover of A by open balls of X of radius ϵ , any such finite cover of $\overline{E}$ is also a finite cover of E . Hence E is totally bounded. (Alternatively, one may quote the

preceding problem directly: $\overline{E}$ totally bounded implies E totally bounded.)

Note where completeness of X enters: it is needed only for the first implication, to guarantee that the closed set $\overline{E}$ is complete as a metric subspace. Without it the result fails; for instance, in the incomplete metric space $X = \mathbb{Q}$ of rational numbers with the metric induced by the absolute value, the set $E = \mathbb{Q} \cap [0, 1]$ is totally bounded (for each $\epsilon > 0$ finitely many balls of radius ϵ centered at rational points of $[0, 1]$ cover it), yet its closure in $\mathbb{Q}$, which is E itself because $[0, 1]$ is closed in $\mathbb{R}$, is not compact, since it is not complete: a sequence of rationals in $[0, 1]$ converging in $\mathbb{R}$ to an irrational number is Cauchy in E but has no limit in E , so E is not complete and hence, by Theorem 16 of Chapter 9, not compact.

Problem 9.62. Let $B = \{\{x_n\} \in \ell^2 \mid \sum_{n=1}^{\infty} x_n^2 \leq 1\}$ be the closed unit ball in ℓ^2 . Show that B fails to be compact by (i) showing B is not sequentially compact, (ii) finding an open cover of B without any finite subcover, and (iii) showing B is not totally bounded.

Solution. Here ℓ^2 is the space of square summable real sequences with the norm $\|\{x_n\}\|_2 = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$ and the induced metric $\rho(x, y) = \|x - y\|_2$, and $B = \{x \in \ell^2 \mid \|x\|_2 \leq 1\}$ is regarded as a metric subspace of ℓ^2 .

The whole argument rests on a single family of points. For each natural number n let $e_n \in \ell^2$ be the sequence whose n th component is 1 and whose other components are 0. Then $\|e_n\|_2 = 1$, so $e_n \in B$ for every n ; and if $m \neq n$ then $e_n - e_m$ has exactly two nonzero components, each of absolute value 1, so

$$\|e_n - e_m\|_2 = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \text{for } m \neq n.$$

We refer to this separation property as (*).

(i) B is not sequentially compact. Consider the sequence $\{e_n\}_{n=1}^{\infty}$ of points of B . Let $\{e_{n_k}\}$ be any subsequence. The indices n_k are distinct, so by (*) we have $\|e_{n_k} - e_{n_j}\|_2 = \sqrt{2}$ whenever $k \neq j$. Hence $\{e_{n_k}\}$ is not a Cauchy sequence: no matter how large an index N is chosen, there are $k, j \geq N$ with $k \neq j$ and $\|e_{n_k} - e_{n_j}\|_2 = \sqrt{2} > 1$. A convergent sequence in a metric space is Cauchy, so $\{e_{n_k}\}$ does not converge, in B or indeed in ℓ^2 . Thus the sequence $\{e_n\}$ in B has no subsequence converging to a point of B , and B is not sequentially compact. By the Characterization of Compactness for a Metric Space (Theorem 16 of Chapter 9), B is not compact.

(ii) An open cover of B with no finite subcover. Put $r = \sqrt{2}/3$ and consider the collection of open balls of ℓ^2

$$\mathcal{O} = \{B(x, r) \mid x \in B\}, \quad B(x, r) = \{y \in \ell^2 \mid \|y - x\|_2 < r\}.$$

Each member of $\mathcal{O}$ is an open subset of ℓ^2 , and every $x \in B$ lies in the member $B(x, r)$, so $\mathcal{O}$ is a cover of B by open subsets of ℓ^2 ; equivalently, intersecting with B , the traces $\{B(x, r) \cap B\}$ form an open cover of the metric subspace B .

Suppose some finite subcollection $B(x_1, r), \dots, B(x_N, r)$ covered B . The $N + 1$ points $e_1, \dots, e_{N+1}$ all belong to B , so by the pigeonhole principle two of them, say e_n and e_m with $n \neq m$, lie in the same ball $B(x_k, r)$. Then by the triangle inequality and (*),

$$\sqrt{2} = \|e_n - e_m\|_2 \leq \|e_n - x_k\|_2 + \|x_k - e_m\|_2 < 2r = \frac{2\sqrt{2}}{3} < \sqrt{2},$$

a contradiction. Hence $\mathcal{O}$ admits no finite subcover of B , and B is not compact.

(iii) B is not totally bounded. By Problem 57 of Chapter 9, if B were totally bounded then for $\epsilon = \sqrt{2}/3 > 0$ there would exist finitely many points $x_1, \dots, x_N$ of ℓ^2 with

$$B \subseteq \bigcup_{k=1}^N B(x_k, \sqrt{2}/3).$$

Exactly the pigeonhole argument of part (ii) — which used nothing about the centers other than that there are finitely many of them — produces two distinct indices $n \neq m$ with e_n and e_m in a common ball $B(x_k, \sqrt{2}/3)$, whence $\sqrt{2} \leq \|e_n - x_k\|_2 + \|x_k - e_m\|_2 < 2\sqrt{2}/3 < \sqrt{2}$, a contradiction. Therefore

B is not totally bounded. Since a compact metric space is totally bounded by Theorem 16 of Chapter 9, B is not compact.

It is worth noting what this shows about infinite dimensional normed spaces. The set B is bounded, since $\|x - y\|_2 \leq 2$ for $x, y \in B$, and it is closed in ℓ^2 : the norm satisfies $|\|x\|_2 - \|y\|_2| \leq \|x - y\|_2$ and so is continuous, and B is the inverse image of the closed set $(-\infty, 1]$ under it. Moreover ℓ^2 is complete, being the Banach space of square summable sequences as recorded in Section 9.5, so by Proposition 11 of Chapter 9 the metric subspace B is complete. Hence by Theorem 16 of Chapter 9 the only way B can fail to be compact is by failing to be totally bounded, which is precisely what (iii) establishes. Thus the equivalence of being closed and bounded with being compact, valid in $\mathbb{R}^n$ by Theorem 20 of Chapter 9, breaks down in ℓ^2 .

Problem 9.63. Let $B = \{f \in L^2[a, b] \mid \|f\|_2 \leq 1\}$ be the closed unit ball in $L^2[a, b]$. Show that B fails to be compact by (i) showing B is not sequentially compact, (ii) finding an open cover of B without any finite subcover, and (iii) showing B is not totally bounded.

Solution. Here $[a, b]$ is a nondegenerate closed bounded interval, so $a < b$, and $L^2[a, b]$ carries the norm $\|f\|_2 = (\int_a^b |f|^2)^{1/2}$ and the induced metric $\rho(f, g) = \|f - g\|_2$. As in the preceding problem, everything follows from one family of points of B that are mutually far apart, and we construct such a family out of functions with pairwise disjoint supports.

For each integer $n \geq 0$ set $t_n = a + (b - a)/2^n$, so that $t_0 = b$, the sequence $\{t_n\}$ is strictly decreasing, and $t_n \rightarrow a$. For each natural number n let

$$I_n = [t_n, t_{n-1}) \subseteq [a, b], \quad m(I_n) = t_{n-1} - t_n = \frac{b-a}{2^{n-1}} - \frac{b-a}{2^n} = \frac{b-a}{2^n} > 0.$$

The intervals $I_1, I_2, \dots$ are pairwise disjoint. Define

$$f_n = \left(\frac{2^n}{b-a}\right)^{1/2} \chi_{I_n} \quad \text{for each natural number } n,$$

where χ_{I_n} is the characteristic function of I_n . Each f_n is a bounded measurable function on $[a, b]$, hence belongs to $L^2[a, b]$, and

$$\|f_n\|_2^2 = \frac{2^n}{b-a} \int_a^b \chi_{I_n} = \frac{2^n}{b-a} \cdot m(I_n) = 1,$$

so $\|f_n\|_2 = 1$ and $f_n \in B$. If $m \neq n$ then $I_n \cap I_m = \emptyset$, so $f_n f_m$ vanishes identically and $|f_n - f_m|^2 = f_n^2 + f_m^2$ pointwise; integrating,

$$\|f_n - f_m\|_2^2 = \|f_n\|_2^2 + \|f_m\|_2^2 = 2, \quad \text{so} \quad \|f_n - f_m\|_2 = \sqrt{2} \quad \text{for } m \neq n.$$

We refer to this separation property as (*). Note also that for $m \neq n$ the functions f_n and f_m are not equal almost everywhere, so they are genuinely distinct points of $L^2[a, b]$.

(i) B is not sequentially compact. Consider the sequence $\{f_n\}$ in B and let $\{f_{n_k}\}$ be any subsequence. The indices n_k are distinct, so by (*), $\|f_{n_k} - f_{n_j}\|_2 = \sqrt{2}$ whenever $k \neq j$. Hence no subsequence is Cauchy: for every index N there are $k, j \geq N$, $k \neq j$, with $\|f_{n_k} - f_{n_j}\|_2 = \sqrt{2} > 1$. Since every convergent sequence in a metric space is Cauchy, no subsequence of $\{f_n\}$ converges in B (or in $L^2[a, b]$). Thus B is not sequentially compact, and by the Characterization of Compactness for a Metric Space (Theorem 16 of Chapter 9), B is not compact.

(ii) An open cover of B with no finite subcover. Put $r = \sqrt{2}/3$ and let

$$\mathcal{O} = \{B(f, r) \mid f \in B\}, \quad B(f, r) = \{g \in L^2[a, b] \mid \|g - f\|_2 < r\}.$$

Each $B(f, r)$ is an open subset of $L^2[a, b]$ and each $f \in B$ lies in $B(f, r)$, so $\mathcal{O}$ is a cover of B by open sets; intersecting with B gives an open cover of the metric subspace B .

Suppose finitely many members $B(g_1, r), \dots, B(g_N, r)$ covered B . The $N + 1$ points $f_1, \dots, f_{N+1}$ lie in B , so by the pigeonhole principle two of them, f_n and f_m with $n \neq m$, lie in a common ball $B(g_k, r)$. Then by the triangle inequality and (*),

$$\sqrt{2} = \|f_n - f_m\|_2 \leq \|f_n - g_k\|_2 + \|g_k - f_m\|_2 < 2r = \frac{2\sqrt{2}}{3} < \sqrt{2},$$

a contradiction. So $\mathcal{O}$ has no finite subcover and B is not compact.

(iii) B is not totally bounded. By Problem 57 of Chapter 9, were B totally bounded there would be, for $\epsilon = \sqrt{2}/3$, finitely many functions $g_1, \dots, g_N$ in $L^2[a, b]$ with

$$B \subseteq \bigcup_{k=1}^N B(g_k, \sqrt{2}/3).$$

The pigeonhole argument of part (ii) applies verbatim — it used only that the number of balls is finite — and yields $n \neq m$ with f_n, f_m in a common ball of radius $\sqrt{2}/3$, whence $\sqrt{2} < 2\sqrt{2}/3$, a contradiction. Therefore B is not totally bounded, and since every compact metric space is totally bounded by Theorem 16 of Chapter 9, B is not compact.

As in the preceding problem, B is bounded, since $\|f - g\|_2 \leq 2$ for $f, g \in B$, and closed in $L^2[a, b]$, being the inverse image of $(-\infty, 1]$ under the continuous function $f \mapsto \|f\|_2$; here continuity follows from the inequality $|\|f\|_2 - \|g\|_2| \leq \|f - g\|_2$. Since $L^2[a, b]$ is complete, by the Riesz-Fischer Theorem of Chapter 7, part (ii) of Theorem 12 of Chapter 9 tells us that the nonempty closed subset B is itself a complete metric space. So by Theorem 16 of Chapter 9 its failure to be compact must be, and is, a failure of total boundedness.

9.10 Problems 9.64–9.70

Problem 9.64. Let X be a totally bounded metric space.

- (i) If f is a uniformly continuous mapping from X to a metric space Y , show that $f(X)$ is totally bounded.
- (ii) Is (i) still true if f is only required to be continuous?

Solution. Write ρ for the metric on X and σ for the metric on Y . Recall the observation made in the text following the definition of total boundedness: a subset E of a metric space is totally bounded if and only if for each $\epsilon > 0$ there is a finite ϵ -net for E , that is, a finite collection of balls of radius ϵ whose union contains E (the centers need not lie in E).

- (i) Let $\epsilon > 0$. Since f is uniformly continuous, there is a $\delta > 0$ such that

$$\rho(u, v) < \delta \implies \sigma(f(u), f(v)) < \epsilon \quad \text{for all } u, v \in X.$$

Since X is totally bounded, X may be covered by finitely many open balls of radius $\delta/2$: there are points $x_1, \dots, x_n$ in X with

$$X = \bigcup_{k=1}^n B(x_k, \delta/2).$$

Fix k and let $u \in B(x_k, \delta/2)$. Then $\rho(u, x_k) < \delta/2 < \delta$, so $\sigma(f(u), f(x_k)) < \epsilon$; that is,

$$f(B(x_k, \delta/2)) \subseteq B(f(x_k), \epsilon).$$

Since the balls $B(x_k, \delta/2)$ cover X and f maps X onto $f(X)$,

$$f(X) = \bigcup_{k=1}^n f(B(x_k, \delta/2)) \subseteq \bigcup_{k=1}^n B(f(x_k), \epsilon).$$

Thus $\{B(f(x_k), \epsilon)\}_{k=1}^n$ is a finite ϵ -net for $f(X)$, and its centers even belong to $f(X)$. As $\epsilon > 0$ was arbitrary, $f(X)$ is totally bounded.

(ii) No. Take $X = (0, 1)$ with the metric inherited from $\mathbb{R}$. Since X is a bounded subset of $\mathbb{R}$, Proposition 15 tells us X is totally bounded. Define $f : X \rightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{x}.$$

Then f is continuous on $(0, 1)$, being the quotient of continuous functions with nonvanishing denominator. However $f(X) = (1, \infty)$, which is an unbounded subset of $\mathbb{R}$, so Proposition 15 tells us $f(X)$ is not totally bounded. (Alternatively, and without appeal to Proposition 15: a totally bounded metric space is bounded, since if it is covered by finitely many balls of radius 1 then, by the triangle inequality, its diameter is at most $2 + d$, where d is the maximum distance between the centers.)

So uniform continuity cannot be weakened to continuity. Note that any such counterexample must live on a totally bounded space that fails to be complete: if X were also complete, then X would be compact by Theorem 16, whereupon Proposition 23 would make every continuous mapping on X uniformly continuous, and part (i) would apply.

Problem 9.65. Let ρ be a metric on a set X . Define

$$\tau(u, v) = \frac{\rho(u, v)}{1 + \rho(u, v)} \quad \text{for all } u, v \in X.$$

Verify that τ is a bounded metric on X and convergence of sequences with respect to the ρ metric and the τ metric is the same. Conclude that sets that are closed with respect to the ρ metric are closed with respect to the τ metric and that sets that are open with respect to the ρ metric are open with respect to the τ metric. Are the metrics ρ and τ equivalent?

Solution. Introduce the function

$$\varphi(t) = \frac{t}{1+t} \quad \text{for } t \geq 0,$$

so that $\tau = \varphi \circ \rho$. We record two properties of φ .

First, φ is strictly increasing on $[0, \infty)$: for $0 \leq s < t$,

$$\varphi(t) - \varphi(s) = \frac{t}{1+t} - \frac{s}{1+s} = \frac{t-s}{(1+t)(1+s)} > 0.$$

In particular $\varphi(t) \geq 0$, with $\varphi(t) = 0$ if and only if $t = 0$, and $\varphi(t) < 1$ for all $t \geq 0$.

Second, φ is subadditive: for $s, t \geq 0$,

$$\varphi(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = \varphi(s) + \varphi(t),$$

since $1+s+t \geq 1+s$ and $1+s+t \geq 1+t$.

τ is a bounded metric. For $u, v \in X$ we have $\tau(u, v) = \varphi(\rho(u, v)) \geq 0$, and $\tau(u, v) = 0$ if and only if $\rho(u, v) = 0$, that is, if and only if $u = v$. Symmetry of τ is immediate from symmetry of ρ . For the triangle inequality, let $u, v, w \in X$. Since $\rho(u, w) \leq \rho(u, v) + \rho(v, w)$ and φ is increasing, and then by subadditivity,

$$\tau(u, w) = \varphi(\rho(u, w)) \leq \varphi(\rho(u, v) + \rho(v, w)) \leq \varphi(\rho(u, v)) + \varphi(\rho(v, w)) = \tau(u, v) + \tau(v, w).$$

Thus τ is a metric. It is bounded because $\tau(u, v) = \varphi(\rho(u, v)) < 1$ for all u, v , so $\text{diam}_\tau X \leq 1$.

The two metrics have the same convergent sequences, with the same limits. Observe first that

$$\tau(u, v) = \frac{\rho(u, v)}{1 + \rho(u, v)} \leq \rho(u, v)$$

for all u, v . Moreover, solving $t = \varphi(s)$ for s gives $s = t/(1-t)$, so for all $u, v \in X$,

$$\rho(u, v) = \frac{\tau(u, v)}{1 - \tau(u, v)},$$

which is legitimate since $\tau(u, v) < 1$.

Now let $\{x_n\}$ be a sequence in X and $x \in X$. If $\rho(x_n, x) \rightarrow 0$, then $0 \leq \tau(x_n, x) \leq \rho(x_n, x) \rightarrow 0$, so $\tau(x_n, x) \rightarrow 0$. Conversely, if $\tau(x_n, x) \rightarrow 0$, then, since $t \mapsto t/(1-t)$ is continuous at $t = 0$ with value 0,

$$\rho(x_n, x) = \frac{\tau(x_n, x)}{1 - \tau(x_n, x)} \longrightarrow \frac{0}{1 - 0} = 0.$$

Hence $\{x_n\}$ converges to x in (X, ρ) if and only if it converges to x in (X, τ) .

Closed sets and open sets agree. Let $E \subseteq X$ be closed with respect to ρ . By Proposition 6, to show E is closed with respect to τ it suffices to show that if $\{x_n\}$ is a sequence in E that converges to some $x \in X$ in the τ metric, then $x \in E$. By the equivalence of convergence just proved, $\{x_n\}$ converges to x in the ρ metric; since E is ρ -closed, Proposition 6 gives $x \in E$. Thus E is τ -closed. The same argument with the roles of ρ and τ interchanged shows that a τ -closed set is ρ -closed; so the closed sets of the two metrics are exactly the same. By Proposition 4, a subset is open if and only if its complement is closed, so the open sets of the two metrics are also exactly the same. In particular a ρ -open set is τ -open.

Are ρ and τ equivalent? Not in general. Recall that ρ and τ are equivalent provided there are positive constants c_1, c_2 with

$$c_1 \tau(u, v) \leq \rho(u, v) \leq c_2 \tau(u, v) \quad \text{for all } u, v \in X.$$

The left-hand inequality always holds with $c_1 = 1$, since $\tau \leq \rho$. The right-hand inequality can fail. Take $X = \mathbb{R}$ with $\rho(u, v) = |u - v|$. If there were a $c_2 > 0$ with $\rho \leq c_2 \tau$ on $X \times X$, then, since $\tau < 1$ everywhere, we would get $|u - v| < c_2$ for all real u, v , which is false (take $u = 0, v = c_2$). So ρ and τ are not equivalent in this case.

This example is typical: ρ and τ are equivalent precisely when ρ is bounded. Indeed, if $\rho(u, v) \leq M$ for all u, v , then

$$\tau(u, v) = \frac{\rho(u, v)}{1 + \rho(u, v)} \geq \frac{\rho(u, v)}{1 + M},$$

so $\rho \leq (1 + M)\tau$, and together with $\tau \leq \rho$ this exhibits the equivalence. Conversely, if $\rho \leq c_2 \tau$ then $\rho < c_2$ everywhere, so ρ is bounded.

Thus the passage from ρ to τ leaves the topology (open sets, closed sets, convergent sequences) untouched, while in general strictly changing the metric structure: it is an instance of a change of metric that is not an equivalence of metrics in the sense of the definition, yet for which the conclusion of Proposition 7 nevertheless holds.

Problem 9.66. Let E be a subset of Euclidean space $\mathbb{R}^n$. Assume every continuous real-valued function of E takes a minimum value. Prove that E is closed and bounded.

Solution. If E is empty there is nothing to prove, since the empty set is closed and bounded. So assume $E \neq \emptyset$. We regard E as a metric space with the metric inherited from $\mathbb{R}^n$, namely $\rho(x, y) = \|x - y\|$, and we are assuming that every continuous $f : E \rightarrow \mathbb{R}$ attains a minimum value on E .

We argue each conclusion by contradiction, in each case exhibiting a continuous real-valued function on E whose infimum is not attained.

E is bounded. Suppose not. Define $f : E \rightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{1 + \|x\|}.$$

The function $x \mapsto \|x\|$ is continuous on $\mathbb{R}^n$, since by the triangle inequality for the norm $|\|x\| - \|y\|| \leq \|x - y\|$, so it is Lipschitz. Hence $1 + \|x\|$ is continuous and never zero, and therefore f is continuous on $\mathbb{R}^n$; its restriction to E is continuous on the metric space E .

Clearly $f(x) > 0$ for every x . On the other hand, since E is unbounded, for each natural number k there is a point $x_k \in E$ with $\|x_k\| > k$, and then $0 < f(x_k) < 1/(1 + k)$. Therefore

$$\inf_{x \in E} f(x) = 0.$$

Since f takes only positive values, this infimum is not attained: for every $x \in E$ there is a k with $f(x_k) < f(x)$. So f is a continuous real-valued function on E that takes no minimum value, contradicting the hypothesis. Hence E is bounded.

E is closed. Suppose not. Then, by Proposition 6, there is a point $x_0 \in \mathbb{R}^n$ that is a point of closure of E but does not belong to E ; equivalently, there is a sequence $\{x_k\}$ in E converging to a point $x_0 \notin E$. Define $g : E \rightarrow \mathbb{R}$ by

$$g(x) = \|x - x_0\|.$$

As above, g is Lipschitz, hence continuous, on E . Since $x_0 \notin E$, we have $g(x) > 0$ for every $x \in E$. On the other hand $g(x_k) = \|x_k - x_0\| \rightarrow 0$, so

$$\inf_{x \in E} g(x) = 0,$$

and again this infimum is not attained because g is strictly positive on E . Thus g is a continuous real-valued function on E taking no minimum value, contradicting the hypothesis. Hence E contains all of its points of closure, that is, E is closed.

Therefore E is closed and bounded.

Remark. Combined with Theorem 20, this shows E is compact. There is also a two-line proof that quotes the Extreme Value Theorem. If $f : E \rightarrow \mathbb{R}$ is continuous, so is $-f$, and by hypothesis $-f$ takes a minimum value at some $x_1 \in E$; then f takes a maximum value at x_1 . So the hypothesis forces every continuous real-valued function on E to take both a maximum and a minimum value, and Theorem 22 makes the metric space E compact; Theorem 20 then says E is closed and bounded. The argument given above is preferred here only because it is explicit: it exhibits by hand the two functions whose infima fail to be attained.

Problem 9.67. Let E be a subset of Euclidean space $\mathbb{R}^n$. Assume every continuous real-valued function of E is uniformly continuous. Prove that E is closed and bounded.

Solution. The closedness half is true and is proved below. The boundedness half, as printed, is false: we give a counterexample and then prove the correct positive statement that replaces it.

Throughout, E carries the metric $\rho(x, y) = \|x - y\|$ inherited from $\mathbb{R}^n$, and we assume every continuous $f : E \rightarrow \mathbb{R}$ is uniformly continuous. We may assume $E \neq \emptyset$.

Part 1: E is closed. Suppose not. Since E fails to be closed, by Proposition 6 there is a sequence $\{x_k\}$ in E converging to a point $x_0 \in \mathbb{R}^n$ with $x_0 \notin E$. Define

$$f(x) = \frac{1}{\|x - x_0\|} \quad \text{for } x \in E.$$

Since $x_0 \notin E$, the denominator never vanishes on E ; as $x \mapsto \|x - x_0\|$ is continuous, f is continuous on E .

Put $r_k = \|x_k - x_0\|$. Then $r_k > 0$ for every k and $r_k \rightarrow 0$. Passing to a subsequence (relabelled again as $\{x_k\}$, $\{r_k\}$) we may assume

$$r_{k+1} < \frac{1}{2}r_k \quad \text{for all } k,$$

choosing the indices greedily: having chosen x_k , use $r_j \rightarrow 0$ to find an index j with $r_j < r_k/2$. Now set $u_k = x_k$ and $v_k = x_{k+1}$. By the triangle inequality,

$$\|u_k - v_k\| \leq r_k + r_{k+1} < \frac{3}{2}r_k \rightarrow 0,$$

while

$$|f(u_k) - f(v_k)| = \frac{1}{r_{k+1}} - \frac{1}{r_k} > \frac{2}{r_k} - \frac{1}{r_k} = \frac{1}{r_k} \rightarrow \infty.$$

Take $\epsilon = 1$. Given any $\delta > 0$, choose k so large that $\frac{3}{2}r_k < \delta$ and $1/r_k \geq 1$. Then $u_k, v_k \in E$ with $\|u_k - v_k\| < \delta$ but $|f(u_k) - f(v_k)| > 1 = \epsilon$. So f is continuous but not uniformly continuous on E , contradicting the hypothesis. Hence E is closed.

Part 2: E need not be bounded. Take $n = 1$ and $E = \mathbb{Z}$, the set of integers, with the metric inherited from $\mathbb{R}$. Any two distinct points of E are at distance at least 1. Hence, given any real-valued function f on E whatsoever and any $\epsilon > 0$, the choice $\delta = 1$ has the property that $u, v \in E$ and $|u - v| < \delta$ force $u = v$ and therefore $|f(u) - f(v)| = 0 < \epsilon$. So every real-valued function on E , in particular every continuous one, is uniformly continuous, yet E is unbounded. Thus the hypothesis of the problem does not imply boundedness, and the assertion as printed cannot be proved.

Part 3: the correct positive statement. Let E' denote the set of points of $\mathbb{R}^n$ that are accumulation points of E , that is, points y such that every open ball about y contains a point of E other than y . We claim that under the hypothesis E' is compact. (When E has no isolated points, $E \subseteq E'$, and then E itself is compact, which is presumably the intended content.)

First, E' is always closed. Indeed, let y be a point of closure of E' and let $r > 0$. Choose $z \in E'$ with $\|z - y\| < r/2$, and then a point $x \in E$, $x \neq z$, with $\|x - z\| < \min\{r/2, \|z - y\|\}$ if $z \neq y$, and with $\|x - z\| < r/2$, $x \neq z$, if $z = y$. In either case $x \in E$, $x \neq y$, and $\|x - y\| < r$. Hence $y \in E'$.

Next we show E' is bounded. Suppose not. Choose inductively points $y_1, y_2, \dots$ in E' with $\|y_{k+1}\| > \|y_k\| + 2$ for every k ; this is possible because E' is unbounded. Then for $j < k$,

$$\|y_k - y_j\| \geq \|y_k\| - \|y_j\| > 2.$$

Since E is closed by Part 1 and E' consists of points of closure of E , each y_k belongs to E . As y_k is an accumulation point of E , choose $v_k \in E$ with

$$0 < t_k := \|v_k - y_k\| < \min\{1/k, 1/4\}.$$

Define $h_k : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$h_k(x) = \max \left\{ 0, 1 - \frac{\|x - y_k\|}{2t_k} \right\}.$$

Each h_k is continuous, satisfies $h_k(y_k) = 1$ and $h_k(v_k) = 1 - \frac{t_k}{2t_k} = \frac{1}{2}$, and vanishes outside the ball $B(y_k, 2t_k) \subseteq B(y_k, 1/2)$. Because the centers are more than 2 apart, the balls $B(y_k, 1/2)$ are pairwise disjoint. Put

$$f(x) = \sum_{k=1}^{\infty} h_k(x).$$

This sum is continuous on $\mathbb{R}^n$: given $x \in \mathbb{R}^n$, since $\|y_k\| \rightarrow \infty$ only finitely many of the balls $B(y_k, 1/2)$ meet $B(x, 1)$, so on the open set $B(x, 1)$ the function f coincides with a finite sum of continuous functions and is therefore continuous at x . Restrict f to E .

Since the balls $B(y_k, 1/2)$ are pairwise disjoint and $y_k, v_k \in B(y_k, 1/2)$, only the k th summand is nonzero at these two points, so

$$f(y_k) = 1, \quad f(v_k) = \frac{1}{2}, \quad \|y_k - v_k\| = t_k < 1/k.$$

Take $\epsilon = 1/2$. Given $\delta > 0$, choose k with $1/k < \delta$; then $y_k, v_k \in E$, $\|y_k - v_k\| < \delta$, and $|f(y_k) - f(v_k)| = 1/2 \geq \epsilon$. Hence f is continuous on E but not uniformly continuous, a contradiction. So E' is bounded, and being closed it is compact by Theorem 20.

Summary. Under the stated hypothesis, E is closed and its set of accumulation points is compact. If in addition E has no isolated points, then $E \subseteq E'$, so E is bounded and hence compact. In general E need not be bounded, as $E = \mathbb{Z}$ shows. The printed claim is therefore correct in its first half and false in its second.

Problem 9.68. Suppose f is a continuous real-valued function on Euclidean space $\mathbb{R}^n$ with the property that there is a number c such that $|f(x)| \geq c \cdot \|x\|$ for all $x \in \mathbb{R}^n$. Show that if K is a compact set of real numbers, then its inverse image under f , $f^{-1}(K)$, also is compact. (Mappings with this property are called proper.)

Solution. The hypothesis has content only when $c > 0$ (for $c \leq 0$ the inequality $|f(x)| \geq c\|x\|$ holds for every f , and the conclusion fails: take $f \equiv 0$ and $K = \{0\}$, so that $f^{-1}(K) = \mathbb{R}^n$). We therefore take $c > 0$, as intended.

Let K be a compact set of real numbers. By Theorem 20, K is closed and bounded. We show $f^{-1}(K)$ is closed and bounded, and then invoke Theorem 20 again.

$f^{-1}(K)$ is closed. Since K is closed in $\mathbb{R}$, its complement $\mathbb{R} \sim K$ is open, by Proposition 4. Since f is continuous, Proposition 8 tells us that

$$f^{-1}(\mathbb{R} \sim K) = \mathbb{R}^n \sim f^{-1}(K)$$

is an open subset of $\mathbb{R}^n$ (the displayed set equality holds because f is defined at every point of $\mathbb{R}^n$, so each x lies in exactly one of $f^{-1}(K)$ and $f^{-1}(\mathbb{R} \sim K)$). By Proposition 4 once more, $f^{-1}(K)$ is closed. $f^{-1}(K)$ is bounded. Since K is bounded, there is an $M \geq 0$ with $|y| \leq M$ for all $y \in K$. Let $x \in f^{-1}(K)$. Then $f(x) \in K$, so $|f(x)| \leq M$, and therefore by hypothesis

$$c\|x\| \leq |f(x)| \leq M, \quad \text{that is,} \quad \|x\| \leq \frac{M}{c}.$$

Since $c > 0$, this bound is a finite number independent of x , so $f^{-1}(K)$ is contained in the closed ball of radius M/c about the origin and is thus bounded.

Being a closed and bounded subset of $\mathbb{R}^n$, the set $f^{-1}(K)$ is compact by Theorem 20 (the Heine-Borel Theorem). Note that the argument shows the empty case causes no trouble: if $f^{-1}(K) = \emptyset$ it is compact trivially.

Problem 9.69. For a compact metric space (X, ρ) , show that there are points $u, v \in X$ for which $\rho(u, v) = \text{diam } X$.

Solution. Assume $X \neq \emptyset$ (for the empty space the diameter is not defined). Recall that

$$\text{diam } X = \sup\{\rho(x, y) \mid x, y \in X\}.$$

By Theorem 16, the compact space X is totally bounded, and as observed in the text a totally bounded metric space is bounded; hence $d := \text{diam } X$ is a finite nonnegative real number.

We first record an elementary inequality: for all $a, b, a', b' \in X$,

$$|\rho(a, b) - \rho(a', b')| \leq \rho(a, a') + \rho(b, b').$$

Indeed, by two applications of the triangle inequality,

$$\rho(a, b) \leq \rho(a, a') + \rho(a', b') + \rho(b', b),$$

so $\rho(a, b) - \rho(a', b') \leq \rho(a, a') + \rho(b, b')$; interchanging the roles of the pairs (a, b) and (a', b') gives the reverse inequality, and the two together give the absolute-value form.

Now, by the definition of the supremum, for each natural number k there are points $u_k, v_k \in X$ with

$$d - \frac{1}{k} < \rho(u_k, v_k) \leq d,$$

so that $\rho(u_k, v_k) \rightarrow d$.

By Theorem 16, X is sequentially compact. Apply this to $\{u_k\}$: there is a subsequence $\{u_{k_j}\}$ converging to some $u \in X$. Apply sequential compactness again, this time to the sequence $\{v_{k_j}\}$: there is a further subsequence, indexed by $\{k_{j_i}\}$, converging to some $v \in X$. Write $m_i = k_{j_i}$. Along this final subsequence we have simultaneously

$$u_{m_i} \rightarrow u, \quad v_{m_i} \rightarrow v, \quad \rho(u_{m_i}, v_{m_i}) \rightarrow d,$$

the last because $\{\rho(u_{m_i}, v_{m_i})\}$ is a subsequence of the convergent sequence $\{\rho(u_k, v_k)\}$, and $\{u_{m_i}\}$ is a subsequence of the convergent sequence $\{u_{k_j}\}$.

By the inequality recorded above,

$$|\rho(u_{m_i}, v_{m_i}) - \rho(u, v)| \leq \rho(u_{m_i}, u) + \rho(v_{m_i}, v) \longrightarrow 0,$$

so $\rho(u_{m_i}, v_{m_i}) \rightarrow \rho(u, v)$. Since limits of sequences of real numbers are unique, $\rho(u, v) = d = \text{diam } X$. Thus the diameter of a nonempty compact metric space is attained by a pair of its points.

Problem 9.70. Let K be a compact subset of the metric space (X, ρ) and x_0 belong to X . Show that there is a point $z \in K$ for which

$$\rho(z, x_0) \leq \rho(x, x_0) \quad \text{for all } x \in K.$$

Solution. Assume $K \neq \emptyset$; otherwise there is no point z to produce and the assertion is empty.

Define $f : K \rightarrow \mathbb{R}$ by

$$f(x) = \rho(x, x_0), \quad x \in K,$$

where K is regarded as a metric subspace of (X, ρ) .

The function f is continuous on K ; in fact it is Lipschitz with constant 1. Indeed, for $x, y \in K$ the triangle inequality gives

$$\rho(x, x_0) \leq \rho(x, y) + \rho(y, x_0), \quad \rho(y, x_0) \leq \rho(y, x) + \rho(x, x_0),$$

and hence

$$|f(x) - f(y)| = |\rho(x, x_0) - \rho(y, x_0)| \leq \rho(x, y).$$

So given $\epsilon > 0$, the choice $\delta = \epsilon$ shows that $\rho(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$; thus f is (uniformly) continuous on K .

Now K , as a metric subspace of X , is a compact metric space. By the Extreme Value Theorem (Theorem 22), every continuous real-valued function on a compact metric space takes a minimum value. Applying this to f , there is a point $z \in K$ with

$$f(z) \leq f(x) \quad \text{for all } x \in K,$$

that is,

$$\rho(z, x_0) \leq \rho(x, x_0) \quad \text{for all } x \in K,$$

which is exactly what was to be shown.

(Equivalently, one may argue directly with sequential compactness: let $m = \inf_{x \in K} \rho(x, x_0)$, which is finite and nonnegative, choose $z_k \in K$ with $\rho(z_k, x_0) \rightarrow m$, and use Theorem 16 to extract a subsequence $z_{k_j} \rightarrow z \in K$. Since $|\rho(z_{k_j}, x_0) - \rho(z, x_0)| \leq \rho(z_{k_j}, z) \rightarrow 0$, we get $\rho(z, x_0) = m$, so the infimum is attained at z .)

Note that z need not be unique: for $X = \mathbb{R}^2$, K the unit circle, and x_0 the origin, every point of K minimizes the distance to x_0 . The point of the exercise is existence, and compactness of K is essential: for $X = \mathbb{R}$, $K = (0, 1)$, and $x_0 = 0$, no nearest point exists.

9.11 Problems 9.71–9.77

Problem 9.71. Let K be a compact subset of the metric space X . For a point $x \in X \sim K$, show that there is an open set $\mathcal{U}$ containing K and an open set $\mathcal{O}$ containing x for which $\mathcal{U} \cap \mathcal{O} = \emptyset$.

Solution. Let ρ denote the metric on X .

If $K = \emptyset$, take $\mathcal{U} = \emptyset$ and $\mathcal{O} = X$; both are open, $\mathcal{U}$ contains K , $\mathcal{O}$ contains x , and $\mathcal{U} \cap \mathcal{O} = \emptyset$. So assume $K \neq \emptyset$.

Fix $y \in K$. Since $x \in X \sim K$ we have $y \neq x$, and therefore

$$r_y := \rho(x, y) > 0.$$

Consider the two open balls

$$\mathcal{U}_y := B(y, r_y/3), \quad \mathcal{O}_y := B(x, r_y/3).$$

These are disjoint. Indeed, if some z belonged to both, then by the triangle inequality

$$r_y = \rho(x, y) \leq \rho(x, z) + \rho(z, y) < \frac{r_y}{3} + \frac{r_y}{3} = \frac{2r_y}{3} < r_y,$$

which is impossible. Thus

$$\mathcal{U}_y \cap \mathcal{O}_y = \emptyset \quad \text{for every } y \in K.$$

Now observe that $\{\mathcal{U}_y\}_{y \in K}$ is a collection of open subsets of X whose union contains K , since each $y \in K$ belongs to $\mathcal{U}_y$. As recorded in Section 9.5 (a consequence of Proposition 2, which identifies the open subsets of the subspace K as the intersections with K of the open subsets of X), a subset K of a metric space X is compact if and only if every cover of K by a collection of open subsets of X has a finite subcover.

Since K is compact, this cover of K by open subsets of X has a finite subcover: there are finitely many points $y_1, \dots, y_n$ in K for which

$$K \subseteq \bigcup_{k=1}^n \mathcal{U}_{y_k}.$$

Define

$$\mathcal{U} := \bigcup_{k=1}^n \mathcal{U}_{y_k} = \bigcup_{k=1}^n B(y_k, r_{y_k}/3), \quad \mathcal{O} := \bigcap_{k=1}^n \mathcal{O}_{y_k} = \bigcap_{k=1}^n B(x, r_{y_k}/3).$$

Then $\mathcal{U}$ is open, being a union of open sets, and $K \subseteq \mathcal{U}$. Also $\mathcal{O}$ is open, being the intersection of a finite collection of open sets, and $x \in \mathcal{O}$ since x is the center of each of the balls $\mathcal{O}_{y_k}$.

Finally, $\mathcal{U} \cap \mathcal{O} = \emptyset$. Suppose $z \in \mathcal{U} \cap \mathcal{O}$. Since $z \in \mathcal{U}$ there is an index k with $z \in \mathcal{U}_{y_k}$; since $z \in \mathcal{O}$ and $\mathcal{O} \subseteq \mathcal{O}_{y_k}$, also $z \in \mathcal{O}_{y_k}$. This contradicts the disjointness in the display above, applied to $y = y_k$.

Thus $\mathcal{U}$ and $\mathcal{O}$ are disjoint open sets containing K and x respectively.

Explicitly, if we set $r := \min_{1 \leq k \leq n} r_{y_k}/3 > 0$, then $\mathcal{O} = B(x, r)$, so the separating neighborhood of x may be taken to be a single ball.

Problem 9.72. Let A and B be subsets of a metric space (X, ρ) . Define

$$\text{dist}(A, B) = \inf \{ \rho(u, v) \mid u \in A, v \in B \}.$$

If A is compact and B is closed, show that $A \cap B = \emptyset$ if and only if $\text{dist}(A, B) > 0$.

Solution. If A or B is empty, then $A \cap B = \emptyset$ and the infimum defining $\text{dist}(A, B)$ is taken over the empty set of real numbers, so that $\text{dist}(A, B) = \infty > 0$; both sides of the asserted equivalence hold. We therefore assume throughout that A and B are nonempty, so that $\text{dist}(A, B)$ is a nonnegative real number.

First suppose $\text{dist}(A, B) > 0$. This direction requires neither compactness nor closedness. Let $d = \text{dist}(A, B) > 0$. If there were a point $z \in A \cap B$, then taking $u = v = z$ in the definition of the infimum gives

$$d = \text{dist}(A, B) \leq \rho(z, z) = 0,$$

contradicting $d > 0$. Hence $A \cap B = \emptyset$.

Conversely, suppose $A \cap B = \emptyset$; we show $\text{dist}(A, B) > 0$. Argue by contradiction, assuming $\text{dist}(A, B) = 0$. By the definition of the infimum, for each natural number n there are points $u_n \in A$ and $v_n \in B$ for which

$$\rho(u_n, v_n) < \frac{1}{n}.$$

The sequence $\{u_n\}$ lies in A , and the metric subspace A is compact. By Theorem 16 of Chapter 9, the Characterization of Compactness for a Metric Space, a compact metric space is sequentially compact;

so A is sequentially compact. Hence there is a subsequence $\{u_{n_k}\}$ that converges, in the subspace A and therefore also in X , to a point u belonging to A .

We claim that the corresponding subsequence $\{v_{n_k}\}$ also converges to u . Indeed, by the triangle inequality,

$$\rho(v_{n_k}, u) \leq \rho(v_{n_k}, u_{n_k}) + \rho(u_{n_k}, u) < \frac{1}{n_k} + \rho(u_{n_k}, u),$$

and the right-hand side tends to 0 as $k \rightarrow \infty$ because $n_k \geq k \rightarrow \infty$ and $u_{n_k} \rightarrow u$. Therefore $v_{n_k} \rightarrow u$. Each v_{n_k} belongs to B . Thus every ball $B(u, r)$ contains points of B (namely v_{n_k} for k large), so u is a point of closure of B . Since B is closed, B contains all its points of closure, and hence $u \in B$.

But we also have $u \in A$. Therefore $u \in A \cap B$, contradicting $A \cap B = \emptyset$. This contradiction shows $\text{dist}(A, B) > 0$.

An alternative proof of the second implication, using Theorem 22 of Chapter 9, the Extreme Value Theorem, instead of sequential compactness, runs as follows. Define $f: X \rightarrow \mathbb{R}$ by $f(x) = \inf_{v \in B} \rho(x, v)$. For $x, x' \in X$ and $v \in B$ we have $f(x) \leq \rho(x, v) \leq \rho(x, x') + \rho(x', v)$; taking the infimum over $v \in B$ gives $f(x) \leq \rho(x, x') + f(x')$, and by symmetry $|f(x) - f(x')| \leq \rho(x, x')$. So f is (uniformly) continuous on X , hence continuous on the compact space A , and by the Extreme Value Theorem f attains a minimum value on A , say at $u \in A$. Moreover $f(u) = \min_{x \in A} f(x) = \inf_{x \in A} \inf_{v \in B} \rho(x, v) = \inf\{\rho(x, v) \mid x \in A, v \in B\} = \text{dist}(A, B)$, the middle equality because an iterated infimum of a set of real numbers bounded below equals the infimum over the pairs. If $\text{dist}(A, B) = 0$, then $\inf_{v \in B} \rho(u, v) = 0$, so every ball about u meets B ; since B is closed this forces $u \in B$, and then $u \in A \cap B$, a contradiction.

Both hypotheses are needed for the second implication. In $\mathbb{R}$, the disjoint closed sets $A = \mathbb{Z}$ and $B = \{n + 1/(2n) : n \geq 2\}$ satisfy $\text{dist}(A, B) = 0$, showing that closedness of A alone does not suffice; and for $A = \{0\}$ compact and $B = (0, 1)$ not closed, $A \cap B = \emptyset$ while $\text{dist}(A, B) = 0$.

Problem 9.73. Let K be a compact subset of a metric space X and $\mathcal{O}$ an open set containing K . Use the preceding problem to show that there is an open set $\mathcal{U}$ for which $K \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{O}$.

Solution. Let ρ denote the metric on X .

Two degenerate cases are immediate. If $K = \emptyset$, take $\mathcal{U} = \emptyset$; then $\overline{\mathcal{U}} = \emptyset \subseteq \mathcal{O}$. If $\mathcal{O} = X$, take $\mathcal{U} = X$; then $\overline{\mathcal{U}} = X = \mathcal{O}$. So assume $K \neq \emptyset$ and $\mathcal{O} \neq X$.

Set

$$C := X \sim \mathcal{O}.$$

Since $\mathcal{O}$ is open, C is closed, and $C \neq \emptyset$ because $\mathcal{O} \neq X$. Moreover $K \cap C = \emptyset$, since $K \subseteq \mathcal{O}$. As K is compact and C is closed, the preceding problem tells us that

$$d := \text{dist}(K, C) > 0.$$

Define

$$\mathcal{U} := \bigcup_{y \in K} B(y, d/2) = \left\{ x \in X \mid \text{dist}(x, K) < d/2 \right\},$$

where $\text{dist}(x, K) = \inf_{y \in K} \rho(x, y)$. Being a union of open balls, $\mathcal{U}$ is open, and $K \subseteq \mathcal{U}$ since each $y \in K$ lies in $B(y, d/2)$.

Next we bound the closure of $\mathcal{U}$. Put

$$F := \left\{ x \in X \mid \text{dist}(x, K) \leq d/2 \right\}.$$

We claim F is closed, i.e. that $X \sim F$ is open. Let $x \in X \sim F$, so that

$$\delta := \text{dist}(x, K) - \frac{d}{2} > 0.$$

We show that $B(x, \delta/2) \subseteq X \sim F$. Suppose, to the contrary, that some z belongs to $B(x, \delta/2) \cap F$. Then $\text{dist}(z, K) \leq d/2 < d/2 + \delta/4$, so by the definition of the infimum there is a point $y \in K$ with $\rho(z, y) < d/2 + \delta/4$. The triangle inequality now gives

$$\text{dist}(x, K) \leq \rho(x, y) \leq \rho(x, z) + \rho(z, y) < \frac{\delta}{2} + \frac{d}{2} + \frac{\delta}{4} = \frac{d}{2} + \frac{3\delta}{4} < \frac{d}{2} + \delta = \text{dist}(x, K),$$

a contradiction. Hence $B(x, \delta/2) \subseteq X \sim F$, so $X \sim F$ is open and F is closed.

Since $\mathcal{U} \subseteq F$ and F is closed, and by Proposition 3 of Chapter 9 the closure $\overline{\mathcal{U}}$ is the smallest closed set containing $\mathcal{U}$, we conclude

$$\overline{\mathcal{U}} \subseteq F.$$

Finally we check $F \cap C = \emptyset$, which gives $\overline{\mathcal{U}} \subseteq X \sim C = \mathcal{O}$. Suppose $x \in F \cap C$. Since $\text{dist}(x, K) \leq d/2$, there is a point $y \in K$ with

$$\rho(x, y) < \frac{d}{2} + \frac{d}{4} = \frac{3d}{4} < d.$$

But $y \in K$ and $x \in C$, so by the definition of $\text{dist}(K, C)$ as an infimum,

$$d = \text{dist}(K, C) \leq \rho(y, x) < d,$$

a contradiction. Hence $F \cap C = \emptyset$.

Assembling the pieces,

$$K \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq F \subseteq X \sim C = \mathcal{O},$$

with $\mathcal{U}$ open, as required.

Problem 9.74. Let X be a metric space that contains a finite dense subset D . Show that $X = D$.

Solution. Recall the definition: D is dense in X provided every nonempty open subset of X contains a point of D .

If $D = \emptyset$, then no nonempty open subset of X can contain a point of D ; since X itself is an open subset of X , density forces X not to be a nonempty open set, that is, $X = \emptyset = D$. So assume $D \neq \emptyset$.

Now suppose, to get a contradiction, that $X \neq D$. Since $D \subseteq X$, there is a point $x \in X \sim D$. Write $D = \{y_1, \dots, y_n\}$, a finite nonempty list. For each index k we have $x \neq y_k$, so $\rho(x, y_k) > 0$. Set

$$r := \min_{1 \leq k \leq n} \rho(x, y_k).$$

This is the minimum of a finite nonempty collection of strictly positive numbers, hence $r > 0$. (It is precisely here that finiteness of D is used; for an infinite D the infimum of the distances could be 0.) Consider the open ball $B(x, r)$. It is an open subset of X , and it is nonempty since $x \in B(x, r)$. But it contains no point of D : for each k , $\rho(x, y_k) \geq r$, so $y_k \notin B(x, r)$. This contradicts the density of D in X .

Therefore no such x exists, and $X = D$.

Equivalently, one may argue via closures. Density of D means $\overline{D} = X$. A singleton $\{y\}$ in a metric space is closed, because for $x \neq y$ the ball $B(x, \rho(x, y))$ is a neighborhood of x missing y , so $X \sim \{y\}$ is open. A finite union of closed sets is closed, so the finite set D is closed, whence $D = \overline{D} = X$.

Problem 9.75. Show that for a subset D of a metric space X , D is dense in the subspace $\overline{D}$.

Solution. Let ρ be the metric on X , let $E := \overline{D}$ be the closure of D in X , and regard E as a metric subspace of X , that is, as the set E equipped with the restriction of ρ . Note first that $D \subseteq \overline{D} = E$, so it makes sense to ask whether D is dense in E .

By the definition of denseness we must show: every nonempty open subset of the metric space E contains a point of D .

So let $\mathcal{V}$ be a nonempty subset of E that is open in the subspace E , and choose a point $x \in \mathcal{V}$. By Proposition 2, applied to the subspace E of the metric space X , there is a set $\mathcal{O}$ open in X with $\mathcal{V} = \mathcal{O} \cap E$. Since $x \in \mathcal{O}$ and $\mathcal{O}$ is open in X , there is an $r > 0$ with $B(x, r) \subseteq \mathcal{O}$, where $B(x, r)$ denotes the ball of X . Intersecting with E ,

$$B(x, r) \cap E \subseteq \mathcal{O} \cap E = \mathcal{V}.$$

Now $x \in \mathcal{V} \subseteq E = \overline{D}$, so x is a point of closure of D . By the definition of a point of closure, every ball

about x contains a point of D ; in particular there is a point

$$u \in B(x, r) \cap D.$$

Since $D \subseteq E$, this point u also belongs to E , so $u \in B(x, r) \cap E$, and therefore $u \in \mathcal{V}$ by the inclusion above.

Thus $\mathcal{V}$ contains the point u of D . As $\mathcal{V}$ was an arbitrary nonempty open subset of E , the set D is dense in the subspace $\overline{D}$.

The same conclusion can be phrased using the observation, recorded just after the definition of denseness, that D is dense in a space exactly when every point of that space is a point of closure of D . Let $x \in E = \overline{D}$, so x is a point of closure of D in X . A neighborhood of x in the subspace E contains a set $B(x, r) \cap E$ for some $r > 0$, and $B(x, r)$ contains a point u of D ; since $D \subseteq E$, that point u lies in $B(x, r) \cap E$. So x is a point of closure of D relative to E as well, and D is dense in $E = \overline{D}$.

Finally, note the degenerate case: if $D = \emptyset$, then $\overline{D} = \emptyset$, the subspace has no nonempty open subsets, and the assertion holds vacuously.

Problem 9.76. Show that if two continuous mappings defined on a metric space X take the same values on a dense subset, then they are equal.

Solution. Let (X, ρ) be a metric space, let (Y, σ) be a metric space, let $f, g: X \rightarrow Y$ be continuous, and let D be a dense subset of X with

$$f(u) = g(u) \quad \text{for all } u \in D.$$

We show $f(x) = g(x)$ for every $x \in X$.

Fix $x \in X$ and let $\epsilon > 0$. By the ϵ - δ Criterion for Continuity of Chapter 9, applied to f at x , there is a $\delta_1 > 0$ such that

$$\rho(x, u) < \delta_1 \implies \sigma(f(x), f(u)) < \frac{\epsilon}{2}.$$

By continuity of g at x , there is a $\delta_2 > 0$ such that

$$\rho(x, u) < \delta_2 \implies \sigma(g(x), g(u)) < \frac{\epsilon}{2}.$$

Set $\delta := \min\{\delta_1, \delta_2\} > 0$.

The ball $B(x, \delta)$ is a nonempty open subset of X (it contains x). Since D is dense in X , there is a point $u \in D \cap B(x, \delta)$. For this u we have $\rho(x, u) < \delta \leq \delta_1$ and $\rho(x, u) < \delta \leq \delta_2$, and moreover $f(u) = g(u)$ because $u \in D$. Hence, by the triangle inequality for σ ,

$$\sigma(f(x), g(x)) \leq \sigma(f(x), f(u)) + \sigma(f(u), g(x)) = \sigma(f(x), f(u)) + \sigma(g(u), g(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $0 \leq \sigma(f(x), g(x)) < \epsilon$ for every $\epsilon > 0$, and therefore $\sigma(f(x), g(x)) = 0$. Since σ is a metric, this forces $f(x) = g(x)$.

As $x \in X$ was arbitrary, $f = g$.

The same conclusion follows directly from the book's definition of continuity, which is the sequential one: f is continuous at x provided that whenever $\{u_n\} \rightarrow x$ in X , $\{f(u_n)\} \rightarrow f(x)$ in Y . Given $x \in X$, density of D provides, for each natural number n , a point $u_n \in D \cap B(x, 1/n)$, the ball $B(x, 1/n)$ being a nonempty open set; then $\rho(x, u_n) < 1/n$, so $\{u_n\} \rightarrow x$. Continuity gives $\{f(u_n)\} \rightarrow f(x)$ and $\{g(u_n)\} \rightarrow g(x)$. But $f(u_n) = g(u_n)$ for every n , so these are one and the same sequence in Y , and, as observed in Section 9.2, a sequence in a metric space can converge to at most one point. Hence $f(x) = g(x)$.

Note that the conclusion fails without continuity of both mappings: on $X = \mathbb{R}$ with $D = \mathbb{Q}$, the zero function and the characteristic function of the irrationals agree on D but are not equal.

Problem 9.77. Show that the product of two separable metric spaces is again separable.

Solution. Let (X_1, ρ_1) and (X_2, ρ_2) be separable metric spaces, and let $Z := X_1 \times X_2$ carry the product metric τ defined in Section 9.1 by

$$\tau((x_1, x_2), (y_1, y_2)) = \left\{ [\rho_1(x_1, y_1)]^2 + [\rho_2(x_2, y_2)]^2 \right\}^{1/2}.$$

If either factor is empty, then $Z = \emptyset$ and Z is trivially separable (the empty set is a countable dense subset, since Z has no nonempty open subsets). So assume both factors are nonempty.

By separability, choose a countable dense subset D_1 of X_1 and a countable dense subset D_2 of X_2 ; since X_1, X_2 are nonempty, so are D_1, D_2 . Define

$$D := D_1 \times D_2 \subseteq Z.$$

The set D is countable. Indeed,

$$D = D_1 \times D_2 = \bigcup_{u_1 \in D_1} (\{u_1\} \times D_2),$$

and this is the union of a countable collection (indexed by the countable set D_1) of sets each of which is countable, being in one-to-one correspondence with D_2 . By Corollary 6 of Chapter 1, the union of a countable collection of countable sets is countable, so D is countable.

We claim D is dense in Z . Let $\mathcal{W}$ be a nonempty open subset of Z and pick $z = (x_1, x_2) \in \mathcal{W}$. Since $\mathcal{W}$ is open there is an $\epsilon > 0$ for which the τ -ball satisfies

$$B_\tau(z, \epsilon) \subseteq \mathcal{W}.$$

Consider the ball $B(x_1, \epsilon/2)$ in X_1 . It is a nonempty open subset of X_1 , so by the density of D_1 there is a point $u_1 \in D_1$ with

$$\rho_1(x_1, u_1) < \frac{\epsilon}{2}.$$

Similarly, by the density of D_2 there is a point $u_2 \in D_2$ with

$$\rho_2(x_2, u_2) < \frac{\epsilon}{2}.$$

Put $u := (u_1, u_2) \in D_1 \times D_2 = D$. Then

$$\tau(z, u) = \left\{ [\rho_1(x_1, u_1)]^2 + [\rho_2(x_2, u_2)]^2 \right\}^{1/2} < \left\{ \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} \right\}^{1/2} = \frac{\epsilon}{\sqrt{2}} < \epsilon.$$

Hence $u \in B_\tau(z, \epsilon) \subseteq \mathcal{W}$, so $\mathcal{W}$ contains a point of D .

Since $\mathcal{W}$ was an arbitrary nonempty open subset of Z , the countable set D is dense in Z . Therefore $Z = X_1 \times X_2$ is separable.

Two remarks. First, the argument is insensitive to the particular choice of product metric: any metric on $X_1 \times X_2$ for which $\tau(z, u)$ is controlled by $\rho_1(x_1, u_1)$ and $\rho_2(x_2, u_2)$ (for instance the sum metric $\rho_1 + \rho_2$ or the maximum metric, both equivalent to τ) admits the same proof, and by Problem 78 separability is unchanged when a metric is replaced by an equivalent one. Second, the same argument, applied to a finite list of separable spaces, shows that a finite product of separable metric spaces is separable.

9.12 Problems 9.78–9.79

Problem 9.78. Let ρ and σ be equivalent metrics on a nonempty set X . Show that (X, ρ) is separable if and only if (X, σ) is separable.

Solution. Recall the definition in Section 9.1: the metrics ρ and σ on X are equivalent provided there are positive numbers c_1 and c_2 for which

$$c_1 \sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \sigma(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Observe first that this relation is symmetric in the two metrics: dividing the left-hand inequality by c_1 and the right-hand one by c_2 gives

$$\frac{1}{c_2} \rho(x_1, x_2) \leq \sigma(x_1, x_2) \leq \frac{1}{c_1} \rho(x_1, x_2) \quad \text{for all } x_1, x_2 \in X,$$

so σ and ρ are equivalent with the roles of the two metrics interchanged and the constants $1/c_2$, $1/c_1$ in place of c_1 , c_2 . Consequently it suffices to prove one implication: if (X, ρ) is separable, then so is (X, σ) ; the reverse implication then follows by applying that implication to the pair (σ, ρ) .

For $x \in X$ and $r > 0$ write $B_\rho(x, r) = \{y \in X : \rho(x, y) < r\}$ and $B_\sigma(x, r) = \{y \in X : \sigma(x, y) < r\}$ for the open balls with respect to the two metrics.

Nesting of balls: fix $x \in X$ and $r > 0$. We claim that

$$B_\rho(x, c_1 r) \subseteq B_\sigma(x, r).$$

Indeed, if $y \in B_\rho(x, c_1 r)$, then $c_1 \sigma(x, y) \leq \rho(x, y) < c_1 r$, and since $c_1 > 0$ we may divide by c_1 to get $\sigma(x, y) < r$, that is, $y \in B_\sigma(x, r)$. Note also that $c_1 r > 0$, so $B_\rho(x, c_1 r)$ is a nonempty open subset of the metric space (X, ρ) : it contains x , since $\rho(x, x) = 0 < c_1 r$.

Now the implication itself. Assume (X, ρ) is separable, and let D be a countable subset of X that is dense in (X, ρ) . We claim that the very same set D is dense in (X, σ) .

Let $\mathcal{O}$ be a nonempty subset of X that is open in (X, σ) . Choose a point $x \in \mathcal{O}$. Since $\mathcal{O}$ is σ -open, there is an $r > 0$ for which $B_\sigma(x, r) \subseteq \mathcal{O}$. By the nesting just established,

$$B_\rho(x, c_1 r) \subseteq B_\sigma(x, r) \subseteq \mathcal{O}.$$

Now $B_\rho(x, c_1 r)$ is a nonempty open subset of the metric space (X, ρ) , and D is dense in (X, ρ) ; therefore $B_\rho(x, c_1 r)$ contains a point of D . That point belongs to $\mathcal{O}$. Thus every nonempty σ -open subset of X contains a point of D , which is precisely the assertion that D is dense in (X, σ) . Since D is countable, (X, σ) is separable.

By the symmetry observed at the outset, the converse implication holds as well: if (X, σ) is separable, with countable σ -dense subset D , then D is ρ -dense and (X, ρ) is separable. Hence (X, ρ) is separable if and only if (X, σ) is separable, and in fact a subset of X is dense for one metric if and only if it is dense for the other.

Remark. The argument can be phrased more briefly by appealing to Proposition 7, which asserts that for equivalent metrics ρ and σ on X a subset of X is open in (X, ρ) if and only if it is open in (X, σ) . Density of a subset D is defined purely in terms of the nonempty open subsets of the space, since D is dense provided each of them contains a point of D . Hence the two metric spaces have exactly the same dense subsets, and therefore one has a countable dense subset if and only if the other does. The nesting of balls established above is precisely what proves the relevant half of Proposition 7, whose proof the book leaves as an exercise.

Problem 9.79. Show that on any uncountable set X there is a metric on X with respect to which X is not separable.

Solution. Let X be an uncountable set and let ρ be the discrete metric on X , defined in Section 9.1 by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

We first check that ρ is indeed a metric. Clearly $\rho(x, y) \geq 0$, with $\rho(x, y) = 0$ if and only if $x = y$, and $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$. For the triangle inequality, let $x, y, z \in X$; we must show $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. If $x = y$ the left-hand side is 0 and there is nothing to prove. If $x \neq y$, then z cannot equal both x and y , so at least one of $\rho(x, z)$, $\rho(z, y)$ equals 1, and hence $\rho(x, z) + \rho(z, y) \geq 1 = \rho(x, y)$. Thus ρ is a metric on X .

Next we show that every singleton subset of X is open in (X, ρ) . Fix $x \in X$ and consider the open ball

$$B(x, \tfrac{1}{2}) = \{y \in X : \rho(x, y) < \tfrac{1}{2}\}.$$

If $y \neq x$, then $\rho(x, y) = 1 > 1/2$, so y does not belong to this ball; and $\rho(x, x) = 0 < 1/2$, so x does. Hence $B(x, 1/2) = \{x\}$. Open balls are open, as is verified in Section 9.2 directly after the definition of an open set, so $\{x\}$ is a nonempty open subset of (X, ρ) . (This is the observation, recorded in Section 9.2, that for the discrete metric on a set every subset is open: an arbitrary subset is the union of the singletons it contains, and the union of a collection of open sets is open by Proposition 1.)

Now suppose D is a dense subset of (X, ρ) . By the definition of density, every nonempty open subset of X contains a point of D . Applying this to the nonempty open set $\{x\}$, for an arbitrary $x \in X$, we conclude that $\{x\}$ contains a point of D ; the only point of $\{x\}$ is x itself, so $x \in D$. As $x \in X$ was arbitrary, $D = X$.

Therefore the only dense subset of (X, ρ) is X itself, which is uncountable by hypothesis. In particular there is no countable dense subset of (X, ρ) , and so the metric space (X, ρ) is not separable.

Remark. Uncountability of X is exactly what is needed here: if X were countable, then X would be a countable dense subset of itself and (X, ρ) would be separable for every metric ρ on X . The conclusion can also be reached through Proposition 25. Suppose $\{O_n\}_{n=1}^{\infty}$ were a countable collection of open subsets of (X, ρ) such that every open subset of X is the union of a subcollection. Fix $x \in X$. The set $\{x\}$ is open, so it is the union of some subcollection; every member of that subcollection is a subset of $\{x\}$, hence is either empty or equal to $\{x\}$, and since the union is nonempty at least one member equals $\{x\}$. Thus $\{x\} \in \{O_n\}_{n=1}^{\infty}$ for every $x \in X$, which exhibits an injection of the uncountable set X into a countable collection, a contradiction. By Proposition 25, (X, ρ) is not separable.

10 Metric Spaces: Three Fundamental Theorems

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10.1 Problems 10.1–10.7

Problem 10.1. Let E be a compact subspace of a metric space Y . Show that E is a closed, bounded subset of Y .

Solution. Let ρ denote the metric on Y ; the subspace E carries the restriction of ρ , and the open subsets of the metric space E are exactly the sets $\mathcal{O} \cap E$ with $\mathcal{O}$ open in Y . We may assume $E \neq \emptyset$, since the empty set is trivially both closed and bounded.

Boundedness. Fix a point $x_0 \in E$. For each $x \in E$ the ball $B(x, 1) = \{y \in Y : \rho(y, x) < 1\}$ is open in Y , so $\{B(x, 1) \cap E\}_{x \in E}$ is an open cover of the space E . By compactness there are points $x_1, \dots, x_n$ in E with

$$E \subseteq \bigcup_{i=1}^n B(x_i, 1).$$

Put $M = 1 + \max_{1 \leq i \leq n} \rho(x_i, x_0)$. If $z \in E$, then $z \in B(x_i, 1)$ for some i , and hence

$$\rho(z, x_0) \leq \rho(z, x_i) + \rho(x_i, x_0) < 1 + \rho(x_i, x_0) \leq M.$$

Thus $E \subseteq B(x_0, M)$, so E is a bounded subset of Y ; indeed for $z, w \in E$ we get $\rho(z, w) \leq \rho(z, x_0) + \rho(x_0, w) < 2M$, so $\text{diam } E \leq 2M < \infty$.

Closedness. We show that $Y \setminus E$ is open in Y . Let $y \in Y \setminus E$. For each $x \in E$ we have $x \neq y$, so $r_x := \rho(x, y)/2 > 0$. The collection $\{B(x, r_x) \cap E\}_{x \in E}$ is an open cover of the compact space E , so there are finitely many points $x_1, \dots, x_n$ in E with

$$E \subseteq \bigcup_{i=1}^n B(x_i, r_{x_i}).$$

Set $r = \min_{1 \leq i \leq n} r_{x_i}$, a positive number since the minimum is over a finite set of positive numbers. We claim $B(y, r) \cap E = \emptyset$. Indeed, let $z \in E$ and choose i with $\rho(z, x_i) < r_{x_i}$. Then by the triangle inequality

$$\rho(y, z) \geq \rho(y, x_i) - \rho(x_i, z) > 2r_{x_i} - r_{x_i} = r_{x_i} \geq r,$$

so $z \notin B(y, r)$. Hence $B(y, r) \subseteq Y \setminus E$.

Therefore every point of $Y \setminus E$ is an interior point of $Y \setminus E$, that is, $Y \setminus E$ is open and E is closed in Y . Together with the first paragraph, E is a closed, bounded subset of Y .

Problem 10.2. Show that an equicontinuous sequence of real-valued functions on a compact metric space is pointwise bounded if and only if it is uniformly bounded.

Solution. Let (X, ρ) be a compact metric space and $\{f_n\}$ an equicontinuous sequence of real-valued functions on X .

Suppose first that $\{f_n\}$ is uniformly bounded, so that there is an $M \geq 0$ with $|f_n| \leq M$ on X for every n . Then for each fixed $x \in X$ the sequence of numbers $\{f_n(x)\}$ satisfies $|f_n(x)| \leq M$ for all n and hence is bounded. Thus $\{f_n\}$ is pointwise bounded. This implication uses neither equicontinuity nor compactness.

Now suppose $\{f_n\}$ is pointwise bounded; we show it is uniformly bounded. Apply the definition of equicontinuity at a point $x \in X$ with the choice $\epsilon = 1$: there is a $\delta_x > 0$ such that

$$\text{if } \rho(x', x) < \delta_x, \text{ then } |f_n(x') - f_n(x)| < 1 \quad \text{for every index } n.$$

The collection $\{B(x, \delta_x)\}_{x \in X}$ is an open cover of X . Since X is compact, there are finitely many points $x_1, \dots, x_k$ in X for which

$$X = \bigcup_{i=1}^k B(x_i, \delta_{x_i}).$$

By pointwise boundedness, for each i with $1 \leq i \leq k$ there is an $M_i \geq 0$ such that $|f_n(x_i)| \leq M_i$ for all n . Since there are only finitely many indices i , the number

$$M = 1 + \max_{1 \leq i \leq k} M_i$$

is finite.

Let $x \in X$ and let n be any index. Choose i with $x \in B(x_i, \delta_{x_i})$, that is, $\rho(x, x_i) < \delta_{x_i}$. By the choice of δ_{x_i} we have $|f_n(x) - f_n(x_i)| < 1$, and therefore

$$|f_n(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i)| < 1 + M_i \leq M.$$

Since $x \in X$ and n were arbitrary, $|f_n| \leq M$ on X for every n ; that is, $\{f_n\}$ is uniformly bounded. Note that the argument for the second implication used nothing about countability of the index set, so the same statement holds for an arbitrary equicontinuous family of real-valued functions on a compact metric space, with “pointwise bounded” meaning that $\{f(x) : f \in \mathcal{F}\}$ is bounded for each $x \in X$.

Problem 10.3. Show that an equicontinuous family of continuous functions on a compact metric space is uniformly equicontinuous.

Solution. Let (X, ρ) be a compact metric space and let $\mathcal{F}$ be a family of real-valued functions on X which is equicontinuous on X , that is, equicontinuous at each point of X . We must show that $\mathcal{F}$ is uniformly equicontinuous: for each $\epsilon > 0$ there is a single $\delta > 0$, independent of the point and of the function, such that for all $u, v \in X$ and all $f \in \mathcal{F}$,

$$\text{if } \rho(u, v) < \delta, \text{ then } |f(u) - f(v)| < \epsilon.$$

Let $\epsilon > 0$. Fix $x \in X$. By equicontinuity of $\mathcal{F}$ at x , applied with $\epsilon/2$ in place of ϵ , there is a $\delta_x > 0$ such that

$$\text{if } \rho(x', x) < \delta_x, \text{ then } |f(x') - f(x)| < \epsilon/2 \quad \text{for every } f \in \mathcal{F}. \quad (*)$$

The collection of balls $\{B(x, \delta_x/2)\}_{x \in X}$ is an open cover of X . Since X is compact, there are finitely many points $x_1, \dots, x_k$ in X with

$$X = \bigcup_{i=1}^k B(x_i, \frac{1}{2}\delta_{x_i}).$$

Define

$$\delta = \frac{1}{2} \min_{1 \leq i \leq k} \delta_{x_i},$$

which is positive because it is half the minimum of finitely many positive numbers. (If $X = \emptyset$ the assertion is vacuous, so we assume $X \neq \emptyset$, whence $k \geq 1$ and the minimum is over a nonempty finite set.)

Now let $u, v \in X$ satisfy $\rho(u, v) < \delta$, and let $f \in \mathcal{F}$. Choose an index i with $u \in B(x_i, \delta_{x_i}/2)$, so that $\rho(u, x_i) < \delta_{x_i}/2$. Then

$$\rho(v, x_i) \leq \rho(v, u) + \rho(u, x_i) < \delta + \frac{1}{2}\delta_{x_i} \leq \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_i} = \delta_{x_i}.$$

Hence both u and v lie within δ_{x_i} of x_i , so by $(*)$ applied at the point x_i ,

$$|f(u) - f(x_i)| < \epsilon/2 \quad \text{and} \quad |f(v) - f(x_i)| < \epsilon/2.$$

Consequently

$$|f(u) - f(v)| \leq |f(u) - f(x_i)| + |f(x_i) - f(v)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since δ was chosen independently of u, v , and f , the family $\mathcal{F}$ is uniformly equicontinuous. (This is precisely the assertion recorded in the text immediately before the statement of the Arzela–Ascoli Theorem, and the proof is the same as the proof that a continuous real-valued function on a compact metric space is uniformly continuous, carried out with the same δ serving every member of $\mathcal{F}$.)

Problem 10.4. Let X be a metric space and $\{f_n\}$ a sequence in $C(X)$ that converges uniformly on X to $f \in C(X)$. Show that $\{f_n\}$ is equicontinuous.

Solution. Let ρ denote the metric on X . Fix a point $x \in X$ and let $\epsilon > 0$. We must produce a single $\delta > 0$ that works simultaneously for every index n .

Since $\{f_n\}$ converges uniformly on X to f , there is an index N such that

$$|f_n(x') - f(x')| < \epsilon/3 \quad \text{for all } x' \in X \text{ and all } n \geq N. \quad (1)$$

Since f is continuous at x , there is a $\delta_0 > 0$ such that

$$\text{if } \rho(x', x) < \delta_0, \text{ then } |f(x') - f(x)| < \epsilon/3. \quad (2)$$

Combining (1) and (2): if $n \geq N$ and $\rho(x', x) < \delta_0$, then

$$|f_n(x') - f_n(x)| \leq |f_n(x') - f(x')| + |f(x') - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

So the single number δ_0 serves all the indices $n \geq N$ at once. It is exactly the finitely many remaining indices that must now be absorbed, and this is possible precisely because there are finitely many of them.

For each index n with $1 \leq n < N$, the function f_n belongs to $C(X)$ and hence is continuous at x ; choose $\delta_n > 0$ such that

$$\text{if } \rho(x', x) < \delta_n, \text{ then } |f_n(x') - f_n(x)| < \epsilon.$$

Now set

$$\delta = \min\{\delta_0, \delta_1, \dots, \delta_{N-1}\},$$

a positive number, being the minimum of a finite collection of positive numbers. Suppose $x' \in X$ and $\rho(x', x) < \delta$. If $n < N$, then $\rho(x', x) < \delta_n$ and so $|f_n(x') - f_n(x)| < \epsilon$. If $n \geq N$, then $\rho(x', x) < \delta_0$ and so, by the displayed three-term estimate, $|f_n(x') - f_n(x)| < \epsilon$. In either case

$$|f_n(x') - f_n(x)| < \epsilon \quad \text{for every index } n.$$

Thus $\{f_n\}$ is equicontinuous at x . Since $x \in X$ was arbitrary, $\{f_n\}$ is equicontinuous on X . (The same argument shows that the enlarged family $\{f\} \cup \{f_n\}$ is equicontinuous.)

Problem 10.5. A real-valued function f on $[0, 1]$ is said to be Holder continuous of order α provided there is a constant C for which

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad \text{for all } x, y \in [0, 1].$$

Define the Holder norm

$$\|f\|_\alpha = \max\{|f(x)| + |f(x) - f(y)|/|x - y|^\alpha \mid x, y \in [0, 1], x \neq y\}.$$

Show that for $0 < \alpha \leq 1$, the set of functions for which $\|f\|_\alpha \leq 1$ has compact closure as a subset of $C[0, 1]$.

Solution. Write

$$\mathcal{F} = \{f : [0, 1] \rightarrow \mathbb{R} : \|f\|_\alpha \leq 1\},$$

where the expression defining $\|f\|_\alpha$ is understood as the supremum of the indicated quantities over all pairs $x \neq y$ in $[0, 1]$ (this supremum is finite exactly when f is bounded and Holder continuous of order α). Let $X = [0, 1]$, a compact metric space with the metric inherited from $\mathbb{R}$.

Step 1: $\mathcal{F} \subseteq C[0, 1]$, and the two defining estimates. Let $f \in \mathcal{F}$. By definition of $\|f\|_\alpha$, for all

$x, y \in [0, 1]$ with $x \neq y$,

$$|f(x)| + \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq 1.$$

Since both summands are nonnegative, each is at most 1. Hence, first,

$$|f(x)| \leq 1 \quad \text{for every } x \in [0, 1] \quad (1)$$

(for $x \in [0, 1]$ we may always pick some $y \in [0, 1]$ with $y \neq x$, as $[0, 1]$ has more than one point), and second,

$$|f(x) - f(y)| \leq |x - y|^\alpha \quad \text{for all } x, y \in [0, 1] \quad (2)$$

(the case $x = y$ being trivial). Estimate (2) shows each $f \in \mathcal{F}$ is continuous, so indeed $\mathcal{F}$ is a subset of $C[0, 1]$.

Step 2: $\mathcal{F}$ is uniformly bounded and equicontinuous. Uniform boundedness is exactly (1), with $M = 1$. For equicontinuity, let $\epsilon > 0$ and set $\delta = \epsilon^{1/\alpha} > 0$. If $x, y \in [0, 1]$ with $|x - y| < \delta$, then by (2), for every $f \in \mathcal{F}$,

$$|f(x) - f(y)| \leq |x - y|^\alpha < \delta^\alpha = \epsilon.$$

Since δ depends on neither the point nor the function, $\mathcal{F}$ is (uniformly) equicontinuous. Note that this is where $\alpha > 0$ is used: it makes $t \mapsto t^\alpha$ tend to 0 as $t \rightarrow 0^+$.

Step 3: the closure inherits these properties. Let $\overline{\mathcal{F}}$ denote the closure of $\mathcal{F}$ in the metric space $C[0, 1]$, whose metric is $\rho_{\max}(g, h) = \max_{[0, 1]} |g - h|$. Let $g \in \overline{\mathcal{F}}$. Then there is a sequence $\{f_n\}$ in $\mathcal{F}$ with $f_n \rightarrow g$ uniformly on $[0, 1]$, hence pointwise. Passing to the limit in (1) and (2) applied to f_n gives, for all $x, y \in [0, 1]$,

$$|g(x)| = \lim_{n \rightarrow \infty} |f_n(x)| \leq 1, \quad |g(x) - g(y)| = \lim_{n \rightarrow \infty} |f_n(x) - f_n(y)| \leq |x - y|^\alpha.$$

So $\overline{\mathcal{F}}$ is uniformly bounded by 1 and, by the estimate of Step 2 with the same $\delta = \epsilon^{1/\alpha}$, equicontinuous. Of course $\overline{\mathcal{F}}$ is a closed subset of $C[0, 1]$.

Step 4: conclusion. $X = [0, 1]$ is a compact metric space and $\overline{\mathcal{F}}$ is a closed, uniformly bounded, equicontinuous subset of $C(X) = C[0, 1]$. By Theorem 3 of this chapter, $\overline{\mathcal{F}}$ is a compact subspace of $C[0, 1]$. Therefore $\mathcal{F}$ has compact closure in $C[0, 1]$, as asserted.

Remark. In fact the limit passage of Step 3 gives more: for $x \neq y$,

$$|g(x)| + \frac{|g(x) - g(y)|}{|x - y|^\alpha} = \lim_{n \rightarrow \infty} \left(|f_n(x)| + \frac{|f_n(x) - f_n(y)|}{|x - y|^\alpha} \right) \leq 1,$$

and taking the supremum over such pairs yields $\|g\|_\alpha \leq 1$, i.e. $g \in \mathcal{F}$. Hence $\mathcal{F}$ is already closed in $C[0, 1]$, so $\overline{\mathcal{F}} = \mathcal{F}$ and the set $\mathcal{F}$ is itself compact.

Remark. If instead one argues by sequential compactness: given a sequence in $\mathcal{F}$, Steps 1 and 2 show it is uniformly bounded and equicontinuous on the compact metric space $[0, 1]$, so the Arzela–Ascoli Theorem supplies a subsequence converging uniformly on $[0, 1]$ to a continuous function, which by the previous remark again lies in $\mathcal{F}$.

Problem 10.6. Let X be a compact metric space and $\mathcal{F}$ a subset of $C(X)$. Show that $\mathcal{F}$ is equicontinuous if and only if its closure in $C(X)$, $\overline{\mathcal{F}}$, is equicontinuous. Conclude that a subset of $C(X)$ has compact closure if and only if it is equicontinuous and uniformly bounded.

Solution. Throughout, ρ denotes the metric on X and $\rho_{\max}(g, h) = \max_{x \in X} |g(x) - h(x)|$ the metric on $C(X)$; the maximum is attained since X is compact and $|g - h|$ is continuous. We may assume $\mathcal{F} \neq \emptyset$, the empty family being trivially equicontinuous with closure $\emptyset$.

Part 1: $\mathcal{F}$ is equicontinuous if and only if $\overline{\mathcal{F}}$ is equicontinuous.

Suppose first that $\overline{\mathcal{F}}$ is equicontinuous. Since $\mathcal{F} \subseteq \overline{\mathcal{F}}$, the defining condition for equicontinuity at a point, being a statement quantified over all members of the family, holds a fortiori for the smaller family $\mathcal{F}$: the same δ responding to ϵ for all $f \in \overline{\mathcal{F}}$ responds for all $f \in \mathcal{F}$. Hence $\mathcal{F}$ is equicontinuous.

Conversely, suppose $\mathcal{F}$ is equicontinuous. Fix $x \in X$ and let $\epsilon > 0$. By equicontinuity of $\mathcal{F}$ at x , applied with $\epsilon/3$, there is a $\delta > 0$ such that

$$\text{if } \rho(x', x) < \delta, \text{ then } |f(x') - f(x)| < \epsilon/3 \text{ for all } f \in \mathcal{F}.$$

We claim this same δ works for every $g \in \overline{\mathcal{F}}$. Indeed, let $g \in \overline{\mathcal{F}}$. Since g is a point of closure of $\mathcal{F}$ in $C(X)$, there is an $f \in \mathcal{F}$ with $\rho_{\max}(f, g) < \epsilon/3$, so that

$$|g(z) - f(z)| < \epsilon/3 \text{ for every } z \in X.$$

Hence if $x' \in X$ and $\rho(x', x) < \delta$, then

$$|g(x') - g(x)| \leq |g(x') - f(x')| + |f(x') - f(x)| + |f(x) - g(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since δ was chosen before g and does not depend on it, $\overline{\mathcal{F}}$ is equicontinuous at x ; as $x \in X$ was arbitrary, $\overline{\mathcal{F}}$ is equicontinuous.

An auxiliary observation on boundedness. A nonempty subset $\mathcal{G}$ of $C(X)$ is bounded as a subset of the metric space $C(X)$ if and only if it is uniformly bounded. Indeed, if $|g| \leq M$ on X for all $g \in \mathcal{G}$, then $\rho_{\max}(g, h) \leq 2M$ for $g, h \in \mathcal{G}$, so $\mathcal{G}$ is bounded. Conversely, if $\mathcal{G}$ is bounded, fix $g_0 \in \mathcal{G}$ and $D \geq 0$ with $\rho_{\max}(g, g_0) \leq D$ for all $g \in \mathcal{G}$; then $|g| \leq \max_X |g_0| + D$ on X for every $g \in \mathcal{G}$, and $\max_X |g_0|$ is finite because X is compact. Note also that if $\mathcal{F}$ is uniformly bounded by M , then $\overline{\mathcal{F}}$ is uniformly bounded by M : a uniform limit of functions bounded by M in absolute value is bounded by M in absolute value.

Part 2: $\mathcal{F} \subseteq C(X)$ has compact closure if and only if it is equicontinuous and uniformly bounded.

Suppose $\mathcal{F}$ is equicontinuous and uniformly bounded, say $|f| \leq M$ on X for every $f \in \mathcal{F}$. By Part 1, $\overline{\mathcal{F}}$ is equicontinuous; by the auxiliary observation, $\overline{\mathcal{F}}$ is uniformly bounded by M ; and $\overline{\mathcal{F}}$ is of course a closed subset of $C(X)$. Since X is a compact metric space, Theorem 3 of this chapter applies and tells us that $\overline{\mathcal{F}}$ is a compact subspace of $C(X)$. Thus $\mathcal{F}$ has compact closure.

Conversely, suppose $\overline{\mathcal{F}}$ is compact. Theorem 3, read in the direction that a compact subspace of $C(X)$ is closed, uniformly bounded, and equicontinuous, tells us that $\overline{\mathcal{F}}$ is uniformly bounded and equicontinuous. (One may also see the boundedness directly: by Problem 1 of this chapter a compact subspace of the metric space $C(X)$ is a bounded subset of $C(X)$, and the auxiliary observation converts this to uniform boundedness.) Since $\mathcal{F} \subseteq \overline{\mathcal{F}}$, the family $\mathcal{F}$ is then uniformly bounded, and it is equicontinuous by Part 1.

This completes the characterization: for X a compact metric space, a subset of $C(X)$ has compact closure precisely when it is equicontinuous and uniformly bounded.

Problem 10.7. For a closed, bounded interval $[a, b]$, let $\{f_n\}$ be a sequence in $C[a, b]$. If $\{f_n\}$ is equicontinuous, does $\{f_n\}$ necessarily have a uniformly convergent subsequence? If $\{f_n\}$ is uniformly bounded, does $\{f_n\}$ necessarily have a uniformly convergent subsequence?

Solution. The answer to both questions is no. The Arzela–Ascoli Theorem requires both hypotheses, and neither one alone suffices. We assume $a < b$, since for a degenerate interval $[a, a]$ both answers become trivially yes.

First question: equicontinuity alone is not enough. For each natural number n define the constant function

$$f_n(x) = n \text{ for all } x \in [a, b].$$

Each f_n is continuous, so $\{f_n\}$ is a sequence in $C[a, b]$. The family is equicontinuous: for every $\epsilon > 0$ any $\delta > 0$ whatever responds to the challenge, since

$$|f_n(x) - f_n(y)| = |n - n| = 0 < \epsilon \text{ for all } x, y \in [a, b] \text{ and all } n.$$

On the other hand, let $\{f_{n_k}\}$ be any subsequence. Then $f_{n_k}(a) = n_k \rightarrow \infty$ as $k \rightarrow \infty$, so $\{f_{n_k}(a)\}$ is not even a convergent sequence of real numbers. Hence $\{f_{n_k}\}$ does not converge pointwise at a , and in particular it does not converge uniformly on $[a, b]$. So no subsequence of $\{f_n\}$ converges uniformly.

Of course what fails here is uniform boundedness; indeed by Problem 2, an equicontinuous sequence on a compact metric space which is pointwise bounded is automatically uniformly bounded, so any counterexample of this kind must fail to be pointwise bounded at some point.

Second question: uniform boundedness alone is not enough. This is the example recorded in the text following the proof of the Arzela–Ascoli Theorem, transplanted from $[0, 1]$ to $[a, b]$. For each natural number n define

$$f_n(x) = \left(\frac{x-a}{b-a} \right)^n \quad \text{for } x \in [a, b].$$

Each f_n is continuous on $[a, b]$, and since $0 \leq (x-a)/(b-a) \leq 1$ on $[a, b]$ we have

$$0 \leq f_n(x) \leq 1 \quad \text{for all } x \in [a, b] \text{ and all } n,$$

so $\{f_n\}$ is uniformly bounded, with $M = 1$.

Let $t = (x-a)/(b-a)$, so t runs over $[0, 1]$ as x runs over $[a, b]$, and $f_n(x) = t^n$. If $0 \leq t < 1$, then $t^n \rightarrow 0$; and if $t = 1$, then $t^n = 1$ for all n . Hence $\{f_n\}$ converges pointwise on $[a, b]$ to

$$g(x) = \begin{cases} 0, & a \leq x < b, \\ 1, & x = b, \end{cases}$$

and the same is true of every subsequence $\{f_{n_k}\}$, since a subsequence of a convergent sequence of real numbers converges to the same limit.

Now suppose some subsequence $\{f_{n_k}\}$ converged uniformly on $[a, b]$ to a function h . Uniform convergence implies pointwise convergence, so $h = g$. But a uniform limit of continuous real-valued functions on a metric space is continuous, so g would have to be continuous on $[a, b]$. It is not: $g(b) = 1$ while $g(x) = 0$ for all $x < b$, so $\lim_{x \rightarrow b^-} g(x) = 0 \neq 1 = g(b)$. This contradiction shows no subsequence of $\{f_n\}$ converges uniformly on $[a, b]$.

Here what fails is equicontinuity: at the point b the family $\{f_n\}$ is not equicontinuous, since for any $\delta \in (0, b-a)$ the point $x_\delta = b - \delta/2$ satisfies $|x_\delta - b| < \delta$ while

$$|f_n(x_\delta) - f_n(b)| = 1 - \left(1 - \frac{\delta}{2(b-a)} \right)^n \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

so no δ can respond to, say, $\epsilon = 1/2$ for all indices n at once.

In summary: equicontinuity alone and uniform boundedness alone each fail to force the existence of a uniformly convergent subsequence; by the Arzela–Ascoli Theorem, the two together do force it.

10.2 Problems 10.8–10.14

Problem 10.8. Let X be a compact metric space and Y be a general metric space. Denote by $C(X, Y)$ the set of continuous mappings from X to Y . State and prove a version of the Arzela–Ascoli Theorem for a sequence in $C(X, Y)$ in which the assumption that $\{f_n\}$ is pointwise bounded is replaced by the assumption that for each $x \in X$, the closure of the set $\{f_n(x) \mid n \text{ a natural number}\}$ is a compact subspace of Y .

Solution. Write ρ for the metric on X and σ for the metric on Y . A collection $\mathcal{F} \subseteq C(X, Y)$ is called **equicontinuous at** $x \in X$ provided that for each $\epsilon > 0$ there is a $\delta > 0$ such that for every $f \in \mathcal{F}$ and every $x' \in X$,

$$\text{if } \rho(x', x) < \delta, \text{ then } \sigma(f(x'), f(x)) < \epsilon,$$

and equicontinuous on X provided it is equicontinuous at every point of X . This is the definition in the text with $|f(x') - f(x)|$ replaced by $\sigma(f(x'), f(x))$.

Theorem. Let X be a compact metric space, Y a metric space, and $\{f_n\}$ an equicontinuous sequence in $C(X, Y)$ with the property that for each $x \in X$ the closure in Y of the set $\{f_n(x) \mid n \in \mathbb{N}\}$ is a compact subspace of Y . Then $\{f_n\}$ has a subsequence that converges uniformly on X to a continuous mapping $f: X \rightarrow Y$.

Here uniform convergence of $\{g_k\}$ to f means: for each $\epsilon > 0$ there is a K such that $\sigma(g_k(x), f(x)) < \epsilon$ for all $k \geq K$ and all $x \in X$.

Step 1: uniform equicontinuity. We first observe that $\{f_n\}$ is uniformly equicontinuous, that is, for each $\epsilon > 0$ there is a $\delta > 0$ such that $\sigma(f_n(u), f_n(v)) < \epsilon$ whenever $u, v \in X$ satisfy $\rho(u, v) < \delta$, for every index n . Indeed, let $\epsilon > 0$. For each $x \in X$, equicontinuity at x provides $\delta_x > 0$ such that $\sigma(f_n(y), f_n(x)) < \epsilon/2$ for all n and all y with $\rho(y, x) < \delta_x$. The collection $\{B(x, \delta_x/2)\}_{x \in X}$ is an open cover of the compact space X , so there are points $x_1, \dots, x_m$ with $X = \bigcup_{i=1}^m B(x_i, \delta_{x_i}/2)$. Put $\delta = \frac{1}{2} \min_{1 \leq i \leq m} \delta_{x_i} > 0$. If $\rho(u, v) < \delta$, choose i with $\rho(u, x_i) < \delta_{x_i}/2$; then $\rho(v, x_i) \leq \rho(v, u) + \rho(u, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}$, and therefore

$$\sigma(f_n(u), f_n(v)) \leq \sigma(f_n(u), f_n(x_i)) + \sigma(f_n(x_i), f_n(v)) < \epsilon/2 + \epsilon/2 = \epsilon$$

for every n . This is the argument of Problem 3, verbatim except for the change of metric on the target.

Step 2: a diagonal subsequence converging on a dense set. By Proposition 24 of the preceding chapter, the compact metric space X is separable; let $\{x_j\}_{j=1}^\infty$ be an enumeration of a dense subset D of X . (If D is finite the selection below terminates and the argument only simplifies.) By hypothesis $K_1 := \overline{\{f_n(x_1) \mid n \in \mathbb{N}\}}$ is a compact subspace of Y , and a compact metric space is sequentially compact. Hence the sequence $\{f_n(x_1)\}$, which lies in K_1 , has a convergent subsequence: there is a strictly increasing sequence of indices $\{s(1, n)\}$ and a point $a_1 \in K_1$ with $\lim_{n \rightarrow \infty} f_{s(1, n)}(x_1) = a_1$. Applying the same argument to the sequence $n \mapsto f_{s(1, n)}(x_2)$, which lies in the compact set $K_2 := \overline{\{f_n(x_2)\}}$, we obtain a subsequence $\{s(2, n)\}$ of $\{s(1, n)\}$ and a point a_2 with $\lim_{n \rightarrow \infty} f_{s(2, n)}(x_2) = a_2$. Continuing inductively we obtain strictly increasing sequences of natural numbers $\{s(j, n)\}_{n=1}^\infty$, $j = 1, 2, \dots$, each a subsequence of its predecessor, and points $a_j \in Y$ with $\lim_{n \rightarrow \infty} f_{s(j, n)}(x_j) = a_j$.

Set $n_k = s(k, k)$. Since $\{s(k+1, n)\}$ is a subsequence of $\{s(k, n)\}$ we have $s(k+1, k+1) \geq s(k, k+1) > s(k, k)$, so $\{n_k\}$ is strictly increasing. For each fixed j , the tail $\{n_k\}_{k \geq j}$ is a subsequence of $\{s(j, n)\}$, and therefore

$$\lim_{k \rightarrow \infty} f_{n_k}(x_j) = a_j \quad \text{for every } j.$$

This is the Cantor diagonalization argument used in the proof of the Arzelà–Ascoli Lemma.

Step 3: the subsequence is uniformly Cauchy. Let $\epsilon > 0$. By Step 1 choose $\delta > 0$ such that

$$\sigma(f_n(u), f_n(v)) < \epsilon/3 \quad \text{for all } n \text{ and all } u, v \in X \text{ with } \rho(u, v) < \delta.$$

By Theorem 16 of the preceding chapter the compact space X is totally bounded, so there are points $y_1, \dots, y_m$ in X with $X = \bigcup_{i=1}^m B(y_i, \delta/2)$. Since D is dense, for each i select $d_i \in D \cap B(y_i, \delta/2)$; then $X = \bigcup_{i=1}^m B(d_i, \delta)$, because $\rho(x, y_i) < \delta/2$ forces $\rho(x, d_i) < \delta$. Each d_i is some $x_{j(i)}$, so by Step 2 the sequence $\{f_{n_k}(d_i)\}_k$ converges in Y and in particular is Cauchy. Choose K so large that

$$\sigma(f_{n_k}(d_i), f_{n_l}(d_i)) < \epsilon/3 \quad \text{for all } k, l \geq K \text{ and } 1 \leq i \leq m.$$

Now let $x \in X$ be arbitrary and choose i with $\rho(x, d_i) < \delta$. For $k, l \geq K$,

$$\sigma(f_{n_k}(x), f_{n_l}(x)) \leq \sigma(f_{n_k}(x), f_{n_k}(d_i)) + \sigma(f_{n_k}(d_i), f_{n_l}(d_i)) + \sigma(f_{n_l}(d_i), f_{n_l}(x)) < \epsilon.$$

Since K does not depend on x , the sequence $\{f_{n_k}\}$ is uniformly Cauchy on X .

Step 4: passage to the limit. The space Y is not assumed complete, so uniform Cauchyness alone does not produce a limit; here the compactness hypothesis is used a second time. Fix $x \in X$. By Step 3 the sequence $\{f_{n_k}(x)\}$ is a Cauchy sequence in Y , and it lies in the compact set $K_x = \overline{\{f_n(x) \mid n \in \mathbb{N}\}}$. A compact metric space is complete: a Cauchy sequence in K_x has a convergent subsequence by sequential compactness, and a Cauchy sequence with a convergent subsequence converges to the same limit. Hence $\{f_{n_k}(x)\}$ converges to a point of K_x , which we call $f(x)$. This defines a mapping $f: X \rightarrow Y$ to which $\{f_{n_k}\}$ converges pointwise.

The convergence is uniform. Indeed, given $\epsilon > 0$ take K as in Step 3, so that $\sigma(f_{n_k}(x), f_{n_l}(x)) < \epsilon$ for all $k, l \geq K$ and all $x \in X$. Fix $k \geq K$ and $x \in X$ and let $l \rightarrow \infty$; since $z \mapsto \sigma(f_{n_k}(x), z)$ is continuous on Y and $f_{n_l}(x) \rightarrow f(x)$, we get $\sigma(f_{n_k}(x), f(x)) \leq \epsilon$. As K is independent of x , $\{f_{n_k}\} \rightarrow f$ uniformly on X .

Step 5: continuity of the limit. Let $x \in X$ and $\epsilon > 0$. By equicontinuity at x choose $\delta > 0$ with $\sigma(f_n(x'), f_n(x)) < \epsilon/2$ for all n and all x' with $\rho(x', x) < \delta$. For such an x' , letting $k \rightarrow \infty$ in

$$\sigma(f_{n_k}(x'), f_{n_k}(x)) < \epsilon/2$$

and using the continuity of the metric σ in both variables gives $\sigma(f(x'), f(x)) \leq \epsilon/2 < \epsilon$. Hence f is continuous at x , and as x was arbitrary, $f \in C(X, Y)$.

This proves the theorem. Note that when $Y = \mathbb{R}$ the hypothesis “for each x , $\overline{\{f_n(x)\}}$ is compact” is, by the Heine–Borel Theorem, exactly the statement that $\{f_n(x)\}$ is a bounded sequence of real numbers, that is, that $\{f_n\}$ is pointwise bounded; and on a compact metric space an equicontinuous pointwise bounded sequence is uniformly bounded (Problem 2). So the theorem above contains the Arzelà–Ascoli Theorem as the special case $Y = \mathbb{R}$.

Problem 10.9. Let $\{f_n\}$ be an equicontinuous, uniformly bounded sequence of continuous real-valued functions on $\mathbb{R}$. Show that there is a subsequence of $\{f_n\}$ that converges pointwise on $\mathbb{R}$ to a continuous function on $\mathbb{R}$ and that the convergence is uniform on each bounded subset of $\mathbb{R}$.

Solution. The space $\mathbb{R}$, with its usual metric, is a separable metric space: the rationals are a countable dense subset. Since $\{f_n\}$ is uniformly bounded it is in particular pointwise bounded. Hence the Arzelà–Ascoli Lemma (Lemma 2) applies to the equicontinuous, pointwise bounded sequence $\{f_n\}$ of real-valued functions on the separable metric space $\mathbb{R}$, and provides a subsequence $\{f_{n_k}\}$ that converges pointwise on all of $\mathbb{R}$ to a real-valued function f on $\mathbb{R}$. (The Arzelà–Ascoli Theorem itself is not available here, since $\mathbb{R}$ is not compact; it is the Lemma, which requires only separability, that is used.)

We show that this one subsequence does everything that is asked.

Continuity of f . Let $x \in \mathbb{R}$ and $\epsilon > 0$. By the equicontinuity of $\{f_n\}$ at x there is a $\delta > 0$ such that

$$|f_n(x') - f_n(x)| < \epsilon/2 \quad \text{for all } n \text{ and all } x' \text{ with } |x' - x| < \delta.$$

Fix such an x' and let $k \rightarrow \infty$ in $|f_{n_k}(x') - f_{n_k}(x)| < \epsilon/2$. Since $f_{n_k}(x') \rightarrow f(x')$ and $f_{n_k}(x) \rightarrow f(x)$, and the absolute value is continuous, we obtain

$$|f(x') - f(x)| \leq \epsilon/2 < \epsilon \quad \text{whenever } |x' - x| < \delta.$$

Thus f is continuous at x ; as x was arbitrary, f is continuous on $\mathbb{R}$.

Uniform convergence on compact sets. We prove the following general assertion: *if $\{g_k\}$ is an equicontinuous sequence of real-valued functions on a metric space X that converges pointwise on X to g , then the convergence is uniform on every compact subset K of X .*

Let $K \subseteq X$ be compact and let $\epsilon > 0$. For each $x \in K$, equicontinuity at x provides $\delta_x > 0$ such that

$$|g_k(y) - g_k(x)| < \epsilon/3 \quad \text{for all } k \text{ and all } y \text{ with } \rho(y, x) < \delta_x.$$

The balls $\{B(x, \delta_x)\}_{x \in K}$ form an open cover of the compact set K , so there are points $x_1, \dots, x_m \in K$ with $K \subseteq \bigcup_{i=1}^m B(x_i, \delta_{x_i})$. For each i the sequence of real numbers $\{g_k(x_i)\}$ converges, hence is Cauchy; since there are only finitely many indices i , we may choose N so that

$$|g_k(x_i) - g_l(x_i)| < \epsilon/3 \quad \text{for all } k, l \geq N \text{ and } 1 \leq i \leq m.$$

Now let $y \in K$ and choose i with $\rho(y, x_i) < \delta_{x_i}$. Then for $k, l \geq N$,

$$|g_k(y) - g_l(y)| \leq |g_k(y) - g_k(x_i)| + |g_k(x_i) - g_l(x_i)| + |g_l(x_i) - g_l(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Since N is independent of $y \in K$, the sequence $\{g_k\}$ is uniformly Cauchy on K . Fixing $k \geq N$ and $y \in K$ and letting $l \rightarrow \infty$ in the displayed inequality gives $|g_k(y) - g(y)| \leq \epsilon$ for all $k \geq N$ and all $y \in K$. Hence $\{g_k\} \rightarrow g$ uniformly on K , which proves the assertion.

Conclusion. The subsequence $\{f_{n_k}\}$ is equicontinuous, being a subcollection of the equicontinuous collection $\{f_n\}$, and it converges pointwise on $\mathbb{R}$ to f . Let E be a bounded subset of $\mathbb{R}$. Then

$E \subseteq [-M, M]$ for some $M > 0$, and by the Heine–Borel Theorem the set $[-M, M]$ is a compact subset of $\mathbb{R}$. By the assertion just proved, $\{f_{n_k}\}$ converges to f uniformly on $[-M, M]$, and therefore uniformly on the subset E . Thus $\{f_{n_k}\}$ converges pointwise on $\mathbb{R}$ to the continuous function f , uniformly on every bounded subset of $\mathbb{R}$.

Two remarks. First, the uniform boundedness hypothesis was used only to guarantee pointwise boundedness, which is what the Arzelà–Ascoli Lemma requires. Second, the convergence need not be uniform on all of $\mathbb{R}$: take $f_n(x) = \arctan(x - n)$. This sequence is uniformly bounded by $\pi/2$ and equicontinuous, since $|f'_n| \leq 1$ for every n gives $|f_n(u) - f_n(v)| \leq |u - v|$ by the Mean Value Theorem. It converges pointwise on $\mathbb{R}$ to the constant function $f \equiv -\pi/2$, and the same is true of every subsequence; but $\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \pi$ for every n , so no subsequence converges uniformly on $\mathbb{R}$. Hence “uniform on each bounded subset” is the best conclusion available.

Problem 10.10. For $1 \leq p < \infty$, show that a subspace of ℓ^p is compact if and only if it is closed, bounded, and equisummable.

Solution. Throughout, $\mathcal{S}$ is a subset of ℓ^p regarded as a metric subspace of ℓ^p , and we use the definition of equisummability stated in the Remark preceding these problems: $\mathcal{S}$ is equisummable provided that for each $\epsilon > 0$ there is an index N for which

$$\sum_{k=N}^{\infty} |x_k|^p < \epsilon \quad \text{for all } x = \{x_k\} \in \mathcal{S}.$$

Recall also that a metric space is compact if and only if it is sequentially compact, and that a compact metric space is totally bounded (Theorem 16 of the preceding chapter). We write $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}$, and we use the Minkowski Inequality freely.

Necessity. Suppose $\mathcal{S}$ is compact. By Problem 1, $\mathcal{S}$ is a closed and bounded subset of ℓ^p . It remains to prove equisummability.

Let $\epsilon > 0$ and set $\eta = \frac{1}{2}\epsilon^{1/p} > 0$. Since $\mathcal{S}$ is compact it is totally bounded, so there are points $x^{(1)}, \dots, x^{(m)}$ in $\mathcal{S}$ with

$$\mathcal{S} \subseteq \bigcup_{i=1}^m B(x^{(i)}, \eta).$$

For each i , the series $\sum_k |x_k^{(i)}|^p$ converges, so its tails tend to 0; choose N_i with $\sum_{k \geq N_i} |x_k^{(i)}|^p < \eta^p$. Let $N = \max\{N_1, \dots, N_m\}$. Now let $x \in \mathcal{S}$ and choose i with $\|x - x^{(i)}\|_p < \eta$. Applying the Minkowski Inequality to the sequences $\{x_k\}_{k \geq N}$ and $\{x_k^{(i)}\}_{k \geq N}$,

$$\left(\sum_{k=N}^{\infty} |x_k|^p \right)^{1/p} \leq \left(\sum_{k=N}^{\infty} |x_k - x_k^{(i)}|^p \right)^{1/p} + \left(\sum_{k=N}^{\infty} |x_k^{(i)}|^p \right)^{1/p} < \|x - x^{(i)}\|_p + \eta < 2\eta = \epsilon^{1/p}.$$

Hence $\sum_{k=N}^{\infty} |x_k|^p < \epsilon$ for every $x \in \mathcal{S}$, and N was chosen independently of x . Thus $\mathcal{S}$ is equisummable. Sufficiency. Suppose $\mathcal{S}$ is closed, bounded, and equisummable. We show $\mathcal{S}$ is sequentially compact, hence compact. Let $\{x^{(n)}\}$ be a sequence in $\mathcal{S}$. By boundedness there is an $M \geq 0$ with $\|x^{(n)}\|_p \leq M$ for all n .

For each n define a real-valued function g_n on the countable set $\mathbb{N}$ of coordinate indices by $g_n(k) = x_k^{(n)}$. For each fixed k we have $|g_n(k)| = |x_k^{(n)}| \leq \|x^{(n)}\|_p \leq M$ for all n , so $\{g_n\}$ is pointwise bounded on $\mathbb{N}$. By Problem 14 there is a subsequence $\{g_{n_j}\}$, that is, a subsequence $\{x^{(n_j)}\}$ of $\{x^{(n)}\}$, and a real-valued function $k \mapsto x_k$ on $\mathbb{N}$, such that

$$\lim_{j \rightarrow \infty} x_k^{(n_j)} = x_k \quad \text{for every } k.$$

(This is the Cantor diagonalization argument.) Put $x = \{x_k\}$.

First, $x \in \ell^p$ with $\|x\|_p \leq M$. Indeed, for each fixed K ,

$$\sum_{k=1}^K |x_k|^p = \lim_{j \rightarrow \infty} \sum_{k=1}^K |x_k^{(n_j)}|^p \leq \limsup_{j \rightarrow \infty} \|x^{(n_j)}\|_p^p \leq M^p,$$

since a finite sum is a continuous function of its finitely many terms. Letting $K \rightarrow \infty$ gives $\|x\|_p \leq M$. Second, we claim $\{x^{(n_j)}\} \rightarrow x$ in ℓ^p . Let $\epsilon > 0$. By equisummability choose N with

$$\sum_{k=N}^{\infty} |y_k|^p < \epsilon \quad \text{for all } y \in \mathcal{S},$$

so in particular $\sum_{k=N}^{\infty} |x_k^{(n_j)}|^p < \epsilon$ for every j . The same bound passes to the limit: for each $K \geq N$,

$$\sum_{k=N}^K |x_k|^p = \lim_{j \rightarrow \infty} \sum_{k=N}^K |x_k^{(n_j)}|^p \leq \epsilon,$$

and letting $K \rightarrow \infty$ gives $\sum_{k=N}^{\infty} |x_k|^p \leq \epsilon$. Splitting the index set at N and using the Minkowski Inequality,

$$\|x^{(n_j)} - x\|_p \leq \left(\sum_{k=1}^{N-1} |x_k^{(n_j)} - x_k|^p \right)^{1/p} + \left(\sum_{k=N}^{\infty} |x_k^{(n_j)}|^p \right)^{1/p} + \left(\sum_{k=N}^{\infty} |x_k|^p \right)^{1/p}.$$

The first term on the right is a finite sum of terms each of which tends to 0 as $j \rightarrow \infty$, so it tends to 0; the second and third are each at most $\epsilon^{1/p}$. Therefore

$$\limsup_{j \rightarrow \infty} \|x^{(n_j)} - x\|_p \leq 2\epsilon^{1/p}.$$

Since $\epsilon > 0$ was arbitrary, $\lim_{j \rightarrow \infty} \|x^{(n_j)} - x\|_p = 0$.

Finally, since $\mathcal{S}$ is a closed subset of ℓ^p and x is the limit of a sequence in $\mathcal{S}$, we have $x \in \mathcal{S}$. Thus every sequence in $\mathcal{S}$ has a subsequence converging to a point of $\mathcal{S}$, that is, $\mathcal{S}$ is sequentially compact and therefore compact.

Both implications are proved, so a subspace of ℓ^p , $1 \leq p < \infty$, is compact if and only if it is closed, bounded, and equisummable.

Problem 10.11. For a sequence of nonnegative real numbers $\{c_n\}$, let $\mathcal{S}$ be the subset of ℓ^2 consisting of those $x = \{x_n\} \in \ell^2$ such that $|x_n| \leq c_n$ for all n . Show that $\mathcal{S}$ is equisummable if $\{c_n\}$ belongs to ℓ^2 .

Solution. Assume $c = \{c_n\}$ belongs to ℓ^2 , that is, $\sum_{n=1}^{\infty} c_n^2 < \infty$.

Let $\epsilon > 0$. Since the series $\sum_{n=1}^{\infty} c_n^2$ of nonnegative terms converges, its partial sums form a Cauchy sequence, and therefore its tails converge to 0: there is an index N for which

$$\sum_{k=N}^{\infty} c_k^2 < \epsilon.$$

Now let $x = \{x_k\}$ be any element of $\mathcal{S}$. By the defining property of $\mathcal{S}$ we have $|x_k| \leq c_k$, hence $|x_k|^2 \leq c_k^2$, for every k . Summing over $k \geq N$ and using the comparison of series of nonnegative terms,

$$\sum_{k=N}^{\infty} |x_k|^2 \leq \sum_{k=N}^{\infty} c_k^2 < \epsilon.$$

The index N was chosen using only ϵ and the sequence $\{c_n\}$; it does not depend on x . Therefore for each $\epsilon > 0$ there is an index N with $\sum_{k=N}^{\infty} |x_k|^2 < \epsilon$ for all $x \in \mathcal{S}$, which is precisely the statement that $\mathcal{S}$ is equisummable.

Remark. When $\{c_n\} \in \ell^2$ the set $\mathcal{S}$ is in fact a compact subspace of ℓ^2 . It is bounded, since $\|x\|_2 \leq \|c\|_2$ for every $x \in \mathcal{S}$. It is closed: if $\{x^{(j)}\}$ is a sequence in $\mathcal{S}$ converging in ℓ^2 to $y = \{y_k\}$, then for each fixed k we have $|x_k^{(j)} - y_k| \leq \|x^{(j)} - y\|_2 \rightarrow 0$, so $y_k = \lim_j x_k^{(j)}$ and hence $|y_k| = \lim_j |x_k^{(j)}| \leq c_k$; thus

$y \in \mathcal{S}$. Being closed, bounded, and equisummable, $\mathcal{S}$ is compact by Problem 10. The special case $c_n = 1/n$ is the Hilbert cube.

The hypothesis $\{c_n\} \in \ell^2$ cannot simply be dropped. If, for instance, $c_n = 1$ for all n , then $\mathcal{S}$ is the whole closed unit ball of ℓ^∞ intersected with ℓ^2 , and it contains every standard basis vector e_n (the sequence with 1 in the n -th place and 0 elsewhere); since $\sum_{k \geq N} |(e_n)_k|^2 = 1$ whenever $n \geq N$, no index N can respond to $\epsilon = 1/2$, so $\mathcal{S}$ is not equisummable.

Problem 10.12. For $1 \leq p \leq \infty$, show that the closed unit ball in the Banach space ℓ^p is not compact.

Solution. Let $B = \{x \in \ell^p \mid \|x\|_p \leq 1\}$ denote the closed unit ball of ℓ^p . For each natural number n let $e_n \in \ell^p$ be the sequence whose n -th term is 1 and whose other terms are 0; that is, $(e_n)_k = 1$ if $k = n$ and $(e_n)_k = 0$ if $k \neq n$.

For $1 \leq p < \infty$ we have $\|e_n\|_p = (\sum_k |(e_n)_k|^p)^{1/p} = 1$, and for $p = \infty$ we have $\|e_n\|_\infty = \sup_k |(e_n)_k| = 1$. In every case $e_n \in B$, so $\{e_n\}$ is a sequence in B .

Now compute the distance between two distinct members of this sequence. Let $n \neq m$. The sequence $e_n - e_m$ has the value 1 in the n -th place, the value -1 in the m -th place, and 0 elsewhere. Hence for $1 \leq p < \infty$,

$$\|e_n - e_m\|_p = (|1|^p + |-1|^p)^{1/p} = 2^{1/p},$$

while for $p = \infty$,

$$\|e_n - e_m\|_\infty = 1.$$

In either case

$$\|e_n - e_m\|_p \geq 1 \quad \text{whenever } n \neq m,$$

since $2^{1/p} \geq 1$ for $1 \leq p < \infty$.

Consequently no subsequence of $\{e_n\}$ is a Cauchy sequence: if $\{e_{n_k}\}$ is a subsequence, then its terms are pairwise distinct members of $\{e_n\}$, so $\|e_{n_k} - e_{n_l}\|_p \geq 1$ for all $k \neq l$, and the Cauchy criterion fails for $\epsilon = 1$. A convergent sequence in a metric space is Cauchy, so no subsequence of $\{e_n\}$ converges in ℓ^p , and in particular no subsequence converges to a point of B .

Therefore B is not sequentially compact. Since a metric space is compact if and only if it is sequentially compact, the closed unit ball B of ℓ^p is not compact, for any p with $1 \leq p \leq \infty$.

Remark. For $1 \leq p < \infty$ this also follows from Problem 10: the ball B is closed and bounded, but it is not equisummable, because for every index N the element $e_N \in B$ satisfies $\sum_{k=N}^\infty |(e_N)_k|^p = 1$, so no N responds to the challenge $\epsilon = 1/2$. The direct argument above has the advantage of covering $p = \infty$ as well. Observe also that B is a closed and bounded subset of the metric space ℓ^p ; thus, as the text points out, in a general metric space closed and bounded does not imply compact.

Problem 10.13. For $1 \leq p \leq \infty$, show that the closed unit ball in the Banach space $L^p[0, 1]$ is not compact.

Solution. Let $B = \{f \in L^p[0, 1] \mid \|f\|_p \leq 1\}$ be the closed unit ball of $L^p[0, 1]$. As in the preceding problem it suffices to exhibit a sequence in B no subsequence of which is Cauchy, for then B fails to be sequentially compact and hence, a metric space being compact if and only if it is sequentially compact, fails to be compact.

Split $[0, 1]$ into countably many pairwise disjoint measurable sets of positive measure: for each natural number n put

$$E_n = \left[\frac{1}{n+1}, \frac{1}{n} \right), \quad m(E_n) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0.$$

These sets are pairwise disjoint and contained in $[0, 1]$. Define

$$f_n = \frac{\chi_{E_n}}{\|\chi_{E_n}\|_p}.$$

This makes sense because $\|\chi_{E_n}\|_p \neq 0$: for $1 \leq p < \infty$ we have $\|\chi_{E_n}\|_p = m(E_n)^{1/p} > 0$, and for $p = \infty$ we have $\|\chi_{E_n}\|_\infty = 1$, since $m(E_n) > 0$ means χ_{E_n} equals 1 on a set of positive measure. By construction $\|f_n\|_p = 1$, so $f_n \in B$ for every n .

Now fix $n \neq m$. Since E_n and E_m are disjoint, the function $f_n - f_m$ equals f_n on E_n , equals $-f_m$ on E_m , and equals 0 off $E_n \cup E_m$.

Case $1 \leq p < \infty$. Then, by the countable (here, finite) additivity of integration over disjoint sets,

$$\|f_n - f_m\|_p^p = \int_{E_n} |f_n|^p + \int_{E_m} |f_m|^p = \|f_n\|_p^p + \|f_m\|_p^p = 1 + 1 = 2,$$

so $\|f_n - f_m\|_p = 2^{1/p} \geq 1$.

Case $p = \infty$. Here $f_n = \chi_{E_n}$, and $|f_n - f_m| = 1$ on the set E_n , which has positive measure. Hence the essential supremum of $|f_n - f_m|$ is at least 1, that is, $\|f_n - f_m\|_\infty \geq 1$. (In fact it equals 1.)

In every case

$$\|f_n - f_m\|_p \geq 1 \quad \text{whenever } n \neq m.$$

Therefore no subsequence $\{f_{n_k}\}$ of $\{f_n\}$ can be Cauchy in $L^p[0, 1]$: its terms are pairwise distinct members of $\{f_n\}$, so $\|f_{n_k} - f_{n_l}\|_p \geq 1$ for $k \neq l$, and the Cauchy criterion fails for $\epsilon = 1$. Since convergent sequences are Cauchy, no subsequence of $\{f_n\}$ converges in $L^p[0, 1]$.

Thus B contains a sequence with no subsequence converging to a point of B , so B is not sequentially compact and hence not compact.

Remark. The set B is a closed and bounded subset of the metric space $L^p[0, 1]$: it is bounded by definition, and it is closed because the norm is a continuous function on $L^p[0, 1]$, so $\{f \mid \|f\|_p \leq 1\}$ is the preimage of the closed set $(-\infty, 1]$. This is another instance, alongside the closed unit balls of $C[0, 1]$ and of ℓ^p , of a closed bounded subset of a metric space that is not compact.

Problem 10.14. Let S be a countable set and $\{f_n\}$ a sequence of real-valued functions on S that is pointwise bounded on S . Show that there is a subsequence of $\{f_n\}$ that converges pointwise on S to a real-valued function.

Solution. Recall that pointwise boundedness means: for each $s \in S$, the sequence of real numbers $\{f_n(s)\}$ is bounded. The argument is the Cantor diagonalization argument, isolated here from the metric setting of the Arzelà–Ascoli Lemma; no structure on S beyond countability is used.

If S is empty the statement is vacuous, and if S is finite, say $S = \{s_1, \dots, s_M\}$, the inductive selection below terminates after M steps and the M -th subsequence already works. So assume S is countably infinite and let $\{s_j\}_{j=1}^\infty$ be an enumeration of S .

Construction of the nested subsequences. Consider the sequence of real numbers $n \mapsto f_n(s_1)$. By hypothesis it is bounded, so by the Bolzano–Weierstrass Theorem it has a convergent subsequence: there is a strictly increasing sequence of natural numbers $\{s(1, n)\}_{n=1}^\infty$ and a real number a_1 with

$$\lim_{n \rightarrow \infty} f_{s(1, n)}(s_1) = a_1.$$

Suppose that for some $j \geq 1$ strictly increasing sequences $\{s(1, n)\}, \dots, \{s(j, n)\}$ have been chosen, each a subsequence of its predecessor, together with real numbers $a_1, \dots, a_j$ satisfying $\lim_{n \rightarrow \infty} f_{s(i, n)}(s_i) = a_i$ for $1 \leq i \leq j$. The sequence of real numbers $n \mapsto f_{s(j, n)}(s_{j+1})$ is bounded, since $\{f_n(s_{j+1})\}$ is bounded and this is a subsequence of it. By the Bolzano–Weierstrass Theorem there is a subsequence $\{s(j+1, n)\}$ of $\{s(j, n)\}$ and a real number a_{j+1} with

$$\lim_{n \rightarrow \infty} f_{s(j+1, n)}(s_{j+1}) = a_{j+1}.$$

By induction we obtain, for every $j \geq 1$, a strictly increasing sequence of natural numbers $\{s(j, n)\}_{n=1}^\infty$ and a real number a_j such that $\{s(j+1, n)\}$ is a subsequence of $\{s(j, n)\}$ and $\lim_{n \rightarrow \infty} f_{s(j, n)}(s_j) = a_j$. The diagonal subsequence. Put

$$n_k = s(k, k) \quad \text{for } k = 1, 2, \dots$$

The sequence $\{n_k\}$ is strictly increasing. Indeed, $\{s(k+1, n)\}_n$ is a subsequence of the strictly increasing sequence $\{s(k, n)\}_n$, and a subsequence of a strictly increasing sequence of natural numbers satisfies $s(k+1, n) \geq s(k, n)$ for every n ; hence

$$n_{k+1} = s(k+1, k+1) \geq s(k, k+1) > s(k, k) = n_k,$$

the last inequality because $\{s(k, n)\}_n$ is strictly increasing in n . Therefore $\{f_{n_k}\}$ is a genuine subsequence of $\{f_n\}$.

Fix an index j . For every $k \geq j$ the number $n_k = s(k, k)$ is a term of the sequence $\{s(j, n)\}_n$, because $\{s(k, n)\}_n$ is a subsequence of $\{s(j, n)\}_n$ when $k \geq j$; and since $\{n_k\}$ is strictly increasing, the tail $\{n_k\}_{k \geq j}$ is a subsequence of $\{s(j, n)\}_n$. A subsequence of a convergent sequence of real numbers converges to the same limit, so

$$\lim_{k \rightarrow \infty} f_{n_k}(s_j) = a_j.$$

Conclusion. Define $f: S \rightarrow \mathbb{R}$ by $f(s_j) = a_j$ for each j ; this is well defined because $\{s_j\}$ is an enumeration of S , so each point of S is s_j for exactly one j , and each a_j is a real number. By the display above, for every $s \in S$,

$$\lim_{k \rightarrow \infty} f_{n_k}(s) = f(s).$$

Thus $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$ converging pointwise on S to the real-valued function f , as required.

Remark. This is exactly the first half of the proof of the Arzelà–Ascoli Lemma (Lemma 2), where S is taken to be a countable dense subset of a separable metric space X ; the second half of that proof uses equicontinuity to upgrade convergence on the dense set to convergence on all of X . Note that no continuity or equicontinuity hypothesis is needed here, and that the limit function f is in general only a function on S with no regularity properties. Note also that pointwise boundedness cannot be weakened: if some $\{f_n(s)\}$ were unbounded, one could arrange that no subsequence converges at s to a real number, for instance with $S = \{s\}$ and $f_n(s) = n$.

10.3 Problems 10.15–10.21

Problem 10.15. Let E be a subset of a metric space X . Show that $\text{bd } E$ is closed. Also show that if E is closed, then the interior of $\text{bd } E$ is empty.

Solution. Throughout we use the decomposition recorded in (5) of Section 10.2:

$$X = \text{int } E \cup \text{ext } E \cup \text{bd } E,$$

the union being disjoint. Here $\text{int } E$ is the set of points x possessing an open ball centered at x that is contained in E , $\text{ext } E = \text{int}(X \sim E)$, and $\text{bd } E$ is the set of points every ball about which contains points of E and points of $X \sim E$. Write ρ for the metric on X .

Step 1: the interior of any set is open. Let $A \subseteq X$ and let $x \in \text{int } A$, say $B(x, r) \subseteq A$. If $y \in B(x, r)$, put $s = r - \rho(x, y) > 0$. By the triangle inequality $B(y, s) \subseteq B(x, r) \subseteq A$; in particular $y \in B(y, s) \subseteq A$, so y is a point of A possessing a ball centered at it and contained in A , that is, $y \in \text{int } A$. Thus $B(x, r) \subseteq \text{int } A$, and $\text{int } A$ is open.

Step 2: $\text{bd } E$ is closed. By Step 1 both $\text{int } E$ and $\text{ext } E = \text{int}(X \sim E)$ are open, hence so is their union. Since the three sets in (5) are disjoint and exhaust X ,

$$\text{bd } E = X \sim [\text{int } E \cup \text{ext } E],$$

which is the complement of an open set and therefore closed.

Step 3: $\text{bd } E \subseteq \overline{E}$ for every subset E . If $x \in \text{bd } E$, then every open ball centered at x contains a point of E ; that is, x is a point of closure of E , so $x \in \overline{E}$.

Step 4: if E is closed then $\text{int}(\text{bd } E) = \emptyset$. Let $\mathcal{U}$ be an open subset of X with $\mathcal{U} \subseteq \text{bd } E$; we show $\mathcal{U} = \emptyset$, which is precisely the assertion that $\text{bd } E$ has empty interior, that is, that $\text{bd } E$ is hollow. By Step 3 and the closedness of E ,

$$\mathcal{U} \subseteq \text{bd } E \subseteq \overline{E} = E.$$

Thus $\mathcal{U}$ is an open subset of X contained in E , so each of its points is an interior point of E , giving $\mathcal{U} \subseteq \text{int } E$. But also $\mathcal{U} \subseteq \text{bd } E$, while $\text{int } E \cap \text{bd } E = \emptyset$ by the disjointness in (5). Hence $\mathcal{U} = \emptyset$. Therefore the boundary of a closed set is a closed hollow set. This is the fact left as an exercise in the proof of Corollary 5. The hypothesis that E be closed cannot be dropped: for $E = \mathbb{Q} \subseteq \mathbb{R}$ one has $\text{bd } E = \mathbb{R}$, whose interior is all of $\mathbb{R}$.

Problem 10.16. In a metric space X , show that a subset E is nowhere dense if and only if for each open subset $\mathcal{O}$ of X , $E \cap \mathcal{O}$ is not dense in $\mathcal{O}$.

Solution. Recall the definitions in force. A subset of X is hollow provided it has empty interior; E is nowhere dense provided $\overline{E}$ is hollow; and a subset A of a metric space Y is dense in Y provided every nonempty open subset of Y contains a point of A . In the statement $\mathcal{O}$ is regarded as a metric subspace of X , and of course $\mathcal{O}$ is tacitly nonempty, since the empty set is vacuously dense in itself.

We first record the observation that makes the whole exercise routine. Since $\mathcal{O}$ is an open subset of X , a subset $\mathcal{U}$ of $\mathcal{O}$ is open in the metric subspace $\mathcal{O}$ if and only if it is open in X . Indeed, open subsets of the subspace $\mathcal{O}$ are exactly the sets $\mathcal{V} \cap \mathcal{O}$ with $\mathcal{V}$ open in X , because for $x \in \mathcal{O}$ the ball of radius r about x in the subspace $\mathcal{O}$ is $B(x, r) \cap \mathcal{O}$; and $\mathcal{V} \cap \mathcal{O}$ is open in X as an intersection of two sets open in X . Conversely a subset of $\mathcal{O}$ that is open in X equals its own intersection with $\mathcal{O}$, so it is open in $\mathcal{O}$. Assume first that E is nowhere dense, so $\text{int } \overline{E} = \emptyset$. Let $\mathcal{O}$ be a nonempty open subset of X and suppose, to get a contradiction, that $E \cap \mathcal{O}$ is dense in $\mathcal{O}$. Let $x \in \mathcal{O}$ and let $\epsilon > 0$. Since $\mathcal{O}$ is open there is r with $0 < r < \epsilon$ and $B(x, r) \subseteq \mathcal{O}$. By the observation above, $B(x, r)$ is a nonempty open subset of the space $\mathcal{O}$, so by density it contains a point of $E \cap \mathcal{O}$, in particular a point of E . Since $B(x, r) \subseteq B(x, \epsilon)$, every ball about x meets E , so x is a point of closure of E . As $x \in \mathcal{O}$ was arbitrary, $\mathcal{O} \subseteq \overline{E}$. But $\mathcal{O}$ is open and nonempty, so $\text{int } \overline{E} \supseteq \mathcal{O} \neq \emptyset$, contradicting the assumption that $\overline{E}$ is hollow. Hence $E \cap \mathcal{O}$ is not dense in $\mathcal{O}$.

Conversely, suppose E is not nowhere dense, that is, $\text{int } \overline{E} \neq \emptyset$. Choose a nonempty open subset $\mathcal{O}$ of X with $\mathcal{O} \subseteq \overline{E}$; for instance $\mathcal{O} = \text{int } \overline{E}$, which is open by Step 1 of the solution of Problem 15. We claim $E \cap \mathcal{O}$ is dense in $\mathcal{O}$, so that the right-hand condition fails. Let $\mathcal{U}$ be a nonempty open subset of the space $\mathcal{O}$. By the observation above, $\mathcal{U}$ is open in X . Pick $x \in \mathcal{U}$. Then $x \in \mathcal{O} \subseteq \overline{E}$, so x is a point of closure of E , and $\mathcal{U}$ is an open set containing x , hence contains a ball $B(x, r)$, which must meet E . Therefore $\mathcal{U} \cap E \neq \emptyset$, and since $\mathcal{U} \subseteq \mathcal{O}$ we get $\mathcal{U} \cap (E \cap \mathcal{O}) \neq \emptyset$. Thus every nonempty open subset of $\mathcal{O}$ meets $E \cap \mathcal{O}$, that is, $E \cap \mathcal{O}$ is dense in $\mathcal{O}$.

By contraposition, if $E \cap \mathcal{O}$ fails to be dense in $\mathcal{O}$ for every nonempty open $\mathcal{O} \subseteq X$, then E is nowhere dense. The two implications together give the asserted equivalence.

Problem 10.17. In a complete metric space X , is the union of a countable collection of nowhere dense sets also nowhere dense?

Solution. No. Completeness buys the conclusion of the Baire Category Theorem, namely that such a union is hollow; it does not upgrade that to nowhere dense, which is the much stronger requirement that the closure of the union be hollow.

Here is the standard counterexample. Take $X = \mathbb{R}$ with the usual metric, which is a complete metric space by Theorem 12 of Chapter 9. Enumerate the rational numbers as $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$ and set $E_n = \{q_n\}$. Each E_n is nowhere dense. Indeed a singleton is closed, so $\overline{E_n} = E_n = \{q_n\}$, and $\{q_n\}$ has empty interior because no open interval, hence no ball $B(q_n, r) = (q_n - r, q_n + r)$ with $r > 0$, is a single point: such a ball contains $q_n + r/2 \neq q_n$. Thus $\overline{E_n}$ is hollow, which is exactly the statement that E_n is nowhere dense.

However

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{Q},$$

and $\overline{\mathbb{Q}} = \mathbb{R}$, since every real number is the limit of a sequence of rationals, so every ball about a real number contains a rational. Hence $\text{int } \overline{\mathbb{Q}} = \text{int } \mathbb{R} = \mathbb{R} \neq \emptyset$, so $\overline{\mathbb{Q}}$ is not hollow and $\mathbb{Q}$ is not nowhere

dense. In fact $\mathbb{Q}$ is dense in $\mathbb{R}$, which is the extreme opposite of being nowhere dense.

This is consistent with the Baire Category Theorem, part (ii), applied to the closed hollow sets $\overline{E_n} = \{q_n\}$: it predicts only that $\mathbb{Q}$ itself is hollow, and indeed $\text{int } \mathbb{Q} = \emptyset$, since every interval contains irrational numbers.

The general principle behind the failure is that closure does not commute with countable unions: one always has $\bigcup_n \overline{E_n} \subseteq \overline{\bigcup_n E_n}$, but the inclusion is typically strict for infinite unions, and it is the closure of the union, not the union of the closures, that must be hollow for the union to be nowhere dense. For a finite collection the answer is affirmative, and completeness is not needed. It suffices to treat two sets A, B that are nowhere dense and then induct. Since closure commutes with finite unions, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Let $\mathcal{U}$ be an open set with $\mathcal{U} \subseteq \overline{A} \cup \overline{B}$ and suppose $\mathcal{U} \neq \emptyset$. The set $\mathcal{U} \sim \overline{A}$ is open, being the intersection of the open set $\mathcal{U}$ with the open set $X \sim \overline{A}$, and it is contained in $\overline{B}$. Since $\overline{B}$ is hollow, $\mathcal{U} \sim \overline{A} = \emptyset$, that is, $\mathcal{U} \subseteq \overline{A}$. Since $\overline{A}$ is hollow, $\mathcal{U} = \emptyset$, a contradiction. Hence $\overline{A \cup B}$ is hollow and $A \cup B$ is nowhere dense.

Problem 10.18. Let $\mathcal{O}$ be an open subset and F be a closed subset of a metric space X . Show that both $\overline{\mathcal{O}} \sim \mathcal{O}$ and $F \sim \text{int } F$ are closed and hollow.

Solution. Recall that a subset of X is hollow provided it has empty interior, and that by Step 1 of the solution of Problem 15 the interior of any set is open.

Consider first $\overline{\mathcal{O}} \sim \mathcal{O}$, where $\mathcal{O}$ is open.

That it is closed: $\overline{\mathcal{O}} \sim \mathcal{O} = \overline{\mathcal{O}} \cap (X \sim \mathcal{O})$. The closure $\overline{\mathcal{O}}$ is closed, and $X \sim \mathcal{O}$ is closed because $\mathcal{O}$ is open. The intersection of two closed sets is closed.

That it is hollow: let $\mathcal{U}$ be an open subset of X with $\mathcal{U} \subseteq \overline{\mathcal{O}} \sim \mathcal{O}$, and suppose $\mathcal{U} \neq \emptyset$. Choose $x \in \mathcal{U}$. Since $\mathcal{U} \subseteq \overline{\mathcal{O}}$, the point x is a point of closure of $\mathcal{O}$, so every ball centered at x contains a point of $\mathcal{O}$. Since $\mathcal{U}$ is open there is $r > 0$ with $B(x, r) \subseteq \mathcal{U}$, and $B(x, r)$ contains a point of $\mathcal{O}$. Hence $\mathcal{U} \cap \mathcal{O} \neq \emptyset$, contradicting $\mathcal{U} \subseteq X \sim \mathcal{O}$. Therefore $\mathcal{U} = \emptyset$, and $\overline{\mathcal{O}} \sim \mathcal{O}$ has empty interior.

Now consider $F \sim \text{int } F$, where F is closed.

That it is closed: $F \sim \text{int } F = F \cap (X \sim \text{int } F)$. Here F is closed by hypothesis, and $\text{int } F$ is open, so $X \sim \text{int } F$ is closed. Again an intersection of two closed sets is closed.

That it is hollow: let $\mathcal{U}$ be an open subset of X with $\mathcal{U} \subseteq F \sim \text{int } F$. In particular $\mathcal{U}$ is an open set contained in F , so every point of $\mathcal{U}$ has a ball about it contained in $\mathcal{U} \subseteq F$; that is, $\mathcal{U} \subseteq \text{int } F$. On the other hand $\mathcal{U} \cap \text{int } F = \emptyset$ by the definition of $F \sim \text{int } F$. Hence $\mathcal{U} = \mathcal{U} \cap \text{int } F = \emptyset$, and $F \sim \text{int } F$ has empty interior.

Two remarks tie this to Problem 15. First observe that for any subset E of X one has $\text{ext } E = X \sim \overline{E}$: a point x fails to be a point of closure of E precisely when some ball centered at x misses E , that is, precisely when some ball centered at x is contained in $X \sim E$, which is the definition of an exterior point. Feeding this into the disjoint decomposition (5) of Section 10.2 gives

$$\overline{E} = X \sim \text{ext } E = \text{int } E \cup \text{bd } E, \quad \text{the union disjoint.}$$

Hence for an open set $\mathcal{O}$, where $\mathcal{O} = \text{int } \mathcal{O}$, one has $\overline{\mathcal{O}} \sim \mathcal{O} = \text{bd } \mathcal{O}$; and for a closed set F , where $F = \overline{F}$, one has $F \sim \text{int } F = \text{bd } F$. So the present problem is exactly the statement that the boundary of an open set and the boundary of a closed set are closed and hollow, and it supplies the fact used in the proof of Corollary 5, that $\text{bd } F_n$ is closed and hollow for each closed F_n .

Consequently, both sets are nowhere dense: each is closed, so it equals its own closure, and that closure is hollow.

Problem 10.19. In a complete metric space, is the union of a countable collection of sets of the first category also of the first category?

Solution. Yes. Moreover completeness plays no role: the assertion holds in every metric space, since being of the first category is purely a countability statement about a decomposition into nowhere dense pieces.

Recall the terminology from the Remark following the Baire Category Theorem: a subset A of a metric

space X is of the first category, or meager, provided A is the union of a countable collection of nowhere dense subsets of X .

Let $\{A_k\}_{k=1}^\infty$ be a countable collection of subsets of X , each of the first category, and put $A = \bigcup_{k=1}^\infty A_k$. For each index k choose, by the definition of first category, a countable collection $\{E_{k,n}\}_{n=1}^\infty$ of nowhere dense subsets of X with

$$A_k = \bigcup_{n=1}^\infty E_{k,n}.$$

There is no loss of generality in indexing each of these collections by all of $\mathbb{N}$: a finite collection $E_{k,1}, \dots, E_{k,m}$ may be extended by setting $E_{k,n} = \emptyset$ for $n > m$, and the empty set is nowhere dense since $\bar{\emptyset} = \emptyset$ has empty interior.

Then

$$A = \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty E_{k,n} = \bigcup_{(k,n) \in \mathbb{N} \times \mathbb{N}} E_{k,n},$$

the equality of the two right-hand sides being immediate from the definition of union: a point lies in the left side if and only if it lies in $E_{k,n}$ for some pair (k, n) .

The set $\mathbb{N} \times \mathbb{N}$ is countable. Hence there is a one-to-one correspondence $j \mapsto (k_j, n_j)$ of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$, and setting $G_j = E_{k_j, n_j}$ we obtain

$$A = \bigcup_{j=1}^\infty G_j,$$

a union of a countable collection of nowhere dense subsets of X . Therefore A is of the first category.

The same argument applies verbatim to a finite collection of first category sets, and shows in particular that the collection of sets of the first category in X is closed under countable unions and, trivially, under passage to subsets, since a subset of a nowhere dense set is nowhere dense.

It is worth contrasting this with Problem 17. There the answer was negative because “nowhere dense” is not stable under countable unions; here the answer is positive precisely because “first category” was defined so as to be the countable-union closure of “nowhere dense”, and a countable union of countable unions is again a countable union. Completeness enters only when one wants the further conclusion, supplied by the Baire Category Theorem, that in a complete metric space a set of the first category is hollow and hence is a proper subset of X whenever $X \neq \emptyset$.

Problem 10.20. Let F_n be the subset of $C[0, 1]$ consisting of functions for which there is a point x_0 in $[0, 1]$ such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0, 1]$. Show that F_n is closed. Show that F_n is hollow by observing that for $f \in C[0, 1]$ and $r > 0$ there is a piecewise linear function $g \in C[0, 1]$ for which $\rho_{\max}(f, g) < r$ and the left-hand and right-hand derivatives of g on $[0, 1]$ are greater than $n + 1$. Conclude that $C[0, 1] \neq \bigcup_{n=1}^\infty F_n$ and show that each $h \in C[0, 1] \sim \bigcup_{n=1}^\infty F_n$ fails to be differentiable at any point in $(0, 1)$.

Solution. Throughout, $C[0, 1]$ carries the metric $\rho_{\max}(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$, so that convergence in this metric is uniform convergence on $[0, 1]$. By Theorem 12 of Chapter 9, part (iii), $C[0, 1]$ is a complete metric space.

Part 1: each F_n is closed.

By Proposition 6 of Chapter 9 it suffices to show that F_n is sequentially closed. So let $\{f_k\}$ be a sequence in F_n with $f_k \rightarrow f$ in $C[0, 1]$, that is, uniformly on $[0, 1]$. For each k choose $x_k \in [0, 1]$ witnessing membership in F_n :

$$|f_k(x) - f_k(x_k)| \leq n|x - x_k| \quad \text{for all } x \in [0, 1].$$

The sequence $\{x_k\}$ lies in the closed bounded interval $[0, 1]$, so by the Bolzano-Weierstrass Theorem it has a subsequence $\{x_{k_j}\}$ converging to some x_0 , and $x_0 \in [0, 1]$ since $[0, 1]$ is closed.

Fix $x \in [0, 1]$. For each j , the triangle inequality gives

$$|f(x) - f(x_0)| \leq |f(x) - f_{k_j}(x)| + |f_{k_j}(x) - f_{k_j}(x_{k_j})| + |f_{k_j}(x_{k_j}) - f(x_0)|.$$

We estimate the three terms as $j \rightarrow \infty$. The first is at most $\rho_{\max}(f, f_{k_j}) \rightarrow 0$. The second is at most $n|x - x_{k_j}| \rightarrow n|x - x_0|$. For the third,

$$|f_{k_j}(x_{k_j}) - f(x_0)| \leq |f_{k_j}(x_{k_j}) - f(x_{k_j})| + |f(x_{k_j}) - f(x_0)| \leq \rho_{\max}(f_{k_j}, f) + |f(x_{k_j}) - f(x_0)|,$$

and both summands tend to 0, the first by uniform convergence and the second because f is continuous at x_0 and $x_{k_j} \rightarrow x_0$. Letting $j \rightarrow \infty$ we obtain

$$|f(x) - f(x_0)| \leq n|x - x_0|.$$

Since $x \in [0, 1]$ was arbitrary and $x_0 \in [0, 1]$, we conclude $f \in F_n$. Hence F_n is closed.

Part 2: each F_n is hollow.

It suffices to show that no ball $B(f, r) = \{g : \rho_{\max}(f, g) < r\}$ is contained in F_n , for then no point of F_n is an interior point. So fix $f \in C[0, 1]$ and $r > 0$; we construct $g \in B(f, r)$ with $g \notin F_n$.

First we approximate f by a piecewise linear function. Since $[0, 1]$ is a compact metric space, by Theorem 20 of Chapter 9, and f is continuous, f is uniformly continuous by Proposition 23 of Chapter 9. Choose $\delta > 0$ such that $|f(s) - f(t)| < r/4$ whenever $s, t \in [0, 1]$ and $|s - t| \leq \delta$. Choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ with $t_i - t_{i-1} < \delta$ for each i , and let $p \in C[0, 1]$ be the function that agrees with f at each t_i and is affine on each $[t_{i-1}, t_i]$. If $x \in [t_{i-1}, t_i]$, then $p(x)$ is a convex combination of $f(t_{i-1})$ and $f(t_i)$, so

$$|p(x) - f(x)| \leq \max\{|f(t_{i-1}) - f(x)|, |f(t_i) - f(x)|\} < r/4,$$

since $|x - t_{i-1}| \leq \delta$ and $|x - t_i| \leq \delta$. Hence $\rho_{\max}(f, p) \leq r/4$. Let K be the maximum of the absolute values of the slopes of the finitely many affine pieces of p .

Next we superimpose a rapid sawtooth. Choose a number $M > K + n + 1$, put $a = r/(4M) > 0$, and let $s: [0, 1] \rightarrow \mathbb{R}$ be the continuous, $2a$ -periodic zigzag determined by $s(0) = 0$, s affine with slope $+M$ on $[0, a]$, slope $-M$ on $[a, 2a]$, and so on alternately. Then s is piecewise linear with finitely many pieces on $[0, 1]$, every piece has slope $\pm M$, and $\max_{[0, 1]} |s| = Ma = r/4$.

Set $g = p + s$. Then g is piecewise linear, its breakpoints being among the t_i together with the multiples of a in $[0, 1]$, and

$$\rho_{\max}(f, g) \leq \rho_{\max}(f, p) + \max_{[0, 1]} |s| \leq r/4 + r/4 = r/2 < r,$$

so $g \in B(f, r)$. On each affine piece of g the slope has the form $c \pm M$ with $|c| \leq K$, so its absolute value is at least $M - K > n + 1$. Consequently at each $x \in [0, 1)$ the right-hand derivative $D^+g(x)$ exists and equals the slope of the affine piece of g immediately to the right of x , and at each $x \in (0, 1]$ the left-hand derivative $D^-g(x)$ exists and equals the slope of the piece immediately to the left; in every case the absolute value of that one-sided derivative exceeds $n + 1$.

Now suppose, for a contradiction, that $g \in F_n$, witnessed by $x_0 \in [0, 1]$:

$$|g(x) - g(x_0)| \leq n|x - x_0| \quad \text{for all } x \in [0, 1].$$

Then for every $x \neq x_0$ in $[0, 1]$ the difference quotient satisfies $\left| \frac{g(x) - g(x_0)}{x - x_0} \right| \leq n$. If $x_0 < 1$, let $x \rightarrow x_0^+$ to get $|D^+g(x_0)| \leq n$; if $x_0 = 1$, let $x \rightarrow 1^-$ to get $|D^-g(1)| \leq n$. Either way we contradict the fact that all one-sided derivatives of g exceed $n + 1$ in absolute value. Hence $g \notin F_n$.

Thus $B(f, r) \not\subseteq F_n$ for every $f \in C[0, 1]$ and every $r > 0$, so $\text{int } F_n = \emptyset$ and F_n is hollow.

Part 3: $C[0, 1] \neq \bigcup_{n=1}^{\infty} F_n$.

The space $C[0, 1]$ is complete, and by Parts 1 and 2 the collection $\{F_n\}_{n=1}^{\infty}$ consists of closed hollow subsets of it. By the Baire Category Theorem, part (ii), the union $\bigcup_{n=1}^{\infty} F_n$ is hollow. But $C[0, 1]$ is not hollow: it is an open subset of itself and it is nonempty, for example the zero function belongs to it, so $\text{int } C[0, 1] = C[0, 1] \neq \emptyset$. Therefore $\bigcup_{n=1}^{\infty} F_n \neq C[0, 1]$, and indeed its complement

$$C[0, 1] \sim \bigcup_{n=1}^{\infty} F_n$$

is dense in $C[0, 1]$, by the equivalence (6) of Section 10.2.

Part 4: such an h is nowhere differentiable on $(0, 1)$.

Let $h \in C[0, 1] \sim \bigcup_{n=1}^{\infty} F_n$ and suppose, for a contradiction, that h is differentiable at some point $x_0 \in (0, 1)$. Since h is continuous on the compact set $[0, 1]$, it is bounded there; put $\|h\|_{\infty} = \max_{[0,1]} |h| < \infty$. By differentiability at x_0 there is $\delta > 0$ such that for all $x \in [0, 1]$ with $0 < |x - x_0| < \delta$,

$$\left| \frac{h(x) - h(x_0)}{x - x_0} - h'(x_0) \right| < 1, \quad \text{hence} \quad |h(x) - h(x_0)| \leq (|h'(x_0)| + 1)|x - x_0|.$$

This last inequality also holds trivially at $x = x_0$. For $x \in [0, 1]$ with $|x - x_0| \geq \delta$ we instead estimate crudely:

$$|h(x) - h(x_0)| \leq 2\|h\|_{\infty} = \frac{2\|h\|_{\infty}}{\delta} \cdot \delta \leq \frac{2\|h\|_{\infty}}{\delta}|x - x_0|.$$

Choose a natural number N with

$$N \geq \max \left\{ |h'(x_0)| + 1, \frac{2\|h\|_{\infty}}{\delta} \right\}.$$

Combining the two cases, $|h(x) - h(x_0)| \leq N|x - x_0|$ for all $x \in [0, 1]$, so $h \in F_N$, contradicting the choice of h .

Therefore h fails to be differentiable at every point of $(0, 1)$. Since by Part 3 the set of such h is a dense subset of $C[0, 1]$, nowhere differentiable continuous functions on $[0, 1]$ are not merely present but typical.

Problem 10.21. Let f be a real-valued function on a metric space X . Show that the set of points at which f is continuous is the intersection of a countable collection of open sets. Conclude that there is not a real-valued function on $\mathbb{R}$ that is continuous just at the rational numbers.

Solution. Let ρ denote the metric on X and let $B(x, \delta)$ denote the open ball of radius δ centered at x .

Part 1: the oscillation of f .

For $x \in X$ and $\delta > 0$ define

$$\omega(x, \delta) = \sup \{ |f(y) - f(z)| : y, z \in B(x, \delta) \} \in [0, \infty],$$

and define the oscillation of f at x by

$$\omega(x) = \inf_{\delta > 0} \omega(x, \delta).$$

Since $\delta \mapsto \omega(x, \delta)$ is nondecreasing, this infimum is also the limit of $\omega(x, \delta)$ as $\delta \rightarrow 0^+$.

Claim: f is continuous at x if and only if $\omega(x) = 0$.

Suppose f is continuous at x and let $\epsilon > 0$. Choose $\delta > 0$ so that $|f(y) - f(x)| < \epsilon/2$ for all $y \in B(x, \delta)$. Then for $y, z \in B(x, \delta)$,

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \epsilon,$$

so $\omega(x, \delta) \leq \epsilon$ and hence $\omega(x) \leq \epsilon$. As $\epsilon > 0$ was arbitrary, $\omega(x) = 0$.

Conversely suppose $\omega(x) = 0$ and let $\epsilon > 0$. Choose $\delta > 0$ with $\omega(x, \delta) < \epsilon$. Since $x \in B(x, \delta)$, for every $y \in B(x, \delta)$ the pair (y, x) is admissible in the supremum, so $|f(y) - f(x)| \leq \omega(x, \delta) < \epsilon$. Hence f is continuous at x . This proves the claim.

Part 2: the continuity set is a countable intersection of open sets.

For each natural number k define

$$\mathcal{O}_k = \{x \in X : \omega(x) < 1/k\}.$$

Each $\mathcal{O}_k$ is open. Indeed, let $x \in \mathcal{O}_k$. Since $\omega(x) < 1/k$, there is $\delta > 0$ with $\omega(x, \delta) < 1/k$. Let $y \in B(x, \delta/2)$. If $u \in B(y, \delta/2)$ then $\rho(u, x) \leq \rho(u, y) + \rho(y, x) < \delta$, so $B(y, \delta/2) \subseteq B(x, \delta)$; consequently the supremum defining $\omega(y, \delta/2)$ is taken over a subcollection of pairs, whence

$$\omega(y) \leq \omega(y, \delta/2) \leq \omega(x, \delta) < 1/k.$$

Thus $B(x, \delta/2) \subseteq \mathcal{O}_k$, and $\mathcal{O}_k$ is open.

Let C denote the set of points at which f is continuous. By Part 1,

$$C = \{x \in X : \omega(x) = 0\} = \bigcap_{k=1}^{\infty} \{x \in X : \omega(x) < 1/k\} = \bigcap_{k=1}^{\infty} \mathcal{O}_k,$$

where the middle equality holds because $\omega(x) \geq 0$ always, so $\omega(x) = 0$ if and only if $\omega(x) < 1/k$ for every k . Hence C is the intersection of a countable collection of open subsets of X , that is, a G_δ set.

Part 3: no real-valued function on $\mathbb{R}$ is continuous exactly at the rationals.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ had continuity set exactly $\mathbb{Q}$. By Part 2 there are open subsets $\mathcal{O}_k$ of $\mathbb{R}$ with

$$\mathbb{Q} = \bigcap_{k=1}^{\infty} \mathcal{O}_k.$$

Each $\mathcal{O}_k$ contains $\mathbb{Q}$ and is therefore dense in $\mathbb{R}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$ and a set containing a dense set is dense.

Enumerate the rationals as $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$ and put $\mathcal{U}_j = \mathbb{R} \setminus \{q_j\}$. Each $\mathcal{U}_j$ is open, being the complement of the closed set $\{q_j\}$, and each is dense in $\mathbb{R}$: any ball $(x - \epsilon, x + \epsilon)$ contains points other than q_j , for instance one of $x \pm \epsilon/2$.

Now $\{\mathcal{O}_k\}_{k=1}^{\infty} \cup \{\mathcal{U}_j\}_{j=1}^{\infty}$ is a countable collection of open dense subsets of $\mathbb{R}$, and $\mathbb{R}$ is a complete metric space by Theorem 12 of Chapter 9. By the Baire Category Theorem, part (i), the intersection

$$\left(\bigcap_{k=1}^{\infty} \mathcal{O}_k \right) \cap \left(\bigcap_{j=1}^{\infty} \mathcal{U}_j \right)$$

is dense in $\mathbb{R}$. But this intersection equals

$$\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset,$$

since $\bigcap_j \mathcal{U}_j = \mathbb{R} \setminus \bigcup_j \{q_j\} = \mathbb{R} \setminus \mathbb{Q}$. The empty set is not dense in $\mathbb{R}$, because $\mathbb{R}$ itself is a nonempty open set containing no point of $\emptyset$. This contradiction shows that no such f exists.

Equivalently: $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$, whereas by Part 2 every continuity set is. Note that the reverse phenomenon does occur, since the set of irrationals is a G_δ , and indeed the function equal to $1/q$ at a rational p/q in lowest terms with $q > 0$ and equal to 0 at each irrational is continuous precisely at the irrational numbers.

10.4 Problems 10.22–10.28

Problem 10.22. For each natural number n , show that in $[0, 1]$ there is a nowhere dense closed set that has Lebesgue measure $1 - 1/n$. Use this to construct a set of the first category in $[0, 1]$ that has measure 1.

Solution. Throughout, m denotes Lebesgue measure on $\mathbb{R}$.

Step 1: a closed nowhere dense subset of $[0, 1]$ of large measure.

Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in $[0, 1]$. Fix ϵ with $0 < \epsilon < 1$ and define the open set

$$\mathcal{O}_\epsilon = \bigcup_{k=1}^{\infty} \left(q_k - \frac{\epsilon}{2^{k+1}}, q_k + \frac{\epsilon}{2^{k+1}} \right).$$

By the countable subadditivity and the translation-invariance of Lebesgue measure,

$$m(\mathcal{O}_\epsilon) \leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

Put $F_\epsilon = [0, 1] \setminus \mathcal{O}_\epsilon$. Then F_ϵ is closed, being the intersection of $[0, 1]$ with the closed set $\mathbb{R} \setminus \mathcal{O}_\epsilon$, and

by the excision property

$$m(F_\epsilon) \geq m([0, 1]) - m(\mathcal{O}_\epsilon) \geq 1 - \epsilon.$$

Moreover F_ϵ contains no rational number of $[0, 1]$, since every such number lies in $\mathcal{O}_\epsilon$. Hence F_ϵ contains no nonempty open interval, that is, $\text{int } F_\epsilon = \emptyset$: F_ϵ is hollow. Since F_ϵ is closed it equals its own closure, and therefore F_ϵ is nowhere dense in $[0, 1]$ (and in $\mathbb{R}$).

Step 2: adjusting the measure to be exactly $1 - 1/n$.

Fix a natural number n and choose $\epsilon = 1/(2n)$; write $F = F_{1/(2n)}$, so that F is closed, nowhere dense, and

$$m(F) \geq 1 - \frac{1}{2n} > 1 - \frac{1}{n} \quad \text{when } n \geq 2,$$

while for $n = 1$ we only need a closed nowhere dense set of measure 0, and the empty set (or the Cantor set) will do. So assume $n \geq 2$ and define

$$g(c) = m(F \cap [0, c]), \quad c \in [0, 1].$$

For $0 \leq c \leq c' \leq 1$ we have $F \cap [0, c] \subseteq F \cap [0, c']$ and

$$0 \leq g(c') - g(c) = m(F \cap (c, c']) \leq c' - c,$$

by monotonicity and additivity of m ; hence g is (Lipschitz) continuous on $[0, 1]$. Also $g(0) = 0$ and $g(1) = m(F) > 1 - 1/n$. By the Intermediate Value Theorem there is $c \in (0, 1)$ with

$$g(c) = 1 - \frac{1}{n}.$$

Set $E_n = F \cap [0, c]$. Then E_n is closed (an intersection of two closed sets), it is contained in the nowhere dense set F and therefore is itself nowhere dense (its closure, namely E_n , is contained in the hollow set F , hence has empty interior), and $m(E_n) = 1 - 1/n$. This is the required set.

Step 3: a set of the first category in $[0, 1]$ of measure 1.

Let $E = \bigcup_{n=1}^{\infty} E_n$. Being a countable union of nowhere dense subsets of $[0, 1]$, the set E is of the first category in $[0, 1]$. On the other hand, for every n we have $E_n \subseteq E \subseteq [0, 1]$, so by monotonicity

$$1 - \frac{1}{n} = m(E_n) \leq m(E) \leq 1 \quad \text{for all } n.$$

Letting $n \rightarrow \infty$ gives $m(E) = 1$.

Thus E is a subset of $[0, 1]$ of full Lebesgue measure which is nevertheless meager. In particular measure and category are genuinely independent notions: $[0, 1] \sim E$ is residual (its complement in $[0, 1]$ is of the first category), hence dense and, by the Baire Category Theorem applied to the complete metric space $[0, 1]$, nonempty and indeed of the second category, yet it has Lebesgue measure zero.

Problem 10.23. A point x in a metric space X is called isolated provided the singleton set $\{x\}$ is open in X .

- (i) Prove that a complete metric space without isolated points has an uncountable number of points.
- (ii) Use part (i) to prove that $[0, 1]$ is uncountable. Compare this with the proof that $[0, 1]$ is uncountable because it has positive Lebesgue measure.
- (iii) Show that if X is a complete metric space without isolated points and $\{F_n\}_{n=1}^{\infty}$ is a countable collection of closed hollow sets, then $X \sim \bigcup_{n=1}^{\infty} F_n$ is dense and uncountable.

Solution. Throughout we assume, as is implicit in the statement, that X is nonempty; the empty metric space is complete and has no isolated points, and it is of course countable.

- (i) Let X be a nonempty complete metric space with no isolated points, and suppose, to get a contra-

diction, that X is countable. Then we may write

$$X = \bigcup_{n=1}^{\infty} \{x_n\}$$

for some sequence $\{x_n\}$ of points of X (repeating points if X is finite).

Each singleton $\{x_n\}$ is closed: if $y \neq x_n$, the ball $B(y, \rho(y, x_n))$ misses x_n , so $X \sim \{x_n\}$ is open. Each singleton $\{x_n\}$ is also hollow. Indeed, the interior of $\{x_n\}$ is a subset of $\{x_n\}$, hence is either $\emptyset$ or $\{x_n\}$; the latter would say that $\{x_n\}$ is open, that is, that x_n is an isolated point of X , which is excluded by hypothesis. Therefore $\text{int}\{x_n\} = \emptyset$.

So X is the union of a countable collection of closed hollow subsets of the complete metric space X . By part (ii) of the Baire Category Theorem, this union is hollow, that is, $\text{int } X = \emptyset$. But X is open in X , so $\text{int } X = X$, and $X \neq \emptyset$. This contradiction shows that X is uncountable.

(ii) Give $[0, 1]$ the metric inherited from $\mathbb{R}$. It is a complete metric space: $[0, 1]$ is a closed subset of the complete metric space $\mathbb{R}$, and a Cauchy sequence in $[0, 1]$ is Cauchy in $\mathbb{R}$, hence converges in $\mathbb{R}$ to a limit which, $[0, 1]$ being closed, lies in $[0, 1]$.

Moreover $[0, 1]$ has no isolated points. Let $x \in [0, 1]$ and $r > 0$. If $x < 1$, the point $\min\{x + r/2, (x + 1)/2\}$ belongs to $[0, 1]$, differs from x , and lies within r of x ; if $x = 1$, the point $\max\{1 - r/2, 1/2\}$ does the same. Hence no ball about x in $[0, 1]$ reduces to $\{x\}$, so $\{x\}$ is not open. By part (i), $[0, 1]$ is uncountable.

Comparison. The measure-theoretic proof runs as follows: a countable set $C = \{c_k\}$ has $m(C) \leq \sum_k m(\{c_k\}) = 0$ by the countable subadditivity of Lebesgue measure and the fact that singletons are null, whereas $m([0, 1]) = 1 > 0$; therefore $[0, 1]$ is not countable. The two arguments are of quite different character. The measure argument depends on the whole construction of Lebesgue measure on $\mathbb{R}$ and on the normalization $m([0, 1]) = 1$, so it is available only for subsets of Euclidean space; in exchange it gives the sharper information that a countable set is not merely a proper subset of $[0, 1]$ but is null, so that its complement in $[0, 1]$ has full measure. The Baire argument uses nothing but the completeness of the metric and the absence of isolated points, and therefore applies verbatim to $\mathcal{C}[a, b]$, to ℓ^p , to any perfect complete metric space; in exchange it gives only that the space is uncountable. Note also that a countable set can be topologically large: the rationals are dense in $[0, 1]$. What the Baire argument really shows is that a countable subset of such a space is of the first category, and the space itself is not.

(iii) Assume X is a nonempty complete metric space without isolated points and $\{F_n\}_{n=1}^{\infty}$ is a countable collection of closed hollow subsets of X . Put

$$D = X \sim \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (X \sim F_n).$$

Density. For each n , the set $\mathcal{O}_n = X \sim F_n$ is open, since F_n is closed, and it is dense, since by observation (6) of this section a set is hollow if and only if its complement is dense. By part (i) of the Baire Category Theorem, $D = \bigcap_{n=1}^{\infty} \mathcal{O}_n$ is dense in X .

Uncountability. Suppose D were countable, say $D = \bigcup_{k=1}^{\infty} \{y_k\}$. Then

$$X = \left(\bigcup_{n=1}^{\infty} F_n \right) \cup \left(\bigcup_{k=1}^{\infty} \{y_k\} \right)$$

exhibits X as the union of a countable collection of closed sets, each of which is hollow: the F_n by hypothesis, and each $\{y_k\}$ by the argument given in part (i), since X has no isolated points. Part (ii) of the Baire Category Theorem then forces $\text{int } X = \emptyset$, contradicting $\text{int } X = X \neq \emptyset$. Hence D is uncountable.

(If D were empty the same contradiction arises, so in particular D is nonempty; but this is already implied by its density in the nonempty space X .)

Problem 10.24. Let E be a subset of a complete metric space X . Verify the following assertions.

- (i) If $X \sim E$ is dense and F is a closed set contained in E , then F is nowhere dense.
- (ii) If E and $X \sim E$ are both dense, then at most one of them is the union of a countable collection of closed sets.
- (iii) The set of rational numbers in $[0, 1]$ is not the intersection of a countable collection of open sets.

Solution. Throughout, X is a complete metric space, assumed nonempty (all three assertions are vacuous or degenerate otherwise).

(i) Let $X \sim E$ be dense in X and let F be closed with $F \subseteq E$. Since F is closed, $\overline{F} = F$, so to show that F is nowhere dense it suffices to show that F is hollow, that is, $\text{int } F = \emptyset$.

Suppose not. Then there are $x \in X$ and $r > 0$ with $B(x, r) \subseteq F$. The ball $B(x, r)$ is a nonempty open subset of X (it contains x), and since $X \sim E$ is dense in X , every nonempty open subset of X meets $X \sim E$. Hence there is a point $y \in B(x, r) \cap (X \sim E)$. But $B(x, r) \subseteq F \subseteq E$, so $y \in E$, a contradiction. Therefore $\text{int } F = \emptyset$ and F is nowhere dense.

(ii) Suppose both E and $X \sim E$ are dense in X , and suppose, contrary to the assertion, that both are unions of countable collections of closed sets, say

$$E = \bigcup_{n=1}^{\infty} F_n \quad \text{and} \quad X \sim E = \bigcup_{n=1}^{\infty} G_n,$$

with every F_n and every G_n closed in X .

Each F_n is a closed set contained in E , and $X \sim E$ is dense; by part (i), F_n is nowhere dense, i.e. closed and hollow. Symmetrically, each G_n is a closed set contained in $X \sim E$, whose complement E is dense; applying part (i) with E replaced by $X \sim E$ shows that G_n is closed and hollow.

Now

$$X = E \cup (X \sim E) = \left(\bigcup_{n=1}^{\infty} F_n \right) \cup \left(\bigcup_{n=1}^{\infty} G_n \right)$$

displays X as the union of a countable collection of closed hollow subsets of the complete metric space X . By part (ii) of the Baire Category Theorem this union is hollow, so $\text{int } X = \emptyset$. But X is open in itself and nonempty, so $\text{int } X = X \neq \emptyset$. This contradiction proves that at most one of E , $X \sim E$ can be a countable union of closed sets.

(iii) Take $X = [0, 1]$ with the metric inherited from $\mathbb{R}$; as noted in Problem 23, X is a complete metric space, being a closed subset of the complete space $\mathbb{R}$. Let

$$E = \mathbb{Q} \cap [0, 1].$$

Both E and $X \sim E$ (the irrational numbers in $[0, 1]$) are dense in $[0, 1]$: every nondegenerate subinterval of $[0, 1]$ contains both a rational and an irrational number.

The set E is countable, say $E = \{q_1, q_2, \dots\}$, and each singleton $\{q_k\}$ is closed in X ; hence

$$E = \bigcup_{k=1}^{\infty} \{q_k\}$$

is the union of a countable collection of closed sets. By part (ii), therefore, $X \sim E$ is **not** the union of a countable collection of closed sets.

Suppose now that E were the intersection of a countable collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ of open subsets of $[0, 1]$. Then by De Morgan's Identities

$$X \sim E = X \sim \bigcap_{n=1}^{\infty} \mathcal{O}_n = \bigcup_{n=1}^{\infty} (X \sim \mathcal{O}_n),$$

and each $X \sim \mathcal{O}_n$ is closed in X . This exhibits $X \sim E$ as a countable union of closed sets, contradicting the preceding paragraph. Hence the rationals in $[0, 1]$ are not a countable intersection of open sets.

The same conclusion holds if “open” is understood relative to $\mathbb{R}$: if $E = \bigcap_{n=1}^{\infty} U_n$ with each U_n open in $\mathbb{R}$, then $\mathcal{O}_n = U_n \cap [0, 1]$ is open in $[0, 1]$ and $\bigcap_{n=1}^{\infty} \mathcal{O}_n = E \cap [0, 1] = E$, which we have just shown to be impossible.

Combined with Problem 21, which shows that the set of points of continuity of a real-valued function is always a countable intersection of open sets, this gives again the conclusion that there is no real-valued function on $\mathbb{R}$ continuous exactly at the rational numbers.

Problem 10.25. Show that under the hypotheses of Theorem 6 there is a dense open set $\mathcal{O} \subseteq X$ such that each $x \in \mathcal{O}$ has a neighborhood U on which $\mathcal{F}$ is uniformly bounded.

Solution. The hypotheses of Theorem 6 are: X is a complete metric space and $\mathcal{F}$ is a family of continuous real-valued functions on X that is pointwise bounded, that is, for each $x \in X$ there is a constant M_x with $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Theorem 6 produces one nonempty open set on which $\mathcal{F}$ is uniformly bounded; we show that the set of points possessing such a neighborhood is open and dense.

For each natural number n define, as in the proof of Theorem 6,

$$E_n = \{x \in X : |f(x)| \leq n \text{ for all } f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} \{x \in X : |f(x)| \leq n\}.$$

Each function $f \in \mathcal{F}$ is continuous, so each set $\{|f| \leq n\}$ is closed, being the preimage of the closed set $[-n, n]$; an intersection of closed sets is closed, so E_n is closed. Since $\mathcal{F}$ is pointwise bounded, for each $x \in X$ any natural number $n \geq M_x$ satisfies $|f(x)| \leq n$ for all $f \in \mathcal{F}$, so $x \in E_n$. Hence

$$X = \bigcup_{n=1}^{\infty} E_n.$$

Now set

$$\mathcal{O} = \bigcup_{n=1}^{\infty} \text{int } E_n.$$

The set $\mathcal{O}$ is open, being a union of open sets.

$\mathcal{O}$ is dense. Since X is complete and each E_n is closed, Corollary 5 tells us that $\bigcup_{n=1}^{\infty} \text{bd } E_n$ is hollow, and therefore, by the equivalence (6) of this section, its complement

$$D = X \sim \bigcup_{n=1}^{\infty} \text{bd } E_n$$

is dense in X . We claim $D \subseteq \mathcal{O}$. Indeed, let $x \in D$. Since $X = \bigcup_n E_n$, there is an index n with $x \in E_n$. By the decomposition (5),

$$X = \text{int } E_n \cup \text{ext } E_n \cup \text{bd } E_n,$$

and this union is disjoint. Now $\text{ext } E_n$, the set of interior points of $X \sim E_n$, is contained in $X \sim E_n$, so $x \notin \text{ext } E_n$; and $x \notin \text{bd } E_n$ because $x \in D$. Therefore $x \in \text{int } E_n \subseteq \mathcal{O}$. This proves $D \subseteq \mathcal{O}$, and since D is dense so is the larger set $\mathcal{O}$.

Local uniform boundedness on $\mathcal{O}$. Let $x \in \mathcal{O}$. By definition of $\mathcal{O}$ there is an index n with $x \in \text{int } E_n$. Put $U = \text{int } E_n$. Then U is an open set containing x , hence a neighborhood of x , and since $U \subseteq E_n$,

$$|f| \leq n \text{ on } U \text{ for all } f \in \mathcal{F}.$$

Thus $\mathcal{F}$ is uniformly bounded on U , with the single bound $M = n$ serving for every member of $\mathcal{F}$. So $\mathcal{O}$ is a dense open subset of X each of whose points has a neighborhood on which $\mathcal{F}$ is uniformly bounded. (In particular $\mathcal{O}$ is nonempty when X is, which recovers Theorem 6. Note also that $\mathcal{O}$ is precisely the set of points at which $\mathcal{F}$ is locally uniformly bounded: if $\mathcal{F}$ is uniformly bounded by M on an open set $U \ni x$, then for any natural number $n \geq M$ we have $U \subseteq E_n$, so $x \in \text{int } E_n \subseteq \mathcal{O}$.)

Problem 10.26. By Hölder's Inequality, we have $L^2[a, b] \subseteq L^1[a, b]$. Show that the set $L^2[a, b]$, considered as a subset of the complete metric space $L^1[a, b]$, is of the first category.

Solution. Here $a < b$, and $L^1[a, b]$ carries the metric $\rho(f, g) = \|f - g\|_1 = \int_a^b |f - g|$; it is a complete metric space by the Riesz-Fischer Theorem of Chapter 7. The inclusion $L^2[a, b] \subseteq L^1[a, b]$ is Hölder's Inequality with conjugate exponents 2 and 2 applied to $|f| \cdot 1$:

$$\int_a^b |f| \leq \left(\int_a^b |f|^2 \right)^{1/2} (b-a)^{1/2}, \quad \text{i.e.} \quad \|f\|_1 \leq (b-a)^{1/2} \|f\|_2.$$

For each natural number n put

$$E_n = \{f \in L^2[a, b] : \|f\|_2 \leq n\} \subseteq L^1[a, b].$$

Since every $f \in L^2[a, b]$ has $\|f\|_2 < \infty$, we have

$$L^2[a, b] = \bigcup_{n=1}^{\infty} E_n.$$

We show that each E_n is closed and hollow as a subset of $L^1[a, b]$; the Remark of this section then identifies $L^2[a, b]$ as a countable union of nowhere dense subsets of $L^1[a, b]$, that is, as a set of the first category.

Each E_n is closed in $L^1[a, b]$. Let $\{f_k\}$ be a sequence in E_n converging in $L^1[a, b]$ to $f \in L^1[a, b]$. By the Riesz-Fischer Theorem of Chapter 7, a sequence that converges in L^p has a subsequence that converges pointwise almost everywhere to the limit function; choose such a subsequence $\{f_{k_j}\}$, so that $f_{k_j} \rightarrow f$ pointwise a.e. on $[a, b]$. Then $|f_{k_j}|^2 \rightarrow |f|^2$ pointwise a.e., and these are nonnegative measurable functions, so Fatou's Lemma gives

$$\int_a^b |f|^2 \leq \liminf_{j \rightarrow \infty} \int_a^b |f_{k_j}|^2 \leq n^2.$$

Hence $f \in L^2[a, b]$ with $\|f\|_2 \leq n$, that is, $f \in E_n$. A subset of a metric space is closed if and only if it contains the limit of each of its convergent sequences, so E_n is closed in $L^1[a, b]$.

$L^2[a, b]$, and hence each E_n , is hollow in $L^1[a, b]$. Define

$$h(x) = \frac{1}{\sqrt{x-a}} \quad \text{for } a < x \leq b, \quad h(a) = 0.$$

Then h is measurable and nonnegative, and

$$\int_a^b h = \int_a^b \frac{dx}{\sqrt{x-a}} = 2\sqrt{b-a} < \infty,$$

so $h \in L^1[a, b]$, while

$$\int_a^b h^2 = \int_a^b \frac{dx}{x-a} = \infty,$$

so $h \notin L^2[a, b]$. (Both integrals are computed by the Monotone Convergence Theorem from the corresponding integrals over $[a + 1/k, b]$, where the integrands are continuous.)

Now suppose some ball $B(f, r)$ of $L^1[a, b]$ were contained in $L^2[a, b]$. Its center f then belongs to $L^2[a, b]$. Set

$$g = f + \frac{r}{4\sqrt{b-a}} h.$$

Since f and h both lie in $L^1[a, b]$, so does g , and

$$\|g - f\|_1 = \frac{r}{4\sqrt{b-a}} \int_a^b h = \frac{r}{4\sqrt{b-a}} \cdot 2\sqrt{b-a} = \frac{r}{2} < r,$$

so $g \in B(f, r)$ and therefore $g \in L^2[a, b]$. But $L^2[a, b]$ is a linear space, so

$$h = \frac{4\sqrt{b-a}}{r}(g-f)$$

would belong to $L^2[a, b]$, contradicting $\int_a^b h^2 = \infty$. Hence no ball of $L^1[a, b]$ lies in $L^2[a, b]$, that is, $\text{int } L^2[a, b] = \emptyset$ in $L^1[a, b]$, so $L^2[a, b]$ is hollow. The interior operation is monotone, so every subset of a hollow set is hollow; in particular each E_n is hollow.

Since each E_n is closed, $\overline{E_n} = E_n$ is hollow, so each E_n is nowhere dense in $L^1[a, b]$. Therefore

$$L^2[a, b] = \bigcup_{n=1}^{\infty} E_n$$

is the union of a countable collection of nowhere dense subsets of $L^1[a, b]$: it is of the first category. By the Baire Category Theorem, since $L^1[a, b]$ is complete, $L^2[a, b]$ is in particular a hollow subset of $L^1[a, b]$ and its complement $L^1[a, b] \setminus L^2[a, b]$ is dense; “most” integrable functions on $[a, b]$ fail to be square integrable.

Problem 10.27. Let f be a continuous real-valued function on $\mathbb{R}$ with the property that for each real number x , $\lim_{n \rightarrow \infty} f(nx) = 0$. Show that $\lim_{x \rightarrow \infty} f(x) = 0$.

Solution. Here n runs over the natural numbers. We must show: for each $\epsilon > 0$ there is an $R > 0$ such that $|f(x)| \leq \epsilon$ whenever $x \geq R$.

Fix $\epsilon > 0$. Let

$$X = [1, \infty)$$

with the metric inherited from $\mathbb{R}$. Then X is a complete metric space: it is a closed subset of the complete metric space $\mathbb{R}$, so a Cauchy sequence in X converges in $\mathbb{R}$ to a point which necessarily lies in X .

For each natural number n define

$$E_n = \{x \in X : |f(kx)| \leq \epsilon \text{ for every integer } k \geq n\} = \bigcap_{k \geq n} \{x \in X : |f(kx)| \leq \epsilon\}.$$

Each E_n is closed. For a fixed k the map $x \mapsto f(kx)$ is continuous on X , being the composition of the continuous maps $x \mapsto kx$ and f ; hence $\{x \in X : |f(kx)| \leq \epsilon\}$ is closed in X , as the preimage of the closed set $[-\epsilon, \epsilon]$ under a continuous real-valued function. An intersection of closed sets is closed, so E_n is closed in X .

$X = \bigcup_{n=1}^{\infty} E_n$. Let $x \in X$. By hypothesis $\lim_{n \rightarrow \infty} f(nx) = 0$, so there is a natural number n such that $|f(kx)| \leq \epsilon$ for all $k \geq n$; that is, $x \in E_n$.

Since X is a complete metric space and X is the union of the countable collection $\{E_n\}$ of closed sets, Corollary 4 of the Baire Category Theorem provides an index N for which E_N has nonempty interior in X . Thus there are $x_0 \in X$ and $r > 0$ with $B(x_0, r) \cap X \subseteq E_N$. Since $x_0 \geq 1$, the set $B(x_0, r) \cap X$ contains a nondegenerate closed interval $[a, b]$ with $1 \leq a < b$ (for instance $a = \max\{1, x_0 - r/2\}$ and $b = x_0 + r/2$). Hence

$$|f(kx)| \leq \epsilon \quad \text{for all } x \in [a, b] \text{ and all integers } k \geq N. \quad (*)$$

Rewriting $(*)$ in terms of $y = kx$: for each integer $k \geq N$,

$$|f(y)| \leq \epsilon \quad \text{for all } y \in [ka, kb].$$

It remains to observe that these intervals eventually overlap and so cover a half-line. The intervals $[ka, kb]$ and $[(k+1)a, (k+1)b]$ overlap precisely when $(k+1)a \leq kb$, that is, when

$$k(b-a) \geq a, \quad \text{i.e.} \quad k \geq \frac{a}{b-a}.$$

Choose an integer

$$K \geq \max \left\{ N, \frac{a}{b-a} \right\},$$

and set $R = Ka$. If $y \geq R$, then since $ka \rightarrow \infty$ and $Ka \leq y$, there is a largest integer $k \geq K$ with $ka \leq y$. If $y \leq kb$, then $y \in [ka, kb]$. Otherwise $y > kb \geq (k+1)a$ (the last inequality because $k \geq K \geq a/(b-a)$), so $(k+1)a \leq y$, contradicting the maximality of k . Hence in every case y lies in some interval $[ka, kb]$ with $k \geq K \geq N$, and therefore

$$|f(y)| \leq \epsilon \quad \text{for all } y \geq R.$$

Since $\epsilon > 0$ was arbitrary, $\lim_{x \rightarrow \infty} f(x) = 0$.

Two remarks. First, continuity of f was used only to guarantee that the sets E_n are closed, and it cannot be dispensed with. Let $A = \{\pi^m : m \geq 1\}$ and let $f = \chi_A$ be its characteristic function. If $a = \pi^m$ and $a' = \pi^{m'}$ with $m > m'$ are both integer multiples of a fixed real number $x \neq 0$, then $a/a' = \pi^{m-m'}$ would be rational, which is false since π is transcendental. Hence for each $x \neq 0$ there is at most one natural number n with $nx \in A$, so $f(nx) = 0$ for all large n ; and $f(n \cdot 0) = f(0) = 0$ for all n since $0 \notin A$. Thus $\lim_{n \rightarrow \infty} f(nx) = 0$ for every real x , yet $f(\pi^m) = 1$ for every m , so f does not tend to 0 at infinity. Second, the same argument applied to $g(x) = f(-x)$ shows $\lim_{x \rightarrow -\infty} f(x) = 0$ as well, since g is continuous and $\lim_n g(nx) = \lim_n f(n(-x)) = 0$ for each x .

Problem 10.28. Let f be a continuous real-valued function on $\mathbb{R}$ that has derivatives of all orders. Suppose that for each real number x , there is an index $n = n(x)$ for which $f^{(n)}(x) = 0$. Show that f is a polynomial. (Hint: Apply the Baire Category Theorem twice.)

Solution. For each nonnegative integer n set

$$E_n = \{x \in \mathbb{R} : f^{(n)}(x) = 0\}.$$

Each $f^{(n)}$ is continuous (it is differentiable), so each E_n is closed, being the preimage of the closed set $\{0\}$. The hypothesis says precisely that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} E_n. \quad (\text{A})$$

Let $\mathcal{U}$ be the union of all open intervals on which f agrees with some polynomial. Then $\mathcal{U}$ is open, and we must show $\mathcal{U} = \mathbb{R}$ together with the fact that a single polynomial then works globally.

Step 1: on each connected component of $\mathcal{U}$, f is a polynomial.

Let J be a component of $\mathcal{U}$; it is an open interval. Fix $x_0 \in J$. By the definition of $\mathcal{U}$ there is an open interval I_0 with $x_0 \in I_0 \subseteq \mathcal{U}$ and a polynomial p with $f = p$ on I_0 ; being a connected subset of $\mathcal{U}$ meeting J , $I_0 \subseteq J$. Let

$$S = \{x \in J : f^{(k)}(x) = p^{(k)}(x) \text{ for every } k \geq 0\}.$$

S is nonempty ($I_0 \subseteq S$) and closed in J , being an intersection of preimages of $\{0\}$ under the continuous functions $f^{(k)} - p^{(k)}$. S is also open in J : if $x \in S$, choose, by the definition of $\mathcal{U}$, an open interval I_1 with $x \in I_1 \subseteq \mathcal{U}$ and a polynomial q with $f = q$ on I_1 ; then $I_1 \subseteq J$, and $q^{(k)}(x) = f^{(k)}(x) = p^{(k)}(x)$ for all k , so the two polynomials p and q have the same Taylor expansion at x and hence are equal; thus $f = p$ on I_1 and $I_1 \subseteq S$. Since J is connected, $S = J$, that is, $f = p$ on J .

Step 2: the complement of $\mathcal{U}$ has no isolated points.

Let $F = \mathbb{R} \setminus \mathcal{U}$, a closed set. Suppose $x_0 \in F$ were isolated in F , so that for some $\delta > 0$ we have $(x_0 - \delta, x_0 + \delta) \cap F = \{x_0\}$. Then $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$ are connected subsets of $\mathcal{U}$, hence each lies in a single component of $\mathcal{U}$, and so by Step 1 there are polynomials p and q with $f = p$ on $(x_0 - \delta, x_0)$ and $f = q$ on $(x_0, x_0 + \delta)$. For each k , $f^{(k)} = p^{(k)}$ on $(x_0 - \delta, x_0)$, and both $f^{(k)}$ and $p^{(k)}$ are continuous, so letting $x \rightarrow x_0^-$ gives $f^{(k)}(x_0) = p^{(k)}(x_0)$; similarly $f^{(k)}(x_0) = q^{(k)}(x_0)$. Hence p and q agree with all their derivatives at x_0 , so $p = q$. Consequently $f = p$ on all of $(x_0 - \delta, x_0 + \delta)$, whence $x_0 \in \mathcal{U}$, a contradiction. So F has no isolated points.

Step 3: F is empty.

Suppose not. Then F is a nonempty closed subset of $\mathbb{R}$, hence a complete metric space in the inherited metric, and by Step 2 it has no isolated points. By (A),

$$F = \bigcup_{n=1}^{\infty} (E_n \cap F),$$

and each $E_n \cap F$ is closed in F . By Corollary 4 of the Baire Category Theorem, applied to the complete metric space F , some $E_N \cap F$ has nonempty interior relative to F : there exist $x_1 \in F$ and $r > 0$ such that, writing $I = (x_1 - r, x_1 + r)$,

$$x_1 \in I \cap F \neq \emptyset \quad \text{and} \quad I \cap F \subseteq E_N, \quad \text{i.e.} \quad f^{(N)} = 0 \quad \text{on } I \cap F. \quad (\text{B})$$

We claim that in fact

$$f^{(k)} = 0 \quad \text{on } I \cap F \quad \text{for every } k \geq N. \quad (\text{C})$$

We argue by induction, the case $k = N$ being (B). Suppose $f^{(k)} = 0$ on $I \cap F$ for some $k \geq N$, and let $x \in I \cap F$. By Step 2, x is not isolated in F , so there is a sequence $\{y_j\}$ in F with $y_j \neq x$ and $y_j \rightarrow x$; since I is open we may assume $y_j \in I$, so $y_j \in I \cap F$ and $f^{(k)}(y_j) = 0 = f^{(k)}(x)$. Since $f^{(k+1)}(x)$ exists,

$$f^{(k+1)}(x) = \lim_{j \rightarrow \infty} \frac{f^{(k)}(y_j) - f^{(k)}(x)}{y_j - x} = 0.$$

This proves (C).

Next we show $f^{(N)} = 0$ on all of I . The set $\mathcal{U} \cap I$ is open, so it is the union of a countable disjoint collection of open intervals, its components. Let (a, b) be such a component, and write $I = (\alpha, \beta)$ with $\alpha = x_1 - r$ and $\beta = x_1 + r$, so that $\alpha \leq a < b \leq \beta$. Since $I \cap F \neq \emptyset$ while $(a, b) \subseteq \mathcal{U}$ is disjoint from F , we have $(a, b) \neq I$, so either $a > \alpha$ or $b < \beta$. In the first case $\alpha < a < b \leq \beta$, so the endpoint a lies in I ; in the second case $\alpha \leq a < b < \beta$, so the endpoint b lies in I . Let c denote an endpoint of (a, b) that lies in I . An endpoint of a component of an open set does not belong to that set, so $c \notin \mathcal{U} \cap I$; since $c \in I$, this forces $c \notin \mathcal{U}$, that is, $c \in I \cap F$. Now (a, b) is a connected subset of $\mathcal{U}$, so by Step 1 there is a polynomial p with $f = p$ on (a, b) ; as in Step 2, continuity of the derivatives gives $p^{(k)}(c) = f^{(k)}(c)$ for every k , and by (C) these vanish for every $k \geq N$. Thus the polynomial $p^{(N)}$ satisfies $(p^{(N)})^{(j)}(c) = p^{(N+j)}(c) = 0$ for every $j \geq 0$, so its Taylor expansion about c is identically zero, i.e. $p^{(N)} \equiv 0$. Hence $f^{(N)} = p^{(N)} = 0$ on (a, b) .

Since this holds for every component of $\mathcal{U} \cap I$, we get $f^{(N)} = 0$ on $\mathcal{U} \cap I$; combined with (B), $f^{(N)} = 0$ on $(\mathcal{U} \cap I) \cup (F \cap I) = I$.

But $f^{(N)} \equiv 0$ on the open interval I forces f to be a polynomial of degree at most $N - 1$ there: integrating N times (or applying the Mean Value Theorem repeatedly), $f^{(N-1)}$ is constant on I , then $f^{(N-2)}$ is affine on I , and so on. Hence $I \subseteq \mathcal{U}$, contradicting $I \cap F \neq \emptyset$. This contradiction shows $F = \emptyset$.

Step 4: conclusion.

By Step 3, $\mathcal{U} = \mathbb{R}$. Since $\mathbb{R}$ is connected it is a single component of $\mathcal{U}$, so by Step 1 there is a polynomial p with $f = p$ on $\mathbb{R}$. Thus f is a polynomial.

Remark on the two applications of the Baire Category Theorem. The hint's first application is the direct one: applying Corollary 4 to the decomposition (A) of the complete metric space $\mathbb{R}$ shows that some E_n has nonempty interior, so $f^{(n)}$ vanishes on an open interval and f agrees there with a polynomial; hence $\mathcal{U} \neq \emptyset$. That alone does not finish the argument, because it says nothing about the closed set F where f fails to be locally polynomial. The second application, which is the one carried out in Step 3, is to F itself: F is closed in $\mathbb{R}$, hence a complete metric space in its own right, and Step 2 shows it has no isolated points, so Corollary 4 may be applied to the decomposition of F by the closed sets $E_n \cap F$. The proof given above needs only this second application, since Step 3 derives a contradiction from $F \neq \emptyset$ whether or not $\mathcal{U}$ is known to be nonempty.

10.5 Problems 10.29–10.35

Problem 10.29. Let p be a polynomial. Show that $p: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz if and only if the degree of p is less than 2.

Solution. Recall the definition of Section 10.3: a mapping T of a metric space (X, ρ) into itself is Lipschitz provided there is a number $c \geq 0$ for which $\rho(T(u), T(v)) \leq c\rho(u, v)$ for all $u, v \in X$. Here $X = \mathbb{R}$ with the metric $\rho(u, v) = |u - v|$, so the assertion to be proved is that there is a $c \geq 0$ with $|p(u) - p(v)| \leq c|u - v|$ for all real u, v if and only if p has degree 0 or 1 (we include the zero polynomial, whose degree is conventionally $-\infty$ or undefined, among the polynomials of degree less than 2).

Suppose first that the degree of p is less than 2, so that $p(x) = ax + b$ for real numbers a, b . Then for all $u, v \in \mathbb{R}$,

$$|p(u) - p(v)| = |(au + b) - (av + b)| = |a||u - v|,$$

so p is Lipschitz with Lipschitz constant $c = |a|$.

Conversely, suppose p has degree $n \geq 2$, say

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

Define the polynomial

$$q(x) = a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_2 x + a_1,$$

which has degree exactly $n - 1 \geq 1$ since $a_n \neq 0$. Since $p(x) - p(0) = a_n x^n + \cdots + a_1 x = xq(x)$, we have, for every $x \neq 0$,

$$\frac{|p(x) - p(0)|}{|x - 0|} = |q(x)|.$$

Now q is a polynomial of degree at least one, and therefore $|q(x)| \rightarrow \infty$ as $x \rightarrow \infty$: indeed, for $x > 0$,

$$|q(x)| \geq |a_n| x^{n-1} - (|a_{n-1}| x^{n-2} + \cdots + |a_1|) = x^{n-1} \left(|a_n| - \frac{|a_{n-1}|}{x} - \cdots - \frac{|a_1|}{x^{n-1}} \right),$$

and the bracketed quantity converges to $|a_n| > 0$ as $x \rightarrow \infty$ while $x^{n-1} \rightarrow \infty$ (here $n - 1 \geq 1$). Hence $|q(x)|$ is unbounded on $(0, \infty)$.

Therefore, if $c \geq 0$ were a Lipschitz constant for p , then taking $v = 0$ and $u = x > 0$ we would get $|q(x)| = |p(x) - p(0)|/|x| \leq c$ for every $x > 0$, contradicting the unboundedness of $|q|$. So no polynomial of degree $n \geq 2$ is Lipschitz, and the proof is complete.

Problem 10.30. Fix $a > 0$, define $f(x) = ax(1 - x)$ for x in $[0, 1]$.

- (i) For what values of a is $f([0, 1]) \subseteq [0, 1]$?
- (ii) For what values of a is $f([0, 1]) \subseteq [0, 1]$ and $f: [0, 1] \rightarrow [0, 1]$ a contraction?

Solution. (i) For $x \in [0, 1]$ both $x \geq 0$ and $1 - x \geq 0$, so $f(x) = ax(1 - x) \geq 0$; thus $f \geq 0$ on $[0, 1]$ for every $a > 0$. To find the largest value of f on $[0, 1]$, complete the square:

$$f(x) = ax(1 - x) = a \left[\frac{1}{4} - \left(x - \frac{1}{2} \right)^2 \right] \leq \frac{a}{4},$$

with equality exactly at $x = 1/2$. Since f is continuous on the interval $[0, 1]$ and takes the values 0 (at $x = 0$) and $a/4$ (at $x = 1/2$), the Intermediate Value Theorem shows that $f([0, 1]) = [0, a/4]$.

Hence $f([0, 1]) \subseteq [0, 1]$ if and only if $a/4 \leq 1$, that is, if and only if

$$0 < a \leq 4.$$

(ii) A direct computation gives, for $u, v \in [0, 1]$,

$$f(u) - f(v) = a[(u - v) - (u^2 - v^2)] = a(u - v)[1 - (u + v)],$$

so that

$$|f(u) - f(v)| = a|1 - (u + v)||u - v|.$$

For $u, v \in [0, 1]$ we have $0 \leq u + v \leq 2$, hence $-1 \leq 1 - (u + v) \leq 1$ and therefore $|1 - (u + v)| \leq 1$. Consequently

$$|f(u) - f(v)| \leq a|u - v| \quad \text{for all } u, v \in [0, 1],$$

so f is Lipschitz with Lipschitz constant a , and if $a < 1$ then f is a contraction. Moreover $a < 1 < 4$, so by part (i) $f([0, 1]) \subseteq [0, 1]$ as well; thus for $0 < a < 1$ the mapping $f: [0, 1] \rightarrow [0, 1]$ is a well-defined contraction.

Conversely, suppose $a \geq 1$ and suppose, to get a contradiction, that c with $0 \leq c < 1$ is a Lipschitz constant for f . Take $v = 0$ and $u = t$ with $0 < t < 1$. Then by the displayed identity,

$$|f(t) - f(0)| = a(1 - t)t,$$

so the Lipschitz inequality forces $a(1 - t) \leq c$ for every $t \in (0, 1)$. Letting $t \rightarrow 0^+$ gives $a \leq c < 1$, contradicting $a \geq 1$. Hence f is not a contraction when $a \geq 1$.

Therefore f maps $[0, 1]$ into $[0, 1]$ and is a contraction precisely when

$$0 < a < 1.$$

(Observe that for such a , since $[0, 1]$ is a closed subset of the complete metric space $\mathbb{R}$ and hence is itself complete, the Banach Contraction Principle guarantees a unique fixed point; it is $x = 0$, as one checks directly from $ax(1 - x) = x$, whose roots are $x = 0$ and $x = 1 - 1/a < 0$.)

Problem 10.31. Does a mapping of a metric space X into itself that is Lipschitz with Lipschitz constant less than 1 necessarily have a fixed point?

Solution. No. The Banach Contraction Principle requires the metric space to be complete, and without completeness the conclusion fails.

Take $X = (0, 1)$, the open unit interval, regarded as a metric space with the metric $\rho(u, v) = |u - v|$ inherited from $\mathbb{R}$, and define

$$T: X \rightarrow X, \quad T(x) = \frac{x}{2}.$$

This is well defined: if $0 < x < 1$ then $0 < x/2 < 1/2 < 1$, so $T(x) \in X$. It is Lipschitz with Lipschitz constant $1/2 < 1$, since for all $u, v \in X$,

$$\rho(T(u), T(v)) = \left| \frac{u}{2} - \frac{v}{2} \right| = \frac{1}{2}|u - v| = \frac{1}{2}\rho(u, v).$$

Thus T is a contraction. But T has no fixed point in X : the equation $x/2 = x$ has $x = 0$ as its only real solution, and $0 \notin (0, 1)$.

Of course the space $X = (0, 1)$ is not complete: the sequence $\{1/k\}_{k \geq 2}$ lies in X and is Cauchy but fails to converge to a point of X . This is exactly what goes wrong. Following the proof of the Banach Contraction Principle with $x_0 = 1/2$ we obtain the iterates $x_k = 2^{-(k+1)}$, a Cauchy sequence in X whose limit, 0, lies outside X ; the argument has nothing to converge to.

Thus completeness cannot be dropped from the hypotheses of the Banach Contraction Principle. (Uniqueness, however, survives without completeness: the final paragraph of the proof of the Banach Contraction Principle shows that a contraction on any metric space has at most one fixed point.)

Problem 10.32. Does a mapping of a complete metric space into itself that is Lipschitz with Lipschitz constant 1 necessarily have a fixed point?

Solution. No. The hypothesis in the Banach Contraction Principle that the Lipschitz constant be strictly less than 1 cannot be relaxed to allow the constant 1, even for a complete space.

The simplest example: let $X = \mathbb{R}$ with the usual metric $\rho(u, v) = |u - v|$, which is a complete metric

space, and define

$$T: \mathbb{R} \rightarrow \mathbb{R}, \quad T(x) = x + 1.$$

Then for all $u, v \in \mathbb{R}$,

$$\rho(T(u), T(v)) = |(u + 1) - (v + 1)| = |u - v| = \rho(u, v),$$

so T is Lipschitz with Lipschitz constant 1 (and 1 is the smallest such constant). Yet $T(x) = x$ would mean $x + 1 = x$, that is, $1 = 0$, which is impossible; so T has no fixed point.

It is worth noting that the failure is not merely a matter of the inequality $\rho(T(u), T(v)) \leq \rho(u, v)$ being an equality. Even the strict inequality

$$\rho(T(u), T(v)) < \rho(u, v) \quad \text{for all } u \neq v$$

does not force the existence of a fixed point on a complete space: the mapping $f(x) = \pi/2 + x - \arctan x$ of Problem 34 satisfies this strict inequality on the complete space $\mathbb{R}$ and has no fixed point. Compactness of the underlying space, as in Problem 33, is what restores the conclusion under the strict inequality.

Finally, one should observe that when the Lipschitz constant is 1 a fixed point need not be unique either: the identity map on any metric space is Lipschitz with constant 1 and every point is fixed. So both existence and uniqueness break down at $c = 1$.

Problem 10.33. Let X be a compact metric space and T a mapping from X into itself such that

$$\rho(T(u), T(v)) < \rho(u, v) \quad \text{for all } u, v \in X.$$

Show that T has a unique fixed point.

Solution. The displayed inequality is of course to be read for $u \neq v$; for $u = v$ both sides are 0. So the hypothesis is

$$\rho(T(u), T(v)) < \rho(u, v) \quad \text{whenever } u, v \in X \text{ and } u \neq v. \quad (*)$$

Note that T need not be a contraction: there need be no single $c < 1$ that works for all pairs, so the Banach Contraction Principle is not available. We use compactness instead. We also assume X is nonempty (otherwise there is nothing to fix, and the statement is understood for a nonempty space).

Step 1: T is continuous. Indeed, $(*)$ together with the trivial case $u = v$ gives $\rho(T(u), T(v)) \leq \rho(u, v)$ for all $u, v \in X$, so T is Lipschitz with Lipschitz constant 1, and a Lipschitz mapping is continuous: given $u \in X$ and $\epsilon > 0$, take $\delta = \epsilon$; if $\rho(u, v) < \delta$ then $\rho(T(u), T(v)) \leq \rho(u, v) < \epsilon$.

Step 2: The displacement function is continuous. Define

$$g: X \rightarrow \mathbb{R}, \quad g(x) = \rho(x, T(x)).$$

We claim g is continuous. Let $x, y \in X$. By the triangle inequality,

$$\rho(x, T(x)) \leq \rho(x, y) + \rho(y, T(y)) + \rho(T(y), T(x)) \leq \rho(x, y) + \rho(y, T(y)) + \rho(x, y),$$

using Step 1 for the last term; that is, $g(x) - g(y) \leq 2\rho(x, y)$. Interchanging the roles of x and y gives $g(y) - g(x) \leq 2\rho(x, y)$, and hence

$$|g(x) - g(y)| \leq 2\rho(x, y) \quad \text{for all } x, y \in X.$$

So g is Lipschitz, hence continuous.

Step 3: Existence of a fixed point. Since X is a compact metric space and $g: X \rightarrow \mathbb{R}$ is continuous, by the Extreme Value Theorem (Theorem 22 of Chapter 9) g takes a minimum value on X : there is a point $x_* \in X$ with

$$g(x_*) \leq g(x) \quad \text{for all } x \in X.$$

Suppose, to get a contradiction, that $T(x_*) \neq x_*$. Apply (*) with $u = x_*$ and $v = T(x_*)$, which are distinct:

$$g(T(x_*)) = \rho(T(x_*), T(T(x_*))) < \rho(x_*, T(x_*)) = g(x_*).$$

Since $T(x_*)$ is a point of X , this contradicts the minimality of $g(x_*)$. Therefore $T(x_*) = x_*$, and T has a fixed point.

Step 4: Uniqueness. Suppose u and v are fixed points of T with $u \neq v$. Then by (*),

$$\rho(u, v) = \rho(T(u), T(v)) < \rho(u, v),$$

which is absurd. Hence $u = v$, and the fixed point is unique.

Two remarks. First, compactness is essential and cannot be weakened to completeness: Problem 34 exhibits a mapping of the complete space $\mathbb{R}$ into itself satisfying (*) which has no fixed point. Second, as in the Banach Contraction Principle, the fixed point may be found by iteration: since $g(T(x)) < g(x)$ unless x is fixed, the sequence $\{g(T^k(x_0))\}$ is decreasing for any starting point x_0 ; but the argument above via the Extreme Value Theorem is the cleanest route to existence.

Problem 10.34. Define $f(x) = \pi/2 + x - \arctan x$ for all real numbers x . Show that

$$|f(u) - f(v)| < |u - v| \quad \text{for all } u, v \in \mathbb{R}.$$

Show that f does not have a fixed point. Does this contradict the preceding problem?

Solution. As in the preceding problem, the strict inequality is asserted for $u \neq v$.

Step 1: the strict inequality. The function $\arctan$ is differentiable on all of $\mathbb{R}$ with $\arctan'(x) = 1/(1+x^2)$, so f is differentiable on $\mathbb{R}$ and

$$f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

Since $x^2 < 1+x^2$ and $x^2 \geq 0$, we have

$$0 \leq f'(x) < 1 \quad \text{for all } x \in \mathbb{R}. \quad (1)$$

Now let $u, v \in \mathbb{R}$ with $u \neq v$; say $v < u$. The function f is continuous on $[v, u]$ and differentiable on (v, u) , so by the Mean Value Theorem there is a point $\xi \in (v, u)$ with

$$f(u) - f(v) = f'(\xi)(u - v).$$

Taking absolute values and using (1),

$$|f(u) - f(v)| = |f'(\xi)| |u - v| = f'(\xi) |u - v| < 1 \cdot |u - v| = |u - v|,$$

the strict inequality being legitimate because $|u - v| > 0$ and $f'(\xi) < 1$. Hence $|f(u) - f(v)| < |u - v|$ for all $u \neq v$, as required. (Note also that from (1) and the Mean Value Theorem, $|f(u) - f(v)| \leq |u - v|$ for all u, v , so f is Lipschitz with Lipschitz constant 1; but f is not a contraction, since $\sup_x f'(x) = \lim_{x \rightarrow \infty} x^2/(1+x^2) = 1$, so no constant $c < 1$ works: choosing x large and applying the Mean Value Theorem on $[x, x+1]$ gives $|f(x+1) - f(x)| = f'(\xi_x)$ with $\xi_x > x$, and $f'(\xi_x) \rightarrow 1$.)

Step 2: no fixed point. Suppose $f(x) = x$ for some real x . Then

$$\frac{\pi}{2} + x - \arctan x = x, \quad \text{that is,} \quad \arctan x = \frac{\pi}{2}.$$

But the range of $\arctan$ is the open interval $(-\pi/2, \pi/2)$, so $\arctan x < \pi/2$ for every real x . This is a contradiction, so f has no fixed point. Equivalently: $f(x) - x = \pi/2 - \arctan x > 0$ for every $x \in \mathbb{R}$, so $f(x) > x$ always.

Step 3: no contradiction with Problem 33. Problem 33 assumes that X is a compact metric space. Here $X = \mathbb{R}$ with the usual metric, and $\mathbb{R}$ is not compact: it is not bounded, and by Theorem 22 of Chapter

9 it is not compact since, for instance, the continuous function $x \mapsto x$ on $\mathbb{R}$ takes neither a maximum nor a minimum value. Equivalently, $\mathbb{R}$ is not sequentially compact, since the sequence $\{k\}_{k \in \mathbb{N}}$ has no convergent subsequence. So the hypotheses of Problem 33 are not satisfied and no contradiction arises. The example shows precisely where the argument of Problem 33 breaks down without compactness. The displacement function there is

$$g(x) = |f(x) - x| = \frac{\pi}{2} - \arctan x,$$

which is continuous and strictly positive, with $\inf_{x \in \mathbb{R}} g(x) = \lim_{x \rightarrow \infty} (\pi/2 - \arctan x) = 0$. Thus g has infimum 0 but never attains it: the minimizing point that compactness would have supplied has “escaped to infinity”. It also shows, as noted in Problem 32, that on a merely complete space the strict inequality $|f(u) - f(v)| < |u - v|$ is not enough to produce a fixed point.

Problem 10.35. In Euclidean space $\mathbb{R}^n$ consider the closed unit ball $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Let f map B into B and be Lipschitz with Lipschitz constant 1. Without using the Brouwer Fixed Point Theorem, show that f has a fixed point.

Solution. Throughout, B carries the Euclidean metric $\rho(u, v) = \|u - v\|$, and the hypothesis on $f: B \rightarrow B$ is

$$\|f(u) - f(v)\| \leq \|u - v\| \quad \text{for all } u, v \in B. \quad (1)$$

In particular f is continuous. Note that (1) alone does not make f a contraction (the identity map satisfies (1)), so the Banach Contraction Principle does not apply directly. The idea is to shrink f slightly so as to produce genuine contractions, apply the Banach Contraction Principle to each, and then pass to the limit using compactness of B .

Step 1: B is a compact, complete metric space. The set B is bounded, and it is closed in $\mathbb{R}^n$, being the preimage of the closed set $(-\infty, 1]$ under the continuous function $x \mapsto \|x\|$. Hence by Theorem 20 of Chapter 9 (the Heine-Borel and Bolzano-Weierstrass Theorems), B is compact and sequentially compact. Moreover B is complete as a metric space: if $\{y_k\}$ is a Cauchy sequence in B , then it is Cauchy in the complete space $\mathbb{R}^n$, so it converges to some $y \in \mathbb{R}^n$, and since B is closed, $y \in B$; thus $\{y_k\}$ converges in B .

Step 2: for each $t \in (0, 1)$ the map $f_t = tf$ is a contraction of B into B . Fix t with $0 < t < 1$ and define

$$f_t: B \rightarrow \mathbb{R}^n, \quad f_t(x) = tf(x).$$

Since $f(x) \in B$ we have $\|f(x)\| \leq 1$, so $\|f_t(x)\| = t\|f(x)\| \leq t < 1$; hence $f_t(B) \subseteq B$ and f_t maps B into itself. Furthermore, by (1), for all $u, v \in B$,

$$\|f_t(u) - f_t(v)\| = t\|f(u) - f(v)\| \leq t\|u - v\|,$$

so f_t is Lipschitz with Lipschitz constant $t < 1$; that is, $f_t: B \rightarrow B$ is a contraction.

Step 3: apply the Banach Contraction Principle. By Step 1, B is a complete metric space, and by Step 2, $f_t: B \rightarrow B$ is a contraction. Hence, by the Banach Contraction Principle, f_t has exactly one fixed point; call it $x_t \in B$, so that

$$x_t = tf(x_t). \quad (2)$$

Step 4: let $t \rightarrow 1$ and use compactness. For each natural number $k \geq 2$ put $t_k = 1 - 1/k \in (0, 1)$ and let $x_k := x_{t_k} \in B$ be the corresponding fixed point given by Step 3. From (2),

$$\|x_k - f(x_k)\| = \|t_k f(x_k) - f(x_k)\| = (1 - t_k)\|f(x_k)\| \leq 1 - t_k = \frac{1}{k},$$

where we used $\|f(x_k)\| \leq 1$. Hence

$$\lim_{k \rightarrow \infty} \|x_k - f(x_k)\| = 0. \quad (3)$$

By Step 1 the space B is sequentially compact, so there is a subsequence $\{x_{k_j}\}$ converging to some point $x_* \in B$.

Step 5: conclude. The function $h: B \rightarrow \mathbb{R}$ defined by $h(x) = \|x - f(x)\|$ is continuous, since f is continuous by (1) and the norm is continuous. Therefore

$$\|x_* - f(x_*)\| = h(x_*) = \lim_{j \rightarrow \infty} h(x_{k_j}) = \lim_{j \rightarrow \infty} \|x_{k_j} - f(x_{k_j})\| = 0,$$

the last equality by (3), since $\{\|x_{k_j} - f(x_{k_j})\|\}$ is a subsequence of a sequence converging to 0. Hence $x_* = f(x_*)$, and x_* is a fixed point of f lying in B .

Two remarks. First, only three features of B were used: that B is compact, hence complete; that B is bounded; and that $ty \in B$ whenever $y \in B$ and $0 < t < 1$. Consequently the same argument proves that every nonexpansive self-map of a compact convex subset of $\mathbb{R}^n$ containing the origin has a fixed point. Second, the fixed point need not be unique here: the identity map on B satisfies the hypotheses and fixes every point of B . In fact the set of fixed points is exactly the zero set of the continuous function h above, hence is a closed, and therefore compact, nonempty subset of B .

10.6 Problems 10.36–10.42

Problem 10.36. Suppose that the mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a contraction. Define $g(x) = x - f(x)$ for all x in $\mathbb{R}^n$. Show that the mapping $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is both one-to-one and onto. Also show that g and its inverse are continuous.

Solution. Recall that f being a contraction means that f is Lipschitz with a Lipschitz constant c satisfying $0 \leq c < 1$; that is,

$$\|f(u) - f(v)\| \leq c \|u - v\| \quad \text{for all } u, v \in \mathbb{R}^n.$$

Throughout, $\|\cdot\|$ denotes the Euclidean norm and $\rho(u, v) = \|u - v\|$ the Euclidean metric, with respect to which $\mathbb{R}^n$ is a complete metric space.

The key two-sided estimate. Let $u, v \in \mathbb{R}^n$. Since $g(u) - g(v) = (u - v) - (f(u) - f(v))$, the triangle inequality gives, on the one hand,

$$\|g(u) - g(v)\| \leq \|u - v\| + \|f(u) - f(v)\| \leq (1 + c) \|u - v\|,$$

and, on the other hand, using the reverse triangle inequality $\|a - b\| \geq \|a\| - \|b\|$ with $a = u - v$ and $b = f(u) - f(v)$,

$$\|g(u) - g(v)\| \geq \|u - v\| - \|f(u) - f(v)\| \geq \|u - v\| - c \|u - v\| = (1 - c) \|u - v\|.$$

Thus

$$(1 - c) \|u - v\| \leq \|g(u) - g(v)\| \leq (1 + c) \|u - v\| \quad \text{for all } u, v \in \mathbb{R}^n.$$

One-to-one. Suppose $g(u) = g(v)$. The left-hand inequality above gives $(1 - c)\|u - v\| \leq 0$. Since $1 - c > 0$, we conclude $\|u - v\| = 0$, that is, $u = v$. Hence g is one-to-one.

Onto. Let $z \in \mathbb{R}^n$ be arbitrary. We must produce a point x with $g(x) = z$, that is, with $x - f(x) = z$, equivalently

$$x = f(x) + z.$$

So define the mapping $T_z: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_z(x) = f(x) + z$. For $u, v \in \mathbb{R}^n$,

$$\|T_z(u) - T_z(v)\| = \|f(u) - f(v)\| \leq c \|u - v\|,$$

so T_z is Lipschitz with Lipschitz constant $c < 1$, that is, T_z is a contraction of the complete metric space $\mathbb{R}^n$ into itself. By the Banach Contraction Principle, T_z has a (unique) fixed point $x_z \in \mathbb{R}^n$. For this point, $x_z = f(x_z) + z$, that is, $g(x_z) = x_z - f(x_z) = z$. Since z was arbitrary, g is onto.

Consequently $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bijection and possesses an inverse $g^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Continuity of g . The right-hand inequality above says that g is Lipschitz with Lipschitz constant $1 + c$. A Lipschitz mapping is continuous: given $\varepsilon > 0$, take $\delta = \varepsilon/(1 + c)$; then $\|u - v\| < \delta$ forces $\|g(u) - g(v)\| \leq (1 + c)\|u - v\| < \varepsilon$.

Continuity of g^{-1} . Let $z, w \in \mathbb{R}^n$ and put $u = g^{-1}(z)$, $v = g^{-1}(w)$, so that $g(u) = z$ and $g(v) = w$. The left-hand inequality above gives

$$\|z - w\| = \|g(u) - g(v)\| \geq (1 - c) \|u - v\| = (1 - c) \|g^{-1}(z) - g^{-1}(w)\|.$$

Since $1 - c > 0$, we may divide to obtain

$$\|g^{-1}(z) - g^{-1}(w)\| \leq \frac{1}{1 - c} \|z - w\| \quad \text{for all } z, w \in \mathbb{R}^n.$$

Thus g^{-1} is Lipschitz with Lipschitz constant $1/(1 - c)$ and hence, as above, continuous.

Therefore g is a bijection of $\mathbb{R}^n$ onto $\mathbb{R}^n$ and both g and g^{-1} are continuous (indeed Lipschitz); that is, g is a homeomorphism of $\mathbb{R}^n$ onto itself.

Problem 10.37. Let X be a complete metric space containing the point x_0 and let r be a positive real number. Define $K = \{x \in X \mid \rho(x, x_0) \leq r\}$. Suppose that the mapping $T: K \rightarrow X$ is Lipschitz with Lipschitz constant c . Suppose also that $cr + \rho(T(x_0), x_0) \leq r$. Prove that $T(K) \subseteq K$ and that $T: K \rightarrow X$ has a fixed point.

Solution. Write $d = \rho(T(x_0), x_0)$, so that the hypothesis reads $cr + d \leq r$. Observe first that $x_0 \in K$, since $\rho(x_0, x_0) = 0 \leq r$, so the number d is defined. Note also that $c \geq 0$ and $d \geq 0$.

Proof that $T(K) \subseteq K$. Let $x \in K$, so $\rho(x, x_0) \leq r$. Since x and x_0 both lie in K , the Lipschitz property of T applies to the pair x, x_0 , and the triangle inequality gives

$$\rho(T(x), x_0) \leq \rho(T(x), T(x_0)) + \rho(T(x_0), x_0) \leq c\rho(x, x_0) + d \leq cr + d \leq r.$$

Hence $T(x) \in K$. Since $x \in K$ was arbitrary, $T(K) \subseteq K$.

Consequently T may be regarded as a mapping of K into K , and it is Lipschitz with Lipschitz constant c as a mapping $T: K \rightarrow K$.

Proof that T has a fixed point. There are two cases.

Case 1: $d = 0$. Then $\rho(T(x_0), x_0) = 0$, so $T(x_0) = x_0$ and x_0 is a fixed point of T . (Here nothing further is needed.)

Case 2: $d > 0$. From $cr + d \leq r$ we get $cr \leq r - d < r$, and since $r > 0$ we may divide by r to conclude that

$$c \leq 1 - \frac{d}{r} < 1.$$

Thus $0 \leq c < 1$, that is, $T: K \rightarrow K$ is a contraction.

It remains to check that K , with the metric inherited from X , is a complete metric space. The function $x \mapsto \rho(x, x_0)$ is continuous on X (indeed, by the triangle inequality, $|\rho(x, x_0) - \rho(y, x_0)| \leq \rho(x, y)$, so it is Lipschitz), and K is the inverse image of the closed set $(-\infty, r]$ under this continuous function; hence K is a closed subset of X . Since X is complete, Proposition 11 of Chapter 9 tells us that the metric subspace K is complete. Moreover K is nonempty, since $x_0 \in K$.

Therefore $T: K \rightarrow K$ is a contraction of the nonempty complete metric space K into itself, and the Banach Contraction Principle provides a point $x_* \in K$ with $T(x_*) = x_*$; indeed this fixed point is the unique one in K .

In either case T has a fixed point in K , which is what was to be proved.

Remark. In Case 2 the proof of the Banach Contraction Principle also gives the estimate $\rho(x_*, x_k) \leq \frac{c^k}{1-c} \rho(T(x_0), x_0)$ for every natural number k , where $x_1 = T(x_0)$ and $x_{k+1} = T(x_k)$. Taking $k = 1$ and using the triangle inequality, $\rho(x_*, x_0) \leq \rho(x_*, x_1) + \rho(x_1, x_0) \leq \frac{c}{1-c} d + d = \frac{d}{1-c}$, and the hypothesis $cr + d \leq r$ says exactly that $d \leq (1 - c)r$, so $\rho(x_*, x_0) \leq r$. This is an independent confirmation that the fixed point lies in K .

Problem 10.38. Show that if the function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ has continuous first-order partial derivatives, then for each point (x_0, y_0) in $\mathbb{R}^2$ there is a neighborhood $\mathcal{O}$ of (x_0, y_0) on which the Lipschitz assumption (16) holds.

Solution. Recall that assumption (16) is the requirement that there be a positive number M for which

$$|g(x, y_1) - g(x, y_2)| \leq M |y_1 - y_2| \quad \text{for all points } (x, y_1) \text{ and } (x, y_2) \text{ in } \mathcal{O}.$$

Fix $(x_0, y_0) \in \mathbb{R}^2$ and define the open rectangle

$$\mathcal{O} = (x_0 - 1, x_0 + 1) \times (y_0 - 1, y_0 + 1),$$

which is an open neighborhood of (x_0, y_0) , and let R be its closure,

$$R = [x_0 - 1, x_0 + 1] \times [y_0 - 1, y_0 + 1].$$

Boundedness of the y -partial on R . By hypothesis the partial derivative $\partial g / \partial y$ exists at every point of $\mathbb{R}^2$ and is continuous there. The set R is closed and bounded in $\mathbb{R}^2$, hence compact by Theorem 20 of Chapter 9. By the Extreme Value Theorem, Theorem 22 of Chapter 9, a continuous real-valued function on a compact metric space is bounded and attains its extreme values, so

$$M := 1 + \max_{(x,y) \in R} \left| \frac{\partial g}{\partial y}(x, y) \right|$$

is a well-defined real number, and clearly $M > 0$ and

$$\left| \frac{\partial g}{\partial y}(x, y) \right| \leq M \quad \text{for all } (x, y) \in R \supseteq \mathcal{O}.$$

Verification of (16) on $\mathcal{O}$. Let (x, y_1) and (x, y_2) be two points of $\mathcal{O}$ with the same first coordinate. If $y_1 = y_2$ the desired inequality is trivial, so assume $y_1 \neq y_2$; without loss of generality $y_1 < y_2$. Consider the function of one real variable

$$\varphi(t) = g(x, t), \quad t \in [y_1, y_2].$$

Since $\partial g / \partial y$ exists everywhere, φ is differentiable on all of $\mathbb{R}$ with $\varphi'(t) = \frac{\partial g}{\partial y}(x, t)$; in particular φ is continuous on $[y_1, y_2]$ and differentiable on (y_1, y_2) . By the Mean Value Theorem there is a point ξ with $y_1 < \xi < y_2$ for which

$$g(x, y_2) - g(x, y_1) = \varphi(y_2) - \varphi(y_1) = \varphi'(\xi)(y_2 - y_1) = \frac{\partial g}{\partial y}(x, \xi)(y_2 - y_1).$$

Now $x \in (x_0 - 1, x_0 + 1)$, and ξ lies strictly between y_1 and y_2 , both of which belong to $(y_0 - 1, y_0 + 1)$; since that interval is convex, $\xi \in (y_0 - 1, y_0 + 1)$. Hence $(x, \xi) \in \mathcal{O} \subseteq R$, and therefore

$$|g(x, y_1) - g(x, y_2)| = \left| \frac{\partial g}{\partial y}(x, \xi) \right| |y_2 - y_1| \leq M |y_1 - y_2|.$$

Since (x, y_1) and (x, y_2) were arbitrary points of $\mathcal{O}$ with a common first coordinate, the Lipschitz assumption (16) holds on $\mathcal{O}$ with the positive constant M .

Remark. The point of choosing $\mathcal{O}$ to be a rectangle (rather than an arbitrary neighborhood) is exactly that the vertical segment joining two of its points with the same first coordinate remains inside $\mathcal{O}$, which is what allows the Mean Value Theorem to be applied without leaving the set. Any convex bounded neighborhood would do equally well. We also note that a function with continuous first-order partial derivatives on $\mathbb{R}^2$ is continuous, so g satisfies both hypotheses of the Picard Local Existence Theorem on $\mathcal{O}$; consequently the initial value problem (14) with initial data (x_0, y_0) has a unique solution on some open interval about x_0 .

Problem 10.39. In case the function $g: \mathcal{O} \rightarrow \mathbb{R}$ has the form $g(x, y) = h(x) + by$, where the function $h: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, prove that the following is an explicit formula for the solution of (14):

$$f(x) = e^{b(x-x_0)}y_0 + \int_{x_0}^x e^{b(x-t)}h(t) dt \quad \text{for all } x \text{ in } I.$$

Solution. Here (14) is the initial value problem

$$\begin{aligned} f'(x) &= g(x, f(x)) = h(x) + bf(x) \quad \text{for all } x \in I, \\ f(x_0) &= y_0, \end{aligned}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^2$ containing (x_0, y_0) and I is an open interval containing x_0 . Recall that, as the text requires, a solution of (14) is a differentiable $f: I \rightarrow \mathbb{R}$ whose graph lies in $\mathcal{O}$, that is, $(x, f(x)) \in \mathcal{O}$ for all $x \in I$, and which satisfies $f'(x) = g(x, f(x))$ on I together with $f(x_0) = y_0$. We prove two things: the displayed formula defines a function f that solves (14) on every open interval I containing x_0 over which the graph of f lies in $\mathcal{O}$, and at least one such interval exists; and every solution of (14) on any open interval I containing x_0 agrees with f there.

Note that the formula itself makes sense for every real x , since the expression $h(x) + by$ is defined for all (x, y) ; the set $\mathcal{O}$ enters only through the requirement that the graph lie in $\mathcal{O}$.

A convenient rewriting. Since h is continuous on $\mathbb{R}$, so is the function $t \mapsto e^{-bt}h(t)$. Define

$$F(x) = \int_{x_0}^x e^{-bt}h(t) dt \quad \text{for } x \in \mathbb{R}.$$

Pulling the factor e^{bx} , which does not depend on the variable of integration, outside the integral, the displayed formula reads

$$f(x) = e^{bx} \left[e^{-bx_0}y_0 + F(x) \right].$$

The formula defines a solution. Because $t \mapsto e^{-bt}h(t)$ is continuous, the Fundamental Theorem of Differential Calculus, invoked in the text just before (15) for exactly this purpose, asserts that F is differentiable at every x with

$$F'(x) = e^{-bx}h(x).$$

Hence f , being a product of the differentiable functions $x \mapsto e^{bx}$ and $x \mapsto e^{-bx_0}y_0 + F(x)$, is differentiable, and by the product rule

$$f'(x) = b e^{bx} \left[e^{-bx_0}y_0 + F(x) \right] + e^{bx} F'(x) = b f(x) + e^{bx} e^{-bx} h(x) = b f(x) + h(x).$$

That is, $f'(x) = g(x, f(x))$ for all x . Moreover $F(x_0) = 0$, so

$$f(x_0) = e^{bx_0} \cdot e^{-bx_0}y_0 = y_0.$$

Thus f satisfies the differential equation and the initial condition on all of $\mathbb{R}$, and hence on any open interval I containing x_0 .

There is an interval about x_0 over which the graph of f lies in $\mathcal{O}$. Indeed, f is differentiable, hence continuous, so the mapping $x \mapsto (x, f(x))$ of $\mathbb{R}$ into $\mathbb{R}^2$ is continuous; since $\mathcal{O}$ is open and contains $(x_0, f(x_0)) = (x_0, y_0)$, the inverse image of $\mathcal{O}$ under this mapping is an open subset of $\mathbb{R}$ containing x_0 , and therefore contains an open interval I about x_0 . On any such I the function f is a solution of (14). In particular, if $\mathcal{O} = \mathbb{R}^2$ we may take $I = \mathbb{R}$.

The solution is unique. Let I be an open interval containing x_0 and let $u: I \rightarrow \mathbb{R}$ be any solution of (14), so u is differentiable on I , $u'(x) = h(x) + bu(x)$ for $x \in I$, and $u(x_0) = y_0$. Introduce the integrating factor and set

$$w(x) = e^{-bx}u(x) \quad \text{for } x \in I.$$

Then w is differentiable on I and

$$w'(x) = -be^{-bx}u(x) + e^{-bx}u'(x) = e^{-bx}[u'(x) - bu(x)] = e^{-bx}h(x).$$

The function w' is therefore continuous on I . Fix $x \in I$ and let J denote the closed, bounded interval with endpoints x_0 and x ; since I is an interval containing both points, $J \subseteq I$. On J the derivative w' is continuous, hence bounded, so by the Mean Value Theorem w is Lipschitz on J and therefore absolutely continuous on J . Theorem 10 of Chapter 6 now gives

$$w(x) - w(x_0) = \int_{x_0}^x w'(t) dt = \int_{x_0}^x e^{-bt} h(t) dt = F(x).$$

(For $x < x_0$ this is the same identity read with the orientation of the interval reversed, the integral $\int_{x_0}^x$ being by convention $-\int_x^{x_0}$.) Since $w(x_0) = e^{-bx_0} u(x_0) = e^{-bx_0} y_0$, we obtain

$$e^{-bx} u(x) = e^{-bx_0} y_0 + F(x),$$

and multiplying through by e^{bx} gives

$$u(x) = e^{b(x-x_0)} y_0 + \int_{x_0}^x e^{b(x-t)} h(t) dt \quad \text{for all } x \in I,$$

which is precisely the displayed formula. Hence every solution of (14) on I coincides with f , and the formula is an explicit formula for the (unique) solution of (14).

Remark. Uniqueness is consistent with the Picard Local Existence Theorem: here $|g(x, y_1) - g(x, y_2)| = |b| |y_1 - y_2|$, so the Lipschitz assumption (16) holds on all of $\mathbb{R}^2$ with $M = |b| + 1$. The integrating-factor argument above, however, gives uniqueness on the whole of I , not merely on a small interval about x_0 , and it exhibits the solution in closed form.

Problem 10.40. Consider the differential equation

$$\begin{aligned} f'(x) &= 3[f(x)]^{2/3} \quad \text{for all } x \in \mathbb{R} \\ f(0) &= 0. \end{aligned}$$

Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ that is identically 0 is a solution and the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 0$, if $x < 0$ and $f(x) = x^3$, if $x \geq 0$, is also a solution. Does this contradict the Picard Existence Theorem?

Solution. Here $g(x, y) = 3y^{2/3}$, where $y^{2/3}$ means $(y^{1/3})^2 = |y|^{2/3}$, the real cube root being used; thus g is defined and continuous on all of $\mathbb{R}^2$, and we may take $\mathcal{O} = \mathbb{R}^2$, $(x_0, y_0) = (0, 0)$.

The zero function is a solution. If $f_1(x) = 0$ for all x , then f_1 is differentiable with $f_1'(x) = 0$, while $3[f_1(x)]^{2/3} = 3 \cdot 0^{2/3} = 0$. Hence $f_1'(x) = 3[f_1(x)]^{2/3}$ for all $x \in \mathbb{R}$, and $f_1(0) = 0$. So f_1 solves the initial value problem on $I = \mathbb{R}$.

The second function is also a solution. Define

$$f_2(x) = \begin{cases} 0, & x < 0, \\ x^3, & x \geq 0. \end{cases}$$

We check differentiability and the equation at each point.

For $x < 0$: on the open interval $(-\infty, 0)$ the function f_2 is identically 0, so $f_2'(x) = 0$, and $3[f_2(x)]^{2/3} = 0$. The equation holds.

For $x > 0$: on the open interval $(0, \infty)$ we have $f_2(x) = x^3$, so $f_2'(x) = 3x^2$, while

$$3[f_2(x)]^{2/3} = 3(x^3)^{2/3} = 3x^2 \quad (x > 0).$$

The equation holds.

At $x = 0$: we compute the difference quotient directly. For $t > 0$,

$$\frac{f_2(t) - f_2(0)}{t - 0} = \frac{t^3}{t} = t^2 \xrightarrow{t \rightarrow 0^+} 0,$$

and for $t < 0$,

$$\frac{f_2(t) - f_2(0)}{t - 0} = \frac{0}{t} = 0 \xrightarrow{t \rightarrow 0^-} 0.$$

Both one-sided limits exist and equal 0, so f_2 is differentiable at 0 with $f_2'(0) = 0$; and $3[f_2(0)]^{2/3} = 3 \cdot 0 = 0$. The equation holds at 0 as well.

Finally $f_2(0) = 0$. Hence f_2 is a solution of the initial value problem on $I = \mathbb{R}$, and $f_2 \neq f_1$ since $f_2(1) = 1 \neq 0 = f_1(1)$. So the initial value problem has (at least) two distinct solutions on every open interval containing 0: indeed, restricted to any open interval I containing 0, f_1 and f_2 remain distinct solutions, because every such I contains points $x > 0$, at which $f_2(x) = x^3 \neq 0$.

Is this a contradiction? No. The Picard Local Existence Theorem asserts existence and uniqueness of a solution on some open interval containing x_0 under two hypotheses: that g be continuous on $\mathcal{O}$, and that the Lipschitz assumption (16) hold, that is, that there be a positive number M with

$$|g(x, y_1) - g(x, y_2)| \leq M|y_1 - y_2| \quad \text{for all } (x, y_1), (x, y_2) \in \mathcal{O}.$$

Our $g(x, y) = 3y^{2/3}$ is continuous, but it fails (16) on every neighborhood $\mathcal{O}$ of the initial point $(0, 0)$. Indeed, let $\mathcal{O}$ be any neighborhood of $(0, 0)$ and choose $\delta > 0$ so small that the square $(-\delta, \delta) \times (-\delta, \delta)$ is contained in $\mathcal{O}$. Suppose (16) held with constant M . Taking $y_1 = y$ with $0 < y < \delta$, $y_2 = 0$ and $x = 0$, both points $(0, y)$ and $(0, 0)$ lie in $\mathcal{O}$ and (16) would give

$$3y^{2/3} = |g(0, y) - g(0, 0)| \leq M|y - 0| = My,$$

that is, $3y^{-1/3} \leq M$ for all $y \in (0, \delta)$. But $3y^{-1/3} \rightarrow +\infty$ as $y \rightarrow 0^+$, so no such finite M exists. Hence hypothesis (16) fails at the initial point, the Picard Theorem does not apply, and there is no contradiction.

Remark. The failure of uniqueness here is not an isolated accident: for each $a \geq 0$ the function equal to 0 for $x \leq a$ and to $(x - a)^3$ for $x > a$ is, by the same computation, a solution of the initial value problem, so it in fact has a continuum of solutions.

Problem 10.41. For a positive number ϵ , consider the differential equation

$$\begin{aligned} f'(x) &= (1/\epsilon)[1 + (f(x))^2] \quad \text{for all } x \in \mathbb{R} \\ f(0) &= 0. \end{aligned}$$

Show that on the interval $I = (-\epsilon(\pi/2), \epsilon(\pi/2))$ there is a unique solution of this differential equation that is defined by $f(x) = \tan(x/\epsilon)$ and there is no solution in an interval strictly containing I .

Solution. Here $g(x, y) = (1/\epsilon)(1 + y^2)$, which is continuous on $\mathcal{O} = \mathbb{R}^2$, and the initial data is $(x_0, y_0) = (0, 0)$. Write $b = \epsilon\pi/2$, so $I = (-b, b)$.

Step 1: the function $f(x) = \tan(x/\epsilon)$ is a solution on I . If $x \in I$ then $x/\epsilon \in (-\pi/2, \pi/2)$, which is the interval on which the tangent function is defined and differentiable; hence $f(x) = \tan(x/\epsilon)$ is a well-defined differentiable function on I . By the chain rule and the identity $\frac{d}{du} \tan u = \sec^2 u = 1 + \tan^2 u$,

$$f'(x) = \frac{1}{\epsilon} \sec^2(x/\epsilon) = \frac{1}{\epsilon} [1 + \tan^2(x/\epsilon)] = \frac{1}{\epsilon} [1 + (f(x))^2] \quad \text{for all } x \in I.$$

Moreover $f(0) = \tan 0 = 0$. Thus f solves the initial value problem on I .

Step 2: uniqueness on I . Let $u: I \rightarrow \mathbb{R}$ be any solution, so u is differentiable on I , $u(0) = 0$, and

$$u'(x) = \frac{1}{\epsilon} [1 + (u(x))^2] \quad \text{for all } x \in I.$$

Define $\varphi: I \rightarrow \mathbb{R}$ by

$$\varphi(x) = \arctan(u(x)) - \frac{x}{\epsilon}.$$

Since $\arctan$ is differentiable on $\mathbb{R}$ with derivative $1/(1+t^2)$, the chain rule gives, for every $x \in I$,

$$\varphi'(x) = \frac{u'(x)}{1+(u(x))^2} - \frac{1}{\epsilon} = \frac{1}{1+(u(x))^2} \cdot \frac{1+(u(x))^2}{\epsilon} - \frac{1}{\epsilon} = 0.$$

A function with identically vanishing derivative on an interval is constant there (an immediate consequence of the Mean Value Theorem). Since $\varphi(0) = \arctan(0) - 0 = 0$, we conclude $\varphi \equiv 0$ on I , that is,

$$\arctan(u(x)) = \frac{x}{\epsilon} \quad \text{for all } x \in I.$$

For $x \in I$ we have $x/\epsilon \in (-\pi/2, \pi/2)$, and $\arctan$ is a one-to-one mapping of $\mathbb{R}$ onto $(-\pi/2, \pi/2)$ with inverse the restriction of $\tan$ to $(-\pi/2, \pi/2)$. Applying $\tan$ to both sides therefore gives

$$u(x) = \tan(x/\epsilon) = f(x) \quad \text{for all } x \in I.$$

Hence f is the unique solution on I .

Step 3: there is no solution on an interval strictly containing I . Suppose, to get a contradiction, that J is an interval with $I \subsetneq J$ and that $u: J \rightarrow \mathbb{R}$ is a solution of the initial value problem on J , that is, u is differentiable on J (one-sidedly at an endpoint of J that belongs to J), satisfies the equation at each point of J , and $u(0) = 0$. We do not assume J is open. Note $0 \in I \subseteq J$.

Choose a point $z \in J \setminus I$. Since $z \notin I = (-b, b)$, either $z \geq b$ or $z \leq -b$. Suppose first that $z \geq b$. Because J is an interval containing both 0 and z , it contains the whole segment $[0, z]$, and in particular $b \in J$.

The restriction of u to I is a solution of the initial value problem on I , so by Step 2,

$$u(x) = \tan(x/\epsilon) \quad \text{for all } x \in I = (-b, b).$$

On the other hand, u is differentiable at the point $b \in J$ (at worst from the left, if b happens to be the right endpoint of J), hence continuous there from the left, so

$$\lim_{x \rightarrow b^-} u(x) = u(b) \in \mathbb{R},$$

a finite real number. In particular, taking the tolerance 1 in the definition of this limit, there is a $\delta > 0$, which we may also require to satisfy $\delta < 2b$ so that $(b - \delta, b) \subseteq (-b, b) = I$, with $|u(x) - u(b)| < 1$, and hence $|u(x)| < |u(b)| + 1$, for all $x \in (b - \delta, b)$; that is, u is bounded on $(b - \delta, b)$. But $u(x) = \tan(x/\epsilon)$ on $(b - \delta, b)$ by the display above, and $\tan(x/\epsilon) \rightarrow +\infty$ as $x \rightarrow b^-$, because $x/\epsilon \rightarrow (\pi/2)^-$ and $\tan t \rightarrow +\infty$ as $t \rightarrow (\pi/2)^-$. So u is unbounded on $(b - \delta, b)$. This is a contradiction.

The case $z \leq -b$ is symmetric: then $[z, 0] \subseteq J$ forces $-b \in J$, so u is continuous at $-b$ from the right and hence bounded on some interval $(-b, -b + \delta) \subseteq I$, yet there $u(x) = \tan(x/\epsilon) \rightarrow -\infty$ as $x \rightarrow (-b)^+$; again a contradiction.

Therefore no interval strictly containing I carries a solution, and I is the maximal interval of existence.

Remark. This does not conflict with the Picard Local Existence Theorem, which is purely local: on any bounded rectangle $\mathcal{O} = (-a, a) \times (-R, R)$ the function $g(x, y) = (1/\epsilon)(1 + y^2)$ satisfies (16) with $M = 2R/\epsilon$, since

$$|g(x, y_1) - g(x, y_2)| = \frac{|y_1^2 - y_2^2|}{\epsilon} = \frac{|y_1 + y_2|}{\epsilon} |y_1 - y_2| \leq \frac{2R}{\epsilon} |y_1 - y_2|,$$

so the theorem does guarantee a unique solution near $x = 0$. What the example shows is that no such local theorem can be upgraded to global existence: the solution escapes to infinity in the finite time $\epsilon\pi/2$, and this escape time can be made arbitrarily small by taking ϵ small.

Problem 10.42. Let I be an open interval in $\mathbb{R}$ and suppose that the function $h: I \rightarrow \mathbb{R}$ has the property that there are points $x_1 < x_2$ in I and a number c such that $h(x_1) < c < h(x_2)$ but c does

not belong to $h(I)$. Prove that there is no solution to the differential equation (14) by arguing that if $f: I \rightarrow \mathbb{R}$ is a solution, then the continuous function $f(x) - cx$ fails to attain a minimum value on the interval $[x_1, x_2]$.

Solution. The differential equation (14) in the special case at hand is the one in which g is independent of its second variable, $g(x, y) = h(x)$; a solution is thus a differentiable function $f: I \rightarrow \mathbb{R}$ with

$$f'(x) = h(x) \quad \text{for all } x \in I$$

(together with an initial condition $f(x_0) = y_0$, which will play no role). We must show no such f exists.

Suppose, to the contrary, that $f: I \rightarrow \mathbb{R}$ is differentiable with $f' = h$ on I . Since x_1 and x_2 belong to the interval I and $x_1 < x_2$, the closed bounded interval $[x_1, x_2]$ is contained in I . Define

$$\psi(x) = f(x) - cx \quad \text{for } x \in I.$$

Then ψ is differentiable on I , hence continuous on I , and

$$\psi'(x) = f'(x) - c = h(x) - c \quad \text{for all } x \in I.$$

Because $c \notin h(I)$, we have

$$\psi'(x) = h(x) - c \neq 0 \quad \text{for every } x \in I.$$

We now show that ψ attains no minimum value on $[x_1, x_2]$, by ruling out each of the three possible locations of a minimizer.

The minimum is not attained at x_1 . We have $\psi'(x_1) = h(x_1) - c < 0$. By the definition of the derivative,

$$\lim_{t \rightarrow x_1} \frac{\psi(t) - \psi(x_1)}{t - x_1} = \psi'(x_1) < 0,$$

so there is a $\delta > 0$ such that the difference quotient is negative whenever $0 < |t - x_1| < \delta$ and $t \in I$. Choose such a t with $x_1 < t < \min\{x_1 + \delta, x_2\}$; this is possible since $x_1 < x_2$. Then $t - x_1 > 0$ and the quotient is negative, so $\psi(t) < \psi(x_1)$ with $t \in [x_1, x_2]$. Hence $\psi(x_1)$ is not the minimum value of ψ on $[x_1, x_2]$.

The minimum is not attained at x_2 . We have $\psi'(x_2) = h(x_2) - c > 0$, so there is a $\delta' > 0$ such that

$$\frac{\psi(t) - \psi(x_2)}{t - x_2} > 0 \quad \text{whenever } 0 < |t - x_2| < \delta', \quad t \in I.$$

Choose such a t with $\max\{x_1, x_2 - \delta'\} < t < x_2$. Then $t - x_2 < 0$, and since the quotient is positive, the numerator must be negative: $\psi(t) < \psi(x_2)$, with $t \in [x_1, x_2]$. Hence $\psi(x_2)$ is not the minimum value of ψ on $[x_1, x_2]$.

The minimum is not attained at an interior point. Suppose $x_* \in (x_1, x_2)$ were a point at which ψ attains its minimum over $[x_1, x_2]$. Then x_* is an interior point of $[x_1, x_2]$ at which the differentiable function ψ has a local minimum, and therefore $\psi'(x_*) = 0$. Indeed, for $t > x_*$ near x_* the quotient $[\psi(t) - \psi(x_*)]/(t - x_*)$ is ≥ 0 , so letting $t \rightarrow x_*^+$ gives $\psi'(x_*) \geq 0$; for $t < x_*$ near x_* that quotient is ≤ 0 , so letting $t \rightarrow x_*^-$ gives $\psi'(x_*) \leq 0$. Hence $\psi'(x_*) = 0$, contradicting the fact that ψ' never vanishes on I .

Combining the three cases: the continuous function $\psi(x) = f(x) - cx$ attains no minimum value on $[x_1, x_2]$.

But $[x_1, x_2]$ is a closed, bounded, nonempty subset of $\mathbb{R}$, hence a compact metric space by Theorem 20 of Chapter 9, and the restriction of ψ to it is continuous; by the Extreme Value Theorem, Theorem 22 of Chapter 9, a continuous real-valued function on a compact metric space attains a minimum value. This is a contradiction.

Therefore no differentiable $f: I \rightarrow \mathbb{R}$ with $f' = h$ on I exists. Since any solution of (14) defined on all of I would in particular be such a function, the differential equation (14) has no solution on I for such an h , whatever the initial data.

Remark. What this argument really proves is Darboux's Theorem: the derivative of a differentiable function on an interval has the intermediate value property, even though it need not be continuous. Consequently, as remarked in the text preceding equation (15), if the image $h(I)$ fails to be an interval, then h cannot be the derivative of any function on I , and (14) has no solution.

10.7 Problems 10.43–10.44

Problem 10.43. Use the preceding exercise to prove the following theorem of Darboux: Let I be an open interval in $\mathbb{R}$ and suppose that the function $f : I \rightarrow \mathbb{R}$ is differentiable. Then the image of the derivative $f' : I \rightarrow \mathbb{R}$ is an interval.

Solution. Throughout, $J \subseteq \mathbb{R}$ is called an interval provided it has the intermediate value property

$$u, v \in J \text{ and } u < c < v \implies c \in J.$$

This is equivalent to the usual definition: if J has this property and we set $\alpha = \inf J$ and $\beta = \sup J$ (allowing $\pm\infty$), then $(\alpha, \beta) \subseteq J \subseteq [\alpha, \beta]$, so that J is one of the intervals with endpoints α and β ; the empty set and one-point sets are degenerate intervals. Thus, to prove that $f'(I)$ is an interval it suffices to show that no number c that lies strictly between two values of f' can fail to be a value of f' . Recall the content of the preceding exercise. Let I be an open interval and let $h : I \rightarrow \mathbb{R}$ have the property that there are points $x_1 < x_2$ in I and a number c with

$$h(x_1) < c < h(x_2) \quad \text{but} \quad c \notin h(I).$$

Then the differential equation (14) with $g(x, y) = h(x)$, that is, the equation

$$\varphi'(x) = h(x) \quad \text{for all } x \in I, \quad \varphi(x_0) = y_0,$$

has no solution. Indeed, suppose $\varphi : I \rightarrow \mathbb{R}$ were a solution, so that φ is differentiable on I with $\varphi' = h$ there. Define the continuous function

$$\psi(x) = \varphi(x) - cx \quad \text{for } x \in [x_1, x_2] \subseteq I.$$

Since $[x_1, x_2]$ is compact and ψ is continuous, the Extreme Value Theorem provides a point $x_* \in [x_1, x_2]$ at which ψ attains its minimum value on $[x_1, x_2]$. Now ψ is differentiable on I with

$$\psi'(x) = h(x) - c \quad \text{for all } x \in I,$$

and since $c \notin h(I)$ we have $\psi'(x) \neq 0$ for every $x \in I$. Observe:

- $x_* \neq x_1$. Indeed, $\psi'(x_1) = h(x_1) - c < 0$, so since

$$\lim_{t \rightarrow x_1^+} \frac{\psi(t) - \psi(x_1)}{t - x_1} = \psi'(x_1) < 0,$$

there is a $t \in (x_1, x_2)$ for which the difference quotient is negative; as $t - x_1 > 0$ this forces $\psi(t) < \psi(x_1)$, so the minimum is not attained at x_1 .

- $x_* \neq x_2$. Indeed, $\psi'(x_2) = h(x_2) - c > 0$, so since

$$\lim_{t \rightarrow x_2^-} \frac{\psi(t) - \psi(x_2)}{t - x_2} = \psi'(x_2) > 0,$$

there is a $t \in (x_1, x_2)$ for which the difference quotient is positive; as $t - x_2 < 0$ this forces $\psi(t) < \psi(x_2)$, so the minimum is not attained at x_2 .

Therefore x_* belongs to the open interval (x_1, x_2) . A differentiable function attains an interior minimum only at a point where its derivative vanishes: for $t > x_*$ near x_* the quotient $[\psi(t) - \psi(x_*)]/(t - x_*)$ is nonnegative, whence $\psi'(x_*) \geq 0$, while for $t < x_*$ near x_* the same quotient is nonpositive, whence $\psi'(x_*) \leq 0$. Thus $\psi'(x_*) = 0$, that is, $h(x_*) = c$ with $x_* \in I$. This contradicts the assumption that $c \notin h(I)$. Hence no such solution φ exists. This is precisely the assertion of the preceding exercise:

the argument shows that a solution φ would force the continuous function $\varphi(x) - cx$ to fail to attain a minimum value on $[x_1, x_2]$, which the Extreme Value Theorem forbids.

We now prove Darboux's theorem. Let $f : I \rightarrow \mathbb{R}$ be differentiable on the open interval I and put $h = f' : I \rightarrow \mathbb{R}$. Suppose, by way of contradiction, that $h(I) = f'(I)$ is not an interval. By the characterization recorded at the outset, there are points $p, q \in I$ and a number c with

$$f'(p) < c < f'(q) \quad \text{and} \quad c \notin f'(I).$$

Note that $p \neq q$. There are two cases.

Case 1: $p < q$. Take $x_1 = p$ and $x_2 = q$. Then $h = f'$ satisfies exactly the hypothesis of the preceding exercise: $x_1 < x_2$ lie in I , $h(x_1) < c < h(x_2)$, and $c \notin h(I)$. Consequently the differential equation

$$\varphi'(x) = h(x) \quad \text{for all } x \in I, \quad \varphi(x_0) = f(x_0)$$

(where x_0 is any point of I) has no solution. But f itself is a solution of this equation, since f is differentiable on I with $f' = h$ and $f(x_0) = f(x_0)$. This contradiction shows that Case 1 cannot occur.

Case 2: $q < p$. Consider the reflected function $F = -f : I \rightarrow \mathbb{R}$, which is differentiable on I with $F' = -f'$, and consider the number $c' = -c$. Then, with $x_1 = q < x_2 = p$,

$$F'(x_1) = -f'(q) < -c = c' < -f'(p) = F'(x_2),$$

and $c' \notin F'(I) = -f'(I)$, because $c' = -c \in -f'(I)$ would mean $c \in f'(I)$. So F' satisfies the hypothesis of the preceding exercise with the number c' , and exactly as in Case 1 the function F is itself a solution of the equation $\varphi' = F'$ on I with $\varphi(x_0) = F(x_0)$, which the preceding exercise forbids. This contradiction shows that Case 2 cannot occur either.

Since neither case can occur, no number c lying strictly between two values of f' can fail to lie in $f'(I)$. Therefore $f'(I)$ has the intermediate value property and hence is an interval.

Problem 10.44. State and prove a form of the Picard Existence Theorem for systems of differential equations in the following context: $\mathcal{O}$ is an open subset of $\mathbb{R} \times \mathbb{R}^n$, $\mathbf{g} : \mathcal{O} \rightarrow \mathbb{R}^n$ is continuous, the point $(x_0, \mathbf{y}_0)$ is in $\mathcal{O}$, and the system of differential equations is

$$\begin{aligned} \mathbf{f}'(x) &= \mathbf{g}(x, \mathbf{f}(x)) \quad \text{for all } x \in I \\ \mathbf{f}(x_0) &= \mathbf{y}_0. \end{aligned}$$

(Hint: Approximate $\mathbf{g}$ by a Lipschitz mapping and then use the Arzela-Ascoli Theorem.)

Solution. Notation. For $\mathbf{u} \in \mathbb{R}^n$ let $|\mathbf{u}|$ denote the Euclidean norm; the same symbol is used for the Euclidean norm on $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$. For a compact interval J , let $C(J, \mathbb{R}^n)$ denote the set of continuous mappings $\mathbf{f} : J \rightarrow \mathbb{R}^n$, metrized by

$$\rho_{\max}(\mathbf{f}_1, \mathbf{f}_2) = \max_{x \in J} |\mathbf{f}_1(x) - \mathbf{f}_2(x)|.$$

Derivatives and integrals of $\mathbb{R}^n$ -valued functions are taken componentwise. Two elementary facts will be used. First, $C(J, \mathbb{R}^n)$ is a complete metric space: a $\rho_{\max}$ -Cauchy sequence is uniformly Cauchy, hence converges pointwise since $\mathbb{R}^n$ is complete, and the convergence is then uniform, so the limit is continuous; this is verbatim the proof, given in the preceding chapter, that $C(J)$ is complete, with $\mathbb{R}$ replaced by the complete space $\mathbb{R}^n$. Second, for continuous $\mathbf{h} : J \rightarrow \mathbb{R}^n$ and $x, x_0 \in J$,

$$\left| \int_{x_0}^x \mathbf{h}(t) dt \right| \leq \left| \int_{x_0}^x |\mathbf{h}(t)| dt \right|;$$

for $x \geq x_0$ this follows by setting $\mathbf{v} = \int_{x_0}^x \mathbf{h}$ and computing $|\mathbf{v}|^2 = \int_{x_0}^x \mathbf{v} \cdot \mathbf{h}(t) dt \leq |\mathbf{v}| \int_{x_0}^x |\mathbf{h}(t)| dt$, and the case $x < x_0$ follows by reversing the limits.

Statement of the theorem.

The Picard Local Existence Theorem for Systems. Let $\mathcal{O}$ be an open subset of $\mathbb{R} \times \mathbb{R}^n$ containing the point $(x_0, \mathbf{y}_0)$ and suppose $\mathbf{g} : \mathcal{O} \rightarrow \mathbb{R}^n$ is continuous. Then there is an open interval I containing x_0 and a differentiable mapping $\mathbf{f} : I \rightarrow \mathbb{R}^n$ whose graph lies in $\mathcal{O}$ and which satisfies

$$\mathbf{f}'(x) = \mathbf{g}(x, \mathbf{f}(x)) \quad \text{for all } x \in I, \quad \mathbf{f}(x_0) = \mathbf{y}_0.$$

If, in addition, there is an $M > 0$ for which the Lipschitz property in the second variable holds uniformly with respect to the first,

$$|\mathbf{g}(x, \mathbf{y}_1) - \mathbf{g}(x, \mathbf{y}_2)| \leq M|\mathbf{y}_1 - \mathbf{y}_2| \quad \text{for all } (x, \mathbf{y}_1), (x, \mathbf{y}_2) \text{ in } \mathcal{O},$$

then the solution is unique in the strong sense that any two solutions defined on a common interval containing x_0 agree there.

Reduction to an integral equation. Exactly as in the scalar case treated in this section, if I is an open interval containing x_0 and $\mathbf{f} : I \rightarrow \mathbb{R}^n$ is continuous with $(x, \mathbf{f}(x)) \in \mathcal{O}$ for all $x \in I$, then $t \mapsto \mathbf{g}(t, \mathbf{f}(t))$ is continuous on I , and by the Fundamental Theorem of Differential Calculus applied to each component, $\mathbf{f}$ is a solution of the system if and only if

$$\mathbf{f}(x) = \mathbf{y}_0 + \int_{x_0}^x \mathbf{g}(t, \mathbf{f}(t)) dt \quad \text{for all } x \in I.$$

We refer to this as the integral form of the system; it is the exact analogue of equation (15).

The compact rectangle. Since $\mathcal{O}$ is open and contains $(x_0, \mathbf{y}_0)$, we may choose positive numbers a and b so that the compact set

$$R = [x_0 - a, x_0 + a] \times \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - \mathbf{y}_0| \leq b\}$$

is contained in $\mathcal{O}$. Since $\mathbf{g}$ is continuous and R is compact, $\mathbf{g}$ is bounded and uniformly continuous on R . Set

$$K = 1 + \max_{(x, \mathbf{y}) \in R} |\mathbf{g}(x, \mathbf{y})|, \quad \ell = \min\{a, b/K\}, \quad I_\ell = [x_0 - \ell, x_0 + \ell].$$

Let X_ℓ be the subspace of $C(I_\ell, \mathbb{R}^n)$ consisting of those $\mathbf{f}$ with $|\mathbf{f}(x) - \mathbf{y}_0| \leq b$ for all $x \in I_\ell$; that is, the continuous mappings whose graph lies in R . Then X_ℓ is nonempty, since it contains the constant mapping $x \mapsto \mathbf{y}_0$, and it is a closed subset of the complete metric space $C(I_\ell, \mathbb{R}^n)$: if $\mathbf{f}_j \rightarrow \mathbf{f}$ uniformly with $|\mathbf{f}_j - \mathbf{y}_0| \leq b$ pointwise, then $|\mathbf{f} - \mathbf{y}_0| \leq b$ pointwise. Hence X_ℓ is itself a complete metric space.

Step 1: the Lipschitz case, on an interval whose length does not depend on the Lipschitz constant. Suppose $\mathbf{h} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, $|\mathbf{h}| \leq K$ on R , and there is an $M > 0$ with

$$|\mathbf{h}(x, \mathbf{y}_1) - \mathbf{h}(x, \mathbf{y}_2)| \leq M|\mathbf{y}_1 - \mathbf{y}_2| \quad \text{for all } (x, \mathbf{y}_1), (x, \mathbf{y}_2) \in R.$$

We claim there is exactly one $\mathbf{f} \in X_\ell$ with $\mathbf{f}(x) = \mathbf{y}_0 + \int_{x_0}^x \mathbf{h}(t, \mathbf{f}(t)) dt$ on I_ℓ , with ℓ as fixed above. For $\mathbf{f} \in X_\ell$ define

$$T(\mathbf{f})(x) = \mathbf{y}_0 + \int_{x_0}^x \mathbf{h}(t, \mathbf{f}(t)) dt \quad \text{for } x \in I_\ell.$$

The integrand is continuous, so $T(\mathbf{f}) \in C(I_\ell, \mathbb{R}^n)$, and for $x \in I_\ell$,

$$|T(\mathbf{f})(x) - \mathbf{y}_0| \leq \left| \int_{x_0}^x |\mathbf{h}(t, \mathbf{f}(t))| dt \right| \leq \ell K \leq b,$$

since $\ell \leq b/K$. Hence $T(X_\ell) \subseteq X_\ell$.

The estimate of this section shows T is a contraction only when $\ell M < 1$, and below the Lipschitz constants M will be forced to tend to ∞ ; so instead of $\rho_{\max}$ we use the equivalent weighted metric

$$\rho_*(\mathbf{f}_1, \mathbf{f}_2) = \max_{x \in I_\ell} e^{-2M|x-x_0|} |\mathbf{f}_1(x) - \mathbf{f}_2(x)|.$$

Since $e^{-2M\ell}\rho_{\max} \leq \rho_* \leq \rho_{\max}$, the metrics ρ_* and $\rho_{\max}$ have the same Cauchy sequences and the same convergent sequences, so (X_ℓ, ρ_*) is also a complete metric space. For $\mathbf{f}_1, \mathbf{f}_2 \in X_\ell$ and $x \in I_\ell$, using the Lipschitz property and $|\mathbf{f}_1(t) - \mathbf{f}_2(t)| \leq e^{2M|t-x_0|}\rho_*(\mathbf{f}_1, \mathbf{f}_2)$,

$$|T(\mathbf{f}_1)(x) - T(\mathbf{f}_2)(x)| \leq \left| \int_{x_0}^x M |\mathbf{f}_1(t) - \mathbf{f}_2(t)| dt \right| \leq M\rho_*(\mathbf{f}_1, \mathbf{f}_2) \left| \int_{x_0}^x e^{2M|t-x_0|} dt \right|,$$

and the last integral equals $(e^{2M|x-x_0|} - 1)/(2M)$. Therefore

$$e^{-2M|x-x_0|}|T(\mathbf{f}_1)(x) - T(\mathbf{f}_2)(x)| \leq \frac{1}{2}\rho_*(\mathbf{f}_1, \mathbf{f}_2) \quad \text{for all } x \in I_\ell,$$

so $\rho_*(T(\mathbf{f}_1), T(\mathbf{f}_2)) \leq \frac{1}{2}\rho_*(\mathbf{f}_1, \mathbf{f}_2)$: the mapping $T : X_\ell \rightarrow X_\ell$ is a contraction with Lipschitz constant $1/2$, whatever M may be. By the Banach Contraction Principle, T has exactly one fixed point in X_ℓ , which is the asserted solution of the integral equation. This proves the claim, and note carefully that ℓ depends only on a, b and K , not on M .

Step 2: approximation of $\mathbf{g}$ by Lipschitz mappings. Let $u : R \rightarrow \mathbb{R}$ be continuous, and for a natural number k define

$$u_k(z) = \inf_{w \in R} [u(w) + k|z - w|] \quad \text{for } z \in \mathbb{R}^{n+1}.$$

Since u is bounded on the compact set R , the infimum is a real number, with $\min_R u \leq u_k(z) \leq u(w) + k|z - w|$ for every $w \in R$. For $z, z' \in \mathbb{R}^{n+1}$ and $w \in R$ we have $u(w) + k|z - w| \leq u(w) + k|z' - w| + k|z - z'|$; taking the infimum over w gives $u_k(z) \leq u_k(z') + k|z - z'|$, and by symmetry

$$|u_k(z) - u_k(z')| \leq k|z - z'| \quad \text{for all } z, z' \in \mathbb{R}^{n+1},$$

so u_k is Lipschitz on all of $\mathbb{R}^{n+1}$ with Lipschitz constant k . Moreover $u_k \leq u$ on R (take $w = z$). We claim $u_k \rightarrow u$ uniformly on R . Let $\epsilon > 0$. Since R is compact, u is uniformly continuous, so there is $\delta > 0$ such that $|u(w) - u(z)| < \epsilon$ whenever $w, z \in R$ and $|w - z| \leq \delta$. Let $k \geq (\max_R u - \min_R u)/\delta$ and let $z \in R$. For $w \in R$ with $|z - w| \leq \delta$ we have $u(w) + k|z - w| \geq u(w) \geq u(z) - \epsilon$, while for $w \in R$ with $|z - w| > \delta$ we have

$$u(w) + k|z - w| \geq \min_R u + k\delta \geq \min_R u + (\max_R u - \min_R u) = \max_R u \geq u(z).$$

Taking the infimum over $w \in R$ gives $u_k(z) \geq u(z) - \epsilon$, and hence $0 \leq u(z) - u_k(z) \leq \epsilon$ for all $z \in R$. This proves the uniform convergence.

Now write $\mathbf{g} = (g^1, \dots, g^n)$ on R and apply the preceding construction to each component, obtaining

$$\mathbf{g}_k = (g_k^1, \dots, g_k^n) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n,$$

which is continuous on $\mathbb{R}^{n+1}$, satisfies $|\mathbf{g}_k(z) - \mathbf{g}_k(z')| \leq \sqrt{n}k|z - z'|$ for all z, z' (combine the componentwise estimates), and converges to $\mathbf{g}$ uniformly on R , since each component does. In particular, with $M_k = \sqrt{n}k$,

$$|\mathbf{g}_k(x, \mathbf{y}_1) - \mathbf{g}_k(x, \mathbf{y}_2)| \leq M_k|\mathbf{y}_1 - \mathbf{y}_2| \quad \text{for all } (x, \mathbf{y}_1), (x, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^n.$$

Choose an index N so that $\sup_R |\mathbf{g}_k - \mathbf{g}| \leq 1$ for $k \geq N$; then $|\mathbf{g}_k| \leq K$ on R for such k , with K as fixed above. Discarding the first N terms, we may assume this holds for all k .

Step 3: solving the approximating systems and passing to the limit. By Step 1 applied to $\mathbf{h} = \mathbf{g}_k$ (which is bounded by K on R and Lipschitz in the second variable with constant M_k), for each k there is $\mathbf{f}_k \in X_\ell$ with

$$\mathbf{f}_k(x) = \mathbf{y}_0 + \int_{x_0}^x \mathbf{g}_k(t, \mathbf{f}_k(t)) dt \quad \text{for all } x \in I_\ell,$$

where, crucially, the interval I_ℓ is the same for every k . Since $\mathbf{f}_k \in X_\ell$, we have $|\mathbf{f}_k(x) - \mathbf{y}_0| \leq b$ for all x and k , so the sequence $\{\mathbf{f}_k\}$ is uniformly bounded. Furthermore, for $x, x' \in I_\ell$,

$$|\mathbf{f}_k(x) - \mathbf{f}_k(x')| = \left| \int_{x'}^x \mathbf{g}_k(t, \mathbf{f}_k(t)) dt \right| \leq K|x - x'|,$$

because $(t, \mathbf{f}_k(t)) \in R$ and $|\mathbf{g}_k| \leq K$ there. Thus $\{\mathbf{f}_k\}$ is uniformly Lipschitz with constant K , and in particular equicontinuous on the compact metric space I_ℓ .

Write $\mathbf{f}_k = (f_k^1, \dots, f_k^n)$. Each scalar sequence $\{f_k^j\}_k$ is uniformly bounded and equicontinuous on I_ℓ , since $|f_k^j(x) - f_k^j(x')| \leq |\mathbf{f}_k(x) - \mathbf{f}_k(x')| \leq K|x - x'|$. Apply the Arzela-Ascoli Theorem to $\{f_k^1\}$ to extract a subsequence converging uniformly on I_ℓ ; apply it again to the second components of that subsequence, and so on through $j = n$. After these n successive extractions we obtain a subsequence, which we relabel $\{\mathbf{f}_k\}$, all of whose component sequences converge uniformly on I_ℓ . Hence there is a continuous $\mathbf{f} : I_\ell \rightarrow \mathbb{R}^n$ with $\mathbf{f}_k \rightarrow \mathbf{f}$ uniformly on I_ℓ . Since $|\mathbf{f}_k(x) - \mathbf{y}_0| \leq b$ for every k , passing to the limit gives $|\mathbf{f}(x) - \mathbf{y}_0| \leq b$, so $\mathbf{f} \in X_\ell$ and the graph of $\mathbf{f}$ lies in $R \subseteq \mathcal{O}$.

We now pass to the limit in the integral equation. Let $\epsilon > 0$. By the uniform continuity of $\mathbf{g}$ on R there is $\delta > 0$ such that $|\mathbf{g}(z) - \mathbf{g}(z')| < \epsilon/2$ whenever $z, z' \in R$ and $|z - z'| < \delta$. Choose N' so that for $k \geq N'$ we have both $\sup_R |\mathbf{g}_k - \mathbf{g}| < \epsilon/2$ and $\rho_{\max}(\mathbf{f}_k, \mathbf{f}) < \delta$. Then for $k \geq N'$ and all $t \in I_\ell$, since $(t, \mathbf{f}_k(t))$ and $(t, \mathbf{f}(t))$ both lie in R and are at distance less than δ ,

$$|\mathbf{g}_k(t, \mathbf{f}_k(t)) - \mathbf{g}(t, \mathbf{f}(t))| \leq |\mathbf{g}_k(t, \mathbf{f}_k(t)) - \mathbf{g}(t, \mathbf{f}_k(t))| + |\mathbf{g}(t, \mathbf{f}_k(t)) - \mathbf{g}(t, \mathbf{f}(t))| < \epsilon.$$

Thus $\mathbf{g}_k(\cdot, \mathbf{f}_k(\cdot)) \rightarrow \mathbf{g}(\cdot, \mathbf{f}(\cdot))$ uniformly on I_ℓ , and therefore, for each fixed $x \in I_\ell$,

$$\left| \int_{x_0}^x \mathbf{g}_k(t, \mathbf{f}_k(t)) dt - \int_{x_0}^x \mathbf{g}(t, \mathbf{f}(t)) dt \right| \leq \ell \cdot \sup_{t \in I_\ell} |\mathbf{g}_k(t, \mathbf{f}_k(t)) - \mathbf{g}(t, \mathbf{f}(t))| \rightarrow 0.$$

Letting $k \rightarrow \infty$ in the integral equation for $\mathbf{f}_k$ therefore yields

$$\mathbf{f}(x) = \mathbf{y}_0 + \int_{x_0}^x \mathbf{g}(t, \mathbf{f}(t)) dt \quad \text{for all } x \in I_\ell.$$

By the equivalence between the system and its integral form, the restriction of $\mathbf{f}$ to the open interval $I = (x_0 - \ell, x_0 + \ell)$ is differentiable there and satisfies

$$\mathbf{f}'(x) = \mathbf{g}(x, \mathbf{f}(x)) \quad \text{for all } x \in I, \quad \mathbf{f}(x_0) = \mathbf{y}_0.$$

This proves the existence assertion.

Step 4: uniqueness under the Lipschitz assumption. Assume now, in addition, that $|\mathbf{g}(x, \mathbf{y}_1) - \mathbf{g}(x, \mathbf{y}_2)| \leq M|\mathbf{y}_1 - \mathbf{y}_2|$ for all $(x, \mathbf{y}_1), (x, \mathbf{y}_2)$ in $\mathcal{O}$. Let J be an interval containing x_0 and let $\mathbf{f}, \mathbf{h} : J \rightarrow \mathbb{R}^n$ both be solutions of the system, both with graphs in $\mathcal{O}$. By the integral form of the system, for $x \in J$,

$$|\mathbf{f}(x) - \mathbf{h}(x)| = \left| \int_{x_0}^x [\mathbf{g}(t, \mathbf{f}(t)) - \mathbf{g}(t, \mathbf{h}(t))] dt \right| \leq M \left| \int_{x_0}^x |\mathbf{f}(t) - \mathbf{h}(t)| dt \right|.$$

Fix a compact interval $[x_0, x_0 + r] \subseteq J$ and let $A = \max_{[x_0, x_0 + r]} |\mathbf{f} - \mathbf{h}|$, a finite number since $|\mathbf{f} - \mathbf{h}|$ is continuous. We show by induction that

$$|\mathbf{f}(x) - \mathbf{h}(x)| \leq A \frac{M^m(x - x_0)^m}{m!} \quad \text{for all } x \in [x_0, x_0 + r] \text{ and all } m \geq 0.$$

The case $m = 0$ is the definition of A . If the inequality holds for m , then for $x \in [x_0, x_0 + r]$,

$$|\mathbf{f}(x) - \mathbf{h}(x)| \leq M \int_{x_0}^x A \frac{M^m(t - x_0)^m}{m!} dt = A \frac{M^{m+1}(x - x_0)^{m+1}}{(m+1)!},$$

which is the case $m+1$. Since $M^m r^m / m! \rightarrow 0$ as $m \rightarrow \infty$, we conclude $\mathbf{f} = \mathbf{h}$ on $[x_0, x_0 + r]$. The same argument applied on $[x_0 - r, x_0]$ (or applied to $x \mapsto \mathbf{f}(2x_0 - x)$, $\mathbf{h}(2x_0 - x)$, which solve the system with $\mathbf{g}$ replaced by $-\mathbf{g}(2x_0 - x, \mathbf{y})$) gives $\mathbf{f} = \mathbf{h}$ to the left of x_0 as well. As r was arbitrary, $\mathbf{f} = \mathbf{h}$ on J . This proves uniqueness and completes the proof of the theorem.

Remark. Under the Lipschitz assumption the whole of Step 2 and Step 3 is unnecessary: Step 1 applied directly to $\mathbf{h} = \mathbf{g}$ (extended in any fashion, since only its values on R are used) produces the solution at once from the Banach Contraction Principle, exactly as in the scalar Picard Local Existence Theorem proved in this section. Note also that no uniqueness is asserted in the merely continuous case, and none can be: Problem 40 exhibits a continuous g for which the initial value problem has two distinct solutions.

11 Topological Spaces: General Properties

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11.1 Problems 11.1–11.7

Problem 11.1. Show that the discrete topology for a nonempty set X is a metric topology.

Solution. Recall that the discrete topology on X is the collection $\mathcal{T}$ of all subsets of X , and that the metric topology induced by a metric ρ on X consists of those sets $\mathcal{O}$ with the property that each point of $\mathcal{O}$ is the center of an open ball contained in $\mathcal{O}$.

Define the discrete metric ρ on X by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

First we check that ρ is a metric. It is nonnegative and real-valued, $\rho(x, y) = 0$ if and only if $x = y$ by definition, and ρ is symmetric since the condition $x = y$ is symmetric in x and y . For the triangle inequality, let x, y, z belong to X ; we must show $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. If $x = z$, the left-hand side is 0 and the inequality is clear since $\rho \geq 0$. If $x \neq z$, then y cannot equal both x and z , so at least one of $\rho(x, y)$, $\rho(y, z)$ equals 1; hence the right-hand side is at least 1 = $\rho(x, z)$. Thus (X, ρ) is a metric space.

Now compute the open balls. For x in X and $0 < r \leq 1$,

$$B(x, r) = \{y \in X : \rho(x, y) < r\} = \{x\},$$

since the only y with $\rho(x, y) < 1$ is $y = x$. In particular $B(x, 1/2) = \{x\}$ for every x .

Let $\mathcal{T}_\rho$ denote the metric topology induced by ρ . If $\mathcal{O}$ is any subset of X and x belongs to $\mathcal{O}$, then $B(x, 1/2) = \{x\} \subseteq \mathcal{O}$, so $\mathcal{O}$ contains an open ball centered at each of its points; therefore $\mathcal{O}$ belongs to $\mathcal{T}_\rho$. (For $\mathcal{O} = \emptyset$ the requirement is vacuous.) Hence every subset of X is open in the metric topology, that is, $\mathcal{T} \subseteq \mathcal{T}_\rho$. The reverse inclusion is immediate because $\mathcal{T}$ consists of all subsets of X , so $\mathcal{T}_\rho \subseteq \mathcal{T}$. Therefore $\mathcal{T}_\rho = \mathcal{T}$: the discrete topology is the metric topology induced by the discrete metric, and in particular it is a metric topology.

Problem 11.2. Show that the discrete topology on a set has a unique base.

Solution. Let X be a nonempty set with the discrete topology $\mathcal{T}$, so that every subset of X is open. Put

$$\mathcal{B}_0 = \{\{x\} : x \in X\},$$

the collection of one-point subsets of X . We show that $\mathcal{B}_0$ is a base for $\mathcal{T}$ and that $\mathcal{B}_0$ is contained in every base for $\mathcal{T}$; consequently $\mathcal{B}_0$ is the one and only base that carries no redundant sets, that is, the unique smallest base.

$\mathcal{B}_0$ is a base. Each $\{x\}$ is open, since all subsets are open. Let x belong to X and let U be a neighborhood of x . Then $\{x\}$ belongs to $\mathcal{B}_0$ and $x \in \{x\} \subseteq U$. Thus $\{\{x\}\}$ is a base for the topology at x , and $\mathcal{B}_0$ contains a base at each point of X : it is a base for $\mathcal{T}$. Equivalently, every nonempty open set $\mathcal{O}$ is the union of the subcollection $\{\{x\} : x \in \mathcal{O}\}$ of $\mathcal{B}_0$, and $\emptyset$ is the union of the empty subcollection. Every base contains $\mathcal{B}_0$. Let $\mathcal{B}$ be any base for $\mathcal{T}$ and let x belong to X . Since $\{x\}$ is open, it is a neighborhood of x , so by the definition of a base there is a set B in $\mathcal{B}$ with $x \in B \subseteq \{x\}$. Hence $B = \{x\}$, and so $\{x\}$ belongs to $\mathcal{B}$. As x was arbitrary, $\mathcal{B}_0 \subseteq \mathcal{B}$.

Therefore $\mathcal{B}_0$ is a base and it is contained in every base for $\mathcal{T}$: it is the smallest base, and it is the unique smallest one, since a smallest element of a collection ordered by inclusion is unique.

A remark on the word *unique* is in order, since with the definition of base used in the text a topology generally has many bases; the book says so explicitly just after Proposition 2. Here too: if X has at least two points, then $\mathcal{B}_0 \cup \{X\}$ is a collection of open sets containing a base at each point, hence is also a base, and it differs from $\mathcal{B}_0$. So uniqueness cannot mean that only one collection qualifies. What is true, and what the above proves, is that the discrete topology has a unique irredundant base, namely $\mathcal{B}_0$. Call a base $\mathcal{B}$ irredundant provided no member of $\mathcal{B}$ is the union of a subcollection of the

remaining members. If $\mathcal{B}$ is an irredundant base for the discrete topology, then $\mathcal{B}_0 \subseteq \mathcal{B}$ by the preceding paragraph; and if S were a member of $\mathcal{B}$ other than a singleton, then $S = \bigcup \{\{x\} : x \in S\}$ exhibits S as the union of a subcollection of $\mathcal{B} \sim \{S\}$ (the empty subcollection in case $S = \emptyset$), contradicting irredundancy. Hence $\mathcal{B} = \mathcal{B}_0$. Every base whatsoever is obtained from $\mathcal{B}_0$ by adjoining sets that are already unions of members of $\mathcal{B}_0$ and so contribute nothing new.

Problem 11.3. Regarding Proposition 2, show that if $\mathcal{B}$ is a base for a topology, then properties (i) and (ii) hold.

Solution. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{T}$ be a base for $\mathcal{T}$; that is, $\mathcal{B}$ is a collection of open sets which contains a base for the topology at each point of X . Unwinding the definition of a base at a point, this says precisely:

for each x in X and each neighborhood U of x , there is a set B in $\mathcal{B}$ with $x \in B \subseteq U$. (*)

Indeed, if $\mathcal{B}_x \subseteq \mathcal{B}$ is a base at x , its members are neighborhoods of x and one of them is contained in U .

Property (i): $\mathcal{B}$ covers X , that is, $X = \bigcup_{B \in \mathcal{B}} B$. Since every member of $\mathcal{B}$ is a subset of X , the union is contained in X . Conversely, let x belong to X . The whole set X is open, hence is a neighborhood of x , so by (*) there is B in $\mathcal{B}$ with $x \in B \subseteq X$. Thus x belongs to the union. Therefore $X = \bigcup_{B \in \mathcal{B}} B$.

Property (ii): if B_1 and B_2 belong to $\mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a set B in $\mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$. Since $\mathcal{B} \subseteq \mathcal{T}$, both B_1 and B_2 are open, and by the defining property of a topology the intersection of a finite collection of open sets is open; hence $B_1 \cap B_2$ is open. As x belongs to $B_1 \cap B_2$, this set is a neighborhood of x , so by (*) there is B in $\mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$, as required.

This completes the converse half of Proposition 2, which was left as an exercise there: a base for a topology necessarily satisfies (i) and (ii), and Proposition 2 shows conversely that any collection satisfying (i) and (ii) is a base for exactly one topology on X , namely the collection of $\emptyset$ together with all unions of subcollections of $\mathcal{B}$.

Problem 11.4. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies for a nonempty set X . Show that $\mathcal{T}_1 = \mathcal{T}_2$ if and only if there are bases $\mathcal{B}_1$ for $\mathcal{T}_1$ and $\mathcal{B}_2$ for $\mathcal{T}_2$ that are related as follows at each point x in X : for each neighborhood $\mathcal{N}_1$ of x belonging to $\mathcal{B}_1$, there is a neighborhood $\mathcal{N}_2$ of x belonging to $\mathcal{B}_2$ for which $\mathcal{N}_2 \subseteq \mathcal{N}_1$, and for each neighborhood $\mathcal{N}_2$ of x belonging to $\mathcal{B}_2$, there is a neighborhood $\mathcal{N}_1$ of x belonging to $\mathcal{B}_1$ for which $\mathcal{N}_1 \subseteq \mathcal{N}_2$.

Solution. Call the displayed condition the interlacing condition for the pair $\mathcal{B}_1, \mathcal{B}_2$.

Necessity. Suppose $\mathcal{T}_1 = \mathcal{T}_2 =: \mathcal{T}$. A topology is a base for itself: it is a collection of open sets, and for x in X and U a neighborhood of x we may take $B = U$, which belongs to $\mathcal{T}$ and satisfies $x \in B \subseteq U$. So set $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{T}$, a base for $\mathcal{T}_1$ and a base for $\mathcal{T}_2$. Now fix x in X . If $\mathcal{N}_1$ belongs to $\mathcal{B}_1$ and contains x , take $\mathcal{N}_2 = \mathcal{N}_1$, which belongs to $\mathcal{B}_2$, contains x , and satisfies $\mathcal{N}_2 \subseteq \mathcal{N}_1$; symmetrically in the other direction. So the interlacing condition holds.

Sufficiency. Suppose $\mathcal{B}_1$ is a base for $\mathcal{T}_1$, $\mathcal{B}_2$ is a base for $\mathcal{T}_2$, and the interlacing condition holds at every point of X . We show $\mathcal{T}_1 \subseteq \mathcal{T}_2$; the reverse inclusion follows by interchanging the roles of the indices 1 and 2, since the interlacing condition is symmetric in them.

Let $\mathcal{O}$ belong to $\mathcal{T}_1$; we may assume $\mathcal{O}$ is nonempty, since $\emptyset$ belongs to every topology. Let x belong to $\mathcal{O}$. Then $\mathcal{O}$ is a $\mathcal{T}_1$ -neighborhood of x , so, $\mathcal{B}_1$ being a base for $\mathcal{T}_1$, there is a set $\mathcal{N}_1$ in $\mathcal{B}_1$ with $x \in \mathcal{N}_1 \subseteq \mathcal{O}$. Thus $\mathcal{N}_1$ is a neighborhood of x belonging to $\mathcal{B}_1$, and the interlacing condition provides a neighborhood $\mathcal{N}_2$ of x belonging to $\mathcal{B}_2$ with $\mathcal{N}_2 \subseteq \mathcal{N}_1$. Denote this set by $\mathcal{N}_2^x$; it satisfies

$$x \in \mathcal{N}_2^x \subseteq \mathcal{N}_1 \subseteq \mathcal{O}, \quad \mathcal{N}_2^x \in \mathcal{B}_2 \subseteq \mathcal{T}_2.$$

Consequently

$$\mathcal{O} = \bigcup_{x \in \mathcal{O}} \mathcal{N}_2^x,$$

because each $\mathcal{N}_2^x$ is contained in $\mathcal{O}$ and each x in $\mathcal{O}$ lies in $\mathcal{N}_2^x$. This exhibits $\mathcal{O}$ as a union of a collection of $\mathcal{T}_2$ -open sets, and the union of any collection of open sets is open; hence $\mathcal{O}$ belongs to

$\mathcal{T}_2$. (Equivalently, by Proposition 1, $\mathcal{O}$ is $\mathcal{T}_2$ -open because each of its points has a $\mathcal{T}_2$ -neighborhood contained in it.)
Therefore $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and by symmetry $\mathcal{T}_2 \subseteq \mathcal{T}_1$, so $\mathcal{T}_1 = \mathcal{T}_2$.

Problem 11.5. Let E be a subset of a topological space X .

- (i) A point $x \in X$ is called an interior point of E provided there is a neighborhood of x that is contained in E : the collection of interior points of E is called the interior of E and denoted by $\text{int } E$. Show that $\text{int } E$ is always open and E is open if and only if $E = \text{int } E$.
- (ii) A point $x \in X$ is called an exterior point of E provided there is a neighborhood of x that is contained in $X \sim E$: the collection of exterior points of E is called the exterior of E and denoted by $\text{ext } E$. Show that $\text{ext } E$ is always open and E is open if and only if $\overline{E} \sim E \subseteq \text{ext } E$.
- (iii) A point $x \in X$ is called a boundary point of E provided every neighborhood of x contains points in E and points in $X \sim E$: the collection of boundary points of E is called the boundary of E and denoted by $\text{bd } E$. Show that (i) $\text{bd } E$ is always closed, (ii) E is open if and only if $E \cap \text{bd } E = \emptyset$, and (iii) E is closed if and only if $\text{bd } E \subseteq E$.

Solution. Throughout, a neighborhood of a point means an open set containing that point.

Part (i). First, $\text{int } E \subseteq E$: if x is an interior point of E , some neighborhood U of x satisfies $x \in U \subseteq E$, so $x \in E$.

$\text{int } E$ is open. Let x belong to $\text{int } E$ and choose a neighborhood U of x with $U \subseteq E$. If y belongs to U , then U is itself a neighborhood of y contained in E , so y is an interior point of E . Hence $U \subseteq \text{int } E$, and U is a neighborhood of x contained in $\text{int } E$. Since x was an arbitrary point of $\text{int } E$, Proposition 1 tells us $\text{int } E$ is open.

If E is open, then for each x in E the set E itself is a neighborhood of x contained in E , so $E \subseteq \text{int } E$; with the reverse inclusion above, $E = \text{int } E$. Conversely, if $E = \text{int } E$, then E is open because $\text{int } E$ is open.

Part (ii). By definition $\text{ext } E = \text{int}(X \sim E)$: a point has a neighborhood contained in $X \sim E$ exactly when it is an interior point of $X \sim E$. Hence, by part (i) applied to the set $X \sim E$, the set $\text{ext } E$ is open.

Next observe

$$\text{ext } E \cap \overline{E} = \emptyset.$$

Indeed, if x belongs to $\text{ext } E$, there is a neighborhood U of x with $U \subseteq X \sim E$, that is, $U \cap E = \emptyset$; so x has a neighborhood containing no point of E and therefore x is not a point of closure of E .

Now consider the asserted criterion $\overline{E} \sim E \subseteq \text{ext } E$. Since $\overline{E} \sim E \subseteq \overline{E}$ and $\overline{E}$ is disjoint from $\text{ext } E$, the inclusion $\overline{E} \sim E \subseteq \text{ext } E$ holds if and only if $\overline{E} \sim E = \emptyset$, that is, if and only if $\overline{E} \subseteq E$. Since always $E \subseteq \overline{E}$, this says exactly $\overline{E} = E$, i.e. that E is closed. So what the stated inclusion characterizes is closedness, not openness: the word *open* in the printed statement should read *closed*.

That the statement as printed is false is shown by an example. In $X = \mathbb{R}$ with the Euclidean topology take $E = (0, 1)$. Then E is open, while $\overline{E} = [0, 1]$, so $\overline{E} \sim E = \{0, 1\}$, and 0 does not belong to $\text{ext } E$ because every neighborhood of 0 contains points of $(0, 1)$. Hence the inclusion $\overline{E} \sim E \subseteq \text{ext } E$ fails for this open set. Conversely the inclusion can hold for a set that is not open: take $E = [0, 1]$ in $\mathbb{R}$, for which $\overline{E} \sim E = \emptyset$.

Thus the correct assertion, which we have proved, is: $\text{ext } E$ is always open, and $\overline{E} \sim E \subseteq \text{ext } E$ if and only if E is closed.

Part (iii). We first record that X is the disjoint union

$$X = \text{int } E \cup \text{bd } E \cup \text{ext } E,$$

the three sets being pairwise disjoint. Indeed, a point x of X fails to be a boundary point exactly when some neighborhood of x contains no point of E (so $x \in \text{ext } E$) or some neighborhood of x contains no point of $X \sim E$, i.e. is contained in E (so $x \in \text{int } E$). Hence $\text{bd } E = X \sim (\text{int } E \cup \text{ext } E)$ and the three sets cover X . They are pairwise disjoint: $\text{bd } E$ is disjoint from the other two by what was just said; and if x belonged to $\text{int } E \cap \text{ext } E$ there would be neighborhoods $U \subseteq E$ and $V \subseteq X \sim E$ of x , whence $x \in U \cap V \subseteq E \cap (X \sim E) = \emptyset$, a contradiction.

(iii)(i) $\text{bd } E$ is closed. By the displayed identity, $X \sim \text{bd } E = \text{int } E \cup \text{ext } E$, which is the union of two open sets by parts (i) and (ii), hence open. By Proposition 4, a subset of X is closed if and only if its complement is open; therefore $\text{bd } E$ is closed.

(iii)(ii) E is open if and only if $E \cap \text{bd } E = \emptyset$. Suppose E is open. If x belongs to E , then E is a neighborhood of x contained in E , so $x \in \text{int } E$, and since $\text{int } E$ and $\text{bd } E$ are disjoint, $x \notin \text{bd } E$. Hence $E \cap \text{bd } E = \emptyset$. Conversely, suppose $E \cap \text{bd } E = \emptyset$ and let x belong to E . Then $x \notin \text{bd } E$ by hypothesis, and $x \notin \text{ext } E$ since $\text{ext } E \subseteq X \sim E$. By the trichotomy above, $x \in \text{int } E$. Thus $E \subseteq \text{int } E$, so $E = \text{int } E$ and E is open by part (i).

(iii)(iii) E is closed if and only if $\text{bd } E \subseteq E$. We claim $\overline{E} = \text{int } E \cup \text{bd } E$. Indeed, x belongs to $\overline{E}$ if and only if every neighborhood of x contains a point of E , that is, if and only if $x \notin \text{ext } E$; and by the trichotomy the complement of $\text{ext } E$ is $\text{int } E \cup \text{bd } E$. Since $\text{int } E \subseteq E \subseteq \overline{E}$, it follows that

$$\overline{E} = \text{int } E \cup \text{bd } E = E \cup \text{bd } E.$$

Hence E is closed, i.e. $\overline{E} = E$, if and only if $E \cup \text{bd } E = E$, which holds if and only if $\text{bd } E \subseteq E$.

Problem 11.6. Let A and B be subsets of a topological space X . Show that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. Also, show that $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$ and $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$.

Solution. Monotonicity. Suppose $A \subseteq B$ and let x belong to $\overline{A}$. Let U be any neighborhood of x . Since x is a point of closure of A , there is a point of A in U ; that point belongs to B as well, since $A \subseteq B$. So every neighborhood of x contains a point of B , that is, x belongs to $\overline{B}$. Hence $\overline{A} \subseteq \overline{B}$.

The union. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, monotonicity gives $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$, so

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

For the reverse inclusion, argue by contraposition. Let x be a point of X that does not belong to $\overline{A \cup B}$. Since $x \notin \overline{A}$ there is a neighborhood U of x with $U \cap A = \emptyset$, and since $x \notin \overline{B}$ there is a neighborhood V of x with $V \cap B = \emptyset$. Then $U \cap V$ is open, being the intersection of a finite collection of open sets, and it contains x ; thus $U \cap V$ is a neighborhood of x , and

$$(U \cap V) \cap (A \cup B) \subseteq (U \cap A) \cup (V \cap B) = \emptyset.$$

So some neighborhood of x misses $A \cup B$, whence $x \notin \overline{A \cup B}$. This proves $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, and therefore $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

(Alternatively: by Proposition 3 the sets $\overline{A}$ and $\overline{B}$ are closed, so by Proposition 5 their union $\overline{A} \cup \overline{B}$ is closed, being the union of a finite collection of closed sets; it contains $A \cup B$, so by the minimality assertion of Proposition 3 it contains $\overline{A \cup B}$.)

The intersection. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, monotonicity gives $\overline{A \cap B} \subseteq \overline{A}$ and $\overline{A \cap B} \subseteq \overline{B}$, hence

$$\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}.$$

This inclusion can be strict, so no equality holds in general. For instance, in $\mathbb{R}$ with the Euclidean topology take $A = (0, 1)$ and $B = (1, 2)$. Then $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$, while $\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\}$.

Problem 11.7. Let $\mathcal{O}$ be an open subset of a topological space X . For a subset E of X , show that $\mathcal{O}$ is disjoint from E if and only if it is disjoint from $\overline{E}$.

Solution. Suppose first that $\mathcal{O} \cap \overline{E} = \emptyset$. Since $E \subseteq \overline{E}$ (every point of E is a point of closure of E , because each of its neighborhoods contains the point itself), we get $\mathcal{O} \cap E \subseteq \mathcal{O} \cap \overline{E} = \emptyset$. So $\mathcal{O}$ is disjoint from E . Observe that openness of $\mathcal{O}$ was not needed here.

Conversely, suppose $\mathcal{O} \cap E = \emptyset$, and let x be a point of $\mathcal{O}$. Because $\mathcal{O}$ is open and contains x , the set $\mathcal{O}$ is a neighborhood of x ; and it contains no point of E , since $\mathcal{O} \cap E = \emptyset$. Thus x possesses a

neighborhood containing no point of E , so x is not a point of closure of E , that is, $x \notin \overline{E}$. As x was an arbitrary point of $\mathcal{O}$, we conclude $\mathcal{O} \cap \overline{E} = \emptyset$.

Therefore $\mathcal{O} \cap E = \emptyset$ if and only if $\mathcal{O} \cap \overline{E} = \emptyset$.

Equivalently, this says $\mathcal{O} \subseteq X \sim E$ implies $\mathcal{O} \subseteq X \sim \overline{E}$, i.e. $X \sim \overline{E}$ is the largest open subset of $X \sim E$; this is consistent with Proposition 3, which asserts that $\overline{E}$ is the smallest closed set containing E , and with Proposition 4, which identifies open sets as complements of closed sets.

11.2 Problems 11.8–11.14

Problem 11.8. For a collection $\mathcal{S}$ of subsets of a nonempty set X , show that there is a topology $\mathcal{T}$ on X that contains the collection $\mathcal{S}$ and has the property that any other topology that contains $\mathcal{S}$ also contains $\mathcal{T}$: it is the topology with the fewest sets that contains $\mathcal{S}$.

Solution. Let Σ denote the family of all topologies on X that contain $\mathcal{S}$. This family is nonempty: the discrete topology $\mathcal{P}(X)$, consisting of all subsets of X , is a topology on X and it certainly contains $\mathcal{S}$. Define

$$\mathcal{T} = \bigcap_{\mathcal{T}' \in \Sigma} \mathcal{T}',$$

the collection of those subsets of X that belong to every topology containing $\mathcal{S}$.

We claim $\mathcal{T}$ is a topology on X .

First, $\emptyset$ and X belong to every topology on X , hence to every member of Σ , hence to $\mathcal{T}$.

Second, let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be any collection of members of $\mathcal{T}$ and fix $\mathcal{T}' \in \Sigma$. Each $\mathcal{O}_\lambda$ belongs to $\mathcal{T}'$ by the definition of $\mathcal{T}$, and $\mathcal{T}'$ is a topology, so $\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ belongs to $\mathcal{T}'$. Since $\mathcal{T}' \in \Sigma$ was arbitrary, $\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ belongs to $\mathcal{T}$.

Third, let $\mathcal{O}_1, \dots, \mathcal{O}_n$ belong to $\mathcal{T}$ and fix $\mathcal{T}' \in \Sigma$. Each $\mathcal{O}_k$ belongs to $\mathcal{T}'$, and a topology is closed under finite intersections, so $\bigcap_{k=1}^n \mathcal{O}_k$ belongs to $\mathcal{T}'$. As $\mathcal{T}'$ was arbitrary, $\bigcap_{k=1}^n \mathcal{O}_k$ belongs to $\mathcal{T}$.

Therefore $\mathcal{T}$ is a topology on X .

Moreover $\mathcal{S} \subseteq \mathcal{T}$: if $S \in \mathcal{S}$, then S belongs to every $\mathcal{T}' \in \Sigma$ by the very definition of Σ , and hence $S \in \mathcal{T}$.

Finally, $\mathcal{T}$ has the asserted minimality. If $\mathcal{T}''$ is any topology on X with $\mathcal{S} \subseteq \mathcal{T}''$, then $\mathcal{T}'' \in \Sigma$, and an intersection of a family of sets is contained in each member of the family; thus $\mathcal{T} \subseteq \mathcal{T}''$. So $\mathcal{T}$ is contained in every topology containing $\mathcal{S}$, that is, it is the topology with the fewest sets that contains $\mathcal{S}$. Uniqueness is immediate: if $\mathcal{T}_1$ and $\mathcal{T}_2$ both contain $\mathcal{S}$ and both have this minimality property, then $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \subseteq \mathcal{T}_1$, so $\mathcal{T}_1 = \mathcal{T}_2$.

It is worth recording the concrete description of $\mathcal{T}$, since it is what one uses in practice. Let $\mathcal{S}_0 = \mathcal{S} \cup \{X\}$; this collection covers X . Let

$$\mathcal{B} = \{S_1 \cap S_2 \cap \dots \cap S_n : n \in \mathbb{N}, S_k \in \mathcal{S}_0\}$$

be the collection of intersections of finite subcollections of $\mathcal{S}_0$. Then $\mathcal{B}$ covers X (it contains X), and $\mathcal{B}$ is closed under finite intersection, so property (ii) of Proposition 2 holds trivially: if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, take $B = B_1 \cap B_2 \in \mathcal{B}$. By Proposition 2, $\mathcal{B}$ is a base for a topology $\mathcal{T}_{\mathcal{B}}$ on X , consisting of $\emptyset$ together with all unions of subcollections of $\mathcal{B}$. Clearly $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}}$. On the other hand, any topology $\mathcal{T}''$ containing $\mathcal{S}$ contains X , hence contains $\mathcal{S}_0$, hence contains $\mathcal{B}$ (closure under finite intersections), hence contains all unions of subcollections of $\mathcal{B}$ together with $\emptyset$, that is, $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}''$. Thus $\mathcal{T}_{\mathcal{B}}$ has exactly the minimality property proved above, and by the uniqueness just noted, $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. In other words, the smallest topology containing $\mathcal{S}$ is precisely the topology for which $\mathcal{S} \cup \{X\}$ is a subbase.

Problem 11.9. (The Sorgenfrey Line) Show that the collection of intervals of the form $[a, b)$, where $a < b$, is a base for a topology for the set of real numbers $\mathbb{R}$. The set of real numbers $\mathbb{R}$ with this topology is called the **Sorgenfrey Line**.

Solution. Let

$$\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}.$$

By Proposition 2, it suffices to verify the two properties (i) and (ii) there.

(i) $\mathcal{B}$ covers $\mathbb{R}$. Let $x \in \mathbb{R}$. Then $x < x + 1$, so $[x, x + 1)$ belongs to $\mathcal{B}$, and $x \in [x, x + 1)$. Hence $\mathbb{R} = \bigcup_{B \in \mathcal{B}} B$.

(ii) Suppose $B_1 = [a, b)$ and $B_2 = [c, d)$ belong to $\mathcal{B}$ and $x \in B_1 \cap B_2$. Set

$$e = \max\{a, c\}, \quad f = \min\{b, d\}.$$

Since $x \in [a, b)$ we have $a \leq x < b$, and since $x \in [c, d)$ we have $c \leq x < d$. Consequently $e = \max\{a, c\} \leq x$ and $x < \min\{b, d\} = f$. In particular $e \leq x < f$, so $e < f$ and therefore $[e, f)$ belongs to $\mathcal{B}$ and contains x . Finally

$$[e, f) \subseteq [a, b) \quad \text{and} \quad [e, f) \subseteq [c, d),$$

because $a \leq e$ and $f \leq b$ give $[e, f) \subseteq [a, b)$, and $c \leq e$, $f \leq d$ give $[e, f) \subseteq [c, d)$. Thus $x \in [e, f) \subseteq B_1 \cap B_2$ with $[e, f) \in \mathcal{B}$, which is exactly property (ii).

By Proposition 2, $\mathcal{B}$ is a base for a (unique) topology on $\mathbb{R}$, namely the topology whose members are $\emptyset$ together with all unions of subcollections of $\mathcal{B}$. This is the Sorgenfrey Line.

Two remarks that identify the topology more concretely. Since

$$(a, b) = \bigcup_{n \geq N} [a + \frac{1}{n}, b)$$

for N large enough that $a + 1/N < b$, every Euclidean open interval is open in this topology; as the Euclidean open intervals form a base for the Euclidean topology on $\mathbb{R}$, the Sorgenfrey topology contains the Euclidean topology. The containment is strict, since $[0, 1)$ is not Euclidean open: no Euclidean neighborhood of the point 0 is contained in $[0, 1)$, because every such neighborhood contains points less than 0.

Problem 11.10. (The Moore Plane) Consider the upper half plane, $\mathbb{R}^{2,+} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. For points (x, y) with $y > 0$, take as a basic open neighborhood a usual Euclidean open ball centered at (x, y) and contained in the upper half plane. As a basic open neighborhood of a point $(x, 0)$ take the set consisting of the point itself and all the points in an open Euclidean ball in the upper half plane that is tangent to the real line at $(x, 0)$. Show that this collection of sets is a base. The set $\mathbb{R}^{2,+}$ with this topology is called the **Moore Plane**.

Solution. Write $B(p, r)$ for the usual Euclidean open ball in $\mathbb{R}^2$ of center p and radius $r > 0$. The described collection is $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\mathcal{B}_1 = \{B((x, y), r) : y > 0, 0 < r \leq y\},$$

$$\mathcal{B}_2 = \{T(x, r) := \{(x, 0)\} \cup B((x, r), r) : x \in \mathbb{R}, r > 0\}.$$

Here $\mathcal{B}_1$ is exactly the family of Euclidean balls centered at a point of positive height and contained in $\mathbb{R}^{2,+}$: indeed the points of $B((x, y), r)$ have second coordinate lying in $(y-r, y+r)$, so $B((x, y), r) \subseteq \mathbb{R}^{2,+}$ if and only if $r \leq y$. And $B((x, r), r)$ is the open ball lying in the open upper half plane and tangent to the real line at $(x, 0)$.

Two elementary observations will be used repeatedly.

(a) **Every set in $\mathcal{B}_1$ consists of points of positive height.** If $0 < r \leq y$ and $(u, v) \in B((x, y), r)$ then $|v - y| < r \leq y$, so $v > 0$.

(b) **Tangent balls at a fixed point of tangency are nested:** if $0 < r \leq s$ then $B((x, r), r) \subseteq B((x, s), s)$. Indeed, expanding the defining inequality,

$$(u, v) \in B((x, r), r) \iff (u - x)^2 + (v - r)^2 < r^2 \iff (u - x)^2 + v^2 < 2rv,$$

and the last inequality forces $v > 0$. Hence $(u - x)^2 + v^2 < 2rv \leq 2sv$, which is equivalent to

$(u - x)^2 + (v - s)^2 < s^2$, that is, $(u, v) \in B((x, s), s)$. Consequently $T(x, r) \subseteq T(x, s)$ whenever $0 < r \leq s$.

We verify the two conditions of Proposition 2 for $\mathcal{B}$ regarded as a collection of subsets of $X = \mathbb{R}^{2,+}$. Note first that each member of $\mathcal{B}$ is indeed a subset of $\mathbb{R}^{2,+}$, by the description of $\mathcal{B}_1$ above and by (b).

(i) $\mathcal{B}$ covers X . If $p = (x, y)$ with $y > 0$, then $p \in B((x, y), y) \in \mathcal{B}_1$. If $p = (x, 0)$, then $p \in T(x, 1) \in \mathcal{B}_2$.
(ii) The intersection condition. Let $B_1, B_2 \in \mathcal{B}$ and let $p \in B_1 \cap B_2$. We produce $B \in \mathcal{B}$ with $p \in B \subseteq B_1 \cap B_2$. There are two cases.

Case 1: $p = (x, y)$ with $y > 0$. For $i = 1, 2$ define a Euclidean open subset U_i of $\mathbb{R}^2$ containing p with $U_i \subseteq B_i$, as follows. If $B_i \in \mathcal{B}_1$, put $U_i = B_i$, which is a Euclidean open ball. If $B_i = T(x_i, r_i) \in \mathcal{B}_2$, then $p \neq (x_i, 0)$ because p has positive height, so $p \in B((x_i, r_i), r_i)$; put $U_i = B((x_i, r_i), r_i)$, again a Euclidean open ball, and $U_i \subseteq B_i$. In both cases U_i is Euclidean open and $p \in U_i \subseteq B_i$.

Since $U_1 \cap U_2$ is a Euclidean open set containing p , there is $\varepsilon > 0$ with $B(p, \varepsilon) \subseteq U_1 \cap U_2$. Put $r = \min\{\varepsilon, y\} > 0$. Then $B(p, r) \in \mathcal{B}_1$, since p has height $y > 0$ and $r \leq y$; moreover

$$p \in B(p, r) \subseteq B(p, \varepsilon) \subseteq U_1 \cap U_2 \subseteq B_1 \cap B_2.$$

So $B = B(p, r)$ works.

Case 2: $p = (x, 0)$. By observation (a), no member of $\mathcal{B}_1$ contains a point of height 0; hence neither B_1 nor B_2 lies in $\mathcal{B}_1$, and we may write $B_i = T(x_i, r_i)$ for $i = 1, 2$. Again by (b) the ball part $B((x_i, r_i), r_i)$ contains only points of positive height, so the only point of $T(x_i, r_i)$ of height 0 is $(x_i, 0)$. Since $p = (x, 0) \in T(x_i, r_i)$, we conclude $x_i = x$ for $i = 1, 2$. Thus $B_1 = T(x, r_1)$ and $B_2 = T(x, r_2)$. Put $r = \min\{r_1, r_2\} > 0$. By the nesting property (b), $T(x, r) \subseteq T(x, r_1)$ and $T(x, r) \subseteq T(x, r_2)$, so

$$p = (x, 0) \in T(x, r) \subseteq B_1 \cap B_2,$$

and $T(x, r) \in \mathcal{B}_2 \subseteq \mathcal{B}$. So $B = T(x, r)$ works.

Both conditions of Proposition 2 hold, so $\mathcal{B}$ is a base for a unique topology on $\mathbb{R}^{2,+}$, whose open sets are $\emptyset$ together with the unions of subcollections of $\mathcal{B}$. This is the Moore Plane.

Problem 11.11. (Kuratowski 14-subset problem)

- (i) Let E be a subset of a topological space X . Show that at most 14 different sets can be obtained from E by repeated use of complementation and closure.
- (ii) Give an example in $\mathbb{R}^2$ where there are 14 different sets coming from a suitable E .

Solution. Throughout, X is a topological space and, for $A \subseteq X$, we write

$$c(A) = X \sim A, \quad k(A) = \overline{A}, \quad i(A) = X \sim \overline{X \sim A} = ckc(A).$$

Operators are composed as functions, so in a word such as ckc the rightmost letter acts first. We freely use the following facts.

- $cc = \text{id}$, since $X \sim (X \sim A) = A$.
- $kk = k$, since $\overline{A}$ is closed and a closed set equals its own closure (Proposition 3).
- k is monotone: if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$ (Problem 6).
- $i(A)$ is the largest open subset of A : it is open because $\overline{X \sim A}$ is closed (Proposition 3) and complements of closed sets are open (Proposition 4); it is contained in A because $X \sim A \subseteq \overline{X \sim A}$; and if $\mathcal{O}$ is open with $\mathcal{O} \subseteq A$, then $X \sim \mathcal{O}$ is closed (Proposition 4) and contains $X \sim A$, so $\overline{X \sim A} \subseteq X \sim \mathcal{O}$ by Proposition 3, i.e. $\mathcal{O} \subseteq i(A)$. (This is the interior of Problem 5(i).)

(i) At most 14 sets.

Step 1: reduction to alternating words. Every set obtainable from E is $w(E)$ for some word w in the two letters c and k . Using $cc = \text{id}$ and $kk = k$ we may delete any pair of adjacent equal letters (replacing kk by k , deleting cc), which strictly shortens the word; iterating, every word equals an **alternating** word, one in which no two consecutive letters are equal. For each length $n \geq 1$ there are exactly two alternating words, one beginning with c and one beginning with k ; together with the empty word id this gives $1 + 2n$ words of length at most n .

Step 2: the Kuratowski identity $ckcckck = ckc$. We first prove:

Lemma. If U is open, then $\overline{i(\overline{U})} = \overline{U}$.

Proof. Since $i(\overline{U}) \subseteq \overline{U}$, monotonicity of closure and $kk = k$ give $\overline{i(\overline{U})} \subseteq \overline{\overline{U}} = \overline{U}$. Conversely, U is an open subset of $\overline{U}$, so by the maximality of the interior, $U \subseteq i(\overline{U})$; taking closures, $\overline{U} \subseteq \overline{i(\overline{U})}$. The two inclusions give the equality. $\square$

Now let $A \subseteq X$ be arbitrary and put $U = ck(A) = X \sim \overline{A}$, which is open by Proposition 4. Then $kck(A) = \overline{U}$, and since $kckc = k \circ (ckc) = k \circ i$,

$$kckckck(A) = (kckc)(kck(A)) = ki(\overline{U}) = \overline{i(\overline{U})} = \overline{U} = kck(A),$$

the fourth equality by the Lemma. Hence $kckckck = kck$ as operators.

Step 3: no alternating word of length ≥ 8 is needed. Multiplying the identity of Step 2 on the left and/or right by c and using $cc = \text{id}$:

$$kckckckc = kckc, \quad ckckckck = ckck.$$

Let w be an alternating word of length $n \geq 8$ and write $w = uv$, where u consists of the first (leftmost) eight letters. Since w alternates, u is either $kckckckc$ or $ckckckck$, so u equals a word u' of length 4 ($kckc$ or $ckck$, respectively). Then $w = u'v$ is a word of length $n - 4 < n$. By Step 1 it equals an alternating word of length at most $n - 4$. Induction on n shows every word equals an alternating word of length at most 7.

Step 4: the count. By Steps 1 and 3, every set obtainable from E is $w(E)$ for one of the alternating words of length at most 7, of which there are $1 + 2 \cdot 7 = 15$, namely

$$\text{id}, c, k, ck, kc, ckc, kck, ckck, kckc, ckckc, kckck, ckckck, kckckc, ckckckc, kckckck.$$

By Step 2 the last of these coincides with kck . Hence at most 14 distinct sets arise.

(ii) A subset of $\mathbb{R}^2$ realizing 14.

We first exhibit a subset of $\mathbb{R}$ realizing 14 and then transport it to $\mathbb{R}^2$.

Let

$$A = (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbb{Q}) \subseteq \mathbb{R},$$

with the usual topology. Note that the words $\text{id}, k, ckc = i, ckck = ik, kckck = kik, kckc = ki, ckckckc = iki$ are seven of the fourteen words of Step 4, and prefixing each with c (and using $cc = \text{id}$) produces exactly the remaining seven. So the fourteen sets are the seven sets

$$A, \quad kA, \quad ikA, \quad kikA, \quad iA, \quad kiA, \quad ikiA$$

together with their complements. Computing directly:

$$\begin{aligned} A &= (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbb{Q}), \\ kA &= [0, 2] \cup \{3\} \cup [4, 5], \\ ikA &= (0, 2) \cup (4, 5), \\ kikA &= [0, 2] \cup [4, 5], \\ iA &= (0, 1) \cup (1, 2), \\ kiA &= [0, 2], \\ ikiA &= (0, 2). \end{aligned}$$

(For kA : every point of $[0, 2]$ is a limit of points of $(0, 1) \cup (1, 2)$, and every point of $[4, 5]$ is a limit of rationals in $[4, 5]$. For iA : the sets $\{3\}$ and $[4, 5] \cap \mathbb{Q}$ have empty interior, since every interval about such a point meets the complement of A , and $1 \notin A$.)

These seven sets are pairwise distinct. Only A and kA contain the point 3, and these two differ because $1 \in kA$ while $1 \notin A$. Among the remaining five, exactly ikA and $kikA$ meet the interval $(4, 5)$, the other three being subsets of $[0, 2]$; and $ikA \neq kikA$ because $0 \in kikA \sim ikA$. Finally, among iA , kiA and $ikiA$: the point 1 belongs to kiA and to $ikiA$ but not to iA , so iA differs from the other two, and $0 \in kiA \sim ikiA$, so $kiA \neq ikiA$.

Furthermore no one of the seven equals the complement of another: each of the seven is a subset of $[0, 5]$, hence bounded, whereas the complement of a subset of $[0, 5]$ contains $(5, \infty)$ and is therefore unbounded. Consequently the fourteen sets are pairwise distinct, and A realizes the bound in $\mathbb{R}$.

Now set

$$E = A \times \mathbb{R} \subseteq \mathbb{R}^2,$$

with $\mathbb{R}^2$ carrying its usual topology. Write $\Phi(S) = S \times \mathbb{R}$ for $S \subseteq \mathbb{R}$. Then:

- Φ is injective: $S \times \mathbb{R} = S' \times \mathbb{R}$ forces $S = S'$.
- Φ commutes with complementation: $\mathbb{R}^2 \sim (S \times \mathbb{R}) = (\mathbb{R} \sim S) \times \mathbb{R}$.
- Φ commutes with closure: $\overline{S \times \mathbb{R}} = \overline{S} \times \mathbb{R}$. Indeed, the Euclidean balls $B((x, y), r)$, $r > 0$, form a base at (x, y) for the topology of $\mathbb{R}^2$, and for each $r > 0$,

$$B((x, y), r) \cap (S \times \mathbb{R}) \neq \emptyset \iff \text{there is } u \in S \text{ with } |u - x| < r.$$

For if (u, v) belongs to the left-hand intersection then $u \in S$ and $|u - x| < r$; conversely if $u \in S$ and $|u - x| < r$ then the point (u, y) lies in both $B((x, y), r)$ and $S \times \mathbb{R}$. Hence (x, y) is a point of closure of $S \times \mathbb{R}$ if and only if every $r > 0$ admits such a u , that is, if and only if x is a point of closure of S in $\mathbb{R}$. Thus $\overline{S \times \mathbb{R}} = \overline{S} \times \mathbb{R}$.

By induction on the length of a word w in c and k , these three properties give $w(E) = w(A) \times \mathbb{R} = \Phi(w(A))$ for every such w . Since the fourteen sets $w(A)$ are distinct and Φ is injective, the fourteen sets $w(E)$ are distinct. Thus $E = A \times \mathbb{R}$ is a subset of $\mathbb{R}^2$ from which exactly 14 different sets are obtained by repeated complementation and closure.

Problem 11.12. Show that if F is a closed subset of a normal space X , then the subspace F is normal. Is it necessary to assume that F is closed?

Solution. Recall that the subspace F of X carries the topology whose open sets are the sets $\mathcal{O} \cap F$ with $\mathcal{O}$ open in X , and that a space is normal provided it is Tychonoff and each two disjoint closed subsets can be separated by disjoint neighborhoods.

Step 1: F is Tychonoff. By Proposition 6, X is Tychonoff if and only if each singleton set is closed. Let $x \in F$. Since X is normal, hence Tychonoff, $\{x\}$ is closed in X , so $X \sim \{x\}$ is open in X (Proposition 4) and therefore $(X \sim \{x\}) \cap F = F \sim \{x\}$ is open in the subspace F . Thus $\{x\}$ is closed in F . By Proposition 6 applied to the space F , the subspace F is Tychonoff.

Step 2: closed subsets of the subspace F are closed in X . Let A be closed in the subspace F . Then $F \sim A$ is open in F , so $F \sim A = \mathcal{O} \cap F$ for some open $\mathcal{O} \subseteq X$. Hence

$$A = F \sim (\mathcal{O} \cap F) = F \cap (X \sim \mathcal{O}).$$

Both F and $X \sim \mathcal{O}$ are closed in X (the latter by Proposition 4, since $\mathcal{O}$ is open), and by Proposition 5 the intersection of closed sets is closed. Therefore A is closed in X . It is here, and only here, that the hypothesis that F is closed in X is used.

Step 3: separation. Let A and B be disjoint closed subsets of the subspace F . By Step 2 they are disjoint closed subsets of X . Since X is normal, there are disjoint open subsets $\mathcal{O}_1, \mathcal{O}_2$ of X with $A \subseteq \mathcal{O}_1$ and $B \subseteq \mathcal{O}_2$. Then $\mathcal{O}_1 \cap F$ and $\mathcal{O}_2 \cap F$ are open in the subspace F , they contain A and B respectively (as $A, B \subseteq F$), and

$$(\mathcal{O}_1 \cap F) \cap (\mathcal{O}_2 \cap F) \subseteq \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset.$$

So A and B are separated by disjoint neighborhoods in F . Together with Step 1, the subspace F is normal.

Is closedness necessary? Yes: normality is not hereditary, so the hypothesis cannot simply be deleted. Note first that no cheap counterexample is possible. If Y is any subspace of a normal space X , then Y is Tychonoff (the argument of Step 1 used nothing about F) and in fact regular: if C is closed in Y and $y \in Y \sim C$, write $C = C' \cap Y$ with C' closed in X ; then $y \notin C'$, and since a normal space is regular, there are disjoint open $\mathcal{O}_1 \ni y$ and $\mathcal{O}_2 \supseteq C'$ in X , whence $\mathcal{O}_1 \cap Y$ and $\mathcal{O}_2 \cap Y$ are disjoint neighborhoods of y and C in Y . So a counterexample must be a regular space that fails to be normal, and such spaces are necessarily somewhat elaborate.

The standard example is the **deleted Tychonoff plank**; producing it requires machinery from later in the book, which we cite explicitly. Let ω_1 denote the first uncountable ordinal and ω the first infinite ordinal, and give the ordinal intervals $[0, \omega_1]$ and $[0, \omega]$ their order topologies, that is, the topologies for which the rays $[0, \delta)$ and $(\gamma, \lambda]$ form a subbase, λ denoting the largest element of the interval in question. Each of these two spaces is Hausdorff: two distinct ordinals $\gamma < \delta$ of $[0, \lambda]$ are separated by the disjoint open sets $[0, \gamma + 1)$ and $(\gamma, \lambda]$. Each is also compact: if $\mathcal{U}$ is an open cover of a closed ordinal interval $[0, \lambda]$ and the set of $\mu \leq \lambda$ for which $[0, \mu]$ admits a finite subcover failed to contain λ , its complement in $[0, \lambda]$ would have a least element μ_0 ; a member U of $\mathcal{U}$ containing μ_0 contains a basic neighborhood of μ_0 , and every basic neighborhood of μ_0 , being a finite intersection of the subbasic rays, contains either $[0, \mu_0]$ or a set $(\gamma, \mu_0]$ with $\gamma < \mu_0$. In the first case U alone covers $[0, \mu_0]$; in the second, $\gamma < \mu_0$, so by the minimality of μ_0 the set $[0, \gamma]$ has a finite subcover, and adjoining U to it covers $[0, \gamma] \cup (\gamma, \mu_0] = [0, \mu_0]$. Either way we contradict the choice of μ_0 . Hence, by the Tychonoff Product Theorem of Section 12.2, the product

$$X = [0, \omega_1] \times [0, \omega]$$

with the product topology is compact, and it is Hausdorff, since two distinct points of a product differ in a coordinate and may be separated there. By Theorem 18 of Chapter 11, a compact Hausdorff space is normal, so X is normal. Let

$$T = X \sim \{(\omega_1, \omega)\},$$

an open, hence non-closed, subspace of X . Put

$$A = \{\omega_1\} \times [0, \omega), \quad B = [0, \omega_1] \times \{\omega\}.$$

These are disjoint, and each is closed in T , since $A = (\{\omega_1\} \times [0, \omega]) \cap T$ and $B = ([0, \omega_1] \times \{\omega\}) \cap T$ are intersections with T of closed subsets of X .

We claim A and B cannot be separated by disjoint neighborhoods in T ; hence T is not normal. Suppose U and V are open in T with $A \subseteq U$ and $B \subseteq V$. For each $n < \omega$ the point (ω_1, n) lies in U ; since $\{n\}$ is open in $[0, \omega]$ (each $n < \omega$ is an isolated point) and the intervals $(\alpha, \omega_1]$ form a base at ω_1 , there is an ordinal $\alpha_n < \omega_1$ with

$$(\alpha_n, \omega_1] \times \{n\} \subseteq U.$$

Let $\alpha = \sup_{n < \omega} \alpha_n$. A countable supremum of countable ordinals is again a countable ordinal, so $\alpha < \omega_1$. Choose any β with $\alpha < \beta < \omega_1$. Then $(\beta, n) \in U$ for every $n < \omega$.

On the other hand $(\beta, \omega) \in B \subseteq V$, and the sets $(\gamma, \beta] \times (m, \omega]$ with $\gamma < \beta$ and $m < \omega$ form a base at (β, ω) in the product topology; hence there are such γ and m with $(\gamma, \beta] \times (m, \omega] \subseteq V$. In particular $(\beta, m + 1) \in V$. But $(\beta, m + 1) \in U$ as well. So $U \cap V \neq \emptyset$, and no such disjoint pair exists.

Thus T is a non-normal subspace of the normal space X , and the hypothesis that F be closed cannot be dropped. (Of course closedness is not **necessary** for a particular subspace to be normal: every open subspace of $\mathbb{R}$, for instance, is metrizable and hence normal by Proposition 7. The point is that closedness cannot be omitted from the general theorem.)

Problem 11.13. Let X be a topological space. Show that X is Hausdorff if and only if the diagonal $D = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ is a closed subset of $X \times X$.

Solution. Give $X \times X$ the product topology, for which the collection

$$\mathcal{B} = \{\mathcal{O}_1 \times \mathcal{O}_2 : \mathcal{O}_1, \mathcal{O}_2 \text{ open in } X\}$$

is a base (the Example following Proposition 2). We use repeatedly the elementary equivalence

$$(\mathcal{O}_1 \times \mathcal{O}_2) \cap D = \emptyset \iff \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset,$$

which holds because $(\mathcal{O}_1 \times \mathcal{O}_2) \cap D = \{(z, z) : z \in \mathcal{O}_1 \cap \mathcal{O}_2\}$.

Suppose X is Hausdorff. By Proposition 4 it suffices to show $(X \times X) \sim D$ is open. Let (x_1, x_2) belong to $(X \times X) \sim D$, so that $x_1 \neq x_2$. By the Hausdorff separation property there are disjoint

neighborhoods $\mathcal{O}_1$ of x_1 and $\mathcal{O}_2$ of x_2 . Then $\mathcal{O}_1 \times \mathcal{O}_2$ is open in $X \times X$, contains (x_1, x_2) , and, since $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, the displayed equivalence gives $(\mathcal{O}_1 \times \mathcal{O}_2) \cap D = \emptyset$, that is,

$$(x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \subseteq (X \times X) \sim D.$$

Thus every point of $(X \times X) \sim D$ has an open neighborhood contained in $(X \times X) \sim D$, so $(X \times X) \sim D$ is the union of a collection of open sets and hence is open. By Proposition 4, D is closed.

Suppose D is closed. Let x_1 and x_2 be two distinct points of X . Then $(x_1, x_2) \notin D$, and by Proposition 4 the set $(X \times X) \sim D$ is open. Since $\mathcal{B}$ is a base for the product topology, there is a basic open set $\mathcal{O}_1 \times \mathcal{O}_2$ with

$$(x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \subseteq (X \times X) \sim D.$$

Thus $\mathcal{O}_1$ is a neighborhood of x_1 , $\mathcal{O}_2$ is a neighborhood of x_2 , and $(\mathcal{O}_1 \times \mathcal{O}_2) \cap D = \emptyset$; by the displayed equivalence, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. Hence x_1 and x_2 are separated by disjoint neighborhoods. As $x_1 \neq x_2$ were arbitrary, X is Hausdorff.

Problem 11.14. Consider the set of real numbers with the topology consisting of the empty-set and sets of the form $(-\infty, c)$, $c \in \mathbb{R}$. Show that this space is Tychonoff but not Hausdorff.

Solution. Write $X = \mathbb{R}$ and

$$\mathcal{T} = \{\emptyset\} \cup \{(-\infty, c) : c \in \mathbb{R}\} \cup \{\mathbb{R}\}.$$

The whole set $\mathbb{R}$ must be adjoined, since $\mathbb{R} = \bigcup_{c \in \mathbb{R}} (-\infty, c)$ and a topology contains the unions of its members; with this understanding (one may think of $\mathbb{R}$ as the case $c = \infty$) the collection is indeed a topology:

- $\emptyset$ and $\mathbb{R}$ belong to $\mathcal{T}$.
- **Unions.** Let $\{(-\infty, c_\lambda)\}_{\lambda \in \Lambda}$ be a nonempty subfamily and put $s = \sup_\lambda c_\lambda \in (-\infty, \infty]$. If $s = \infty$, the union is $\mathbb{R}$; otherwise the union is $(-\infty, s)$, since $x < c_\lambda$ for some λ exactly when $x < s$. Adjoining $\emptyset$ or $\mathbb{R}$ to the family changes nothing. So $\mathcal{T}$ is closed under arbitrary unions.
- **Finite intersections.** $(-\infty, c_1) \cap (-\infty, c_2) = (-\infty, \min\{c_1, c_2\})$, and intersecting with $\emptyset$ or $\mathbb{R}$ again stays in $\mathcal{T}$.

The space is not Hausdorff. Any two nonempty members of $\mathcal{T}$ intersect: if $\mathcal{O}_1, \mathcal{O}_2$ are nonempty and open, each contains a half-line $(-\infty, c_i)$ with $c_i \in \mathbb{R}$ (taking c_i arbitrary if $\mathcal{O}_i = \mathbb{R}$), so $\mathcal{O}_1 \cap \mathcal{O}_2 \supseteq (-\infty, \min\{c_1, c_2\}) \neq \emptyset$. Hence no two points possess disjoint neighborhoods, and since X has more than one point, the Hausdorff separation property fails badly. (In particular, by Problem 13 the diagonal is not closed in $X \times X$.)

The Tychonoff separation property, as defined in this section, also fails; the exercise is misstated. Royden's Tychonoff separation property requires, for each two points u and v , a neighborhood of u not containing v **and** a neighborhood of v not containing u . Take $u < v$. Every neighborhood of v is either $\mathbb{R}$ or a set $(-\infty, c)$ with $c > v$, and in either case it contains u , since $u < v < c$. So there is no neighborhood of v that excludes u , and the Tychonoff separation property fails.

Equivalently, one may argue through Proposition 6, which says that X is Tychonoff if and only if every singleton is closed. Here no singleton is closed: for $x \in \mathbb{R}$ the set $\mathbb{R} \sim \{x\}$ is neither $\emptyset$, nor $\mathbb{R}$, nor a half-line $(-\infty, c)$ (it contains points $> x$ but not x), hence is not open, so $\{x\}$ is not closed by Proposition 4. In fact one computes the closures exactly:

$$\overline{\{x\}} = [x, \infty).$$

Indeed, if $y \geq x$ then every neighborhood of y is $\mathbb{R}$ or $(-\infty, c)$ with $c > y \geq x$, and so contains x ; thus y is a point of closure of $\{x\}$. If $y < x$, choose c with $y < c \leq x$; then $(-\infty, c)$ is a neighborhood of y missing x , so y is not a point of closure.

The conclusion is not an artifact of Royden's particular use of the word. Under the other common use, in which "Tychonoff" means completely regular, the space still fails: the only continuous real-valued functions on X are the constants. Indeed, suppose $f: X \rightarrow \mathbb{R}$ is continuous and $f(x_1) < f(x_2)$ for some x_1, x_2 ; choose t with $f(x_1) < t < f(x_2)$. The set $f^{-1}((t, \infty))$ is open, contains x_2 , and omits x_1 ,

so it is neither $\emptyset$ nor $\mathbb{R}$ and hence equals some $(-\infty, c)$ with $x_2 < c \leq x_1$, giving $x_2 < x_1$. The set $f^{-1}((-\infty, t))$ is open, contains x_1 , and omits x_2 , so by the same reasoning $x_1 < x_2$. This contradiction shows f is constant. Since the closed sets are $\emptyset$, $\mathbb{R}$ and the half-lines $[c, \infty)$, no constant function can separate the point y from the closed set $[c, \infty)$ when $y < c$.

What is true. The space does satisfy the weaker (Kolmogorov) separation condition: for each two distinct points $u < v$, there is an open set containing one of them and not the other, namely $(-\infty, v)$, which contains u and not v . This is presumably the intended content of the exercise: the space distinguishes points by open sets, yet it is not Hausdorff. Indeed, by the two paragraphs above, it is not even Tychonoff in the sense of this section. Note that this is consistent with the chain of inclusions

$$\mathcal{T}_{\text{metric}} \subseteq \mathcal{T}_{\text{normal}} \subseteq \mathcal{T}_{\text{regular}} \subseteq \mathcal{T}_{\text{Hausdorff}} \subseteq \mathcal{T}_{\text{Tychonoff}}$$

displayed in this section: a space that fails to be Tychonoff must fail to be Hausdorff, so the present space cannot serve as an example of a Tychonoff space that is not Hausdorff. For a genuine such example see Problem 15, the Zariski topology, in which singletons **are** closed while any two nonempty open sets meet.

11.3 Problems 11.15–11.21

Problem 11.15. (Zariski Topology) In $\mathbb{R}^n$ let $\mathcal{B}$ be the family of sets $\{x \in \mathbb{R}^n \mid p(x) \neq 0\}$, where p is a polynomial in n variables. Let $\mathcal{T}$ be the topology on X that has $\mathcal{B}$ as a subbase. Show that $\mathcal{T}$ is a topology for $\mathbb{R}^n$ that is Tychonoff but not Hausdorff.

Solution. Throughout, $n \geq 1$, and for a polynomial p in n real variables write

$$\mathcal{O}_p = \{x \in \mathbb{R}^n \mid p(x) \neq 0\},$$

so that $\mathcal{B} = \{\mathcal{O}_p \mid p \text{ a polynomial}\}$.

Step 1: $\mathcal{B}$ is in fact a base, and $\mathcal{T}$ is the collection of unions of members of $\mathcal{B}$.

By Proposition 2, a collection $\mathcal{B}$ of subsets of a set X is a base for a topology on X provided $\mathcal{B}$ covers X and, for B_1, B_2 in $\mathcal{B}$ and $x \in B_1 \cap B_2$, there is $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$. Both hold here. For the covering property, take p to be the constant polynomial 1; then $\mathcal{O}_p = \mathbb{R}^n$. For the intersection property, observe that for polynomials p, q ,

$$\mathcal{O}_p \cap \mathcal{O}_q = \{x \mid p(x) \neq 0 \text{ and } q(x) \neq 0\} = \{x \mid p(x)q(x) \neq 0\} = \mathcal{O}_{pq},$$

since a product of two real numbers is nonzero if and only if both factors are nonzero, and pq is again a polynomial. Thus $\mathcal{B}$ is closed under finite intersections, and one may take $B = B_1 \cap B_2$.

Consequently, by the remark following the definition of subbase in Section 11.1, the collection $\mathcal{B}$, which covers $\mathbb{R}^n$, is a subbase for a unique topology $\mathcal{T}$, obtained by first forming intersections of finite subcollections of $\mathcal{B}$ — which by the displayed identity gives back $\mathcal{B}$ itself — and then taking $\emptyset$ together with all unions of subcollections. So

$$\mathcal{T} = \left\{ \bigcup_{\lambda \in \Lambda} \mathcal{O}_{p_\lambda} \mid \{p_\lambda\}_{\lambda \in \Lambda} \text{ a family of polynomials} \right\}.$$

That this is a topology is immediate. The empty set is the union of the empty family, and also $\emptyset = \mathcal{O}_0$; the whole space is $\mathbb{R}^n = \mathcal{O}_1$; a union of sets of the displayed form is again of that form; and the intersection of two of them is

$$\left(\bigcup_{\lambda} \mathcal{O}_{p_\lambda} \right) \cap \left(\bigcup_{\mu} \mathcal{O}_{q_\mu} \right) = \bigcup_{\lambda, \mu} (\mathcal{O}_{p_\lambda} \cap \mathcal{O}_{q_\mu}) = \bigcup_{\lambda, \mu} \mathcal{O}_{p_\lambda q_\mu} \in \mathcal{T}.$$

Hence $\mathcal{T}$ is a topology for $\mathbb{R}^n$. Equivalently, the closed sets of $\mathcal{T}$ are exactly the intersections of zero sets of polynomials.

Step 2: a polynomial that vanishes at every point of $\mathbb{R}^n$ is the zero polynomial.

Argue by induction on n . For $n = 1$ a nonzero polynomial of degree d has at most d real roots, while $\mathbb{R}$ is infinite. Assume the claim for $n - 1$ and write

$$p(x_1, \dots, x_n) = \sum_{k=0}^d c_k(x_1, \dots, x_{n-1}) x_n^k$$

with each c_k a polynomial in $n - 1$ variables. Fix $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Then $t \mapsto p(x_1, \dots, x_{n-1}, t)$ is a one-variable polynomial vanishing for all real t , so each $c_k(x_1, \dots, x_{n-1}) = 0$ by the case $n = 1$. Since $(x_1, \dots, x_{n-1})$ was arbitrary, each c_k vanishes identically on $\mathbb{R}^{n-1}$, hence each c_k is the zero polynomial by the inductive hypothesis, and therefore so is p . In particular, if p is not the zero polynomial then $\mathcal{O}_p \neq \emptyset$.

Step 3: if p and q are nonzero polynomials, then $\mathcal{O}_{pq} \neq \emptyset$.

By Step 2 choose $a, b \in \mathbb{R}^n$ with $p(a) \neq 0$ and $q(b) \neq 0$. If $p(b) \neq 0$ we are done, since then $(pq)(b) \neq 0$. So assume $p(b) = 0$; since $p(a) \neq 0$ this forces $a \neq b$, and we may restrict attention to the line through a and b , setting

$$s(t) = p(a + t(b - a)), \quad r(t) = q(a + t(b - a)).$$

These are polynomials in the single real variable t , and $s(0) = p(a) \neq 0$, $r(1) = q(b) \neq 0$, so neither is the zero polynomial. Hence sr is a nonzero one-variable polynomial (the degree of a product of nonzero one-variable real polynomials is the sum of the degrees, because the product of the leading coefficients is nonzero), so sr has only finitely many roots and there is t_0 with $s(t_0)r(t_0) \neq 0$. The point $x_0 = a + t_0(b - a)$ then satisfies $(pq)(x_0) \neq 0$.

Step 4: $(\mathbb{R}^n, \mathcal{T})$ is Tychonoff.

By Proposition 6 it suffices to show each singleton is closed. Let $x_0 = (a_1, \dots, a_n)$. For $1 \leq i \leq n$ let $p_i(x) = x_i - a_i$. A point x fails to equal x_0 exactly when $x_i \neq a_i$ for some i , so

$$\mathbb{R}^n \sim \{x_0\} = \bigcup_{i=1}^n \mathcal{O}_{p_i},$$

a union of members of $\mathcal{B}$ and hence open. Therefore $\{x_0\}$ is closed and, by Proposition 6, $\mathcal{T}$ has the Tychonoff separation property. (Concretely, given $u \neq v$, the set $\mathbb{R}^n \sim \{v\}$ is a neighborhood of u not containing v , and symmetrically.)

Step 5: $(\mathbb{R}^n, \mathcal{T})$ is not Hausdorff.

We show more: any two nonempty open sets meet. Let $\mathcal{U}$ and $\mathcal{V}$ be nonempty members of $\mathcal{T}$, say $u \in \mathcal{U}$ and $v \in \mathcal{V}$. Since $\mathcal{B}$ is a base there are polynomials p and q with

$$u \in \mathcal{O}_p \subseteq \mathcal{U}, \quad v \in \mathcal{O}_q \subseteq \mathcal{V}.$$

Since $p(u) \neq 0$ and $q(v) \neq 0$, neither p nor q is the zero polynomial, so by Step 3

$$\emptyset \neq \mathcal{O}_{pq} = \mathcal{O}_p \cap \mathcal{O}_q \subseteq \mathcal{U} \cap \mathcal{V}.$$

Now take any two distinct points $u \neq v$ of $\mathbb{R}^n$, which is possible since $n \geq 1$. Every neighborhood of u and every neighborhood of v is a nonempty open set, so any two such neighborhoods meet. Hence u and v cannot be separated by disjoint neighborhoods, and $\mathcal{T}$ is not Hausdorff.

Thus the Zariski topology on $\mathbb{R}^n$ is Tychonoff but not Hausdorff. In particular it witnesses that the inclusion of the Hausdorff topologies in the Tychonoff topologies, in the string of inclusions displayed in Section 11.2, is strict.

Problem 11.16. Show the Sorgenfrey Line and the Moore Plane are Hausdorff (see Problems 9 and 10).

Solution. Recall the two spaces. The Sorgenfrey Line is $\mathbb{R}$ with the topology having as a base the collection of intervals $[a, b)$ with $a < b$ (Problem 9). The Moore Plane is the closed upper half plane

$\mathbb{R}^{2,+} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ with the topology having as a base the following sets (Problem 10): for a point (x, y) with $y > 0$, the ordinary Euclidean open balls $B((x, y), r)$ centered at (x, y) and contained in $\mathbb{R}^{2,+}$, which are exactly those with $0 < r \leq y$: for such r every point of the ball has second coordinate greater than $y - r \geq 0$, so the ball even lies in the open upper half plane, while if $r > y$ one may choose s with $y < s < r$ and then $(x, y - s)$ belongs to the ball but has negative second coordinate. And for a point $(x, 0)$ on the horizontal axis, the sets

$$N_r(x) = \{(x, 0)\} \cup B((x, r), r), \quad r > 0,$$

that is, the point itself together with the open Euclidean disc of radius r lying in the upper half plane and tangent to the real line at $(x, 0)$. Here $B(c, r)$ denotes the open Euclidean ball of $\mathbb{R}^2$ with center c and radius r , and $|\cdot|$ denotes the Euclidean norm.

The Sorgenfrey Line.

Let u and v be distinct real numbers; without loss of generality $u < v$. Consider the basic open sets

$$\mathcal{U} = [u, v), \quad \mathcal{V} = [v, v + 1).$$

Then $u \in \mathcal{U}$ and $v \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$ since every point of $\mathcal{U}$ is strictly less than v while every point of $\mathcal{V}$ is at least v . Thus u and v are separated by disjoint neighborhoods, and the Sorgenfrey Line is Hausdorff.

The Moore Plane.

Let $p \neq q$ be points of $\mathbb{R}^{2,+}$ and put $d = |p - q| > 0$. We first record an estimate for the tangent discs: if $z \in N_r(x)$ then

$$|z - (x, 0)| \leq |z - (x, r)| + |(x, r) - (x, 0)| < r + r = 2r$$

when $z \in B((x, r), r)$, and $|z - (x, 0)| = 0$ when $z = (x, 0)$. Hence

$$N_r(x) \subseteq B((x, 0), 2r) \quad \text{for every } r > 0. \quad (*)$$

Also note that if $A \subseteq B(p, \rho)$ and $C \subseteq B(q, \sigma)$ with $\rho + \sigma \leq d$, then $A \cap C = \emptyset$: a common point z would give $d = |p - q| \leq |p - z| + |z - q| < \rho + \sigma \leq d$, a contradiction. We use this repeatedly with $\rho = \sigma = d/2$.

Case 1: both points lie off the axis, say $p = (a, s)$ and $q = (b, t)$ with $s > 0$ and $t > 0$. Put

$$\rho = \min\{d/2, s\}, \quad \sigma = \min\{d/2, t\}.$$

Then $B(p, \rho)$ and $B(q, \sigma)$ are basic open sets of the Moore Plane containing p and q respectively (their radii do not exceed the second coordinates of their centers, so they lie in the open upper half plane), and since $\rho + \sigma \leq d/2 + d/2 = d$ they are disjoint.

Case 2: $p = (a, 0)$ lies on the axis and $q = (b, t)$ with $t > 0$. Choose $r = d/4$ and $\sigma = \min\{d/2, t\}$. By $(*)$, $N_r(a) \subseteq B(p, 2r) = B(p, d/2)$, while $B(q, \sigma) \subseteq B(q, d/2)$. Since $d/2 + d/2 = d$, the observation above gives

$$N_r(a) \cap B(q, \sigma) = \emptyset.$$

These are basic open sets containing p and q respectively.

Case 3: both points lie on the axis, say $p = (a, 0)$ and $q = (b, 0)$ with $a \neq b$, so $d = |a - b|$. Put $r = d/2$ and consider the tangent discs $N_r(a)$ and $N_r(b)$. Their discs $B((a, r), r)$ and $B((b, r), r)$ have centers at Euclidean distance $|a - b| = d = r + r$, so they are disjoint: a common point z would give $2r = |(a, r) - (b, r)| \leq |(a, r) - z| + |z - (b, r)| < r + r$, a contradiction. Moreover $(a, 0) \notin B((b, r), r)$, because $|(a, 0) - (b, r)| = \sqrt{d^2 + r^2} > r$; symmetrically $(b, 0) \notin B((a, r), r)$; and $(a, 0) \neq (b, 0)$. Assembling,

$$N_r(a) \cap N_r(b) = \emptyset,$$

and these are basic open sets containing p and q respectively.

In every case p and q have disjoint neighborhoods, so the Moore Plane is Hausdorff.

Problem 11.17. A topological space is said to be a Lindelof space or to have the Lindelof property provided each open cover of X has a countable subcover. Show that if X is second countable, then it is Lindelof.

Solution. Let X be second countable, and let $\{\mathcal{B}_n\}_{n=1}^\infty$ be a countable base for the topology of X . Let $\mathcal{F}$ be an open cover of X ; we must extract a countable subcover.

Let

$$N = \{n \in \mathbb{N} \mid \text{there is some } \mathcal{U} \in \mathcal{F} \text{ with } \mathcal{B}_n \subseteq \mathcal{U}\}.$$

For each $n \in N$ use the countable axiom of choice to select one member $\mathcal{U}_n \in \mathcal{F}$ with $\mathcal{B}_n \subseteq \mathcal{U}_n$. The collection

$$\mathcal{F}' = \{\mathcal{U}_n \mid n \in N\}$$

is a subcollection of $\mathcal{F}$ indexed by a subset of $\mathbb{N}$, hence is countable. We claim it covers X .

Indeed, let $x \in X$. Since $\mathcal{F}$ is a cover of X there is $\mathcal{U} \in \mathcal{F}$ with $x \in \mathcal{U}$. Since $\mathcal{U}$ is open and $\{\mathcal{B}_n\}$ is a base for the topology, by the definition of a base there is an index n with

$$x \in \mathcal{B}_n \subseteq \mathcal{U}.$$

The existence of this $\mathcal{U}$ witnesses that $n \in N$, so $\mathcal{U}_n$ is defined and $\mathcal{B}_n \subseteq \mathcal{U}_n$. Therefore

$$x \in \mathcal{B}_n \subseteq \mathcal{U}_n \in \mathcal{F}'.$$

Since $x \in X$ was arbitrary, $\mathcal{F}'$ is a countable subcover of $\mathcal{F}$. Hence X is a Lindelof space.

(The same argument shows a little more: from any base $\mathcal{B}$ for the topology one gets a subcover indexed by a subset of $\mathcal{B}$, so a space with a base of cardinality κ has the property that every open cover admits a subcover of cardinality at most κ .)

Problem 11.18. Let X be an uncountable set of points, and let $\mathcal{T}$ consist of $\emptyset$ and all subsets of X that have finite complements. Show that $\mathcal{T}$ is a topology for X and that the space $(X, \mathcal{T})$ is not first countable.

Solution. This $\mathcal{T}$ is the so-called finite complement, or cofinite, topology.

Part 1: $\mathcal{T}$ is a topology.

By definition $\emptyset \in \mathcal{T}$. Also $X \in \mathcal{T}$, since $X \sim X = \emptyset$ is finite.

Closure under arbitrary unions. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be a family of members of $\mathcal{T}$ and put $\mathcal{O} = \bigcup_\lambda \mathcal{O}_\lambda$. If every $\mathcal{O}_\lambda$ is empty (in particular if $\Lambda = \emptyset$), then $\mathcal{O} = \emptyset \in \mathcal{T}$. Otherwise fix λ_0 with $\mathcal{O}_{\lambda_0} \neq \emptyset$, so that $X \sim \mathcal{O}_{\lambda_0}$ is finite. By De Morgan's identities,

$$X \sim \mathcal{O} = \bigcap_{\lambda \in \Lambda} (X \sim \mathcal{O}_\lambda) \subseteq X \sim \mathcal{O}_{\lambda_0},$$

and a subset of a finite set is finite. Hence $X \sim \mathcal{O}$ is finite and $\mathcal{O} \in \mathcal{T}$.

Closure under finite intersections. It suffices to treat two sets $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$. If either is empty then $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \in \mathcal{T}$. Otherwise both complements are finite and

$$X \sim (\mathcal{O}_1 \cap \mathcal{O}_2) = (X \sim \mathcal{O}_1) \cup (X \sim \mathcal{O}_2)$$

is a union of two finite sets, hence finite; so $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}$.

Therefore $\mathcal{T}$ is a topology for X .

Part 2: $(X, \mathcal{T})$ is not first countable.

We show that no point of X has a countable base of neighborhoods; in particular X is not first countable.

Fix $x \in X$ and suppose, to get a contradiction, that $\{\mathcal{B}_n\}_{n=1}^\infty$ is a countable base at x : each $\mathcal{B}_n$ is an open set containing x , and every open set containing x contains some $\mathcal{B}_n$.

Since $x \in \mathcal{B}_n$, each $\mathcal{B}_n$ is nonempty, so by the definition of $\mathcal{T}$ the set $F_n = X \sim \mathcal{B}_n$ is finite. Put

$$F = \bigcup_{n=1}^{\infty} F_n.$$

A countable union of finite sets is countable (Chapter 1, Corollary 6), so F is countable, and hence so is $F \cup \{x\}$. Since X is uncountable, $X \sim (F \cup \{x\})$ is nonempty; choose a point y in it. Thus $y \neq x$ and $y \notin F_n$, that is $y \in \mathcal{B}_n$, for every n .

Now consider $\mathcal{O} = X \sim \{y\}$. Its complement $\{y\}$ is finite, so $\mathcal{O} \in \mathcal{T}$, and $x \in \mathcal{O}$ because $y \neq x$. Since $\{\mathcal{B}_n\}$ is a base at x , there must be an index n with

$$x \in \mathcal{B}_n \subseteq \mathcal{O} = X \sim \{y\},$$

which forces $y \notin \mathcal{B}_n$. This contradicts the fact that y belongs to every $\mathcal{B}_n$.

Hence there is no countable base at x , and $(X, \mathcal{T})$ is not first countable at any of its points. Since second countable spaces are first countable, $(X, \mathcal{T})$ is not second countable either. Observe also that $(X, \mathcal{T})$ is Tychonoff: each singleton $\{z\}$ is closed because its complement $X \sim \{z\}$ has the finite complement $\{z\}$ and so is open, and Proposition 6 applies. It is not Hausdorff, since any two nonempty open sets meet: their complements are finite while X is infinite, so the complement of their intersection is finite and the intersection cannot be empty.

Problem 11.19. Show that a second countable space is separable and every subspace of a second countable space is second countable.

Solution. Let X be a second countable topological space and let $\{\mathcal{B}_n\}_{n \in N}$, with $N \subseteq \mathbb{N}$, be a countable base for its topology. Discarding those $\mathcal{B}_n$ that are empty changes nothing (a base remains a base after deleting the empty set, since the empty set contains no point and so is never needed to exhibit a point inside an open set), so we may and do assume every $\mathcal{B}_n$ is nonempty.

Part 1: X is separable.

For each $n \in N$ use the countable axiom of choice to select a point $x_n \in \mathcal{B}_n$, and set

$$E = \{x_n \mid n \in N\}.$$

Then E is countable, being the image of the countable set N . We claim E is dense in X , that is, every nonempty open subset of X contains a point of E .

Let $\mathcal{O}$ be a nonempty open subset of X and pick $x \in \mathcal{O}$. Since $\{\mathcal{B}_n\}$ is a base, there is $n \in N$ with $x \in \mathcal{B}_n \subseteq \mathcal{O}$. Then $x_n \in \mathcal{B}_n \subseteq \mathcal{O}$, so $\mathcal{O}$ contains the point x_n of E . Hence E is dense and X is separable.

(Equivalently, in the language of the text: every point of X is a point of closure of E , so $\overline{E} = X$. Indeed if $x \in X$ and $\mathcal{U}$ is any neighborhood of x , the argument just given produces a point of E in $\mathcal{U}$.)

Part 2: every subspace of X is second countable.

Let $Y \subseteq X$ carry the subspace topology, whose open sets are exactly the sets $\mathcal{O} \cap Y$ with $\mathcal{O}$ open in X . Define

$$\mathcal{B}'_n = \mathcal{B}_n \cap Y \quad (n \in N).$$

Each $\mathcal{B}'_n$ is open in Y , and $\{\mathcal{B}'_n\}_{n \in N}$ is a countable collection. We claim it is a base for the subspace topology of Y .

Let $\mathcal{V}$ be open in Y and let $y \in \mathcal{V}$. Write $\mathcal{V} = \mathcal{O} \cap Y$ with $\mathcal{O}$ open in X . Then $y \in \mathcal{O}$, so since $\{\mathcal{B}_n\}$ is a base for X there is $n \in N$ with $y \in \mathcal{B}_n \subseteq \mathcal{O}$. Intersecting with Y and using $y \in Y$,

$$y \in \mathcal{B}_n \cap Y = \mathcal{B}'_n \subseteq \mathcal{O} \cap Y = \mathcal{V}.$$

Thus every open subset of Y is a union of members of $\{\mathcal{B}'_n\}$, so this countable collection is a base for the topology of Y , and Y is second countable.

Remark. Separability, unlike second countability, is not inherited by subspaces: Problem 20 exhibits the Moore Plane, which is separable, together with its subspace $\mathbb{R} \times \{0\}$, which is not. Combined with Part 2 this gives another proof that the Moore Plane is not second countable.

Problem 11.20. Show that the Moore Plane is separable (see Problem 10). Show that the subspace $\mathbb{R} \times \{0\}$ of the Moore Plane is not separable. Conclude that the Moore Plane is not metrizable and not second countable.

Solution. Recall from Problem 10 that the Moore Plane is $\mathbb{R}^{2,+} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ with the base consisting of: the Euclidean open balls $B((x, y), r)$ with $y > 0$ and $0 < r \leq y$ (these are precisely the Euclidean balls centered at (x, y) and contained in $\mathbb{R}^{2,+}$, and each of them in fact lies in the open upper half plane); and, for each $x \in \mathbb{R}$ and $r > 0$, the tangent-disc sets

$$N_r(x) = \{(x, 0)\} \cup B((x, r), r).$$

Here $B(c, \rho)$ is the ordinary Euclidean open disc.

Part 1: the Moore Plane is separable.

Let

$$E = \{(p, q) \mid p, q \in \mathbb{Q}, q > 0\},$$

the set of points with rational coordinates lying strictly above the axis. Being in one-to-one correspondence with a subset of $\mathbb{Q} \times \mathbb{Q}$, the set E is countable.

We check E is dense, that is, meets every nonempty open set; since every nonempty open set contains a basic open set, it suffices to check the two kinds of basic sets. Note first that if $\mathcal{D}$ is any nonempty open subset of $\mathbb{R}^2$ (in the Euclidean topology) contained in the open upper half plane, then $\mathcal{D}$ contains a point of E : $\mathcal{D}$ contains a Euclidean ball, and $\mathbb{Q} \times \mathbb{Q}$ is dense in the Euclidean plane, so $\mathcal{D}$ contains a point (p, q) with p, q rational, and $q > 0$ because $\mathcal{D}$ lies in the open upper half plane.

Now, a basic set $B((x, y), r)$ with $y > 0$ and $r \leq y$ is a nonempty Euclidean open set contained in the open upper half plane, so it meets E . A basic set $N_r(x)$ contains the Euclidean open disc $B((x, r), r)$, which is nonempty and lies in the open upper half plane (if $|z - (x, r)| < r$ and $z = (z_1, z_2)$ then $|z_2 - r| < r$, so $z_2 > 0$); hence $N_r(x)$ meets E as well.

So every nonempty open subset of the Moore Plane contains a point of E ; equivalently every point of $\mathbb{R}^{2,+}$ is a point of closure of E , so $\overline{E} = \mathbb{R}^{2,+}$. Thus the Moore Plane is separable.

Part 2: the subspace $L = \mathbb{R} \times \{0\}$ is not separable.

We claim the subspace topology that L inherits from the Moore Plane is the discrete topology. Fix $a \in \mathbb{R}$ and consider the basic open set $N_1(a) = \{(a, 0)\} \cup B((a, 1), 1)$. As noted above, every point of $B((a, 1), 1)$ has strictly positive second coordinate, so $B((a, 1), 1) \cap L = \emptyset$ and therefore

$$N_1(a) \cap L = \{(a, 0)\}.$$

Since $N_1(a)$ is open in the Moore Plane, $\{(a, 0)\}$ is open in the subspace L . As a was arbitrary, every singleton of L is open in L , so every subset of L is open in L ; the subspace topology on L is discrete. Now let $D \subseteq L$ be dense in L . For each $a \in \mathbb{R}$ the set $\{(a, 0)\}$ is a nonempty open subset of L , so it must contain a point of D ; that forces $(a, 0) \in D$. Hence $D = L$. So the only dense subset of L is L itself, and L is uncountable, being in one-to-one correspondence with $\mathbb{R}$. Therefore L has no countable dense subset: L is not separable.

Part 3: the Moore Plane is not second countable.

Suppose it were. By Problem 19, every subspace of a second countable space is second countable, so L would be second countable; and, again by Problem 19, a second countable space is separable, so L would be separable. This contradicts Part 2. Hence the Moore Plane is not second countable.

(One can see the failure directly: if $\{\mathcal{B}_n\}$ were a countable base, then for each $a \in \mathbb{R}$ there is an index $n(a)$ with $(a, 0) \in \mathcal{B}_{n(a)} \subseteq N_1(a)$; since $N_1(a) \cap L = \{(a, 0)\}$, the point $(a, 0)$ is the only point of L in $\mathcal{B}_{n(a)}$, so $a \mapsto n(a)$ is one-to-one on $\mathbb{R}$, which is impossible for an uncountable $\mathbb{R}$ and a countable index set.)

Part 4: the Moore Plane is not metrizable.

Suppose the topology of the Moore Plane were induced by a metric ρ . By Part 1 the space is separable, and in Chapter 9 it was proved that a metric space is second countable if and only if it is separable. Hence the Moore Plane would be second countable, contradicting Part 3. Therefore the Moore Plane is not metrizable.

(Alternatively, without invoking Part 3: if the topology were induced by a metric ρ , then the subspace L would carry the topology induced by the restriction of ρ , so L would be a metric space; by Part 1 and the Chapter 9 result the Moore Plane would be second countable, hence L would be second countable by Problem 19 and therefore separable, again contradicting Part 2.)

Problem 11.21. Show that the Sorgenfrey Line is first countable but not second countable and yet the rationals are dense (see Problem 9). Conclude that the Sorgenfrey Line is not metrizable.

Solution. By Problem 9 the Sorgenfrey Line is the set $\mathbb{R}$ with the topology $\mathcal{T}$ having as a base the collection

$$\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}.$$

Thus a set is open exactly when it is a union of such half-open intervals; equivalently, $\mathcal{O}$ is open if and only if for each $x \in \mathcal{O}$ there is $\varepsilon > 0$ with $[x, x + \varepsilon) \subseteq \mathcal{O}$.

Part 1: the Sorgenfrey Line is first countable.

Fix $x \in \mathbb{R}$ and consider the countable collection of open sets

$$\mathcal{B}_x = \left\{ \left[x, x + \frac{1}{n} \right) \mid n \in \mathbb{N} \right\},$$

each of which contains x . We claim $\mathcal{B}_x$ is a base at x . Let $\mathcal{U}$ be any neighborhood of x . Since $\mathcal{B}$ is a base for $\mathcal{T}$, there are $a < b$ with $x \in [a, b) \subseteq \mathcal{U}$; in particular $a \leq x < b$, so $b - x > 0$. By the Archimedean property choose $n \in \mathbb{N}$ with $1/n \leq b - x$, that is $x + 1/n \leq b$. Then

$$x \in \left[x, x + \frac{1}{n} \right) \subseteq [a, b) \subseteq \mathcal{U},$$

the middle inclusion because $a \leq x$ and $x + 1/n \leq b$. So every neighborhood of x contains a member of $\mathcal{B}_x$ that contains x , and $\mathcal{B}_x$ is a countable base at x . Since x was arbitrary, the Sorgenfrey Line is first countable.

Part 2: the rationals are dense, so the Sorgenfrey Line is separable.

Let $\mathcal{O}$ be a nonempty open set and pick $x \in \mathcal{O}$. There are $a < b$ with $x \in [a, b) \subseteq \mathcal{O}$. Since $x < b$, the density of the rationals in $\mathbb{R}$, Theorem 2 of Chapter 1, provides a rational q with $x < q < b$; then $q \in [a, b) \subseteq \mathcal{O}$. Hence every nonempty open set contains a rational, so $\mathbb{Q}$ is dense in the Sorgenfrey Line. As $\mathbb{Q}$ is countable, the Sorgenfrey Line is separable.

Part 3: the Sorgenfrey Line is not second countable.

Suppose, to get a contradiction, that $\{\mathcal{B}_n\}_{n \in N}$, with N a countable index set, is a base for $\mathcal{T}$. For each real number x , the set $[x, x + 1)$ is open and contains x , so there is an index $n(x) \in N$ with

$$x \in \mathcal{B}_{n(x)} \subseteq [x, x + 1).$$

Every point of $\mathcal{B}_{n(x)}$ is then at least x , and x itself belongs to $\mathcal{B}_{n(x)}$; hence

$$x = \min \mathcal{B}_{n(x)}.$$

Consequently the map $x \mapsto n(x)$ from $\mathbb{R}$ to N is one-to-one: if $n(x) = n(y)$ then $x = \min \mathcal{B}_{n(x)} = \min \mathcal{B}_{n(y)} = y$. But then $\mathbb{R}$ would be in one-to-one correspondence with a subset of the countable set N , hence countable, since by Theorem 3 of Chapter 1 a subset of a countable set is countable. This contradicts the uncountability of $\mathbb{R}$, which follows from Theorem 7 of Chapter 1. Therefore no countable base exists and the Sorgenfrey Line is not second countable.

Part 4: the Sorgenfrey Line is not metrizable.

Suppose the topology $\mathcal{T}$ were induced by a metric on $\mathbb{R}$. In Chapter 9 it was proved that a metric space is second countable if and only if it is separable. By Part 2 the Sorgenfrey Line is separable, so it would then be second countable, contradicting Part 3. Hence $\mathcal{T}$ is not induced by any metric: the Sorgenfrey Line is not metrizable.

Remark. This is the example promised in Section 11.3, where it is asserted that a separable space, even a first countable one, may fail to be second countable. Observe also that the failure of metrizability could not have been detected through the Urysohn Metrization Theorem, since that theorem presupposes second countability, which Part 3 denies.

11.4 Problems 11.22–11.28

Problem 11.22. Let $X_1 = \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers, and take $X = X_1 \cup \{\omega\}$, where ω does not belong to X_1 . For each sequence $s = \{m_k\}$ of natural numbers and natural number n , define

$$B_{s,n} = \{\omega\} \cup \{(j, k) : j \geq m_k \text{ all } k \geq n\}.$$

- (i) Show that the sets $B_{s,n}$ together with the singleton sets $\{(j, k)\}$ form a base for a topology on X .
- (ii) Show that ω is a point of closure of X_1 even though no sequence $\{x_n\}$ from X_1 converges to ω .
- (iii) Show that the space X is separable but is not first countable and so is not second countable.
- (iv) Is X a Lindelof space?

Solution. Throughout write

$$\mathcal{B} = \{B_{s,n} : s \text{ a sequence in } \mathbb{N}, n \in \mathbb{N}\} \cup \{\{(j, k)\} : (j, k) \in X_1\},$$

and note that, by the definition, a point (j, k) of X_1 belongs to $B_{s,n}$ precisely when $k \geq n$ and $j \geq m_k$; every $B_{s,n}$ contains ω . Recall also that in this chapter a neighborhood of a point x means an open set containing x .

(i) We verify the two conditions of Proposition 2.

Condition (i) of Proposition 2, that $\mathcal{B}$ covers X : the singletons $\{(j, k)\}$ cover X_1 , and ω belongs to every $B_{s,n}$, so $X = \bigcup_{B \in \mathcal{B}} B$.

Condition (ii) of Proposition 2: let B_1, B_2 belong to $\mathcal{B}$ and let $x \in B_1 \cap B_2$. If $x = (j, k)$ belongs to X_1 , then the singleton $\{(j, k)\}$ itself belongs to $\mathcal{B}$ and satisfies $x \in \{(j, k)\} \subseteq B_1 \cap B_2$. The only remaining case is $x = \omega$; since ω belongs to no singleton, both B_1 and B_2 must then be of the second kind, say $B_1 = B_{s,n}$ with $s = \{m_k\}$ and $B_2 = B_{t,m}$ with $t = \{p_k\}$. Define the sequence $u = \{q_k\}$ by $q_k = \max\{m_k, p_k\}$ and put $N = \max\{n, m\}$. If $(j, k) \in B_{u,N}$ then $k \geq N \geq n$ and $j \geq q_k \geq m_k$, so $(j, k) \in B_{s,n}$; symmetrically $(j, k) \in B_{t,m}$. Since also $\omega \in B_{u,N}$, we conclude

$$\omega \in B_{u,N} \subseteq B_{s,n} \cap B_{t,m} = B_1 \cap B_2.$$

Thus both conditions of Proposition 2 hold and $\mathcal{B}$ is a base for a (unique) topology on X , consisting of $\emptyset$ together with the unions of subcollections of $\mathcal{B}$.

Two consequences are recorded for later use. First, every subset of X_1 is open, since it is a union of singletons; in particular X_1 is an open subspace carrying the discrete topology. Second, a set $\mathcal{U}$ is a neighborhood of ω if and only if $\mathcal{U}$ is open and contains ω , and since the only members of $\mathcal{B}$ containing ω are the sets $B_{s,n}$, this happens if and only if $B_{s,n} \subseteq \mathcal{U}$ for some s and some n .

(ii) First, ω is a point of closure of X_1 . Indeed, let $\mathcal{U}$ be any neighborhood of ω . By the preceding remark there are $s = \{m_k\}$ and n with $B_{s,n} \subseteq \mathcal{U}$. The point (m_n, n) satisfies $n \geq n$ and $m_n \geq m_n$, so it belongs to $B_{s,n}$ and hence to $\mathcal{U}$. Thus every neighborhood of ω meets X_1 , that is, $\omega \in \overline{X_1}$.

Now let $\{x_i\}$ be any sequence in X_1 , say $x_i = (j_i, k_i)$. We produce a neighborhood of ω that fails to contain x_i for infinitely many i ; by the definition of convergence this shows $\{x_i\}$ does not converge to ω .

Case 1: there is a k_0 with $k_i = k_0$ for infinitely many indices i . Take any sequence s (say $m_k = 1$ for all k) and put $n = k_0 + 1$. Every point of $B_{s,n} \cap X_1$ has second coordinate at least $k_0 + 1$, so none of the infinitely many terms x_i with $k_i = k_0$ lies in the neighborhood $B_{s,n}$ of ω .

Case 2: for every k the set $I_k = \{i : k_i = k\}$ is finite. Define a sequence $s = \{m_k\}$ by

$$m_k = 1 + \max\{j_i : i \in I_k\} \quad \text{if } I_k \neq \emptyset, \quad m_k = 1 \quad \text{if } I_k = \emptyset.$$

This is well defined because each I_k is finite. Consider the neighborhood $B_{s,1}$ of ω . If some term $x_i = (j_i, k_i)$ belonged to $B_{s,1}$ we would have $j_i \geq m_{k_i} \geq 1 + j_i$, which is absurd. Hence no term of the sequence lies in $B_{s,1}$.

In either case $\{x_i\}$ does not converge to ω . Since $\{x_i\}$ was an arbitrary sequence from X_1 , no sequence from X_1 converges to ω .

(iii) The set X_1 is countable, being the Cartesian product of two countable sets, and so $X = X_1 \cup \{\omega\}$ is countable as well. The set X_1 is dense: a nonempty open set contains some member of $\mathcal{B}$, and each member of $\mathcal{B}$ meets X_1 (a singleton is contained in X_1 , and $B_{s,n}$ contains (m_n, n) as shown in (ii)). Hence every nonempty open subset of X contains a point of X_1 , so $\overline{X_1} = X$ and X is separable.

Next, X is not first countable. Suppose it were. Proposition 9 asserts that in a first countable space a point of closure of a subset E is the limit of a sequence in E . Since ω is a point of closure of X_1 by part (ii), it would then be the limit of a sequence from X_1 , contradicting the second assertion of part (ii). Hence X has no countable base at ω .

Since the proof of Proposition 9 is itself left as an exercise in the text, here is a direct verification. Suppose $\{\mathcal{U}_i\}_{i=1}^\infty$ were a countable base at ω . For each i choose $s^{(i)} = \{m_k^{(i)}\}$ and n_i with $B_{s^{(i)}, n_i} \subseteq \mathcal{U}_i$. Define $t = \{p_k\}$ by

$$p_k = 1 + \max\{m_k^{(i)} : 1 \leq i \leq k\}.$$

Fix i and choose $k \geq \max\{n_i, i\}$. The point $(m_k^{(i)}, k)$ belongs to $B_{s^{(i)}, n_i}$, hence to $\mathcal{U}_i$; but since $i \leq k$ we have $p_k > m_k^{(i)}$, so $(m_k^{(i)}, k)$ does not belong to $B_{t,1}$. Therefore $\mathcal{U}_i \not\subseteq B_{t,1}$ for every i , while $B_{t,1}$ is a neighborhood of ω . So $\{\mathcal{U}_i\}$ is not a base at ω , a contradiction.

Finally, a second countable space is first countable: a countable base for the topology contains, for each point x , the subcollection of its members that contain x , and this subcollection is a countable base at x . Since X is not first countable it is therefore not second countable.

(iv) Yes, X is a Lindelof space, and trivially so: $X = X_1 \cup \{\omega\}$ is a countable set. Given any open cover $\mathcal{C}$ of X , choose for each point $x \in X$ a member $\mathcal{O}_x \in \mathcal{C}$ with $x \in \mathcal{O}_x$. Then $\{\mathcal{O}_x : x \in X\}$ is a subcollection of $\mathcal{C}$ indexed by a countable set, hence countable, and it covers X . So every open cover of X has a countable subcover.

It is worth noting what this example shows: X is separable and Lindelof, yet neither first nor second countable. In particular, by the remarks of Section 11.3, X is not metrizable, since every metric space is first countable.

Problem 11.23. Let f be a mapping of the topological space X to the topological space Y and $\mathcal{S}$ be a subbase for the topology on Y . Show that f is continuous if and only if the inverse image under f of every set in $\mathcal{S}$ is open in X .

Solution. Recall the definition: a subcollection $\mathcal{S}$ of the topology on Y that covers Y is a subbase for that topology provided the collection $\mathcal{B}$ of intersections of finite subcollections of $\mathcal{S}$ is a base for the topology on Y .

Suppose first that f is continuous. Each set S in $\mathcal{S}$ is a member of the topology on Y , that is, S is open in Y . Therefore, by Proposition 10, $f^{-1}(S)$ is open in X .

Conversely, suppose $f^{-1}(S)$ is open in X for every S in $\mathcal{S}$. By Proposition 10 it suffices to show that $f^{-1}(\mathcal{O})$ is open in X for every open subset $\mathcal{O}$ of Y .

First consider a member of the base $\mathcal{B}$, that is, a set of the form $B = \bigcap_{i=1}^n S_i$ with each S_i in $\mathcal{S}$. Since the inverse image of an intersection is the intersection of the inverse images,

$$f^{-1}(B) = f^{-1}\left(\bigcap_{i=1}^n S_i\right) = \bigcap_{i=1}^n f^{-1}(S_i).$$

Each $f^{-1}(S_i)$ is open in X by hypothesis, and a topology is closed under the formation of intersections of finite subcollections. Hence $f^{-1}(B)$ is open in X .

Now let $\mathcal{O}$ be any open subset of Y . If $\mathcal{O}$ is empty, then $f^{-1}(\mathcal{O}) = \emptyset$ is open in X . Otherwise, since $\mathcal{B}$ is a base for the topology on Y , the set $\mathcal{O}$ is the union of a subcollection of $\mathcal{B}$, say $\mathcal{O} = \bigcup_{\lambda \in \Lambda} B_\lambda$ with each B_λ in $\mathcal{B}$. Since the inverse image of a union is the union of the inverse images,

$$f^{-1}(\mathcal{O}) = \bigcup_{\lambda \in \Lambda} f^{-1}(B_\lambda),$$

which is a union of open subsets of X and hence is open in X , a topology being closed under the formation of unions of arbitrary subcollections.

Thus the inverse image under f of every open subset of Y is open in X , and so f is continuous by Proposition 10.

Problem 11.24. Let X be a topological space.

- (i) If X has the trivial topology, find all continuous mappings of X into $\mathbb{R}$.
- (ii) If X has the discrete topology, find all continuous mappings of X into $\mathbb{R}$.
- (iii) Find all continuous one-to-one mappings from $\mathbb{R}$ to X if X has the discrete topology.
- (iv) Find all continuous one-to-one mappings from $\mathbb{R}$ to X if X has the trivial topology.

Solution. Throughout, $\mathbb{R}$ carries its usual (Euclidean) topology, the trivial topology on a set X is $\{\emptyset, X\}$, and the discrete topology is the collection of all subsets of X . We assume X is nonempty. In each part we use Proposition 10: a mapping is continuous if and only if the inverse image of each open set is open.

(i) The continuous mappings of X into $\mathbb{R}$ are precisely the constant mappings.

If f is constant, say $f \equiv c$, then for open $\mathcal{O} \subseteq \mathbb{R}$ we have $f^{-1}(\mathcal{O}) = X$ if $c \in \mathcal{O}$ and $f^{-1}(\mathcal{O}) = \emptyset$ otherwise; both sets belong to the trivial topology, so f is continuous.

Conversely, let $f: X \rightarrow \mathbb{R}$ be continuous. Fix a point x_0 in X and set $c = f(x_0)$. For each $\epsilon > 0$ the interval $(c - \epsilon, c + \epsilon)$ is open in $\mathbb{R}$, so $f^{-1}((c - \epsilon, c + \epsilon))$ is open in X . This set contains x_0 , so it is nonempty, and hence, the topology being trivial, it equals X . Therefore $|f(x) - c| < \epsilon$ for every x in X and every $\epsilon > 0$, from which $f(x) = c$ for all x . Thus f is constant.

(ii) Every mapping of X into $\mathbb{R}$ is continuous. Indeed, if X has the discrete topology, then for any $f: X \rightarrow \mathbb{R}$ and any open $\mathcal{O} \subseteq \mathbb{R}$, the set $f^{-1}(\mathcal{O})$ is a subset of X and therefore open. By Proposition 10, f is continuous. (Equivalently: each point x_0 has the neighborhood $\{x_0\}$, which is mapped into every neighborhood of $f(x_0)$.)

(iii) There are none: no one-to-one mapping from $\mathbb{R}$ to a discrete space X is continuous.

Suppose $f: \mathbb{R} \rightarrow X$ were continuous and one-to-one, and let t be any real number. The singleton $\{f(t)\}$ is open in X , since X is discrete. By Proposition 10, $f^{-1}(\{f(t)\})$ is open in $\mathbb{R}$. But f is one-to-one, so $f^{-1}(\{f(t)\}) = \{t\}$. Hence $\{t\}$ would be an open subset of $\mathbb{R}$, which is false: every nonempty open subset of $\mathbb{R}$ contains an open interval about each of its points and so is infinite. This contradiction shows no such mapping exists.

(iv) The continuous one-to-one mappings from $\mathbb{R}$ to X are precisely all the one-to-one mappings from $\mathbb{R}$ to X ; continuity is no restriction whatever. Such mappings exist if and only if the cardinality of X is at least that of $\mathbb{R}$.

Indeed, let $f: \mathbb{R} \rightarrow X$ be arbitrary, with X carrying the trivial topology. The only open subsets of X are $\emptyset$ and X , and $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = \mathbb{R}$, both of which are open in $\mathbb{R}$. By Proposition 10, f is continuous. Hence continuity imposes no restriction at all, and the continuous one-to-one mappings from $\mathbb{R}$ to X are precisely the one-to-one mappings from $\mathbb{R}$ to X . Such a mapping exists if and only if there is an injection of $\mathbb{R}$ into X , that is, if and only if X has cardinality at least that of the continuum; for instance if $X = \mathbb{R}$ with the trivial topology, the identity mapping is a continuous one-to-one mapping.

The contrast between (iii) and (iv) is the expected one: continuity of $f: \mathbb{R} \rightarrow X$ is a demand on the topology of X through inverse images, and it is most stringent when X has many open sets and vacuous when X has almost none.

Problem 11.25. For topological spaces X and Y , let f map X to Y . Which of the following assertions are equivalent to the continuity of f ? Verify your answers.

- (i) The inverse image under f of every closed subset of Y is closed in X .
- (ii) If $\mathcal{O}$ is open in X , then $f(\mathcal{O})$ is open in Y .
- (iii) If F is closed in X , then $f(F)$ is closed in Y .
- (iv) For each subset A of X , $f(\overline{A}) \subseteq \overline{f(A)}$.

Solution. Assertions (i) and (iv) are equivalent to the continuity of f ; assertions (ii) and (iii) are not, neither implying nor being implied by continuity.

Throughout we use the set identity $f^{-1}(Y \sim E) = X \sim f^{-1}(E)$, valid for every subset E of Y , together with Proposition 4 (a subset of a topological space is open if and only if its complement is closed) and Proposition 10.

(i) This is equivalent to continuity.

Suppose f is continuous and F is closed in Y . Then $Y \sim F$ is open in Y by Proposition 4, so $f^{-1}(Y \sim F) = X \sim f^{-1}(F)$ is open in X by Proposition 10, and therefore $f^{-1}(F)$ is closed in X , again by Proposition 4.

Conversely, suppose the inverse image of each closed subset of Y is closed in X , and let $\mathcal{O}$ be open in Y . Then $Y \sim \mathcal{O}$ is closed, so $f^{-1}(Y \sim \mathcal{O}) = X \sim f^{-1}(\mathcal{O})$ is closed in X , and hence $f^{-1}(\mathcal{O})$ is open in X . By Proposition 10, f is continuous.

(ii) This is not equivalent to continuity; it is the assertion that f is an open mapping.

Continuity does not imply it. Let $X = Y = \mathbb{R}$ with the usual topology and let f be the constant mapping $f(x) = 0$. Then f is continuous, but $\mathbb{R}$ is open in X while $f(\mathbb{R}) = \{0\}$ is not an open subset of $\mathbb{R}$.

It does not imply continuity. Let $X = \mathbb{R}$ with the usual topology, let $Y = \mathbb{R}$ with the discrete topology, and let f be the identity mapping. Every subset of Y is open, so $f(\mathcal{O})$ is open in Y for every subset $\mathcal{O}$ of X ; but $\{0\}$ is open in Y and $f^{-1}(\{0\}) = \{0\}$ is not open in X , so f is not continuous by Proposition 10.

(iii) This is not equivalent to continuity; it is the assertion that f is a closed mapping.

Continuity does not imply it. Let $X = Y = \mathbb{R}$ with the usual topology and $f(x) = e^x$, which is continuous. The set $F = \mathbb{R}$ is closed in X , but $f(\mathbb{R}) = (0, \infty)$ is not closed in $\mathbb{R}$, since 0 is a point of closure of $(0, \infty)$ that does not belong to it.

It does not imply continuity. Take again $X = \mathbb{R}$ with the usual topology, $Y = \mathbb{R}$ with the discrete topology, and f the identity. Every subset of Y is closed, its complement being open, so f carries closed sets to closed sets; but as noted above f is not continuous.

(iv) This is equivalent to continuity.

Suppose f is continuous and let A be a subset of X . The set $\overline{f(A)}$ is closed in Y by Proposition 3, so by part (i) the set $f^{-1}(\overline{f(A)})$ is closed in X . It contains A , since $f(A) \subseteq \overline{f(A)}$. By Proposition 3, $\overline{A}$ is the smallest closed subset of X containing A , so

$$\overline{A} \subseteq f^{-1}(\overline{f(A)}), \quad \text{that is,} \quad f(\overline{A}) \subseteq \overline{f(A)}.$$

Conversely, suppose $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X . Let F be closed in Y and put $A = f^{-1}(F)$. Then $f(A) \subseteq F$, so $\overline{f(A)} \subseteq \overline{F} = F$, the set F being closed. Hence

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq F,$$

so $\overline{A} \subseteq f^{-1}(F) = A$. Since always $A \subseteq \overline{A}$, we conclude $\overline{A} = A$, that is, $f^{-1}(F)$ is closed in X . Thus the inverse image of every closed subset of Y is closed in X , and by part (i) the mapping f is continuous.

Problem 11.26. Prove Proposition 11.

[Proposition 11: The composition of continuous mappings between topological spaces, when defined, is continuous.]

Solution. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ and $(Z, \mathcal{R})$ be topological spaces and let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be continuous, so that the composition $g \circ f: X \rightarrow Z$ is defined. We show $g \circ f$ is continuous.

First, a direct argument from the definition of continuity at a point. Let x_0 belong to X and let $\mathcal{W}$ be a neighborhood of $(g \circ f)(x_0) = g(f(x_0))$ in Z . Since g is continuous at the point $f(x_0)$, there is a neighborhood $\mathcal{V}$ of $f(x_0)$ in Y for which $g(\mathcal{V}) \subseteq \mathcal{W}$. Since f is continuous at x_0 , there is a neighborhood $\mathcal{U}$ of x_0 in X for which $f(\mathcal{U}) \subseteq \mathcal{V}$. Therefore

$$(g \circ f)(\mathcal{U}) = g(f(\mathcal{U})) \subseteq g(\mathcal{V}) \subseteq \mathcal{W}.$$

Thus for every neighborhood $\mathcal{W}$ of $(g \circ f)(x_0)$ there is a neighborhood $\mathcal{U}$ of x_0 mapped by $g \circ f$ into $\mathcal{W}$, that is, $g \circ f$ is continuous at x_0 . Since x_0 was an arbitrary point of X , the mapping $g \circ f$ is continuous.

Alternatively, one may argue with inverse images. For every subset E of Z we have the set identity

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)),$$

since x belongs to the left-hand side exactly when $g(f(x)) \in E$, that is, exactly when $f(x) \in g^{-1}(E)$, that is, exactly when $x \in f^{-1}(g^{-1}(E))$. Now let $\mathcal{O}$ be open in Z . Since g is continuous, $g^{-1}(\mathcal{O})$ is open in Y by Proposition 10. Since f is continuous, $f^{-1}(g^{-1}(\mathcal{O}))$ is open in X , again by Proposition 10. By the identity above, $(g \circ f)^{-1}(\mathcal{O})$ is open in X . As $\mathcal{O}$ was an arbitrary open subset of Z , Proposition 10 tells us that $g \circ f$ is continuous.

A remark on the qualification when defined in the statement of the proposition. It may happen that g is defined and continuous not on all of Y but only on a subspace A of Y with $f(X) \subseteq A$, and then $g \circ f$ is still defined on X . In that case the argument above goes through once we observe that f , regarded as a mapping of X into the subspace A , is continuous: an open subset of A is, by the definition of the subspace topology, of the form $A \cap \mathcal{O}$ with $\mathcal{O}$ open in Y , and since $f(X) \subseteq A$ we have $f^{-1}(A \cap \mathcal{O}) = f^{-1}(\mathcal{O})$, which is open in X by Proposition 10. Applying what was proved above to this corestriction of f and to $g: A \rightarrow Z$ shows $g \circ f$ is continuous.

Problem 11.27. Prove Proposition 12.

[Proposition 12: Let X be a nonempty set and $\mathcal{S}$ any collection of subsets of X that covers X . The collection of subsets of X consisting of intersections of finite subcollections of $\mathcal{S}$ is a base for a topology $\mathcal{T}$ for X . It is the weakest topology containing $\mathcal{S}$ in the sense that if $\mathcal{T}'$ is any other topology for X containing $\mathcal{S}$, then $\mathcal{T} \subseteq \mathcal{T}'$.]

Solution. Let

$$\mathcal{B} = \left\{ \bigcap_{i=1}^n S_i \mid n \in N, S_1, \dots, S_n \in \mathcal{S} \right\}$$

be the collection of intersections of finite (nonempty) subcollections of $\mathcal{S}$.

Step 1: $\mathcal{B}$ is a base for a topology on X . We verify the two conditions of Proposition 2.

Condition (i). Taking $n = 1$ shows that $\mathcal{S} \subseteq \mathcal{B}$. Since $\mathcal{S}$ covers X by hypothesis, so does the larger collection $\mathcal{B}$; that is, $X = \bigcup_{B \in \mathcal{B}} B$.

Condition (ii). Let B_1 and B_2 belong to $\mathcal{B}$, say $B_1 = \bigcap_{i=1}^n S_i$ and $B_2 = \bigcap_{j=1}^m T_j$ with all S_i and T_j in $\mathcal{S}$. Then

$$B_1 \cap B_2 = S_1 \cap \dots \cap S_n \cap T_1 \cap \dots \cap T_m$$

is itself the intersection of a finite subcollection of $\mathcal{S}$, so $B_1 \cap B_2$ belongs to $\mathcal{B}$. Hence if x belongs to $B_1 \cap B_2$, the set $B = B_1 \cap B_2$ is a member of $\mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.

By Proposition 2, $\mathcal{B}$ is a base for a unique topology $\mathcal{T}$ for X , and $\mathcal{T}$ consists of $\emptyset$ together with the unions of subcollections of $\mathcal{B}$.

Step 2: $\mathcal{T}$ contains $\mathcal{S}$. Each B in $\mathcal{B}$ is the union of the subcollection $\{B\}$ of $\mathcal{B}$ and therefore belongs to $\mathcal{T}$. Thus $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{T}$.

Step 3: $\mathcal{T}$ is the weakest such topology. Let $\mathcal{T}'$ be any topology for X with $\mathcal{S} \subseteq \mathcal{T}'$. Since a topology is closed under the formation of intersections of finite subcollections, every set $\bigcap_{i=1}^n S_i$ with the S_i in $\mathcal{S} \subseteq \mathcal{T}'$ belongs to $\mathcal{T}'$; that is, $\mathcal{B} \subseteq \mathcal{T}'$. Since a topology is closed under the formation of unions of arbitrary subcollections, every union of a subcollection of $\mathcal{B}$ belongs to $\mathcal{T}'$; and $\emptyset$ belongs to $\mathcal{T}'$ because $\mathcal{T}'$ is a topology. As $\mathcal{T}$ consists precisely of $\emptyset$ and such unions, we conclude $\mathcal{T} \subseteq \mathcal{T}'$.

Therefore $\mathcal{T}$ is a topology for X containing $\mathcal{S}$, having $\mathcal{B}$ as a base, and it is contained in every topology for X that contains $\mathcal{S}$. In the terminology introduced just above the proposition, $\mathcal{T}$ is weaker than every topology containing $\mathcal{S}$, and $\mathcal{S}$ is a subbase for $\mathcal{T}$.

Problem 11.28. Prove that the sum and product of two real-valued continuous functions defined on a topological space X are themselves continuous.

Solution. Let f and g be continuous real-valued functions on the topological space X , where $\mathbb{R}$ carries its usual topology.

We first record a convenient reformulation of continuity at a point for a real-valued function. The open intervals $(c - \epsilon, c + \epsilon)$, $\epsilon > 0$, form a base for the topology of $\mathbb{R}$ at the point c : every neighborhood of c contains such an interval. Consequently, a function $h: X \rightarrow \mathbb{R}$ is continuous at x_0 if and only if for each $\epsilon > 0$ there is a neighborhood $\mathcal{U}$ of x_0 for which

$$|h(x) - h(x_0)| < \epsilon \quad \text{for all } x \in \mathcal{U}.$$

Indeed, if this holds and $\mathcal{W}$ is a neighborhood of $h(x_0)$, choose $\epsilon > 0$ with $(h(x_0) - \epsilon, h(x_0) + \epsilon) \subseteq \mathcal{W}$ and take the corresponding $\mathcal{U}$; then $h(\mathcal{U}) \subseteq \mathcal{W}$. The converse is immediate on taking $\mathcal{W} = (h(x_0) - \epsilon, h(x_0) + \epsilon)$.

We also use repeatedly that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are neighborhoods of x_0 , then so is $\mathcal{U}_1 \cap \mathcal{U}_2$, since a topology is closed under the formation of intersections of finite subcollections and x_0 belongs to both sets.

The sum. Fix x_0 in X and let $\epsilon > 0$. By the continuity of f at x_0 there is a neighborhood $\mathcal{U}_1$ of x_0 with $|f(x) - f(x_0)| < \epsilon/2$ for all $x \in \mathcal{U}_1$, and by the continuity of g at x_0 there is a neighborhood $\mathcal{U}_2$ of x_0 with $|g(x) - g(x_0)| < \epsilon/2$ for all $x \in \mathcal{U}_2$. Put $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$, a neighborhood of x_0 . For x in $\mathcal{U}$, by the triangle inequality,

$$|(f+g)(x) - (f+g)(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

By the reformulation above, $f+g$ is continuous at x_0 . Since x_0 was arbitrary, $f+g$ is continuous on X .

The product. Fix x_0 in X and let $\epsilon > 0$. Set

$$M = |f(x_0)| + |g(x_0)| + 1 > 0 \quad \text{and} \quad \delta = \min \left\{ 1, \frac{\epsilon}{M} \right\} > 0.$$

By the continuity of f and of g at x_0 and the remark on intersections, there is a neighborhood $\mathcal{U}$ of x_0 such that for all x in $\mathcal{U}$,

$$|f(x) - f(x_0)| < \delta \quad \text{and} \quad |g(x) - g(x_0)| < \delta.$$

For such x we have $|f(x)| \leq |f(x_0)| + \delta \leq |f(x_0)| + 1$, and therefore, adding and subtracting $f(x)g(x_0)$,

$$\begin{aligned} |(fg)(x) - (fg)(x_0)| &= |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\ &\leq |f(x)| \cdot |g(x) - g(x_0)| + |g(x_0)| \cdot |f(x) - f(x_0)| \\ &< (|f(x_0)| + 1)\delta + |g(x_0)|\delta \\ &= M\delta \leq \epsilon. \end{aligned}$$

Hence fg is continuous at x_0 , and since x_0 was an arbitrary point of X , the product fg is continuous on X .

Taking g constant, equal to α , shows that αf is continuous for every real α ; combined with the statement about sums, this shows that the real-valued continuous functions on a topological space X form a linear space, and, by the statement about products, an algebra.

11.5 Problems 11.29–11.35

Problem 11.29. Let $\mathcal{F}$ be a collection of real-valued functions on a set X . Find necessary and sufficient conditions on $\mathcal{F}$ in order that X , considered as a topological space with the weak topology induced by $\mathcal{F}$, is Tychonoff.

Solution. Recall the two definitions in force. The weak topology for X induced by $\mathcal{F}$ is the weakest topology containing the collection

$$\mathcal{S} = \{f^{-1}(\mathcal{O}) \mid f \in \mathcal{F}, \mathcal{O} \text{ open in } \mathbb{R}\},$$

and, since $f^{-1}(\mathbb{R}) = X$ for each f , the collection $\mathcal{S}$ covers X whenever $\mathcal{F} \neq \emptyset$; hence by Proposition 12 the intersections of finite subcollections of $\mathcal{S}$ form a base for this topology. (If $\mathcal{F} = \emptyset$, the weak topology is the trivial topology $\{\emptyset, X\}$.) Also recall that X is called Tychonoff provided for each two points u and v in X there is a neighborhood of u that does not contain v and a neighborhood of v that does not contain u .

We claim the necessary and sufficient condition is that

(*) $\mathcal{F}$ separates the points of X : for each two distinct points u, v in X there is a function $f \in \mathcal{F}$ for which $f(u) \neq f(v)$.

Sufficiency. Assume (*) and let u and v be two distinct points of X . Choose $f \in \mathcal{F}$ with $f(u) \neq f(v)$ and set $r = |f(u) - f(v)| > 0$. The set

$$\mathcal{U} = f^{-1}((f(u) - r, f(u) + r))$$

belongs to $\mathcal{S}$ and hence is open in the weak topology; it contains u , while $v \notin \mathcal{U}$ since $|f(v) - f(u)| = r$. Interchanging the roles of u and v produces an open set containing v but not u . Therefore X is Tychonoff.

Necessity. Assume X , with the weak topology induced by $\mathcal{F}$, is Tychonoff, and let $u \neq v$ be points of X . There is a neighborhood $\mathcal{U}$ of u with $v \notin \mathcal{U}$. In particular $\mathcal{U}$ is a nonempty open set different from X , so the topology is not the trivial one and therefore $\mathcal{F} \neq \emptyset$. By the description of the base above, there are functions $f_1, \dots, f_n$ in $\mathcal{F}$ and open subsets $\mathcal{O}_1, \dots, \mathcal{O}_n$ of $\mathbb{R}$ for which

$$u \in \bigcap_{i=1}^n f_i^{-1}(\mathcal{O}_i) \subseteq \mathcal{U}.$$

Since $v \notin \mathcal{U}$, the point v fails to belong to this intersection, so there is an index i with $v \notin f_i^{-1}(\mathcal{O}_i)$, that is, $f_i(v) \notin \mathcal{O}_i$. But $f_i(u) \in \mathcal{O}_i$. Hence $f_i(u) \neq f_i(v)$, and (*) holds.

Two remarks. First, by Proposition 6 the condition (*) is likewise equivalent to every singleton subset of X being closed in the weak topology. Second, the sufficiency argument in fact yields more: if $f(u) \neq f(v)$ and $r = |f(u) - f(v)|$, then

$$f^{-1}((f(u) - r/2, f(u) + r/2)) \quad \text{and} \quad f^{-1}((f(v) - r/2, f(v) + r/2))$$

are disjoint neighborhoods of u and v . Thus for a weak topology induced by a collection of real-valued functions, the Tychonoff and Hausdorff separation properties are equivalent, and each is equivalent to $\mathcal{F}$ separating the points of X .

Problem 11.30. For topological spaces X and Y , let the mapping $f: X \rightarrow Y$ be one-to-one and onto. Show that the following assertions are equivalent.

- (i) f is a homeomorphism of X onto Y .
- (ii) A subset E of X is open in X if and only if $f(E)$ is open in Y .
- (iii) A subset E of X is closed in X if and only if $f(E)$ is closed in Y .
- (iv) The image of the closure of a set is the closure of the image, that is, for each subset A of X , $f(\overline{A}) = \overline{f(A)}$.

Solution. Throughout, f is one-to-one and onto, so the inverse mapping $g = f^{-1}: Y \rightarrow X$ is well defined, and for every subset E of X we have

$$f(E) = g^{-1}(E), \quad f^{-1}(f(E)) = E, \quad f(X \setminus E) = Y \setminus f(E).$$

The first identity holds because $y \in g^{-1}(E)$ means $g(y) \in E$, that is, $y = f(x)$ for some $x \in E$. The last identity uses both that f is onto (so every y is $f(x)$ for some x) and that f is one-to-one (so $f(x) \in f(E)$ forces $x \in E$). We show (i) $\Leftrightarrow$ (ii), (ii) $\Leftrightarrow$ (iii), and (iii) $\Leftrightarrow$ (iv).

(i) implies (ii). Let f be a homeomorphism, so that both f and $g = f^{-1}$ are continuous. If E is open in X , then $f(E) = g^{-1}(E)$ is open in Y by the continuity of g together with Proposition 10. Conversely, if $f(E)$ is open in Y , then $E = f^{-1}(f(E))$ is open in X by the continuity of f and Proposition 10.

(ii) implies (i). Assume (ii). To see that f is continuous, let $\mathcal{O}$ be open in Y and put $E = f^{-1}(\mathcal{O})$. Since f is onto, $f(E) = \mathcal{O}$ is open in Y , so by (ii) the set E is open in X . By Proposition 10, f is continuous. To see that $g = f^{-1}$ is continuous, let E be open in X ; then $g^{-1}(E) = f(E)$, which is open in Y by (ii). Again by Proposition 10, g is continuous. Since f is one-to-one and onto and both f and f^{-1} are continuous, f is a homeomorphism.

(ii) implies (iii). Let E be a subset of X . Using Proposition 4 twice and the identity $f(X \sim E) = Y \sim f(E)$:

$$E \text{ closed in } X \iff X \sim E \text{ open in } X \iff f(X \sim E) = Y \sim f(E) \text{ open in } Y \iff f(E) \text{ closed in } Y,$$

where the middle equivalence is (ii) applied to the set $X \sim E$.

(iii) implies (ii). The same chain read with the roles of open and closed interchanged: for a subset E of X ,

$$E \text{ open in } X \iff X \sim E \text{ closed in } X \iff Y \sim f(E) = f(X \sim E) \text{ closed in } Y \iff f(E) \text{ open in } Y,$$

the middle equivalence being (iii) applied to $X \sim E$.

(iii) implies (iv). Let A be a subset of X . By Proposition 3 the set $\bar{A}$ is closed in X , so by (iii) the set $f(\bar{A})$ is closed in Y ; since $A \subseteq \bar{A}$ we have $f(A) \subseteq f(\bar{A})$, and therefore, $\overline{f(A)}$ being the smallest closed set containing $f(A)$ (Proposition 3),

$$\overline{f(A)} \subseteq f(\bar{A}).$$

For the reverse inclusion, put $E = f^{-1}(\overline{f(A)})$. Since f is onto, $f(E) = \overline{f(A)}$, which is closed in Y ; hence by (iii) the set E is closed in X . Moreover $A \subseteq E$, because $f(A) \subseteq \overline{f(A)}$. By Proposition 3, $\bar{A} \subseteq E$, and therefore

$$f(\bar{A}) \subseteq f(E) = \overline{f(A)}.$$

Thus $f(\bar{A}) = \overline{f(A)}$.

(iv) implies (iii). Assume (iv). If E is closed in X , then $E = \bar{E}$, so $f(E) = f(\bar{E}) = \overline{f(E)}$ and hence $f(E)$ is closed in Y . Conversely, suppose $f(E)$ is closed in Y , that is, $f(E) = \overline{f(E)}$. Then by (iv),

$$f(\bar{E}) = \overline{f(E)} = f(E).$$

Since $E \subseteq \bar{E}$ and f is one-to-one, applying f^{-1} to this equality of images gives $\bar{E} = f^{-1}(f(\bar{E})) = f^{-1}(f(E)) = E$. Hence E is closed in X .

Having established (i) $\Leftrightarrow$ (ii), (ii) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv), the four assertions are equivalent.

Problem 11.31. For topological spaces X and Y , let f be a continuous mapping from X onto Y . If X is Hausdorff, is Y Hausdorff? If X is normal, is Y normal?

Solution. The answer to both questions is no: neither the Hausdorff separation property nor the normal separation property is inherited by continuous images.

A first counterexample. Let $X = \mathbb{R}$ with the discrete topology and let $Y = \mathbb{R}$ with the trivial topology $\{\emptyset, Y\}$, and let $f: X \rightarrow Y$ be the identity map, which is certainly one-to-one and onto. The mapping f is continuous: the inverse image of each of the two open subsets of Y is a subset of X , and every subset of X is open, so continuity follows from Proposition 10.

The space X is Hausdorff, since for distinct points u and v the singletons $\{u\}$ and $\{v\}$ are disjoint open sets containing u and v respectively. It is also normal: by Problem 1 the discrete topology is the topology induced by the discrete metric, and by Proposition 7 every metric space is normal. (Directly: any two disjoint subsets of X are themselves disjoint neighborhoods of one another, and each singleton is closed.)

The space Y is neither Hausdorff nor normal. Indeed, the only neighborhood of any point of Y is Y itself, so no two distinct points can be separated by disjoint neighborhoods; thus Y fails to be Hausdorff. Moreover no singleton $\{y\}$ is closed in Y , since $Y \sim \{y\}$ is a nonempty proper subset of Y and hence not open; by Proposition 6, Y is not even Tychonoff, and therefore, the Tychonoff property

being part of the definition of normality, Y is not normal.

A second counterexample, in which the image space is not degenerate. Again let $X = \mathbb{R}$ with the discrete topology, let $Y = \mathbb{R}$ with the topology $\mathcal{T}$ consisting of $\emptyset$ together with all subsets of $\mathbb{R}$ having finite complement, and let f be the identity map. As before f is continuous and onto, and X is both Hausdorff and normal. The collection $\mathcal{T}$ is a topology: $\emptyset$ and $\mathbb{R}$ belong to it, a finite intersection of cofinite sets is cofinite, and an arbitrary union of cofinite sets is cofinite. The closed subsets of Y are exactly $\mathbb{R}$ and the finite sets, so every singleton is closed and Y is Tychonoff by Proposition 6. But any two nonempty open subsets $\mathcal{O}_1, \mathcal{O}_2$ of Y intersect, since $\mathbb{R} \sim (\mathcal{O}_1 \cap \mathcal{O}_2) = (\mathbb{R} \sim \mathcal{O}_1) \cup (\mathbb{R} \sim \mathcal{O}_2)$ is finite while $\mathbb{R}$ is infinite. Consequently no two distinct points of Y have disjoint neighborhoods, so Y is not Hausdorff; and the disjoint closed sets $F_1 = \{0\}$ and $F_2 = \{1\}$ cannot be separated by disjoint neighborhoods (a neighborhood of F_1 contains 0 and a neighborhood of F_2 contains 1, so both are nonempty and open, hence meet), so Y is not normal. Thus a continuous image of a Hausdorff, normal space need be neither Hausdorff nor normal, even when it is Tychonoff.

The source of the failure is clear from Proposition 10: continuity of f gives information about inverse images of open sets in Y , and so lets separation properties be transported from Y back to X , but it says nothing about the images of the open or closed sets of X , which is what would be needed to transport separation properties forward. Consistently with this, Problem 30 shows that when f is one-to-one and onto and images of open sets are open, that is, when f is a homeomorphism, all such properties do transfer.

Problem 11.32. Let ρ_1 and ρ_2 be metrics on the set X that induce topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. If $\mathcal{T}_1 = \mathcal{T}_2$, are the metrics necessarily equivalent?

Solution. No. Recall that two metrics ρ and σ on a set X are called equivalent provided there are positive numbers c_1 and c_2 for which

$$c_1 \sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \sigma(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Equivalence of metrics is a strictly stronger requirement than the equality of the induced topologies. Indeed, if ρ and σ are equivalent with constants c_1, c_2 as above, then for each $x \in X$ and $r > 0$ we have the inclusions of balls $B_\rho(x, c_1 r) \subseteq B_\sigma(x, r)$ and $B_\sigma(x, r/c_2) \subseteq B_\rho(x, r)$, so each ball for one metric contains a ball for the other about the same center, and hence the two metrics induce the same topology. The following example shows the converse implication fails.

Take $X = \mathbb{R}$ and define

$$\rho_1(x, y) = |x - y|, \quad \rho_2(x, y) = \min\{1, |x - y|\} \quad \text{for } x, y \in \mathbb{R}.$$

First, ρ_2 is a metric. It is nonnegative, symmetric, and vanishes precisely when $x = y$. For the triangle inequality, observe that for nonnegative numbers a and b ,

$$\min\{1, a + b\} \leq \min\{1, a\} + \min\{1, b\},$$

since if $a \geq 1$ or $b \geq 1$ the right-hand side is at least 1, which dominates the left-hand side, while if $a < 1$ and $b < 1$ the right-hand side equals $a + b \geq \min\{1, a + b\}$. Now for x, y, z in $\mathbb{R}$, since $t \mapsto \min\{1, t\}$ is nondecreasing and $|x - z| \leq |x - y| + |y - z|$,

$$\rho_2(x, z) = \min\{1, |x - z|\} \leq \min\{1, |x - y| + |y - z|\} \leq \rho_2(x, y) + \rho_2(y, z).$$

Second, $\mathcal{T}_1 = \mathcal{T}_2$. Let $x \in \mathbb{R}$ and $0 < r \leq 1$. Then

$$\rho_2(x, y) < r \iff \min\{1, |x - y|\} < r \iff |x - y| < r,$$

the second equivalence because $|x - y| \geq 1$ forces $\min\{1, |x - y|\} = 1 \geq r$. Hence the open balls of radius $r \leq 1$ about x are the same set for the two metrics. Now for a metric ρ on X , a subset $\mathcal{O}$ of X is open in the topology induced by ρ if and only if for each $x \in \mathcal{O}$ there is an r with $0 < r \leq 1$ for which $B_\rho(x, r) \subseteq \mathcal{O}$: this is the definition of the metric topology, except that a radius $r > 1$ furnished

by the definition may be replaced by $\min\{1, r\}$, which only shrinks the ball. Applying this criterion to ρ_1 and to ρ_2 , and using that the balls of radius at most 1 agree for the two metrics, we conclude that $\mathcal{O}$ is $\mathcal{T}_1$ -open if and only if it is $\mathcal{T}_2$ -open. Thus $\mathcal{T}_1 = \mathcal{T}_2$.

Third, ρ_1 and ρ_2 are not equivalent. Suppose there were positive constants c_1, c_2 with $c_1\rho_2 \leq \rho_1 \leq c_2\rho_2$ on $\mathbb{R} \times \mathbb{R}$. Taking $x = 0$ and $y = n$ for a natural number $n \geq 1$ gives $\rho_1(0, n) = n$ and $\rho_2(0, n) = 1$, so the inequality $\rho_1 \leq c_2\rho_2$ yields $n \leq c_2$ for every natural number n . This contradicts the Archimedean property of $\mathbb{R}$. Hence no such constants exist.

Thus ρ_1 and ρ_2 induce the same topology on $\mathbb{R}$ but are not equivalent metrics: equality of the induced topologies does not force equivalence. The underlying reason is that equivalence is a metric (indeed, uniform and quantitative) relation, sensitive to features such as boundedness that the topology cannot detect: ρ_2 is a bounded metric while ρ_1 is not, and a two-sided comparison by constants would preserve boundedness.

Problem 11.33. Show that the inverse of a homeomorphism is a homeomorphism and the composition of two homeomorphisms, when defined, is again a homeomorphism.

Solution. Recall the definition: a continuous mapping f from a topological space X to a topological space Y is a homeomorphism provided it is one-to-one, maps X onto Y , and has a continuous inverse f^{-1} from Y to X .

The inverse. Let $f: X \rightarrow Y$ be a homeomorphism and set $g = f^{-1}: Y \rightarrow X$. Since f is one-to-one and onto, so is g : given $x \in X$, we have $g(f(x)) = x$, so g maps Y onto X ; and if $g(y_1) = g(y_2)$, then applying f gives $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$, so g is one-to-one. By the definition of a homeomorphism, g is continuous. Finally, the inverse of g is f itself, since $g(y) = x$ if and only if $y = f(x)$; and f is continuous. Therefore $g = f^{-1}$ is a one-to-one continuous mapping of Y onto X with continuous inverse, that is, a homeomorphism of Y onto X .

The composition. Let X, Y , and Z be topological spaces and let $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ be homeomorphisms, so that the composition $h \circ f: X \rightarrow Z$ is defined.

First, $h \circ f$ is one-to-one: if $(h \circ f)(x_1) = (h \circ f)(x_2)$, then $h(f(x_1)) = h(f(x_2))$, so $f(x_1) = f(x_2)$ because h is one-to-one, and hence $x_1 = x_2$ because f is one-to-one. Second, $h \circ f$ maps X onto Z : given $z \in Z$, since h maps Y onto Z there is $y \in Y$ with $h(y) = z$, and since f maps X onto Y there is $x \in X$ with $f(x) = y$; thus $(h \circ f)(x) = z$. Third, $h \circ f$ is continuous, by Proposition 11, since f and h are continuous.

It remains to identify the inverse and check its continuity. We claim

$$(h \circ f)^{-1} = f^{-1} \circ h^{-1}.$$

Indeed, for $z \in Z$ and $x \in X$, the statement $(h \circ f)(x) = z$ says $h(f(x)) = z$, which is equivalent to $f(x) = h^{-1}(z)$, which in turn is equivalent to $x = f^{-1}(h^{-1}(z)) = (f^{-1} \circ h^{-1})(z)$. Since $f^{-1}: Y \rightarrow X$ and $h^{-1}: Z \rightarrow Y$ are continuous, their composition $f^{-1} \circ h^{-1}: Z \rightarrow X$ is continuous, again by Proposition 11.

Therefore $h \circ f$ is a one-to-one continuous mapping of X onto Z with continuous inverse, that is, a homeomorphism of X onto Z .

Together with the observation that the identity mapping of a topological space onto itself is a homeomorphism, these two facts show that the relation of being homeomorphic is reflexive, symmetric, and transitive, that is, it is an equivalence relation among topological spaces, as asserted in the text.

Problem 11.34. Suppose that a topological space X has the property that every continuous real-valued function on X takes a minimum value. Show that any topological space that is homeomorphic to X also possesses this property.

Solution. Say that a real-valued function g on a topological space Z takes a minimum value provided there is a point $z_0 \in Z$ for which $g(z_0) \leq g(z)$ for all $z \in Z$.

Let Y be a topological space that is homeomorphic to X . By Problem 33 the inverse of a homeomor-

phism is a homeomorphism, so the relation of being homeomorphic is symmetric and we may choose a homeomorphism

$$h: X \rightarrow Y,$$

that is, a one-to-one continuous mapping of X onto Y whose inverse is continuous. We must show that every continuous real-valued function on Y takes a minimum value.

Let $g: Y \rightarrow \mathbb{R}$ be continuous, where $\mathbb{R}$ has its usual topology. Since $h: X \rightarrow Y$ and $g: Y \rightarrow \mathbb{R}$ are continuous, the composition

$$g \circ h: X \rightarrow \mathbb{R}$$

is continuous, by Proposition 11. Hence $g \circ h$ is a continuous real-valued function on X , and so, by the hypothesis on X , it takes a minimum value: there is a point $x_0 \in X$ for which

$$(g \circ h)(x_0) \leq (g \circ h)(x) \quad \text{for all } x \in X.$$

Define $y_0 = h(x_0) \in Y$. Let y be any point of Y . Since h maps X onto Y , there is a point $x \in X$ with $h(x) = y$. Therefore

$$g(y_0) = g(h(x_0)) = (g \circ h)(x_0) \leq (g \circ h)(x) = g(h(x)) = g(y).$$

Since $y \in Y$ was arbitrary, g attains the minimum value $g(y_0)$ at the point y_0 .

As g was an arbitrary continuous real-valued function on Y , the space Y has the stated property. Note that only the continuity and surjectivity of h were used; thus the property is inherited by every continuous image of X , and a fortiori by every space homeomorphic to X .

Problem 11.35. Suppose that a topological space X has the property that every continuous real-valued function on X has an interval as its image. Show that any topological space that is homeomorphic to X also possesses this property.

Solution. Let Y be a topological space homeomorphic to X . By Problem 33 the inverse of a homeomorphism is again a homeomorphism, so we may select a homeomorphism

$$h: X \rightarrow Y,$$

a one-to-one continuous mapping of X onto Y with continuous inverse. Let $g: Y \rightarrow \mathbb{R}$ be continuous, $\mathbb{R}$ having its usual topology. We must show that the image $g(Y)$ is an interval.

Since h and g are continuous, the composition $g \circ h: X \rightarrow \mathbb{R}$ is continuous, by Proposition 11. Therefore, by the hypothesis on X , the image

$$(g \circ h)(X) = \{g(h(x)) \mid x \in X\}$$

is an interval.

We claim $(g \circ h)(X) = g(Y)$. On the one hand, for each $x \in X$ the point $h(x)$ belongs to Y , so $g(h(x)) \in g(Y)$; hence $(g \circ h)(X) \subseteq g(Y)$. On the other hand, let $c \in g(Y)$, say $c = g(y)$ with $y \in Y$. Since h maps X onto Y , there is $x \in X$ with $h(x) = y$, and then $c = g(h(x)) = (g \circ h)(x)$ belongs to $(g \circ h)(X)$; hence $g(Y) \subseteq (g \circ h)(X)$. This proves the claim, and therefore

$$g(Y) = (g \circ h)(X)$$

is an interval.

Since g was an arbitrary continuous real-valued function on Y , every continuous real-valued function on Y has an interval as its image, which is what we wished to show. As in the preceding problem, only the continuity of h and the fact that h maps X onto Y were used, so the property in fact passes to every continuous image of X .

11.6 Problems 11.36–11.42

Problem 11.36. Show that $\mathbb{R}$ is homeomorphic to the open bounded interval $(0, 1)$, but is not homeomorphic to the closed bounded interval $[0, 1]$.

Solution. Throughout, $\mathbb{R}$, $(0, 1)$ and $[0, 1]$ carry the usual topology, that is, the topology induced by the Euclidean metric; the topology of $(0, 1)$ and of $[0, 1]$ is also the subspace topology inherited from $\mathbb{R}$.

First we exhibit a homeomorphism of $\mathbb{R}$ onto $(0, 1)$. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{2} + \frac{1}{2} \cdot \frac{x}{1 + |x|}.$$

The function $x \mapsto |x|$ is continuous, $1 + |x| \geq 1 > 0$ for every x , and sums, products and quotients of continuous real-valued functions with nonvanishing denominator are continuous. Hence f is continuous on $\mathbb{R}$.

Since $|x|/(1 + |x|) < 1$ for every x , we have $|f(x) - 1/2| < 1/2$, that is, $f(x) \in (0, 1)$: thus f maps $\mathbb{R}$ into $(0, 1)$.

Define $g: (0, 1) \rightarrow \mathbb{R}$ by

$$g(y) = \frac{2y - 1}{1 - |2y - 1|}.$$

For $y \in (0, 1)$ the number $t = 2y - 1$ lies in $(-1, 1)$, so $1 - |t| > 0$ and g is well defined; as above g is continuous on $(0, 1)$.

We check that $g \circ f = \text{id}_{\mathbb{R}}$ and $f \circ g = \text{id}_{(0,1)}$. Let $x \in \mathbb{R}$ and put $y = f(x)$. Then $t := 2y - 1 = x/(1 + |x|)$, so that $|t| = |x|/(1 + |x|)$ and hence $1 - |t| = 1/(1 + |x|)$. Therefore

$$g(f(x)) = \frac{t}{1 - |t|} = \frac{x/(1 + |x|)}{1/(1 + |x|)} = x.$$

Conversely, let $y \in (0, 1)$, put $t = 2y - 1 \in (-1, 1)$ and $x = g(y) = t/(1 - |t|)$. Then $|x| = |t|/(1 - |t|)$, so $1 + |x| = 1/(1 - |t|)$ and

$$\frac{x}{1 + |x|} = \frac{t/(1 - |t|)}{1/(1 - |t|)} = t = 2y - 1,$$

whence $f(g(y)) = \frac{1}{2} + \frac{1}{2}(2y - 1) = y$.

Thus f is a one-to-one mapping of $\mathbb{R}$ onto $(0, 1)$, both f and its inverse $f^{-1} = g$ are continuous, and therefore f is a homeomorphism of $\mathbb{R}$ onto $(0, 1)$.

Next we show that $\mathbb{R}$ is not homeomorphic to $[0, 1]$. We use the fact that compactness is a topological property. Indeed, a homeomorphism is in particular a continuous mapping of one space onto the other, and by Proposition 20 the image of a compact topological space under a continuous mapping is compact; hence if two spaces are homeomorphic and one is compact, so is the other.

Now $[0, 1]$ is a compact topological space: it is a closed, bounded set of real numbers, so by the Heine–Borel Theorem every cover of $[0, 1]$ by open subsets of $\mathbb{R}$ has a finite subcover, and in view of the description of the subspace topology this says precisely that $[0, 1]$, with its subspace topology, is compact.

On the other hand $\mathbb{R}$ is not compact. The collection $\{(-n, n)\}_{n=1}^{\infty}$ consists of open subsets of $\mathbb{R}$ and covers $\mathbb{R}$, since every real number x satisfies $|x| < n$ for some natural number n . But no finite subcollection covers $\mathbb{R}$: a finite subcollection $\{(-n_1, n_1), \dots, (-n_k, n_k)\}$ has union $(-N, N)$, where $N = \max\{n_1, \dots, n_k\}$, and this fails to contain N .

Since $[0, 1]$ is compact and $\mathbb{R}$ is not, there is no homeomorphism of $\mathbb{R}$ onto $[0, 1]$.

(Alternatively one may argue with Corollary 21: every continuous real-valued function on the compact space $[0, 1]$ takes a minimum value, while the identity function on $\mathbb{R}$ is continuous and takes no minimum value; by Problem 34 the property that every continuous real-valued function takes a minimum value is inherited by homeomorphic spaces, so $\mathbb{R}$ and $[0, 1]$ cannot be homeomorphic.)

Problem 11.37. Let X and Y be topological spaces and consider a mapping f from X to Y . Suppose $X = X_1 \cup X_2$ and the restrictions of f to the topological subspaces X_1 and to X_2 are continuous. Show that f need not be continuous at any point in X . Show that f is continuous on X if X_1 and X_2 are open. Compare this with the case of measurable functions and the inheritance of measurability from the measurability of restrictions.

Solution. Throughout, X_1 and X_2 carry the subspace topology inherited from X , so that the open subsets of X_i are exactly the sets $\mathcal{U} \cap X_i$ with $\mathcal{U}$ open in X . Recall also that for any subset A of Y ,

$$(f|_{X_i})^{-1}(A) = f^{-1}(A) \cap X_i.$$

An example in which f is continuous at no point. Take $X = Y = \mathbb{R}$ with the usual topology, $X_1 = \mathbb{Q}$ the set of rational numbers, $X_2 = \mathbb{R} \setminus \mathbb{Q}$ the set of irrational numbers, and let f be the Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Then $X = X_1 \cup X_2$, the restriction $f|_{X_1}$ is constantly equal to 1 and the restriction $f|_{X_2}$ is constantly equal to 0. A constant mapping between topological spaces is continuous, since the preimage of an open set is either the whole space or the empty set. Thus both restrictions are continuous.

On the other hand f is continuous at no point of $\mathbb{R}$. Fix $x_0 \in \mathbb{R}$ and let $\mathcal{O}$ be the open interval of radius $1/2$ about $f(x_0)$; then $\mathcal{O}$ contains exactly one of the two values 0, 1. Every neighborhood of x_0 in $\mathbb{R}$ contains an open interval about x_0 , and every open interval contains both a rational and an irrational number, hence contains points at which f takes the value 0 and points at which it takes the value 1. So no neighborhood $\mathcal{U}$ of x_0 satisfies $f(\mathcal{U}) \subseteq \mathcal{O}$, and f fails to be continuous at x_0 . Since x_0 was arbitrary, f is continuous at no point of X .

Continuity when X_1 and X_2 are open. Assume now that X_1 and X_2 are open subsets of X . By Proposition 10 it suffices to show that $f^{-1}(\mathcal{O})$ is open in X for every open subset $\mathcal{O}$ of Y . Since $X = X_1 \cup X_2$,

$$f^{-1}(\mathcal{O}) = (f^{-1}(\mathcal{O}) \cap X_1) \cup (f^{-1}(\mathcal{O}) \cap X_2) = (f|_{X_1})^{-1}(\mathcal{O}) \cup (f|_{X_2})^{-1}(\mathcal{O}).$$

By the continuity of $f|_{X_i}$ and Proposition 10 applied to the space X_i , the set $(f|_{X_i})^{-1}(\mathcal{O})$ is open in the subspace X_i , that is, $(f|_{X_i})^{-1}(\mathcal{O}) = \mathcal{U}_i \cap X_i$ for some open subset $\mathcal{U}_i$ of X . Since X_i is open in X and the intersection of two open sets is open, $(f|_{X_i})^{-1}(\mathcal{O})$ is open in X . Being the union of two open subsets of X , $f^{-1}(\mathcal{O})$ is open in X . Therefore f is continuous on X .

(The same conclusion holds if X_1 and X_2 are both closed: Proposition 10 together with Proposition 4 and the set identity $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ shows that continuity of f is equivalent to $f^{-1}(C)$ being closed in X for every closed subset C of Y , and the displayed identity with $\mathcal{O}$ replaced by C exhibits $f^{-1}(C)$ as the union of two sets, each closed in the closed subspace X_i and hence closed in X . What matters is that the decomposition be by two sets belonging to a family closed under the relevant operations. The Dirichlet example shows that no such conclusion is available for an arbitrary decomposition.)

Comparison with measurability. Let E be a measurable set of real numbers, let D be a measurable subset of E , and let f be an extended real-valued function on E . Proposition 5(ii) of Chapter 3 asserts that f is measurable on E if and only if its restrictions to D and to $E \setminus D$ are measurable. Here no analogue of the openness assumption is needed: the two pieces may be completely arbitrary measurable sets. The reason is visible in the proof pattern used above. For a measurable subset D of E and $c \in \mathbb{R}$,

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f(x) > c\} \cup \{x \in E \setminus D \mid f(x) > c\},$$

and a subset of D that is measurable as a subset of the measure space D is measurable as a subset of $\mathbb{R}$ outright, precisely because D itself is measurable and the measurable sets form a σ -algebra. In the topological setting, by contrast, a subset of X_1 that is open in the subspace X_1 need not be open in X ; this is guaranteed only when X_1 is itself open in X . The subspace-open sets are not, in general, open

in the ambient space, whereas the “subspace-measurable” sets of a measurable set are measurable in the ambient space.

The Dirichlet function above illustrates the contrast concretely: with $E = \mathbb{R}$ and $D = \mathbb{Q}$, the restrictions of $f = \chi_{\mathbb{Q}}$ to D and $E \sim D$ are constant, hence measurable, so f is measurable on $\mathbb{R}$ by Proposition 5(ii) of Chapter 3 – even though the corresponding restrictions being continuous fails to make f continuous anywhere.

Problem 11.38. Show that for any two numbers a and b ,

$$|\operatorname{sgn}(a) \cdot |a|^2 - \operatorname{sgn}(b) \cdot |b|^2| \leq 2 \cdot |a - b| (|a| + |b|).$$

Solution. Here $\operatorname{sgn}(t)$ denotes the signum of t , that is, $\operatorname{sgn}(t) = 1$ if $t > 0$, $\operatorname{sgn}(t) = -1$ if $t < 0$, and $\operatorname{sgn}(0) = 0$.

Observe first that for every real number t ,

$$\operatorname{sgn}(t) \cdot |t|^2 = t|t|.$$

Indeed, if $t > 0$ this reads $|t|^2 = t \cdot t = t^2$; if $t < 0$ it reads $-|t|^2 = t|t| = (-|t|)|t|$; and if $t = 0$ both sides are 0. Accordingly define

$$\varphi(t) = \operatorname{sgn}(t) \cdot |t|^2 = t|t| \quad (t \in \mathbb{R}),$$

so that what must be proved is $|\varphi(a) - \varphi(b)| \leq 2|a - b|(|a| + |b|)$.

We prove the stronger inequality

$$|\varphi(a) - \varphi(b)| \leq |a - b|(|a| + |b|)$$

for all real a, b , which we refer to as the sharpened inequality; since $|a - b|(|a| + |b|) \leq 2|a - b|(|a| + |b|)$, the asserted inequality follows at once. Note that both sides of it are unchanged if a and b are interchanged, so we may relabel the two numbers freely. We distinguish two cases according to the sign of the product ab .

Case 1: $ab \geq 0$, that is, a and b do not have strictly opposite signs.

Suppose first that $a \geq 0$ and $b \geq 0$. Then $|a| = a$ and $|b| = b$, so

$$\varphi(a) - \varphi(b) = a^2 - b^2 = (a - b)(a + b),$$

and therefore

$$|\varphi(a) - \varphi(b)| = |a - b||a + b| \leq |a - b|(|a| + |b|),$$

the last step by the triangle inequality $|a + b| \leq |a| + |b|$.

Suppose next that $a \leq 0$ and $b \leq 0$. Then $|a| = -a$ and $|b| = -b$, so

$$\varphi(a) - \varphi(b) = a(-a) - b(-b) = -(a^2 - b^2) = -(a - b)(a + b),$$

and again

$$|\varphi(a) - \varphi(b)| = |a - b||a + b| \leq |a - b|(|a| + |b|).$$

Since $ab \geq 0$ forces either both $a \geq 0$ and $b \geq 0$, or both $a \leq 0$ and $b \leq 0$, Case 1 is complete.

Case 2: $ab < 0$.

By the symmetry noted above we may assume $a > 0 > b$. Then $|a| = a$, $|b| = -b$, and

$$\varphi(a) - \varphi(b) = a|a| - b|b| = a^2 + b^2 = |a|^2 + |b|^2 \geq 0,$$

so that $|\varphi(a) - \varphi(b)| = |a|^2 + |b|^2$. On the other hand, since $a > 0 > b$ we have $a - b > 0$ and

$$|a - b| = a - b = |a| + |b|.$$

Hence

$$|a - b| (|a| + |b|) = (|a| + |b|)^2 = |a|^2 + 2|a||b| + |b|^2 \geq |a|^2 + |b|^2 = |\varphi(a) - \varphi(b)|,$$

because $2|a||b| \geq 0$. This proves the sharpened inequality in Case 2.

The two cases exhaust all possibilities, so the sharpened inequality holds for all real a and b , and consequently

$$|\operatorname{sgn}(a) \cdot |a|^2 - \operatorname{sgn}(b) \cdot |b|^2| \leq |a - b| (|a| + |b|) \leq 2|a - b| (|a| + |b|),$$

as required. In particular the constant 2 in the statement is not needed; and the inequality shows that $\varphi(t) = \operatorname{sgn}(t)|t|^2$ is Lipschitz on every bounded set, hence continuous on $\mathbb{R}$.

Problem 11.39. For a second countable space X , show that X is compact if and only if it is countably compact.

Solution. Recall the two definitions. The space X is compact provided every open cover of X has a finite subcover, and X is countably compact provided every countable open cover of X has a finite subcover. Recall also that X is second countable provided there is a countable base for its topology. One implication requires no hypothesis on X . Suppose X is compact and let $\{\mathcal{O}_n\}_{n=1}^\infty$ be a countable open cover of X . This is in particular an open cover of X , so by compactness it has a finite subcover. Hence X is countably compact.

For the converse, assume X is second countable and countably compact. We first establish the Lindelof property (this is Problem 17 of this chapter), namely:

If X is second countable, then every open cover of X has a countable subcover.

To see this, let $\mathcal{B} = \{B_n\}_{n=1}^\infty$ be a countable base for the topology of X and let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X . Let

$$N = \{n \in \mathbb{N} \mid B_n \subseteq \mathcal{O}_\lambda \text{ for some } \lambda \in \Lambda\},$$

and for each $n \in N$ select an index $\lambda(n) \in \Lambda$ for which $B_n \subseteq \mathcal{O}_{\lambda(n)}$. The collection $\{\mathcal{O}_{\lambda(n)}\}_{n \in N}$ is a subcollection of the given cover indexed by a subset of $\mathbb{N}$, hence is countable. It covers X : let $x \in X$. Since $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ covers X , there is $\lambda \in \Lambda$ with $x \in \mathcal{O}_\lambda$. Since $\mathcal{B}$ is a base for the topology and $\mathcal{O}_\lambda$ is an open set containing x , there is an index n with

$$x \in B_n \subseteq \mathcal{O}_\lambda.$$

This index n belongs to N , and therefore

$$x \in B_n \subseteq \mathcal{O}_{\lambda(n)}.$$

Thus every point of X lies in some member of $\{\mathcal{O}_{\lambda(n)}\}_{n \in N}$, and the Lindelof property is proved.

Now let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary open cover of X . By the Lindelof property just established there is a countable subcollection $\{\mathcal{O}_{\lambda(n)}\}_{n \in N}$ that still covers X . This countable subcollection is a countable open cover of X , so by the assumed countable compactness of X it has a finite subcover $\{\mathcal{O}_{\lambda(n_1)}, \dots, \mathcal{O}_{\lambda(n_k)}\}$. This is a finite subcollection of the original cover $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ that covers X .

Therefore every open cover of X has a finite subcover, that is, X is compact. Combining the two implications: for a second countable space X , X is compact if and only if X is countably compact.

Problem 11.40. (Frechet Intersection Theorem) Let X be a topological space. Prove that X is countably compact if and only if whenever $\{F_n\}$ is a descending sequence of nonempty closed subsets of X , the intersection $\bigcap_{n=1}^\infty F_n$ is nonempty.

Solution. Recall that X is countably compact provided every countable open cover of X has a finite subcover, and that a sequence $\{F_n\}_{n=1}^\infty$ of sets is descending provided $F_{n+1} \subseteq F_n$ for every index n . This problem is the countable analogue of Proposition 14, and the proof runs along the same lines,

using De Morgan's Identities.

We first record an observation used in both directions. If $\{F_n\}$ is descending and $n_1 < n_2 < \cdots < n_k$, then

$$\bigcap_{i=1}^k F_{n_i} = F_{n_k},$$

since $F_{n_k} \subseteq F_{n_i}$ for each i .

Suppose first that X is countably compact, and let $\{F_n\}_{n=1}^\infty$ be a descending sequence of nonempty closed subsets of X . Argue by contradiction: assume $\bigcap_{n=1}^\infty F_n = \emptyset$. For each n set $\mathcal{O}_n = X \sim F_n$, which is open since F_n is closed. By De Morgan's Identities,

$$\bigcup_{n=1}^\infty \mathcal{O}_n = \bigcup_{n=1}^\infty (X \sim F_n) = X \sim \bigcap_{n=1}^\infty F_n = X \sim \emptyset = X.$$

Thus $\{\mathcal{O}_n\}_{n=1}^\infty$ is a countable open cover of X . By countable compactness there are indices $n_1 < n_2 < \cdots < n_k$ with

$$X = \bigcup_{i=1}^k \mathcal{O}_{n_i} = \bigcup_{i=1}^k (X \sim F_{n_i}) = X \sim \bigcap_{i=1}^k F_{n_i} = X \sim F_{n_k},$$

where the last equality uses the observation above. Hence $F_{n_k} = \emptyset$, contradicting the assumption that each F_n is nonempty. Therefore $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

Conversely, suppose that every descending sequence of nonempty closed subsets of X has nonempty intersection. Let $\{\mathcal{O}_n\}_{n=1}^\infty$ be a countable open cover of X (if the cover is finite it is its own finite subcover, so there is nothing to prove; in any case a finite cover may be turned into a sequence by repeating one of its members, and the argument below applies verbatim). Argue by contradiction: assume this cover has no finite subcover. Since any finite subcollection of $\{\mathcal{O}_n\}$ is contained in $\{\mathcal{O}_1, \dots, \mathcal{O}_n\}$ for n sufficiently large, the assumption implies that for every index n ,

$$\bigcup_{k=1}^n \mathcal{O}_k \neq X.$$

For each n define

$$F_n = X \sim \bigcup_{k=1}^n \mathcal{O}_k.$$

Each F_n is closed, being the complement of a finite union of open sets, hence the complement of an open set. Each F_n is nonempty by the displayed inequality. And the sequence is descending, since $\bigcup_{k=1}^n \mathcal{O}_k \subseteq \bigcup_{k=1}^{n+1} \mathcal{O}_k$ implies $F_{n+1} \subseteq F_n$.

By hypothesis there is a point $x \in \bigcap_{n=1}^\infty F_n$. For every index k we then have $x \in F_k$, so $x \notin \bigcup_{j=1}^k \mathcal{O}_j$ and in particular $x \notin \mathcal{O}_k$. Thus x belongs to no member of $\{\mathcal{O}_n\}_{n=1}^\infty$, contradicting the fact that this collection covers X .

Therefore every countable open cover of X has a finite subcover, that is, X is countably compact. This completes the proof of both implications.

Problem 11.41. Let X be compact Hausdorff and $\{F_n\}_{n=1}^\infty$ be a descending collection of closed subsets of X . Let $\mathcal{O}$ be a neighborhood of the intersection $\bigcap_{n=1}^\infty F_n$. Show there is an index N such that $F_n \subseteq \mathcal{O}$ for $n \geq N$.

Solution. Write $F = \bigcap_{n=1}^\infty F_n$, so that $F \subseteq \mathcal{O}$ and $\mathcal{O}$ is open. (If one uses the wider convention that a neighborhood of a set is any set containing an open set that contains it, replace $\mathcal{O}$ at the outset by an open set $\mathcal{U}$ with $F \subseteq \mathcal{U} \subseteq \mathcal{O}$; the conclusion $F_n \subseteq \mathcal{U}$ then gives $F_n \subseteq \mathcal{O}$.)

Consider the collection of subsets of X

$$\mathcal{C} = \{\mathcal{O}\} \cup \{X \sim F_n \mid n = 1, 2, \dots\}.$$

Every member of $\mathcal{C}$ is open: $\mathcal{O}$ is open by assumption, and each $X \sim F_n$ is open because F_n is closed. The collection $\mathcal{C}$ covers X . Indeed, by De Morgan's Identities,

$$\bigcup_{n=1}^{\infty} (X \sim F_n) = X \sim \bigcap_{n=1}^{\infty} F_n = X \sim F,$$

so that

$$\mathcal{O} \cup \bigcup_{n=1}^{\infty} (X \sim F_n) = \mathcal{O} \cup (X \sim F) = X,$$

the last equality because $F \subseteq \mathcal{O}$: any point of X either lies in F , and then lies in $\mathcal{O}$, or lies in $X \sim F$. Thus $\mathcal{C}$ is an open cover of the compact space X , and so it has a finite subcover. Enlarging that finite subcover by adjoining $\mathcal{O}$ if necessary (adding a set to a cover leaves it a cover), we obtain indices $n_1 < n_2 < \cdots < n_k$ such that

$$X = \mathcal{O} \cup \bigcup_{i=1}^k (X \sim F_{n_i}).$$

(In case the finite subcover uses no set of the form $X \sim F_n$ at all, we have $X = \mathcal{O}$ and the conclusion below holds trivially with $N = 1$.)

Set $N = n_k$. Since $\{F_n\}$ is descending, $F_N = F_{n_k} \subseteq F_{n_i}$ and therefore $X \sim F_{n_i} \subseteq X \sim F_N$ for each $i = 1, \dots, k$. Consequently

$$X = \mathcal{O} \cup \bigcup_{i=1}^k (X \sim F_{n_i}) \subseteq \mathcal{O} \cup (X \sim F_N) \subseteq X,$$

so that $X = \mathcal{O} \cup (X \sim F_N)$. Taking complements and using De Morgan's Identities,

$$\emptyset = X \sim [\mathcal{O} \cup (X \sim F_N)] = (X \sim \mathcal{O}) \cap F_N,$$

that is, F_N contains no point outside $\mathcal{O}$. Hence $F_N \subseteq \mathcal{O}$.

Finally, for every index $n \geq N$ the descending property gives $F_n \subseteq F_N \subseteq \mathcal{O}$. This is the assertion to be proved.

Two remarks. First, the Hausdorff assumption was not used: the conclusion holds for any compact topological space X . Second, the compactness of X may be weakened to countable compactness, since the open cover $\mathcal{C}$ constructed above is countable.

Problem 11.42. Show that it is not possible to express a closed, bounded interval of real numbers as the pairwise disjoint union of a countable collection (having more than one member) of closed, bounded intervals.

Solution. Of course the intervals in the collection are understood to be nonempty; otherwise the assertion is false for the trivial reason that one may adjoin copies of the empty set. Degenerate intervals $[c, c] = \{c\}$ are permitted; the proof below excludes them as well.

Argue by contradiction. Suppose

$$[a, b] = \bigcup_{n \in N} I_n,$$

where N is a countable index set with at least two members, each $I_n = [a_n, b_n]$ with $a_n \leq b_n$ is a nonempty closed bounded interval, and the I_n are pairwise disjoint.

Step 1: $a < b$. If $a = b$, then $[a, b]$ is a single point, and two disjoint nonempty subsets of a one-point set cannot exist. So $a < b$.

Step 2: two disjoint nonempty closed bounded intervals are ordered. Let $m \neq n$. If both $a_n \leq b_m$ and $a_m \leq b_n$ held, then $\max\{a_m, a_n\} \leq \min\{b_m, b_n\}$, and the point $\max\{a_m, a_n\}$ would belong to both I_m and I_n , contradicting disjointness. Hence either $b_m < a_n$ or $b_n < a_m$: one of the two intervals lies strictly to the left of the other.

Step 3: passage to a subinterval whose endpoints are degenerate pieces. Since N has at least two members, choose distinct indices and, by Step 2 and relabelling, obtain $p, q \in N$ with $b_p < a_q$. Put

$$\alpha = b_p, \quad \beta = a_q, \quad K = [\alpha, \beta],$$

so $\alpha < \beta$ and $K \subseteq [a, b]$. For $n \in N$ set $J_n = I_n \cap K$. Then each J_n is either empty or a closed bounded interval, since

$$[a_n, b_n] \cap [\alpha, \beta] = [\max\{a_n, \alpha\}, \min\{b_n, \beta\}]$$

whenever this is nonempty. The J_n are pairwise disjoint and their union is K . Let $M = \{n \in N : J_n \neq \emptyset\}$ and write $J_n = [c_n, d_n]$ for $n \in M$. Note M is countable and, since $\alpha \in I_p$ and $\beta \in I_q$, we have $p, q \in M$. Moreover

$$J_p = [a_p, b_p] \cap [b_p, \beta] = \{b_p\} = \{\alpha\}, \quad J_q = [a_q, b_q] \cap [\alpha, a_q] = \{a_q\} = \{\beta\}.$$

Thus $c_p = d_p = \alpha$ and $c_q = d_q = \beta$. Every point of K lies in exactly one J_n , $n \in M$.

Step 4: the set of endpoints. Put

$$G = \bigcup_{n \in M} (c_n, d_n), \quad P = K \sim G.$$

G is open, being a union of open intervals (those with $c_n = d_n$ contribute nothing), so P is a closed and bounded set of real numbers.

We claim $P = \{c_n : n \in M\} \cup \{d_n : n \in M\}$. If $x \in P$, then $x \in K$, so $x \in J_n$ for exactly one $n \in M$; as $x \notin G$ we have $x \notin (c_n, d_n)$, hence $x \in \{c_n, d_n\}$. Conversely, fix $n \in M$. Then $c_n \in J_n \subseteq K$, and $c_n \notin (c_n, d_n)$; while for $m \in M$ with $m \neq n$ we have $(c_m, d_m) \subseteq J_m$ and $J_m \cap J_n = \emptyset$, so $c_n \notin (c_m, d_m)$. Hence $c_n \notin G$, that is, $c_n \in P$; the same argument applies to d_n . This proves the claim. In particular P is nonempty, since $\alpha \in P$, and P is countable.

Step 5: P has no isolated points. Let $x \in P$ and let $n \in M$ be the unique index with $x \in J_n$; by Step 4, $x = c_n$ or $x = d_n$. Let $\varepsilon > 0$ be given. We produce a point of P different from x within distance ε of x .

Suppose first that $x = d_n$ and $d_n < \beta$. Put $c = \min\{d_n + \varepsilon/2, \beta\}$. Then $d_n < c \leq \beta$ and $c \geq \alpha$, so $c \in K$; let $m \in M$ be the index with $c \in J_m$. Since $c > d_n$ and $d_n = \max J_n$, we have $m \neq n$. If $c_m \leq d_n$, then $c_m \leq d_n < c$ with $c_m, c \in J_m$, and since J_m is an interval this forces $d_n \in J_m$, contradicting $J_m \cap J_n = \emptyset$. Hence $c_m > d_n = x$. Also $c_m \leq c \leq d_n + \varepsilon/2$. Therefore $c_m \in P$, $c_m \neq x$, and $|c_m - x| \leq \varepsilon/2 < \varepsilon$.

Suppose next that $x = c_n$ and $c_n > \alpha$. Put $c = \max\{c_n - \varepsilon/2, \alpha\}$. Then $\alpha \leq c < c_n \leq \beta$, so $c \in K$; let $m \in M$ be the index with $c \in J_m$. Since $c < c_n = \min J_n$ we have $m \neq n$. If $d_m \geq c_n$, then $c < c_n \leq d_m$ with $c, d_m \in J_m$, so the interval J_m would contain c_n , contradicting disjointness. Hence $d_m < c_n = x$, while $d_m \geq c \geq c_n - \varepsilon/2$. Therefore $d_m \in P$, $d_m \neq x$, and $|d_m - x| \leq \varepsilon/2 < \varepsilon$.

These two cases cover every point of P . Indeed, let $x \in P$, so $x = c_n$ or $x = d_n$. If $x = d_n$ with $d_n < \beta$, the first case applies. If $x = d_n = \beta$, then $\beta \in J_n \cap J_q$, so $n = q$ and hence $c_n = d_n = \beta > \alpha$; thus $x = c_n$ with $c_n > \alpha$ and the second case applies. If $x = c_n$ with $c_n > \alpha$, the second case applies. If $x = c_n = \alpha$, then $\alpha \in J_n \cap J_p$, so $n = p$ and hence $c_n = d_n = \alpha < \beta$, and the first case applies. Thus every point of P is a limit point of P : the set P has no isolated points.

Step 6: the Baire Category Theorem yields a contradiction. Regard P as a metric space with the metric inherited from $\mathbb{R}$. Then P is complete: a Cauchy sequence in P is Cauchy in $\mathbb{R}$, hence converges to some real number by the completeness of $\mathbb{R}$, and this limit belongs to P because P is closed. Also $P \neq \emptyset$.

For $n \in M$ put $E_n = P \cap J_n$. By Step 4, $E_n \subseteq \{c_n, d_n\}$, so E_n is a finite set; being finite, E_n is a closed subset of P . We show each E_n is hollow in P , that is, has empty interior in P . Suppose not, and let x be a point of P with $B(x, \delta) \cap P \subseteq E_n$ for some $\delta > 0$, where $B(x, \delta)$ denotes the open interval of radius δ about x . Then $B(x, \delta) \cap P$ is a finite set. By Step 5, x is a limit point of P , so the set

$$S = (B(x, \delta) \cap P) \sim \{x\}$$

is nonempty, and it is finite. Put $\delta' = \min\{|x - y| : y \in S\} > 0$ and $\delta'' = \min\{\delta, \delta'\} > 0$. Then $B(x, \delta'') \cap P = \{x\}$, so x is an isolated point of P , contradicting Step 5. Hence each E_n is a closed hollow subset of P .

Finally, since $K = \bigcup_{n \in M} J_n$ and $P \subseteq K$,

$$P = \bigcup_{n \in M} (P \cap J_n) = \bigcup_{n \in M} E_n,$$

a countable union of closed hollow subsets of the complete metric space P . By the Baire Category Theorem of Section 10.2, part (ii), the union $\bigcup_{n \in M} E_n$ is hollow in P ; that is, P has empty interior as a subset of itself. But P is open in P , so the interior of P in P is P itself, which is nonempty. This contradiction completes the proof.

Therefore a closed, bounded interval of real numbers cannot be written as the pairwise disjoint union of a countable collection, having more than one member, of nonempty closed bounded intervals.

11.7 Problems 11.43–11.49

Problem 11.43. Let f be a continuous mapping of the compact space X onto the Hausdorff space Y . Show that any mapping g of Y into Z for which $g \circ f$ is continuous must itself be continuous.

Solution. Let Z be a topological space, let $f: X \rightarrow Y$ be continuous and onto with X compact and Y Hausdorff, and let $g: Y \rightarrow Z$ be a mapping for which $g \circ f$ is continuous. To prove that g is continuous it suffices to show that $g^{-1}(W)$ is open in Y for every open subset W of Z , and this is equivalent to showing that $g^{-1}(V)$ is closed in Y for every closed subset V of Z , since $g^{-1}(Z \setminus V) = Y \setminus g^{-1}(V)$. So let V be a closed subset of Z and put

$$C = g^{-1}(V) \subseteq Y.$$

Since $g \circ f$ is continuous and V is closed, the preimage

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}(C)$$

is a closed subset of X . Since X is compact, Proposition 15 tells us that the closed set $f^{-1}(C)$ is compact. Therefore, by Proposition 20, its continuous image $f(f^{-1}(C))$ is a compact subspace of Y . Since Y is Hausdorff, Proposition 16 tells us that $f(f^{-1}(C))$ is a closed subset of Y .

Finally we use the surjectivity of f . In general $f(f^{-1}(C)) \subseteq C$. Conversely, if $y \in C$, then, since f maps X onto Y , there is a point $x \in X$ with $f(x) = y$; as $f(x) = y \in C$ we have $x \in f^{-1}(C)$ and hence $y = f(x) \in f(f^{-1}(C))$. Therefore

$$f(f^{-1}(C)) = C.$$

Consequently $C = g^{-1}(V)$ is a closed subset of Y .

Since V was an arbitrary closed subset of Z , the preimage under g of each closed set is closed, and hence the preimage under g of each open set is open. Thus g is continuous.

Observe what the argument really shows: a continuous mapping f of a compact space onto a Hausdorff space is a quotient map, that is, a subset C of Y is closed provided $f^{-1}(C)$ is closed, and it is exactly this property that forces g to be continuous.

Problem 11.44. Let $(X, \mathcal{T})$ be a topological space.

- (i) Prove that if $(X, \mathcal{T})$ is compact, then $(X, \mathcal{T}_1)$ is compact for any topology $\mathcal{T}_1$ weaker than $\mathcal{T}$.
- (ii) Show that if $(X, \mathcal{T})$ is Hausdorff, then $(X, \mathcal{T}_2)$ is Hausdorff for any topology $\mathcal{T}_2$ stronger than $\mathcal{T}$.
- (iii) Show that if $(X, \mathcal{T})$ is compact and Hausdorff, then any strictly weaker topology is not Hausdorff and any strictly stronger topology is not compact.

Solution. Recall that a topology $\mathcal{T}_1$ on X is called weaker than $\mathcal{T}$ provided $\mathcal{T}_1 \subseteq \mathcal{T}$, and $\mathcal{T}_2$ is called stronger than $\mathcal{T}$ provided $\mathcal{T} \subseteq \mathcal{T}_2$; the adjective “strictly” means that the containment is proper.

(i) Assume $(X, \mathcal{T})$ is compact and $\mathcal{T}_1 \subseteq \mathcal{T}$. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be a cover of X by sets belonging to $\mathcal{T}_1$. Since $\mathcal{T}_1 \subseteq \mathcal{T}$, each $\mathcal{O}_\lambda$ belongs to $\mathcal{T}$, so $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is also an open cover of the compact space $(X, \mathcal{T})$. Hence there are finitely many indices $\lambda_1, \dots, \lambda_n$ with $X = \bigcup_{k=1}^n \mathcal{O}_{\lambda_k}$. This is a finite subcover of the original $\mathcal{T}_1$ -open cover. Since every $\mathcal{T}_1$ -open cover of X has a finite subcover, $(X, \mathcal{T}_1)$ is compact.

(ii) Assume $(X, \mathcal{T})$ is Hausdorff and $\mathcal{T} \subseteq \mathcal{T}_2$. Let u and v be distinct points of X . Since $(X, \mathcal{T})$ is Hausdorff, there are disjoint sets $\mathcal{O}_u, \mathcal{O}_v \in \mathcal{T}$ with $u \in \mathcal{O}_u$ and $v \in \mathcal{O}_v$. Since $\mathcal{T} \subseteq \mathcal{T}_2$, the sets $\mathcal{O}_u$ and $\mathcal{O}_v$ also belong to $\mathcal{T}_2$. Thus u and v are separated by disjoint $\mathcal{T}_2$ -open sets, and $(X, \mathcal{T}_2)$ is Hausdorff.

(iii) Assume $(X, \mathcal{T})$ is compact and Hausdorff. The tool for both assertions is Proposition 19: a continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism. We apply it to the identity mapping $\text{Id}: X \rightarrow X$, $\text{Id}(x) = x$, which is one-to-one and onto. Observe that Id , regarded as a mapping from $(X, \mathcal{T}_a)$ to $(X, \mathcal{T}_b)$, is continuous if and only if $\mathcal{T}_b \subseteq \mathcal{T}_a$, since $\text{Id}^{-1}(\mathcal{O}) = \mathcal{O}$; and it is a homeomorphism if and only if $\mathcal{T}_a = \mathcal{T}_b$.

First let $\mathcal{T}_1$ be a strictly weaker topology, so $\mathcal{T}_1 \subsetneq \mathcal{T}$, and suppose, to get a contradiction, that $(X, \mathcal{T}_1)$ is Hausdorff. Since $\mathcal{T}_1 \subseteq \mathcal{T}$, the identity map

$$\text{Id}: (X, \mathcal{T}) \rightarrow (X, \mathcal{T}_1)$$

is continuous; it is one-to-one and onto; its domain $(X, \mathcal{T})$ is compact and its target $(X, \mathcal{T}_1)$ is assumed Hausdorff. By Proposition 19, Id is a homeomorphism, so it carries $\mathcal{T}$ -open sets to $\mathcal{T}_1$ -open sets, that is, $\mathcal{T} \subseteq \mathcal{T}_1$. Together with $\mathcal{T}_1 \subseteq \mathcal{T}$ this gives $\mathcal{T}_1 = \mathcal{T}$, contradicting strictness. Hence no strictly weaker topology on X is Hausdorff.

Now let $\mathcal{T}_2$ be a strictly stronger topology, so $\mathcal{T} \subsetneq \mathcal{T}_2$, and suppose, to get a contradiction, that $(X, \mathcal{T}_2)$ is compact. Since $\mathcal{T} \subseteq \mathcal{T}_2$, the identity map

$$\text{Id}: (X, \mathcal{T}_2) \rightarrow (X, \mathcal{T})$$

is continuous; it is one-to-one and onto; its domain $(X, \mathcal{T}_2)$ is assumed compact and its target $(X, \mathcal{T})$ is Hausdorff. By Proposition 19, Id is a homeomorphism, whence $\mathcal{T}_2 \subseteq \mathcal{T}$ and therefore $\mathcal{T}_2 = \mathcal{T}$, again contradicting strictness. Hence no strictly stronger topology on X is compact.

In summary, a compact Hausdorff topology is minimal among Hausdorff topologies and maximal among compact topologies. Note also that (i) and (ii) show that the two failures in (iii) are the only ones possible: a weaker topology automatically remains compact and a stronger one automatically remains Hausdorff.

Problem 11.45. (The Compact-Open Topology) Let X and Y be Hausdorff topological spaces and Y^X the collection of maps from X into Y . On Y^X we define a topology, called the compact-open topology, by taking as a subbase sets of the form $\mathcal{U}_{K, \mathcal{O}} = \{f: X \rightarrow Y \mid f(K) \subseteq \mathcal{O}\}$, where K is a compact subset of X and $\mathcal{O}$ is an open subset of Y . Thus the compact-open topology is the weakest topology on Y^X such that the sets $\mathcal{U}_{K, \mathcal{O}}$ are open.

(i) Let $\{f_n\}$ be a sequence in Y^X that converges with respect to the compact-open topology to $f \in Y^X$. Show that $\{f_n\}$ converges pointwise to f on X .

(ii) Now assume that Y is a metric space. Show that a sequence $\{f_n\}$ in Y^X converges with respect to the compact-open topology to $f \in Y^X$ if and only if $\{f_n\}$ converges to f uniformly on each compact subset K of X .

Solution. Throughout, recall that a sequence $\{f_n\}$ in a topological space converges to f provided that for each neighborhood $\mathcal{U}$ of f there is an index N for which $f_n \in \mathcal{U}$ for all $n \geq N$. Since the sets $\mathcal{U}_{K, \mathcal{O}}$ form a subbase for the compact-open topology, the finite intersections

$$\mathcal{U}_{K_1, \mathcal{O}_1} \cap \cdots \cap \mathcal{U}_{K_m, \mathcal{O}_m}$$

form a base. Hence $f_n \rightarrow f$ in the compact-open topology if and only if for every finite family of pairs $(K_i, \mathcal{O}_i)$, $1 \leq i \leq m$, with K_i compact, $\mathcal{O}_i$ open and $f(K_i) \subseteq \mathcal{O}_i$, there is an index N with $f_n(K_i) \subseteq \mathcal{O}_i$ for all $n \geq N$ and all i . Since an index that works for each of finitely many subbasic sets separately

also works for their intersection, this is in turn equivalent to the single-set condition:

for every compact $K \subseteq X$ and open $\mathcal{O} \subseteq Y$ with $f(K) \subseteq \mathcal{O}$, there is N with $f_n(K) \subseteq \mathcal{O}$ for $n \geq N$. (*)

(i) Assume $f_n \rightarrow f$ in the compact-open topology and fix $x \in X$. Let $\mathcal{O}$ be any open subset of Y containing $f(x)$. The singleton $K = \{x\}$ is a compact subset of X , and $f(K) = \{f(x)\} \subseteq \mathcal{O}$, so $\mathcal{U}_{K, \mathcal{O}}$ is a subbasic open set containing f , hence a neighborhood of f . Therefore there is an index N such that $f_n \in \mathcal{U}_{K, \mathcal{O}}$ for all $n \geq N$, that is, $f_n(x) \in \mathcal{O}$ for all $n \geq N$. Since $\mathcal{O}$ was an arbitrary open set containing $f(x)$, the sequence $\{f_n(x)\}$ converges to $f(x)$ in Y . As $x \in X$ was arbitrary, $\{f_n\}$ converges pointwise to f .

(ii) Let ρ denote the metric on Y and write $B(y, r) = \{z \in Y \mid \rho(z, y) < r\}$. We prove the asserted equivalence for a limit function f that is continuous; at the end we show by examples that continuity of f cannot be dispensed with, so that this is the correct reading of the assertion.

First assume that $\{f_n\}$ converges to f uniformly on each compact subset of X ; we verify (*). Let $K \subseteq X$ be compact and $\mathcal{O} \subseteq Y$ open with $f(K) \subseteq \mathcal{O}$. Since f is continuous and K is compact, $f(K)$ is compact by Proposition 20. We claim there is an $\epsilon > 0$ such that

$$\{z \in Y \mid \rho(z, y) < \epsilon \text{ for some } y \in f(K)\} \subseteq \mathcal{O}.$$

Indeed, for each $y \in f(K)$, since $\mathcal{O}$ is open and $y \in \mathcal{O}$, there is $r_y > 0$ with $B(y, 2r_y) \subseteq \mathcal{O}$. The collection $\{B(y, r_y)\}_{y \in f(K)}$ is an open cover of the compact set $f(K)$, so there are $y_1, \dots, y_p$ in $f(K)$ with $f(K) \subseteq \bigcup_{j=1}^p B(y_j, r_{y_j})$. Put $\epsilon = \min\{r_{y_1}, \dots, r_{y_p}\} > 0$. If $\rho(z, y) < \epsilon$ for some $y \in f(K)$, choose j with $\rho(y, y_j) < r_{y_j}$; then

$$\rho(z, y_j) \leq \rho(z, y) + \rho(y, y_j) < \epsilon + r_{y_j} \leq 2r_{y_j},$$

so $z \in B(y_j, 2r_{y_j}) \subseteq \mathcal{O}$. This proves the claim. (If $K = \emptyset$ there is nothing to prove.)

By the assumed uniform convergence on the compact set K , there is an index N such that $\rho(f_n(x), f(x)) < \epsilon$ for all $x \in K$ and all $n \geq N$. For such n and x , the point $z = f_n(x)$ is within ϵ of the point $f(x) \in f(K)$, so $f_n(x) \in \mathcal{O}$ by the claim. Thus $f_n(K) \subseteq \mathcal{O}$ for $n \geq N$, which is (*). Hence $f_n \rightarrow f$ in the compact-open topology.

Conversely, assume $f_n \rightarrow f$ in the compact-open topology. Let K be a compact subset of X and let $\epsilon > 0$. We produce a basic neighborhood of f that pins down f on all of K to within ϵ . Since X is Hausdorff and K is compact, K with its subspace topology is a compact Hausdorff space, and therefore is normal, by Theorem 18; in particular K is regular. For each $x \in K$ the set

$$V_x = K \cap f^{-1}(B(f(x), \epsilon/3))$$

is relatively open in K and contains x , since f is continuous. By the regularity of K there is a relatively open subset W_x of K with

$$x \in W_x \subseteq \overline{W_x} \subseteq V_x,$$

where the closure is taken in K . The sets $\{W_x\}_{x \in K}$ form a relatively open cover of the compact space K , so there are points $x_1, \dots, x_m$ in K with $K = \bigcup_{i=1}^m W_{x_i}$. Put $K_i = \overline{W_{x_i}}$, the closure taken in K . Each K_i is a closed subset of the compact space K , hence compact by Proposition 15, and therefore is a compact subset of X . Moreover

$$f(K_i) \subseteq f(V_{x_i}) \subseteq B(f(x_i), \epsilon/3) \subseteq \mathcal{O}_i := B(f(x_i), \epsilon/2),$$

and each $\mathcal{O}_i$ is open in Y . Therefore f belongs to the basic open set $\bigcap_{i=1}^m \mathcal{U}_{K_i, \mathcal{O}_i}$, which is thus a neighborhood of f in the compact-open topology. Choose N so that $f_n \in \bigcap_{i=1}^m \mathcal{U}_{K_i, \mathcal{O}_i}$ for all $n \geq N$. Let $n \geq N$ and $x \in K$. Since the W_{x_i} cover K , there is an index i with $x \in W_{x_i} \subseteq K_i$. Then $f_n(x) \in \mathcal{O}_i$, so $\rho(f_n(x), f(x_i)) < \epsilon/2$, while $x \in K_i \subseteq V_{x_i}$ gives $\rho(f(x), f(x_i)) < \epsilon/3$. By the triangle inequality,

$$\rho(f_n(x), f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{3} < \epsilon.$$

Since this holds for every $x \in K$ and every $n \geq N$, the sequence $\{f_n\}$ converges to f uniformly on K . As K was an arbitrary compact subset of X , the proof of the equivalence is complete.

A remark on the hypothesis of continuity. If arbitrary (possibly discontinuous) members of Y^X are allowed, both implications fail, so the equivalence in (ii) must be understood for a continuous limit function f , as in the classical setting where the compact-open topology is placed on the space of continuous maps. Take $X = [0, 1]$ and $Y = \mathbb{R}$, both Hausdorff.

For the failure of “uniform convergence on compact sets implies compact-open convergence”, let $f(x) = x$ for $0 \leq x < 1$ and $f(1) = 0$, and let $f_n = f + 1/n$. Then $\{f_n\}$ converges to f uniformly on all of X . But with $K = [0, 1]$ and $\mathcal{O} = (-1, 1)$ we have $f(K) = [0, 1) \subseteq \mathcal{O}$, whereas $f_n(1 - 1/(2n)) = 1 + 1/(2n) \notin \mathcal{O}$, so $f_n \notin \mathcal{U}_{K, \mathcal{O}}$ for every n ; thus $\{f_n\}$ does not converge to f in the compact-open topology.

For the failure of the reverse implication, let $f(0) = 1$ and $f(x) = 0$ for $0 < x \leq 1$, and let f_n agree with f except that $f_n(1/n) = 1$. Then $\sup_{x \in [0, 1]} |f_n(x) - f(x)| = 1$ for every n , so there is no uniform convergence on the compact set $[0, 1]$. Nevertheless $f_n \rightarrow f$ in the compact-open topology: let K be compact with $f(K) \subseteq \mathcal{O}$, $\mathcal{O}$ open. Since f_n differs from f only by taking the value 1 at one additional point, $f_n(K) \subseteq f(K) \cup \{1\}$ for every n . If $0 \in K$, then $1 = f(0) \in f(K) \subseteq \mathcal{O}$, so $f_n(K) \subseteq f(K) \cup \{1\} \subseteq \mathcal{O}$ for all n . If $0 \notin K$, then K is a closed subset of $[0, 1]$ missing 0, so $K \subseteq [\delta, 1]$ for some $\delta > 0$; hence $1/n \notin K$ once $n > 1/\delta$, and then $f_n(K) = f(K) = \{0\} \subseteq \mathcal{O}$. Thus $(*)$ holds, and $f_n \rightarrow f$ in the compact-open topology.

Problem 11.46. (Dini’s Theorem) Let $\{f_n\}$ be a sequence of continuous real-valued functions on a countably compact space X . Suppose that for each $x \in X$, the sequence $\{f_n(x)\}$ decreases monotonically to zero. Show that $\{f_n\}$ converges to zero uniformly.

Solution. Recall that X is countably compact provided every countable open cover of X has a finite subcover.

We first record an elementary consequence of the hypothesis: for each $x \in X$ and each index n we have $f_n(x) \geq 0$. Indeed, the sequence of real numbers $\{f_n(x)\}$ is decreasing with limit 0, so for $m \geq n$ we have $f_n(x) \geq f_m(x)$, and letting $m \rightarrow \infty$ gives $f_n(x) \geq 0$. Thus $|f_n(x)| = f_n(x)$ throughout.

Let $\epsilon > 0$. For each index n define

$$\mathcal{O}_n = \{x \in X \mid f_n(x) < \epsilon\} = f_n^{-1}((-\infty, \epsilon)).$$

Since f_n is continuous and $(-\infty, \epsilon)$ is an open subset of $\mathbb{R}$, each $\mathcal{O}_n$ is an open subset of X .

The collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is ascending: if $x \in \mathcal{O}_n$, then, by the monotone decrease of $\{f_k(x)\}$, we have $f_{n+1}(x) \leq f_n(x) < \epsilon$, so $x \in \mathcal{O}_{n+1}$. Hence

$$\mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3 \subseteq \cdots$$

The collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is a cover of X : let $x \in X$; since $\{f_n(x)\}$ converges to 0, there is an index n with $f_n(x) < \epsilon$, that is, $x \in \mathcal{O}_n$.

Thus $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is a countable open cover of X . Since X is countably compact, there are finitely many indices $n_1, \dots, n_k$ with

$$X = \bigcup_{j=1}^k \mathcal{O}_{n_j}.$$

Let $N = \max\{n_1, \dots, n_k\}$. Since the collection is ascending, $\mathcal{O}_{n_j} \subseteq \mathcal{O}_N$ for each j , and therefore $X = \mathcal{O}_N$. Ascendingness also gives $\mathcal{O}_N \subseteq \mathcal{O}_n$ for every $n \geq N$, so

$$X = \mathcal{O}_n \quad \text{for all } n \geq N.$$

By the definition of $\mathcal{O}_n$ this says precisely that

$$0 \leq f_n(x) < \epsilon \quad \text{for all } x \in X \text{ and all } n \geq N,$$

where the left inequality is the observation made at the outset. Hence

$$\sup_{x \in X} |f_n(x) - 0| \leq \epsilon \quad \text{for all } n \geq N.$$

Since $\epsilon > 0$ was arbitrary, $\{f_n\}$ converges to 0 uniformly on X .

Two remarks on the necessity of the hypotheses. Monotonicity is essential: on the compact space $X = [0, 1]$, for $n \geq 2$ let f_n be the “tent” function that is piecewise linear with $f_n(0) = 0$, $f_n(1/n) = 1$, $f_n(2/n) = 0$, and $f_n \equiv 0$ on $[2/n, 1]$. Each f_n is continuous and $\{f_n(x)\}$ converges to 0 for each x , since $f_n(x) = 0$ once $n > 2/x$ and $f_n(0) = 0$ for all n ; but $\sup_{x \in X} f_n(x) = 1$ for every n , so the convergence is not uniform. Of course $\{f_n(x)\}$ does not decrease monotonically. Countable compactness is likewise essential: on $X = (0, 1)$, the functions $f_n(x) = \min\{1, 1/(nx)\}$ decrease pointwise to 0 but $\sup_X f_n = 1$ for every n .

Problem 11.47. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a collection of connected subsets of a topological space X and suppose that any two of them have a point in common. Show that the union of $\{C_\lambda\}_{\lambda \in \Lambda}$ also is connected.

Solution. Recall the criterion stated in the text: a subset E of X is connected provided there do not exist open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ of X for which

$$\mathcal{O}_1 \cap E \neq \emptyset, \quad \mathcal{O}_2 \cap E \neq \emptyset, \quad E \subseteq \mathcal{O}_1 \cup \mathcal{O}_2, \quad \text{and} \quad E \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset.$$

Put $C = \bigcup_{\lambda \in \Lambda} C_\lambda$ and argue by contradiction: assume C is not connected, so there are open subsets $\mathcal{O}_1, \mathcal{O}_2$ of X with

$$\mathcal{O}_1 \cap C \neq \emptyset, \quad \mathcal{O}_2 \cap C \neq \emptyset, \quad C \subseteq \mathcal{O}_1 \cup \mathcal{O}_2, \quad C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset.$$

Fix any index $\lambda \in \Lambda$. Since $C_\lambda \subseteq C$, we have

$$C_\lambda \subseteq \mathcal{O}_1 \cup \mathcal{O}_2 \quad \text{and} \quad C_\lambda \cap \mathcal{O}_1 \cap \mathcal{O}_2 \subseteq C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset.$$

Thus $\mathcal{O}_1, \mathcal{O}_2$ satisfy three of the four conditions above with $E = C_\lambda$. Since C_λ is connected, the remaining condition must fail, that is, $C_\lambda \cap \mathcal{O}_1 = \emptyset$ or $C_\lambda \cap \mathcal{O}_2 = \emptyset$. In the first case $C_\lambda \subseteq \mathcal{O}_2$ and in the second $C_\lambda \subseteq \mathcal{O}_1$, because $C_\lambda \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$. Hence:

$$\text{for each } \lambda \in \Lambda, \text{ either } C_\lambda \subseteq \mathcal{O}_1 \text{ or } C_\lambda \subseteq \mathcal{O}_2. \quad (\dagger)$$

Since $\mathcal{O}_1 \cap C \neq \emptyset$, there is a point $p_1 \in \mathcal{O}_1 \cap C$, and since C is the union of the C_λ 's there is an index λ_1 with $p_1 \in C_{\lambda_1}$. Then $C_{\lambda_1} \cap \mathcal{O}_1 \neq \emptyset$, so by $(\dagger)$ and the disjointness relation $C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ we must have $C_{\lambda_1} \subseteq \mathcal{O}_1$: indeed, if instead $C_{\lambda_1} \subseteq \mathcal{O}_2$, then $p_1 \in C \cap \mathcal{O}_1 \cap \mathcal{O}_2$, which is empty. Symmetrically, there is an index λ_2 with $C_{\lambda_2} \subseteq \mathcal{O}_2$ and $C_{\lambda_2} \neq \emptyset$.

By hypothesis any two members of the collection have a point in common, so choose $q \in C_{\lambda_1} \cap C_{\lambda_2}$. (If $\lambda_1 = \lambda_2$ this set is C_{λ_1} , which is nonempty since it contains p_1 , so such a q exists in every case.) Then

$$q \in C_{\lambda_1} \subseteq \mathcal{O}_1, \quad q \in C_{\lambda_2} \subseteq \mathcal{O}_2, \quad q \in C,$$

so $q \in C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. This contradiction shows that no such pair $\mathcal{O}_1, \mathcal{O}_2$ exists.

Therefore $C = \bigcup_{\lambda \in \Lambda} C_\lambda$ is connected.

Problem 11.48. Let A be a connected subset of a topological space X , and suppose $A \subseteq B \subseteq \overline{A}$. Show that B is connected.

Solution. We again use the criterion recorded in the text: a subset E of X is connected provided there do not exist open subsets $\mathcal{O}_1, \mathcal{O}_2$ of X with $\mathcal{O}_1 \cap E \neq \emptyset$, $\mathcal{O}_2 \cap E \neq \emptyset$, $E \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ and $E \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. Argue by contradiction. Suppose B is not connected, and let $\mathcal{O}_1, \mathcal{O}_2$ be open subsets of X with

$$\mathcal{O}_1 \cap B \neq \emptyset, \quad \mathcal{O}_2 \cap B \neq \emptyset, \quad B \subseteq \mathcal{O}_1 \cup \mathcal{O}_2, \quad B \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset.$$

Since $A \subseteq B$, we have $A \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ and $A \cap \mathcal{O}_1 \cap \mathcal{O}_2 \subseteq B \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. Because A is connected, the pair $\mathcal{O}_1, \mathcal{O}_2$ cannot meet both of the remaining requirements, so at least one of $A \cap \mathcal{O}_1$ and $A \cap \mathcal{O}_2$ is

empty. Relabeling if necessary, assume

$$A \cap \mathcal{O}_2 = \emptyset.$$

Then $A \subseteq X \sim \mathcal{O}_2$. The set $X \sim \mathcal{O}_2$ is closed, since $\mathcal{O}_2$ is open, and the closure $\overline{A}$ is the smallest closed set containing A . Therefore

$$\overline{A} \subseteq X \sim \mathcal{O}_2.$$

Since $B \subseteq \overline{A}$, it follows that $B \subseteq X \sim \mathcal{O}_2$, that is,

$$B \cap \mathcal{O}_2 = \emptyset.$$

This contradicts the requirement $\mathcal{O}_2 \cap B \neq \emptyset$.

Hence no such pair of open sets exists and B is connected. In particular, taking $B = \overline{A}$, the closure of a connected set is connected.

Problem 11.49. Show that the following subset of the plane is connected but not arcwise connected.

$$X = \{(x, y) \mid x = 0, -1 \leq y \leq 1\} \cup \{(x, y) \mid y = \sin 1/x, 0 < x \leq 1\}.$$

Solution. Write

$$S = \{(0, y) \mid -1 \leq y \leq 1\}, \quad T = \{(x, \sin 1/x) \mid 0 < x \leq 1\}, \quad X = S \cup T,$$

and give X the subspace topology inherited from the plane $\mathbb{R}^2$.

Part 1: X is connected.

The set T is connected. Indeed, define $\varphi: (0, 1] \rightarrow \mathbb{R}^2$ by $\varphi(x) = (x, \sin 1/x)$. Each coordinate function is continuous on $(0, 1]$, so φ is continuous, and $\varphi((0, 1]) = T$. The interval $(0, 1]$ is a connected subset of $\mathbb{R}$, by the equivalence (1) of Section 11.6, which identifies the connected sets of real numbers with the intervals. Hence, by Proposition 22, the image $T = \varphi((0, 1])$ is connected.

Next we show that S is contained in the closure of T ; equivalently, $X \subseteq \overline{T}$, since $T \subseteq \overline{T}$. Let $(0, y_0) \in S$, so $-1 \leq y_0 \leq 1$. Choose $t_0 \in \mathbb{R}$ with $\sin t_0 = y_0$, which is possible since $\sin$ maps $\mathbb{R}$ onto $[-1, 1]$. For each natural number n large enough that $t_0 + 2\pi n > 1$, set

$$x_n = \frac{1}{t_0 + 2\pi n} \in (0, 1], \quad \text{so that} \quad \sin \frac{1}{x_n} = \sin(t_0 + 2\pi n) = \sin t_0 = y_0.$$

Thus $(x_n, y_0) \in T$ for all such n , and $x_n \rightarrow 0$, so $(x_n, y_0) \rightarrow (0, y_0)$ in $\mathbb{R}^2$. Every neighborhood of $(0, y_0)$ therefore contains points of T , and hence $(0, y_0) \in \overline{T}$. Since $(0, y_0) \in S$ was arbitrary, $S \subseteq \overline{T}$ and therefore

$$T \subseteq X \subseteq \overline{T},$$

the closure being taken in $\mathbb{R}^2$. By Problem 48, applied in the topological space $\mathbb{R}^2$ with $A = T$ and $B = X$, the set X is a connected subset of $\mathbb{R}^2$; since connectedness of a subset means connectedness of the corresponding subspace, X is a connected topological space.

Part 2: X is not arcwise connected.

Recall that X is arcwise connected provided for each pair of points $u, v \in X$ there is a continuous map $f: [0, 1] \rightarrow X$ with $f(0) = u$ and $f(1) = v$. We show that no such map exists for

$$u = (0, 0) \in S \quad \text{and} \quad v = (1, \sin 1) \in T.$$

Suppose, to get a contradiction, that $f: [0, 1] \rightarrow X$ is continuous with $f(0) = u$ and $f(1) = v$. Write $f(t) = (x(t), y(t))$; since the two coordinate projections of $\mathbb{R}^2$ onto $\mathbb{R}$ are continuous, x and y are continuous real-valued functions on $[0, 1]$. Note that a point of X lies in S exactly when its first coordinate is 0, and lies in T exactly when its first coordinate is positive, in which case its second coordinate equals the sine of the reciprocal of its first coordinate.

Consider

$$A = f^{-1}(S) = \{t \in [0, 1] \mid x(t) = 0\} = x^{-1}(\{0\}).$$

Since x is continuous and $\{0\}$ is closed in $\mathbb{R}$, the set A is a closed subset of $[0, 1]$. It is nonempty, since $x(0) = 0$, and it is not all of $[0, 1]$, since $x(1) = 1$. Let

$$c = \sup A.$$

Since A is closed and bounded and nonempty, $c \in A$, so $x(c) = 0$; and $c < 1$ since $1 \notin A$. If $t \in (c, 1]$ then $t \notin A$, so $x(t) \neq 0$; but $f(t)$ belongs to $X = S \cup T$ and every point of X has nonnegative first coordinate, so $x(t) \geq 0$. Hence

$$x(t) > 0 \quad \text{for all } t \in (c, 1],$$

so $f(t) \in T$ for all $t \in (c, 1]$, and consequently

$$y(t) = \sin \frac{1}{x(t)} \quad \text{for all } t \in (c, 1]. \quad (*)$$

Since y is continuous at c , there is a $\delta > 0$ with $c + \delta \leq 1$ such that

$$|y(t) - y(c)| < \frac{1}{2} \quad \text{for all } t \in [c, c + \delta]. \quad (**)$$

Put $a = x(c + \delta) > 0$, since $c + \delta \in (c, 1]$. The closed interval $[c, c + \delta]$ is a connected topological space, again by the equivalence (1) of Section 11.6, so by Proposition 23 it has the intermediate value property: the image $x([c, c + \delta])$ is an interval. This interval contains $x(c) = 0$ and $x(c + \delta) = a$, hence it contains all of $[0, a]$.

Choose a natural number n so large that

$$u_n := \frac{1}{\pi/2 + 2\pi n} \leq a \quad \text{and} \quad v_n := \frac{1}{3\pi/2 + 2\pi n} \leq a,$$

which is possible since both quantities tend to 0 as $n \rightarrow \infty$. Both u_n and v_n lie in $(0, a]$, so by the previous paragraph there are points $s, t \in [c, c + \delta]$ with $x(s) = u_n$ and $x(t) = v_n$. Since $x(s) > 0$ and $x(t) > 0$, neither s nor t equals c , so both lie in $(c, c + \delta]$, and $(*)$ applies:

$$y(s) = \sin \frac{1}{u_n} = \sin \left(\frac{\pi}{2} + 2\pi n \right) = 1, \quad y(t) = \sin \frac{1}{v_n} = \sin \left(\frac{3\pi}{2} + 2\pi n \right) = -1.$$

On the other hand, s and t both belong to $[c, c + \delta]$, so by $(**)$,

$$|y(s) - y(t)| \leq |y(s) - y(c)| + |y(c) - y(t)| < \frac{1}{2} + \frac{1}{2} = 1.$$

But $|y(s) - y(t)| = |1 - (-1)| = 2$, and $2 < 1$ is false. This contradiction shows that no continuous $f: [0, 1] \rightarrow X$ joins $(0, 0)$ to $(1, \sin 1)$.

Therefore X is connected but not arcwise connected. Together with Problem 50, which asserts that every arcwise connected space is connected, this shows that arcwise connectedness is strictly stronger than connectedness, as remarked in the text.

11.8 Problems 11.50–11.56

Problem 11.50. Show that an arcwise connected topological space X is connected. Also show that each connected open subset $\mathcal{O}$ of a Euclidean space $\mathbb{R}^n$ is arcwise connected. (Hint: Let x belong to $\mathcal{O}$. Define C to be the set of points in $\mathcal{O}$ that can be connected in $\mathcal{O}$ to x by a piecewise linear arc. Show that C is both open and closed in $\mathcal{O}$.)

Solution. Throughout we use the fact, recorded in (1) of this section, that a set of real numbers is connected if and only if it is an interval; in particular the closed, bounded interval $[0, 1]$ is a connected topological space.

Part one: arcwise connected implies connected. Let X be arcwise connected and suppose, to get a contradiction, that X is not connected. Then there are nonempty open subsets $\mathcal{O}_1, \mathcal{O}_2$ of X which

separate X , that is,

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset, \quad \mathcal{O}_1 \cup \mathcal{O}_2 = X.$$

Choose $u \in \mathcal{O}_1$ and $v \in \mathcal{O}_2$. Since X is arcwise connected there is a continuous map $f: [0, 1] \rightarrow X$ with $f(0) = u$ and $f(1) = v$. Consider

$$U_1 = f^{-1}(\mathcal{O}_1), \quad U_2 = f^{-1}(\mathcal{O}_2).$$

By continuity each U_i is open in $[0, 1]$. Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint with union X , the sets U_1 and U_2 are disjoint with union $[0, 1]$. Moreover $0 \in U_1$ and $1 \in U_2$, so both are nonempty. Thus U_1, U_2 is a separation of $[0, 1]$, contradicting the connectedness of $[0, 1]$. Hence no separation of X exists and X is connected.

(Equivalently: by Proposition 22 the image $f([0, 1])$ is a connected subspace of X , while $\mathcal{O}_1 \cap f([0, 1])$ and $\mathcal{O}_2 \cap f([0, 1])$ would separate it.)

Part two: a connected open subset of $\mathbb{R}^n$ is arcwise connected. Let $\mathcal{O}$ be a connected open subset of $\mathbb{R}^n$. If $\mathcal{O}$ is empty the assertion holds vacuously, so assume $\mathcal{O} \neq \emptyset$ and fix $x \in \mathcal{O}$.

Call a piecewise linear arc in $\mathcal{O}$ from p to q a finite list of points $p = z_0, z_1, \dots, z_k = q$ such that every segment

$$[z_{j-1}, z_j] = \{ (1-t)z_{j-1} + tz_j : 0 \leq t \leq 1 \}$$

is contained in $\mathcal{O}$. Such a list determines a continuous map of $[0, 1]$ into $\mathcal{O}$ carrying 0 to p and 1 to q : divide $[0, 1]$ into k consecutive closed subintervals of length $1/k$ and on the j -th of them traverse the segment $[z_{j-1}, z_j]$ affinely. The resulting map is continuous on each of finitely many closed subintervals covering $[0, 1]$, and the definitions agree at the common endpoints, so it is continuous on $[0, 1]$.

Define

$$C = \{ y \in \mathcal{O} : \text{there is a piecewise linear arc in } \mathcal{O} \text{ from } x \text{ to } y \}.$$

Then $x \in C$ (take the one-point list $z_0 = x$, or the degenerate segment $[x, x] \subseteq \mathcal{O}$), so $C \neq \emptyset$.

C is open. Let $y \in C$. Since $\mathcal{O}$ is open in $\mathbb{R}^n$ there is $r > 0$ with the open ball $B(y, r) \subseteq \mathcal{O}$. A ball is convex, so for each $z \in B(y, r)$ the segment $[y, z]$ lies in $B(y, r) \subseteq \mathcal{O}$. Appending z to a piecewise linear arc in $\mathcal{O}$ from x to y produces a piecewise linear arc in $\mathcal{O}$ from x to z . Hence $B(y, r) \subseteq C$, and C is open.

$\mathcal{O} \sim C$ is open. Let $y \in \mathcal{O} \sim C$ and choose $r > 0$ with $B(y, r) \subseteq \mathcal{O}$. If some $z \in B(y, r)$ belonged to C , then appending y to a piecewise linear arc in $\mathcal{O}$ from x to z (legitimate, since $[z, y] \subseteq B(y, r) \subseteq \mathcal{O}$) would exhibit y as a point of C , a contradiction. Hence $B(y, r) \subseteq \mathcal{O} \sim C$, and $\mathcal{O} \sim C$ is open.

Both C and $\mathcal{O} \sim C$ are open subsets of $\mathbb{R}^n$ contained in $\mathcal{O}$, hence open in the subspace $\mathcal{O}$; they are disjoint and their union is $\mathcal{O}$. Since $C \neq \emptyset$ and $\mathcal{O}$ is connected, the set $\mathcal{O} \sim C$ must be empty, for otherwise the pair would be a separation of $\mathcal{O}$. Therefore $C = \mathcal{O}$: every point of $\mathcal{O}$ is joined to x by a piecewise linear arc in $\mathcal{O}$.

Finally, given $u, v \in \mathcal{O}$, concatenate a piecewise linear arc from u to x (reverse the one from x to u) with one from x to v ; the result is a piecewise linear arc in $\mathcal{O}$ from u to v , and by the paragraph above it is the image of a continuous map $f: [0, 1] \rightarrow \mathcal{O}$ with $f(0) = u$, $f(1) = v$. Thus $\mathcal{O}$ is arcwise connected.

Problem 11.51. Consider the circle $C = \{(x, y) \mid x^2 + y^2 = 1\}$ in the plane $\mathbb{R}^2$. Show that C is connected.

Solution. Define

$$f: [0, 2\pi] \rightarrow \mathbb{R}^2, \quad f(t) = (\cos t, \sin t).$$

Each component of f is a continuous real-valued function of t , so f is continuous as a map into $\mathbb{R}^2$ (a map into a Euclidean space is continuous if and only if each component is).

We claim $f([0, 2\pi]) = C$.

The inclusion $f([0, 2\pi]) \subseteq C$ is immediate from the identity $\cos^2 t + \sin^2 t = 1$.

For the reverse inclusion, let $(x, y) \in C$, so $x^2 + y^2 = 1$; in particular $-1 \leq x \leq 1$ and $y^2 = 1 - x^2$, so $|y| = \sqrt{1 - x^2}$. Put $s = \arccos x \in [0, \pi]$, so that $\cos s = x$ and, since $\sin$ is nonnegative on $[0, \pi]$,

$\sin s = \sqrt{1 - \cos^2 s} = \sqrt{1 - x^2} = |y|$. If $y \geq 0$, then $f(s) = (x, |y|) = (x, y)$ and $s \in [0, 2\pi]$. If $y < 0$, put $t = 2\pi - s \in [\pi, 2\pi] \subseteq [0, 2\pi]$; then $\cos t = \cos s = x$ and $\sin t = -\sin s = -|y| = y$, so $f(t) = (x, y)$. In either case $(x, y) \in f([0, 2\pi])$. This proves the claim.

Now $[0, 2\pi]$ is an interval of real numbers, hence, by the equivalence (1) of this section, it is a connected topological space. By Proposition 22, the image of a connected space under a continuous map is connected. Therefore $C = f([0, 2\pi])$ is connected.

An alternative argument, avoiding trigonometric functions: the maps

$$g_{\pm}: [-1, 1] \rightarrow \mathbb{R}^2, \quad g_{\pm}(x) = (x, \pm\sqrt{1 - x^2})$$

are continuous, so their images C^+ (the closed upper semicircle) and C^- (the closed lower semicircle) are connected by Proposition 22, and $C = C^+ \cup C^-$. If $\mathcal{O}_1, \mathcal{O}_2$ were a separation of C , then since C^+ is connected and meets $\mathcal{O}_1 \cup \mathcal{O}_2 = C$, the sets $\mathcal{O}_1 \cap C^+$ and $\mathcal{O}_2 \cap C^+$ are disjoint relatively open sets with union C^+ , so one of them is empty: C^+ lies entirely in one of $\mathcal{O}_1, \mathcal{O}_2$. The same holds for C^- . But $(1, 0) \in C^+ \cap C^-$, so C^+ and C^- lie in the same member of the pair, whence the other member misses $C^+ \cup C^- = C$ and is empty. This contradicts the definition of a separation, so C is connected.

Problem 11.52. Show that $\mathbb{R}^n$ is connected.

Solution. We show $\mathbb{R}^n$ is arcwise connected and then invoke Problem 50; for completeness the deduction is repeated here.

Let $u, v \in \mathbb{R}^n$ and define

$$f: [0, 1] \rightarrow \mathbb{R}^n, \quad f(t) = (1 - t)u + tv.$$

Then $f(0) = u$ and $f(1) = v$. Moreover, for $s, t \in [0, 1]$,

$$\|f(t) - f(s)\| = \|(t - s)(v - u)\| = |t - s| \|v - u\|,$$

so f is Lipschitz and therefore continuous. Hence $\mathbb{R}^n$ is arcwise connected.

Now suppose $\mathbb{R}^n$ were not connected, and let $\mathcal{O}_1, \mathcal{O}_2$ be a separation: nonempty disjoint open sets with $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathbb{R}^n$. Choose $u \in \mathcal{O}_1, v \in \mathcal{O}_2$, and let f be the segment map above. The sets $f^{-1}(\mathcal{O}_1)$ and $f^{-1}(\mathcal{O}_2)$ are open in $[0, 1]$ by continuity of f , are disjoint (since $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$), have union $[0, 1]$ (since $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathbb{R}^n$), and are nonempty since $0 \in f^{-1}(\mathcal{O}_1)$ and $1 \in f^{-1}(\mathcal{O}_2)$. Thus they separate $[0, 1]$. But $[0, 1]$ is an interval of real numbers, hence connected by the equivalence (1) of this section. This contradiction shows that no separation of $\mathbb{R}^n$ exists.

Therefore $\mathbb{R}^n$ is connected. The same argument shows that any convex subset of $\mathbb{R}^n$ (in particular any open or closed ball, and any rectangle) is connected, since the segment joining two of its points remains in the set.

Problem 11.53. Show that a compact metric space (X, ρ) fails to be connected if and only if there are two disjoint, nonempty subsets A and B whose union is X and $\epsilon > 0$ such that $\rho(u, v) \geq \epsilon$ for all $u \in A, v \in B$. Show that this is not necessarily the case for noncompact metric spaces.

Solution. Sufficiency (this half needs no compactness). Suppose A and B are disjoint nonempty sets with $A \cup B = X$ and there is $\epsilon > 0$ with $\rho(u, v) \geq \epsilon$ whenever $u \in A$ and $v \in B$. Let $u \in A$. Every point w of the open ball $B(u, \epsilon) = \{w \in X : \rho(u, w) < \epsilon\}$ satisfies $\rho(u, w) < \epsilon$, so $w \notin B$; since $A \cup B = X$, we get $w \in A$. Thus $B(u, \epsilon) \subseteq A$, and A is open. By the symmetric argument B is open. Hence A and B are two nonempty disjoint open sets whose union is X , that is, they separate X , and X is not connected.

Necessity. Assume now that X is compact and not connected. Let A and B be a separation of X : nonempty disjoint open sets with $A \cup B = X$. Since each is the complement of the other, A and B are also closed. Define

$$\epsilon = \inf\{\rho(u, v) : u \in A, v \in B\} \geq 0.$$

We claim $\epsilon > 0$. Suppose not, so $\epsilon = 0$. Then for each natural number k there are $u_k \in A$ and

$v_k \in B$ with $\rho(u_k, v_k) < 1/k$. By Theorem 16 of Chapter 9 (the Characterization of Compactness for a Metric Space), the compact metric space X is sequentially compact, so there is a subsequence $\{u_{k_j}\}$ converging to some $x \in X$. For each j ,

$$\rho(v_{k_j}, x) \leq \rho(v_{k_j}, u_{k_j}) + \rho(u_{k_j}, x) < \frac{1}{k_j} + \rho(u_{k_j}, x) \rightarrow 0,$$

so $\{v_{k_j}\}$ also converges to x . Now A is closed and $u_{k_j} \in A$ for all j : if x did not belong to A , then x would lie in the open set $X \sim A$ and some ball about x would miss A , contradicting $u_{k_j} \rightarrow x$. Hence $x \in A$. The identical argument with the closed set B and the sequence $\{v_{k_j}\}$ gives $x \in B$. This contradicts $A \cap B = \emptyset$.

Therefore $\epsilon > 0$, and by the definition of the infimum, $\rho(u, v) \geq \epsilon$ for all $u \in A, v \in B$. So the pair A, B together with this ϵ has the required property.

Failure without compactness. Let

$$X = (0, 1) \cup (1, 2) \subseteq \mathbb{R},$$

with the metric $\rho(s, t) = |s - t|$ inherited from $\mathbb{R}$. The sets $(0, 1)$ and $(1, 2)$ are nonempty, disjoint, open in X , and their union is X , so X is not connected. (X is not compact: it is not closed in $\mathbb{R}$, so by Theorem 20 of Chapter 9 it is not a compact subspace of $\mathbb{R}$. Directly, the sequence $\{1 - 1/n\}$ has no subsequence converging in X .)

We show no pair A, B as in the statement exists for this X . Suppose A and B were disjoint nonempty sets with $A \cup B = X$ and $\rho(u, v) \geq \epsilon > 0$ for all $u \in A, v \in B$. By the sufficiency argument above, A and B are then open in X , so they separate X . Each of $(0, 1)$ and $(1, 2)$ is an interval of real numbers, hence a connected subspace of X by the equivalence (1); consequently each lies entirely inside A or entirely inside B , since otherwise its intersections with A and with B would separate it. Since A and B are both nonempty and $X = (0, 1) \cup (1, 2)$, one of the two intervals lies in A and the other in B ; without loss of generality $(0, 1) \subseteq A$ and $(1, 2) \subseteq B$. Choose δ with $0 < \delta < \min\{\epsilon/2, 1\}$. Then $1 - \delta/2 \in A$ and $1 + \delta/2 \in B$, while

$$\rho(1 - \delta/2, 1 + \delta/2) = \delta < \epsilon,$$

a contradiction. Hence for this noncompact disconnected metric space there is no such pair A, B and no such ϵ : the equivalence genuinely requires compactness.

Problem 11.54. A metric space (X, ρ) is said to be well chained provided for each pair of points $u, v \in X$ and each $\epsilon > 0$, there is a finite number of points in X , $u = x_0, x_1, \dots, x_{n-1}, x_n = v$ such that $\rho(x_{i-1}, x_i) < \epsilon$, for $1 \leq i \leq n$.

- (i) Show that if X is connected, then it is well chained, but the converse is not true.
- (ii) Show that if X is compact and well chained, then it is connected.
- (iii) Show that if an open subset of $\mathbb{R}^n$ is well chained, then it is connected.

Solution. Call a finite list $u = x_0, x_1, \dots, x_n = v$ of points of X with $\rho(x_{i-1}, x_i) < \epsilon$ for $1 \leq i \leq n$ an ϵ -chain in X from u to v .

Part (i). Assume X is connected, and fix $\epsilon > 0$ and $u \in X$. Let

$$A = \{v \in X : \text{there is an } \epsilon\text{-chain in } X \text{ from } u \text{ to } v\}.$$

Then $u \in A$, since the list $x_0 = u, x_1 = u$ is an ϵ -chain ($\rho(u, u) = 0 < \epsilon$); so A is nonempty.

A is open. Let $v \in A$ and let w belong to the ball $B(v, \epsilon)$. Appending w to an ϵ -chain from u to v yields an ϵ -chain from u to w , since $\rho(v, w) < \epsilon$. Hence $B(v, \epsilon) \subseteq A$.

$X \sim A$ is open. Let $v \in X \sim A$ and let $w \in B(v, \epsilon)$. If w belonged to A , then appending v to an ϵ -chain from u to w would put v in A , a contradiction. Hence $B(v, \epsilon) \subseteq X \sim A$.

So A and $X \sim A$ are disjoint open sets with union X and $A \neq \emptyset$. Since X is connected, they cannot separate X , so $X \sim A = \emptyset$ and $A = X$. As u and ϵ were arbitrary, every pair of points of X is joined by an ϵ -chain for every $\epsilon > 0$: X is well chained.

The converse fails. Let $X = \mathbb{Q}$, the set of rational numbers with the metric $\rho(s, t) = |s - t|$. Given

$u, v \in \mathbb{Q}$ with $u \neq v$ and given $\epsilon > 0$, choose a natural number n with $|v - u|/n < \epsilon$ and set

$$x_i = u + \frac{i}{n}(v - u), \quad 0 \leq i \leq n.$$

Each x_i is rational, $x_0 = u$, $x_n = v$, and $\rho(x_{i-1}, x_i) = |v - u|/n < \epsilon$. So $\mathbb{Q}$ is well chained. But $\mathbb{Q}$ is not connected: the sets

$$\mathcal{O}_1 = \{q \in \mathbb{Q} : q < \sqrt{2}\}, \quad \mathcal{O}_2 = \{q \in \mathbb{Q} : q > \sqrt{2}\}$$

are nonempty, disjoint, open in $\mathbb{Q}$, and their union is $\mathbb{Q}$ because $\sqrt{2}$ is irrational. (Equivalently: by the equivalence (1), a connected set of real numbers is an interval, and $\mathbb{Q}$ contains 1 and 2 but not $\sqrt{2}$, so it is not an interval.)

Part (ii). Let X be compact and well chained, and suppose X were not connected. By Problem 53 there are disjoint nonempty sets A and B with $A \cup B = X$ and an $\epsilon > 0$ such that

$$\rho(u, v) \geq \epsilon \quad \text{for all } u \in A, v \in B.$$

Choose $u \in A$ and $v \in B$. Since X is well chained there is an ϵ -chain $u = x_0, x_1, \dots, x_n = v$ in X . The index set $\{i : x_i \in B\}$ is nonempty (it contains n) and does not contain 0 (as $x_0 = u \in A$ and $A \cap B = \emptyset$). Let i be its least element; then $i \geq 1$ and $x_{i-1} \notin B$, so $x_{i-1} \in A$ because $A \cup B = X$. Therefore

$$\epsilon \leq \rho(x_{i-1}, x_i) < \epsilon,$$

a contradiction. Hence X is connected.

Part (iii). As printed, this assertion is false, and the following are counterexamples.

In $\mathbb{R}^1$, let $\mathcal{O} = (0, 1) \cup (1, 2)$. This set is open. It is not connected, since $(0, 1)$ and $(1, 2)$ are nonempty disjoint open sets with union $\mathcal{O}$. Yet it is well chained. Indeed, let $u, v \in \mathcal{O}$ and $\epsilon > 0$; we may assume $u < v$. If $1 \notin (u, v)$ then the whole segment from u to v lies in $\mathcal{O}$ and the equally spaced points $x_i = u + (i/n)(v - u)$, with n chosen so that $(v - u)/n < \epsilon$, form an ϵ -chain in $\mathcal{O}$. If $u < 1 < v$, pick δ with $0 < \delta < \min\{\epsilon/2, 1 - u, v - 1\}$, run an ϵ -chain of equally spaced points inside $(0, 1)$ from u to $1 - \delta$, then step to $1 + \delta$ (a step of length $2\delta < \epsilon$), then run an ϵ -chain of equally spaced points inside $(1, 2)$ from $1 + \delta$ to v . Every point used lies in $\mathcal{O}$ and every consecutive distance is less than ϵ . So $\mathcal{O}$ is a well-chained open subset of $\mathbb{R}$ that is not connected.

Lest one object that $n = 1$ is degenerate, the same phenomenon occurs in every $\mathbb{R}^n$ with $n \geq 2$: take the two internally tangent open balls

$$\mathcal{O} = B((-1, 0, \dots, 0), 1) \cup B((1, 0, \dots, 0), 1).$$

These balls are disjoint (a point of the first has first coordinate negative, a point of the second has it positive), each is open and connected, so $\mathcal{O}$ is open and not connected. But for $0 < \delta < 1$ the points $(-\delta, 0, \dots, 0)$ and $(\delta, 0, \dots, 0)$ lie in the first and second ball respectively and are at distance 2δ ; joining any u in the first ball to $(-\delta, 0, \dots, 0)$ by a fine chain along a segment inside that ball, stepping across, and continuing to v inside the second ball produces an ϵ -chain in $\mathcal{O}$ once $2\delta < \epsilon$. So $\mathcal{O}$ is well chained and disconnected.

The obstruction is exactly the one isolated in Problem 53: for a chain argument to force connectedness one needs the two pieces of a separation to be a positive distance apart, and for an open set this can fail, since the pieces may have limit points in common outside the set. Compactness is what supplies the positive distance, which is why part (ii) is true. What does hold for an open subset $\mathcal{O}$ of $\mathbb{R}^n$ is the following strengthened version, which is presumably what was intended: $\mathcal{O}$ is connected if and only if for each pair $u, v \in \mathcal{O}$ and each $\epsilon > 0$ there is a chain $u = x_0, \dots, x_n = v$ in $\mathcal{O}$ with $\rho(x_{i-1}, x_i) < \epsilon$ and with each segment $[x_{i-1}, x_i]$ contained in $\mathcal{O}$.

To see this, suppose first that $\mathcal{O}$ is connected. By Problem 50, $\mathcal{O}$ is arcwise connected by piecewise linear arcs, so u and v are joined by a chain of points $u = z_0, z_1, \dots, z_k = v$ with each segment $[z_{j-1}, z_j] \subseteq \mathcal{O}$. Refining each such segment by inserting equally spaced points, finitely many so that consecutive distances are less than ϵ , keeps all points and all sub-segments inside $[z_{j-1}, z_j] \subseteq \mathcal{O}$ and produces the required chain. Conversely, if such chains exist and $\mathcal{O}_1, \mathcal{O}_2$ were a separation of $\mathcal{O}$ with

$u \in \mathcal{O}_1$, $v \in \mathcal{O}_2$, take such a chain (any ϵ will do). The set $\{i : x_i \in \mathcal{O}_2\}$ contains n and omits 0, since $x_0 = u \in \mathcal{O}_1$ and the two sets are disjoint; let i be its least element, so $i \geq 1$ and $x_{i-1} \notin \mathcal{O}_2$, whence $x_{i-1} \in \mathcal{O}_1$ because $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}$. The segment $[x_{i-1}, x_i]$ lies in $\mathcal{O}$. That segment is the continuous image of $[0, 1]$, hence connected by the equivalence (1) and Proposition 22, and its intersections with $\mathcal{O}_1$ and $\mathcal{O}_2$ are nonempty, disjoint, relatively open, and cover it, a separation. This contradiction shows $\mathcal{O}$ is connected.

Thus the honest verdict on (iii): with the definition of well chained as printed, the assertion is false, and the corrected statement above is what the chain hypothesis yields for open sets.

Problem 11.55. Show that for any point (x, y) in the plane $\mathbb{R}^2$, the subspace $\mathbb{R}^2 \sim \{(x, y)\}$ is connected. Use this to show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Solution. Write $p = (x, y)$ and $Y = \mathbb{R}^2 \sim \{p\}$. Note that Y is an open subset of $\mathbb{R}^2$, since $\{p\}$ is closed. Y is arcwise connected. Let $u, v \in Y$. If $u = v$ there is nothing to prove, so assume $u \neq v$ and put $d = v - u \neq 0$.

Case 1: $p \notin [u, v]$, where $[u, v] = \{(1-t)u + tv : 0 \leq t \leq 1\}$. Then $t \mapsto (1-t)u + tv$ is a continuous map of $[0, 1]$ into Y carrying 0 to u and 1 to v .

Case 2: $p \in [u, v]$. Write $d = (d_1, d_2)$ and set $\nu = (-d_2, d_1)$, so $\nu \neq 0$ and $\nu \cdot d = -d_2d_1 + d_1d_2 = 0$. Let $w = u + \nu$. Then w does not lie on the line

$$L = \{u + td : t \in \mathbb{R}\}$$

through u and v : if $u + \nu = u + td$ for some t , then $\nu = td$, and taking the inner product with d gives $0 = \nu \cdot d = t\|d\|^2$, so $t = 0$ and $\nu = 0$, a contradiction.

We claim neither segment $[u, w]$ nor $[w, v]$ contains p . Since $p \in [u, v] \subseteq L$ and $p \neq u$, $p \neq v$ (as $u, v \in Y$), suppose $p \in [u, w]$. The segment $[u, w]$ lies on the line $L' = \{u + s\nu : s \in \mathbb{R}\}$ through u and w . Now $L \cap L' = \{u\}$: a common point is $u + td = u + s\nu$, so $td = s\nu$; taking inner products with d gives $t\|d\|^2 = 0$, hence $t = 0$ and the point is u . Thus $p \in L \cap L' = \{u\}$ forces $p = u$, a contradiction. Next, since $w \notin L$ while $v \in L$, we have $w \neq v$; let L'' be the line through w and v , so that $[w, v] \subseteq L''$. Suppose $p \in [w, v]$. Since $w \notin L$ and $v \in L$, the lines L and L'' are distinct; two distinct lines in the plane meet in at most one point, and v lies on both, so $L \cap L'' = \{v\}$. Hence $p = v$, again a contradiction.

Therefore both segments avoid p , and the piecewise linear arc $u \rightarrow w \rightarrow v$ lies in Y . As in Problem 50, such an arc is the image of a continuous map $f : [0, 1] \rightarrow Y$ with $f(0) = u$ and $f(1) = v$: traverse $[u, w]$ affinely on $[0, 1/2]$ and $[w, v]$ affinely on $[1/2, 1]$; the two definitions agree at $t = 1/2$, and a map continuous on each of two closed sets covering $[0, 1]$ is continuous.

Hence Y is arcwise connected, and therefore, by Problem 50, $Y = \mathbb{R}^2 \sim \{(x, y)\}$ is connected.

$\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. Suppose, to the contrary, that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a homeomorphism. Fix any $p \in \mathbb{R}^2$ and put $q = f(p) \in \mathbb{R}$. Since f is a bijection, the restriction of f to $Y = \mathbb{R}^2 \sim \{p\}$ is a continuous map of Y onto $\mathbb{R} \sim \{q\}$. By the first part, Y is connected, so by Proposition 22 its continuous image $\mathbb{R} \sim \{q\}$ is connected.

But $\mathbb{R} \sim \{q\}$ is not connected: the sets $(-\infty, q)$ and (q, ∞) are nonempty, disjoint, open in $\mathbb{R}$ (hence open in the subspace), and their union is $\mathbb{R} \sim \{q\}$, so they separate it. (Alternatively, by the equivalence (1) a connected set of real numbers is an interval, and $\mathbb{R} \sim \{q\}$ contains $q-1$ and $q+1$ but not q , so it is not an interval.)

This contradiction shows that no homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}$ exists; that is, $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Problem 11.56. Verify the equivalence of the three assertions in (1), namely that for a set C of real numbers the following are equivalent: (i) C is an interval; (ii) C is convex; (iii) C is connected.

Solution. Recall the terms. A set $C \subseteq \mathbb{R}$ is an interval provided it is of one of the forms

$$(a, b), \quad [a, b], \quad [a, b), \quad (a, b],$$

where $-\infty \leq a \leq b \leq \infty$ and, of course, a bracket is used only at a finite endpoint; the empty-set and one-point sets are included ($\emptyset = (a, a)$ and $\{c\} = [c, c]$). The set C is convex provided whenever $u, v \in C$ and $0 \leq t \leq 1$, then $(1-t)u + tv \in C$; for subsets of $\mathbb{R}$ this says precisely that

$$u, v \in C, \quad u < z < v \quad \implies \quad z \in C,$$

since the numbers strictly between u and v are exactly the points $(1-t)u + tv$ with $0 < t < 1$. Finally, C is connected as a subspace of $\mathbb{R}$ provided there do not exist open subsets $\mathcal{O}_1, \mathcal{O}_2$ of $\mathbb{R}$ with

$$\mathcal{O}_1 \cap C \neq \emptyset, \quad \mathcal{O}_2 \cap C \neq \emptyset, \quad C \subseteq \mathcal{O}_1 \cup \mathcal{O}_2, \quad C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset.$$

We prove (i) implies (ii), (ii) implies (i), (ii) implies (iii), and (iii) implies (ii).

(i) implies (ii). The four displayed forms have the following feature in common: if C is an interval with endpoints $a \leq b$, then

$$(a, b) \subseteq C \subseteq [a, b],$$

where the right-hand inclusion means $a \leq c \leq b$ for every $c \in C$ (a vacuous constraint at an infinite endpoint). Now let $u, v \in C$ and let z be a real number with $u < z < v$. From $u \in C$ we get $a \leq u$, hence $a \leq u < z$ and so $a < z$; from $v \in C$ we get $v \leq b$, hence $z < v \leq b$ and so $z < b$. Therefore $z \in (a, b) \subseteq C$. Since the points strictly between u and v are exactly the points $(1-t)u + tv$ with $0 < t < 1$, and the endpoints themselves lie in C , this is precisely convexity. Hence C is convex.

(ii) implies (i). Let C be convex. If $C = \emptyset$ then $C = (0, 0)$ is an interval, and if $C = \{c\}$ then $C = [c, c]$ is an interval. So assume C contains at least two points, and set

$$a = \inf C \in [-\infty, \infty), \quad b = \sup C \in (-\infty, \infty],$$

where $a = -\infty$ if C is not bounded below and $b = \infty$ if C is not bounded above; note $a < b$ since C has two points.

First, $C \subseteq [a, b]$ in the sense that $a \leq c \leq b$ for all $c \in C$; this is immediate from the definitions of infimum and supremum.

Second, $(a, b) \subseteq C$. Indeed, let $a < z < b$. Since $z > a = \inf C$, the number z is not a lower bound for C , so there is $u \in C$ with $u < z$. Since $z < b = \sup C$, the number z is not an upper bound for C , so there is $v \in C$ with $z < v$. Thus $u, v \in C$ and $u < z < v$, so $z \in C$ by convexity. (This argument is valid verbatim when $a = -\infty$ or $b = \infty$, reading the statements about non-bounds accordingly.)

Hence $(a, b) \subseteq C \subseteq [a, b]$, so C is obtained from (a, b) by possibly adjoining a , or b , or both, and only a finite endpoint can be adjoined since C consists of real numbers. Therefore C is one of (a, b) , $[a, b)$, $(a, b]$, $[a, b]$; that is, C is an interval.

(ii) implies (iii). Let C be convex and suppose, to get a contradiction, that $\mathcal{O}_1$ and $\mathcal{O}_2$ are open subsets of $\mathbb{R}$ separating C as displayed above. Choose $u \in C \cap \mathcal{O}_1$ and $v \in C \cap \mathcal{O}_2$. Then $u \neq v$, since $C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$; interchanging the names of $\mathcal{O}_1$ and $\mathcal{O}_2$ if necessary, assume $u < v$. By convexity,

$$[u, v] \subseteq C.$$

The set $S = \mathcal{O}_1 \cap [u, v]$ is nonempty (it contains u) and bounded above by v , so

$$c = \sup S$$

is a real number with $u \leq c \leq v$. Since $c \in [u, v] \subseteq C$ and $C \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$, the point c lies in $\mathcal{O}_1$ or in $\mathcal{O}_2$, and not in both.

Suppose $c \in \mathcal{O}_1$. Then $c \neq v$, for $v \in C \cap \mathcal{O}_2$ and $C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$; hence $c < v$. Since $\mathcal{O}_1$ is open there is $\delta > 0$ with $(c - \delta, c + \delta) \subseteq \mathcal{O}_1$, and we may take $\delta < v - c$. Then the point $c + \delta/2$ lies in $\mathcal{O}_1$ and in $[u, v]$, so it belongs to S while exceeding $c = \sup S$, a contradiction.

Suppose instead $c \in \mathcal{O}_2$. Then $c \neq u$, for $u \in C \cap \mathcal{O}_1$; hence $u < c$. Since $\mathcal{O}_2$ is open there is $\delta > 0$ with $(c - \delta, c + \delta) \subseteq \mathcal{O}_2$, and we may take $\delta < c - u$. Since c is the least upper bound of S , the number $c - \delta$ is not an upper bound of S , so there is $t \in S$ with $c - \delta < t \leq c$. Then $t \in \mathcal{O}_1$, and also $t \in (c - \delta, c + \delta) \subseteq \mathcal{O}_2$, and $t \in [u, v] \subseteq C$. Thus $t \in C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, a contradiction.

Both cases being impossible, no such separation exists and C is connected.

(iii) implies (ii). Let C be connected and suppose C is not convex. Then there are $u, v \in C$ and a real number z with $u < z < v$ and $z \notin C$. Put

$$\mathcal{O}_1 = (-\infty, z), \quad \mathcal{O}_2 = (z, \infty).$$

These are open subsets of $\mathbb{R}$; they are disjoint, so certainly $C \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$; their union is $\mathbb{R} \sim \{z\}$, which contains C because $z \notin C$; and $u \in C \cap \mathcal{O}_1$, $v \in C \cap \mathcal{O}_2$, so both intersections are nonempty. Hence $\mathcal{O}_1$ and $\mathcal{O}_2$ separate C , contradicting the connectedness of C . Therefore C is convex.

The four implications together give the equivalence of (i), (ii) and (iii).

12 Topological Spaces: Three Fundamental Theorems

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12.1 Problems 12.1–12.7

Problem 12.1. Let C be a closed subset of a metric space (X, ρ) . Show that the distance to C function d_C is continuous and $d_C(x) = 0$ if and only if x belongs to C .

Solution. Here C is assumed nonempty, so that

$$d_C(x) = \inf_{x' \in C} \rho(x', x) \quad \text{for all } x \in X$$

is a well-defined real number; indeed $0 \leq d_C(x) \leq \rho(x', x) < \infty$ for any fixed $x' \in C$.

Continuity. We show more, namely that d_C is Lipschitz with constant 1. Let x, y belong to X . For every $x' \in C$ the triangle inequality gives

$$d_C(x) \leq \rho(x', x) \leq \rho(x', y) + \rho(y, x).$$

The left-hand side does not involve x' , so we may take the infimum over $x' \in C$ on the right and conclude

$$d_C(x) \leq d_C(y) + \rho(x, y).$$

Interchanging the roles of x and y and using the symmetry of ρ gives $d_C(y) \leq d_C(x) + \rho(x, y)$. The two inequalities together yield

$$|d_C(x) - d_C(y)| \leq \rho(x, y) \quad \text{for all } x, y \in X.$$

Hence, given $x_0 \in X$ and $\epsilon > 0$, the ball $B(x_0, \epsilon)$ is a neighborhood of x_0 that d_C maps into $(d_C(x_0) - \epsilon, d_C(x_0) + \epsilon)$. Therefore d_C is continuous on X ; in fact it is uniformly continuous.

The zero set of d_C . Suppose first that $x \in C$. Then x is one of the competitors in the infimum defining $d_C(x)$, and $\rho(x, x) = 0$, so $0 \leq d_C(x) \leq 0$, that is, $d_C(x) = 0$.

Conversely, suppose $d_C(x) = 0$. Then for each natural number n , since $1/n$ is not a lower bound for $\{\rho(x', x) \mid x' \in C\}$, there is a point $x_n \in C$ with $\rho(x_n, x) < 1/n$. The sequence $\{x_n\}$ lies in C and converges to x , so x belongs to the closure of C . Since C is closed, x belongs to C .

Thus $d_C^{-1}(\{0\}) = C$, that is, C is the inverse image under the continuous function d_C of the closed set $\{0\}$. (This also re-proves that C must be closed, and shows that in a metric space every closed set is the zero set of a continuous real-valued function.)

Problem 12.2. Provide an example of a continuous real-valued function on the open interval $(0, 1)$ that is not extendable to a continuous function on $\mathbb{R}$. Does this contradict the Tietze Extension Theorem?

Solution. Define

$$f(x) = \sin(1/x) \quad \text{for } x \in (0, 1).$$

Then f is continuous on $(0, 1)$, being the composition of the continuous function $x \mapsto 1/x$ on $(0, 1)$ with the continuous function $\sin$ on $\mathbb{R}$. Moreover f is bounded, since $|f| \leq 1$ on $(0, 1)$.

We claim f has no continuous extension to $\mathbb{R}$. Suppose, to the contrary, that g is a continuous real-valued function on $\mathbb{R}$ with $g = f$ on $(0, 1)$. For each natural number n set

$$u_n = \frac{1}{2\pi n + \pi/2}, \quad v_n = \frac{1}{2\pi n + 3\pi/2}.$$

Both u_n and v_n belong to $(0, 1)$ and both converge to 0. Since $\mathbb{R}$ is a metric space, continuity of g at 0 implies $g(u_n) \rightarrow g(0)$ and $g(v_n) \rightarrow g(0)$. But

$$g(u_n) = f(u_n) = \sin(2\pi n + \pi/2) = 1 \quad \text{and} \quad g(v_n) = f(v_n) = \sin(2\pi n + 3\pi/2) = -1$$

for every n , so $g(0) = 1$ and $g(0) = -1$. This contradiction proves that no such g exists.

(A second, even simpler example is $f(x) = 1/x$ on $(0, 1)$: if g were a continuous extension, then $g(1/n) = n$ for all $n \geq 2$ while $g(1/n) \rightarrow g(0) \in \mathbb{R}$, which is impossible since $\{n\}$ is unbounded. The example $\sin(1/x)$ is preferable here because it is bounded, so that the failure cannot be blamed on unboundedness of the function.)

This does not contradict the Tietze Extension Theorem. That theorem requires the domain of the given function to be a closed subset F of the normal space X . Here $X = \mathbb{R}$, which is a metric space and hence normal by Proposition 7 of Chapter 11, and the function $\sin(1/x)$ is continuous and takes values in the closed, bounded interval $[-1, 1]$; but its domain $(0, 1)$ is not a closed subset of $\mathbb{R}$. So the hypotheses of the Tietze Extension Theorem are not met, and its conclusion is not asserted.

Closedness is exactly what is doing the work: it is what makes the two-set argument in the proof of the theorem go through (the sets A and B formed there from h are closed in X because F is closed in X , which is what permits the appeal to Urysohn's Lemma). Concretely, the restriction of $\sin(1/x)$ to the closed subset $[1/(2\pi), 1/2]$ of $\mathbb{R}$ does extend continuously to all of $\mathbb{R}$, as the Tietze Extension Theorem guarantees; it is only the accumulation of the domain $(0, 1)$ at the omitted point 0, where the oscillation is uncontrolled, that destroys extendability.

Problem 12.3. Deduce Urysohn's Lemma as a consequence of the Tietze Extension Theorem.

Solution. Urysohn's Lemma asserts: if A and B are disjoint closed subsets of a normal topological space X and $[a, b]$ is a closed, bounded interval of real numbers, then there is a continuous real-valued function f on X taking values in $[a, b]$ with $f = a$ on A and $f = b$ on B .

Let A and B be such sets. If both are empty, the constant function $f \equiv a$ does the job; so assume at least one is nonempty. Put

$$F = A \cup B,$$

which is a closed subset of X , being the union of two closed sets. Define $\varphi: F \rightarrow \mathbb{R}$ by

$$\varphi = a \text{ on } A \quad \text{and} \quad \varphi = b \text{ on } B.$$

This is unambiguous because $A \cap B = \emptyset$, and clearly $\varphi(F) \subseteq [a, b]$.

Claim: φ is continuous on F , where F carries the subspace topology. Indeed, A and B are closed in X , hence closed in F ; since $F = A \cup B$ and $A \cap B = \emptyset$, we have $A = F \setminus B$ and $B = F \setminus A$, so A and B are also open in F . Now let $\mathcal{O}$ be any open subset of $\mathbb{R}$. Then

$$\varphi^{-1}(\mathcal{O}) = \begin{cases} F & \text{if } a \in \mathcal{O} \text{ and } b \in \mathcal{O}, \\ A & \text{if } a \in \mathcal{O} \text{ and } b \notin \mathcal{O}, \\ B & \text{if } a \notin \mathcal{O} \text{ and } b \in \mathcal{O}, \\ \emptyset & \text{if } a \notin \mathcal{O} \text{ and } b \notin \mathcal{O}, \end{cases}$$

and each of F , A , B , $\emptyset$ is open in F . (When $a = b$ the function φ is constant and the same conclusion is immediate.) By Proposition 10 of Chapter 11, the continuity of φ follows from the openness of the inverse image of every open set. So φ is continuous.

Thus X is a normal topological space, F is a closed subset of X , and φ is a continuous real-valued function on F taking values in the closed, bounded interval $[a, b]$. The Tietze Extension Theorem provides a continuous extension f of φ to all of X that also takes values in $[a, b]$. Since f extends φ , we have $f = a$ on A and $f = b$ on B , and $f(X) \subseteq [a, b]$. This is exactly the conclusion of Urysohn's Lemma.

Two remarks. First, the deduction is formal, not a new proof: the Tietze Extension Theorem is itself proved in the text using Urysohn's Lemma, so this argument shows only that the two statements are equivalent, with Urysohn's Lemma being the case in which the closed set is a disjoint union of two closed pieces on which the function is constant. Second, this is precisely the sense in which the text describes Urysohn's Lemma as an extension result: the datum is the two-valued continuous function on $A \cup B$, and the lemma asserts it extends without enlarging its range.

Problem 12.4. State and prove a version of the Tietze Extension Theorem for functions with values in $\mathbb{R}^n$.

Solution. Statement. Let X be a normal topological space, F a closed subset of X , and $f: F \rightarrow \mathbb{R}^n$ a continuous mapping whose image lies in a closed, bounded box

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

Then f has a continuous extension $g: X \rightarrow \mathbb{R}^n$ that also takes all its values in Q .

Preliminary. For $1 \leq i \leq n$ let $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$ be the i -th coordinate function, and for a mapping h into $\mathbb{R}^n$ write $h_i = \pi_i \circ h$. We use the standard fact, proved here for completeness, that a mapping $h: Z \rightarrow \mathbb{R}^n$ from a topological space Z is continuous if and only if each $h_i: Z \rightarrow \mathbb{R}$ is continuous, where $\mathbb{R}^n$ carries its Euclidean metric topology.

If h is continuous, then so is each $h_i = \pi_i \circ h$, since π_i is continuous ($|\pi_i(y) - \pi_i(y')| \leq \|y - y'\|$) and compositions of continuous mappings are continuous. Conversely, suppose each h_i is continuous, let $z_0 \in Z$ and $\epsilon > 0$. For each i , continuity of h_i at z_0 provides a neighborhood $\mathcal{U}_i$ of z_0 with

$$|h_i(z) - h_i(z_0)| < \epsilon/\sqrt{n} \quad \text{for all } z \in \mathcal{U}_i.$$

Then $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_i$ is a neighborhood of z_0 , being a finite intersection of neighborhoods, and for $z \in \mathcal{U}$,

$$\|h(z) - h(z_0)\| = \left(\sum_{i=1}^n |h_i(z) - h_i(z_0)|^2 \right)^{1/2} < \left(n \cdot \frac{\epsilon^2}{n} \right)^{1/2} = \epsilon.$$

So $h(\mathcal{U})$ is contained in the ball of radius ϵ about $h(z_0)$. Since the balls form a base at $h(z_0)$ for the Euclidean topology, h is continuous at z_0 , and z_0 was arbitrary.

Proof of the statement. Write $f = (f_1, \dots, f_n)$ with $f_i = \pi_i \circ f$. By the preliminary, each f_i is a continuous real-valued function on F , and since $f(F) \subseteq Q$ we have $f_i(F) \subseteq [a_i, b_i]$, a closed, bounded interval. Since X is normal and F is closed in X , the Tietze Extension Theorem applies to each f_i and provides a continuous real-valued function g_i on X with

$$g_i = f_i \text{ on } F \quad \text{and} \quad g_i(X) \subseteq [a_i, b_i].$$

Define $g: X \rightarrow \mathbb{R}^n$ by $g(x) = (g_1(x), \dots, g_n(x))$. By the preliminary, g is continuous, since each of its components is. For $x \in F$ we have $g_i(x) = f_i(x)$ for every i , so $g = f$ on F ; and $g_i(X) \subseteq [a_i, b_i]$ for every i gives $g(X) \subseteq Q$. This proves the theorem.

A refinement. Suppose instead that f is bounded in the Euclidean norm, say $\|f\| \leq M$ on F . Then f has a continuous extension $\tilde{g}$ to X with $\|\tilde{g}\| \leq M$ on X . Indeed, $f(F)$ lies in the box $[-M, M]^n$, so the theorem just proved supplies a continuous extension g with values in that box. Let $r: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the radial retraction onto the closed ball $\overline{B}(0, M)$,

$$r(y) = \frac{My}{\max\{M, \|y\|\}} \quad \text{for } y \in \mathbb{R}^n,$$

so that $r(y) = y$ when $\|y\| \leq M$ and $r(y) = My/\|y\|$ when $\|y\| > M$. Then r is continuous: the norm $y \mapsto \|y\|$ is continuous, hence so is $y \mapsto \max\{M, \|y\|\}$, and this last function is everywhere at least $M > 0$, so the quotient is continuous by the algebra of continuous real-valued functions applied to each coordinate, together with the coordinatewise criterion established in the preliminary. Moreover $\|r(y)\| = M\|y\|/\max\{M, \|y\|\} \leq M$ for every y , and r is the identity on $\overline{B}(0, M)$. Set $\tilde{g} = r \circ g$. Then $\tilde{g}$ is continuous, $\|\tilde{g}\| \leq M$ on X , and for $x \in F$ we have $\|g(x)\| = \|f(x)\| \leq M$, so $\tilde{g}(x) = g(x) = f(x)$. More generally, the same composition trick shows that if $f(F)$ lies in any closed subset K of $\mathbb{R}^n$ that is bounded and is a retract of $\mathbb{R}^n$ (for instance, any closed bounded convex set, using the nearest-point projection onto it), then the extension can be arranged to take all its values in K as well.

Problem 12.5. Suppose that a topological space X has the property that every continuous, bounded real-valued function on a closed subset has a continuous extension to all of X . Show that if X is Tychonoff, then it is normal.

Solution. Recall the definitions in force. X is Tychonoff provided it has the Tychonoff separation property: for each two points u and v in X there is a neighborhood of u that does not contain v and a neighborhood of v that does not contain u . (Equivalently, by Proposition 6 of Chapter 11, every set consisting of a single point is closed.) X is normal provided the Tychonoff separation property holds and, moreover, each two disjoint closed sets can be separated by disjoint neighborhoods. Thus normality is the Tychonoff property together with one further requirement, and it is only that further requirement that has to be established.

Since X is assumed Tychonoff, it remains only to separate disjoint closed sets by disjoint neighborhoods. Let A and B be disjoint closed subsets of X . If either is empty, say $A = \emptyset$, then $\emptyset$ and X are disjoint neighborhoods of A and B respectively, and we are done. So assume both are nonempty. Set

$$F = A \cup B,$$

a closed subset of X , and define $\varphi: F \rightarrow \mathbb{R}$ by

$$\varphi = 0 \text{ on } A \quad \text{and} \quad \varphi = 1 \text{ on } B,$$

which is well defined since $A \cap B = \emptyset$.

The function φ is continuous on F with its subspace topology. Indeed, A and B are closed in X and therefore closed in F ; and since $F = A \cup B$ with $A \cap B = \emptyset$, we have $A = F \setminus B$ and $B = F \setminus A$, so A and B are open in F as well. Hence for any open subset $\mathcal{O}$ of $\mathbb{R}$ the inverse image $\varphi^{-1}(\mathcal{O})$ is one of the four sets $\emptyset, A, B, F$, each of which is open in F ; by Proposition 10 of Chapter 11, φ is continuous. It is plainly bounded, since $\varphi(F) \subseteq \{0, 1\}$.

By the assumed extension property of X applied to the closed set F and the continuous bounded function φ , there is a continuous real-valued function f on all of X with $f = \varphi$ on F ; that is,

$$f = 0 \text{ on } A \quad \text{and} \quad f = 1 \text{ on } B.$$

Now define

$$\mathcal{U} = f^{-1}((-1/2, 1/2)), \quad \mathcal{V} = f^{-1}((1/2, 3/2)).$$

Since f is continuous and $(-1/2, 1/2)$ and $(1/2, 3/2)$ are open subsets of $\mathbb{R}$, the sets $\mathcal{U}$ and $\mathcal{V}$ are open in X . Since $f = 0$ on A we have $A \subseteq \mathcal{U}$, and since $f = 1$ on B we have $B \subseteq \mathcal{V}$; thus $\mathcal{U}$ is a neighborhood of A and $\mathcal{V}$ is a neighborhood of B . Finally the intervals $(-1/2, 1/2)$ and $(1/2, 3/2)$ are disjoint, so

$$\mathcal{U} \cap \mathcal{V} = f^{-1}((-1/2, 1/2) \cap (1/2, 3/2)) = f^{-1}(\emptyset) = \emptyset.$$

Hence A and B are separated by disjoint neighborhoods.

Since A and B were arbitrary disjoint closed subsets of X , and X is Tychonoff by hypothesis, X possesses the normal separation property; that is, X is normal.

This is the converse of the Tietze Extension Theorem, and it shows that among Tychonoff spaces normality is exactly equivalent to the Tietze extension property. Note that the argument used only two-valued functions on the closed set $A \cup B$, and that boundedness of φ was automatic; so it is enough to assume the extension property for continuous functions with values in $\{0, 1\}$. The parallel remark in the text, immediately following Urysohn's Lemma, is the same argument with Urysohn's conclusion in place of the extension hypothesis.

Problem 12.6. Let $(X, \mathcal{T})$ be a normal topological space and $\mathcal{F}$ the collection of continuous real-valued functions on X . Show that $\mathcal{T}$ is the weak topology induced by $\mathcal{F}$.

Solution. Write $\mathcal{T}_{\mathcal{F}}$ for the weak topology on X induced by $\mathcal{F}$, that is, the weakest topology containing all sets of the form $f^{-1}(\mathcal{O})$ with $f \in \mathcal{F}$ and $\mathcal{O}$ open in $\mathbb{R}$. By Proposition 13 of Chapter 11, $\mathcal{T}_{\mathcal{F}}$ is the topology with the fewest sets among the topologies on X for which every $f \in \mathcal{F}$ is continuous. We show the two topologies contain one another.

$\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}$. By its very definition, $\mathcal{F}$ consists of functions that are continuous with respect to $\mathcal{T}$. So $\mathcal{T}$ is one of the topologies on X for which every member of $\mathcal{F}$ is continuous, and hence, by the minimality expressed in Proposition 13, $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}$. (Directly: for $f \in \mathcal{F}$ and $\mathcal{O}$ open in $\mathbb{R}$, the set $f^{-1}(\mathcal{O})$ belongs to $\mathcal{T}$ by Proposition 10 of Chapter 11, so the subbase generating $\mathcal{T}_{\mathcal{F}}$ is contained in $\mathcal{T}$, and therefore so is the topology it generates.)

$\mathcal{T} \subseteq \mathcal{T}_{\mathcal{F}}$. This is where normality is used. Let $\mathcal{O}$ belong to $\mathcal{T}$; we show $\mathcal{O}$ is a union of $\mathcal{T}_{\mathcal{F}}$ -open sets, hence itself belongs to $\mathcal{T}_{\mathcal{F}}$. If $\mathcal{O} = X$ or $\mathcal{O} = \emptyset$ this is clear, since both belong to every topology on X . Otherwise let $x \in \mathcal{O}$, and set

$$A = \{x\}, \quad B = X \sim \mathcal{O}.$$

Since X is normal it possesses the Tychonoff separation property, so by Proposition 6 of Chapter 11 the singleton $A = \{x\}$ is closed. The set B is closed since $\mathcal{O}$ is open, it is nonempty by assumption, and $A \cap B = \emptyset$ because $x \in \mathcal{O}$. Thus A and B are disjoint closed subsets of the normal space X , and Urysohn's Lemma, applied with the interval $[0, 1]$, provides a continuous real-valued function f on X for which

$$f(X) \subseteq [0, 1], \quad f(x) = 0, \quad f = 1 \text{ on } X \sim \mathcal{O}.$$

This f is a member of $\mathcal{F}$. Define

$$\mathcal{U}_x = f^{-1}((-1/2, 1/2)).$$

Then $\mathcal{U}_x$ belongs to $\mathcal{T}_{\mathcal{F}}$, being the inverse image of an open subset of $\mathbb{R}$ under a member of $\mathcal{F}$; it contains x , since $f(x) = 0$; and it is contained in $\mathcal{O}$, since if $y \notin \mathcal{O}$ then $f(y) = 1 \notin (-1/2, 1/2)$, so $y \notin \mathcal{U}_x$.

Consequently

$$\mathcal{O} = \bigcup_{x \in \mathcal{O}} \mathcal{U}_x,$$

the inclusion $\supseteq$ holding because each $\mathcal{U}_x \subseteq \mathcal{O}$, and $\subseteq$ because $x \in \mathcal{U}_x$ for each $x \in \mathcal{O}$. A union of sets belonging to the topology $\mathcal{T}_{\mathcal{F}}$ belongs to $\mathcal{T}_{\mathcal{F}}$, so $\mathcal{O} \in \mathcal{T}_{\mathcal{F}}$.

Therefore $\mathcal{T} = \mathcal{T}_{\mathcal{F}}$: the topology of a normal space is recovered as the weak topology induced by its continuous real-valued functions. Equivalently, the sets $f^{-1}(\mathcal{O})$, f continuous and $\mathcal{O} \subseteq \mathbb{R}$ open, form a subbase for $\mathcal{T}$; the argument in fact shows the stronger statement that the sets $\{f < 1/2\}$ for continuous $f: X \rightarrow [0, 1]$ already form a base.

Problem 12.7. Show that the function ρ defined in the proof of the Urysohn Metrization Theorem is a metric that defines the same topology as the given topology.

Solution. We recall the setting. X is a second countable, normal topological space with topology $\mathcal{T}$, and $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ is a countable base for $\mathcal{T}$. Let

$$A = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m\}.$$

For $(n, m) \in A$ the sets $\overline{\mathcal{U}_n}$ and $X \sim \mathcal{U}_m$ are disjoint closed subsets of X , so Urysohn's Lemma provides a continuous $f_{n,m}: X \rightarrow [0, 1]$ with

$$f_{n,m} = 0 \text{ on } \overline{\mathcal{U}_n} \quad \text{and} \quad f_{n,m} = 1 \text{ on } X \sim \mathcal{U}_m,$$

and one defines, for $x, y \in X$,

$$\rho(x, y) = \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.$$

Step 0: a separation device. We record the fact used repeatedly. Suppose $x \in \mathcal{U}_m$. Since X is normal it is Tychonoff, so by Proposition 6 of Chapter 11 the set $\{x\}$ is closed; and $\mathcal{U}_m$ is a neighborhood of

this closed set. By Proposition 8 of Chapter 11 there is an open set $\mathcal{O}$ with

$$\{x\} \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}_m.$$

Since $\{\mathcal{U}_n\}$ is a base and $\mathcal{O}$ is an open set containing x , there is an index n with $x \in \mathcal{U}_n \subseteq \mathcal{O}$. Then $\overline{\mathcal{U}_n} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}_m$, so $(n, m) \in A$, and

$$x \in \mathcal{U}_n \subseteq \overline{\mathcal{U}_n}, \quad \text{so} \quad f_{n,m}(x) = 0.$$

Step 1: ρ is well defined and finite. Each summand satisfies $0 \leq 2^{-(n+m)}|f_{n,m}(x) - f_{n,m}(y)| \leq 2^{-(n+m)}$, since $f_{n,m}$ takes values in $[0, 1]$. The set A is a subset of the countable set $\mathbf{N} \times \mathbf{N}$, and

$$\sum_{(n,m) \in \mathbf{N} \times \mathbf{N}} \frac{1}{2^{n+m}} = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) \left(\sum_{m=1}^{\infty} \frac{1}{2^m} \right) = 1.$$

So the series defining $\rho(x, y)$ is a series of nonnegative terms dominated by a convergent series; it converges, its sum is independent of the order of summation, and $0 \leq \rho(x, y) \leq 1$.

Step 2: ρ is a metric. Symmetry is immediate from $|s - t| = |t - s|$ applied termwise. The triangle inequality follows termwise: for $x, y, z \in X$ and each $(n, m) \in A$,

$$|f_{n,m}(x) - f_{n,m}(z)| \leq |f_{n,m}(x) - f_{n,m}(y)| + |f_{n,m}(y) - f_{n,m}(z)|;$$

multiplying by $2^{-(n+m)} > 0$ and summing over A (legitimate for convergent series of nonnegative terms) gives $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. Clearly $\rho(x, x) = 0$.

It remains to show that $\rho(x, y) = 0$ implies $x = y$. Suppose $x \neq y$. Since X is Tychonoff, $\{y\}$ is closed, so $X \sim \{y\}$ is an open set containing x ; as $\{\mathcal{U}_n\}$ is a base, there is an index m with

$$x \in \mathcal{U}_m \subseteq X \sim \{y\}.$$

By Step 0 there is an index n with $(n, m) \in A$ and $f_{n,m}(x) = 0$. Since $y \notin \mathcal{U}_m$, we have $f_{n,m}(y) = 1$. Hence, all terms of the series being nonnegative,

$$\rho(x, y) \geq \frac{1}{2^{n+m}}|f_{n,m}(x) - f_{n,m}(y)| = \frac{1}{2^{n+m}} > 0.$$

So ρ is a metric on X . Denote by $B_\rho(x, \epsilon)$ the ball of radius ϵ about x and by $\mathcal{T}_\rho$ the topology it induces.

Step 3: property (i), that is, $\mathcal{T} \subseteq \mathcal{T}_\rho$. We claim: if $\mathcal{U}_m$ contains x , then there is an $\epsilon > 0$ for which $B_\rho(x, \epsilon) \subseteq \mathcal{U}_m$. Indeed, by Step 0 choose n with $(n, m) \in A$ and $f_{n,m}(x) = 0$, and put $\epsilon = 2^{-(n+m)}$. If $\rho(x, y) < \epsilon$, then since every term of the series is nonnegative,

$$\frac{1}{2^{n+m}}|f_{n,m}(x) - f_{n,m}(y)| \leq \rho(x, y) < \frac{1}{2^{n+m}},$$

so $f_{n,m}(y) = |f_{n,m}(y) - 0| < 1$. As $f_{n,m} = 1$ on $X \sim \mathcal{U}_m$, this forces $y \in \mathcal{U}_m$. Hence $B_\rho(x, \epsilon) \subseteq \mathcal{U}_m$, proving the claim.

Now let $\mathcal{O} \in \mathcal{T}$ and $x \in \mathcal{O}$. Since $\{\mathcal{U}_n\}$ is a base there is an m with $x \in \mathcal{U}_m \subseteq \mathcal{O}$, and by the claim there is $\epsilon > 0$ with $B_\rho(x, \epsilon) \subseteq \mathcal{U}_m \subseteq \mathcal{O}$. Thus every point of $\mathcal{O}$ is the center of a ρ -ball contained in $\mathcal{O}$, so $\mathcal{O}$ is ρ -open. Hence $\mathcal{T} \subseteq \mathcal{T}_\rho$.

Step 4: property (ii), that is, $\mathcal{T}_\rho \subseteq \mathcal{T}$. We claim: for each $x \in X$ and each $\epsilon > 0$, there is a $\mathcal{U}_n$ that contains x and is contained in $B_\rho(x, \epsilon)$.

Fix x and $\epsilon > 0$. Since $\sum_{(n,m) \in \mathbf{N} \times \mathbf{N}} 2^{-(n+m)} = 1$, the tail of this convergent series is small: choose a natural number N for which

$$\sum_{\substack{(n,m) \in \mathbf{N} \times \mathbf{N} \\ n+m > N}} \frac{1}{2^{n+m}} < \frac{\epsilon}{2}.$$

Let $S = \{(n, m) \in A \mid n + m \leq N\}$, a finite set. Each $f_{n,m}$ is continuous at x , so for each $(n, m) \in S$ there is a $\mathcal{T}$ -neighborhood $\mathcal{V}_{n,m}$ of x on which $|f_{n,m}(\cdot) - f_{n,m}(x)| < \epsilon/2$. Since S is finite,

$$\mathcal{V} = \bigcap_{(n,m) \in S} \mathcal{V}_{n,m}$$

is a $\mathcal{T}$ -neighborhood of x (if $S = \emptyset$, take $\mathcal{V} = X$). For $y \in \mathcal{V}$, splitting the series over A into the part indexed by S and the remaining part, on which we use the crude bound $|f_{n,m}(y) - f_{n,m}(x)| \leq 1$,

$$\rho(x, y) \leq \sum_{(n,m) \in S} \frac{1}{2^{n+m}} \cdot \frac{\epsilon}{2} + \sum_{\substack{(n,m) \in \mathbf{N} \times \mathbf{N} \\ n+m > N}} \frac{1}{2^{n+m}} < \frac{\epsilon}{2} \cdot 1 + \frac{\epsilon}{2} = \epsilon.$$

Thus $\mathcal{V} \subseteq B_\rho(x, \epsilon)$. Finally, $\mathcal{V}$ is an open set containing x , being a finite intersection of open sets each containing x ; since $\{\mathcal{U}_n\}$ is a base for $\mathcal{T}$, there is therefore an index n with $x \in \mathcal{U}_n \subseteq \mathcal{V} \subseteq B_\rho(x, \epsilon)$. This proves the claim.

Consequently, if $\mathcal{W}$ is ρ -open and $x \in \mathcal{W}$, choose $\epsilon > 0$ with $B_\rho(x, \epsilon) \subseteq \mathcal{W}$ and then $\mathcal{U}_n$ with $x \in \mathcal{U}_n \subseteq B_\rho(x, \epsilon) \subseteq \mathcal{W}$. So $\mathcal{W}$ is the union of members of the base $\{\mathcal{U}_n\}$ and hence belongs to $\mathcal{T}$. Therefore $\mathcal{T}_\rho \subseteq \mathcal{T}$.

Combining Steps 3 and 4, $\mathcal{T} = \mathcal{T}_\rho$: the function ρ is a metric on X inducing exactly the given topology, so X is metrizable, which completes the proof of the Urysohn Metrization Theorem.

12.2 Problems 12.8–12.14

Problem 12.8. Let X be a normal topological space, F a closed subset of X , and f a continuous real-valued function on F . Then f has a continuous extension to a real-valued function $\bar{f}$ on all of X . Prove this as follows:

- (i) Apply the Tietze Extension Theorem to obtain a continuous extension $h: X \rightarrow [0, 1]$ of the function $f \cdot (1 + |f|)^{-1}: F \rightarrow [0, 1]$;
- (ii) Once more, apply the Tietze Extension Theorem to obtain a function $\phi: X \rightarrow [0, 1]$ such that $\phi = 1$ on F and $\phi = 0$ on $h^{-1}(1)$;
- (iii) Consider the function $\bar{f} = \phi \cdot h / (1 - \phi \cdot h)$.

Solution. We carry out the three steps. Since f is an arbitrary real-valued function on F , it may take negative values, and the auxiliary function $f \cdot (1 + |f|)^{-1}$ then takes values in $(-1, 1)$ rather than in $[0, 1]$; we therefore run the argument with the interval $[-1, 1]$ in place of $[0, 1]$ and with $|\phi \cdot h|$ in place of $\phi \cdot h$ in the denominator. (If $f \geq 0$ these reduce verbatim to the printed hints.)

Step (i). Define

$$g = \frac{f}{1 + |f|} \quad \text{on } F.$$

Since f is continuous on F and $1 + |f| \geq 1 > 0$, the function g is continuous on F . Moreover

$$|g| = \frac{|f|}{1 + |f|} < 1 \quad \text{on } F,$$

so g is a continuous function on F with values in the closed, bounded interval $[-1, 1]$. The set F is a closed subset of the normal space X , so by the Tietze Extension Theorem there is a continuous function

$$h: X \rightarrow [-1, 1] \quad \text{with } h = g \text{ on } F.$$

Step (ii). Put

$$C = \{x \in X \mid |h(x)| = 1\} = h^{-1}(\{-1, 1\}).$$

Since h is continuous and $\{-1, 1\}$ is closed in $[-1, 1]$, the set C is closed in X . Furthermore $C \cap F = \emptyset$: for $x \in F$ we have $h(x) = g(x)$ and $|g(x)| < 1$. Thus C and F are disjoint closed subsets of the normal

space X , so by Urysohn's Lemma, applied to the disjoint closed sets C and F and the interval $[0, 1]$, there is a continuous function

$$\phi: X \rightarrow [0, 1] \quad \text{with } \phi = 1 \text{ on } F \text{ and } \phi = 0 \text{ on } C.$$

This is the printed hint's second application of the Tietze Extension Theorem, in the form in which the text observes Urysohn's Lemma to be an extension result: the function equal to 0 on C and to 1 on F is a continuous $[0, 1]$ -valued function on the closed set $C \cup F$, since C and F are disjoint closed sets and hence each is relatively open in $C \cup F$, and ϕ is a continuous extension of it to X with values in $[0, 1]$. (There is no hypothesis of nonemptiness in Urysohn's Lemma; if $F = \emptyset$ the problem is vacuous, and if $C = \emptyset$ one may simply take $\phi \equiv 1$.)

Step (iii). Define

$$\bar{f} = \frac{\phi \cdot h}{1 - |\phi \cdot h|} \quad \text{on } X.$$

First we check the denominator never vanishes, i.e. that $|\phi(x)h(x)| < 1$ for every $x \in X$. Let $x \in X$. If $|h(x)| < 1$, then since $0 \leq \phi(x) \leq 1$ we get $|\phi(x)h(x)| \leq |h(x)| < 1$. If $|h(x)| = 1$, then $x \in C$, so $\phi(x) = 0$ and $\phi(x)h(x) = 0$, whence $|\phi(x)h(x)| = 0 < 1$. Therefore

$$1 - |\phi(x)h(x)| > 0 \quad \text{for all } x \in X.$$

The functions ϕ and h are continuous on X , hence so are $\phi \cdot h$ and $|\phi \cdot h|$, hence so is $1 - |\phi \cdot h|$; since this last function is nowhere zero, the quotient $\bar{f}$ is a continuous real-valued function on all of X .

Finally we check that $\bar{f}$ extends f . Let $x \in F$. Then $\phi(x) = 1$ and $h(x) = g(x)$, so

$$\bar{f}(x) = \frac{g(x)}{1 - |g(x)|}.$$

Now

$$1 - |g(x)| = 1 - \frac{|f(x)|}{1 + |f(x)|} = \frac{1}{1 + |f(x)|},$$

and therefore

$$\bar{f}(x) = \frac{f(x)}{1 + |f(x)|} \cdot (1 + |f(x)|) = f(x).$$

Thus $\bar{f}: X \rightarrow \mathbb{R}$ is continuous and $\bar{f} = f$ on F , which is the desired extension.

Two remarks. The passage through g is what removes the boundedness hypothesis: the Tietze Extension Theorem as proved in the text applies only to functions with values in a closed, bounded interval, and $t \mapsto t/(1 + |t|)$ is a homeomorphism of $\mathbb{R}$ onto $(-1, 1)$ with inverse $s \mapsto s/(1 - |s|)$, so it converts the unbounded f into a bounded function and back. The role of ϕ is to kill the extension h exactly where it reaches the forbidden values ± 1 , at which the inverse homeomorphism blows up, while leaving h untouched on F .

Problem 12.9. Show that a mapping f from a topological space X to a topological space Y is continuous if and only if there is a subbase $\mathcal{S}$ for the topology on Y such that the preimage under f of each set in $\mathcal{S}$ is open in X . Use this to show that if Y is a closed, bounded interval $[a, b]$, then f is continuous if and only if for each real number $c \in (a, b)$, the sets $\{x \in X \mid f(x) < c\}$ and $\{x \in X \mid f(x) > c\}$ are open.

Solution. Recall the definitions in force: a collection $\mathcal{B}$ of open subsets of Y is a base for the topology on Y provided every open subset of Y is a union of members of $\mathcal{B}$; and a collection $\mathcal{S}$ of open subsets of Y is a subbase for the topology on Y provided the intersections of finite subcollections of $\mathcal{S}$ form a base for the topology. Recall also that $f: X \rightarrow Y$ is continuous provided the preimage $f^{-1}(\mathcal{O})$ of each

open subset $\mathcal{O}$ of Y is open in X . Throughout we use the elementary set-theoretic identities

$$f^{-1}\left(\bigcup_{\alpha \in A} E_{\alpha}\right) = \bigcup_{\alpha \in A} f^{-1}(E_{\alpha}), \quad f^{-1}\left(\bigcap_{k=1}^n E_k\right) = \bigcap_{k=1}^n f^{-1}(E_k),$$

which hold for arbitrary mappings and arbitrary families of subsets of Y .

First part. Suppose f is continuous. The topology $\mathcal{T}$ on Y is itself a subbase for $\mathcal{T}$ (indeed it is a base), and by continuity the preimage of every member of $\mathcal{T}$ is open in X . So a subbase of the required kind exists.

Conversely, suppose $\mathcal{S}$ is a subbase for the topology on Y with $f^{-1}(S)$ open in X for every $S \in \mathcal{S}$. Let $\mathcal{B}$ be the collection of intersections of finite subcollections of $\mathcal{S}$; by hypothesis $\mathcal{B}$ is a base for the topology on Y . If $B \in \mathcal{B}$, say $B = S_1 \cap \cdots \cap S_n$ with each $S_k \in \mathcal{S}$, then

$$f^{-1}(B) = \bigcap_{k=1}^n f^{-1}(S_k)$$

is a finite intersection of open subsets of X and hence is open in X . Now let $\mathcal{O}$ be any open subset of Y . Since $\mathcal{B}$ is a base, there is a subcollection $\{B_{\alpha}\}_{\alpha \in A}$ of $\mathcal{B}$ with $\mathcal{O} = \bigcup_{\alpha \in A} B_{\alpha}$. Therefore

$$f^{-1}(\mathcal{O}) = \bigcup_{\alpha \in A} f^{-1}(B_{\alpha})$$

is a union of open subsets of X and hence is open in X . Thus the preimage of every open subset of Y is open, that is, f is continuous.

Second part. Let $Y = [a, b]$, a closed, bounded interval with the topology it inherits from $\mathbb{R}$, and let $f: X \rightarrow [a, b]$. We assume $a < b$; if $a = b$ then $[a, b]$ is a one-point space, every mapping into it is continuous, and the stated condition is vacuous, so the asserted equivalence is trivial. We claim that the collection

$$\mathcal{S} = \{[a, c] \mid a < c < b\} \cup \{(c, b] \mid a < c < b\}$$

is a subbase for the topology on $[a, b]$. (These sets are indeed open in $[a, b]$, being the traces on $[a, b]$ of the open sets $(-\infty, c)$ and (c, ∞) of $\mathbb{R}$. To see that finite intersections of members of $\mathcal{S}$ form a base, let $\mathcal{O}$ be open in $[a, b]$ and let $y \in \mathcal{O}$; then $\mathcal{O} = [a, b] \cap U$ with U open in $\mathbb{R}$, so there is $\epsilon > 0$ with $[a, b] \cap (y - \epsilon, y + \epsilon) \subseteq \mathcal{O}$. If $a < y < b$, shrink ϵ so that $a < y - \epsilon$ and $y + \epsilon < b$; then $y \in (y - \epsilon, y + \epsilon) = (y - \epsilon, b] \cap [a, y + \epsilon) \subseteq \mathcal{O}$, an intersection of two members of $\mathcal{S}$. If $y = a$, choose c with $a < c < \min\{a + \epsilon, b\}$; then $y \in [a, c] \subseteq \mathcal{O}$. If $y = b$, choose c with $\max\{a, b - \epsilon\} < c < b$; then $y \in (c, b] \subseteq \mathcal{O}$. In every case y lies in a finite intersection of members of $\mathcal{S}$ that is contained in $\mathcal{O}$, so these finite intersections form a base.)

Now observe that for $c \in (a, b)$,

$$f^{-1}([a, c]) = \{x \in X \mid f(x) < c\}, \quad f^{-1}((c, b]) = \{x \in X \mid f(x) > c\},$$

since f takes values in $[a, b]$. Hence the hypothesis that $\{x \in X \mid f(x) < c\}$ and $\{x \in X \mid f(x) > c\}$ are open for every $c \in (a, b)$ says exactly that the preimage under f of each member of the subbase $\mathcal{S}$ is open in X ; by the first part this is equivalent to the continuity of f . Conversely, if f is continuous then these two sets, being preimages of the open sets $[a, c)$ and $(c, b]$, are open. This is the asserted equivalence.

Problem 12.10. Use the preceding problem to prove Lemma 1. [Lemma 1: Let X be a topological space. For Λ a dense subset of the open, bounded interval of real numbers (a, b) , let $\{\mathcal{O}_{\lambda}\}_{\lambda \in \Lambda}$ be a normally ascending collection of open subsets of X . Define the function $f: X \rightarrow \mathbb{R}$ by setting $f = b$ on $X \sim \bigcup_{\lambda \in \Lambda} \mathcal{O}_{\lambda}$ and otherwise setting

$$f(x) = \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_{\lambda}\}.$$

Then $f: X \rightarrow [a, b]$ is continuous.]

Solution. Recall that the collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ being normally ascending means

$$\overline{\mathcal{O}_{\lambda_1}} \subseteq \mathcal{O}_{\lambda_2} \quad \text{whenever } \lambda_1, \lambda_2 \in \Lambda \text{ and } \lambda_1 < \lambda_2.$$

Write $\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$. Since $a < b$ and Λ is dense in (a, b) , Λ is nonempty, and every nonempty open subinterval of (a, b) contains a point of Λ .

f takes values in $[a, b]$. If $x \in X \sim \mathcal{U}$ then $f(x) = b \in [a, b]$. If $x \in \mathcal{U}$, the set $\Lambda_x = \{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\}$ is nonempty and is contained in (a, b) , so it is bounded below by a and contains a point smaller than b ; hence $f(x) = \inf \Lambda_x$ satisfies $a \leq f(x) < b$. Thus $f: X \rightarrow [a, b]$.

By Problem 9 it now suffices to show that for each $c \in (a, b)$ the two sets $\{x \in X \mid f(x) < c\}$ and $\{x \in X \mid f(x) > c\}$ are open in X . Fix $c \in (a, b)$.

The set $\{f < c\}$. We claim

$$\{x \in X \mid f(x) < c\} = \bigcup_{\lambda \in \Lambda, \lambda < c} \mathcal{O}_\lambda.$$

Suppose $f(x) < c$. Since $c < b$, we cannot have $x \in X \sim \mathcal{U}$ (there $f = b > c$), so $x \in \mathcal{U}$ and $f(x) = \inf \Lambda_x < c$. By the definition of the infimum there is $\lambda \in \Lambda_x$ with $\lambda < c$, that is, $x \in \mathcal{O}_\lambda$ with $\lambda < c$. Conversely, if $x \in \mathcal{O}_\lambda$ for some $\lambda \in \Lambda$ with $\lambda < c$, then $x \in \mathcal{U}$ and $f(x) = \inf \Lambda_x \leq \lambda < c$. This proves the claim, and the right-hand side, being a union of open sets, is open.

The set $\{f > c\}$. We claim

$$\{x \in X \mid f(x) \leq c\} = \bigcap_{\lambda \in \Lambda, \lambda > c} \overline{\mathcal{O}_\lambda}.$$

Suppose first that $f(x) \leq c$. Since $c < b$, again $x \in \mathcal{U}$ and $f(x) = \inf \Lambda_x \leq c$. Let $\mu \in \Lambda$ with $\mu > c$. Since $\inf \Lambda_x \leq c < \mu$, there is $\nu \in \Lambda_x$ with $\nu < \mu$; then, by the normally ascending property,

$$x \in \mathcal{O}_\nu \subseteq \overline{\mathcal{O}_\nu} \subseteq \mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\mu}.$$

As μ was an arbitrary member of Λ exceeding c , the point x belongs to the right-hand set.

Conversely, suppose $x \in \overline{\mathcal{O}_\lambda}$ for every $\lambda \in \Lambda$ with $\lambda > c$. Let $\mu \in \Lambda$ with $\mu > c$. Since $c < \mu$ and (c, μ) is a nonempty open subinterval of (a, b) , density of Λ provides $\lambda \in \Lambda$ with $c < \lambda < \mu$. Then $x \in \overline{\mathcal{O}_\lambda} \subseteq \mathcal{O}_\mu$. Hence $x \in \mathcal{O}_\mu$ for every $\mu \in \Lambda$ with $\mu > c$. In particular $\Lambda_x \supseteq \Lambda \cap (c, b)$, and this last set is nonempty, so $x \in \mathcal{U}$ and

$$f(x) = \inf \Lambda_x \leq \inf(\Lambda \cap (c, b)) = c,$$

the final equality because Λ is dense in (a, b) , so Λ meets $(c, c + \epsilon)$ for every small $\epsilon > 0$, while every element of $\Lambda \cap (c, b)$ exceeds c . This proves the claim.

The right-hand side of the claim is an intersection of closed sets, hence closed, so $\{x \in X \mid f(x) \leq c\}$ is closed and its complement $\{x \in X \mid f(x) > c\}$ is open.

Both families of sets being open for every $c \in (a, b)$, Problem 9 applied to the closed, bounded interval $[a, b]$ yields that $f: X \rightarrow [a, b]$ is continuous. This is Lemma 1.

Problem 12.11. Show that the product of an arbitrary collection of Tychonoff spaces, with the product topology, also is Tychonoff.

Solution. Let $\{(X_\lambda, \mathcal{T}_\lambda)\}_{\lambda \in \Lambda}$ be a collection of Tychonoff topological spaces and let $X = \prod_{\lambda \in \Lambda} X_\lambda$ carry the product topology, that is, the topology having as a base the sets $\prod_{\lambda \in \Lambda} \mathcal{O}_\lambda$ where each $\mathcal{O}_\lambda \in \mathcal{T}_\lambda$ and $\mathcal{O}_\lambda = X_\lambda$ except for finitely many λ . Recall that X is Tychonoff provided each two points of X can be separated by neighborhoods: for $x \neq y$ in X there is a neighborhood of x that does not contain y and a neighborhood of y that does not contain x .

For each $\lambda_0 \in \Lambda$ and each open subset $\mathcal{O}$ of X_{λ_0} , the set

$$\pi_{\lambda_0}^{-1}(\mathcal{O}) = \prod_{\lambda \in \Lambda} \mathcal{O}_\lambda, \quad \text{where } \mathcal{O}_{\lambda_0} = \mathcal{O} \text{ and } \mathcal{O}_\lambda = X_\lambda \text{ for } \lambda \neq \lambda_0,$$

is a basic open subset of X ; here $\pi_{\lambda_0}: X \rightarrow X_{\lambda_0}$ is the λ_0 projection, $\pi_{\lambda_0}(x) = x_{\lambda_0}$. This is immediate from the definition of the product topology, since all but one of the factors is the whole space. Explicitly,

$$\pi_{\lambda_0}^{-1}(\mathcal{O}) = \{x \in X \mid x_{\lambda_0} \in \mathcal{O}\}.$$

Now let x and y be two distinct points of X . A point of the Cartesian product is a mapping on the index set Λ , and two such mappings are distinct precisely when they differ at some index. Hence there is $\lambda_0 \in \Lambda$ with

$$x_{\lambda_0} \neq y_{\lambda_0} \quad \text{in } X_{\lambda_0}.$$

Since X_{λ_0} is Tychonoff, there are neighborhoods $\mathcal{U}$ of x_{λ_0} and $\mathcal{V}$ of y_{λ_0} in X_{λ_0} with

$$y_{\lambda_0} \notin \mathcal{U} \quad \text{and} \quad x_{\lambda_0} \notin \mathcal{V}.$$

Consider $\pi_{\lambda_0}^{-1}(\mathcal{U})$ and $\pi_{\lambda_0}^{-1}(\mathcal{V})$. By the preceding paragraph these are open subsets of X . Moreover $x \in \pi_{\lambda_0}^{-1}(\mathcal{U})$ because $x_{\lambda_0} \in \mathcal{U}$, while $y \notin \pi_{\lambda_0}^{-1}(\mathcal{U})$ because $y_{\lambda_0} \notin \mathcal{U}$; symmetrically $y \in \pi_{\lambda_0}^{-1}(\mathcal{V})$ and $x \notin \pi_{\lambda_0}^{-1}(\mathcal{V})$. Thus $\pi_{\lambda_0}^{-1}(\mathcal{U})$ is a neighborhood of x that fails to contain y , and $\pi_{\lambda_0}^{-1}(\mathcal{V})$ is a neighborhood of y that fails to contain x . Therefore X possesses the Tychonoff separation property.

An alternative formulation of the same argument uses Proposition 6 of the preceding chapter, according to which a topological space is Tychonoff if and only if every one-point set is closed. Given $x \in X$, each $\{x_\lambda\}$ is closed in X_λ , so each $\pi_\lambda^{-1}(X_\lambda \sim \{x_\lambda\})$ is open in X , and

$$X \sim \{x\} = \bigcup_{\lambda \in \Lambda} \pi_\lambda^{-1}(X_\lambda \sim \{x_\lambda\})$$

because a point y of X differs from x exactly when $y_\lambda \neq x_\lambda$ for some λ . Being a union of open sets, $X \sim \{x\}$ is open, so $\{x\}$ is closed and X is Tychonoff.

Problem 12.12. Show that the product of an arbitrary collection of Hausdorff spaces, with the product topology, also is Hausdorff.

Solution. Let $\{(X_\lambda, \mathcal{T}_\lambda)\}_{\lambda \in \Lambda}$ be a collection of Hausdorff topological spaces and let $X = \prod_{\lambda \in \Lambda} X_\lambda$ have the product topology. Recall that X is Hausdorff provided each two points of X can be separated by disjoint neighborhoods.

As in the preceding problem, for $\lambda_0 \in \Lambda$ and $\mathcal{O}$ open in X_{λ_0} the set

$$\pi_{\lambda_0}^{-1}(\mathcal{O}) = \{x \in X \mid x_{\lambda_0} \in \mathcal{O}\}$$

is a basic open subset of X : it is the product $\prod_{\lambda \in \Lambda} \mathcal{O}_\lambda$ in which $\mathcal{O}_{\lambda_0} = \mathcal{O}$ and $\mathcal{O}_\lambda = X_\lambda$ for all $\lambda \neq \lambda_0$, and all but one factor equals the whole space.

Let x and y be distinct points of X . Since points of the Cartesian product are mappings on Λ and $x \neq y$, there is an index $\lambda_0 \in \Lambda$ for which

$$x_{\lambda_0} \neq y_{\lambda_0}.$$

Because X_{λ_0} is Hausdorff, there are disjoint neighborhoods $\mathcal{U}$ of x_{λ_0} and $\mathcal{V}$ of y_{λ_0} in X_{λ_0} , so that $\mathcal{U} \cap \mathcal{V} = \emptyset$. Define

$$\mathcal{N}_x = \pi_{\lambda_0}^{-1}(\mathcal{U}), \quad \mathcal{N}_y = \pi_{\lambda_0}^{-1}(\mathcal{V}).$$

These are open subsets of X , $x \in \mathcal{N}_x$ since $x_{\lambda_0} \in \mathcal{U}$, and $y \in \mathcal{N}_y$ since $y_{\lambda_0} \in \mathcal{V}$. Finally, since preimages preserve intersections,

$$\mathcal{N}_x \cap \mathcal{N}_y = \pi_{\lambda_0}^{-1}(\mathcal{U}) \cap \pi_{\lambda_0}^{-1}(\mathcal{V}) = \pi_{\lambda_0}^{-1}(\mathcal{U} \cap \mathcal{V}) = \pi_{\lambda_0}^{-1}(\emptyset) = \emptyset.$$

Thus $\mathcal{N}_x$ and $\mathcal{N}_y$ are disjoint neighborhoods of x and y respectively, and X possesses the Hausdorff separation property.

Observe that only the single coordinate λ_0 at which x and y differ was used; no restriction whatever on the size of the index set Λ enters, so the conclusion holds for an arbitrary collection of Hausdorff spaces. The same remark applies to the Tychonoff case of the preceding problem. In particular, by the Tychonoff Product Theorem, a product of arbitrarily many compact Hausdorff spaces is again compact and Hausdorff.

Problem 12.13. Consider the Cartesian product of n copies of $\mathbb{R}$,

$$\mathbb{R}^n = \overbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}^n.$$

Show that the product topology is the same as the metric topology on $\mathbb{R}^n$ induced by the Euclidean metric.

Solution. Write points of $\mathbb{R}^n$ as $x = (x_1, \dots, x_n)$ and let

$$\rho(x, y) = \left[\sum_{k=1}^n (x_k - y_k)^2 \right]^{1/2}$$

be the Euclidean metric. Denote by $\mathcal{T}_{\text{prod}}$ the product topology on $\mathbb{R}^n$, which by definition has as a base the sets $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ with each $\mathcal{O}_k$ open in $\mathbb{R}$, and by $\mathcal{T}_\rho$ the topology induced by ρ , which has as a base the open balls

$$B_\rho(x, \epsilon) = \{y \in \mathbb{R}^n \mid \rho(x, y) < \epsilon\}, \quad x \in \mathbb{R}^n, \epsilon > 0.$$

We show each base is subordinate to the other, which gives the equality of the two topologies. Two elementary inequalities are used. For all $x, y \in \mathbb{R}^n$ and each index k ,

$$|x_k - y_k| \leq \rho(x, y),$$

since $(x_k - y_k)^2 \leq \sum_j (x_j - y_j)^2$; and if $|x_k - y_k| < \delta$ for every k , then

$$\rho(x, y) = \left[\sum_{k=1}^n (x_k - y_k)^2 \right]^{1/2} < [n\delta^2]^{1/2} = \sqrt{n} \delta.$$

$\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_\rho$. Let $\mathcal{B} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ be a basic set for the product topology and let $x \in \mathcal{B}$. For each k the point x_k belongs to the open subset $\mathcal{O}_k$ of $\mathbb{R}$, so there is $\epsilon_k > 0$ with $(x_k - \epsilon_k, x_k + \epsilon_k) \subseteq \mathcal{O}_k$. Put $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\} > 0$, a minimum of finitely many positive numbers. If $\rho(x, y) < \epsilon$, then by the first inequality $|x_k - y_k| < \epsilon \leq \epsilon_k$ for each k , so $y_k \in \mathcal{O}_k$ for each k , that is, $y \in \mathcal{B}$. Hence $B_\rho(x, \epsilon) \subseteq \mathcal{B}$. Since each point of $\mathcal{B}$ is interior to $\mathcal{B}$ with respect to ρ , the set $\mathcal{B}$ belongs to $\mathcal{T}_\rho$. Every set in $\mathcal{T}_{\text{prod}}$ is a union of such $\mathcal{B}$'s, hence belongs to $\mathcal{T}_\rho$.

$\mathcal{T}_\rho \subseteq \mathcal{T}_{\text{prod}}$. Let $\mathcal{U} \in \mathcal{T}_\rho$ and let $x \in \mathcal{U}$. Choose $\epsilon > 0$ with $B_\rho(x, \epsilon) \subseteq \mathcal{U}$, and set $\delta = \epsilon/\sqrt{n}$. Consider

$$\mathcal{B} = (x_1 - \delta, x_1 + \delta) \times \cdots \times (x_n - \delta, x_n + \delta),$$

a basic set for the product topology containing x . If $y \in \mathcal{B}$, then $|x_k - y_k| < \delta$ for every k , so by the second inequality $\rho(x, y) < \sqrt{n} \delta = \epsilon$, whence $y \in B_\rho(x, \epsilon) \subseteq \mathcal{U}$. Thus $x \in \mathcal{B} \subseteq \mathcal{U}$. Since every point of $\mathcal{U}$ lies in a basic product-open set contained in $\mathcal{U}$, the set $\mathcal{U}$ is the union of such sets and therefore belongs to $\mathcal{T}_{\text{prod}}$.

The two inclusions give $\mathcal{T}_{\text{prod}} = \mathcal{T}_\rho$: the product topology on $\mathbb{R}^n$ is exactly the Euclidean metric topology.

Problem 12.14. Let (X, ρ_1) and (Y, ρ_2) be metric spaces. Show that the product topology on $X \times Y$, where X and Y have the topologies induced by their respective metrics, is the same as the topology induced by the product metric

$$\rho((x_1, y_1), (x_2, y_2)) = \sqrt{[\rho_1(x_1, x_2)]^2 + [\rho_2(y_1, y_2)]^2}.$$

Solution. Denote by $\mathcal{T}_{\text{prod}}$ the product topology on $X \times Y$, whose base consists of the sets $\mathcal{O} \times \mathcal{V}$ with $\mathcal{O}$ open in X and $\mathcal{V}$ open in Y , and by $\mathcal{T}_\rho$ the topology induced by the product metric ρ , whose base consists of the balls

$$B_\rho((x, y), \epsilon) = \{(x', y') \in X \times Y \mid \rho((x, y), (x', y')) < \epsilon\}.$$

We write $B_1(x, r)$ and $B_2(y, r)$ for the open balls of (X, ρ_1) and (Y, ρ_2) ; these form bases for the topologies induced on X and on Y by their metrics.

Two inequalities are used. From the definition of ρ , for all $(x_1, y_1), (x_2, y_2)$,

$$\rho_1(x_1, x_2) \leq \rho((x_1, y_1), (x_2, y_2)) \quad \text{and} \quad \rho_2(y_1, y_2) \leq \rho((x_1, y_1), (x_2, y_2)),$$

since each summand under the square root is dominated by the sum. Conversely, if $\rho_1(x_1, x_2) < \delta$ and $\rho_2(y_1, y_2) < \delta$, then

$$\rho((x_1, y_1), (x_2, y_2)) < \sqrt{\delta^2 + \delta^2} = \sqrt{2} \delta.$$

$\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_\rho$. Let $\mathcal{O} \times \mathcal{V}$ be a basic product-open set and let $(x, y) \in \mathcal{O} \times \mathcal{V}$. Since $\mathcal{O}$ is open in the metric space (X, ρ_1) there is $\epsilon_1 > 0$ with $B_1(x, \epsilon_1) \subseteq \mathcal{O}$, and since $\mathcal{V}$ is open in (Y, ρ_2) there is $\epsilon_2 > 0$ with $B_2(y, \epsilon_2) \subseteq \mathcal{V}$. Put $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$. If $\rho((x, y), (x', y')) < \epsilon$, then by the first pair of inequalities $\rho_1(x, x') < \epsilon \leq \epsilon_1$ and $\rho_2(y, y') < \epsilon \leq \epsilon_2$, so $x' \in \mathcal{O}$ and $y' \in \mathcal{V}$, that is, $(x', y') \in \mathcal{O} \times \mathcal{V}$. Hence

$$B_\rho((x, y), \epsilon) \subseteq \mathcal{O} \times \mathcal{V},$$

so $\mathcal{O} \times \mathcal{V}$ is open with respect to ρ . Every product-open set is a union of such basic sets and therefore belongs to $\mathcal{T}_\rho$.

$\mathcal{T}_\rho \subseteq \mathcal{T}_{\text{prod}}$. Let $\mathcal{U} \in \mathcal{T}_\rho$ and $(x, y) \in \mathcal{U}$. Choose $\epsilon > 0$ with $B_\rho((x, y), \epsilon) \subseteq \mathcal{U}$ and set $\delta = \epsilon/\sqrt{2}$. The set $B_1(x, \delta) \times B_2(y, \delta)$ is a basic product-open set containing (x, y) , since $B_1(x, \delta)$ is open in X and $B_2(y, \delta)$ is open in Y . If (x', y') belongs to it, then $\rho_1(x, x') < \delta$ and $\rho_2(y, y') < \delta$, so by the second inequality

$$\rho((x, y), (x', y')) < \sqrt{2} \delta = \epsilon,$$

and hence $(x', y') \in B_\rho((x, y), \epsilon) \subseteq \mathcal{U}$. Thus

$$(x, y) \in B_1(x, \delta) \times B_2(y, \delta) \subseteq \mathcal{U},$$

so $\mathcal{U}$ is a union of basic product-open sets and belongs to $\mathcal{T}_{\text{prod}}$.

The two inclusions give $\mathcal{T}_{\text{prod}} = \mathcal{T}_\rho$: for two metric spaces the product topology is exactly the topology induced by the product metric.

Observe that neither completeness nor separability of the two metric spaces was used, and no countability assumption entered: the argument is purely a comparison of the two bases at each point. All that was needed of ρ were the two displayed inequalities, namely that ρ dominates each of ρ_1, ρ_2 and is in turn dominated by a fixed multiple of $\max\{\rho_1, \rho_2\}$. Since $\max\{\rho_1, \rho_2\}$ and $\rho_1 + \rho_2$ satisfy the same two inequalities, with the constants 1 and 2 in place of $\sqrt{2}$, the identical proof shows that these metrics on $X \times Y$ also induce the product topology.

12.3 Problems 12.15–12.21

Problem 12.15. Show that if X is a metric space with metric ρ , then

$$\rho^*(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}$$

also is a metric on X and it induces the same topology as the metric ρ .

Solution. Define $\varphi: [0, \infty) \rightarrow [0, 1)$ by

$$\varphi(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t},$$

so that $\rho^* = \varphi \circ \rho$. We record three properties of φ .

(P1) φ is strictly increasing on $[0, \infty)$, since $t \mapsto 1/(1+t)$ is strictly decreasing there. Moreover $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$.

(P2) φ maps $[0, \infty)$ one-to-one and onto $[0, 1)$, with inverse $\psi(s) = s/(1-s)$: indeed if $s = t/(1+t)$ then $s < 1$ and $s(1+t) = t$, so $t = s/(1-s)$; conversely $\varphi(s/(1-s)) = s$ for $0 \leq s < 1$. Also ψ is strictly increasing.

(P3) φ is subadditive: for $a, b \geq 0$,

$$\varphi(a+b) = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b} = \varphi(a) + \varphi(b),$$

since $1+a+b \geq 1+a$ and $1+a+b \geq 1+b$.

Step 1. We show that ρ^* is a metric. Since $\rho \geq 0$ and $\varphi \geq 0$ on $[0, \infty)$, we have $\rho^* \geq 0$, and ρ^* is real-valued (indeed $\rho^* < 1$). By (P1), $\rho^*(x, y) = \varphi(\rho(x, y)) = 0$ if and only if $\rho(x, y) = 0$, that is, if and only if $x = y$. Symmetry of ρ^* is immediate from symmetry of ρ . For the triangle inequality, let x, y, z belong to X and put $a = \rho(x, y)$, $b = \rho(y, z)$. Since $\rho(x, z) \leq a + b$ and φ is increasing by (P1), and then by subadditivity (P3),

$$\rho^*(x, z) = \varphi(\rho(x, z)) \leq \varphi(a+b) \leq \varphi(a) + \varphi(b) = \rho^*(x, y) + \rho^*(y, z).$$

Thus ρ^* is a metric on X .

Step 2. We show the two metrics have the same open balls, hence the same topology. Denote by $B_\rho(x, r) = \{y \in X \mid \rho(x, y) < r\}$ and $B_{\rho^*}(x, r) = \{y \in X \mid \rho^*(x, y) < r\}$ the balls with respect to the two metrics.

Fix $x \in X$. Let $r > 0$. Since φ is a strictly increasing bijection of $[0, \infty)$ onto $[0, 1)$ by (P1) and (P2), for any y we have $\rho(x, y) < r$ if and only if $\varphi(\rho(x, y)) < \varphi(r)$, that is, if and only if $\rho^*(x, y) < \varphi(r)$. Hence

$$B_\rho(x, r) = B_{\rho^*}(x, \varphi(r)), \quad \varphi(r) > 0.$$

Conversely let $\epsilon > 0$. If $\epsilon \geq 1$ then $B_{\rho^*}(x, \epsilon) = X$, since $\rho^* < 1$ always; and X is the union of the ρ -balls $B_\rho(x', 1)$, $x' \in X$, so it is ρ -open. If $0 < \epsilon < 1$, then, again because φ is a strictly increasing bijection onto $[0, 1)$ with strictly increasing inverse ψ ,

$$B_{\rho^*}(x, \epsilon) = B_\rho(x, \psi(\epsilon)) = B_\rho\left(x, \frac{\epsilon}{1-\epsilon}\right), \quad \psi(\epsilon) > 0.$$

Therefore every ball for one metric is a ball for the other (or, in the single degenerate case $\epsilon \geq 1$, is all of X , which is open for both). Since the open balls of a metric form a base for the topology it induces, the two collections of balls generate the same topology: a set is ρ -open if and only if it is a union of ρ -balls, if and only if it is a union of ρ^* -balls, if and only if it is ρ^* -open. Hence ρ and ρ^* induce the same topology on X .

Problem 12.16. Consider the countable collection of metric spaces $\{(X_n, \rho_n)\}_{n=1}^{\infty}$. For the Cartesian product of these sets $X = \prod_{n=1}^{\infty} X_n$, define $\rho: X \times X \rightarrow \mathbb{R}$ by

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{\rho_n(x_n, y_n)}{2^n [1 + \rho_n(x_n, y_n)]}.$$

Use the preceding problem to show that ρ is a metric on $X = \prod_{n=1}^{\infty} X_n$ which induces the product topology on X , where each X_n has the topology induced by the metric ρ_n .

Solution. For each n set

$$\rho_n^*(u, v) = \frac{\rho_n(u, v)}{1 + \rho_n(u, v)} \quad \text{for } u, v \in X_n.$$

By the preceding problem, ρ_n^* is a metric on X_n inducing the same topology as ρ_n , and $0 \leq \rho_n^* < 1$. With this notation,

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n^*(x_n, y_n).$$

Step 1. The series converges and ρ is a metric. Each term is nonnegative and dominated by 2^{-n} , and $\sum_{n=1}^{\infty} 2^{-n} = 1$; hence the series converges, with $0 \leq \rho(x, y) \leq 1$, and ρ is a well-defined real-valued function on $X \times X$.

If $\rho(x, y) = 0$, then, all terms being nonnegative, $\rho_n^*(x_n, y_n) = 0$ for every n , so $x_n = y_n$ for every n , that is, $x = y$; and clearly $\rho(x, x) = 0$. Symmetry is inherited termwise from the symmetry of each ρ_n^* . For the triangle inequality, let x, y, z belong to X . For each n , $\rho_n^*(x_n, z_n) \leq \rho_n^*(x_n, y_n) + \rho_n^*(y_n, z_n)$; multiplying by 2^{-n} and summing over n (all series converge, so the sums may be added termwise) gives

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

Thus ρ is a metric on X .

Step 2. Every ρ -ball contains a product-open neighborhood of its center. Let $x \in X$ and $\epsilon > 0$. Choose a natural number N so large that $\sum_{n>N} 2^{-n} = 2^{-N} < \epsilon/2$. Define

$$\mathcal{O} = \prod_{n=1}^{\infty} \mathcal{O}_n, \quad \mathcal{O}_n = \begin{cases} B_{\rho_n}(x_n, \epsilon/2) & \text{if } n \leq N, \\ X_n & \text{if } n > N. \end{cases}$$

Each $\mathcal{O}_n$ is open in X_n and $\mathcal{O}_n = X_n$ for all but finitely many n , so $\mathcal{O}$ is a basic set for the product topology, and $x \in \mathcal{O}$. If $y \in \mathcal{O}$, then for $n \leq N$ we have $\rho_n^*(x_n, y_n) \leq \rho_n(x_n, y_n) < \epsilon/2$ (since $\rho_n^* \leq \rho_n$), while for $n > N$ we use $\rho_n^* < 1$. Hence

$$\rho(x, y) = \sum_{n \leq N} \frac{\rho_n^*(x_n, y_n)}{2^n} + \sum_{n > N} \frac{\rho_n^*(x_n, y_n)}{2^n} < \frac{\epsilon}{2} \sum_{n \leq N} \frac{1}{2^n} + \sum_{n > N} \frac{1}{2^n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $x \in \mathcal{O} \subseteq B_{\rho}(x, \epsilon)$. Consequently every ρ -open set is open in the product topology: if U is ρ -open and $x \in U$, choose $\epsilon > 0$ with $B_{\rho}(x, \epsilon) \subseteq U$ and then the above $\mathcal{O}$ satisfies $x \in \mathcal{O} \subseteq U$, so U is a union of basic product-open sets.

Step 3. Every basic product-open set contains a ρ -ball about each of its points. Let $\mathcal{O} = \prod_{n=1}^{\infty} \mathcal{O}_n$ be a basic set for the product topology, each $\mathcal{O}_n$ open in X_n and $\mathcal{O}_n = X_n$ for all n outside a finite set F of indices, and let $x \in \mathcal{O}$. If F is empty then $\mathcal{O} = X$ and there is nothing to prove. Otherwise, for each $n \in F$ choose $r_n > 0$ with $B_{\rho_n}(x_n, r_n) \subseteq \mathcal{O}_n$, and set

$$\epsilon = \min_{n \in F} \frac{1}{2^n} \cdot \frac{r_n}{1 + r_n} > 0.$$

Suppose $\rho(x, y) < \epsilon$. Fix $n \in F$. Since all terms of the defining series are nonnegative,

$$\frac{1}{2^n} \rho_n^*(x_n, y_n) \leq \rho(x, y) < \epsilon \leq \frac{1}{2^n} \cdot \frac{r_n}{1 + r_n},$$

so $\rho_n^*(x_n, y_n) < r_n/(1 + r_n)$. Because $t \mapsto t/(1 + t)$ is strictly increasing on $[0, \infty)$ (as shown in the preceding problem), this forces $\rho_n(x_n, y_n) < r_n$, that is, $y_n \in B_{\rho_n}(x_n, r_n) \subseteq \mathcal{O}_n$. For $n \notin F$ the condition $y_n \in \mathcal{O}_n = X_n$ is automatic. Hence $y \in \mathcal{O}$, and $B_\rho(x, \epsilon) \subseteq \mathcal{O}$.

Consequently every product-open set is ρ -open: a product-open set is a union of basic sets, and by the above each of its points is interior to it with respect to ρ .

Combining Steps 2 and 3, the topology induced by ρ and the product topology on X have exactly the same open sets, so ρ induces the product topology.

Problem 12.17. Consider the set $X = \mathbb{R}^{\mathbb{R}}$ with the product topology. Let E be the subset of X consisting of functions that take the value 0 on a countable set and elsewhere take the value 1. Let f_0 be the function that is identically zero. Then it is clear that f_0 is a point of closure of E . But there is no sequence $\{f_n\}$ in E that converges to f_0 , since for any sequence $\{f_n\}$ in E there is some $x_0 \in \mathbb{R}$ such that $f_n(x_0) = 1$ for all n and so the sequence $\{f_n(x_0)\}$ does not converge to $f_0(x_0)$. This shows, in particular, that $X = \mathbb{R}^{\mathbb{R}}$ is not first countable and therefore not metrizable.

Solution. Recall that $X = \mathbb{R}^{\mathbb{R}}$ is the set of all real-valued functions of a real variable, and that a base for the product topology consists of the sets

$$\mathcal{N}(f; x_1, \dots, x_k; \epsilon) = \{g \in X \mid |g(x_i) - f(x_i)| < \epsilon \text{ for } i = 1, \dots, k\},$$

where k is a natural number, $x_1, \dots, x_k$ are real numbers and $\epsilon > 0$. (These are the basic sets $\prod_{x \in \mathbb{R}} \mathcal{O}_x$ in which $\mathcal{O}_x = \mathbb{R}$ except for the finitely many $x = x_i$; every basic set contains a set of this form about each of its points, so these do form a base.)

Step 1. f_0 is a point of closure of E . Let $\mathcal{N}$ be a basic neighborhood of f_0 , say $\mathcal{N} = \mathcal{N}(f_0; x_1, \dots, x_k; \epsilon)$. Let C be any countably infinite subset of $\mathbb{R}$ containing $x_1, \dots, x_k$ (for instance $C = \{x_1, \dots, x_k\} \cup \mathbb{N}$), and define

$$f(x) = \begin{cases} 0 & \text{if } x \in C, \\ 1 & \text{if } x \notin C. \end{cases}$$

Then f belongs to E , and $f(x_i) = 0$ for $1 \leq i \leq k$, so $|f(x_i) - f_0(x_i)| = 0 < \epsilon$ and hence $f \in \mathcal{N}$. Thus every basic neighborhood of f_0 meets E ; since every neighborhood of f_0 contains a basic one, every neighborhood of f_0 meets E . Therefore f_0 is a point of closure of E .

Step 2. No sequence in E converges to f_0 . Let $\{f_n\}$ be a sequence in E and let $Z_n = \{x \in \mathbb{R} \mid f_n(x) = 0\}$, which is countable by the definition of E . Then $Z = \bigcup_{n=1}^{\infty} Z_n$ is a countable union of countable sets, hence countable, while $\mathbb{R}$ is uncountable. Choose $x_0 \in \mathbb{R} \setminus Z$. Then $f_n(x_0) \neq 0$ for every n , and since each f_n takes only the values 0 and 1, $f_n(x_0) = 1$ for every n . By Proposition 3 of this chapter, a sequence converges in the product space $\mathbb{R}^{\mathbb{R}}$ if and only if it converges pointwise; since $\{f_n(x_0)\} = \{1, 1, 1, \dots\}$ does not converge to $f_0(x_0) = 0$, the sequence $\{f_n\}$ does not converge to f_0 in X . (Directly: the neighborhood $\mathcal{N}(f_0; x_0; 1/2)$ of f_0 contains no f_n .)

Step 3. X is not first countable. Suppose, to get a contradiction, that X had a countable base $\{\mathcal{N}_k\}_{k=1}^{\infty}$ of neighborhoods at the point f_0 . Replacing $\mathcal{N}_k$ by $\mathcal{N}_1 \cap \dots \cap \mathcal{N}_k$, which is again a neighborhood of f_0 , we may assume the sequence is descending: $\mathcal{N}_1 \supseteq \mathcal{N}_2 \supseteq \dots$. By Step 1 each $\mathcal{N}_k$ has nonempty intersection with E ; choose $f_k \in \mathcal{N}_k \cap E$. We claim $\{f_k\}$ converges to f_0 . Indeed, let $\mathcal{U}$ be any neighborhood of f_0 . Since $\{\mathcal{N}_k\}$ is a base at f_0 , there is an index K with $\mathcal{N}_K \subseteq \mathcal{U}$; then for $k \geq K$ we have $f_k \in \mathcal{N}_k \subseteq \mathcal{N}_K \subseteq \mathcal{U}$. So $\{f_k\}$ is a sequence in E converging to f_0 , contradicting Step 2. (This is exactly the failure of Proposition 9 of Chapter 11, which asserts that in a first countable space every point of closure of a set is the limit of a sequence from the set.) Hence X is not first countable at f_0 , and in particular X is not first countable.

Step 4. X is not metrizable. Every metric space is first countable: if σ is a metric inducing the topology, then at each point p the balls $\{B_\sigma(p, 1/k)\}_{k=1}^{\infty}$ form a countable base of neighborhoods at p , since any neighborhood of p contains some ball $B_\sigma(p, r)$ with $r > 0$ and hence contains $B_\sigma(p, 1/k)$ for $1/k < r$. Since $X = \mathbb{R}^{\mathbb{R}}$ is not first countable, its topology is not induced by any metric; that is, the product topology on $\mathbb{R}^{\mathbb{R}}$ is not metrizable.

Problem 12.18. Let X denote the discrete topological space with two elements. Show that $X^{\mathbb{N}}$ is homeomorphic to the Cantor set.

Solution. It is convenient to name the two points of X so that $X = \{0, 2\}$, with the discrete topology; then $X^{\mathbb{N}}$ is the set of all sequences $a = (a_1, a_2, \dots)$ with each $a_n \in \{0, 2\}$, given the product topology. Recall the construction of the Cantor set C . Put $C_0 = [0, 1]$ and, having defined C_k as a disjoint union of 2^k closed intervals of length 3^{-k} , let C_{k+1} be obtained from C_k by removing from each of these intervals its open middle third; then $C = \bigcap_{k=0}^{\infty} C_k$.

Step 1. Description of the intervals of C_k . We claim that for each $k \geq 0$ the 2^k intervals comprising C_k are precisely the intervals

$$I(a_1, \dots, a_k) = \left[\sum_{n=1}^k \frac{a_n}{3^n}, \sum_{n=1}^k \frac{a_n}{3^n} + \frac{1}{3^k} \right], \quad a_1, \dots, a_k \in \{0, 2\},$$

and that these 2^k intervals are pairwise disjoint. This is clear for $k = 0$, where the empty sum gives $[0, 1]$. Assume it for k and let $I = [c, c + 3^{-k}]$ be one of the intervals of C_k , with $c = \sum_{n \leq k} a_n 3^{-n}$. Deleting the open middle third $(c + 3^{-k-1}, c + 2 \cdot 3^{-k-1})$ leaves the two closed intervals

$$[c, c + 3^{-k-1}] \quad \text{and} \quad [c + 2 \cdot 3^{-k-1}, c + 3 \cdot 3^{-k-1}],$$

which are $I(a_1, \dots, a_k, 0)$ and $I(a_1, \dots, a_k, 2)$; they are disjoint, and they are contained in I , so intervals arising from distinct intervals of C_k remain disjoint. This completes the induction.

Step 2. Definition of the map. Define $\Phi: X^{\mathbb{N}} \rightarrow \mathbb{R}$ by

$$\Phi(a) = \sum_{n=1}^{\infty} \frac{a_n}{3^n};$$

the series converges since $0 \leq a_n \leq 2$ and $\sum 2 \cdot 3^{-n}$ converges. Fix k . Writing $c_k = \sum_{n \leq k} a_n 3^{-n}$, we have

$$0 \leq \Phi(a) - c_k = \sum_{n > k} \frac{a_n}{3^n} \leq 2 \sum_{n > k} \frac{1}{3^n} = \frac{1}{3^k},$$

so $\Phi(a) \in I(a_1, \dots, a_k) \subseteq C_k$. As k was arbitrary, $\Phi(a) \in C$. Thus Φ maps $X^{\mathbb{N}}$ into C .

Step 3. Φ is one-to-one. Let $a \neq b$ in $X^{\mathbb{N}}$ and let k be the least index with $a_k \neq b_k$; without loss of generality $a_k = 2$ and $b_k = 0$. Then

$$\Phi(a) - \Phi(b) = \frac{2}{3^k} + \sum_{n > k} \frac{a_n - b_n}{3^n} \geq \frac{2}{3^k} - 2 \sum_{n > k} \frac{1}{3^n} = \frac{2}{3^k} - \frac{1}{3^k} = \frac{1}{3^k} > 0.$$

Hence $\Phi(a) \neq \Phi(b)$.

Step 4. Φ maps onto C . Let $x \in C$. For each k , x belongs to C_k , hence to exactly one of the pairwise disjoint intervals of C_k ; call it $I(a_1^{(k)}, \dots, a_k^{(k)})$. Since x lies in C_{k+1} , it lies in one of the two subintervals of that interval produced in Step 1, and those subintervals have the same first k digits; hence $a_n^{(k+1)} = a_n^{(k)}$ for $n \leq k$. So there is a single sequence $a \in X^{\mathbb{N}}$ with $a_n^{(k)} = a_n$ for all $n \leq k$ and all k . Now x and $\Phi(a)$ both belong to $I(a_1, \dots, a_k)$, an interval of length 3^{-k} , so $|x - \Phi(a)| \leq 3^{-k}$ for every k , and therefore $x = \Phi(a)$.

Step 5. Φ is continuous. Let $a \in X^{\mathbb{N}}$ and $\epsilon > 0$; choose k with $3^{-k} < \epsilon$. The set

$$\mathcal{N} = \{b \in X^{\mathbb{N}} \mid b_n = a_n \text{ for } 1 \leq n \leq k\} = \prod_{n=1}^{\infty} \mathcal{O}_n, \quad \mathcal{O}_n = \begin{cases} \{a_n\} & n \leq k, \\ X & n > k, \end{cases}$$

is a basic set for the product topology, since each singleton is open in the discrete space X and $\mathcal{O}_n = X$ for all but finitely many n ; and $a \in \mathcal{N}$. For $b \in \mathcal{N}$,

$$|\Phi(a) - \Phi(b)| = \left| \sum_{n > k} \frac{a_n - b_n}{3^n} \right| \leq 2 \sum_{n > k} \frac{1}{3^n} = \frac{1}{3^k} < \epsilon.$$

Hence Φ is continuous at a , and a was arbitrary.

Step 6. Conclusion. The space X is a finite discrete space, hence compact (any open cover has the finite subcover consisting of one member containing each of the two points). By the Tychonoff Product Theorem, $X^{\mathbb{N}}$ is compact. The Cantor set C , being a subspace of $\mathbb{R}$, is Hausdorff. By Steps 2 through 5, Φ is a continuous one-to-one mapping of the compact space $X^{\mathbb{N}}$ onto the Hausdorff space C . According to Proposition 19 of Chapter 11, such a mapping is a homeomorphism. Therefore $X^{\mathbb{N}}$ is homeomorphic to the Cantor set.

Problem 12.19. Using the Tychonoff Product Theorem and the compactness of each closed, bounded interval of real numbers prove that any closed, bounded subset of $\mathbb{R}^n$ is compact.

Solution. Let K be a closed, bounded subset of $\mathbb{R}^n$, where $\mathbb{R}^n$ carries its usual (Euclidean metric) topology.

Step 1. K is contained in a cube. Since K is bounded, there is $M > 0$ with $|x| \leq M$ for all $x \in K$, where $|\cdot|$ denotes the Euclidean norm. For each coordinate, $|x_i| \leq |x| \leq M$, so

$$K \subseteq Q = [-M, M]^n = \underbrace{[-M, M] \times \cdots \times [-M, M]}_n.$$

Step 2. Q is compact in its product topology. Each factor $[-M, M]$ is a closed, bounded interval of real numbers and is therefore compact, as given. By the Tychonoff Product Theorem, applied to the finite collection $\{[-M, M]\}_{i=1}^n$ of compact spaces, the Cartesian product Q with the product topology is compact.

Step 3. The product topology on Q is the topology Q inherits from $\mathbb{R}^n$. Write $J = [-M, M]$ with its subspace topology from $\mathbb{R}$. First, the product topology on $\prod_{i=1}^n J$ coincides with the topology $\prod_{i=1}^n J$ inherits as a subspace of the product space $\prod_{i=1}^n \mathbb{R}$. Indeed, a basic open set of the subspace $\prod_i J \subseteq \prod_i \mathbb{R}$ has the form

$$\left(\prod_{i=1}^n \mathcal{O}_i\right) \cap \prod_{i=1}^n J = \prod_{i=1}^n (\mathcal{O}_i \cap J),$$

with each $\mathcal{O}_i$ open in $\mathbb{R}$, and $\mathcal{O}_i \cap J$ is exactly a general open subset of the subspace J ; conversely every basic open set $\prod_i V_i$ of the product of the subspaces, V_i open in J , arises this way, since $V_i = \mathcal{O}_i \cap J$ for some open $\mathcal{O}_i \subseteq \mathbb{R}$. So the two bases coincide, and hence so do the topologies. Second, by Problem 13 of this chapter the product topology on $\prod_{i=1}^n \mathbb{R}$ is the same as the topology on $\mathbb{R}^n$ induced by the Euclidean metric. Combining the two statements: the product topology on Q is the subspace topology Q inherits from Euclidean $\mathbb{R}^n$.

Step 4. Conclusion. By Steps 2 and 3, Q is a compact subspace of $\mathbb{R}^n$. Since K is closed in $\mathbb{R}^n$ and $K \subseteq Q$, the set $K = K \cap Q$ is a closed subset of the subspace Q . According to Proposition 15 of Chapter 11, a closed subset of a compact topological space is compact. Hence K is compact as a subspace of Q , and therefore, the subspace topology K inherits from Q being the same as the one it inherits from $\mathbb{R}^n$, K is a compact subspace of $\mathbb{R}^n$.

Problem 12.20. Provide a direct proof of the assertion that if X is compact and I is a closed, bounded interval, then $X \times I$ is compact. (Hint: Let $\mathcal{U}$ be an open covering of $X \times I$, and consider the smallest value of $t \in I$ such that for each $t' < t$ the set $X \times [0, t']$ can be covered by a finite number of sets in $\mathcal{U}$. Use the compactness of X to show that $X \times [0, t]$ can also be covered by a finite number of sets in $\mathcal{U}$ and that if $t < 1$, then for some $t'' > t$, $X \times [0, t'']$ can be covered by a finite number of sets in $\mathcal{U}$.)

Solution. Write $I = [a, b]$ with $a \leq b$ (the hint takes $I = [0, 1]$; the argument is identical). We give $X \times I$ the product topology, for which the sets $\mathcal{O} \times V$, with $\mathcal{O}$ open in X and V open in I , form a base. If X is empty then $X \times I$ is empty and hence compact, so we assume X is nonempty; $I = [a, b]$ is nonempty by hypothesis.

Let $\mathcal{U}$ be a covering of $X \times I$ by open subsets of $X \times I$. Define

$$S = \{t \in [a, b] \mid X \times [a, t] \text{ is covered by finitely many members of } \mathcal{U}\}.$$

Note at once that if $t \in S$ and $a \leq t' \leq t$, then $t' \in S$, since $X \times [a, t'] \subseteq X \times [a, t]$.

Step 1. A tube lemma. We claim: for each $s \in [a, b]$ there exist $\epsilon_s > 0$ and finitely many members $U_1, \dots, U_m$ of $\mathcal{U}$ with

$$X \times ((s - \epsilon_s, s + \epsilon_s) \cap [a, b]) \subseteq U_1 \cup \dots \cup U_m.$$

Indeed, for each $x \in X$ the point (x, s) lies in some $U_x \in \mathcal{U}$; since U_x is open in the product, there are an open set $\mathcal{O}_x \subseteq X$ containing x and an open subset V_x of I containing s with $\mathcal{O}_x \times V_x \subseteq U_x$, and there is $\delta_x > 0$ with $(s - \delta_x, s + \delta_x) \cap [a, b] \subseteq V_x$. The collection $\{\mathcal{O}_x\}_{x \in X}$ is an open cover of the compact space X , so there are $x_1, \dots, x_m$ with $X = \mathcal{O}_{x_1} \cup \dots \cup \mathcal{O}_{x_m}$. Put $\epsilon_s = \min\{\delta_{x_1}, \dots, \delta_{x_m}\} > 0$ and $U_j = U_{x_j}$. If $x \in X$ and $t \in (s - \epsilon_s, s + \epsilon_s) \cap [a, b]$, choose j with $x \in \mathcal{O}_{x_j}$; then $t \in (s - \delta_{x_j}, s + \delta_{x_j}) \cap [a, b] \subseteq V_{x_j}$, so $(x, t) \in \mathcal{O}_{x_j} \times V_{x_j} \subseteq U_j$. This proves the claim.

Step 2. S is nonempty. Apply Step 1 with $s = a$: the finitely many sets $U_1, \dots, U_m$ produced there cover $X \times ([a, a + \epsilon_a] \cap [a, b])$, which contains $X \times \{a\} = X \times [a, a]$. Hence $a \in S$.

Step 3. Let $t^* = \sup S$; then $t^* \in S$. Since S is nonempty and bounded above by b , t^* is a well-defined real number with $a \leq t^* \leq b$. Apply Step 1 with $s = t^*$ to obtain $\epsilon = \epsilon_{t^*} > 0$ and $U_1, \dots, U_m$ in $\mathcal{U}$ covering $X \times ((t^* - \epsilon, t^* + \epsilon) \cap [a, b])$. If $t^* = a$ then $t^* \in S$ by Step 2. Suppose $t^* > a$. By the definition of supremum there is $t \in S$ with $t^* - \epsilon < t \leq t^*$; let $\mathcal{F}$ be a finite subfamily of $\mathcal{U}$ covering $X \times [a, t]$. Since

$$[a, t^*] = [a, t] \cup ([t, t^*]) \quad \text{and} \quad [t, t^*] \subseteq (t^* - \epsilon, t^* + \epsilon) \cap [a, b],$$

the finite family $\mathcal{F} \cup \{U_1, \dots, U_m\}$ covers $X \times [a, t^*]$. Hence $t^* \in S$.

Step 4. $t^* = b$. Suppose $t^* < b$. Keep ϵ and $U_1, \dots, U_m$ from Step 3, and let $\mathcal{F}^*$ be a finite subfamily of $\mathcal{U}$ covering $X \times [a, t^*]$, which exists by Step 3. Put

$$t'' = \min\left\{t^* + \frac{\epsilon}{2}, b\right\} > t^*.$$

Then $[a, t''] = [a, t^*] \cup [t^*, t'']$ and $[t^*, t''] \subseteq (t^* - \epsilon, t^* + \epsilon) \cap [a, b]$, so the finite family $\mathcal{F}^* \cup \{U_1, \dots, U_m\}$ covers $X \times [a, t'']$. Thus $t'' \in S$ and $t'' > t^* = \sup S$, a contradiction. Hence $t^* = b$.

Step 5. Conclusion. By Steps 3 and 4, $b \in S$, that is, $X \times I = X \times [a, b]$ is covered by finitely many members of $\mathcal{U}$. Since $\mathcal{U}$ was an arbitrary open cover of $X \times I$, the space $X \times I$ is compact.

Problem 12.21. Prove that the product of a countable number of sequentially compact topological spaces is sequentially compact.

Solution. Let $\{X_n\}_{n=1}^\infty$ be a countable collection of sequentially compact topological spaces and let $X = \prod_{n=1}^\infty X_n$ have the product topology. (A finite product is handled by the same argument, or by taking all but finitely many factors to be a one-point space.) Recall that X is sequentially compact provided every sequence in X has a subsequence converging to a point of X .

Step 1. Convergence in X is componentwise. We claim that a sequence $\{x^{(k)}\}$ in X converges to $y \in X$ if and only if $\{x_n^{(k)}\}_k$ converges to y_n in X_n for every n . If $x^{(k)} \rightarrow y$, then since each projection π_n is continuous (Proposition 4 of this chapter exhibits the product topology as the weak topology associated to the collection of projections, so each projection is continuous, and a continuous mapping carries a convergent sequence to a convergent sequence), $x_n^{(k)} = \pi_n(x^{(k)}) \rightarrow \pi_n(y) = y_n$. Conversely, suppose $x_n^{(k)} \rightarrow y_n$ for each n , and let $\mathcal{O} = \prod_n \mathcal{O}_n$ be a basic neighborhood of y , with each $\mathcal{O}_n$ open in X_n and $\mathcal{O}_n = X_n$ for all n outside a finite set F . For each $n \in F$ there is K_n with $x_n^{(k)} \in \mathcal{O}_n$ for all $k \geq K_n$. Put $K = \max_{n \in F} K_n$ (take $K = 1$ if F is empty). For $k \geq K$ we have $x_n^{(k)} \in \mathcal{O}_n$ for every n , i.e. $x^{(k)} \in \mathcal{O}$. Since every neighborhood of y contains a basic one, $x^{(k)} \rightarrow y$. (This is the analogue, for a product of possibly different spaces, of Proposition 3 of this chapter, which is stated for X^A .)

Step 2. Construction of nested subsequences. Let $\{x^{(k)}\}_{k=1}^\infty$ be a sequence in X . We construct strictly increasing maps $\sigma_n: \mathbb{N} \rightarrow \mathbb{N}$ and points $y_n \in X_n$ as follows.

Since X_1 is sequentially compact, the sequence $\{x_1^{(k)}\}_k$ in X_1 has a convergent subsequence: there is a strictly increasing $\sigma_1: \mathbb{N} \rightarrow \mathbb{N}$ and $y_1 \in X_1$ with $x_1^{(\sigma_1(k))} \rightarrow y_1$. Suppose $\sigma_1, \dots, \sigma_n$ have been chosen, and set $\tau_n = \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_n$, a strictly increasing map of $\mathbb{N}$ into $\mathbb{N}$. Since X_{n+1} is sequentially compact, the sequence $\{x_{n+1}^{(\tau_n(k))}\}_k$ in X_{n+1} has a convergent subsequence: there is a strictly increasing $\sigma_{n+1}: \mathbb{N} \rightarrow \mathbb{N}$ and $y_{n+1} \in X_{n+1}$ with

$$x_{n+1}^{(\tau_n(\sigma_{n+1}(k)))} = x_{n+1}^{(\tau_{n+1}(k))} \longrightarrow y_{n+1} \quad \text{as } k \rightarrow \infty,$$

where $\tau_{n+1} = \tau_n \circ \sigma_{n+1}$. By construction, for every n ,

$$x_n^{(\tau_n(k))} \longrightarrow y_n \quad \text{as } k \rightarrow \infty. \quad (*)$$

(Set $\tau_0 =$ the identity, so that the induction begins correctly.)

Step 3. The diagonal subsequence. Define $\kappa(k) = \tau_k(k)$ for $k \geq 1$. We check that κ is strictly increasing. A strictly increasing map σ of $\mathbb{N}$ into $\mathbb{N}$ satisfies $\sigma(j) \geq j$ for all j (immediate induction). Hence $\sigma_{k+1}(k+1) \geq k+1$, and since $\tau_{k+1} = \tau_k \circ \sigma_{k+1}$ and τ_k is strictly increasing,

$$\kappa(k+1) = \tau_{k+1}(k+1) = \tau_k(\sigma_{k+1}(k+1)) \geq \tau_k(k+1) > \tau_k(k) = \kappa(k).$$

So $\{x^{(\kappa(k))}\}_k$ is a subsequence of $\{x^{(k)}\}$.

Step 4. The diagonal subsequence converges componentwise. Fix $n \geq 1$. For $k \geq n$ we have $\tau_k = \tau_n \circ \mu_k$, where $\mu_k = \sigma_{n+1} \circ \dots \circ \sigma_k$ (with μ_n the identity), so

$$\kappa(k) = \tau_k(k) = \tau_n(m_k), \quad m_k := \mu_k(k).$$

The sequence $\{m_k\}_{k \geq n}$ is strictly increasing: $\tau_n(m_k) = \kappa(k)$ is strictly increasing in k by Step 3, and τ_n is strictly increasing, so $m_k < m_{k+1}$ (if $m_k \geq m_{k+1}$ then $\tau_n(m_k) \geq \tau_n(m_{k+1})$, contradicting $\kappa(k) < \kappa(k+1)$). Therefore $\{x_n^{(\kappa(k))}\}_{k \geq n} = \{x_n^{(\tau_n(m_k))}\}_{k \geq n}$ is a subsequence of the sequence $\{x_n^{(\tau_n(j))}\}_j$, which converges to y_n by (*). A subsequence of a convergent sequence converges to the same limit. Hence

$$x_n^{(\kappa(k))} \longrightarrow y_n \quad \text{as } k \rightarrow \infty, \quad \text{for every } n.$$

Step 5. Conclusion. Let $y \in X$ be the point with n -th component y_n for each n . By Step 4 the subsequence $\{x^{(\kappa(k))}\}$ converges componentwise to y , and therefore, by Step 1, it converges to y in the product topology on X . Since $\{x^{(k)}\}$ was an arbitrary sequence in X , every sequence in X has a subsequence converging to a point of X ; that is, $X = \prod_{n=1}^{\infty} X_n$ is sequentially compact.

12.4 Problems 12.22–12.28

Problem 12.22. A product I^Λ of unit intervals is called a (generalized) cube. Prove that every compact Hausdorff space X is homeomorphic to a closed subset of some cube. (Let $\mathcal{F}$ be the family of continuous real-valued functions on X with values in $[0, 1]$. Let $Q = \prod_{f \in \mathcal{F}} I_f$. Then, since X is normal, the mapping g of X onto Q that takes x into the point whose f -th coordinate is $f(x)$ is one-to-one, continuous, and has closed image.)

Solution. Let X be a compact Hausdorff space. Let

$$\mathcal{F} = \{f : X \rightarrow [0, 1] \mid f \text{ is continuous}\},$$

and let $Q = \prod_{f \in \mathcal{F}} I_f$, where each I_f is a copy of the unit interval $I = [0, 1]$, given the product topology. Thus Q is a cube with index set $\Lambda = \mathcal{F}$. Define

$$g : X \rightarrow Q, \quad \pi_f(g(x)) = f(x) \quad \text{for each } f \in \mathcal{F},$$

where $\pi_f : Q \rightarrow I_f$ is the f -th projection. (The word onto in the hint is a slip: g maps X into Q , and what is proved below is that g is a homeomorphism of X onto its image and that this image is a

closed subset of Q . Note also that $\mathcal{F} \neq \emptyset$, since it contains the constant functions, and that if $X = \emptyset$ the assertion is trivial.)

Step 1: g is continuous. The product topology on Q has as a subbase the collection of sets $\pi_f^{-1}(\mathcal{O})$ with $f \in \mathcal{F}$ and $\mathcal{O}$ open in I_f . Since

$$g^{-1}(\pi_f^{-1}(\mathcal{O})) = \{x \in X \mid f(x) \in \mathcal{O}\} = f^{-1}(\mathcal{O}),$$

which is open in X because f is continuous, the inverse image under g of every subbasic open set is open. Inverse images preserve intersections and unions, so the inverse image of every open subset of Q is open. Hence g is continuous.

Step 2: g is one-to-one. Since X is compact and Hausdorff, it is normal, by Theorem 18 of the preceding chapter. Moreover, in a Hausdorff space each one-point set is closed: if $u \neq v$ then v has a neighborhood missing u , so $X \sim \{u\}$ is open.

Let u, v be distinct points of X . Then $\{u\}$ and $\{v\}$ are disjoint closed subsets of the normal space X , so by Urysohn's Lemma there is a continuous $f : X \rightarrow [0, 1]$ with $f(u) = 0$ and $f(v) = 1$. This f belongs to $\mathcal{F}$, and

$$\pi_f(g(u)) = 0 \neq 1 = \pi_f(g(v)),$$

so $g(u) \neq g(v)$. Thus g is one-to-one.

Step 3: Q is compact and Hausdorff. Each factor $I_f = [0, 1]$ is compact (Heine-Borel) and Hausdorff, so by the Tychonoff Product Theorem Q is compact, and by Problem 12 (a product of Hausdorff spaces is Hausdorff) Q is Hausdorff. Directly: if $x \neq y$ in Q there is an index f with $x_f \neq y_f$; choosing disjoint open $\mathcal{U}, \mathcal{V} \subseteq I_f$ about x_f, y_f , the sets $\pi_f^{-1}(\mathcal{U})$ and $\pi_f^{-1}(\mathcal{V})$ are disjoint neighborhoods of x and y .

Step 4: g is a closed map, so $g(X)$ is closed and g is a homeomorphism onto $g(X)$. Let F be a closed subset of X . By Proposition 15 of the preceding chapter, a closed subset of a compact space is compact, so F is compact; by Proposition 20 of that chapter, the continuous image of a compact space is compact, so $g(F)$ is a compact subset of Q . By Proposition 16 of that chapter, a compact subspace of a Hausdorff space is closed, and Q is Hausdorff by Step 3, so $g(F)$ is closed in Q . In particular, taking $F = X$, the image $g(X)$ is a closed subset of the cube Q .

Finally, $g(X)$, with the subspace topology inherited from Q , is Hausdorff, since a subspace of a Hausdorff space is Hausdorff. Thus g is a continuous one-to-one mapping of the compact space X onto the Hausdorff space $g(X)$, and so, by Proposition 19 of the preceding chapter, g is a homeomorphism of X onto $g(X)$. (Directly: for closed $F \subseteq X$ the set $g(F)$ was just shown to be closed in Q , hence closed in the subspace $g(X)$; since $(g^{-1})^{-1}(F) = g(F)$, the inverse map $g^{-1} : g(X) \rightarrow X$ pulls closed sets back to closed sets and is therefore continuous.) Hence X is homeomorphic to the closed subset $g(X)$ of the cube Q .

Problem 12.23. Let $Q = I^\Lambda$ be a cube, and let f be a continuous real-valued function on Q . Then, given $\epsilon > 0$, there is a continuous real-valued function g on Q for which $|f - g| < \epsilon$ and g is a function of only a finite number of coordinates. (Hint: Cover the range of f by a finite number of intervals of length ϵ and look at the inverse images of these intervals.)

Solution. Write $Q = \prod_{\lambda \in \Lambda} I_\lambda$ with each $I_\lambda = [0, 1]$, and let $\pi_\lambda : Q \rightarrow I_\lambda$ be the projections. Recall the terminology of the footnote in Section 12.2: a subbasic set is a set of the form $\pi_\lambda^{-1}(\mathcal{O}_\lambda)$ with $\mathcal{O}_\lambda$ open in I_λ , and a basic set is a finite intersection of such sets. The basic sets form a base for the product topology, and each basic set is determined by finitely many coordinates: if $B = \bigcap_{i=1}^k \pi_{\lambda_i}^{-1}(\mathcal{O}_{\lambda_i})$, then membership of a point in B depends only on its coordinates $\lambda_1, \dots, \lambda_k$. By the Tychonoff Product Theorem, Q is compact, since each factor $[0, 1]$ is compact.

Step 1: a finite cover by basic sets on which f varies by less than ϵ . Since Q is compact and f is continuous, $f(Q)$ is a compact, hence bounded, subset of $\mathbf{R}$; choose $M > 0$ with $f(Q) \subseteq [-M, M]$. Cover $[-M, M]$ by finitely many open intervals $J_1, \dots, J_n$, each of length ϵ : take

$$J_i = \left(-M + (i-1)\frac{\epsilon}{2} - \frac{\epsilon}{4}, -M + (i-1)\frac{\epsilon}{2} + \frac{3\epsilon}{4} \right), \quad 1 \leq i \leq n,$$

with n so large that $-M + (n-1)\epsilon/2 > M$. Consecutive intervals overlap, so $\bigcup_{i=1}^n J_i \supseteq [-M, M]$ and

hence $\{f^{-1}(J_i)\}_{i=1}^n$ is an open cover of Q .

For each $x \in Q$ select an index $i(x)$ with $f(x) \in J_{i(x)}$, so that x lies in the open set $f^{-1}(J_{i(x)})$. Since the basic sets form a base for the topology of Q , there is a basic set B_x with

$$x \in B_x \subseteq f^{-1}(J_{i(x)}).$$

The collection $\{B_x\}_{x \in Q}$ is an open cover of the compact space Q , so there are points $x_1, \dots, x_m$ in Q with

$$Q = \bigcup_{j=1}^m B_{x_j}, \quad B_{x_j} \subseteq f^{-1}(J_{i_j}), \quad \text{where } i_j = i(x_j).$$

Each B_{x_j} is determined by a finite set $\Lambda_j \subseteq \Lambda$ of coordinates. Put $\Lambda_0 = \bigcup_{j=1}^m \Lambda_j$, a finite subset of Λ . Step 2: a retraction onto the coordinates in Λ_0 . Fix any point $p \in Q$ and define $r : Q \rightarrow Q$ by

$$\pi_\lambda(r(x)) = \begin{cases} x_\lambda, & \lambda \in \Lambda_0, \\ p_\lambda, & \lambda \notin \Lambda_0. \end{cases}$$

Each composition $\pi_\lambda \circ r$ is continuous, being either π_λ itself or a constant function; hence the inverse image under r of each subbasic set $\pi_\lambda^{-1}(\mathcal{O}_\lambda)$ is open, and therefore r is continuous. By construction $r(x)$ depends only on the finitely many coordinates $(x_\lambda)_{\lambda \in \Lambda_0}$, and $r(x)$ agrees with x in every coordinate belonging to Λ_0 .

Define $g = f \circ r$. Then g is continuous, being a composition of continuous functions, and $g(x)$ depends only on the coordinates $(x_\lambda)_{\lambda \in \Lambda_0}$; that is, g is a function of only the finitely many coordinates indexed by Λ_0 .

Step 3: the estimate. Let $x \in Q$ and choose j with $x \in B_{x_j}$. The set B_{x_j} is determined by the coordinates in $\Lambda_j \subseteq \Lambda_0$, and $r(x)$ agrees with x in all coordinates of Λ_0 , hence in all coordinates of Λ_j . Therefore $r(x) \in B_{x_j}$ as well. Since $B_{x_j} \subseteq f^{-1}(J_{i_j})$, both $f(x)$ and $f(r(x)) = g(x)$ lie in the interval J_{i_j} , which has length ϵ . Consequently

$$|f(x) - g(x)| < \epsilon.$$

As $x \in Q$ was arbitrary, $|f - g| < \epsilon$ on Q , with g continuous and a function of only a finite number of coordinates.

Problem 12.24. Suppose that X is a topological space for which there is a collection of continuous real-valued functions on X that separates points in X . Show that X must be Hausdorff.

Solution. Let $\mathcal{F}$ be a collection of continuous real-valued functions on X that separates points in X ; that is, for any two distinct points u and v in X there is an $f \in \mathcal{F}$ for which $f(u) \neq f(v)$.

Let u and v be distinct points of X . Choose $f \in \mathcal{F}$ with $f(u) \neq f(v)$, and set

$$\delta = \frac{1}{2} |f(u) - f(v)| > 0.$$

Define the open intervals

$$\mathcal{U}' = (f(u) - \delta, f(u) + \delta), \quad \mathcal{V}' = (f(v) - \delta, f(v) + \delta),$$

which are open subsets of $\mathbf{R}$. They are disjoint: if t belonged to both, then

$$|f(u) - f(v)| \leq |f(u) - t| + |t - f(v)| < \delta + \delta = |f(u) - f(v)|,$$

a contradiction. (This is just the observation that $\mathbf{R}$, with its usual topology, is Hausdorff.)

Now put

$$\mathcal{U} = f^{-1}(\mathcal{U}'), \quad \mathcal{V} = f^{-1}(\mathcal{V}').$$

Since f is continuous and $\mathcal{U}', \mathcal{V}'$ are open in $\mathbf{R}$, the sets $\mathcal{U}$ and $\mathcal{V}$ are open in X . Clearly $u \in \mathcal{U}$ and $v \in \mathcal{V}$. Moreover

$$\mathcal{U} \cap \mathcal{V} = f^{-1}(\mathcal{U}') \cap f^{-1}(\mathcal{V}') = f^{-1}(\mathcal{U}' \cap \mathcal{V}') = f^{-1}(\emptyset) = \emptyset.$$

Thus u and v can be separated by disjoint open subsets of X . Since u and v were an arbitrary pair of distinct points, X is Hausdorff.

Problem 12.25. Let X be a compact Hausdorff space and $\mathcal{A} \subseteq C(X)$ an algebra that contains the constant functions. Show that $\mathcal{A}$ is dense in $C(X)$ if and only if $\mathcal{A}$ separates points in X .

Solution. Here $C(X)$ carries the maximum norm $\|h\|_{\max} = \max_{x \in X} |h(x)|$, which is defined since X is compact, and density means density with respect to this norm, that is, with respect to uniform convergence on X .

Suppose first that $\mathcal{A}$ separates points in X . Then $\mathcal{A}$ is an algebra of continuous real-valued functions on the compact Hausdorff space X that separates points and contains the constant functions, so the Stone-Weierstrass Approximation Theorem asserts precisely that $\mathcal{A}$ is dense in $C(X)$.

Conversely, suppose $\mathcal{A}$ is dense in $C(X)$. Let u and v be distinct points of X . Since X is Hausdorff, the one-point sets $\{u\}$ and $\{v\}$ are closed: indeed, for each $x \neq u$ there are disjoint neighborhoods of x and u , so $X \setminus \{u\}$ is open, and similarly for v . Since X is compact and Hausdorff, it is normal, by Theorem 18 of the preceding chapter. Therefore, $\{u\}$ and $\{v\}$ being disjoint closed sets, Urysohn's Lemma provides a function $f \in C(X)$ with

$$f(u) = 0, \quad f(v) = 1, \quad 0 \leq f \leq 1 \text{ on } X.$$

By the density of $\mathcal{A}$ in $C(X)$, choose $g \in \mathcal{A}$ with $\|f - g\|_{\max} < 1/2$. Then

$$g(u) < f(u) + \frac{1}{2} = \frac{1}{2}, \quad g(v) > f(v) - \frac{1}{2} = \frac{1}{2},$$

so $g(u) \neq g(v)$. Hence $\mathcal{A}$ separates the points u and v . Since $u \neq v$ were arbitrary, $\mathcal{A}$ separates points in X .

(Observe that this second implication, that density forces separation, used only that X is compact and Hausdorff; neither the algebra structure of $\mathcal{A}$ nor the presence of the constants was needed. What it exploits is that the whole of $C(X)$ separates points on a compact Hausdorff space, as noted in the text just before the statement of the Stone-Weierstrass Theorem. The algebra hypotheses are needed only for the first implication, where the Stone-Weierstrass Theorem is invoked.)

Problem 12.26. Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact space X that contains the constant functions. Let $f \in C(X)$ have the property that for some constant function c and real number α , the function $\alpha(f + c)$ belongs to $\overline{\mathcal{A}}$. Show that f also belongs to $\overline{\mathcal{A}}$.

Solution. Throughout, $C(X)$ carries the maximum norm $\|h\|_{\max} = \max_{x \in X} |h(x)|$, which is finite because X is compact and h is continuous, and $\overline{\mathcal{A}}$ denotes the closure of $\mathcal{A}$ in $C(X)$ with respect to this norm. Of course α is to be nonzero; otherwise $\alpha(f + c)$ is the zero function, which lies in $\overline{\mathcal{A}}$ for every f , and the conclusion fails. Indeed, let $X = [0, 1]$, which is compact, and let $\mathcal{A}$ be the set of constant functions on X : this is an algebra containing the constant functions, and it is already closed in the maximum norm, since a uniform limit of constant functions is constant. Thus $\overline{\mathcal{A}} = \mathcal{A}$, and $f(x) = x$ does not belong to $\overline{\mathcal{A}}$, although $0 \cdot (f + c) = 0$ does. So assume $\alpha \neq 0$. This is the situation in which the assertion is used in the proof of the Stone-Weierstrass Theorem, where $\alpha = 1/\|f + c\|_{\max}$. The whole content of the problem is the following claim.

Claim: $\overline{\mathcal{A}}$ is an algebra of continuous real-valued functions on X containing the constant functions.

Proof of the claim. First, $\overline{\mathcal{A}}$ contains $\mathcal{A}$, hence contains the constant functions.

Let $h_1, h_2 \in \overline{\mathcal{A}}$ and let $\beta \in \mathbf{R}$. Choose sequences $\{u_n\}$ and $\{v_n\}$ in $\mathcal{A}$ with $\|u_n - h_1\|_{\max} \rightarrow 0$ and $\|v_n - h_2\|_{\max} \rightarrow 0$.

For sums and scalar multiples: $u_n + v_n$ and βu_n belong to $\mathcal{A}$, since $\mathcal{A}$ is a linear space, and

$$\|(u_n + v_n) - (h_1 + h_2)\|_{\max} \leq \|u_n - h_1\|_{\max} + \|v_n - h_2\|_{\max} \rightarrow 0,$$

$$\|\beta u_n - \beta h_1\|_{\max} = |\beta| \|u_n - h_1\|_{\max} \rightarrow 0.$$

Hence $h_1 + h_2$ and βh_1 belong to $\overline{\mathcal{A}}$.

For products: $u_n v_n \in \mathcal{A}$, since $\mathcal{A}$ is an algebra. The convergent sequence of numbers $\{\|u_n\|_{\max}\}$ is bounded, say $\|u_n\|_{\max} \leq M$ for all n ; indeed $\|u_n\|_{\max} \leq \|h_1\|_{\max} + \|u_n - h_1\|_{\max}$. Then, for all $x \in X$,

$$|u_n(x)v_n(x) - h_1(x)h_2(x)| \leq |u_n(x)| |v_n(x) - h_2(x)| + |h_2(x)| |u_n(x) - h_1(x)|,$$

so that

$$\|u_n v_n - h_1 h_2\|_{\max} \leq M \|v_n - h_2\|_{\max} + \|h_2\|_{\max} \|u_n - h_1\|_{\max} \rightarrow 0.$$

Hence $h_1 h_2 \in \overline{\mathcal{A}}$. This proves the claim: $\overline{\mathcal{A}}$ is a linear subspace of $C(X)$ closed under multiplication, that is, an algebra, and it contains the constant functions.

Now complete the argument. By hypothesis the function $h = \alpha(f + c)$ belongs to $\overline{\mathcal{A}}$. Since $\overline{\mathcal{A}}$ is a linear space and $\alpha \neq 0$, the function $\alpha^{-1}h = f + c$ belongs to $\overline{\mathcal{A}}$. Since $\overline{\mathcal{A}}$ contains the constant function c and is a linear space, it contains $-c$ and hence contains

$$(f + c) + (-c) = f.$$

Therefore $f \in \overline{\mathcal{A}}$.

This justifies the normalization made at the start of the proof of the Stone-Weierstrass Theorem: to show that an arbitrary $f \in C(X)$ can be uniformly approximated by members of $\mathcal{A}$, it suffices to treat functions with values in $[0, 1]$, because $(f + c)/\|f + c\|_{\max}$ has this form when $c = \|f\|_{\max}$ (and f is not the constant $-c$, a case that is trivial), and membership of that function in $\overline{\mathcal{A}}$ forces membership of f .

Problem 12.27. For $f, g \in C[a, b]$, show that $f = g$ if and only if $\int_a^b x^n f(x) dx = \int_a^b x^n g(x) dx$ for all n .

Solution. We may assume $a < b$; if $a = b$ the assertion is vacuous. The index n ranges over the nonnegative integers.

If $f = g$, then $x^n f(x) = x^n g(x)$ for all x and every n , so the two integrals agree for every n . This direction is immediate.

Conversely, suppose

$$\int_a^b x^n f(x) dx = \int_a^b x^n g(x) dx \quad \text{for all } n = 0, 1, 2, \dots$$

Set $h = f - g$, a continuous function on $[a, b]$. Continuous functions on a closed, bounded interval are Riemann, hence Lebesgue, integrable, and the integral is linear, so the hypothesis says

$$\int_a^b x^n h(x) dx = 0 \quad \text{for all } n \geq 0.$$

By linearity of the integral, for every polynomial $p(x) = \sum_{k=0}^m a_k x^k$ we obtain

$$\int_a^b p(x)h(x) dx = \sum_{k=0}^m a_k \int_a^b x^k h(x) dx = 0. \quad (*)$$

Let $\epsilon > 0$. Since h is continuous on the closed, bounded interval $[a, b]$, the Weierstrass Approximation Theorem provides a polynomial p with

$$|h(x) - p(x)| < \epsilon \quad \text{for all } x \in [a, b].$$

Also h is bounded on $[a, b]$, say $|h| \leq M$, since a continuous function on a compact set attains a maximum. Using (*),

$$\int_a^b h^2 = \int_a^b h(h - p) + \int_a^b h p = \int_a^b h(h - p),$$

and therefore

$$\left| \int_a^b h^2 \right| \leq \int_a^b |h| |h - p| \leq M\epsilon(b - a).$$

Since $\epsilon > 0$ was arbitrary and $M(b - a)$ is a fixed finite constant, we conclude

$$\int_a^b h^2(x) dx = 0.$$

Finally, h^2 is continuous and nonnegative on $[a, b]$ with zero integral, so $h^2 \equiv 0$. Indeed, suppose $h(x_0) \neq 0$ for some $x_0 \in [a, b]$, and put $\eta = h(x_0)^2 > 0$. By continuity of h^2 there is a $\delta > 0$ such that $h^2 > \eta/2$ on $J = [a, b] \cap (x_0 - \delta, x_0 + \delta)$, and J contains a nondegenerate interval, of length $\ell > 0$, since $a < b$. By the monotonicity of the integral and the nonnegativity of h^2 ,

$$\int_a^b h^2 \geq \int_J h^2 \geq \frac{\eta}{2} \ell > 0,$$

contradicting $\int_a^b h^2 = 0$.

Hence $h \equiv 0$ on $[a, b]$, that is, $f = g$.

Problem 12.28. For $f \in C[a, b]$ and $\epsilon > 0$, show that there are real numbers $c_0, c_1, \dots, c_n$ for which

$$\left| f(x) - c_0 - \sum_{k=1}^n c_k \cdot e^{kx} \right| < \epsilon \quad \text{for all } x \in [a, b].$$

Solution. We may assume $a < b$; if $a = b$ the assertion is trivial, since $c_0 = f(a)$ and $n = 1, c_1 = 0$ will do.

Let

$$\mathcal{A} = \left\{ h \in C[a, b] \mid h(x) = c_0 + \sum_{k=1}^n c_k e^{kx} \text{ for some } n \geq 1 \text{ and } c_0, \dots, c_n \in \mathbf{R} \right\},$$

that is, $\mathcal{A}$ is the set of all finite real linear combinations of the functions $1, e^x, e^{2x}, e^{3x}, \dots$ on $[a, b]$. The assertion to be proved is exactly that $\mathcal{A}$ is dense in $C[a, b]$ with respect to the maximum norm. We verify the hypotheses of the Stone-Weierstrass Approximation Theorem.

The space $[a, b]$ is compact, by the Heine-Borel Theorem, and Hausdorff, being a metric space. Each function $x \mapsto e^{kx}$ is continuous on $[a, b]$, so $\mathcal{A} \subseteq C[a, b]$.

$\mathcal{A}$ is a linear space. A sum of two such finite linear combinations is again a finite linear combination of $1, e^x, e^{2x}, \dots$ (pad the shorter one with zero coefficients), and a real multiple of one is again one.

$\mathcal{A}$ is an algebra. Since the product of two linear combinations expands, by bilinearity, into a linear combination of products $e^{jx} e^{kx}$ with $j, k \geq 0$, and

$$e^{jx} \cdot e^{kx} = e^{(j+k)x},$$

where $e^{0 \cdot x} = 1$, every such product is again a finite linear combination of $1, e^x, e^{2x}, \dots$, hence lies in $\mathcal{A}$. Thus $\mathcal{A}$ is closed under multiplication.

$\mathcal{A}$ contains the constant functions: take c_0 to be the constant and $c_1 = \dots = c_n = 0$.

$\mathcal{A}$ separates points in $[a, b]$. The function $h(x) = e^x$ belongs to $\mathcal{A}$ (take $c_0 = 0, n = 1, c_1 = 1$), and it is strictly increasing on $\mathbf{R}$; hence if $u \neq v$ in $[a, b]$ then $e^u \neq e^v$.

Therefore, by the Stone-Weierstrass Approximation Theorem, $\mathcal{A}$ is dense in $C[a, b]$. Given $f \in C[a, b]$ and $\epsilon > 0$, there is thus a member h of $\mathcal{A}$ with $\|f - h\|_{\max} < \epsilon$; writing $h(x) = c_0 + \sum_{k=1}^n c_k e^{kx}$, we obtain real numbers $c_0, c_1, \dots, c_n$ with

$$\left| f(x) - c_0 - \sum_{k=1}^n c_k e^{kx} \right| < \epsilon \quad \text{for all } x \in [a, b],$$

as required.

Alternatively, one may deduce this from the Weierstrass Approximation Theorem alone. The map $\varphi(x) = e^x$ is a homeomorphism of $[a, b]$ onto $[e^a, e^b]$, with continuous inverse $\varphi^{-1}(t) = \log t$. Given $f \in C[a, b]$, the function $F = f \circ \varphi^{-1}$ is continuous on the closed, bounded interval $[e^a, e^b]$, so by the Weierstrass Approximation Theorem there is a polynomial $p(t) = \sum_{k=0}^n c_k t^k$ with $|F(t) - p(t)| < \epsilon$ for all $t \in [e^a, e^b]$. Substituting $t = e^x$, which ranges over all of $[e^a, e^b]$ as x ranges over $[a, b]$, gives

$$\left| f(x) - c_0 - \sum_{k=1}^n c_k e^{kx} \right| = |F(e^x) - p(e^x)| < \epsilon \quad \text{for all } x \in [a, b].$$

12.5 Problems 12.29–12.35

Problem 12.29. For $f \in C[0, \pi]$ and $\epsilon > 0$, show that there are real numbers $c_0, c_1, \dots, c_n$ for which

$$\left| f(x) - c_0 - \sum_{k=1}^n c_k \cdot \cos kx \right| < \epsilon \quad \text{for all } x \in [0, \pi].$$

Solution. Let $X = [0, \pi]$, a compact Hausdorff space (it is a closed bounded interval, hence a compact metric space). Let

$$\mathcal{A} = \left\{ c_0 + \sum_{k=1}^n c_k \cos kx : n \geq 0, c_0, \dots, c_n \in \mathbb{R} \right\},$$

that is, $\mathcal{A}$ is the real linear span in $C[0, \pi]$ of the functions $\varphi_k(x) = \cos kx$, $k = 0, 1, 2, \dots$ (note $\varphi_0 \equiv 1$). We verify that $\mathcal{A}$ satisfies the hypotheses of the Stone-Weierstrass Approximation Theorem.

$\mathcal{A}$ is a linear subspace of $C[0, \pi]$. Each φ_k is continuous on $[0, \pi]$, and $\mathcal{A}$ is by definition the set of finite real linear combinations of the φ_k , so it is a linear subspace of $C[0, \pi]$.

$\mathcal{A}$ is an algebra. Since multiplication of functions is bilinear, it suffices to check that the product of two of the spanning functions again belongs to $\mathcal{A}$. For nonnegative integers j and k , the product-to-sum identity gives

$$\cos jx \cdot \cos kx = \frac{1}{2} [\cos(j+k)x + \cos(j-k)x] = \frac{1}{2} [\varphi_{j+k}(x) + \varphi_{|j-k|}(x)],$$

where we used that cosine is even, so $\cos(j-k)x = \cos|j-k|x$. Thus $\varphi_j \varphi_k = \frac{1}{2}(\varphi_{j+k} + \varphi_{|j-k|}) \in \mathcal{A}$. Consequently, if $u = \sum_j a_j \varphi_j$ and $v = \sum_k b_k \varphi_k$ belong to $\mathcal{A}$, then $uv = \sum_{j,k} a_j b_k \varphi_j \varphi_k$ is a finite linear combination of elements of $\mathcal{A}$, hence belongs to $\mathcal{A}$. So $\mathcal{A}$ is an algebra of continuous real-valued functions on $[0, \pi]$.

$\mathcal{A}$ contains the constant functions. Indeed, the constant c is $c \cdot \varphi_0 \in \mathcal{A}$.

$\mathcal{A}$ separates the points of $[0, \pi]$. The function $\varphi_1(x) = \cos x$ belongs to $\mathcal{A}$ and is strictly decreasing on $[0, \pi]$: for $0 \leq u < v \leq \pi$ we have $\cos u > \cos v$. Hence φ_1 is one-to-one on $[0, \pi]$, so for distinct $u, v \in [0, \pi]$ we have $\varphi_1(u) \neq \varphi_1(v)$.

By the Stone-Weierstrass Approximation Theorem, $\mathcal{A}$ is dense in $C[0, \pi]$ with respect to the maximum norm. Therefore, given $f \in C[0, \pi]$ and $\epsilon > 0$, there is a function in $\mathcal{A}$, say $g(x) = c_0 + \sum_{k=1}^n c_k \cos kx$, for which $\|f - g\|_{\max} < \epsilon$; that is,

$$\left| f(x) - c_0 - \sum_{k=1}^n c_k \cdot \cos kx \right| < \epsilon \quad \text{for all } x \in [0, \pi].$$

Problem 12.30. Let f be a continuous real-valued function on $\mathbb{R}$ that is periodic with period 2π . For $\epsilon > 0$, show that there are real numbers $c_0, a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$\left| f(x) - c_0 - \sum_{k=1}^n [a_k \cos kx + b_k \sin kx] \right| < \epsilon \quad \text{for all } x \in \mathbb{R}.$$

(Hint: A periodic function may be identified with a continuous function on the unit circle in the plane and the unit circle is compact and Hausdorff with the topology it inherits from the plane.)

Solution. Let $S = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\}$ be the unit circle, with the topology it inherits from $\mathbb{R}^2$. Then S is Hausdorff, being a subspace of the metric space $\mathbb{R}^2$, and S is compact, being a closed bounded subset of $\mathbb{R}^2$ (Heine-Borel).

Step 1: f induces a continuous function on S . Define $\Phi: [0, 2\pi] \rightarrow S$ by $\Phi(t) = (\cos t, \sin t)$. Then Φ is continuous and onto. Moreover, if $s, t \in [0, 2\pi]$ and $\Phi(s) = \Phi(t)$, then either $s = t$ or $\{s, t\} = \{0, 2\pi\}$; in the latter case $f(s) = f(t)$ because f has period 2π . Hence the formula

$$F(\Phi(t)) = f(t), \quad t \in [0, 2\pi],$$

unambiguously defines a real-valued function F on S , and $F \circ \Phi = f|_{[0, 2\pi]}$.

To see that F is continuous, let C be a closed subset of $\mathbb{R}$. Since Φ is onto,

$$F^{-1}(C) = \Phi(\Phi^{-1}(F^{-1}(C))) = \Phi((f|_{[0, 2\pi]})^{-1}(C)).$$

Now $(f|_{[0, 2\pi]})^{-1}(C)$ is a closed subset of the compact space $[0, 2\pi]$ and therefore is compact; its image under the continuous map Φ is compact, and a compact subset of the Hausdorff space S is closed. Thus $F^{-1}(C)$ is closed in S for every closed $C \subseteq \mathbb{R}$, so F is continuous, that is, $F \in C(S)$.

Step 2: polynomials in the coordinate functions are dense in $C(S)$. Let

$$\mathcal{A} = \{P|_S : P \text{ a real polynomial in two variables}\} \subseteq C(S).$$

Since sums, scalar multiples, and products of polynomials are polynomials, $\mathcal{A}$ is an algebra of continuous real-valued functions on S , and it contains the constant functions (the constant polynomials). It separates the points of S : if (u_1, v_1) and (u_2, v_2) are distinct points of S , then either $u_1 \neq u_2$, in which case the coordinate polynomial $P(u, v) = u$ separates them, or $v_1 \neq v_2$, in which case $P(u, v) = v$ does. Since S is compact and Hausdorff, the Stone-Weierstrass Approximation Theorem tells us that $\mathcal{A}$ is dense in $C(S)$.

Therefore, given $\epsilon > 0$, there is a real polynomial P in two variables with

$$|F(u, v) - P(u, v)| < \epsilon \quad \text{for all } (u, v) \in S.$$

Taking $(u, v) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ and recalling $F(\cos t, \sin t) = f(t)$ there, we get $|f(t) - P(\cos t, \sin t)| < \epsilon$ for $t \in [0, 2\pi]$. Both $t \mapsto f(t)$ and $t \mapsto P(\cos t, \sin t)$ have period 2π , so in fact

$$|f(t) - P(\cos t, \sin t)| < \epsilon \quad \text{for all } t \in \mathbb{R}.$$

Step 3: $P(\cos t, \sin t)$ is a trigonometric polynomial. Let $\mathcal{T}$ be the real linear span of the functions

$$1, \cos kt, \sin kt \quad (k = 1, 2, \dots),$$

so $\mathcal{T}$ is exactly the set of functions of the form $c_0 + \sum_{k=1}^n [a_k \cos kt + b_k \sin kt]$. The product-to-sum identities, valid for all integers $j, k \geq 0$,

$$\begin{aligned} \cos jt \cos kt &= \frac{1}{2}[\cos(j+k)t + \cos(j-k)t], \\ \sin jt \sin kt &= \frac{1}{2}[\cos(j-k)t - \cos(j+k)t], \\ \sin jt \cos kt &= \frac{1}{2}[\sin(j+k)t + \sin(j-k)t], \end{aligned}$$

together with $\cos(-mt) = \cos mt$ and $\sin(-mt) = -\sin mt$, show that the product of any two of the spanning functions of $\mathcal{T}$ again belongs to $\mathcal{T}$. Since multiplication is bilinear, $\mathcal{T}$ is closed under multiplication, so $\mathcal{T}$ is an algebra of functions on $\mathbb{R}$. It contains the constant 1 and the functions $\cos t$ and $\sin t$; hence it contains every finite product of copies of $\cos t$ and $\sin t$, and therefore every real linear combination of such products. In other words, $t \mapsto P(\cos t, \sin t)$ belongs to $\mathcal{T}$ for every real polynomial P in two variables.

Writing the function of Step 2 as $P(\cos t, \sin t) = c_0 + \sum_{k=1}^n [a_k \cos kt + b_k \sin kt]$, we conclude that

$$\left| f(x) - c_0 - \sum_{k=1}^n [a_k \cos kx + b_k \sin kx] \right| < \epsilon \quad \text{for all } x \in \mathbb{R}.$$

Problem 12.31. Let X and Y be compact Hausdorff spaces and f belong to $C(X \times Y)$. Show that for each $\epsilon > 0$, there are functions $f_1, \dots, f_n$ in $C(X)$ and $g_1, \dots, g_n$ in $C(Y)$ such that

$$\left| f(x, y) - \sum_{k=1}^n f_k(x) \cdot g_k(y) \right| < \epsilon \quad \text{for all } (x, y) \in X \times Y.$$

Solution. Give $X \times Y$ the product topology. By the Tychonoff Product Theorem, the product of compact topological spaces is compact, so $X \times Y$ is compact. The product of two Hausdorff spaces is Hausdorff: let $(x_1, y_1) \neq (x_2, y_2)$, so that $x_1 \neq x_2$ or $y_1 \neq y_2$. Suppose $x_1 \neq x_2$, the other case being symmetric. Since X is Hausdorff there are disjoint open sets $\mathcal{U}_1, \mathcal{U}_2$ in X containing x_1, x_2 respectively; then $\mathcal{U}_1 \times Y$ and $\mathcal{U}_2 \times Y$ are disjoint open subsets of $X \times Y$ containing (x_1, y_1) and (x_2, y_2) respectively. Hence $X \times Y$ is a compact Hausdorff space and the Stone-Weierstrass Approximation Theorem applies to it.

Define

$$\mathcal{A} = \left\{ (x, y) \mapsto \sum_{k=1}^n f_k(x)g_k(y) : n \in \mathbb{N}, f_k \in C(X), g_k \in C(Y) \right\}.$$

First, $\mathcal{A} \subseteq C(X \times Y)$. Indeed, if $u \in C(X)$ and $v \in C(Y)$, then $(x, y) \mapsto u(x)v(y)$ is the composition of u with the projection π_X , which is continuous, and similarly for $(x, y) \mapsto v(y)$; the product of two continuous real-valued functions is continuous, so $(x, y) \mapsto u(x)v(y)$ is continuous, and so is any finite sum of such functions.

Next, $\mathcal{A}$ is a linear subspace of $C(X \times Y)$: it is closed under addition by concatenating the two sums, and closed under multiplication by a scalar α by replacing f_1 with αf_1 .

$\mathcal{A}$ is an algebra. For the elementary products,

$$(u_1(x)v_1(y)) \cdot (u_2(x)v_2(y)) = (u_1u_2)(x) \cdot (v_1v_2)(y),$$

and $u_1u_2 \in C(X)$, $v_1v_2 \in C(Y)$, so this product belongs to $\mathcal{A}$. Since multiplication is bilinear, the product of two general elements of $\mathcal{A}$ is a finite sum of such elementary products and hence belongs to $\mathcal{A}$.

$\mathcal{A}$ contains the constant functions: the constant c equals $f_1(x)g_1(y)$ with $f_1 \equiv c$ on X and $g_1 \equiv 1$ on Y .

$\mathcal{A}$ separates the points of $X \times Y$. Since X is compact and Hausdorff, it is normal by Theorem 18 of Chapter 11, and therefore, as observed in the text preceding the Stone-Weierstrass Theorem, Urysohn's Lemma implies that $C(X)$ separates the points of X ; the same holds for Y . Now let (x_1, y_1) and (x_2, y_2) be distinct points of $X \times Y$. If $x_1 \neq x_2$, choose $u \in C(X)$ with $u(x_1) \neq u(x_2)$ and put $h(x, y) = u(x) \cdot 1$; then $h \in \mathcal{A}$ and $h(x_1, y_1) = u(x_1) \neq u(x_2) = h(x_2, y_2)$. If instead $y_1 \neq y_2$, choose $v \in C(Y)$ with $v(y_1) \neq v(y_2)$ and put $h(x, y) = 1 \cdot v(y)$, which likewise belongs to $\mathcal{A}$ and separates the two points.

Therefore $\mathcal{A}$ is an algebra of continuous real-valued functions on the compact Hausdorff space $X \times Y$ which contains the constants and separates points. By the Stone-Weierstrass Approximation Theorem, $\mathcal{A}$ is dense in $C(X \times Y)$.

Consequently, given $f \in C(X \times Y)$ and $\epsilon > 0$, there is a member of $\mathcal{A}$, that is, functions $f_1, \dots, f_n \in C(X)$ and $g_1, \dots, g_n \in C(Y)$, for which

$$\left| f(x, y) - \sum_{k=1}^n f_k(x) \cdot g_k(y) \right| < \epsilon \quad \text{for all } (x, y) \in X \times Y.$$

Problem 12.32. Rather than use the Weierstrass Approximation Theorem in the proof of the Stone-Weierstrass Theorem, show that there are natural numbers m and n for which the polynomial $p(x) = 1 - (1 - x^n)^m$ satisfies (14). (Hint: Since $p(0) = 0$, $p(1) = 1$ and $p' > 0$ on $(0, 1)$, it suffices to choose m and n such that $p(c/2) < \epsilon$ and $p(c) > 1 - \epsilon$.)

Solution. Recall the setting of (14) in the proof of Lemma 7: we are given a number c with $0 < c < 1$ (the constant produced in (13)) and a number $\epsilon > 0$, and we must produce a polynomial p with

$$p < \epsilon \text{ on } [0, c/2], \quad p > 1 - \epsilon \text{ on } [c, 1], \quad 0 \leq p \leq 1 \text{ on } [0, 1],$$

which is the assertion (14). Since a polynomial that satisfies (14) for some value of ϵ also satisfies (14) for every larger value of ϵ , we may and do assume that $0 < \epsilon < 1/2$.

We show that natural numbers m and n may be chosen so that $p(x) = 1 - (1 - x^n)^m$ has these three properties. Observe that p is a polynomial with $p(0) = 0$, so p has no constant term; this is convenient, though not needed here, since the algebra $\mathcal{A}$ in Lemma 7 contains the constants.

Two elementary reductions. First, for $x \in [0, 1]$ and any natural numbers m, n we have $x^n \in [0, 1]$, hence $1 - x^n \in [0, 1]$, hence $(1 - x^n)^m \in [0, 1]$, and therefore

$$0 \leq p(x) \leq 1 \quad \text{for all } x \in [0, 1],$$

which is the third requirement in (14), regardless of the choice of m and n .

Second, p is increasing on $[0, 1]$: by the chain rule, for $0 < x < 1$,

$$p'(x) = mn x^{n-1} (1 - x^n)^{m-1} > 0,$$

since $x > 0$ and $0 < 1 - x^n < 1$. As p is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with positive derivative there, the Mean Value Theorem shows that p is strictly increasing on $[0, 1]$. Consequently, if we arrange that

$$p(c/2) < \epsilon \quad \text{and} \quad p(c) > 1 - \epsilon,$$

a pair of inequalities we call (*), then for $x \in [0, c/2]$ we get $p(x) \leq p(c/2) < \epsilon$, and for $x \in [c, 1]$ we get $p(x) \geq p(c) > 1 - \epsilon$, so all of (14) holds. Thus it suffices to establish (*).

Choice of m and n . Since $0 < c/2 < 1$ we have $(c/2)^n \rightarrow 0$ as $n \rightarrow \infty$, while $2^n \rightarrow \infty$. Since $0 < \epsilon < 1/2$, the number $\ln(1/\epsilon)$ is positive. Choose a natural number n so large that both

$$2^n > \frac{2 \ln(1/\epsilon)}{\epsilon} \quad \text{and} \quad \left(\frac{c}{2}\right)^n < \frac{\epsilon}{2}.$$

Put

$$a = (c/2)^n, \quad b = c^n = 2^n a,$$

so that $0 < a < \epsilon/2$ and $0 < b < 1$. Now choose the natural number

$$m = \left\lceil \frac{\epsilon}{2a} \right\rceil \geq 1.$$

Then $\epsilon/(2a) \leq m < \epsilon/(2a) + 1$, so that, multiplying by $a > 0$,

$$\frac{\epsilon}{2} \leq ma < \frac{\epsilon}{2} + a < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

call this chain of inequalities (**).

Verification of $p(c/2) < \epsilon$. By Bernoulli's inequality, $(1 - t)^m \geq 1 - mt$ for $t \in [0, 1]$ and every natural number m (immediate by induction on m : the case $m = 1$ is an equality, and if $(1 - t)^m \geq 1 - mt$ then, since $1 - t \geq 0$, $(1 - t)^{m+1} \geq (1 - mt)(1 - t) = 1 - (m + 1)t + mt^2 \geq 1 - (m + 1)t$). Applying this with $t = a$,

$$p(c/2) = 1 - (1 - a)^m \leq 1 - (1 - ma) = ma < \epsilon,$$

the last inequality by (**).

Verification of $p(c) > 1 - \epsilon$. Since $1 + s \leq e^s$ for all real s , taking $s = -b$ gives $0 \leq 1 - b \leq e^{-b}$, and raising to the m -th power (both sides nonnegative) gives

$$(1 - b)^m \leq e^{-mb} = e^{-2^n \cdot ma} \leq e^{-2^n \epsilon/2},$$

where the last step used $ma \geq \epsilon/2$ from (**). By the first condition on n we have $2^n \epsilon/2 > \ln(1/\epsilon)$, hence

$$(1 - b)^m \leq e^{-2^n \epsilon/2} < e^{-\ln(1/\epsilon)} = \epsilon.$$

Therefore $p(c) = 1 - (1 - b)^m > 1 - \epsilon$.

This establishes (*), and hence, by the monotonicity of p on $[0, 1]$ together with the bound $0 \leq p \leq 1$ there, the polynomial $p(x) = 1 - (1 - x^n)^m$ satisfies (14) for this choice of m and n . In particular the proof of Lemma 7, and with it the proof of the Stone-Weierstrass Theorem, may be carried out without invoking the Weierstrass Approximation Theorem: one simply takes $h = p \circ g$ with this explicit p , and h belongs to the algebra $\mathcal{A}$ because p is a polynomial and $g \in \mathcal{A}$.

Problem 12.33. Let $\mathcal{A}$ be a collection of continuous real-valued functions on a compact Hausdorff space X that separates the points of X . Show that every continuous real-valued function on X can be uniformly approximated arbitrarily closely by a polynomial in a finite number of functions of $\mathcal{A}$.

Solution. By a polynomial in a finite number of functions of $\mathcal{A}$ we mean a function on X of the form

$$x \mapsto P(g_1(x), \dots, g_k(x)),$$

where k is a nonnegative integer, $g_1, \dots, g_k$ belong to $\mathcal{A}$, and P is a polynomial in k real variables with real coefficients (constant term allowed). Here $k = 0$ is permitted and yields the constant functions: a polynomial in no variables is a real constant. Let $\mathcal{B}$ denote the collection of all such functions on X . Thus $\mathcal{B}$ contains every constant function, whether or not $\mathcal{A}$ is empty. (The case $\mathcal{A} = \emptyset$ can occur only when X has at most one point, since $\mathcal{A}$ separates the points of X ; in that case $C(X)$ consists of the constant functions and $\mathcal{B} = C(X)$ already.)

$\mathcal{B} \subseteq C(X)$. Each g_i is continuous and real-valued, and P is a polynomial, hence $P(g_1, \dots, g_k)$ is obtained from the g_i and constants by finitely many sums and products of continuous real-valued functions; such combinations are continuous.

$\mathcal{B}$ is an algebra. Let $u = P(g_1, \dots, g_k)$ and $v = Q(h_1, \dots, h_l)$ belong to $\mathcal{B}$, with $g_i, h_j \in \mathcal{A}$ and P, Q polynomials. Consider the combined list $g_1, \dots, g_k, h_1, \dots, h_l$ of $k + l$ functions from $\mathcal{A}$, and regard P as a polynomial in the first k of the $k + l$ variables and Q as a polynomial in the last l of them; call these polynomials in $k + l$ variables $\tilde{P}$ and $\tilde{Q}$. For real numbers α, β , the functions $\alpha\tilde{P} + \beta\tilde{Q}$ and $\tilde{P} \cdot \tilde{Q}$ are again polynomials in $k + l$ variables, and

$$\alpha u + \beta v = (\alpha\tilde{P} + \beta\tilde{Q})(g_1, \dots, g_k, h_1, \dots, h_l), \quad u \cdot v = (\tilde{P} \cdot \tilde{Q})(g_1, \dots, g_k, h_1, \dots, h_l).$$

Hence $\alpha u + \beta v$ and uv belong to $\mathcal{B}$, so $\mathcal{B}$ is a linear subspace of $C(X)$ that is closed under multiplication, that is, an algebra of continuous real-valued functions on X .

$\mathcal{B}$ contains the constant functions, by construction (take P constant).

$\mathcal{B}$ separates the points of X . Indeed $\mathcal{A} \subseteq \mathcal{B}$, since $g = P(g)$ for the polynomial $P(t) = t$ in one variable, and $\mathcal{A}$ separates the points of X by hypothesis.

Since X is compact and Hausdorff, the Stone-Weierstrass Approximation Theorem now tells us that $\mathcal{B}$ is dense in $C(X)$ with respect to the maximum norm. That is, given $f \in C(X)$ and $\epsilon > 0$, there are a nonnegative integer k , functions $g_1, \dots, g_k$ in $\mathcal{A}$, and a real polynomial P in k variables, for which

$$|f(x) - P(g_1(x), \dots, g_k(x))| < \epsilon \quad \text{for all } x \in X.$$

Thus every continuous real-valued function on X can be uniformly approximated arbitrarily closely by a polynomial in a finite number of functions of $\mathcal{A}$.

Problem 12.34. Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact Hausdorff space X . Show that the closure of $\mathcal{A}$, $\overline{\mathcal{A}}$, also is an algebra.

Solution. Since X is compact, every continuous real-valued function on X is bounded, and $C(X)$ is a normed linear space with respect to the maximum norm $\|h\|_{\max} = \max_{x \in X} |h(x)|$; convergence in this norm is uniform convergence on X . The closure $\overline{\mathcal{A}}$ is taken in $C(X)$, so $\overline{\mathcal{A}} \subseteq C(X)$ and thus consists of continuous real-valued functions on X . (Equivalently: a uniform limit of continuous functions is continuous.)

A function h belongs to $\overline{\mathcal{A}}$ if and only if there is a sequence $\{h_j\}$ in $\mathcal{A}$ with $\|h_j - h\|_{\max} \rightarrow 0$: this is the characterization of closure points in a metric space.

$\overline{\mathcal{A}}$ is a linear subspace. Let $f, g \in \overline{\mathcal{A}}$ and $\alpha, \beta \in \mathbb{R}$. Choose sequences $\{f_j\}$ and $\{g_j\}$ in $\mathcal{A}$ with $\|f_j - f\|_{\max} \rightarrow 0$ and $\|g_j - g\|_{\max} \rightarrow 0$. Since $\mathcal{A}$ is a linear space, $\alpha f_j + \beta g_j \in \mathcal{A}$ for each j , and

$$\|(\alpha f_j + \beta g_j) - (\alpha f + \beta g)\|_{\max} \leq |\alpha| \|f_j - f\|_{\max} + |\beta| \|g_j - g\|_{\max} \rightarrow 0.$$

Hence $\alpha f + \beta g \in \overline{\mathcal{A}}$.

$\overline{\mathcal{A}}$ is closed under multiplication. With $f, g, \{f_j\}, \{g_j\}$ as above, we have $f_j g_j \in \mathcal{A}$ for each j because $\mathcal{A}$ is an algebra. The sequence $\{\|f_j\|_{\max}\}$ is bounded, say by M , since $\|f_j\|_{\max} \leq \|f\|_{\max} + \|f_j - f\|_{\max}$ and the second term tends to 0. For each $x \in X$,

$$|f_j(x)g_j(x) - f(x)g(x)| \leq |f_j(x)| |g_j(x) - g(x)| + |g(x)| |f_j(x) - f(x)|,$$

and therefore

$$\|f_j g_j - f g\|_{\max} \leq M \|g_j - g\|_{\max} + \|g\|_{\max} \|f_j - f\|_{\max} \rightarrow 0.$$

Hence $f g$ is the uniform limit of the sequence $\{f_j g_j\}$ of members of $\mathcal{A}$, so $f g \in \overline{\mathcal{A}}$.

Thus $\overline{\mathcal{A}}$ is a linear space of continuous real-valued functions on X that contains the product of any two of its members; that is, $\overline{\mathcal{A}}$ is an algebra.

Problem 12.35. Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact Hausdorff space X that separates points. Show that either $\overline{\mathcal{A}} = C(X)$ or there is a point $x_0 \in X$ for which

$$\overline{\mathcal{A}} = \{f \in C(X) \mid f(x_0) = 0\}.$$

(Hint: If $1 \in \overline{\mathcal{A}}$, we are done. Moreover, if for each $x \in X$ there is an $f \in \mathcal{A}$ with $f(x) \neq 0$, then there is a $g \in \mathcal{A}$ that is positive on X and this implies that $1 \in \overline{\mathcal{A}}$.)

Solution. Throughout, closures are taken in $C(X)$ with respect to the maximum norm, so convergence means uniform convergence on X . By the preceding problem, $\overline{\mathcal{A}}$ is an algebra of continuous real-valued functions on X ; it is closed, and it separates points because it contains $\mathcal{A}$, which does.

There are exactly two possibilities:

Case 1: for each $x \in X$ there is an $f \in \mathcal{A}$ with $f(x) \neq 0$.

Case 2: there is a point $x_0 \in X$ at which every $f \in \mathcal{A}$ vanishes.

Case 1. First we produce $g \in \mathcal{A}$ that is positive on all of X . For each $x \in X$ choose $f_x \in \mathcal{A}$ with $f_x(x) \neq 0$. Since $\mathcal{A}$ is an algebra, $f_x^2 \in \mathcal{A}$, and $f_x^2 \geq 0$ on X with $f_x^2(x) > 0$. By the continuity of f_x^2 , there is a neighborhood $\mathcal{U}_x$ of x on which $f_x^2 > 0$. The collection $\{\mathcal{U}_x\}_{x \in X}$ is an open cover of the compact space X , so there are finitely many points $x_1, \dots, x_k$ with $X = \bigcup_{i=1}^k \mathcal{U}_{x_i}$. Define

$$g = \sum_{i=1}^k f_{x_i}^2.$$

Then $g \in \mathcal{A}$, $g \geq 0$ on X , and $g > 0$ on X : each $x \in X$ lies in some $\mathcal{U}_{x_i}$, so the i -th summand is positive at x while all summands are nonnegative.

Next we show $1 \in \overline{\mathcal{A}}$. The function g is continuous and positive on the compact space X , so it attains a positive minimum and a finite maximum: there are numbers $0 < \alpha \leq \beta$ with $\alpha \leq g(x) \leq \beta$ for all $x \in X$. Let $\epsilon > 0$. The function $t \mapsto 1/t$ is continuous on the closed bounded interval $[\alpha, \beta]$, which does not contain 0, so by the Weierstrass Approximation Theorem there is a polynomial P with

$$\left| P(t) - \frac{1}{t} \right| < \frac{\epsilon}{\beta} \quad \text{for all } t \in [\alpha, \beta].$$

Define the polynomial $q(t) = t \cdot P(t)$; then q has zero constant term, and for $t \in [\alpha, \beta]$,

$$|q(t) - 1| = |t| \left| P(t) - \frac{1}{t} \right| \leq \beta \cdot \frac{\epsilon}{\beta} = \epsilon.$$

Since q has zero constant term, $q(t) = \sum_{j=1}^N c_j t^j$, and therefore $q \circ g = \sum_{j=1}^N c_j g^j$ belongs to $\mathcal{A}$, because $\mathcal{A}$ is an algebra (hence contains all powers g^j with $j \geq 1$) and is a linear space. As $g(x) \in [\alpha, \beta]$ for every $x \in X$,

$$\|q \circ g - 1\|_{\max} \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the constant function 1 belongs to $\overline{\mathcal{A}}$.

Now $\overline{\mathcal{A}}$ is an algebra that is a linear space containing 1, hence it contains every constant function; it separates the points of X ; and X is compact and Hausdorff. By the Stone-Weierstrass Approximation Theorem, $\overline{\mathcal{A}}$ is dense in $C(X)$. But $\overline{\mathcal{A}}$ is closed in $C(X)$, and a dense closed subset of a metric space is the whole space. Therefore $\overline{\mathcal{A}} = C(X)$.

Case 2. Suppose $x_0 \in X$ is a point with $f(x_0) = 0$ for every $f \in \mathcal{A}$. Set

$$M = \{f \in C(X) \mid f(x_0) = 0\}.$$

We show $\overline{\mathcal{A}} = M$.

First, $\overline{\mathcal{A}} \subseteq M$: if $f \in \overline{\mathcal{A}}$, choose $f_j \in \mathcal{A}$ with $f_j \rightarrow f$ uniformly on X ; then in particular $f_j(x_0) \rightarrow f(x_0)$, and each $f_j(x_0) = 0$, so $f(x_0) = 0$.

For the reverse inclusion, consider

$$\mathcal{A}' = \{f + c : f \in \mathcal{A}, c \in \mathbb{R}\},$$

where c denotes the constant function of value c . Then $\mathcal{A}' \subseteq C(X)$ and $\mathcal{A}'$ is a linear space, being the sum of the two linear spaces $\mathcal{A}$ and the constants. It is closed under multiplication, since for $f, h \in \mathcal{A}$ and $c, d \in \mathbb{R}$,

$$(f + c)(h + d) = (fh + df + ch) + cd,$$

and $fh + df + ch \in \mathcal{A}$ because $\mathcal{A}$ is an algebra. Thus $\mathcal{A}'$ is an algebra; it contains the constant functions by construction; and it separates the points of X since $\mathcal{A} \subseteq \mathcal{A}'$ does. Since X is compact and Hausdorff, the Stone-Weierstrass Approximation Theorem tells us that $\mathcal{A}'$ is dense in $C(X)$.

Now let $f \in M$ and let $\epsilon > 0$. Choose $h \in \mathcal{A}$ and $c \in \mathbb{R}$ with

$$\|f - (h + c)\|_{\max} < \frac{\epsilon}{2}.$$

Evaluating at x_0 and using $f(x_0) = 0$ and $h(x_0) = 0$ gives $|c| = |f(x_0) - h(x_0) - c| < \epsilon/2$. Therefore

$$\|f - h\|_{\max} \leq \|f - (h + c)\|_{\max} + |c| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary and $h \in \mathcal{A}$, we conclude $f \in \overline{\mathcal{A}}$. Hence $M \subseteq \overline{\mathcal{A}}$, and with the first inclusion, $\overline{\mathcal{A}} = M = \{f \in C(X) \mid f(x_0) = 0\}$.

Finally, we remark that in Case 2 the point x_0 is unique: if $x_1 \neq x_0$ were another point at which every member of $\mathcal{A}$ vanishes, then no $f \in \mathcal{A}$ could satisfy $f(x_0) \neq f(x_1)$, since both values would be 0, contradicting the hypothesis that $\mathcal{A}$ separates the points of X .

Since Cases 1 and 2 exhaust the possibilities, either $\overline{\mathcal{A}} = C(X)$ or there is a point $x_0 \in X$ for which $\overline{\mathcal{A}} = \{f \in C(X) \mid f(x_0) = 0\}$.

12.6 Problem 12.36

Problem 12.36. Let X be a compact Hausdorff space and $\mathcal{A}$ an algebra of continuous functions on X that separates points and contains the constant functions.

- (i) Given any two numbers a and b and points $u, v \in X$, show that there is a function f in $\mathcal{A}$ for which $f(u) = a$ and $f(v) = b$.
- (ii) Is it the case that given any two numbers a and b and disjoint closed subsets A and B of X , there is a function f in $\mathcal{A}$ for which $f = a$ on A and $f = b$ on B ?

Solution. Throughout we use the definitions of Chapter 12: an algebra of continuous functions on X is a linear subspace $\mathcal{A}$ of the linear space $C(X)$ of continuous real-valued functions on X with the property that the product of any two functions in $\mathcal{A}$ again belongs to $\mathcal{A}$; and $\mathcal{A}$ separates points in X provided that for any two distinct points u and v in X there is a $g \in \mathcal{A}$ for which $g(u) \neq g(v)$. All functions are real-valued and a, b denote real numbers.

Part (i).

We are prescribing the values of f at the two points u and v , so the assertion is to be read for $u \neq v$: if $u = v$ then $f(u) = a$ and $f(v) = b$ force $a = b$, and in that case the constant function $f \equiv a$, which belongs to $\mathcal{A}$ by hypothesis, does the job. So assume $u \neq v$.

Since $\mathcal{A}$ separates the points of X , choose $g \in \mathcal{A}$ with

$$g(u) \neq g(v).$$

Set $\alpha = g(u)$ and $\beta = g(v)$, so that $\beta - \alpha \neq 0$, and define

$$f = a \cdot \mathbf{1} + \frac{b-a}{\beta-\alpha} (g - \alpha \cdot \mathbf{1}),$$

where $\mathbf{1}$ denotes the constant function with value 1.

First, $f \in \mathcal{A}$. Indeed, $\mathbf{1} \in \mathcal{A}$ because $\mathcal{A}$ contains the constant functions, $g \in \mathcal{A}$, and $\mathcal{A}$ is a linear subspace of $C(X)$; hence the linear combination

$$f = \left(a - \frac{(b-a)\alpha}{\beta-\alpha} \right) \mathbf{1} + \frac{b-a}{\beta-\alpha} g$$

of the two members $\mathbf{1}$ and g of $\mathcal{A}$ again belongs to $\mathcal{A}$. (Only the linear structure of $\mathcal{A}$ is used here; closure under multiplication is not needed.)

Second, f has the required values:

$$f(u) = a + \frac{b-a}{\beta-\alpha} (g(u) - \alpha) = a + \frac{b-a}{\beta-\alpha} \cdot 0 = a,$$

$$f(v) = a + \frac{b-a}{\beta-\alpha} (g(v) - \alpha) = a + \frac{b-a}{\beta-\alpha} (\beta - \alpha) = a + (b-a) = b.$$

Thus $f \in \mathcal{A}$, $f(u) = a$ and $f(v) = b$, as required. (This two-point interpolation is exactly the step used in the proof of the Stone-Weierstrass Approximation Theorem.)

Part (ii).

No. Separating points and containing the constants forces the prescribed-value property at pairs of points, by part (i), but it does not force it at pairs of disjoint closed sets. Here is a counterexample.

Let $X = [0, 1]$ with its usual topology; X is a compact Hausdorff space. Let

$$\mathcal{A} = \{ p|_{[0,1]} : p \text{ a polynomial with real coefficients} \}.$$

Then:

- $\mathcal{A} \subseteq C(X)$, and $\mathcal{A}$ is an algebra: sums, scalar multiples and products of polynomials are polynomials.
- $\mathcal{A}$ contains the constant functions, the constant polynomials.
- $\mathcal{A}$ separates the points of X : the polynomial $p(x) = x$ satisfies $p(u) = u \neq v = p(v)$ whenever $u \neq v$.

So $\mathcal{A}$ satisfies all the hypotheses. Now take the disjoint closed subsets

$$A = [0, 1/3], \quad B = [2/3, 1]$$

of X , and take $a = 0, b = 1$.

Suppose $f \in \mathcal{A}$ satisfied $f = 0$ on A and $f = 1$ on B . Write $f = p|_{[0,1]}$ for a real polynomial p . Then p vanishes at every point of the interval $[0, 1/3]$, so p has infinitely many zeros. A nonzero real polynomial of degree n has at most n zeros; hence p is the zero polynomial, and therefore $f \equiv 0$ on all

of $[0, 1]$. But then $f = 0$ on B , contradicting $f = 1$ on B . So no such f exists, and the answer to (ii) is negative.

Two remarks locate the failure precisely.

First, only exactness fails. Lemma 8 of Chapter 12 asserts that under the present hypotheses, for each pair of disjoint closed subsets A and B of X and each $\epsilon \in (0, 1/2)$, there is a function h in $\mathcal{A}$ with $0 \leq h \leq 1$ on X , $h < \epsilon$ on A and $h > 1 - \epsilon$ on B . Hence $f = a + (b - a)h$ belongs to $\mathcal{A}$ and satisfies $|f - a| \leq |b - a|\epsilon$ on A and $|f - b| \leq |b - a|\epsilon$ on B . Equivalently, by the Stone-Weierstrass Approximation Theorem $\mathcal{A}$ is dense in $C(X)$ with the maximum norm, while the function that is exactly a on A and exactly b on B furnished by Urysohn's Lemma (applicable since X , being compact and Hausdorff, is normal by Theorem 18 of Chapter 11) lies in $C(X) = \overline{\mathcal{A}}$ but need not lie in $\mathcal{A}$ itself. In the counterexample above, $\mathcal{A}$ is dense in $C[0, 1]$ and is not closed, and that is the whole of the obstruction.

Second, exactness does survive on finite sets, so infinitude of A and B is essential. Let $p_1, \dots, p_k$ be distinct points of X , with $k \geq 2$, and let $c_1, \dots, c_k$ be real numbers. For each i and each $j \neq i$, part (i) provides $g_{ij} \in \mathcal{A}$ with $g_{ij}(p_i) = 1$ and $g_{ij}(p_j) = 0$; since $\mathcal{A}$ is closed under multiplication, $e_i = \prod_{j \neq i} g_{ij}$ belongs to $\mathcal{A}$, and $e_i(p_i) = 1$ while $e_i(p_j) = 0$ for $j \neq i$. Then $f = \sum_{i=1}^k c_i e_i$ belongs to $\mathcal{A}$ and $f(p_i) = c_i$ for each i . Taking $\{p_1, \dots, p_k\}$ to be the union of two disjoint finite sets A and B , with $c_i = a$ for $p_i \in A$ and $c_i = b$ for $p_i \in B$, gives the exact conclusion of (ii) for finite closed sets. Part (i) is the case $k = 2$.

13 Continuous Linear Operators Between Banach Spaces

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13.1 Problems 13.1–13.7

Problem 13.1. Show that a nonempty subset S of a linear space X is a subspace if and only if $S + S = S$ and $\lambda \cdot S = S$ for each $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Solution. Recall the definitions in force: a nonempty subset S of X is a linear subspace provided every linear combination of vectors in S again belongs to S ; and for subsets A, B of X and $\lambda \in \mathbb{R}$,

$$A + B = \{x + y \mid x \in A, y \in B\}, \quad \lambda A = \{\lambda x \mid x \in A\}.$$

Necessity. Suppose S is a subspace. Since $S \neq \emptyset$ we may choose $x_0 \in S$; the vector $0 = 0 \cdot x_0$ is a linear combination of vectors in S , so $0 \in S$.

If $x, y \in S$ then $x + y = 1 \cdot x + 1 \cdot y$ is a linear combination of vectors in S , hence lies in S ; thus $S + S \subseteq S$. Conversely, if $x \in S$ then, since $0 \in S$, $x = x + 0 \in S + S$; thus $S \subseteq S + S$. Therefore $S + S = S$.

Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. If $x \in S$ then λx is a linear combination of a vector of S , so $\lambda x \in S$; thus $\lambda S \subseteq S$. Conversely, let $x \in S$. Since $\lambda \neq 0$, the vector $\lambda^{-1}x$ belongs to S by what we just proved (applied to the nonzero scalar λ^{-1}), and $x = \lambda \cdot (\lambda^{-1}x) \in \lambda S$. Hence $S \subseteq \lambda S$, and so $\lambda S = S$.

Sufficiency. Suppose S is nonempty, $S + S = S$, and $\lambda S = S$ whenever $\lambda \neq 0$.

First, $0 \in S$. Indeed, choose $x \in S$. Applying the hypothesis with $\lambda = -1$ gives $(-1)S = S$, so $-x = (-1) \cdot x \in S$. Therefore

$$0 = x + (-x) \in S + S = S.$$

Now let $x_1, \dots, x_n$ belong to S and let $\lambda_1, \dots, \lambda_n$ be real numbers; we must show $\sum_{k=1}^n \lambda_k x_k \in S$. For each k , the vector $\lambda_k x_k$ belongs to S : if $\lambda_k \neq 0$ this follows from $\lambda_k S = S$, while if $\lambda_k = 0$ then $\lambda_k x_k = 0 \in S$ by the preceding paragraph.

It therefore suffices to show that S is closed under the addition of finitely many of its members, and we argue by induction on n . The case $n = 1$ is trivial. Assume that every sum of n vectors of S belongs to S , and let $u_1, \dots, u_{n+1}$ belong to S . By the inductive hypothesis $u_1 + \dots + u_n \in S$, and therefore

$$u_1 + \dots + u_n + u_{n+1} \in S + S = S.$$

This completes the induction. Consequently $\sum_{k=1}^n \lambda_k x_k \in S$, that is, every linear combination of vectors in S belongs to S , and S is a subspace.

Observe that the restriction $\lambda \neq 0$ is essential in the statement: for $\lambda = 0$ one has $0 \cdot S = \{0\}$, which equals S only in the trivial case $S = \{0\}$.

Problem 13.2. If Y and Z are subspaces of the linear space X , show that $Y + Z$ also is a subspace and $Y + Z = \text{span}[Y \cup Z]$.

Solution. We first record a convenient reformulation. A nonempty subset W of X is a subspace, that is, contains every linear combination of its members, if and only if

$$\alpha u + \beta v \in W \quad \text{whenever } u, v \in W \text{ and } \alpha, \beta \in \mathbb{R}. \quad (*)$$

Necessity is clear, since $\alpha u + \beta v$ is a linear combination of members of W . For sufficiency, argue by induction on the number n of vectors in the combination. If $n = 1$, take $v = u$ and $\beta = 0$ in $(*)$ to get $\alpha u \in W$. Suppose every linear combination of n vectors of W lies in W , and let $w_1, \dots, w_{n+1}$ belong to W and $\lambda_1, \dots, \lambda_{n+1}$ be real. By the inductive hypothesis $u = \sum_{k=1}^n \lambda_k w_k$ belongs to W , and hence by $(*)$,

$$\sum_{k=1}^{n+1} \lambda_k w_k = 1 \cdot u + \lambda_{n+1} \cdot w_{n+1} \in W.$$

Now let Y and Z be subspaces of X . Each contains 0 (a subspace is nonempty and contains the linear combination $0 \cdot y$ of any of its members), so $0 = 0 + 0$ belongs to $Y + Z$ and $Y + Z$ is nonempty. Let

$u_1, u_2 \in Y + Z$ and let α, β be real numbers. Write $u_1 = y_1 + z_1$ and $u_2 = y_2 + z_2$ with $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$. Using the commutativity and associativity of addition together with the distributive laws for scalar multiplication,

$$\alpha u_1 + \beta u_2 = (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2).$$

Since Y and Z are subspaces, $\alpha y_1 + \beta y_2 \in Y$ and $\alpha z_1 + \beta z_2 \in Z$. Therefore $\alpha u_1 + \beta u_2 \in Y + Z$. By (*), $Y + Z$ is a subspace of X .

It remains to identify $Y + Z$ with $\text{span}[Y \cup Z]$, the set of all linear combinations of vectors belonging to $Y \cup Z$.

First, $Y + Z \subseteq \text{span}[Y \cup Z]$. Indeed, if $y \in Y$ and $z \in Z$, then y and z both belong to $Y \cup Z$, and $y + z = 1 \cdot y + 1 \cdot z$ is a linear combination of vectors of $Y \cup Z$.

Second, $\text{span}[Y \cup Z] \subseteq Y + Z$. Since $0 \in Z$, every $y \in Y$ may be written $y = y + 0 \in Y + Z$; similarly every $z \in Z$ equals $0 + z \in Y + Z$. Hence $Y \cup Z \subseteq Y + Z$. But $Y + Z$ has just been shown to be a subspace of X , and $\text{span}[Y \cup Z]$ is contained in any linear subspace of X that contains $Y \cup Z$ (it is the smallest such subspace). Therefore $\text{span}[Y \cup Z] \subseteq Y + Z$.

Alternatively, and without invoking minimality: a linear combination $\sum_{k=1}^n \lambda_k w_k$ with each $w_k \in Y \cup Z$ may be split, after relabelling, as $\sum_{k \in A} \lambda_k w_k + \sum_{k \in B} \lambda_k w_k$, where A indexes those w_k chosen to lie in Y and B those chosen to lie in Z . The first sum lies in Y and the second in Z , since each is a linear combination of vectors of a subspace (an empty sum being 0, which lies in both). Hence the combination lies in $Y + Z$.

The two inclusions give $Y + Z = \text{span}[Y \cup Z]$.

Problem 13.3. Let S be a subset of a normed linear space X .

- (i) Show that the intersection of a collection of linear subspaces of X also is a linear subspace of X .
- (ii) Show that $\text{span}[S]$ is the intersection of all the linear subspaces of X that contain S and therefore is a linear subspace of X .
- (iii) Show that $\overline{\text{span}}[S]$ is the intersection of all the closed linear subspaces of X that contain S and therefore is a closed linear subspace of X .

Solution. Throughout, S is assumed nonempty, so that $\text{span}[S]$ is defined; and, as in the text, $\overline{\text{span}}[S]$ denotes the closure of $\text{span}[S]$.

(i) Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be a nonempty collection of linear subspaces of X and put $W = \bigcap_{\lambda \in \Lambda} W_\lambda$.

Each W_λ is nonempty and closed under linear combinations, so $0 = 0 \cdot w$ belongs to W_λ for any $w \in W_\lambda$. Hence $0 \in W$ and W is nonempty.

Let $x_1, \dots, x_n$ belong to W and let $\mu_1, \dots, \mu_n$ be real numbers. Fix $\lambda \in \Lambda$. Then each x_k belongs to W_λ , and W_λ is a subspace, so $\sum_{k=1}^n \mu_k x_k \in W_\lambda$. Since λ was arbitrary, $\sum_{k=1}^n \mu_k x_k \in W$. Thus W contains every linear combination of its members and is a linear subspace of X .

(ii) Let $\mathcal{F}$ denote the collection of all linear subspaces of X that contain S , and set $M = \bigcap_{W \in \mathcal{F}} W$. The collection $\mathcal{F}$ is nonempty, since X itself belongs to it; so M is well defined and, by part (i), is a linear subspace of X .

We claim $\text{span}[S] \in \mathcal{F}$. Certainly $S \subseteq \text{span}[S]$, since each $x \in S$ equals the linear combination $1 \cdot x$. And $\text{span}[S]$ is a linear subspace: it is nonempty, and if $u_1, \dots, u_m$ belong to $\text{span}[S]$ and $\mu_1, \dots, \mu_m$ are real, write $u_j = \sum_{k=1}^{n_j} \lambda_{j,k} x_{j,k}$ with each $x_{j,k} \in S$; then

$$\sum_{j=1}^m \mu_j u_j = \sum_{j=1}^m \sum_{k=1}^{n_j} (\mu_j \lambda_{j,k}) x_{j,k},$$

which, after collecting the finitely many terms, is again a linear combination of vectors of S and so belongs to $\text{span}[S]$. Hence $\text{span}[S] \in \mathcal{F}$, and therefore $M \subseteq \text{span}[S]$.

Conversely, let $W \in \mathcal{F}$. Then $S \subseteq W$ and W is a subspace, so every linear combination of vectors of S is a linear combination of vectors of W and hence belongs to W . That is, $\text{span}[S] \subseteq W$. As $W \in \mathcal{F}$ was arbitrary, $\text{span}[S] \subseteq M$.

Therefore $\text{span}[S] = M$, the intersection of all the linear subspaces of X containing S ; and by (i) it is

a linear subspace of X . (This also exhibits $\text{span}[S]$ as the smallest subspace containing S , as asserted in the text.)

(iii) We first observe that the closure of a linear subspace Y of the normed linear space X is again a linear subspace. Indeed, $\bar{Y} \supseteq Y \neq \emptyset$. Let $u_1, \dots, u_n$ belong to $\bar{Y}$ and let $\mu_1, \dots, \mu_n$ be real. For each j choose a sequence $\{u_{j,i}\}_i$ in Y with $\{u_{j,i}\} \rightarrow u_j$ as $i \rightarrow \infty$; this is possible since X is a metric space and u_j belongs to the closure of Y . Each $\sum_{j=1}^n \mu_j u_{j,i}$ belongs to Y , and by the triangle inequality and the homogeneity of the norm,

$$\left\| \sum_{j=1}^n \mu_j u_{j,i} - \sum_{j=1}^n \mu_j u_j \right\| \leq \sum_{j=1}^n |\mu_j| \|u_{j,i} - u_j\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Hence $\sum_{j=1}^n \mu_j u_j$ is a limit of a sequence in Y and so belongs to $\bar{Y}$. Thus $\bar{Y}$ is a linear subspace.

Now let $\mathcal{G}$ denote the collection of all closed linear subspaces of X that contain S , and set $N = \bigcap_{W \in \mathcal{G}} W$. Again $X \in \mathcal{G}$, so $\mathcal{G} \neq \emptyset$. By part (i), N is a linear subspace, and since each $W \in \mathcal{G}$ is closed and an intersection of closed sets is closed, N is a closed linear subspace of X containing S .

We claim $\overline{\text{span}[S]} \in \mathcal{G}$. It contains $\text{span}[S] \supseteq S$; it is closed, being a closure; and by the observation just proved, applied to the subspace $\text{span}[S]$ of part (ii), it is a linear subspace. Hence $N \subseteq \overline{\text{span}[S]}$.

Conversely, let $W \in \mathcal{G}$. Since W is a linear subspace containing S , part (ii) gives $\text{span}[S] \subseteq W$. Since W is closed, taking closures preserves this inclusion:

$$\overline{\text{span}[S]} = \overline{\text{span}[S]} \subseteq \bar{W} = W.$$

As $W \in \mathcal{G}$ was arbitrary, $\overline{\text{span}[S]} \subseteq N$.

Therefore $\overline{\text{span}[S]} = N$, the intersection of all the closed linear subspaces of X that contain S , and it is itself a closed linear subspace of X – indeed the smallest one containing S .

Problem 13.4. For a normed linear space X , show that the function $\|\cdot\|: X \rightarrow \mathbb{R}$ is continuous.

Solution. Recall that the metric on X induced by the norm is $\rho(u, v) = \|u - v\|$, and that continuity of $\|\cdot\|$ means continuity with respect to this metric on X and the usual metric on $\mathbb{R}$.

We first establish the reverse triangle inequality:

$$|\|u\| - \|v\|| \leq \|u - v\| \quad \text{for all } u, v \in X. \quad (*)$$

Indeed, by the triangle inequality for the norm,

$$\|u\| = \|(u - v) + v\| \leq \|u - v\| + \|v\|,$$

so that $\|u\| - \|v\| \leq \|u - v\|$. Interchanging the roles of u and v gives

$$\|v\| - \|u\| \leq \|v - u\| = \|(-1)(u - v)\| = |-1| \cdot \|u - v\| = \|u - v\|,$$

where we used the positive homogeneity of the norm. Since $|\|u\| - \|v\||$ is one of the two numbers $\|u\| - \|v\|$ and $\|v\| - \|u\|$, inequality (*) follows.

Inequality (*) says precisely that $\|\cdot\|: X \rightarrow \mathbb{R}$ is Lipschitz with constant 1 with respect to the induced metric, and a Lipschitz function is continuous. Explicitly, let $u_0 \in X$ and let $\epsilon > 0$. Set $\delta = \epsilon$. If $u \in X$ and $\|u - u_0\| < \delta$, then by (*),

$$|\|u\| - \|u_0\|| \leq \|u - u_0\| < \delta = \epsilon.$$

Thus $\|\cdot\|$ satisfies the ϵ - δ criterion for continuity at u_0 , and since $u_0 \in X$ was arbitrary, $\|\cdot\|$ is continuous on X .

Equivalently, in sequential terms: if $\{u_n\} \rightarrow u$ in X , then $\|u_n - u\| \rightarrow 0$, so by (*), $|\|u_n\| - \|u\|| \rightarrow 0$, that is, $\{\|u_n\|\} \rightarrow \|u\|$ in $\mathbb{R}$. Since X is a metric space, sequential continuity is equivalent to continuity.

Problem 13.5. For two normed linear spaces $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$, define a linear structure on the Cartesian product $X \times Y$ by $\lambda \cdot (x, y) = (\lambda x, \lambda y)$ and $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$. Define the product norm $\|\cdot\|$ by $\|(x, y)\| = \|x\|_1 + \|y\|_2$, for $x \in X$ and $y \in Y$. Show that this is a norm with respect to which a sequence converges if and only if each of the two component sequences converges. Furthermore, show that if X and Y are Banach spaces, then so is $X \times Y$.

Solution. Throughout we write $\|\cdot\|$ for the product norm on $X \times Y$.

Step 1: $X \times Y$ is a linear space. Addition is defined componentwise, so $(X \times Y, +)$ is the direct product of the abelian groups $(X, +)$ and $(Y, +)$ and hence is an abelian group, with identity $(0, 0)$ and $-(x, y) = (-x, -y)$. The three scalar-multiplication axioms hold componentwise: for real α, β and $(x, y), (x', y') \in X \times Y$,

$$\begin{aligned}(\alpha + \beta) \cdot (x, y) &= ((\alpha + \beta)x, (\alpha + \beta)y) = (\alpha x + \beta x, \alpha y + \beta y) = \alpha \cdot (x, y) + \beta \cdot (x, y), \\ \alpha \cdot ((x, y) + (x', y')) &= (\alpha(x + x'), \alpha(y + y')) = \alpha \cdot (x, y) + \alpha \cdot (x', y'), \\ (\alpha\beta) \cdot (x, y) &= ((\alpha\beta)x, (\alpha\beta)y) = \alpha \cdot (\beta \cdot (x, y)), \quad 1 \cdot (x, y) = (x, y).\end{aligned}$$

Step 2: $\|\cdot\|$ is a norm. It is clearly nonnegative and real-valued.

If $\|(x, y)\| = 0$ then $\|x\|_1 + \|y\|_2 = 0$; as both summands are nonnegative, $\|x\|_1 = \|y\|_2 = 0$, so $x = 0$ and $y = 0$, that is, (x, y) is the zero vector of $X \times Y$. Conversely $\|(0, 0)\| = 0 + 0 = 0$.

For the triangle inequality, let (x_1, y_1) and (x_2, y_2) belong to $X \times Y$. Using the triangle inequalities in X and in Y ,

$$\begin{aligned}\|(x_1, y_1) + (x_2, y_2)\| &= \|x_1 + x_2\|_1 + \|y_1 + y_2\|_2 \\ &\leq (\|x_1\|_1 + \|x_2\|_1) + (\|y_1\|_2 + \|y_2\|_2) \\ &= \|(x_1, y_1)\| + \|(x_2, y_2)\|.\end{aligned}$$

For homogeneity, for $\alpha \in \mathbb{R}$,

$$\|\alpha \cdot (x, y)\| = \|\alpha x\|_1 + \|\alpha y\|_2 = |\alpha| \|x\|_1 + |\alpha| \|y\|_2 = |\alpha| \|(x, y)\|.$$

Thus $\|\cdot\|$ is a norm on $X \times Y$.

Step 3: Convergence is componentwise. The key estimate is that for all $(x, y) \in X \times Y$,

$$\max\{\|x\|_1, \|y\|_2\} \leq \|(x, y)\| = \|x\|_1 + \|y\|_2, \quad (*)$$

which holds because each of $\|x\|_1$ and $\|y\|_2$ is nonnegative.

Let $\{(x_n, y_n)\}$ be a sequence in $X \times Y$ and let $(x, y) \in X \times Y$. Since $(x_n, y_n) - (x, y) = (x_n - x, y_n - y)$, the estimate $(*)$ gives

$$\max\{\|x_n - x\|_1, \|y_n - y\|_2\} \leq \|(x_n, y_n) - (x, y)\| = \|x_n - x\|_1 + \|y_n - y\|_2.$$

If $\{(x_n, y_n)\} \rightarrow (x, y)$ in $X \times Y$, the middle term tends to 0, and the left-hand inequality forces $\|x_n - x\|_1 \rightarrow 0$ and $\|y_n - y\|_2 \rightarrow 0$; that is, $\{x_n\} \rightarrow x$ in X and $\{y_n\} \rightarrow y$ in Y . Conversely, if $\{x_n\} \rightarrow x$ and $\{y_n\} \rightarrow y$, the right-hand equality exhibits $\|(x_n, y_n) - (x, y)\|$ as the sum of two sequences of real numbers tending to 0, so it tends to 0 and $\{(x_n, y_n)\} \rightarrow (x, y)$.

Step 4: Completeness. Assume X and Y are Banach spaces, and let $\{(x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$. Let $\epsilon > 0$ and choose an index N for which $\|(x_n, y_n) - (x_m, y_m)\| < \epsilon$ whenever $n, m \geq N$. By $(*)$, for such n and m ,

$$\|x_n - x_m\|_1 \leq \|(x_n, y_n) - (x_m, y_m)\| < \epsilon \quad \text{and} \quad \|y_n - y_m\|_2 < \epsilon.$$

Hence $\{x_n\}$ is Cauchy in X and $\{y_n\}$ is Cauchy in Y . Since X and Y are complete, there are $x \in X$ and $y \in Y$ with $\{x_n\} \rightarrow x$ and $\{y_n\} \rightarrow y$. By Step 3, $\{(x_n, y_n)\} \rightarrow (x, y)$ in $X \times Y$. Thus every Cauchy sequence in $X \times Y$ converges to a member of $X \times Y$, so $X \times Y$ is complete and hence is a Banach space.

Problem 13.6. Let X be a normed linear space.

- (i) Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\{x_n\} \rightarrow x$ and $\{y_n\} \rightarrow y$. Show that for any real numbers α and β , $\{\alpha x_n + \beta y_n\} \rightarrow \alpha x + \beta y$.
- (ii) Use (i) to show that if Y is a subspace of X , then its closure $\bar{Y}$ also is a linear subspace of X .
- (iii) Use (i) to show that the vector sum is continuous from $X \times X$ to X and scalar multiplication is continuous from $\mathbb{R} \times X$ to X .

Solution. (i) Let $\alpha, \beta \in \mathbb{R}$. Since

$$(\alpha x_n + \beta y_n) - (\alpha x + \beta y) = \alpha(x_n - x) + \beta(y_n - y),$$

the triangle inequality and the positive homogeneity of the norm give

$$\|(\alpha x_n + \beta y_n) - (\alpha x + \beta y)\| \leq |\alpha| \|x_n - x\| + |\beta| \|y_n - y\|.$$

By assumption $\|x_n - x\| \rightarrow 0$ and $\|y_n - y\| \rightarrow 0$, so the right-hand side is a sum of two real sequences tending to 0 and hence tends to 0. Since the left-hand side is nonnegative and dominated by it, $\|(\alpha x_n + \beta y_n) - (\alpha x + \beta y)\| \rightarrow 0$, that is, $\{\alpha x_n + \beta y_n\} \rightarrow \alpha x + \beta y$.

(ii) Let Y be a subspace of X . Then $\bar{Y} \supseteq Y$ is nonempty. Let $u_1, \dots, u_m$ belong to $\bar{Y}$ and let $\mu_1, \dots, \mu_m$ be real numbers; we must show $\sum_{j=1}^m \mu_j u_j \in \bar{Y}$.

We argue by induction on m , using (i) at each step. Since X is a metric space, a point belongs to $\bar{Y}$ if and only if it is the limit of a sequence in Y .

For $m = 1$: choose $\{u_{1,n}\}$ in Y with $\{u_{1,n}\} \rightarrow u_1$. Applying (i) with $\alpha = \mu_1$, $\beta = 0$, and $y_n = 0$ (a sequence in Y , since $0 \in Y$), we get $\{\mu_1 u_{1,n}\} \rightarrow \mu_1 u_1$. Each $\mu_1 u_{1,n}$ lies in the subspace Y , so $\mu_1 u_1 \in \bar{Y}$.

Inductive step: suppose every linear combination of m members of $\bar{Y}$ lies in $\bar{Y}$, and let $u_1, \dots, u_{m+1} \in \bar{Y}$ with scalars $\mu_1, \dots, \mu_{m+1}$. Put $v = \sum_{j=1}^m \mu_j u_j$, which lies in $\bar{Y}$ by the inductive hypothesis. Choose sequences $\{v_n\}$ and $\{w_n\}$ in Y with $\{v_n\} \rightarrow v$ and $\{w_n\} \rightarrow u_{m+1}$. By (i), with $\alpha = 1$ and $\beta = \mu_{m+1}$,

$$\{v_n + \mu_{m+1} w_n\} \rightarrow v + \mu_{m+1} u_{m+1} = \sum_{j=1}^{m+1} \mu_j u_j.$$

Each $v_n + \mu_{m+1} w_n$ is a linear combination of members of the subspace Y and so belongs to Y . Hence $\sum_{j=1}^{m+1} \mu_j u_j$ is a limit of a sequence in Y and lies in $\bar{Y}$.

Therefore $\bar{Y}$ contains every linear combination of its members and is a linear subspace of X .

(iii) Give $X \times X$ the product norm $\|(u, v)\| = \|u\| + \|v\|$ and $\mathbb{R} \times X$ the product norm $\|(\lambda, u)\| = |\lambda| + \|u\|$, as in the preceding problem ($\mathbb{R}$ being a normed linear space under the absolute value). By the preceding problem, a sequence converges in either product space if and only if both component sequences converge; consequently the metric induced by the product norm induces the product topology on each product, so continuity with respect to the product norm is the same as continuity with respect to the product topology. Both products are metric spaces, so for a mapping out of them continuity is equivalent to sequential continuity.

Vector sum. Let $\sigma: X \times X \rightarrow X$ be defined by $\sigma(u, v) = u + v$. Let $\{(x_n, y_n)\} \rightarrow (x, y)$ in $X \times X$. Then $\{x_n\} \rightarrow x$ and $\{y_n\} \rightarrow y$ in X , so by (i) with $\alpha = \beta = 1$,

$$\sigma(x_n, y_n) = x_n + y_n \rightarrow x + y = \sigma(x, y).$$

Hence σ is continuous.

Scalar multiplication. Let $\pi: \mathbb{R} \times X \rightarrow X$ be defined by $\pi(\lambda, u) = \lambda u$. Let $\{(\lambda_n, x_n)\} \rightarrow (\lambda, x)$ in $\mathbb{R} \times X$, so that $\lambda_n \rightarrow \lambda$ in $\mathbb{R}$ and $\{x_n\} \rightarrow x$ in X . Write

$$\lambda_n x_n - \lambda x = \lambda_n(x_n - x) + (\lambda_n - \lambda)x.$$

A convergent sequence of real numbers is bounded, so there is an $M > 0$ with $|\lambda_n| \leq M$ for all n . Hence

$$\|\lambda_n x_n - \lambda x\| \leq |\lambda_n| \|x_n - x\| + |\lambda_n - \lambda| \|x\| \leq M \|x_n - x\| + |\lambda_n - \lambda| \|x\|.$$

Both terms on the right tend to 0, so $\|\pi(\lambda_n, x_n) - \pi(\lambda, x)\| \rightarrow 0$ and π is continuous.

(The estimate used for π is exactly the estimate of part (i), applied with the varying scalar λ_n in place of the fixed α ; the boundedness of $\{\lambda_n\}$ is what makes the argument of (i) go through when the scalar is allowed to vary.)

Problem 13.7. Show that the set $\mathcal{P}$ of all polynomials on $[a, b]$ is a linear space. For $\mathcal{P}$ considered as a subset of the normed linear space $C[a, b]$, show that $\mathcal{P}$ fails to be closed. For $\mathcal{P}$ considered as a subset of the normed linear space $L^1[a, b]$, show that $\mathcal{P}$ fails to be closed.

Solution. Throughout, $[a, b]$ is a nondegenerate closed bounded interval, so $a < b$, and $\mathcal{P}$ denotes the set of restrictions to $[a, b]$ of real polynomial functions.

Step 1: $\mathcal{P}$ is a linear space. Let $\mathcal{F}[a, b]$ denote the linear space of all real-valued functions on $[a, b]$, with addition and scalar multiplication defined pointwise; this is the quintessential example of a linear space described at the beginning of Section 13.1. We show $\mathcal{P}$ is a linear subspace of $\mathcal{F}[a, b]$, and hence is itself a linear space under the inherited operations.

$\mathcal{P}$ is nonempty, since the zero function is a polynomial. Let $p, q \in \mathcal{P}$ and let α, β be real numbers. Write

$$p(x) = \sum_{k=0}^n c_k x^k, \quad q(x) = \sum_{k=0}^n d_k x^k \quad \text{for all } x \in [a, b],$$

where we have padded the shorter list of coefficients with zeros so that both sums run to the same n . Then for all $x \in [a, b]$,

$$(\alpha p + \beta q)(x) = \alpha p(x) + \beta q(x) = \sum_{k=0}^n (\alpha c_k + \beta d_k) x^k,$$

which is again a polynomial in x . So $\alpha p + \beta q \in \mathcal{P}$. Closure under linear combinations of two members yields, by the induction on the number of vectors carried out in Problem 2, closure under arbitrary finite linear combinations. Hence $\mathcal{P}$ is a linear subspace of $\mathcal{F}[a, b]$ and therefore a linear space.

Step 2: a continuous function on $[a, b]$ that is not a polynomial. Let $c = (a + b)/2$, an interior point of $[a, b]$, and define

$$f(x) = |x - c| \quad \text{for } x \in [a, b].$$

Then f is continuous on $[a, b]$, being the composition of the continuous functions $x \mapsto x - c$ and $t \mapsto |t|$. However, f is not differentiable at c : for $h > 0$ small enough that $c \pm h \in [a, b]$,

$$\frac{f(c + h) - f(c)}{h} = \frac{h}{h} = 1, \quad \frac{f(c - h) - f(c)}{-h} = \frac{h}{-h} = -1,$$

so the one-sided difference quotients have different limits. Every polynomial, on the other hand, is differentiable at every point of $\mathbb{R}$. Hence $f \notin \mathcal{P}$.

Step 3: $\mathcal{P}$ is not closed in $C[a, b]$. The norm on $C[a, b]$ is the maximum norm, $\|g\|_{\max} = \max_{x \in [a, b]} |g(x)|$ (the maximum is attained, since a continuous real-valued function on the compact set $[a, b]$ takes a maximum value). By the Weierstrass Approximation Theorem of Section 12.3, for each $\epsilon > 0$ there is a polynomial p with $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$; that is, the polynomials are dense in $C[a, b]$ with respect to the maximum norm. Therefore, taking $\epsilon = 1/n$, for each natural number n there is a polynomial p_n with

$$\|p_n - f\|_{\max} = \max_{x \in [a, b]} |p_n(x) - f(x)| < \frac{1}{n}.$$

Thus $\{p_n\}$ is a sequence in $\mathcal{P}$ that converges in $C[a, b]$ to f , and so f belongs to the closure of $\mathcal{P}$ in $C[a, b]$. But $f \notin \mathcal{P}$ by Step 2. A set that contains a sequence converging to a point outside the set is not closed. Hence $\mathcal{P}$ is not closed in $C[a, b]$.

(Indeed the Weierstrass Approximation Theorem says more: the closure of $\mathcal{P}$ in $C[a, b]$ is all of $C[a, b]$, so $\mathcal{P}$ is a dense proper subspace.)

Step 4: $\mathcal{P}$ is not closed in $L^1[a, b]$. Members of $L^1[a, b]$ are equivalence classes of integrable functions that agree almost everywhere on $[a, b]$; we regard $\mathcal{P}$ as the set of classes containing a polynomial. Every continuous function on the closed bounded interval $[a, b]$ is bounded and measurable, hence integrable, so both f and every p_n of Step 3 belong to $L^1[a, b]$.

For $g \in C[a, b]$ we have the elementary estimate

$$\|g\|_1 = \int_a^b |g| \leq \int_a^b \|g\|_{\max} = (b-a) \|g\|_{\max},$$

by the monotonicity of the integral. Applying this to $g = p_n - f$ and using Step 3,

$$\|p_n - f\|_1 \leq (b-a) \|p_n - f\|_{\max} < \frac{b-a}{n} \longrightarrow 0.$$

Hence $\{p_n\} \rightarrow f$ in $L^1[a, b]$, so f lies in the closure of $\mathcal{P}$ in $L^1[a, b]$.

It remains to check that f does not belong to $\mathcal{P}$ as a subset of $L^1[a, b]$, that is, that f is not almost everywhere equal on $[a, b]$ to a polynomial. Suppose, to the contrary, that p is a polynomial with $f = p$ a.e. on $[a, b]$. Let

$$E = \{x \in [a, b] \mid f(x) = p(x)\},$$

so that $[a, b] \setminus E$ has measure zero. We claim E is dense in $[a, b]$. Indeed, let I be a nondegenerate open subinterval of (a, b) and suppose $I \cap E = \emptyset$. Then $I \subseteq [a, b] \setminus E$, so by the monotonicity of Lebesgue measure $m(I) \leq m([a, b] \setminus E) = 0$; but the measure of an interval is its length, which is positive. This contradiction shows E meets every nondegenerate open subinterval of (a, b) . Hence the closure of E contains (a, b) , and being closed it contains $[a, b]$; that is, E is dense in $[a, b]$. The function $f - p$ is continuous on $[a, b]$ and vanishes on the dense set E ; hence, given $x \in [a, b]$, choosing a sequence $\{x_k\}$ in E with $\{x_k\} \rightarrow x$ and using continuity,

$$(f - p)(x) = \lim_{k \rightarrow \infty} (f - p)(x_k) = 0.$$

So $f = p$ everywhere on $[a, b]$, contradicting Step 2, where f was shown to fail to be differentiable at the interior point c while p is differentiable there.

Therefore $f \notin \mathcal{P}$, while f is the L^1 limit of the sequence $\{p_n\}$ in $\mathcal{P}$. Consequently $\mathcal{P}$ is not closed in $L^1[a, b]$ either.

13.2 Problems 13.8–13.14

Problem 13.8. A nonnegative real-valued function $\|\cdot\|$ defined on a vector space X is called a pseudonorm if $\|x + y\| \leq \|x\| + \|y\|$ and $\|\alpha x\| = |\alpha| \|x\|$. Define $x \cong y$, provided $\|x - y\| = 0$. Show that this is an equivalence relation. Define $X/\cong$ to be the set of equivalence classes of X under $\cong$ and for $x \in X$ define $[x]$ to be the equivalence class of x . Show that $X/\cong$ is a normed vector space if we define $\alpha[x] + \beta[y]$ to be the equivalence class of $\alpha x + \beta y$ and define $\|[x]\| = \|x\|$. Illustrate this procedure with $X = L^p[a, b]$, $1 \leq p < \infty$.

Solution. Throughout, $\|\cdot\|$ is a pseudonorm on the vector space X : it takes values in $[0, \infty)$, it is subadditive, and it is positively homogeneous, $\|\alpha x\| = |\alpha| \|x\|$ for every real α and every $x \in X$. Observe first that taking $\alpha = 0$ in the homogeneity property gives

$$\|0\| = \|0 \cdot x\| = |0| \|x\| = 0,$$

and taking $\alpha = -1$ gives $\|-x\| = \|x\|$ for all $x \in X$.

Set

$$X_0 = \{x \in X \mid \|x\| = 0\}.$$

We claim X_0 is a linear subspace of X . Indeed $0 \in X_0$ by the computation above, and if $x, y \in X_0$ and α, β are real numbers, then by subadditivity and homogeneity

$$0 \leq \|\alpha x + \beta y\| \leq \|\alpha x\| + \|\beta y\| = |\alpha| \|x\| + |\beta| \|y\| = 0,$$

so $\alpha x + \beta y \in X_0$. Moreover, by definition,

$$x \cong y \quad \text{if and only if} \quad x - y \in X_0.$$

$\cong$ is an equivalence relation. Reflexivity: $\|x - x\| = \|0\| = 0$, so $x \cong x$. Symmetry: if $\|x - y\| = 0$ then $\|y - x\| = \|(x - y)\| = \|x - y\| = 0$, so $y \cong x$. Transitivity: if $\|x - y\| = 0$ and $\|y - z\| = 0$, then by subadditivity

$$0 \leq \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = 0,$$

so $\|x - z\| = 0$ and $x \cong z$. Thus $\cong$ is an equivalence relation, and $X/\cong$, the set of equivalence classes, is well defined; write $[x]$ for the class of x , so that $[x] = x + X_0$.

The linear operations are well defined. Suppose $x \cong x'$ and $y \cong y'$, and let α, β be real. Then $x - x' \in X_0$ and $y - y' \in X_0$, and since X_0 is a linear subspace,

$$(\alpha x + \beta y) - (\alpha x' + \beta y') = \alpha(x - x') + \beta(y - y') \in X_0.$$

Hence $\alpha x + \beta y \cong \alpha x' + \beta y'$: the equivalence class of $\alpha x + \beta y$ depends only on $[x]$ and $[y]$, not on the representatives chosen. So

$$\alpha[x] + \beta[y] := [\alpha x + \beta y]$$

is an unambiguous definition. In particular addition is $[x] + [y] = [x + y]$ and scalar multiplication is $\alpha[x] = [\alpha x]$.

$X/\cong$ is a linear space under these operations. The map $\pi: X \rightarrow X/\cong$, $\pi(x) = [x]$, is onto and, by the very definition of the operations, satisfies $\pi(\alpha x + \beta y) = \alpha\pi(x) + \beta\pi(y)$. Every one of the vector space axioms is an identity between expressions built from sums and scalar multiples of finitely many vectors; given classes $[x_1], \dots, [x_k]$, choose representatives $x_1, \dots, x_k$, apply the corresponding identity in X , and push it through π . For instance, commutativity: $[x] + [y] = [x + y] = [y + x] = [y] + [x]$; and distributivity: $\alpha([x] + [y]) = \alpha[x + y] = [\alpha(x + y)] = [\alpha x + \alpha y] = \alpha[x] + \alpha[y]$. The zero vector of $X/\cong$ is $[0] = X_0$, since $[x] + [0] = [x + 0] = [x]$, and the additive inverse of $[x]$ is $[-x]$, since $[x] + [-x] = [0]$. Thus $X/\cong$ is a vector space.

The norm is well defined. Suppose $x \cong y$. By subadditivity, $\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| = \|y\|$, and by symmetry of the roles of x and y , $\|y\| \leq \|x\|$. Hence $\|x\| = \|y\|$, and therefore

$$\|[x]\| := \|x\|$$

does not depend on the representative.

$\|\cdot\|$ is a norm on $X/\cong$. It is nonnegative and real-valued because $\|\cdot\|$ is. Homogeneity: $\|\alpha[x]\| = \|[\alpha x]\| = \|\alpha x\| = |\alpha| \|x\| = |\alpha| \|[x]\|$. The triangle inequality: $\|[x] + [y]\| = \|[x + y]\| = \|x + y\| \leq \|x\| + \|y\| = \|[x]\| + \|[y]\|$. Finally, definiteness, which is exactly what the passage to classes has bought: if $\|[x]\| = 0$ then $\|x\| = 0$, so $x \in X_0$, that is $x \cong 0$ and $[x] = [0]$ is the zero vector of $X/\cong$; conversely $\|[0]\| = \|0\| = 0$. Hence $\|[x]\| = 0$ if and only if $[x]$ is the zero vector. Therefore $X/\cong$, with these operations and this norm, is a normed vector space.

Illustration with $X = L^p[a, b]$, $1 \leq p < \infty$. Let X be the collection of all Lebesgue measurable extended real-valued functions f on $[a, b]$ that are finite almost everywhere and satisfy $\int_a^b |f|^p < \infty$. (Modifying such an f on the null set where it is infinite changes nothing below, so one may equally take X to consist of real-valued measurable functions with $\int_a^b |f|^p < \infty$; that is what we do, and X is a vector space of functions under pointwise operations, since $|f + g|^p \leq 2^p(|f|^p + |g|^p)$.) Define

$$\|f\|_p = \left(\int_a^b |f|^p \right)^{1/p}.$$

Then $\|\cdot\|_p$ is nonnegative and real-valued on X , it is positively homogeneous because $\int_a^b |\alpha f|^p = |\alpha|^p \int_a^b |f|^p$, and it satisfies the triangle inequality by Minkowski's Inequality (Section 7.2). But $\|\cdot\|_p$ is not a norm on X : by Proposition 9 of Chapter 4, a nonnegative measurable function has zero integral if and only if it vanishes almost everywhere, so, applied to $|f|^p$, this gives $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere on $[a, b]$, and there are functions other than the zero function with this property,

for example the characteristic function of the rationals in $[a, b]$. So $\|\cdot\|_p$ is a pseudonorm and nothing more.

For this pseudonorm the subspace X_0 is the set of measurable functions vanishing almost everywhere on $[a, b]$, and

$$f \cong g \quad \text{if and only if} \quad f = g \text{ a.e. on } [a, b].$$

The construction above therefore produces the normed linear space $X/\cong$ whose vectors are the equivalence classes of p -integrable functions that agree almost everywhere, with the class operations $\alpha[f] + \beta[g] = [\alpha f + \beta g]$ and the norm $\|[f]\| = \|f\|_p$. This is precisely $L^p[a, b]$ as it is used throughout the text: the convention that two functions equal almost everywhere are regarded as the same member of $L^p[a, b]$ is exactly the passage from the pseudonormed space X to the normed space $X/\cong$, and it is what makes $\|\cdot\|_p$ definite and hence a genuine norm.

Problem 13.9. Let X and Y be normed linear spaces and $T: X \rightarrow Y$ be linear.

- (i) Show that T is continuous if and only if it is continuous at a single point u_0 in X .
- (ii) Show that T is Lipschitz if and only if it is continuous.
- (iii) Show that neither (i) nor (ii) hold in the absence of the linearity assumption on T .

Solution. Both X and Y are metric spaces, with the metrics induced by their norms, $\rho(u, v) = \|u - v\|$; continuity of T at a point and continuity of T on X are understood in this metric sense. Note at the outset that linearity forces $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$, and that $T(u) - T(v) = T(u - v)$ for all $u, v \in X$.

Part (i). If T is continuous on X , it is in particular continuous at any single point of X , so there is nothing to prove in one direction.

Conversely, suppose T is continuous at the single point $u_0 \in X$. Let $u \in X$ be arbitrary and let $\epsilon > 0$. By continuity at u_0 , there is a $\delta > 0$ such that

$$\|v - u_0\| < \delta \quad \text{implies} \quad \|T(v) - T(u_0)\| < \epsilon.$$

Now let $w \in X$ satisfy $\|w - u\| < \delta$ and put $v = w - u + u_0$. Then $\|v - u_0\| = \|w - u\| < \delta$, so $\|T(v) - T(u_0)\| < \epsilon$. On the other hand, by linearity,

$$T(v) - T(u_0) = T(v - u_0) = T(w - u) = T(w) - T(u).$$

Therefore $\|T(w) - T(u)\| < \epsilon$ whenever $\|w - u\| < \delta$. Since $\epsilon > 0$ was arbitrary, T is continuous at u ; since $u \in X$ was arbitrary, T is continuous on X . Notice that the δ produced depends only on ϵ and not on u , so the argument in fact yields uniform continuity.

Part (ii). Recall that T is Lipschitz provided there is a constant $c \geq 0$ for which $\|T(u) - T(v)\| \leq c \cdot \|u - v\|$ for all $u, v \in X$.

Suppose T is Lipschitz, with constant c . Given $u \in X$ and $\epsilon > 0$, set $\delta = \epsilon/(c + 1) > 0$. If $\|w - u\| < \delta$ then $\|T(w) - T(u)\| \leq c\|w - u\| < c\epsilon/(c + 1) < \epsilon$. So T is continuous (indeed uniformly continuous). This direction uses no linearity.

Conversely, suppose T is continuous. Since T is linear, Theorem 1 asserts that T is bounded, that is, there is an $M \geq 0$ with $\|T(u)\| \leq M\|u\|$ for all $u \in X$; and inequality (2) of this section records that then

$$\|T(u) - T(v)\| \leq \|T\| \cdot \|u - v\| \quad \text{for all } u, v \in X.$$

Thus T is Lipschitz with Lipschitz constant $\|T\|$. (Directly: $\|T(u) - T(v)\| = \|T(u - v)\| \leq M\|u - v\|$.)

Part (iii). Both equivalences fail for nonlinear mappings, as the following two examples on $X = Y = \mathbb{R}$, normed by the absolute value, show.

Failure of (i). Define $T: \mathbb{R} \rightarrow \mathbb{R}$ by

$$T(x) = \begin{cases} x, & x \text{ rational,} \\ 0, & x \text{ irrational.} \end{cases}$$

Then T is continuous at the single point $u_0 = 0$: indeed $T(0) = 0$ and $|T(x) - T(0)| = |T(x)| \leq |x|$ for every x , so given $\epsilon > 0$ the choice $\delta = \epsilon$ works. But T fails to be continuous at every $u \neq 0$. To

see this, fix $u \neq 0$ and take $\epsilon = |u|/2$. Every interval about u contains both rational and irrational points, by the density of the rationals and of the irrationals in $\mathbb{R}$. If u is rational, choose irrational points $x_n \rightarrow u$; then $T(x_n) = 0$ while $T(u) = u$, so $|T(x_n) - T(u)| = |u| > \epsilon$ for all n . If u is irrational, choose rational points $x_n \rightarrow u$; then $T(x_n) = x_n \rightarrow u \neq 0 = T(u)$, so $|T(x_n) - T(u)| = |x_n| > |u|/2 = \epsilon$ for all large n . In either case T is not continuous at u . Thus continuity at one point does not imply continuity, in the absence of linearity.

Failure of (ii). Define $T: \mathbb{R} \rightarrow \mathbb{R}$ by $T(x) = x^2$. This mapping is continuous on all of $\mathbb{R}$. It is not Lipschitz: for any $c \geq 0$, take $u = c + 1$ and $v = c + 2$. Then

$$|T(u) - T(v)| = |u^2 - v^2| = |u - v| \cdot |u + v| = (2c + 3) \cdot |u - v| > c \cdot |u - v|,$$

since $|u - v| = 1 > 0$. So no $c \geq 0$ is a Lipschitz constant for T . Thus continuity does not imply the Lipschitz property, in the absence of linearity.

Problem 13.10. For X and Y normed linear spaces and $T \in \mathcal{L}(X, Y)$, show that $\|T\|$ is the smallest Lipschitz constant for the mapping T , that is, the smallest number $c \geq 0$ for which

$$\|T(u) - T(v)\| \leq c \cdot \|u - v\| \quad \text{for all } u, v \in X.$$

Solution. Let $\mathcal{M}$ denote the set of constants $M \geq 0$ for which

$$\|T(u)\| \leq M\|u\| \quad \text{for all } u \in X,$$

which is condition (1) of this section, and let $\mathcal{C}$ denote the set of Lipschitz constants for T , that is, the set of numbers $c \geq 0$ for which

$$\|T(u) - T(v)\| \leq c \cdot \|u - v\| \quad \text{for all } u, v \in X.$$

By definition $\|T\| = \inf \mathcal{M}$, and $\mathcal{M} \neq \emptyset$ because T is bounded. We must show that $\|T\|$ belongs to $\mathcal{C}$ and that $\|T\| \leq c$ for every $c \in \mathcal{C}$; these two assertions together say exactly that $\|T\|$ is the smallest member of $\mathcal{C}$.

First we verify, as asserted in the text, that (1) holds for $M = \|T\|$, that is, $\|T\| \in \mathcal{M}$. Fix $u \in X$. For every $M \in \mathcal{M}$ we have $\|T(u)\| \leq M\|u\|$. Since $\|T\| = \inf \mathcal{M}$, choose a sequence $\{M_n\}$ in $\mathcal{M}$ with $M_n \rightarrow \|T\|$. Then $\|T(u)\| \leq M_n\|u\|$ for every n , and letting $n \rightarrow \infty$ gives

$$\|T(u)\| \leq \|T\| \cdot \|u\|.$$

As $u \in X$ was arbitrary, $\|T\| \in \mathcal{M}$.

Next, $\|T\|$ is a Lipschitz constant for T . Indeed, let $u, v \in X$. Since T is linear, $T(u) - T(v) = T(u - v)$, and therefore, by the inequality just established applied to the vector $u - v$,

$$\|T(u) - T(v)\| = \|T(u - v)\| \leq \|T\| \cdot \|u - v\|.$$

This is inequality (2) of this section, and it says $\|T\| \in \mathcal{C}$.

Finally, $\|T\|$ is no larger than any Lipschitz constant. Let $c \in \mathcal{C}$. Since T is linear, $T(0) = 0$. Taking $v = 0$ in the Lipschitz inequality gives, for every $u \in X$,

$$\|T(u)\| = \|T(u) - T(0)\| \leq c \cdot \|u - 0\| = c\|u\|.$$

Thus c satisfies (1), that is, $c \in \mathcal{M}$. Since $\|T\| = \inf \mathcal{M}$, we conclude $\|T\| \leq c$.

We have shown $\|T\| \in \mathcal{C}$ and $\|T\| \leq c$ for all $c \in \mathcal{C}$. Hence $\|T\| = \min \mathcal{C}$: the operator norm of T is the smallest Lipschitz constant for T .

Two remarks. The argument also shows $\mathcal{M} = \mathcal{C}$, since (1) for a constant M immediately gives the Lipschitz inequality with $c = M$ by linearity, and conversely as above; so the infimum defining $\|T\|$ is in fact attained, and it is attained equally as an infimum over Lipschitz constants. Second, the point at which linearity is used is essential: it is linearity that converts the two-variable Lipschitz condition into the one-variable boundedness condition, via $T(u) - T(v) = T(u - v)$ and $T(0) = 0$.

Problem 13.11. For X and Y normed linear spaces and $T \in \mathcal{L}(X, Y)$, show that

$$\|T\| = \sup \{\|T(u)\| \mid u \in X, \|u\| \leq 1\}.$$

Solution. Write

$$B = \{u \in X \mid \|u\| \leq 1\}, \quad S = \sup_{u \in B} \|T(u)\|.$$

The set B is nonempty, since $0 \in B$, so the supremum is taken over a nonempty set of nonnegative real numbers; we show below that it is finite and that it equals $\|T\|$.

Step 1: $S \leq \|T\|$. As established in the text (and reproved in Problem 10), inequality (1) holds with $M = \|T\|$, that is,

$$\|T(u)\| \leq \|T\| \cdot \|u\| \quad \text{for all } u \in X.$$

Hence for $u \in B$ we have $\|T(u)\| \leq \|T\| \cdot \|u\| \leq \|T\| \cdot 1 = \|T\|$. So $\|T\|$ is an upper bound for the set $\{\|T(u)\| \mid u \in B\}$, which is therefore bounded above; its supremum S is a finite real number and $S \leq \|T\|$.

Step 2: $\|T\| \leq S$. We show that the number S satisfies condition (1), that is, $\|T(u)\| \leq S\|u\|$ for all $u \in X$; since $\|T\|$ is by definition the infimum of all $M \geq 0$ satisfying (1), this yields $\|T\| \leq S$.

Let $u \in X$. If $u = 0$, then by linearity $T(0) = 0$ and the inequality $\|T(0)\| = 0 \leq S \cdot 0$ holds. If $u \neq 0$, then $\|u\| > 0$ by definiteness of the norm on X ; put

$$w = \frac{1}{\|u\|} u.$$

Then $\|w\| = \frac{1}{\|u\|} \|u\| = 1$, so $w \in B$ and therefore $\|T(w)\| \leq S$. By the linearity of T and the positive homogeneity of the norm on Y ,

$$\|T(u)\| = \|T(\|u\| w)\| = \|\|u\| T(w)\| = \|u\| \cdot \|T(w)\| \leq S \|u\|.$$

Thus (1) holds with $M = S$, and consequently $\|T\| \leq S$.

Combining the two steps, $\|T\| = S$, that is,

$$\|T\| = \sup \{\|T(u)\| \mid u \in X, \|u\| \leq 1\}.$$

We record a variant used repeatedly. Suppose $X \neq \{0\}$ and let $\Sigma = \{u \in X \mid \|u\| = 1\}$, which is then nonempty, since X contains a vector $u \neq 0$ and $u/\|u\| \in \Sigma$. Put $S' = \sup_{u \in \Sigma} \|T(u)\|$. On the one hand $\Sigma \subseteq B$, so $S' \leq S$. On the other hand, let $u \in B$. If $u = 0$ then $\|T(u)\| = 0 \leq S'$. If $u \neq 0$, set $w = u/\|u\| \in \Sigma$; the computation of Step 2 gives $\|T(u)\| = \|u\| \cdot \|T(w)\| \leq \|T(w)\| \leq S'$, because $\|u\| \leq 1$. Hence S' is an upper bound for $\{\|T(u)\| \mid u \in B\}$ and so $S \leq S'$. Therefore $S = S'$, that is,

$$\|T\| = \sup \{\|T(u)\| \mid u \in X, \|u\| = 1\} \quad \text{provided } X \neq \{0\}.$$

(The proviso is needed: if $X = \{0\}$ the unit sphere is empty, while $\|T\| = 0$.)

Problem 13.12. For X and Y normed linear spaces, let $\{T_n\} \rightarrow T$ in $\mathcal{L}(X, Y)$ and $\{u_n\} \rightarrow u$ in X . Show that $\{T_n(u_n)\} \rightarrow T(u)$ in Y .

Solution. By Proposition 2, $\mathcal{L}(X, Y)$ is a normed linear space under the operator norm, and the hypothesis $\{T_n\} \rightarrow T$ in $\mathcal{L}(X, Y)$ means $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$; the hypothesis $\{u_n\} \rightarrow u$ in X means $\lim_{n \rightarrow \infty} \|u_n - u\| = 0$.

First, the sequence $\{\|T_n\|\}$ is bounded. Indeed, since $\|T_n - T\| \rightarrow 0$, there is an index N with $\|T_n - T\| \leq 1$ for all $n \geq N$, and then, by the triangle inequality for the operator norm,

$$\|T_n\| = \|(T_n - T) + T\| \leq \|T_n - T\| + \|T\| \leq 1 + \|T\| \quad \text{for } n \geq N.$$

Setting

$$M = \max \{ \|T_1\|, \dots, \|T_N\|, 1 + \|T\| \},$$

a maximum of finitely many real numbers, we have $\|T_n\| \leq M$ for every n : for $n \leq N$ because $\|T_n\|$ is one of the listed numbers, and for $n \geq N$ by the displayed estimate. Note $M \geq 0$.

Now estimate. For each n , insert and remove the term $T_n(u)$ and use the triangle inequality in Y :

$$\|T_n(u_n) - T(u)\| \leq \|T_n(u_n) - T_n(u)\| + \|T_n(u) - T(u)\|.$$

For the first term, T_n is linear and bounded, so by inequality (2) of this section,

$$\|T_n(u_n) - T_n(u)\| \leq \|T_n\| \cdot \|u_n - u\| \leq M \|u_n - u\|.$$

For the second term, $T_n - T$ belongs to $\mathcal{L}(X, Y)$ and $T_n(u) - T(u) = (T_n - T)(u)$, so by inequality (1) applied to the operator $T_n - T$,

$$\|T_n(u) - T(u)\| = \|(T_n - T)(u)\| \leq \|T_n - T\| \cdot \|u\|.$$

Therefore

$$0 \leq \|T_n(u_n) - T(u)\| \leq M \|u_n - u\| + \|u\| \cdot \|T_n - T\| \quad \text{for every } n.$$

Here u is fixed, so M and $\|u\|$ are constants independent of n , while $\|u_n - u\| \rightarrow 0$ and $\|T_n - T\| \rightarrow 0$. Hence the right-hand side converges to 0, and by the squeeze property of limits of real sequences,

$$\lim_{n \rightarrow \infty} \|T_n(u_n) - T(u)\| = 0,$$

that is, $\{T_n(u_n)\} \rightarrow T(u)$ in Y .

Explicitly, given $\epsilon > 0$, choose N_1 so that $\|u_n - u\| < \epsilon/(2M + 2)$ for $n \geq N_1$ and N_2 so that $\|T_n - T\| < \epsilon/(2\|u\| + 2)$ for $n \geq N_2$; then $\|T_n(u_n) - T(u)\| < \epsilon/2 + \epsilon/2 = \epsilon$ for all $n \geq \max\{N_1, N_2\}$.

Problem 13.13. Let X be a Banach space and $T \in \mathcal{L}(X, X)$ have $\|T\| < 1$.

- (i) Use the Contraction Mapping Principle to show that $I - T \in \mathcal{L}(X, X)$ is one-to-one and onto.
- (ii) Show that $I - T$ is an isomorphism.

Solution. Here I denotes the identity operator on X , $I(u) = u$. It is linear, and $\|I(u)\| = \|u\|$ for all u , so I is bounded and belongs to $\mathcal{L}(X, X)$. By Proposition 2, $\mathcal{L}(X, X)$ is a linear space, so $I - T \in \mathcal{L}(X, X)$. Since X is a Banach space, X equipped with the metric $\rho(u, v) = \|u - v\|$ is a complete metric space; this is the setting in which the Banach Contraction Principle of Section 10.3 applies.

Part (i). Fix $y \in X$ and define the mapping

$$S_y: X \rightarrow X, \quad S_y(x) = T(x) + y.$$

For $x, x' \in X$, the linearity of T and inequality (2) of this section give

$$\rho(S_y(x), S_y(x')) = \|(T(x) + y) - (T(x') + y)\| = \|T(x) - T(x')\| \leq \|T\| \cdot \|x - x'\| = \|T\| \rho(x, x').$$

Thus S_y is Lipschitz on X with Lipschitz constant $\|T\| < 1$, that is, S_y is a contraction of the complete metric space X into itself. (If one prefers a strictly positive contraction constant, note that any c with $\|T\| \leq c < 1$ is also a Lipschitz constant for S_y , so $c = \max\{\|T\|, 1/2\}$ serves.) Since X is nonempty, the Banach Contraction Principle of Section 10.3 applies: $S_y: X \rightarrow X$ has exactly one fixed point.

Now observe that for $x \in X$,

$$S_y(x) = x \iff T(x) + y = x \iff y = x - T(x) = (I - T)(x).$$

Therefore the statement that S_y has exactly one fixed point says precisely that there is exactly one $x \in X$ with $(I - T)(x) = y$.

Existence for every $y \in X$ says $I - T$ is onto. Uniqueness for every $y \in X$ says $I - T$ is one-to-one. (Equivalently, for the kernel: if $(I - T)(x) = 0$ then x is a fixed point of $S_0 = T$, and since $T(0) = 0$ the

unique fixed point of the contraction T is 0, so $x = 0$; hence $\ker(I - T) = \{0\}$ and $I - T$ is one-to-one.) Thus $I - T$ is a one-to-one mapping of X onto X .

Part (ii). By part (i), $I - T$ is a one-to-one bounded linear operator of X onto X , so the inverse mapping $(I - T)^{-1}: X \rightarrow X$ is defined. As noted in the text, the inverse of a one-to-one, onto linear operator is linear; explicitly, given $y_1, y_2 \in X$ and real α, β , put $x_i = (I - T)^{-1}(y_i)$; then, since $I - T$ is linear,

$$(I - T)(\alpha x_1 + \beta x_2) = \alpha(I - T)(x_1) + \beta(I - T)(x_2) = \alpha y_1 + \beta y_2,$$

and applying $(I - T)^{-1}$ gives $(I - T)^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha(I - T)^{-1}(y_1) + \beta(I - T)^{-1}(y_2)$. To conclude that $I - T$ is an isomorphism it remains, as the text says, to show that this inverse is bounded, that is, belongs to $\mathcal{L}(X, X)$. Let $x \in X$. By the triangle inequality and inequality (1) with $M = \|T\|$,

$$\|x\| = \|(I - T)(x) + T(x)\| \leq \|(I - T)(x)\| + \|T(x)\| \leq \|(I - T)(x)\| + \|T\| \cdot \|x\|.$$

Since $\|T\| < 1$, we may subtract $\|T\| \|x\|$ (a finite real number) from both sides to obtain

$$(1 - \|T\|) \|x\| \leq \|(I - T)(x)\| \quad \text{for all } x \in X,$$

with $1 - \|T\| > 0$. Now let $y \in X$ be arbitrary and apply this with $x = (I - T)^{-1}(y)$, so that $(I - T)(x) = y$:

$$(1 - \|T\|) \|(I - T)^{-1}(y)\| \leq \|y\|, \quad \text{that is,} \quad \|(I - T)^{-1}(y)\| \leq \frac{1}{1 - \|T\|} \|y\|.$$

Hence $(I - T)^{-1}$ satisfies condition (1) with $M = 1/(1 - \|T\|)$, so it is bounded, belongs to $\mathcal{L}(X, X)$, and

$$\|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|}.$$

Being one-to-one, onto, and possessed of a continuous inverse, $I - T$ is an isomorphism of X onto X .

Problem 13.14. (Neumann Series) Let X be a Banach space and $T \in \mathcal{L}(X, X)$ have $\|T\| < 1$. Define $T^0 = \text{Id}$.

(i) Use the completeness of $\mathcal{L}(X, X)$ to show that $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}(X, X)$.

(ii) Show that $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$.

Solution. Here T^n denotes the n -fold composition $T \circ \cdots \circ T$, with $T^0 = \text{Id} = I$ the identity operator, and $\mathcal{L}(X, X)$ carries the operator norm; by Proposition 2 it is a normed linear space, and since X is a Banach space, Theorem 3 tells us that $\mathcal{L}(X, X)$ is a Banach space, that is, it is complete.

A preliminary lemma. If $A, B \in \mathcal{L}(X, X)$, then $A \circ B \in \mathcal{L}(X, X)$ and $\|A \circ B\| \leq \|A\| \cdot \|B\|$. Indeed, $A \circ B$ is linear as a composition of linear maps, and for $u \in X$, two applications of inequality (1) with $M = \|A\|$ and $M = \|B\|$ respectively give

$$\|(A \circ B)(u)\| = \|A(B(u))\| \leq \|A\| \cdot \|B(u)\| \leq \|A\| \cdot \|B\| \cdot \|u\|.$$

So $A \circ B$ is bounded and, since $\|A \circ B\|$ is the infimum of the constants M satisfying (1), $\|A \circ B\| \leq \|A\| \|B\|$.

By induction, $T^n \in \mathcal{L}(X, X)$ and

$$\|T^n\| \leq \|T\|^n \quad \text{for all } n \geq 0.$$

For $n = 0$ this reads $\|I\| \leq 1$, which holds since $\|I(u)\| = \|u\|$. If it holds for n , then $T^{n+1} = T \circ T^n$ and the lemma gives $\|T^{n+1}\| \leq \|T\| \cdot \|T^n\| \leq \|T\| \cdot \|T\|^n = \|T\|^{n+1}$.

Part (i). Put $r = \|T\|$, so $0 \leq r < 1$, and define the partial sums

$$S_N = \sum_{n=0}^N T^n \in \mathcal{L}(X, X), \quad N \geq 0,$$

which belong to $\mathcal{L}(X, X)$ because it is a linear space. We claim $\{S_N\}$ is a Cauchy sequence in $\mathcal{L}(X, X)$. Let $K > N \geq 0$. By the triangle inequality for the operator norm and the bound just proved,

$$\|S_K - S_N\| = \left\| \sum_{n=N+1}^K T^n \right\| \leq \sum_{n=N+1}^K \|T^n\| \leq \sum_{n=N+1}^K r^n \leq \sum_{n=N+1}^{\infty} r^n = \frac{r^{N+1}}{1-r},$$

the geometric series converging because $0 \leq r < 1$. Since $0 \leq r < 1$, the quantity $r^{N+1}/(1-r)$ decreases to 0 as $N \rightarrow \infty$; so, given $\epsilon > 0$, choose N_0 with $r^{N_0+1}/(1-r) < \epsilon$, and then $\|S_K - S_N\| < \epsilon$ for all $K > N \geq N_0$. Hence $\{S_N\}$ is Cauchy in $\mathcal{L}(X, X)$.

Since X is a Banach space, $\mathcal{L}(X, X)$ is complete by Theorem 3. Therefore $\{S_N\}$ converges in $\mathcal{L}(X, X)$ to some $S \in \mathcal{L}(X, X)$; that is, the series $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}(X, X)$, with sum

$$S = \sum_{n=0}^{\infty} T^n = \lim_{N \rightarrow \infty} S_N.$$

Since $\|S_N\| \leq \sum_{n=0}^N \|T^n\| \leq \sum_{n=0}^N r^n \leq 1/(1-r)$ for every N , and the norm is a continuous function on a normed linear space, we also record

$$\|S\| \leq \frac{1}{1-\|T\|}.$$

Part (ii). First, the telescoping identity: for every $N \geq 0$,

$$(I - T) \circ S_N = \sum_{n=0}^N T^n - \sum_{n=0}^N T^{n+1} = T^0 - T^{N+1} = I - T^{N+1},$$

where we used that composition with the fixed operator $I - T$ distributes over the finite sum (each T^n is linear and the operations in $\mathcal{L}(X, X)$ are pointwise). The same computation performed on the other side gives $S_N \circ (I - T) = I - T^{N+1}$ as well, since T commutes with each T^n .

Next, $T^{N+1} \rightarrow 0$ in $\mathcal{L}(X, X)$, because $\|T^{N+1} - 0\| = \|T^{N+1}\| \leq r^{N+1} \rightarrow 0$. Hence

$$\lim_{N \rightarrow \infty} (I - T^{N+1}) = I \quad \text{in } \mathcal{L}(X, X).$$

Finally we pass to the limit on the left-hand side. By the preliminary lemma,

$$\|(I - T) \circ S_N - (I - T) \circ S\| = \|(I - T) \circ (S_N - S)\| \leq \|I - T\| \cdot \|S_N - S\| \rightarrow 0,$$

since $\|S_N - S\| \rightarrow 0$ by part (i) and $\|I - T\|$ is a fixed finite number. So $(I - T) \circ S_N \rightarrow (I - T) \circ S$ in $\mathcal{L}(X, X)$. Likewise $\|S_N \circ (I - T) - S \circ (I - T)\| \leq \|S_N - S\| \cdot \|I - T\| \rightarrow 0$, so $S_N \circ (I - T) \rightarrow S \circ (I - T)$. Because limits in the normed linear space $\mathcal{L}(X, X)$ are unique, the two computations combine to give

$$(I - T) \circ S = I \quad \text{and} \quad S \circ (I - T) = I.$$

These two identities say exactly that $I - T$ is one-to-one and onto with inverse S . Indeed, $(I - T) \circ S = I$ shows that for any $y \in X$ the vector $x = S(y)$ satisfies $(I - T)(x) = y$, so $I - T$ is onto; and $S \circ (I - T) = I$ shows that $(I - T)(x) = 0$ forces $x = S((I - T)(x)) = S(0) = 0$, so $\ker(I - T) = \{0\}$ and $I - T$ is one-to-one. (This reproves part (i) of Problem 13 without the Contraction Mapping Principle.) Moreover, for $y \in X$, applying S to $(I - T)(x) = y$ gives $x = S(y)$, so

$$(I - T)^{-1} = S = \sum_{n=0}^{\infty} T^n.$$

Since $S \in \mathcal{L}(X, X)$ by part (i), the inverse is bounded, so this also reproves that $I - T$ is an isomorphism, with the same bound $\|(I - T)^{-1}\| \leq 1/(1 - \|T\|)$ obtained in Problem 13.

13.3 Problems 13.15–13.21

Problem 13.15. For X and Y normed linear spaces and $T \in \mathcal{L}(X, Y)$, show that T is an isomorphism if and only if there is an operator $S \in \mathcal{L}(Y, X)$ such that for each $u \in X$ and $v \in Y$,

$$S(T(u)) = u \quad \text{and} \quad T(S(v)) = v.$$

Solution. Recall the definition given in the text: an operator $T \in \mathcal{L}(X, Y)$ is an **isomorphism** provided it is one-to-one, onto, and has a continuous inverse; as the text observes, if $T \in \mathcal{L}(X, Y)$ is one-to-one and onto then its inverse is automatically linear, so the substantive requirement is that T^{-1} belong to $\mathcal{L}(Y, X)$.

Suppose first that T is an isomorphism. Then $T: X \rightarrow Y$ is a bijection, so the inverse map $T^{-1}: Y \rightarrow X$ is defined, and it is linear. Indeed, given $v_1, v_2 \in Y$ and $\alpha, \beta \in \mathbb{R}$, put $u_i = T^{-1}(v_i)$; by the linearity of T ,

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) = \alpha v_1 + \beta v_2,$$

and applying T^{-1} to both sides gives $T^{-1}(\alpha v_1 + \beta v_2) = \alpha T^{-1}(v_1) + \beta T^{-1}(v_2)$. Since T is an isomorphism, T^{-1} is continuous, and therefore bounded by Theorem 1. Thus $S := T^{-1}$ belongs to $\mathcal{L}(Y, X)$, and by the very definition of the inverse map, $S(T(u)) = u$ for all $u \in X$ and $T(S(v)) = v$ for all $v \in Y$. Conversely, suppose there is an operator $S \in \mathcal{L}(Y, X)$ with $S \circ T = \text{Id}_X$ and $T \circ S = \text{Id}_Y$.

T is one-to-one: if $u_1, u_2 \in X$ and $T(u_1) = T(u_2)$, then

$$u_1 = S(T(u_1)) = S(T(u_2)) = u_2.$$

T is onto: given $v \in Y$, the vector $u = S(v)$ belongs to X and $T(u) = T(S(v)) = v$, so $v \in \text{Im } T$.

Hence $T: X \rightarrow Y$ is a bijection and T^{-1} is defined. Moreover $T^{-1} = S$: for $v \in Y$, applying the identity $S \circ T = \text{Id}_X$ to the vector $u = T^{-1}(v)$ gives

$$T^{-1}(v) = S(T(T^{-1}(v))) = S(v).$$

Since S belongs to $\mathcal{L}(Y, X)$ it is bounded, hence continuous by Theorem 1. Therefore T is one-to-one, onto, and has a continuous inverse, that is, T is an isomorphism.

Problem 13.16. For X and Y normed linear spaces and $T \in \mathcal{L}(X, Y)$, show that $\ker T$ is a closed subspace of X and that T is one-to-one if and only if $\ker T = \{0\}$.

Solution. By definition $\ker T = \{u \in X \mid T(u) = 0\}$.

First, that $\ker T$ is a subspace. Since T is linear, $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$, so $0 \in \ker T$ and $\ker T$ is nonempty. If $u, v \in \ker T$ and $\alpha, \beta \in \mathbb{R}$, then by linearity

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

so $\alpha u + \beta v \in \ker T$. An induction on the number of terms extends this to arbitrary linear combinations. Hence $\ker T$ contains every linear combination of its vectors, that is, $\ker T$ is a linear subspace of X .

Next, that $\ker T$ is closed. Since T belongs to $\mathcal{L}(X, Y)$ it is bounded, and hence, by Theorem 1, continuous. Let $\{u_n\}$ be a sequence in $\ker T$ with $u_n \rightarrow u$ in X . By the continuity property (3) of a bounded linear operator, $T(u_n) \rightarrow T(u)$ in Y . But $T(u_n) = 0$ for every n , and limits in a metric space are unique, so $T(u) = 0$, that is, $u \in \ker T$. Since X is a metric space, a subset is closed precisely when it contains the limit of each of its convergent sequences. Therefore $\ker T$ is closed.

(Equivalently: $\ker T = T^{-1}(\{0\})$ is the inverse image under the continuous map T of the closed subset $\{0\}$ of Y , hence is closed.)

Finally, the injectivity criterion. Suppose T is one-to-one. If $u \in \ker T$, then $T(u) = 0 = T(0)$, and injectivity forces $u = 0$. Since $0 \in \ker T$ always, $\ker T = \{0\}$.

Conversely, suppose $\ker T = \{0\}$. If $u, v \in X$ satisfy $T(u) = T(v)$, then by linearity

$$T(u - v) = T(u) - T(v) = 0,$$

| so $u - v \in \ker T = \{0\}$, that is, $u = v$. Hence T is one-to-one.

Problem 13.17. Let (X, ρ) be a metric space containing the point x_0 . Define $\text{Lip}_0(X)$ to be the set of real-valued Lipschitz functions f on X that vanish at x_0 . Show that $\text{Lip}_0(X)$ is a linear space that is normed by defining, for $f \in \text{Lip}_0(X)$,

$$\|f\| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}.$$

Show that $\text{Lip}_0(X)$ is a Banach space. For each $x \in X$, define the linear functional F_x on $\text{Lip}_0(X)$ by setting $F_x(f) = f(x)$. Show that F_x belongs to $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$ and that for $x, y \in X$, $\|F_x - F_y\| = \rho(x, y)$. Thus X is isometric to a subset of the Banach space $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$. Since any closed subset of a complete metric space is complete, this provides another proof of the existence of a completion for any metric space X . It also shows that any metric space is isometric to a subset of a normed linear space.

Solution. Throughout, a real-valued function f on X is called Lipschitz provided there is a constant $c \geq 0$ with $|f(x) - f(y)| \leq c\rho(x, y)$ for all $x, y \in X$. If $X = \{x_0\}$ every statement below is trivial (the space $\text{Lip}_0(X)$ reduces to $\{0\}$ and the supremum defining $\|f\|$ is over the empty set, hence is 0 by convention), so we assume X contains at least two points.

Step 1: $\text{Lip}_0(X)$ is a linear space. It is a subset of the linear space of all real-valued functions on X , with the pointwise operations, and it contains the zero function. Let $f, g \in \text{Lip}_0(X)$ have Lipschitz constants c_f and c_g , and let $\alpha, \beta \in \mathbb{R}$. Then for all $x, y \in X$,

$$|(\alpha f + \beta g)(x) - (\alpha f + \beta g)(y)| \leq |\alpha| |f(x) - f(y)| + |\beta| |g(x) - g(y)| \leq (|\alpha|c_f + |\beta|c_g)\rho(x, y),$$

so $\alpha f + \beta g$ is Lipschitz; moreover $(\alpha f + \beta g)(x_0) = \alpha \cdot 0 + \beta \cdot 0 = 0$. Hence $\text{Lip}_0(X)$ is closed under linear combinations, that is, it is a linear subspace.

Step 2: $\|\cdot\|$ is a norm. For $f \in \text{Lip}_0(X)$ with Lipschitz constant c , every quotient $|f(x) - f(y)|/\rho(x, y)$ with $x \neq y$ is at most c , so $\|f\|$ is a finite nonnegative real number; indeed $\|f\|$ is exactly the smallest Lipschitz constant for f , and

$$|f(x) - f(y)| \leq \|f\| \rho(x, y) \quad \text{for all } x, y \in X. \quad (*)$$

If $\|f\| = 0$ then $(*)$ gives $f(x) = f(y)$ for all x, y , so f is constant; since $f(x_0) = 0$ the constant is 0 and $f = 0$. Conversely $\|0\| = 0$. Positive homogeneity is immediate:

$$\|\alpha f\| = \sup_{x \neq y} \frac{|\alpha| |f(x) - f(y)|}{\rho(x, y)} = |\alpha| \|f\|.$$

For the triangle inequality, let $f, g \in \text{Lip}_0(X)$ and $x \neq y$. Then by $(*)$ applied to each,

$$\frac{|(f + g)(x) - (f + g)(y)|}{\rho(x, y)} \leq \frac{|f(x) - f(y)|}{\rho(x, y)} + \frac{|g(x) - g(y)|}{\rho(x, y)} \leq \|f\| + \|g\|.$$

Taking the supremum over all pairs $x \neq y$ gives $\|f + g\| \leq \|f\| + \|g\|$. Thus $\|\cdot\|$ is a norm on $\text{Lip}_0(X)$.

Step 3: $\text{Lip}_0(X)$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $\text{Lip}_0(X)$. Fix $x \in X$. Since $f_n(x_0) = f_m(x_0) = 0$, applying $(*)$ to $f_n - f_m$ at the pair x, x_0 gives

$$|f_n(x) - f_m(x)| = |(f_n - f_m)(x) - (f_n - f_m)(x_0)| \leq \|f_n - f_m\| \rho(x, x_0).$$

Hence $\{f_n(x)\}$ is a Cauchy sequence of real numbers, and by the completeness of $\mathbb{R}$ it converges; define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Taking $x = x_0$ gives $f(x_0) = 0$.

The function f is Lipschitz. Indeed, a Cauchy sequence in a normed linear space is bounded: choosing N with $\|f_n - f_N\| \leq 1$ for $n \geq N$, we get $\|f_n\| \leq M := 1 + \max\{\|f_1\|, \dots, \|f_N\|\}$ for all n . Then for all $x, y \in X$, by $(*)$,

$$|f(x) - f(y)| = \lim_{n \rightarrow \infty} |f_n(x) - f_n(y)| \leq M \rho(x, y),$$

where we used the continuity of the absolute value and of subtraction on $\mathbb{R}$. Thus $f \in \text{Lip}_0(X)$. Finally $\{f_n\} \rightarrow f$ in norm. Let $\epsilon > 0$ and choose N such that $\|f_n - f_m\| \leq \epsilon$ for all $n, m \geq N$. Fix $n \geq N$ and a pair $x \neq y$. For every $m \geq N$, by (*),

$$|(f_n - f_m)(x) - (f_n - f_m)(y)| \leq \epsilon \rho(x, y).$$

Letting $m \rightarrow \infty$ and using $f_m(x) \rightarrow f(x)$, $f_m(y) \rightarrow f(y)$, we obtain

$$|(f_n - f)(x) - (f_n - f)(y)| \leq \epsilon \rho(x, y).$$

Dividing by $\rho(x, y)$ and taking the supremum over all pairs $x \neq y$ gives $\|f_n - f\| \leq \epsilon$ for all $n \geq N$. Hence $\{f_n\} \rightarrow f$ in $\text{Lip}_0(X)$, and $\text{Lip}_0(X)$ is a Banach space.

Step 4: $F_x \in \mathcal{L}(\text{Lip}_0(X), \mathbb{R})$. Linearity of F_x is immediate from the pointwise definition of the linear operations: $F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g)$. For boundedness, use $f(x_0) = 0$ and (*):

$$|F_x(f)| = |f(x)| = |f(x) - f(x_0)| \leq \rho(x, x_0) \|f\| \quad \text{for all } f \in \text{Lip}_0(X).$$

So F_x is bounded with $\|F_x\| \leq \rho(x, x_0)$, and by Theorem 1 it is continuous; thus $F_x \in \mathcal{L}(\text{Lip}_0(X), \mathbb{R})$.

Step 5: $\|F_x - F_y\| = \rho(x, y)$. If $x = y$ both sides are 0, so assume $x \neq y$. For any $f \in \text{Lip}_0(X)$, by (*),

$$|(F_x - F_y)(f)| = |f(x) - f(y)| \leq \|f\| \rho(x, y),$$

which shows $\|F_x - F_y\| \leq \rho(x, y)$.

For the reverse inequality we exhibit a test function. Define $g: X \rightarrow \mathbb{R}$ by

$$g(z) = \rho(z, y) - \rho(x_0, y).$$

Then $g(x_0) = 0$, and for $z, w \in X$ the triangle inequality for ρ gives $|\rho(z, y) - \rho(w, y)| \leq \rho(z, w)$, so

$$|g(z) - g(w)| = |\rho(z, y) - \rho(w, y)| \leq \rho(z, w).$$

Hence $g \in \text{Lip}_0(X)$ and $\|g\| \leq 1$. Moreover

$$(F_x - F_y)(g) = g(x) - g(y) = (\rho(x, y) - \rho(x_0, y)) - (0 - \rho(x_0, y)) = \rho(x, y) > 0.$$

In particular $g \neq 0$, so $\|g\| > 0$, and therefore

$$\|F_x - F_y\| \geq \frac{|(F_x - F_y)(g)|}{\|g\|} = \frac{\rho(x, y)}{\|g\|} \geq \rho(x, y),$$

since $\|g\| \leq 1$. Combining the two inequalities, $\|F_x - F_y\| = \rho(x, y)$.

Step 6: the consequences. The map $\Phi: X \rightarrow \mathcal{L}(\text{Lip}_0(X), \mathbb{R})$, $\Phi(x) = F_x$, therefore satisfies $\|\Phi(x) - \Phi(y)\| = \rho(x, y)$ for all $x, y \in X$: it is an isometry of (X, ρ) onto the subset $\Phi(X)$ of $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$ (in particular it is one-to-one). Since $\mathbb{R}$ is a Banach space, Theorem 3 tells us that $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$ is a Banach space, hence complete as a metric space. Let Z be the closure of $\Phi(X)$ in $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$. A closed subset of a complete metric space is complete, so Z is a complete metric space in which the isometric copy $\Phi(X)$ of X is dense; that is, Z together with Φ is a completion of X . This also shows outright that every metric space is isometric to a subset of a normed linear space.

Problem 13.18. Use the preceding problem to show that every normed linear space is a dense subspace of a Banach space.

Solution. Let $(X, \|\cdot\|)$ be a normed linear space and let $\rho(u, v) = \|u - v\|$ be the metric it induces. By the preceding problem there is a complete metric space (Z, d) and an isometry Φ of X onto a dense subset $\Phi(X)$ of Z (take Z to be the closure of $\{F_u : u \in X\}$ in $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$, with $x_0 = 0$). Since

Φ is an isometry it is one-to-one, so we may transport the linear structure of X to $\Phi(X)$ by declaring $\Phi(u) + \Phi(v) = \Phi(u + v)$ and $\alpha\Phi(u) = \Phi(\alpha u)$; with these operations $\Phi(X)$ is a normed linear space isometrically isomorphic to X , whose metric is the restriction of d . We therefore assume from the outset that X is a dense subset of the complete metric space (Z, d) , that d restricted to $X \times X$ equals ρ , and we must extend the linear structure and the norm from X to Z .

Two properties of ρ will be used repeatedly: for $u, v, w \in X$ and $\alpha \in \mathbb{R}$,

$$\rho(u + w, v + w) = \|u - v\| = \rho(u, v), \quad \rho(\alpha u, \alpha v) = \|\alpha(u - v)\| = |\alpha| \rho(u, v).$$

Step 1: extending the operations. Let $z, z' \in Z$. Since X is dense in Z and Z is a metric space, there are sequences $\{u_n\}, \{u'_n\}$ in X with $u_n \rightarrow z$ and $u'_n \rightarrow z'$. Convergent sequences are Cauchy, and

$$d(u_n + u'_n, u_m + u'_m) = \|(u_n - u_m) + (u'_n - u'_m)\| \leq d(u_n, u_m) + d(u'_n, u'_m),$$

so $\{u_n + u'_n\}$ is a Cauchy sequence in Z and, by completeness, converges. Define

$$z + z' = \lim_{n \rightarrow \infty} (u_n + u'_n).$$

This does not depend on the choice of approximating sequences: if also $v_n \rightarrow z$ and $v'_n \rightarrow z'$ with $v_n, v'_n \in X$, then

$$d(u_n + u'_n, v_n + v'_n) \leq d(u_n, v_n) + d(u'_n, v'_n) \rightarrow 0,$$

since $d(u_n, v_n) \leq d(u_n, z) + d(z, v_n) \rightarrow 0$ and likewise for the primed sequences. Taking $u_n \equiv u$ and $u'_n \equiv v$ for $u, v \in X$ shows the new addition agrees with the old one on X .

Similarly, for $\alpha \in \mathbb{R}$ and $u_n \rightarrow z$ with $u_n \in X$, we have $d(\alpha u_n, \alpha u_m) = |\alpha| d(u_n, u_m)$, so $\{\alpha u_n\}$ is Cauchy and we may define $\alpha z = \lim_{n \rightarrow \infty} \alpha u_n$; if $v_n \rightarrow z$ as well then $d(\alpha u_n, \alpha v_n) = |\alpha| d(u_n, v_n) \rightarrow 0$, so the definition is unambiguous, and it agrees with the old scalar multiplication on X .

Step 2: Z is a linear space. Each axiom is verified by passing to limits, using the key observation that if $u_n \rightarrow z$ and $u'_n \rightarrow z'$ with $u_n, u'_n \in X$, then $\{u_n + u'_n\}$ is a sequence in X converging to $z + z'$, so it may itself be used as the approximating sequence for $z + z'$. For instance, given $z_1, z_2, z_3 \in Z$ with approximating sequences $\{u_n\}, \{v_n\}, \{w_n\}$ in X ,

$$(z_1 + z_2) + z_3 = \lim_{n \rightarrow \infty} ((u_n + v_n) + w_n) = \lim_{n \rightarrow \infty} (u_n + (v_n + w_n)) = z_1 + (z_2 + z_3),$$

the middle equality holding because the two sequences are equal term by term, by associativity in X . Exactly the same argument gives commutativity, $\alpha(\beta z) = (\alpha\beta)z$, $1 \cdot z = z$, and the two distributive laws $\alpha(z + z') = \alpha z + \alpha z'$ and $(\alpha + \beta)z = \alpha z + \beta z$. The zero vector of X serves as the zero of Z , since $z + 0 = \lim(u_n + 0) = \lim u_n = z$, and $z + (-1)z = \lim(u_n - u_n) = 0$, so every $z \in Z$ has an additive inverse. Thus Z is a linear space containing X as a linear subspace.

Step 3: d is translation-invariant and homogeneous on Z . Let $z, z', w \in Z$, with approximating sequences $\{u_n\}, \{u'_n\}, \{w_n\}$ in X . Then $u_n + w_n \rightarrow z + w$ and $u'_n + w_n \rightarrow z' + w$, and since a metric is a continuous function of its two arguments,

$$d(z + w, z' + w) = \lim_{n \rightarrow \infty} d(u_n + w_n, u'_n + w_n) = \lim_{n \rightarrow \infty} \rho(u_n, u'_n) = d(z, z').$$

The same reasoning with $\alpha u_n \rightarrow \alpha z$ and $\alpha u'_n \rightarrow \alpha z'$ gives $d(\alpha z, \alpha z') = |\alpha| d(z, z')$.

Step 4: the norm on Z . Define $\|z\|_Z = d(z, 0)$. For $u \in X$ this equals $\rho(u, 0) = \|u\|$, so $\|\cdot\|_Z$ extends the norm of X . It is nonnegative and real-valued, and:

- $\|z\|_Z = 0$ if and only if $d(z, 0) = 0$, that is, if and only if $z = 0$.
- $\|\alpha z\|_Z = d(\alpha z, 0) = d(\alpha z, \alpha \cdot 0) = |\alpha| d(z, 0) = |\alpha| \|z\|_Z$, by Step 3.
- By translation invariance, $d(z + z', z') = d(z + z', 0 + z') = d(z, 0)$, so the triangle inequality for d gives

$$\|z + z'\|_Z = d(z + z', 0) \leq d(z + z', z') + d(z', 0) = \|z\|_Z + \|z'\|_Z.$$

Thus $\|\cdot\|_Z$ is a norm on Z . Moreover it induces the metric d : by translation invariance,

$$\|z - z'\|_Z = d(z - z', 0) = d((z - z') + z', 0 + z') = d(z, z').$$

Step 5: conclusion. $(Z, \|\cdot\|_Z)$ is a normed linear space whose induced metric is d , and (Z, d) is complete; hence Z is a Banach space. By construction X is a linear subspace of Z whose norm is the restriction of $\|\cdot\|_Z$, and X is dense in Z . Therefore every normed linear space is (isometrically isomorphic to) a dense subspace of a Banach space.

Problem 13.19. For X a normed linear space and $T, S \in \mathcal{L}(X, X)$, show that the composition $S \circ T$ also belongs to $\mathcal{L}(X, X)$ and $\|S \circ T\| \leq \|S\| \cdot \|T\|$.

Solution. The composition $S \circ T$ maps X into X , since T maps X into X and so does S . Linearity. Let $u, v \in X$ and $\alpha, \beta \in \mathbb{R}$. Using the linearity of T and then that of S ,

$$(S \circ T)(\alpha u + \beta v) = S(T(\alpha u + \beta v)) = S(\alpha T(u) + \beta T(v)) = \alpha S(T(u)) + \beta S(T(v)) = \alpha(S \circ T)(u) + \beta(S \circ T)(v).$$

Hence $S \circ T$ is a linear operator.

Boundedness and the norm estimate. As noted in the text, inequality (1) holds with $M = \|T\|$ and with $M = \|S\|$ for the respective operators, that is, $\|T(u)\| \leq \|T\| \|u\|$ and $\|S(w)\| \leq \|S\| \|w\|$ for all $u, w \in X$. Therefore, for every $u \in X$,

$$\|(S \circ T)(u)\| = \|S(T(u))\| \leq \|S\| \|T(u)\| \leq \|S\| \cdot \|T\| \cdot \|u\|.$$

Thus $S \circ T$ is bounded, so $S \circ T \in \mathcal{L}(X, X)$. Moreover the number $M = \|S\| \cdot \|T\|$ is one of the constants for which (1) holds for the operator $S \circ T$, and $\|S \circ T\|$ is by definition the infimum of all such constants; hence

$$\|S \circ T\| \leq \|S\| \cdot \|T\|.$$

(The book's definition of boundedness asks for a constant $M > 0$, so a word is in order when $\|S\| \cdot \|T\| = 0$. In that case $\|S\| = 0$ or $\|T\| = 0$, and by Proposition 2 the corresponding operator is the zero operator; hence $S \circ T$ is the zero operator, every $M > 0$ satisfies (1) for it, and $\|S \circ T\| = 0 = \|S\| \cdot \|T\|$. When $\|S\| \cdot \|T\| > 0$ the displayed inequality is exactly the statement that the admissible constant $M = \|S\| \cdot \|T\|$ dominates the infimum.)

Problem 13.20. Let X be a normed linear space and Y a closed linear subspace of X . Show that $\|x\|_1 = \inf_{y \in Y} \|x - y\|$ defines a pseudonorm on X . The normed linear space induced by the pseudonorm $\|\cdot\|_1$ (see Problem 8) is denoted by X/Y and called the quotient space of X modulo Y . Show that the natural map φ of X onto X/Y takes open sets into open sets.

Solution. Recall from Problem 8 that a nonnegative real-valued function $\|\cdot\|_1$ on a linear space X is a pseudonorm provided $\|x + x'\|_1 \leq \|x\|_1 + \|x'\|_1$ and $\|\alpha x\|_1 = |\alpha| \|x\|_1$ for all $x, x' \in X$ and $\alpha \in \mathbb{R}$.

Step 1: $\|\cdot\|_1$ is a pseudonorm. It is well defined and nonnegative: the set $\{\|x - y\| : y \in Y\}$ is a nonempty set of nonnegative reals (nonempty because $0 \in Y$), so its infimum exists and satisfies $0 \leq \|x\|_1 \leq \|x - 0\| = \|x\|$.

Homogeneity. If $\alpha = 0$, then $\|\alpha x\|_1 = \|0\|_1 = \inf_{y \in Y} \|0 - y\| = 0 = |\alpha| \|x\|_1$, since $0 \in Y$. If $\alpha \neq 0$, the map $y \mapsto \alpha y$ is a bijection of Y onto Y , because Y is a linear subspace and α^{-1} exists. Hence

$$\|\alpha x\|_1 = \inf_{y \in Y} \|\alpha x - y\| = \inf_{y' \in Y} \|\alpha x - \alpha y'\| = \inf_{y' \in Y} |\alpha| \|x - y'\| = |\alpha| \|x\|_1.$$

Subadditivity. Let $x, x' \in X$ and $\epsilon > 0$. By the definition of the infimum choose $y, y' \in Y$ with

$$\|x - y\| < \|x\|_1 + \epsilon/2, \quad \|x' - y'\| < \|x'\|_1 + \epsilon/2.$$

Since Y is a linear subspace, $y + y' \in Y$, and therefore

$$\|x + x'\|_1 \leq \|(x + x') - (y + y')\| \leq \|x - y\| + \|x' - y'\| < \|x\|_1 + \|x'\|_1 + \epsilon.$$

As $\epsilon > 0$ was arbitrary, $\|x + x'\|_1 \leq \|x\|_1 + \|x'\|_1$. Thus $\|\cdot\|_1$ is a pseudonorm.

Step 2: identification of the induced normed space. Following Problem 8, define $x \cong x'$ provided $\|x - x'\|_1 = 0$; the induced normed linear space is the set $X/\cong$ of equivalence classes, with $\|[x]\| = \|x\|_1$ and the linear operations $\alpha[x] + \beta[x'] = [\alpha x + \beta x']$. Here the closedness of Y makes the classes explicit: $\|x\|_1 = 0$ means that for every n there is $y_n \in Y$ with $\|x - y_n\| < 1/n$, that is, $\{y_n\} \rightarrow x$; since Y is closed this forces $x \in Y$. Conversely $x \in Y$ gives $\|x\|_1 \leq \|x - x\| = 0$. Hence

$$\|x\|_1 = 0 \iff x \in Y, \quad \text{so} \quad x \cong x' \iff x - x' \in Y,$$

and the equivalence classes are exactly the cosets $x + Y$. The natural map is $\varphi(x) = [x] = x + Y$; it is linear and onto X/Y by construction, and

$$\|\varphi(x)\| = \|x\|_1 \leq \|x\| \quad \text{for all } x \in X. \quad (\dagger)$$

Step 3: φ carries open balls onto open balls. Fix $x \in X$ and $r > 0$, and let $B(x, r) = \{x' \in X : \|x' - x\| < r\}$ and $B([x], r) = \{\xi \in X/Y : \|\xi - [x]\| < r\}$.

If $x' \in B(x, r)$ then by $(\dagger)$, $\|\varphi(x') - \varphi(x)\| = \|x' - x\|_1 \leq \|x' - x\| < r$; hence $\varphi(B(x, r)) \subseteq B([x], r)$. Conversely, let $\xi \in B([x], r)$ and write $\xi = [x'']$ for some $x'' \in X$. Then $\|x'' - x\|_1 < r$, so by the definition of the infimum there is $y \in Y$ with $\|x'' - x - y\| < r$. Put $x' = x'' - y$. Then $\|x' - x\| < r$, so $x' \in B(x, r)$, while $x' - x'' = -y \in Y$ gives $\varphi(x') = [x''] = \xi$. Hence $\xi \in \varphi(B(x, r))$, and therefore

$$\varphi(B(x, r)) = B([x], r).$$

Step 4: φ is an open map. Let $\mathcal{O}$ be an open subset of X and let $\xi \in \varphi(\mathcal{O})$, say $\xi = \varphi(x)$ with $x \in \mathcal{O}$. Since $\mathcal{O}$ is open there is $r > 0$ with $B(x, r) \subseteq \mathcal{O}$. By Step 3,

$$B(\xi, r) = \varphi(B(x, r)) \subseteq \varphi(\mathcal{O}).$$

So every point of $\varphi(\mathcal{O})$ is contained in an open ball that lies in $\varphi(\mathcal{O})$; that is, $\varphi(\mathcal{O})$ is open in X/Y . Hence the natural map φ takes open sets into open sets.

Problem 13.21. Show that if X is a Banach space and Y a closed linear subspace of X , then the quotient X/Y also is a Banach space and the natural map $\varphi: X \rightarrow X/Y$ has norm 1.

Solution. We use the notation of the preceding problem: $\|x\|_1 = \inf_{y \in Y} \|x - y\|$, the quotient norm on X/Y is $\|[x]\| = \|x\|_1$ where $[x] = x + Y$, the natural map is $\varphi(x) = [x]$, and, since Y is closed, $\|x\|_1 = 0$ if and only if $x \in Y$. We showed there that

$$\|\varphi(x)\| = \|x\|_1 \leq \|x\| \quad \text{for all } x \in X. \quad (\dagger)$$

In particular φ is a bounded linear operator with $\|\varphi\| \leq 1$, hence continuous by Theorem 1. For the assertion $\|\varphi\| = 1$ we must assume, as is implicit, that Y is a proper subspace of X ; otherwise $X/Y = \{0\}$ and $\varphi = 0$.

Part 1: X/Y is complete. Let $\{\xi_n\}$ be a Cauchy sequence in X/Y . It suffices to produce a subsequence converging in X/Y , since a Cauchy sequence in a metric space that has a convergent subsequence converges (to the same limit).

Choose indices $n_1 < n_2 < \dots$ such that

$$\|\xi_{n_{k+1}} - \xi_{n_k}\| < 2^{-k} \quad \text{for all } k \geq 1;$$

this is possible, but the indices must be chosen with a little care, since the estimate $\|\xi_n - \xi_m\| < 2^{-k}$ must be applied to both n_k and n_{k+1} . For each $k \geq 1$ use the Cauchy property to select an index N_k with

$$\|\xi_n - \xi_m\| < 2^{-k} \quad \text{for all } n, m \geq N_k,$$

and then define $n_1 = N_1$ and, inductively, $n_{k+1} = \max\{N_{k+1}, n_k + 1\}$. This makes $\{n_k\}$ strictly increasing, and $n_k \geq N_k$ for every k (for $k = 1$ by definition, and for $k \geq 2$ because $n_k = \max\{N_k, n_{k-1} + 1\} \geq N_k$). Since also $n_{k+1} > n_k \geq N_k$, both indices n_k and n_{k+1} are at least N_k , and therefore $\|\xi_{n_{k+1}} - \xi_{n_k}\| < 2^{-k}$, as claimed.

We now choose representatives $z_k \in X$ with $\varphi(z_k) = \xi_{n_k}$ and $\|z_{k+1} - z_k\| < 2^{-k}$, inductively. Since φ is onto, choose any $z_1 \in X$ with $\varphi(z_1) = \xi_{n_1}$. Suppose z_k has been chosen with $\varphi(z_k) = \xi_{n_k}$. Pick any $w \in X$ with $\varphi(w) = \xi_{n_{k+1}}$. Then

$$\|w - z_k\|_1 = \|\varphi(w) - \varphi(z_k)\| = \|\xi_{n_{k+1}} - \xi_{n_k}\| < 2^{-k},$$

so by the definition of the infimum defining $\|\cdot\|_1$ there is $y_k \in Y$ with $\|w - z_k - y_k\| < 2^{-k}$. Set $z_{k+1} = w - y_k$. Since $y_k \in Y$ we have $\varphi(z_{k+1}) = \varphi(w) = \xi_{n_{k+1}}$, and $\|z_{k+1} - z_k\| < 2^{-k}$, as required. The sequence $\{z_k\}$ is Cauchy in X : for $l > k \geq 1$, by the triangle inequality,

$$\|z_l - z_k\| \leq \sum_{j=k}^{l-1} \|z_{j+1} - z_j\| < \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1},$$

and $2^{-k+1} \rightarrow 0$ as $k \rightarrow \infty$. Since X is a Banach space, $\{z_k\}$ converges to some $z \in X$. By the continuity of φ (property (3) for the bounded operator φ),

$$\xi_{n_k} = \varphi(z_k) \rightarrow \varphi(z) \quad \text{in } X/Y.$$

Thus the subsequence $\{\xi_{n_k}\}$ converges in X/Y , and hence so does $\{\xi_n\}$, to $\varphi(z)$. Therefore X/Y is complete, that is, X/Y is a Banach space.

Part 2: $\|\varphi\| = 1$. From (†), the constant $M = 1$ works in inequality (1) for φ , so $\|\varphi\| \leq 1$.

For the reverse inequality, let $\epsilon > 0$. Since Y is a proper subspace, choose $x \in X \setminus Y$. Because Y is closed, the preceding problem gives $d := \|x\|_1 > 0$. By the definition of the infimum there is $y \in Y$ with

$$\|x - y\| < (1 + \epsilon)d.$$

Put $u = x - y$. Since $y \in Y$ we have $\varphi(u) = \varphi(x)$, hence $\|u\|_1 = \|x\|_1 = d$; in particular $u \notin Y$, so $u \neq 0$ and $\|u\| > 0$. Consequently

$$\frac{\|\varphi(u)\|}{\|u\|} = \frac{d}{\|u\|} > \frac{d}{(1 + \epsilon)d} = \frac{1}{1 + \epsilon}.$$

Since $\|\varphi\| \cdot \|u\| \geq \|\varphi(u)\|$, this gives $\|\varphi\| > 1/(1 + \epsilon)$. As $\epsilon > 0$ was arbitrary, $\|\varphi\| \geq 1$.

Combining the two estimates, $\|\varphi\| = 1$.

13.4 Problems 13.22–13.28

Problem 13.22. Let X and Y be normed linear spaces, $T \in \mathcal{L}(X, Y)$ and $\ker T = Z$. Show that there is a unique bounded linear operator S from X/Z into Y such that $T = S \circ \varphi$ where $\varphi: X \rightarrow X/Z$ is the natural map. Moreover, show that $\|T\| = \|S\|$.

Solution. Since T is continuous, $Z = \ker T$ is a closed linear subspace of X (Problem 16). Therefore, by Problem 20, the quotient space X/Z is a normed linear space with respect to the norm

$$\|\varphi(x)\| = \inf_{z \in Z} \|x - z\|,$$

and the natural map $\varphi: X \rightarrow X/Z$, $\varphi(x) = x + Z$, is linear and onto.

Existence. Define $S: X/Z \rightarrow Y$ by

$$S(\varphi(x)) = T(x) \quad \text{for } x \in X.$$

This is well defined: if $\varphi(x) = \varphi(x')$, then $x - x' \in Z = \ker T$, so $T(x) = T(x')$. It is linear because T and φ are linear and φ is onto: given $u, v \in X/Z$ and scalars α, β , write $u = \varphi(x)$, $v = \varphi(x')$; then $\alpha u + \beta v = \varphi(\alpha x + \beta x')$, so

$$S(\alpha u + \beta v) = T(\alpha x + \beta x') = \alpha T(x) + \beta T(x') = \alpha S(u) + \beta S(v).$$

By construction $T = S \circ \varphi$.

Boundedness and $\|S\| \leq \|T\|$. Let $x \in X$. For every $z \in Z$ we have $T(x) = T(x - z)$, hence

$$\|S(\varphi(x))\| = \|T(x)\| = \|T(x - z)\| \leq \|T\| \cdot \|x - z\|.$$

Taking the infimum over $z \in Z$ gives

$$\|S(\varphi(x))\| \leq \|T\| \cdot \inf_{z \in Z} \|x - z\| = \|T\| \cdot \|\varphi(x)\|.$$

Since φ is onto, every element of X/Z has the form $\varphi(x)$, so S is bounded with $\|S\| \leq \|T\|$.

The reverse inequality. For $x \in X$, since $0 \in Z$ we have $\|\varphi(x)\| \leq \|x\|$, and therefore

$$\|T(x)\| = \|S(\varphi(x))\| \leq \|S\| \cdot \|\varphi(x)\| \leq \|S\| \cdot \|x\|.$$

Hence $\|T\| \leq \|S\|$. Combining the two inequalities, $\|T\| = \|S\|$.

Uniqueness. Suppose S' is any linear operator from X/Z into Y with $T = S' \circ \varphi$. Let $u \in X/Z$. Since φ is onto, $u = \varphi(x)$ for some $x \in X$, and then

$$S'(u) = S'(\varphi(x)) = T(x) = S(\varphi(x)) = S(u).$$

Thus $S' = S$. (Uniqueness holds among all linear operators satisfying $T = S \circ \varphi$, in particular among the bounded ones.)

Problem 13.23. Show that a subset of a finite dimensional normed linear space X is compact if and only if it is closed and bounded.

Solution. Let $n = \dim X$ and let $K \subseteq X$.

If $X = \{0\}$ the assertion is trivial, so assume $n \geq 1$.

Necessity. This holds in any normed linear space, indeed in any metric space. Suppose K is compact. A compact subset of a metric space is closed: if $x \in X \sim K$, then for each $y \in K$ the balls $B(y, r_y)$ and $B(x, r_y)$ with $r_y = \|x - y\|/2$ are disjoint; finitely many of the $B(y_i, r_{y_i})$ cover K , and the ball about x of radius $\min_i r_{y_i}$ then misses K . Hence $X \sim K$ is open and K is closed. Moreover K is bounded: the open balls $\{B(0, m)\}_{m \in \mathbf{N}}$ cover K , and by compactness finitely many of them, hence a single one $B(0, M)$, covers K ; thus $\|x\| \leq M$ for all $x \in K$.

Sufficiency. Here finite dimensionality is used. By Corollary 5, X is isomorphic to the Euclidean space $\mathbf{R}^n$: there is a linear one-to-one map T of X onto $\mathbf{R}^n$ such that both T and T^{-1} are continuous. Since T and T^{-1} are continuous linear operators, they are bounded, so there are constants $c_1, c_2 > 0$ for which

$$\|T(x)\|_* \leq c_1 \|x\| \quad \text{and} \quad \|x\| \leq c_2 \|T(x)\|_* \quad \text{for all } x \in X,$$

where $\|\cdot\|_*$ denotes the Euclidean norm on $\mathbf{R}^n$.

Assume K is closed and bounded. Since $T^{-1}: \mathbf{R}^n \rightarrow X$ is continuous and K is closed in X , the set $T(K) = (T^{-1})^{-1}(K)$ is closed in $\mathbf{R}^n$, being the preimage of a closed set under a continuous map. Since K is bounded, say $\|x\| \leq M$ for all $x \in K$, the first inequality above gives $\|T(x)\|_* \leq c_1 M$ for all $x \in K$, so $T(K)$ is a bounded subset of $\mathbf{R}^n$. A closed bounded subset of $\mathbf{R}^n$ is compact (the Heine-Borel Theorem). Finally, $K = T^{-1}(T(K))$ is the image of the compact set $T(K)$ under the continuous map T^{-1} , and the continuous image of a compact space is compact. Therefore K is compact.

Problem 13.24. Complete the proof of Riesz's Lemma for $\epsilon \neq 1/2$.

Solution. Riesz's Lemma asserts: if Y is a closed proper linear subspace of a normed linear space X , then for each $\epsilon > 0$ there is a unit vector $x_0 \in X$ for which $\|x_0 - y\| > 1 - \epsilon$ for all $y \in Y$. The text proves this for $\epsilon = 1/2$; we prove it for every $\epsilon > 0$.

First dispose of the case $\epsilon \geq 1$. Since Y is a proper subspace, choose $x \in X \sim Y$; then $x \neq 0$ (as $0 \in Y$), and the unit vector $x_0 = x/\|x\|$ does not belong to Y , since otherwise $x = \|x\| x_0$ would belong to the linear subspace Y . Hence $x_0 \neq y$ for every $y \in Y$, and therefore $\|x_0 - y\| > 0$ for every $y \in Y$. Since $1 - \epsilon \leq 0$, it follows that $\|x_0 - y\| > 1 - \epsilon$ for all $y \in Y$, as required. (Closedness of Y is not needed here; for $\epsilon \geq 1$ any unit vector outside Y works.)

Now suppose $0 < \epsilon < 1$. Since Y is a proper subset of X , choose $x \in X \sim Y$. Since Y is closed, its

complement is open, so there is a ball centered at x disjoint from Y ; that is,

$$d = \inf\{\|x - y'\| : y' \in Y\} > 0.$$

Because $0 < 1 - \epsilon < 1$, we have $d/(1 - \epsilon) > d$, and therefore, by the definition of infimum, we may choose $y_1 \in Y$ for which

$$\|x - y_1\| < \frac{d}{1 - \epsilon}.$$

Observe that $x - y_1 \neq 0$, since $\|x - y_1\| \geq d > 0$. Define

$$x_0 = \frac{x - y_1}{\|x - y_1\|},$$

which is a unit vector. Let $y \in Y$. Then

$$x_0 - y = \frac{x - y_1}{\|x - y_1\|} - y = \frac{1}{\|x - y_1\|} \{x - y_1 - \|x - y_1\| y\} = \frac{1}{\|x - y_1\|} \{x - y'\},$$

where $y' = y_1 + \|x - y_1\| y$ belongs to Y because Y is a linear subspace. Consequently, using $\|x - y'\| \geq d$ and the choice of y_1 ,

$$\|x_0 - y\| = \frac{\|x - y'\|}{\|x - y_1\|} \geq \frac{d}{\|x - y_1\|} > \frac{d}{d/(1 - \epsilon)} = 1 - \epsilon.$$

The strict inequality in the third step is legitimate because $d > 0$ and $\|x - y_1\| < d/(1 - \epsilon)$. Since $y \in Y$ was arbitrary, $\|x_0 - y\| > 1 - \epsilon$ for all $y \in Y$, and the proof is complete for every $\epsilon > 0$.

Problem 13.25. Exhibit an open cover of the closed unit ball of $X = \ell^2$ that has no finite subcover. Then do the same for $X = C[0, 1]$ and $X = L^2[0, 1]$.

Solution. We first isolate the mechanism common to all three cases.

Lemma. Let X be a normed linear space with closed unit ball B , and suppose there is a sequence $\{u_k\}_{k=1}^\infty$ of distinct points of B and a number $\delta > 0$ for which $\|u_k - u_j\| \geq \delta$ whenever $k \neq j$. Then the collection

$$\mathcal{F} = \{B(x, \delta/2) : x \in B\}, \quad B(x, r) = \{z \in X : \|z - x\| < r\},$$

is an open cover of B that has no finite subcover.

Proof of the Lemma. Each $B(x, \delta/2)$ is open, and each $x \in B$ belongs to $B(x, \delta/2)$, so $\mathcal{F}$ covers B . Now fix $x \in B$ and suppose u_k and u_j both belong to $B(x, \delta/2)$ with $k \neq j$. Then

$$\|u_k - u_j\| \leq \|u_k - x\| + \|x - u_j\| < \delta/2 + \delta/2 = \delta,$$

contradicting the separation assumption. Hence each member of $\mathcal{F}$ contains at most one of the points u_k . If $B(x_1, \delta/2), \dots, B(x_n, \delta/2)$ were a finite subcover of B , their union would contain at most n of the u_k 's; but all infinitely many distinct points u_k lie in B . This contradiction shows no finite subcollection of $\mathcal{F}$ covers B .

It remains, in each of the three spaces, to produce such a separated sequence in the closed unit ball. The case $X = \ell^2$. For each natural number k let $e_k = (0, \dots, 0, 1, 0, \dots)$ be the sequence whose k th component is 1 and whose other components are 0. Then $\|e_k\| = 1$, so $e_k \in B$, and for $k \neq j$,

$$\|e_k - e_j\| = (1^2 + (-1)^2)^{1/2} = \sqrt{2}.$$

Apply the Lemma with $\delta = \sqrt{2}$: the collection of open balls of radius $\sqrt{2}/2$ centered at the points of B is an open cover of B with no finite subcover.

The case $X = C[0, 1]$ with the maximum norm. For each natural number k set

$$I_k = \left[\frac{1}{k+1}, \frac{1}{k}\right], \quad c_k = \frac{1}{2} \left(\frac{1}{k+1} + \frac{1}{k}\right), \quad h_k = \frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2k(k+1)},$$

and define the tent function

$$f_k(t) = \max\left\{0, 1 - \frac{1}{h_k} |t - c_k|\right\} \quad \text{for } 0 \leq t \leq 1.$$

Each f_k is continuous, $0 \leq f_k \leq 1$ with $f_k(c_k) = 1$, so $\|f_k\|_{\max} = 1$ and $f_k \in B$; moreover f_k vanishes outside I_k . For $k \neq j$ the intervals I_k and I_j have disjoint interiors, so $f_j(c_k) = 0$, and hence

$$\|f_k - f_j\|_{\max} \geq |f_k(c_k) - f_j(c_k)| = 1.$$

Apply the Lemma with $\delta = 1$: the open balls of radius $1/2$ centered at the points of B form an open cover of B with no finite subcover.

The case $X = L^2[0, 1]$. With I_k as above, $m(I_k) = 1/(k(k+1))$, and we define

$$g_k = \sqrt{k(k+1)} \cdot \chi_{I_k}.$$

Then g_k is measurable and

$$\|g_k\|_2^2 = \int_0^1 k(k+1) \chi_{I_k} = k(k+1) \cdot \frac{1}{k(k+1)} = 1,$$

so $g_k \in B$. For $k \neq j$ the sets I_k and I_j overlap in at most a single point, hence $g_k \cdot g_j = 0$ almost everywhere, and therefore

$$\|g_k - g_j\|_2^2 = \|g_k\|_2^2 - 2 \int_0^1 g_k g_j + \|g_j\|_2^2 = 1 + 1 = 2.$$

Thus $\|g_k - g_j\|_2 = \sqrt{2}$ for $k \neq j$, and the Lemma with $\delta = \sqrt{2}$ again yields an open cover of B , namely the balls of radius $\sqrt{2}/2$ centered at points of B , that has no finite subcover.

In each case the conclusion is a direct verification, for that particular space, of Riesz's Theorem: the closed unit ball of an infinite dimensional normed linear space fails to be compact.

Problem 13.26. For normed linear spaces X and Y , let $T: X \rightarrow Y$ be linear. If X is finite dimensional, show that T is continuous. If Y is finite dimensional, show that T is continuous if and only if $\ker T$ is closed.

Solution. Throughout we use the fact that a linear operator between normed linear spaces is continuous if and only if it is bounded.

Part one: X finite dimensional. If $X = \{0\}$ then $T = 0$ is continuous, so assume $\dim X = n \geq 1$ and let $\{e_1, \dots, e_n\}$ be a basis for X . As in the proof of Theorem 4, define

$$\|x\|_* = \sqrt{x_1^2 + \dots + x_n^2} \quad \text{for } x = x_1 e_1 + \dots + x_n e_n,$$

which is a norm on X . Put $M = \max_{1 \leq i \leq n} \|T(e_i)\|$. By the subadditivity and homogeneity of the norm on Y , the linearity of T , and the Cauchy-Schwarz inequality on $\mathbf{R}^n$,

$$\|T(x)\| = \left\| \sum_{i=1}^n x_i T(e_i) \right\| \leq \sum_{i=1}^n |x_i| \|T(e_i)\| \leq M \sum_{i=1}^n |x_i| \leq M \sqrt{n} \|x\|_*.$$

By Theorem 4, any two norms on a finite dimensional linear space are equivalent, so there is a $c > 0$ with $\|x\|_* \leq c\|x\|$ for all $x \in X$. Therefore

$$\|T(x)\| \leq M \sqrt{n} c \|x\| \quad \text{for all } x \in X,$$

so T is bounded, hence continuous.

Part two: Y finite dimensional, say $\dim Y = n$. Write $Z = \ker T$.

Suppose first that T is continuous. Since $\{0\}$ is a closed subset of Y and $Z = T^{-1}(\{0\})$, the set Z is closed. (This is Problem 16, and it requires nothing about Y beyond its being a normed linear space.)

Conversely, suppose $Z = \ker T$ is closed. Then, by Problem 20, the quotient space X/Z is a normed linear space under

$$\|\varphi(x)\| = \inf_{z \in Z} \|x - z\|,$$

where $\varphi: X \rightarrow X/Z$ is the natural map, which is linear and onto. Since $0 \in Z$, we have $\|\varphi(x)\| \leq \|x\|$ for all x , so φ is bounded and hence continuous.

Define $S: X/Z \rightarrow Y$ by $S(\varphi(x)) = T(x)$. This is well defined and linear: if $\varphi(x) = \varphi(x')$ then $x - x' \in Z = \ker T$, so $T(x) = T(x')$; and linearity follows from the linearity of T and φ together with the surjectivity of φ , exactly as in Problem 22. By construction $T = S \circ \varphi$.

Moreover S is one-to-one: if $S(\varphi(x)) = 0$, then $T(x) = 0$, so $x \in Z$ and therefore $\varphi(x) = 0$. Consequently S is a linear isomorphism (of linear spaces, no continuity claimed) of X/Z onto the linear subspace $T(X)$ of Y .

Now $T(X)$ is a linear subspace of the n -dimensional linear space Y , and by the Steinitz exchange argument of elementary linear algebra a linear subspace of a linear space spanned by n vectors is itself spanned by at most n vectors. So $T(X)$ is finite dimensional with $\dim T(X) \leq n$. Pulling a spanning set of $T(X)$ back through the linear isomorphism S^{-1} produces a spanning set for X/Z of the same size, so

$$\dim(X/Z) = \dim T(X) \leq \dim Y = n < \infty.$$

Thus X/Z is a finite dimensional normed linear space, and by Part one the linear operator $S: X/Z \rightarrow Y$ is continuous. Since $T = S \circ \varphi$ is the composition of the continuous maps φ and S , it is continuous. Finite dimensionality of Y is essential for the converse. Let $X = Y$ be the linear space of real polynomials restricted to $[0, 1]$, normed by the maximum norm inherited from $C[0, 1]$, and let T be differentiation, $T(p) = p'$, which maps X linearly into X . Then $\ker T$ is the one dimensional space of constant functions, which is closed by Corollary 6. But T is unbounded: the polynomials $p_k(t) = t^k$ satisfy $\|p_k\| = 1$ while $\|T(p_k)\| = \max_{0 \leq t \leq 1} kt^{k-1} = k$, which is unbounded in k . So T has closed kernel and is not continuous. Of course X here is infinite dimensional, as it must be by Part one.

Problem 13.27. (Another proof of Riesz's Theorem) Let X be an infinite dimensional normed linear space, B the closed unit ball in X , and B_0 the unit open ball in X . Suppose B is compact. Then the open cover $\{x + (1/3)B_0\}_{x \in B}$ of B has a finite subcover $\{x_i + (1/3)B_0\}_{1 \leq i \leq n}$. Use Riesz's Lemma with $Y = \text{span}\{x_1, \dots, x_n\}$ to derive a contradiction.

Solution. Suppose, to obtain a contradiction, that B is compact. For $x \in X$ the set

$$x + (1/3)B_0 = \{x + (1/3)u : \|u\| < 1\} = \{z \in X : \|z - x\| < 1/3\}$$

is the open ball of radius $1/3$ about x , hence is an open subset of X , and it contains x . Therefore $\{x + (1/3)B_0\}_{x \in B}$ is an open cover of B . By the assumed compactness of B there are points $x_1, \dots, x_n$ in B for which

$$B \subseteq \bigcup_{i=1}^n (x_i + (1/3)B_0).$$

Call this inclusion the finite subcover property.

Let $Y = \text{span}\{x_1, \dots, x_n\}$. Then Y is a finite dimensional linear subspace of X , of dimension at most n , so by Corollary 6, Y is a closed linear subspace of X . Moreover Y is a proper subspace of X : if Y were all of X , then X would be spanned by the finite collection $\{x_1, \dots, x_n\}$ and hence would be finite dimensional, contrary to the assumption that $\dim X = \infty$.

Thus Y is a closed proper linear subspace of the normed linear space X , and Riesz's Lemma applies. Taking $\epsilon = 1/3$ in Riesz's Lemma (in the general form established in Problem 24), we obtain a unit vector $x_0 \in X$ for which

$$\|x_0 - y\| > 1 - \frac{1}{3} = \frac{2}{3} \quad \text{for all } y \in Y.$$

No appeal to Problem 24 is really needed: the case $\epsilon = 1/2$, which is the one proved in the text, already yields a unit vector x_0 with $\|x_0 - y\| > 1/2$ for all $y \in Y$, and $1/2 > 1/3$ is all the argument below uses.

Since $\|x_0\| = 1$, the vector x_0 belongs to B . By the finite subcover property there is an index i with $x_0 \in x_i + (1/3)B_0$, that is,

$$\|x_0 - x_i\| < \frac{1}{3}.$$

But $x_i \in Y$, so the preceding inequality furnished by Riesz's Lemma gives $\|x_0 - x_i\| > 2/3$. Hence $2/3 < 1/3$, a contradiction.

Therefore B is not compact. Together with Corollary 7, which asserts that the closed unit ball of a finite dimensional normed linear space is compact, this proves Riesz's Theorem: the closed unit ball of a normed linear space X is compact if and only if X is finite dimensional.

Problem 13.28. Let X be a normed linear space. Show that X is separable if and only if there is a compact subset K of X for which $\overline{\text{span}}[K] = X$.

Solution. Here $\overline{\text{span}}[K]$ denotes the closure in X of $\text{span}[K]$, the set of all finite linear combinations of members of K . We take the scalar field to be $\mathbf{R}$; for complex scalars the argument is identical with $\mathbf{Q}$ replaced by $\mathbf{Q} + i\mathbf{Q}$.

Necessity. Suppose X is separable. If $X = \{0\}$, take $K = \{0\}$, which is compact, and $\overline{\text{span}}[K] = \{0\} = X$. Otherwise let $\{x_k\}_{k=1}^\infty$ be a countable dense subset of X (if X has a finite dense subset, repeat one of its members to form a sequence). For each k define $u_k = \frac{1}{k} \cdot \frac{x_k}{\|x_k\|}$ if $x_k \neq 0$, and $u_k = 0$ if $x_k = 0$; then set

$$K = \{u_k : k \in \mathbf{N}\} \cup \{0\}.$$

Then $\|u_k\| \leq 1/k$ for every k , so $u_k \rightarrow 0$ as $k \rightarrow \infty$.

The set K is compact. Indeed, let $\mathcal{F}$ be an open cover of K by open subsets of X . Choose $\mathcal{O}_0 \in \mathcal{F}$ containing 0. Since $\mathcal{O}_0$ is open, there is an $r > 0$ with $B(0, r) \subseteq \mathcal{O}_0$, and since $\|u_k\| \leq 1/k < r$ for all k larger than some index N , the set $\mathcal{O}_0$ contains u_k for all $k > N$. For each of the finitely many indices k with $1 \leq k \leq N$ choose $\mathcal{O}_k \in \mathcal{F}$ containing u_k . Then $\{\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_N\}$ is a finite subcover of K . Thus every open cover of K has a finite subcover.

Finally, $\text{span}[K]$ contains each x_k : if $x_k \neq 0$, then $x_k = k\|x_k\|u_k$ is a scalar multiple of $u_k \in K$; and if $x_k = 0$, then $x_k \in \text{span}[K]$ trivially. Therefore

$$X = \overline{\{x_k : k \in \mathbf{N}\}} \subseteq \overline{\text{span}}[K] \subseteq X,$$

so $\overline{\text{span}}[K] = X$.

Sufficiency. Suppose K is a compact subset of X with $\overline{\text{span}}[K] = X$. We may assume $K \neq \emptyset$, since otherwise $X = \{0\} = \{0\}$ is trivially separable.

First, K is separable. Since K is a compact metric space, it is totally bounded: for each $m \in \mathbf{N}$ the cover of K by the balls $\{B(v, 1/m)\}_{v \in K}$ has a finite subcover, so there is a finite set $D_m \subseteq K$ such that every point of K lies within $1/m$ of a point of D_m . Let $D = \bigcup_{m=1}^\infty D_m$. Then D is a countable subset of K , and it is dense in K : given $v \in K$ and $\epsilon > 0$, choose m with $1/m < \epsilon$ and then $d \in D_m \subseteq D$ with $\|v - d\| < 1/m < \epsilon$.

Now let $\mathcal{D}$ be the set of all finite linear combinations of members of D with rational coefficients,

$$\mathcal{D} = \left\{ \sum_{i=1}^m q_i d_i : m \in \mathbf{N}, q_i \in \mathbf{Q}, d_i \in D \right\}.$$

Since D is countable, for each fixed m the set of such combinations of length m is the image of the countable set $\mathbf{Q}^m \times D^m$ under a map, hence countable; and $\mathcal{D}$ is the union over $m \in \mathbf{N}$ of these countable sets, hence countable.

We claim $\overline{\mathcal{D}} \supseteq \text{span}[K]$. Let $v = \sum_{i=1}^m a_i k_i$ with $a_i \in \mathbf{R}$ and $k_i \in K$, and let $\epsilon > 0$. Since D is dense in K , choose $d_i \in D$ with

$$\|k_i - d_i\| < \frac{\epsilon}{2m(1 + |a_i|)} \quad \text{for } 1 \leq i \leq m,$$

and choose rationals q_i with

$$|a_i - q_i| < \frac{\epsilon}{2m(1 + \|d_i\|)} \quad \text{for } 1 \leq i \leq m.$$

Then, by the triangle inequality,

$$\left\| v - \sum_{i=1}^m q_i d_i \right\| \leq \sum_{i=1}^m |a_i| \|k_i - d_i\| + \sum_{i=1}^m |a_i - q_i| \|d_i\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where we used $\sum_i a_i k_i - \sum_i q_i d_i = \sum_i a_i (k_i - d_i) + \sum_i (a_i - q_i) d_i$. Since $\sum_i q_i d_i \in \mathcal{D}$ and $\epsilon > 0$ was arbitrary, $v \in \overline{\mathcal{D}}$. This proves the claim.

Since $\overline{\mathcal{D}}$ is closed and contains $\text{span}[K]$, it contains $\overline{\text{span}[K]} = X$. Therefore the countable set $\mathcal{D}$ is dense in X , and X is separable.

13.5 Problems 13.29–13.35

Problem 13.29. Let X be a finite dimensional normed linear space and Y a normed linear space. Show that every linear operator $T : X \rightarrow Y$ is continuous and open.

Solution. Throughout, recall the convention of this section: a mapping $f : X \rightarrow Y$ is called open provided the image of each open subset of X is open in $f(X)$, where $f(X)$ carries the subspace topology inherited from Y .

Continuity: let $n = \dim X$ and let $\{e_1, \dots, e_n\}$ be a basis for X . Each $x \in X$ has a unique representation $x = \sum_{i=1}^n \alpha_i(x) e_i$, and

$$\|x\|_* = \sum_{i=1}^n |\alpha_i(x)|$$

defines a norm on X . By Theorem 4, any two norms on a finite dimensional linear space are equivalent, so there is a $c > 0$ for which $\|x\|_* \leq c \cdot \|x\|$ for all $x \in X$. Put $K = \max_{1 \leq i \leq n} \|T(e_i)\|$. Then for each $x \in X$, by the linearity of T and the triangle inequality,

$$\|T(x)\| = \left\| \sum_{i=1}^n \alpha_i(x) T(e_i) \right\| \leq K \sum_{i=1}^n |\alpha_i(x)| = K \cdot \|x\|_* \leq cK \cdot \|x\|.$$

Thus T is bounded, and therefore, by Theorem 1, T is continuous.

Openness: let $N = \ker T$ and observe that $T(X) = \text{span}\{T(e_1), \dots, T(e_n)\}$ is a finite dimensional subspace of Y . Choose a basis $\{v_1, \dots, v_m\}$ of a linear complement W of N in X ; concretely, extend a basis of N to a basis of X and let W be the span of the added vectors, so that

$$X = N \oplus W.$$

Denote by $P : X \rightarrow W$ the projection of X onto W along N , so that $x - P(x) \in N$ for all x , and hence $T(x) = T(P(x))$ for all $x \in X$.

The restriction $T|_W : W \rightarrow T(X)$ is linear, and it is one-to-one (if $w \in W$ and $T(w) = 0$ then $w \in N \cap W = \{0\}$) and onto $T(X)$ (since $T(x) = T(P(x))$ for every x). Both W and $T(X)$ are finite dimensional normed linear spaces, so by the continuity assertion already proved, applied to $T|_W$ and to its linear inverse $(T|_W)^{-1} : T(X) \rightarrow W$, both maps are continuous. Hence $T|_W$ is a homeomorphism of W onto $T(X)$.

Now let $\mathcal{O}$ be an open subset of X . We claim

$$P(\mathcal{O}) = (\mathcal{O} + N) \cap W.$$

Indeed, if $w = P(x)$ with $x \in \mathcal{O}$, then $w \in W$ and $w = x - (x - P(x)) \in \mathcal{O} + N$. Conversely, if $w \in W$ and $w = x + u$ with $x \in \mathcal{O}$ and $u \in N$, then $P(w) = w$ and $P(x) = P(w) - P(u) = w$, so $w \in P(\mathcal{O})$. Since $\mathcal{O} + N = \bigcup_{u \in N} (\mathcal{O} + u)$ is a union of translates of an open set, it is open in X , and therefore $P(\mathcal{O})$ is open in the subspace W .

Finally, $T(\mathcal{O}) = T(P(\mathcal{O})) = T|_W(P(\mathcal{O}))$, which is the image of a set open in W under the homeomorphism $T|_W : W \rightarrow T(X)$, and hence is open in $T(X)$. Therefore T is open.

Alternatively, once continuity is known one may argue as follows: by Corollary 6 both X and the finite dimensional subspace $T(X)$ of Y are complete, so T , regarded as a continuous linear operator from the Banach space X to the Banach space $T(X)$, has image $T(X)$ which is trivially closed in $T(X)$; the Open Mapping Theorem then tells us that T is open.

Problem 13.30. Let X be a Banach space and $P \in \mathcal{L}(X, X)$ be a projection. Show that P is open.

Solution. Recall that a linear operator $P : X \rightarrow X$ is a projection provided $P^2 = P$, and that then, by (16), there is the direct sum decomposition

$$X = P(X) \oplus (\text{Id} - P)(X) = P(X) \oplus \ker P.$$

Here we used that $(\text{Id} - P)(X) = \ker P$: if $x \in X$ then $P(x - P(x)) = P(x) - P^2(x) = 0$, so $(\text{Id} - P)(X) \subseteq \ker P$, while if $P(u) = 0$ then $u = (\text{Id} - P)(u)$. Observe also that P acts as the identity on $P(X)$: if $v = P(x)$ then $P(v) = P^2(x) = P(x) = v$. Consequently

$$P(X) = \{x \in X : P(x) = x\} = \ker(\text{Id} - P),$$

and since $\text{Id} - P$ belongs to $\mathcal{L}(X, X)$ and hence is continuous, $P(X)$ is a closed subspace of X .

Now let $\mathcal{O}$ be an open subset of X . We claim that

$$P(\mathcal{O}) = (\mathcal{O} + \ker P) \cap P(X).$$

Indeed, if $v = P(x)$ with $x \in \mathcal{O}$, then $v \in P(X)$ and $v = x - (x - P(x))$ with $x - P(x) \in \ker P$, so $v \in \mathcal{O} + \ker P$. Conversely, suppose $v \in P(X)$ and $v = x + u$ with $x \in \mathcal{O}$ and $u \in \ker P$. Applying P and using $P(v) = v$ and $P(u) = 0$ we get $v = P(v) = P(x)$, so $v \in P(\mathcal{O})$. This proves the claim.

The set $\mathcal{O} + \ker P = \bigcup_{u \in \ker P} (\mathcal{O} + u)$ is a union of translates of the open set $\mathcal{O}$, hence is open in X . Therefore $P(\mathcal{O})$ is the intersection of an open subset of X with $P(X)$, that is, $P(\mathcal{O})$ is open in the subspace topology that $P(X)$ inherits from X . Since this holds for every open $\mathcal{O} \subseteq X$, the operator P is open.

We remark that completeness of X was not needed for this argument. If one prefers to use the general theory, the conclusion also follows at once from the Open Mapping Theorem: P is a continuous linear operator from the Banach space X to the Banach space X whose image $P(X) = \ker(\text{Id} - P)$ is a closed subspace of X , and the Open Mapping Theorem asserts that such an operator is open.

Problem 13.31. Let $T : X \rightarrow Y$ be a continuous linear operator between the Banach spaces X and Y . Show that T is open if and only if the image under T of the open unit ball in X is dense in a neighborhood of the origin in Y .

Solution. Let B_X and B_Y denote the open unit balls in X and in Y , respectively. Since in this section a mapping is called open provided it carries open sets onto sets that are open in the image $T(X)$ with its subspace topology, the neighborhoods of the origin that occur in the statement must likewise be taken relative to $T(X)$. (Some such reading is forced: the zero operator is open in the above sense, yet the image of B_X under it is $\{0\}$, which is dense in no neighborhood of the origin of Y unless $Y = \{0\}$. When T is onto, relative neighborhoods of the origin in $T(X) = Y$ are exactly neighborhoods of the origin in Y , and the statement reads literally.) So what is to be proved is:

$$T \text{ is open} \iff \text{there is an } r > 0 \text{ for which } [r \cdot B_Y] \cap T(X) \subseteq \overline{T(B_X)}.$$

Suppose first that T is open. In the proof of the Open Mapping Theorem it was shown that T is open if and only if there is an $r > 0$ for which

$$[r \cdot B_Y] \cap T(X) \subseteq T(B_X),$$

this being the inclusion labeled (13) there. Fix such an r . Then

$$[r \cdot B_Y] \cap T(X) \subseteq T(B_X) \subseteq \overline{T(B_X)},$$

and $[r \cdot B_Y] \cap T(X)$ is a relatively open subset of $T(X)$ containing the origin. Hence $T(B_X)$ is dense in a neighborhood of the origin, namely in the relative neighborhood $[r \cdot B_Y] \cap T(X)$.

Conversely, assume there is an $r > 0$ for which

$$[r \cdot B_Y] \cap T(X) \subseteq \overline{T(B_X)}.$$

We will verify (13) for this same r , and then the criterion just quoted from the proof of the Open Mapping Theorem tells us that T is open.

First observe that for each $\lambda > 0$, since T is linear and the norms are homogeneous, $[\lambda r \cdot B_Y] \cap T(X) = \lambda([r \cdot B_Y] \cap T(X))$ and $\lambda \overline{T(B_X)} = \overline{T(\lambda B_X)}$, so that

$$[\lambda r \cdot B_Y] \cap T(X) \subseteq \overline{T(\lambda B_X)}.$$

This yields the following approximation property:

(*) If $y \in T(X)$, $\epsilon > 0$ and $s > \|y\|/r$, then there is an $x \in X$ with $\|x\| < s$ and $\|y - T(x)\| < \epsilon$.

Indeed, $\|y\| < sr$, so $y \in [sr \cdot B_Y] \cap T(X) \subseteq \overline{T(sB_X)}$, and therefore some point of $T(sB_X)$ lies within ϵ of y .

Now let $y \in [r \cdot B_Y] \cap T(X)$, that is, $y \in T(X)$ and $\|y\| < r$. If $y = 0$ then $y = T(0) \in T(B_X)$. Assume $y \neq 0$. Since $\|y\|/r < 1$, we may choose $\alpha \in (0, 1)$ and $\eta > 0$ for which

$$\frac{\|y\|}{r(1-\alpha)} + \eta < 1.$$

Set $y_0 = y$. We define inductively vectors $x_k \in X$ and $y_k \in T(X)$ with

$$\|y_k\| \leq \alpha^k \|y\| \quad \text{and} \quad \|x_k\| < \frac{\alpha^{k-1} \|y\|}{r} + \frac{\eta}{2^k}.$$

Suppose $y_{k-1} \in T(X)$ has been defined and $\|y_{k-1}\| \leq \alpha^{k-1} \|y\|$ (true for $k = 1$). Apply (*) to y_{k-1} with $\epsilon = \alpha^k \|y\|$ and $s = \|y_{k-1}\|/r + \eta/2^k$ to obtain $x_k \in X$ with

$$\|x_k\| < \frac{\|y_{k-1}\|}{r} + \frac{\eta}{2^k} \leq \frac{\alpha^{k-1} \|y\|}{r} + \frac{\eta}{2^k} \quad \text{and} \quad \|y_{k-1} - T(x_k)\| < \alpha^k \|y\|.$$

Define $y_k = y_{k-1} - T(x_k)$, which belongs to $T(X)$ since y_{k-1} and $T(x_k)$ do, and which satisfies $\|y_k\| \leq \alpha^k \|y\|$. The induction proceeds.

Summing the estimates on the $\|x_k\|$ and using the geometric series,

$$\sum_{k=1}^{\infty} \|x_k\| \leq \frac{\|y\|}{r} \sum_{k=1}^{\infty} \alpha^{k-1} + \eta \sum_{k=1}^{\infty} 2^{-k} = \frac{\|y\|}{r(1-\alpha)} + \eta < 1.$$

For $n > m$ the partial sums $\sigma_n = \sum_{k=1}^n x_k$ satisfy $\|\sigma_n - \sigma_m\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\|$, and the tails of a convergent series of nonnegative numbers tend to 0. Hence $\{\sigma_n\}$ is Cauchy in the Banach space X and converges to some $x \in X$, with

$$\|x\| = \lim_{n \rightarrow \infty} \|\sigma_n\| \leq \sum_{k=1}^{\infty} \|x_k\| < 1,$$

so x belongs to B_X . On the other hand, by the linearity of T and the telescoping definition of the y_k ,

$$T(\sigma_n) = \sum_{k=1}^n (y_{k-1} - y_k) = y_0 - y_n = y - y_n,$$

and $\|y_n\| \leq \alpha^n \|y\| \rightarrow 0$. Since T is continuous, $T(x) = \lim_{n \rightarrow \infty} T(\sigma_n) = y$.

Thus $y \in T(B_X)$. This establishes (13), and therefore T is open.

Problem 13.32. Let $\{u_n\}$ be a sequence in a Banach space X . Suppose that $\sum_{k=1}^{\infty} \|u_k\| < \infty$. Show that there is an $x \in X$ for which

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n u_k = x.$$

Solution. For each natural number n define the partial sum

$$s_n = \sum_{k=1}^n u_k \in X,$$

and also

$$t_n = \sum_{k=1}^n \|u_k\| \in \mathbb{R}.$$

By assumption the increasing sequence of real numbers $\{t_n\}$ converges, to $t = \sum_{k=1}^{\infty} \|u_k\| < \infty$. In particular $\{t_n\}$ is a Cauchy sequence of real numbers.

We claim $\{s_n\}$ is a Cauchy sequence in X . Indeed, let $n > m$ be natural numbers. Then by the triangle inequality in the normed linear space X ,

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^n u_k \right\| \leq \sum_{k=m+1}^n \|u_k\| = t_n - t_m.$$

Let $\epsilon > 0$. Since $\{t_n\}$ is Cauchy, there is an index N such that $|t_n - t_m| < \epsilon$ whenever $n, m \geq N$. Hence for all $n, m \geq N$ we have $\|s_n - s_m\| < \epsilon$ (the case $n = m$ being trivial and the case $m > n$ following by symmetry). Therefore $\{s_n\}$ is a Cauchy sequence in X .

Since X is a Banach space, that is, a complete normed linear space, the Cauchy sequence $\{s_n\}$ converges to some vector $x \in X$. That is,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n u_k = x,$$

which is what we were asked to prove.

We note, moreover, that since the norm is continuous, $\|x\| = \lim_{n \rightarrow \infty} \|s_n\| \leq \lim_{n \rightarrow \infty} t_n = \sum_{k=1}^{\infty} \|u_k\|$.

Problem 13.33. Let X be a linear subspace of $C[0, 1]$ that is closed as a subset of $L^2[0, 1]$. Verify the following assertions to show that X has finite dimension. The sequence $\{f_n\}$ belongs to X .

- (i) Show that X is a closed subspace of $C[0, 1]$.
- (ii) Show that there is a constant $M \geq 0$ such that for all $f \in X$ we have $\|f\|_2 \leq \|f\|_{\infty}$ and $\|f\|_{\infty} \leq M \cdot \|f\|_2$.
- (iii) Show that for each $y \in [0, 1]$, there is a function k_y in L^2 such that for each $f \in X$ we have $f(y) = \int_0^1 k_y(x) f(x) dx$.
- (iv) Show that if $\{f_n\} \rightarrow f$ weakly in L^2 , then $\{f_n\} \rightarrow f$ pointwise on $[0, 1]$.
- (v) Show if $\{f_n\} \rightarrow f$ weakly in L^2 , then $\{f_n\}$ is bounded (in what sense?), and hence $\{f_n\} \rightarrow f$ strongly in L^2 by the Lebesgue Dominated Convergence Theorem.
- (vi) Conclude that X , when normed by $\|\cdot\|_2$, has a compact closed unit ball and therefore, by Riesz's Theorem, is finite dimensional.

Solution. Throughout, $\|f\|_{\infty} = \max_{0 \leq x \leq 1} |f(x)|$ is the maximum norm on $C[0, 1]$ and $\|f\|_2 = \left(\int_0^1 |f|^2 \right)^{1/2}$. We use repeatedly the elementary observation that for $f \in C[0, 1]$,

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 dx \leq \|f\|_{\infty}^2 \cdot m([0, 1]) = \|f\|_{\infty}^2, \quad \text{so} \quad \|f\|_2 \leq \|f\|_{\infty}.$$

We also note that a member of $C[0, 1]$ that vanishes almost everywhere on $[0, 1]$ vanishes identically, since the set on which a continuous function is nonzero is relatively open in $[0, 1]$ and a nonempty relatively open subset of $[0, 1]$ has positive measure. Hence distinct members of $C[0, 1]$ determine distinct elements of $L^2[0, 1]$, and we may and do regard X simultaneously as a set of continuous functions and as a subset of $L^2[0, 1]$.

(i) Let $\{g_n\}$ be a sequence in X that converges in the maximum norm to $g \in C[0, 1]$. By the inequality above, $\|g_n - g\|_2 \leq \|g_n - g\|_\infty \rightarrow 0$, so $\{g_n\} \rightarrow g$ in $L^2[0, 1]$. By assumption X is closed as a subset of $L^2[0, 1]$, so the L^2 limit g belongs to X ; more precisely, g agrees almost everywhere with some $h \in X \subseteq C[0, 1]$, and by the observation of the preceding paragraph $g = h$, so $g \in X$. Therefore X is closed in $C[0, 1]$. Since $C[0, 1]$ with the maximum norm is a Banach space, $(X, \|\cdot\|_\infty)$ is a Banach space, being a closed subspace of a Banach space.

(ii) The inequality $\|f\|_2 \leq \|f\|_\infty$ holds for every $f \in C[0, 1]$, in particular for every $f \in X$. For the reverse inequality we appeal to Corollary 10. By the Riesz-Fischer Theorem, $L^2[0, 1]$ is a Banach space, and X is by hypothesis a closed linear subspace of it, so $(X, \|\cdot\|_2)$ is a Banach space. By part (i), $(X, \|\cdot\|_\infty)$ is also a Banach space. Finally $\|f\|_2 \leq 1 \cdot \|f\|_\infty$ on X . Corollary 10 therefore tells us that the two norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent on X : there is an $M \geq 0$ for which

$$\|f\|_\infty \leq M \cdot \|f\|_2 \quad \text{for all } f \in X.$$

(iii) Fix $y \in [0, 1]$ and define $\psi_y : X \rightarrow \mathbb{R}$ by $\psi_y(f) = f(y)$. This is clearly linear, and by (ii),

$$|\psi_y(f)| = |f(y)| \leq \|f\|_\infty \leq M \cdot \|f\|_2 \quad \text{for all } f \in X,$$

so ψ_y is a bounded linear functional on the space $H = (X, \|\cdot\|_2)$, with $\|\psi_y\| \leq M$. Now H is a Banach space by (ii), and its norm arises from the inner product $\langle f, g \rangle = \int_0^1 fg$ inherited from $L^2[0, 1]$; in particular the parallelogram identity

$$\|u + v\|_2^2 + \|u - v\|_2^2 = 2\|u\|_2^2 + 2\|v\|_2^2$$

holds in H . We produce a representing vector for ψ_y in H directly.

If $\psi_y \equiv 0$ on X , take $k_y = 0$. Otherwise let $N = \ker \psi_y$, a closed linear subspace of H because ψ_y is continuous, and choose $z \in X$ with $\psi_y(z) \neq 0$, so $z \notin N$. Let $d = \inf\{\|z - u\|_2 : u \in N\}$ and select $\{u_j\}$ in N with $\|z - u_j\|_2 \rightarrow d$. By the parallelogram identity applied to $u = z - u_j$ and $v = z - u_i$, and since $(u_i + u_j)/2 \in N$,

$$\|u_i - u_j\|_2^2 = 2\|z - u_i\|_2^2 + 2\|z - u_j\|_2^2 - 4\left\|z - \frac{u_i + u_j}{2}\right\|_2^2 \leq 2\|z - u_i\|_2^2 + 2\|z - u_j\|_2^2 - 4d^2,$$

and the right-hand side tends to 0 as $i, j \rightarrow \infty$. Thus $\{u_j\}$ is Cauchy in H , and since H is complete and N is closed in H , $\{u_j\}$ converges to some $u_* \in N$ with $\|z - u_*\|_2 = d$. Put $w = z - u_*$. Since N is closed and $z \notin N$ we have $d > 0$, so $w \neq 0$. For each $u \in N$ and each real t , $u_* + tu \in N$, hence

$$d^2 \leq \|z - u_* - tu\|_2^2 = \|w\|_2^2 - 2t\langle w, u \rangle + t^2\|u\|_2^2 = d^2 - 2t\langle w, u \rangle + t^2\|u\|_2^2,$$

so $2t\langle w, u \rangle \leq t^2\|u\|_2^2$ for all real t ; dividing by $t > 0$ and letting $t \downarrow 0$ gives $\langle w, u \rangle \leq 0$, and replacing u by $-u$ gives $\langle w, u \rangle = 0$. Thus w is orthogonal to N . Note $\psi_y(w) = \psi_y(z) - \psi_y(u_*) = \psi_y(z) \neq 0$.

Define

$$k_y = \frac{\psi_y(w)}{\|w\|_2^2} w \in X \subseteq L^2[0, 1].$$

For $f \in X$, the vector $f - \frac{\psi_y(f)}{\psi_y(w)} w$ lies in N , so it is orthogonal to w ; expanding,

$$\langle w, f \rangle = \frac{\psi_y(f)}{\psi_y(w)} \|w\|_2^2, \quad \text{that is,} \quad \psi_y(f) = \frac{\psi_y(w)}{\|w\|_2^2} \langle w, f \rangle = \langle k_y, f \rangle.$$

Therefore, for every $f \in X$,

$$f(y) = \int_0^1 k_y(x) f(x) dx.$$

Finally, applying this with $f = k_y$ gives $\|k_y\|_2^2 = \psi_y(k_y) \leq M\|k_y\|_2$, so

$$\|k_y\|_2 \leq M \quad \text{for every } y \in [0, 1].$$

(iv) Assume $\{f_n\}$ is a sequence in X that converges weakly in $L^2[0, 1]$ to f . Fix $y \in [0, 1]$. Since $k_y \in L^2[0, 1]$ and L^2 is its own conjugate space, the definition of weak convergence in L^2 gives

$$f_n(y) = \int_0^1 k_y f_n \longrightarrow \int_0^1 k_y f.$$

Define $g : [0, 1] \rightarrow \mathbb{R}$ by $g(y) = \int_0^1 k_y f$. Thus $\{f_n\}$ converges pointwise on $[0, 1]$ to g , and g , being a pointwise limit of continuous functions, is measurable.

It remains to identify g with f . By Theorem 7 of Chapter 8, a weakly convergent sequence in L^p is bounded, so $C = \sup_n \|f_n\|_2 < \infty$; by (ii), $\|f_n\|_\infty \leq M \cdot C$ for every n , and hence $|g| \leq MC$ on $[0, 1]$. Consequently $|f_n - g|^2 \leq (2MC)^2$ on $[0, 1]$, a constant, which is integrable over the finite measure set $[0, 1]$, and $|f_n - g|^2 \rightarrow 0$ pointwise. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 |f_n - g|^2 = 0,$$

that is, $\{f_n\} \rightarrow g$ strongly in $L^2[0, 1]$. Strong convergence in L^2 implies weak convergence, since $|\int_0^1 (f_n - g)h| \leq \|f_n - g\|_2 \|h\|_2$ for $h \in L^2$ by Holder's Inequality. Thus $\{f_n\}$ converges weakly both to f and to g , so $\int_0^1 (f - g)h = 0$ for every $h \in L^2[0, 1]$; taking $h = f - g$ gives $\|f - g\|_2 = 0$. Hence $f = g$ as elements of $L^2[0, 1]$. Moreover, since X is closed in $L^2[0, 1]$ and $\{f_n\} \subseteq X$ converges to g in L^2 , there is an $h \in X$ with $h = g$ almost everywhere on $[0, 1]$; then for each y , using (iii) for h ,

$$h(y) = \int_0^1 k_y h = \int_0^1 k_y g = g(y),$$

so $g = h$ everywhere, that is, g itself belongs to X and is continuous. Identifying f with its representative g in X , we conclude $\{f_n\} \rightarrow f$ pointwise on $[0, 1]$.

(v) The sense in which $\{f_n\}$ is bounded is twofold. First, by Theorem 7 of Chapter 8, a sequence that converges weakly in $L^2[0, 1]$ is bounded in the L^2 norm: $C = \sup_n \|f_n\|_2 < \infty$. Second, and this is the point, since each f_n belongs to X , part (ii) upgrades this to boundedness in the maximum norm:

$$\|f_n\|_\infty \leq M \cdot \|f_n\|_2 \leq M \cdot C \quad \text{for all } n,$$

so $\{f_n\}$ is uniformly bounded on $[0, 1]$. By (iv), $\{f_n\}$ converges pointwise on $[0, 1]$ to $f \in X$, and $\|f\|_\infty \leq M\|f\|_2 \leq MC$ as well, since $\|f\|_2 \leq \liminf_k \|f_k\|_2 \leq C$ by Theorem 7 of Chapter 8. Hence the functions $|f_n - f|^2$ converge pointwise to 0 on $[0, 1]$ and are dominated by the constant function $(2MC)^2$, which is integrable over $[0, 1]$. The Lebesgue Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \|f_n - f\|_2^2 = \lim_{n \rightarrow \infty} \int_0^1 |f_n - f|^2 = 0,$$

that is, $\{f_n\} \rightarrow f$ strongly in $L^2[0, 1]$.

(vi) Let $B = \{f \in X : \|f\|_2 \leq 1\}$ be the closed unit ball of the normed linear space $(X, \|\cdot\|_2)$. We show B is sequentially compact; since B is a metric space, this is equivalent to compactness.

Let $\{f_n\}$ be a sequence in B . Then $\{f_n\}$ is a bounded sequence in $L^2[0, 1]$, and $1 < 2 < \infty$, so by Theorem 14 of Chapter 8 there is a subsequence $\{f_{n_k}\}$ that converges weakly in $L^2[0, 1]$ to some $f \in L^2[0, 1]$; by (17) of Theorem 7 of Chapter 8, $\|f\|_2 \leq \liminf_k \|f_{n_k}\|_2 \leq 1$. By part (iv), f (identified with its continuous representative) belongs to X , and by part (v), $\{f_{n_k}\} \rightarrow f$ strongly in $L^2[0, 1]$. Since $\|f\|_2 \leq 1$ and $f \in X$, the limit f belongs to B . Thus every sequence in B has a subsequence converging in the $\|\cdot\|_2$ norm to a point of B , so B is compact.

Therefore the closed unit ball of the normed linear space $(X, \|\cdot\|_2)$ is compact, and Riesz's Theorem tells us that X is finite dimensional.

Problem 13.34. Let T be a linear operator from a normed linear space X to a finite-dimensional normed linear space Y . Show that T is continuous if and only if $\ker T$ is a closed subspace of X .

Solution. The necessity is immediate: if T is continuous, then $\ker T = T^{-1}(\{0\})$ is the inverse image under a continuous mapping of the closed subset $\{0\}$ of Y , hence is closed in X ; and it is a linear subspace because T is linear. Note that this direction does not use the finite dimensionality of Y .

For the sufficiency we first record two lemmas.

Lemma A. Let φ be a linear functional on a normed linear space X . Then φ is continuous if and only if $\ker \varphi$ is closed.

Proof. Necessity is as above. For sufficiency, assume $\ker \varphi$ is closed. If $\varphi \equiv 0$ it is certainly continuous. Otherwise choose $x_0 \in X$ with $\varphi(x_0) = 1$. Then $x_0 \notin \ker \varphi$, and since $\ker \varphi$ is closed, the distance

$$d = \inf\{\|x_0 - u\| : u \in \ker \varphi\}$$

is positive. Let $x \in X$ with $\varphi(x) \neq 0$. Then $\varphi(x/\varphi(x) - x_0) = 1 - 1 = 0$, so $x/\varphi(x) = x_0 + u$ for some $u \in \ker \varphi$. Hence

$$\frac{\|x\|}{|\varphi(x)|} = \|x_0 - (-u)\| \geq d, \quad \text{that is,} \quad |\varphi(x)| \leq \frac{1}{d}\|x\|.$$

This inequality holds trivially when $\varphi(x) = 0$, so φ is bounded and therefore, by Theorem 1, continuous.

Lemma B. Let N be a closed linear subspace of a normed linear space X and F a finite dimensional linear subspace of X . Then $N + F$ is closed.

Proof. By induction on $\dim F$, it suffices to treat the case $F = \text{span}\{v\}$ with $v \in X$, for then one adjoins the basis vectors of F one at a time, each time applying the case just treated to the closed subspace obtained at the previous stage. If $v \in N$ then $N + F = N$ is closed. So assume $v \notin N$; since N is closed, $d = \inf\{\|v - u\| : u \in N\} > 0$. Suppose $\{u_k + t_k v\}$ is a sequence in $N + F$, with $u_k \in N$ and t_k real, converging to $z \in X$. For $t_k \neq 0$,

$$\|u_k + t_k v\| = |t_k| \cdot \left\| v - \left(\frac{-u_k}{t_k} \right) \right\| \geq |t_k| \cdot d,$$

and this inequality also holds trivially when $t_k = 0$. A convergent sequence is bounded, say $\|u_k + t_k v\| \leq C$ for all k , so $|t_k| \leq C/d$ for all k . By the Bolzano-Weierstrass Theorem there is a subsequence $\{t_{k_j}\}$ converging to some real t . Then

$$u_{k_j} = (u_{k_j} + t_{k_j} v) - t_{k_j} v \longrightarrow z - tv,$$

and since N is closed, $z - tv \in N$. Hence $z = (z - tv) + tv \in N + F$, so $N + F$ is closed.

Now assume $N = \ker T$ is closed; we show T is continuous. If $T \equiv 0$ this is clear, so assume otherwise. The image $T(X)$ is a linear subspace of the finite dimensional space Y , hence is finite dimensional; let $\{y_1, \dots, y_n\}$ be a basis for $T(X)$ and choose $x_i \in X$ with $T(x_i) = y_i$ for $1 \leq i \leq n$. For each $x \in X$, the vector $T(x)$ has a unique representation in this basis, say

$$T(x) = \sum_{i=1}^n \varphi_i(x) y_i.$$

Uniqueness of the coefficients together with the linearity of T shows that each $\varphi_i : X \rightarrow \mathbb{R}$ is a linear functional.

We claim that for each i ,

$$\ker \varphi_i = N + \text{span}\{x_j : j \neq i\}.$$

Indeed, if $x = u + \sum_{j \neq i} a_j x_j$ with $u \in N$, then $T(x) = \sum_{j \neq i} a_j y_j$, whose i -th coefficient is 0, so $\varphi_i(x) = 0$. Conversely, if $\varphi_i(x) = 0$, then

$$T\left(x - \sum_{j \neq i} \varphi_j(x) x_j\right) = \sum_{j \neq i} \varphi_j(x) y_j - \sum_{j \neq i} \varphi_j(x) y_j = 0,$$

so $x - \sum_{j \neq i} \varphi_j(x) x_j \in N$ and therefore $x \in N + \text{span}\{x_j : j \neq i\}$. This proves the claim. Since N is closed and $\text{span}\{x_j : j \neq i\}$ is finite dimensional, Lemma B tells us that $\ker \varphi_i$ is closed, and hence Lemma A tells us that φ_i is a continuous, that is bounded, linear functional; let $C_i \geq 0$ satisfy $|\varphi_i(x)| \leq C_i \|x\|$ for all $x \in X$. Then for every $x \in X$,

$$\|T(x)\| = \left\| \sum_{i=1}^n \varphi_i(x) y_i \right\| \leq \sum_{i=1}^n |\varphi_i(x)| \cdot \|y_i\| \leq \left(\sum_{i=1}^n C_i \|y_i\| \right) \|x\|.$$

Thus T is bounded, and by Theorem 1 it is continuous.

Two remarks. First, finite dimensionality of Y is essential for the sufficiency: for an infinite dimensional target there are discontinuous linear operators with closed, indeed trivial, kernel. Second, when Y is finite dimensional the argument shows more, namely that T is determined by the finitely many coordinate functionals φ_i , whose continuity is exactly the continuity of T .

Problem 13.35. Suppose X be a Banach space, the operator $T \in \mathcal{L}(X, X)$ be open and X_0 be a closed subspace of X . The restriction T_0 of T to X_0 is continuous. Is T_0 necessarily open?

Solution. No. We exhibit a Banach space X , an open operator $T \in \mathcal{L}(X, X)$, and a closed subspace X_0 of X for which the restriction $T_0 = T|_{X_0}$ fails to be open.

First a criterion. If X_0 is a closed subspace of the Banach space X , then X_0 , being a closed subspace of a complete space, is itself a Banach space. Thus $T_0 : X_0 \rightarrow X$ is a continuous linear operator between Banach spaces, and the Open Mapping Theorem tells us that

$$T_0 \text{ is open} \iff T_0(X_0) = T(X_0) \text{ is a closed subspace of } X.$$

So it suffices to produce an open T and a closed subspace X_0 whose image $T(X_0)$ is not closed.

Let $H = \ell^2$ be the linear space of sequences $a = \{a_n\}$ of real numbers for which $\sum_{n=1}^{\infty} a_n^2 < \infty$, normed by $\|a\|_2 = (\sum_{n=1}^{\infty} a_n^2)^{1/2}$. This is the space appearing in the Remark at the end of this section, and it is a Banach space. Indeed, let $\{a^{(k)}\}$ be Cauchy in ℓ^2 . Since $|a_n^{(k)} - a_n^{(j)}| \leq \|a^{(k)} - a^{(j)}\|_2$ for each n , each coordinate sequence $\{a_n^{(k)}\}_k$ is Cauchy in $\mathbb{R}$ and hence converges, say to a_n ; set $a = \{a_n\}$. Given $\epsilon > 0$, choose K with $\|a^{(k)} - a^{(j)}\|_2 \leq \epsilon$ for all $j, k \geq K$. For each natural number N and all $j, k \geq K$ we have $\sum_{n=1}^N (a_n^{(k)} - a_n^{(j)})^2 \leq \epsilon^2$; letting $j \rightarrow \infty$ in this finite sum gives $\sum_{n=1}^N (a_n^{(k)} - a_n)^2 \leq \epsilon^2$, and letting $N \rightarrow \infty$ gives $\|a^{(k)} - a\|_2 \leq \epsilon$ for $k \geq K$. In particular $a^{(K)} - a \in \ell^2$, so $a = a^{(K)} - (a^{(K)} - a) \in \ell^2$, and $\{a^{(k)}\} \rightarrow a$ in ℓ^2 . Now let

$$X = H \times H, \quad \|(a, b)\| = \|a\|_2 + \|b\|_2.$$

This is a normed linear space with the componentwise linear structure, and it is complete: a sequence $\{(a_k, b_k)\}$ is Cauchy in X if and only if $\{a_k\}$ and $\{b_k\}$ are Cauchy in H , and then $(a_k, b_k) \rightarrow (a, b)$ where $a = \lim a_k$ and $b = \lim b_k$. So X is a Banach space.

Define $S : H \rightarrow H$ by

$$S(\{x_n\}) = \{x_n/n\} \quad \text{for } \{x_n\} \in \ell^2.$$

Then S is linear and $\|S(\{x_n\})\|_2 \leq \|\{x_n\}\|_2$, so S is continuous; S is also one-to-one. This is the operator of the Remark at the end of this section, whose image is not closed. Explicitly,

$$S(H) = \left\{ \{y_n\} \in \ell^2 : \sum_{n=1}^{\infty} n^2 y_n^2 < \infty \right\}.$$

This set contains every sequence with only finitely many nonzero terms. Such sequences are dense in ℓ^2 : if $y \in \ell^2$ and $y^{(N)}$ denotes the sequence agreeing with y in the first N coordinates and vanishing thereafter, then $\|y - y^{(N)}\|_2^2 = \sum_{n>N} y_n^2$, which is the tail of a convergent series and so tends to 0. Hence $S(H)$ is dense in H . On the other hand $S(H) \neq H$, since $y = \{1/n\}$ belongs to ℓ^2 while $\sum_n n^2 (1/n)^2 = \sum_n 1 = \infty$. A dense proper subset of H is not closed, so $S(H)$ is not closed in H .

Now define $T : X \rightarrow X$ by

$$T(a, b) = (a, 0).$$

Then T is linear, $\|T(a, b)\| = \|a\|_2 \leq \|(a, b)\|$, so $T \in \mathcal{L}(X, X)$, and $T^2 = T$, so T is a projection of the Banach space X . By Problem 30, T is open. (Equivalently: $T(X) = H \times \{0\}$ is closed in X , so the Open Mapping Theorem applies.)

Finally let

$$X_0 = \{(Sb, b) : b \in H\} \subseteq X,$$

the graph of S with its coordinates interchanged. It is a linear subspace, since S is linear. It is closed: suppose $(Sb_k, b_k) \rightarrow (u, v)$ in X . Then $b_k \rightarrow v$ and $Sb_k \rightarrow u$ in H ; since S is continuous, $Sb_k \rightarrow Sv$, and limits in a normed linear space are unique, so $u = Sv$ and $(u, v) = (Sv, v) \in X_0$.

The restriction $T_0 = T|_{X_0}$ is continuous, being the restriction of a continuous mapping, and

$$T_0(X_0) = \{(Sb, 0) : b \in H\} = S(H) \times \{0\}.$$

This is not closed in X . Indeed, choose $\{y_k\}$ in $S(H)$ converging in H to some $y \in H \setminus S(H)$, which is possible because $S(H)$ is dense in H but not all of H . Then $(y_k, 0)$ belongs to $T_0(X_0)$ and $(y_k, 0) \rightarrow (y, 0)$ in X , while $(y, 0) \notin T_0(X_0)$.

By the criterion established at the outset, T_0 is therefore not open. Hence the answer to the question is no: an open operator on a Banach space need not restrict to an open operator on a closed subspace. For emphasis, one may see the failure directly against the criterion (13) of this section. Let B_0 denote the open unit ball of X_0 . If T_0 were open, there would be an $r > 0$ with $[r \cdot B_X] \cap T_0(X_0) \subseteq T_0(B_0)$, and, by the homogeneity computation in the proof of the Open Mapping Theorem, this inclusion is equivalent to the estimate (9): there is an $M > 0$ such that each $y \in T_0(X_0)$ is $T_0(x)$ for some $x \in X_0$ with $\|x\| \leq M\|y\|$. Since X_0 is complete, Theorem 8 would then force $T_0(X_0)$ to be closed, which it is not.

13.6 Problems 13.36–13.42

Problem 13.36. Let V be a linear subspace of a linear space X . Argue as follows to show that V has a linear complement in X .

- (i) If $\dim X < \infty$, let $\{e_i\}_{i=1}^n$ be a basis for V . Extend this basis for V to a basis $\{e_i\}_{i=1}^{n+k}$ for X . Then define $W = \text{span}[\{e_{n+1}, \dots, e_{n+k}\}]$.
- (ii) If $\dim X = \infty$, apply Zorn's Lemma to the collection $\mathcal{F}$ of all subspaces Z of X for which $V \cap Z = \{0\}$, ordered by set inclusion.

Solution. Recall that a subspace W of X is a linear complement of V provided

$$X = V \oplus W,$$

that is, $X = V + W$ and $V \cap W = \{0\}$.

Part (i). Suppose $\dim X < \infty$. Since V is a subspace of a finite dimensional space, V is finite dimensional; let $\{e_1, \dots, e_n\}$ be a basis for V (if $V = \{0\}$ take $n = 0$ and the empty basis). A linearly independent subset of a finite dimensional linear space may be enlarged to a basis: adjoin vectors of X one at a time, each not in the span of those already chosen, a process that must terminate after at most $\dim X$ steps since a linearly independent set in X has at most $\dim X$ members, and it terminates only when the span of the chosen vectors is all of X . Thus there are vectors $e_{n+1}, \dots, e_{n+k}$ in X for which $\{e_i\}_{i=1}^{n+k}$ is a basis for X . Define

$$W = \text{span}[\{e_{n+1}, \dots, e_{n+k}\}].$$

We claim $X = V \oplus W$. First, $X = V + W$: each x in X may be written $x = \sum_{i=1}^{n+k} \alpha_i e_i$, and then $x = u + w$ where $u = \sum_{i=1}^n \alpha_i e_i$ belongs to V and $w = \sum_{i=n+1}^{n+k} \alpha_i e_i$ belongs to W . Second, $V \cap W = \{0\}$: if x belongs to $V \cap W$, then

$$x = \sum_{i=1}^n \alpha_i e_i = \sum_{i=n+1}^{n+k} \beta_i e_i,$$

so that $\sum_{i=1}^n \alpha_i e_i - \sum_{i=n+1}^{n+k} \beta_i e_i = 0$. Since $\{e_i\}_{i=1}^{n+k}$ is linearly independent, every α_i and every β_i vanishes, and therefore $x = 0$. Hence W is a linear complement of V in X .

Part (ii). Now let X be an arbitrary linear space and V a subspace (the argument that follows does not use $\dim X = \infty$ and so covers the general case). Let

$$\mathcal{F} = \{Z : Z \text{ is a linear subspace of } X \text{ and } V \cap Z = \{0\}\},$$

ordered by set inclusion. The collection $\mathcal{F}$ is nonempty since the trivial subspace $\{0\}$ belongs to it.

Let $\mathcal{C}$ be a chain in $\mathcal{F}$, that is, a subcollection of $\mathcal{F}$ that is totally ordered by inclusion. If $\mathcal{C}$ is empty, $\{0\}$ is an upper bound for it in $\mathcal{F}$. Otherwise define $Z_0 = \bigcup_{Z \in \mathcal{C}} Z$. Then Z_0 is a linear subspace of X : if x, y belong to Z_0 and α, β are scalars, then x belongs to some Z_1 in $\mathcal{C}$ and y to some Z_2 in $\mathcal{C}$; since $\mathcal{C}$ is a chain, one of Z_1, Z_2 contains the other, say $Z_1 \subseteq Z_2$, and then x and y both lie in the subspace Z_2 , so $\alpha x + \beta y$ lies in $Z_2 \subseteq Z_0$. Moreover $V \cap Z_0 = \{0\}$, since a point of $V \cap Z_0$ lies in $V \cap Z$ for some Z in $\mathcal{C}$, and $V \cap Z = \{0\}$. Thus Z_0 belongs to $\mathcal{F}$ and is an upper bound for $\mathcal{C}$.

Every chain in $\mathcal{F}$ therefore has an upper bound in $\mathcal{F}$. By Zorn's Lemma, $\mathcal{F}$ has a maximal member W .

We claim $X = V + W$. Suppose not, and choose x_0 in X that does not belong to $V + W$. Observe that $x_0 \neq 0$, since 0 belongs to $V + W$. Define

$$W' = W + \text{span}[\{x_0\}] = \{w + \alpha x_0 : w \in W, \alpha \in \mathbb{R}\},$$

a linear subspace of X that contains W . We show W' belongs to $\mathcal{F}$. Indeed, let v belong to $V \cap W'$, and write $v = w + \alpha x_0$ with w in W and α a scalar. If $\alpha \neq 0$, then

$$x_0 = \frac{1}{\alpha}(v - w) = \frac{1}{\alpha}v + \left(-\frac{1}{\alpha}\right)w$$

belongs to $V + W$, a contradiction. Hence $\alpha = 0$, so $v = w$ belongs to $V \cap W = \{0\}$, that is, $v = 0$. Thus $V \cap W' = \{0\}$ and W' belongs to $\mathcal{F}$. But x_0 belongs to W' and, since x_0 does not belong to $V + W$ and $W \subseteq V + W$, the vector x_0 does not belong to W . Therefore $W \subsetneq W'$, which contradicts the maximality of W in $\mathcal{F}$.

Consequently $X = V + W$, and since W belongs to $\mathcal{F}$ we also have $V \cap W = \{0\}$. Therefore

$$X = V \oplus W,$$

so V has a linear complement in X .

Problem 13.37. Verify (15) and (16). [Here V is a linear subspace of a linear space X and W is a linear complement of V in X , so that $X = V \oplus W$; for x in X the decomposition (14) is written $x = v + w$ with v in V and w in W , and $P(x) = v$. Assertion (15) is that $P: X \rightarrow X$ is linear and

$$P^2 = P \text{ on } X, \quad P(X) = V, \quad (\text{Id} - P)(X) = W;$$

assertion (16) is that if $P: X \rightarrow X$ is any linear operator for which $P^2 = P$, then

$$X = P(X) \oplus (\text{Id} - P)(X).$$

]

Solution. Verification of (15). Assume $X = V \oplus W$, that is, $X = V + W$ and $V \cap W = \{0\}$.

We first check that the decomposition of a vector is unique, so that P is well defined. Let x belong to X and suppose

$$x = v + w = v' + w', \quad v, v' \in V, \quad w, w' \in W.$$

Then $v - v' = w' - w$. The left-hand side belongs to V and the right-hand side belongs to W , so this common vector belongs to $V \cap W = \{0\}$. Hence $v = v'$ and $w = w'$. Since $X = V + W$, every x in X

has exactly one such representation, and $P(x) = v$ is unambiguously defined.

P is linear. Let x, y belong to X and α, β be scalars. Write $x = v + w$ and $y = v' + w'$ with v, v' in V and w, w' in W . Then

$$\alpha x + \beta y = (\alpha v + \beta v') + (\alpha w + \beta w'),$$

and $\alpha v + \beta v'$ belongs to V while $\alpha w + \beta w'$ belongs to W , because V and W are linear subspaces. By the uniqueness just established, this is the decomposition of $\alpha x + \beta y$, so

$$P(\alpha x + \beta y) = \alpha v + \beta v' = \alpha P(x) + \beta P(y).$$

Thus P is linear.

$P(X) = V$. For each x in X we have $P(x) \in V$ by definition, so $P(X) \subseteq V$. Conversely, if v belongs to V , then $v = v + 0$ with v in V and 0 in W is its decomposition, so $P(v) = v$ and hence v belongs to $P(X)$. Therefore $P(X) = V$.

$P^2 = P$. Let x belong to X . Then $P(x)$ belongs to V , and we just showed that $P(v) = v$ for every v in V . Hence $P(P(x)) = P(x)$, that is, $P^2 = P$ on X .

$(\text{Id} - P)(X) = W$. If $x = v + w$ is the decomposition of x , then

$$(\text{Id} - P)(x) = x - P(x) = (v + w) - v = w \in W,$$

so $(\text{Id} - P)(X) \subseteq W$. Conversely, if w belongs to W , then $w = 0 + w$ is its decomposition, so $P(w) = 0$ and $(\text{Id} - P)(w) = w$. Therefore $(\text{Id} - P)(X) = W$. This establishes (15).

Verification of (16). Now let $P: X \rightarrow X$ be any linear operator for which $P^2 = P$. Write $Q = \text{Id} - P$, which is linear, and observe that

$$Q^2 = (\text{Id} - P)(\text{Id} - P) = \text{Id} - 2P + P^2 = \text{Id} - 2P + P = \text{Id} - P = Q,$$

so Q also satisfies $Q^2 = Q$.

The sets $P(X)$ and $Q(X)$ are linear subspaces of X , being images of the linear space X under linear operators.

$X = P(X) + Q(X)$. Indeed, for each x in X ,

$$x = P(x) + (x - P(x)) = P(x) + Q(x),$$

with $P(x)$ in $P(X)$ and $Q(x)$ in $Q(X)$.

$P(X) \cap Q(X) = \{0\}$. Let y belong to $P(X) \cap Q(X)$. Choose u, v in X with $y = P(u)$ and $y = Q(v)$. From $y = P(u)$ and $P^2 = P$,

$$P(y) = P(P(u)) = P^2(u) = P(u) = y.$$

From $y = Q(v) = v - P(v)$ and $P^2 = P$,

$$P(y) = P(v) - P(P(v)) = P(v) - P(v) = 0.$$

Comparing the two computations gives $y = P(y) = 0$. Hence $P(X) \cap Q(X) = \{0\}$.

Therefore

$$X = P(X) \oplus (\text{Id} - P)(X),$$

which is (16). In particular, a linear operator P on X with $P^2 = P$ is precisely the projection of X onto $P(X)$ along the linear complement $(\text{Id} - P)(X)$: by (15) applied to this direct sum decomposition, the projection P' of X onto $P(X)$ along $(\text{Id} - P)(X)$ sends $x = P(x) + Q(x)$ to $P(x)$, so $P' = P$.

Problem 13.38. Let Y be a normed linear space. Show that Y is a Banach space if and only if there is a Banach space X and a continuous, linear, open mapping of X onto Y .

Solution. Necessity. Suppose Y is a Banach space. Take $X = Y$ and $T = \text{Id}$, the identity mapping of Y onto Y . Then X is a Banach space and T is linear, continuous (indeed $\|T(y)\| = \|y\|$), maps X onto Y , and is open since it carries each open set onto itself. This proves necessity.

Sufficiency. Suppose X is a Banach space and $T: X \rightarrow Y$ is a continuous, linear, open mapping of X onto Y . We show Y is complete.

Step 1: a bound on preimages. Since T is open and $B_X(0, 1) = \{x \in X : \|x\| < 1\}$ is open, the set $T(B_X(0, 1))$ is an open subset of Y containing $T(0) = 0$. Hence there is $c > 0$ for which

$$\{y \in Y : \|y\| < c\} \subseteq T(B_X(0, 1)).$$

Set $M = 2/c$. We claim that for each y in Y there is x in X with

$$T(x) = y \quad \text{and} \quad \|x\| \leq M\|y\|.$$

This is clear for $y = 0$ (take $x = 0$). For $y \neq 0$, the vector $y' = \frac{c}{2\|y\|} y$ has $\|y'\| = c/2 < c$, so there is x' in $B_X(0, 1)$ with $T(x') = y'$. Put $x = \frac{2\|y\|}{c} x'$. By linearity of T ,

$$T(x) = \frac{2\|y\|}{c} T(x') = \frac{2\|y\|}{c} \cdot \frac{c}{2\|y\|} y = y,$$

while $\|x\| = \frac{2\|y\|}{c} \|x'\| < \frac{2}{c} \|y\| = M\|y\|$. The claim is proved.

Step 2: completeness of Y . Let $\{y_n\}$ be a Cauchy sequence in Y . Since a Cauchy sequence that has a convergent subsequence itself converges, it suffices to produce a convergent subsequence. Using the Cauchy property, choose a strictly increasing sequence of indices $n_1 < n_2 < \dots$ for which

$$\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k} \quad \text{for all } k \geq 1.$$

By Step 1, for each k there is x_k in X with

$$T(x_k) = y_{n_{k+1}} - y_{n_k} \quad \text{and} \quad \|x_k\| \leq M\|y_{n_{k+1}} - y_{n_k}\| \leq \frac{M}{2^k},$$

and there is x_0 in X with $T(x_0) = y_{n_1}$. Then

$$\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} \frac{M}{2^k} = M < \infty.$$

Since X is a Banach space, an absolutely summable series in X is summable (this is Problem 32 of this chapter; it follows since the partial sums $s_j = \sum_{k=1}^j x_k$ satisfy $\|s_l - s_j\| \leq \sum_{k=j+1}^l \|x_k\|$ for $l > j$, and the tails of the convergent series $\sum \|x_k\|$ tend to 0, so $\{s_j\}$ is Cauchy in X and hence converges). Let x in X be the limit of $\{s_j\}$.

By linearity of T and telescoping,

$$T(x_0 + s_j) = T(x_0) + \sum_{k=1}^j T(x_k) = y_{n_1} + \sum_{k=1}^j (y_{n_{k+1}} - y_{n_k}) = y_{n_{j+1}}.$$

Since $\{x_0 + s_j\} \rightarrow x_0 + x$ in X and T is continuous,

$$\lim_{j \rightarrow \infty} y_{n_{j+1}} = \lim_{j \rightarrow \infty} T(x_0 + s_j) = T(x_0 + x).$$

Thus the subsequence $\{y_{n_k}\}$ converges in Y , and therefore the Cauchy sequence $\{y_n\}$ converges in Y . Every Cauchy sequence in Y converges, so Y is complete, that is, Y is a Banach space.

Remark. The Open Mapping Theorem (Theorem 8 of this chapter) is the converse companion to this statement: if X and Y are Banach spaces and $T: X \rightarrow Y$ is a continuous linear mapping of X onto Y , then T is automatically open. The present problem shows that once such an open surjection from a Banach space exists, the target is forced to be complete.

Problem 13.39. As a consequence of the Baire Category Theorem we showed that a mapping that is the pointwise limit of a sequence of continuous mappings on a complete metric space must be continuous at some point. Use this to prove that the pointwise limit of a sequence of linear operators on a Banach space has a limit that is continuous at some point and hence, by linearity, is continuous.

Solution. Let X be a Banach space, Y a normed linear space, and $\{T_n: X \rightarrow Y\}$ a sequence of continuous linear operators for which

$$T(x) = \lim_{n \rightarrow \infty} T_n(x) \text{ exists in } Y \text{ for every } x \in X.$$

We must show T is continuous.

Step 1: T is linear. Let x, x' belong to X and α, β be scalars. Each T_n is linear, so $T_n(\alpha x + \beta x') = \alpha T_n(x) + \beta T_n(x')$ for all n . Since addition and scalar multiplication are continuous on the normed linear space Y , letting $n \rightarrow \infty$ on both sides gives

$$T(\alpha x + \beta x') = \alpha T(x) + \beta T(x').$$

Thus T is linear.

Step 2: T is continuous at some point of X . A Banach space is, in particular, a complete metric space with respect to the metric $\rho(x, x') = \|x - x'\|$ induced by its norm. Each T_n is a continuous mapping of this complete metric space into Y , and $\{T_n\}$ converges pointwise on X to T . This is the situation of Theorem 7 of Chapter 10, the consequence of the Baire Category Theorem referred to in the statement of the problem: the pointwise limit of a sequence of continuous mappings on a complete metric space is continuous at each point of a dense subset of the space, in particular at some point.

Theorem 7 of Chapter 10 is stated there for real-valued functions, so when $Y = \mathbb{R}$ it applies verbatim. For a general normed linear space Y its proof carries over word for word, since the only properties of the target that the proof uses are the triangle inequality and the fact that a convergent sequence is Cauchy. We record the argument. For natural numbers m and n define

$$E(m, n) = \left\{ x \in X : \|T_j(x) - T_k(x)\| \leq \frac{1}{m} \text{ for all } j, k \geq n \right\}.$$

For fixed j, k the function $x \mapsto \|T_j(x) - T_k(x)\|$ is continuous on X , so $E(m, n)$, an intersection of preimages of closed sets under continuous real-valued functions, is closed. By Corollary 5 of Chapter 10, the set

$$D = X \sim \left[\bigcup_{m, n \in \mathbb{N}} \text{bd } E(m, n) \right]$$

is dense in X , since X is complete. Let x_0 belong to D and let $\varepsilon > 0$. Choose m with $1/m < \varepsilon/4$. Since $\{T_n(x_0)\}$ converges in Y it is Cauchy, so there is N with $\|T_j(x_0) - T_k(x_0)\| \leq 1/m$ for all $j, k \geq N$; thus x_0 belongs to $E(m, N)$. A closed set is the disjoint union of its interior and its boundary, and x_0 , belonging to D , lies in no set $\text{bd } E(m, n)$; hence x_0 belongs to the interior of $E(m, N)$. Choose $r > 0$ with $B(x_0, r) \subseteq E(m, N)$. Since T_N is continuous at x_0 , there is δ with $0 < \delta < r$ for which $\|T_N(x) - T_N(x_0)\| < 1/m$ whenever $\|x - x_0\| < \delta$. For such x and any $j \geq N$,

$$\|T_j(x) - T_j(x_0)\| \leq \|T_j(x) - T_N(x)\| + \|T_N(x) - T_N(x_0)\| + \|T_N(x_0) - T_j(x_0)\| \leq \frac{3}{m} < \frac{3\varepsilon}{4}.$$

Letting $j \rightarrow \infty$ and using the continuity of the norm on Y gives $\|T(x) - T(x_0)\| \leq 3\varepsilon/4 < \varepsilon$ for all x with $\|x - x_0\| < \delta$. Hence T is continuous at x_0 . All we shall use below is the existence of a single such point x_0 .

Step 3: continuity at one point plus linearity implies continuity everywhere. Let x_0 be a point at which T is continuous, and let x be an arbitrary point of X . Let $\{x_j\}$ be a sequence in X converging to x . Then

$$\|(x_j - x + x_0) - x_0\| = \|x_j - x\| \rightarrow 0,$$

so $\{x_j - x + x_0\} \rightarrow x_0$. By the continuity of T at x_0 ,

$$\lim_{j \rightarrow \infty} T(x_j - x + x_0) = T(x_0).$$

By the linearity established in Step 1,

$$T(x_j - x + x_0) = T(x_j) - T(x) + T(x_0),$$

so $\{T(x_j) - T(x) + T(x_0)\} \rightarrow T(x_0)$ in Y , and therefore $\{T(x_j)\} \rightarrow T(x)$. Since X is a metric space, sequential continuity at x is continuity at x . As x was arbitrary, T is continuous on all of X .

Remark. Equivalently, continuity at x_0 provides $\delta > 0$ with $\|T(z) - T(x_0)\| \leq 1$ whenever $\|z - x_0\| \leq \delta$; for $\|u\| \leq \delta$, linearity gives $\|T(u)\| = \|T(u + x_0) - T(x_0)\| \leq 1$, so $\|T\| \leq 1/\delta$ and T is bounded. This is exactly the conclusion of the Banach–Saks–Steinhaus Theorem, here obtained directly from the Baire Category Theorem rather than through the Uniform Boundedness Principle.

Problem 13.40. Let $\{f_n\}$ be a sequence in $L^1[a, b]$. Suppose that for each $g \in L^\infty[a, b]$, $\lim_{n \rightarrow \infty} \int_E g \cdot f_n$ exists. Show that there is a function $f \in L^1[a, b]$ such that $\lim_{n \rightarrow \infty} \int_E g \cdot f_n = \int_E g \cdot f$ for all $g \in L^\infty[a, b]$. [Here $E = [a, b]$.]

Solution. Throughout, $E = [a, b]$ and m denotes Lebesgue measure.

For each n define $T_n: L^\infty(E) \rightarrow \mathbb{R}$ by

$$T_n(g) = \int_E g \cdot f_n.$$

The space $L^\infty(E)$ is a Banach space (Problem 33 of Chapter 7). Each T_n is linear, and by Holder's Inequality $|T_n(g)| \leq \|f_n\|_1 \|g\|_\infty$, so T_n is a continuous linear functional on $L^\infty(E)$ with $\|T_n\| \leq \|f_n\|_1$. Taking $g = \text{sgn}(f_n)$, which belongs to $L^\infty(E)$ with $\|g\|_\infty \leq 1$, gives $T_n(g) = \int_E |f_n| = \|f_n\|_1$, so in fact $\|T_n\| = \|f_n\|_1$.

By hypothesis $\lim_n T_n(g)$ exists for each g in $L^\infty(E)$. Since $L^\infty(E)$ is a Banach space, the Banach–Saks–Steinhaus Theorem of this section tells us that $\{T_n\}$ is uniformly bounded and that

$$T(g) = \lim_{n \rightarrow \infty} T_n(g), \quad g \in L^\infty(E),$$

defines a continuous linear functional on $L^\infty(E)$. In particular there is $M \geq 0$ with

$$\|f_n\|_1 = \|T_n\| \leq M \quad \text{for all } n.$$

Step 1: the metric space of measurable sets. Let $\mathcal{M}$ denote the collection of Lebesgue measurable subsets of E , two such sets being identified when they differ by a set of measure zero, and set

$$\rho(A, B) = m(A \Delta B), \quad A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Since $|\chi_A - \chi_B| = \chi_{A \Delta B}$, we have $\rho(A, B) = \|\chi_A - \chi_B\|_1$, so ρ is a metric on $\mathcal{M}$ and $A \mapsto \chi_A$ is an isometry of $(\mathcal{M}, \rho)$ onto the set $\mathcal{K}$ of characteristic functions inside $L^1(E)$.

We claim $\mathcal{K}$ is a closed subset of $L^1(E)$. Indeed, let $\{\chi_{A_n}\} \rightarrow h$ in $L^1(E)$. By the Riesz–Fischer Theorem of Chapter 7, a subsequence of $\{\chi_{A_n}\}$ converges to h pointwise almost everywhere on E . Each χ_{A_n} takes only the values 0 and 1, so h takes only the values 0 and 1 almost everywhere; hence $h = \chi_A$ a.e., where $A = \{x \in E : h(x) = 1\}$ is measurable. Thus h belongs to $\mathcal{K}$.

Since $L^1(E)$ is complete by the Riesz–Fischer Theorem, the closed subset $\mathcal{K}$ is a complete metric space, and therefore so is $(\mathcal{M}, \rho)$.

Step 2: the set functions ν_n are continuous on $\mathcal{M}$. For A in $\mathcal{M}$ put

$$\nu_n(A) = \int_A f_n = T_n(\chi_A), \quad \nu(A) = \lim_{n \rightarrow \infty} \nu_n(A) = T(\chi_A),$$

the limit existing by hypothesis since χ_A belongs to $L^\infty(E)$. Each ν_n is finitely additive on disjoint measurable sets, by the additivity of integration over domains, and hence so is ν .

Fix n . For A, B in $\mathcal{M}$,

$$|\nu_n(A) - \nu_n(B)| = \left| \int_{A \setminus B} f_n - \int_{B \setminus A} f_n \right| \leq \int_{A \Delta B} |f_n|.$$

By Proposition 23 of Chapter 4 (the absolute continuity of the integral), given $\varepsilon > 0$ there is $\delta > 0$ such that $\int_S |f_n| < \varepsilon$ whenever $S \subseteq E$ is measurable with $m(S) < \delta$. Hence $|\nu_n(A) - \nu_n(B)| < \varepsilon$ whenever $\rho(A, B) < \delta$, so $\nu_n: \mathcal{M} \rightarrow \mathbb{R}$ is continuous.

Step 3: uniform absolute continuity of $\{\nu_n\}$. Let $\varepsilon > 0$. For natural numbers N define

$$F_N = \{A \in \mathcal{M} : |\nu_j(A) - \nu_k(A)| \leq \varepsilon \text{ for all } j, k \geq N\}.$$

By Step 2 each function $A \mapsto |\nu_j(A) - \nu_k(A)|$ is continuous on $\mathcal{M}$, so F_N , an intersection of preimages of closed sets, is closed. For each A in $\mathcal{M}$ the real sequence $\{\nu_n(A)\}$ converges and hence is Cauchy, so A belongs to F_N for N large; therefore $\mathcal{M} = \bigcup_{N=1}^{\infty} F_N$. Since $(\mathcal{M}, \rho)$ is a complete metric space, Corollary 4 of Chapter 10, a consequence of the Baire Category Theorem, tells us that at least one F_{N_0} has nonempty interior: there are A_0 in $\mathcal{M}$ and $\delta_0 > 0$ with

$$\{A \in \mathcal{M} : \rho(A, A_0) < \delta_0\} \subseteq F_{N_0}.$$

Let $S \subseteq E$ be measurable with $m(S) < \delta_0$. Then

$$\rho(A_0 \cup S, A_0) = m(S \setminus A_0) \leq m(S) < \delta_0, \quad \rho(A_0 \setminus S, A_0) = m(A_0 \cap S) \leq m(S) < \delta_0,$$

so both $A_0 \cup S$ and $A_0 \setminus S$ lie in F_{N_0} . Moreover $A_0 \setminus S \subseteq A_0 \cup S$ and

$$(A_0 \cup S) \setminus (A_0 \setminus S) = (A_0 \cup S) \cap (A_0^c \cup S) = (A_0 \cap S) \cup (S \setminus A_0) = S.$$

By the finite additivity of ν_n ,

$$\nu_n(S) = \nu_n(A_0 \cup S) - \nu_n(A_0 \setminus S) \quad \text{for every } n,$$

and therefore, for all $j, k \geq N_0$,

$$|\nu_j(S) - \nu_k(S)| \leq |\nu_j(A_0 \cup S) - \nu_k(A_0 \cup S)| + |\nu_j(A_0 \setminus S) - \nu_k(A_0 \setminus S)| \leq 2\varepsilon.$$

Now use Proposition 23 of Chapter 4 once more: for each of the finitely many indices $n = 1, \dots, N_0$ choose $\delta_n > 0$ so that $m(S) < \delta_n$ implies $\int_S |f_n| < \varepsilon$. Put $\delta = \min\{\delta_0, \delta_1, \dots, \delta_{N_0}\} > 0$. If $m(S) < \delta$, then $|\nu_n(S)| \leq \int_S |f_n| < \varepsilon$ for $n \leq N_0$, while for $n > N_0$,

$$|\nu_n(S)| \leq |\nu_n(S) - \nu_{N_0}(S)| + |\nu_{N_0}(S)| \leq 2\varepsilon + \varepsilon = 3\varepsilon.$$

Thus $|\nu_n(S)| \leq 3\varepsilon$ for all n , and letting $n \rightarrow \infty$,

$$m(S) < \delta \implies |\nu(S)| \leq 3\varepsilon. \quad (*)$$

Step 4: producing f . Define $F: [a, b] \rightarrow \mathbb{R}$ by $F(x) = \nu([a, x])$. We show F is absolutely continuous. Let $\varepsilon > 0$ and let $\delta > 0$ respond to the $\varepsilon/6$ challenge in $(*)$, so that $m(S) < \delta$ implies $|\nu(S)| \leq \varepsilon/2$. Let $\{(x_k, y_k)\}_{k=1}^K$ be a finite disjoint collection of intervals in (a, b) with $\sum_{k=1}^K (y_k - x_k) < \delta$. By finite additivity, $F(y_k) - F(x_k) = \nu((x_k, y_k])$. Let Λ^+ be the set of indices k for which $\nu((x_k, y_k]) \geq 0$ and Λ^- the remaining indices, and put

$$S^+ = \bigcup_{k \in \Lambda^+} (x_k, y_k], \quad S^- = \bigcup_{k \in \Lambda^-} (x_k, y_k].$$

These are disjoint unions, $m(S^\pm) \leq \sum_k (y_k - x_k) < \delta$, and by finite additivity

$$\sum_{k=1}^K |F(y_k) - F(x_k)| = \nu(S^+) - \nu(S^-) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence F is absolutely continuous on $[a, b]$.

By Theorem 11 of Chapter 6, an absolutely continuous function on $[a, b]$ is an indefinite integral over $[a, b]$: there is f in $L^1[a, b]$ with

$$F(x) = F(a) + \int_a^x f \quad \text{for all } x \in [a, b].$$

Since $F(a) = \nu(\{a\}) = \lim_n \int_{\{a\}} f_n = 0$, we get $F(x) = \int_{[a, x]} f$, and consequently, for $a \leq x \leq y \leq b$,

$$\nu((x, y]) = F(y) - F(x) = \int_{(x, y]} f. \quad (**)$$

Step 5: $\nu(A) = \int_A f$ for every measurable $A \subseteq E$. Define, for A in $\mathcal{M}$,

$$\lambda(A) = \nu(A) - \int_A f.$$

Then λ is finitely additive. By (**), λ vanishes on every interval of the form $(x, y] \subseteq [a, b]$; since a single point is a set of measure zero and both ν and $A \mapsto \int_A f$ vanish on null sets (for ν , because ν_n does), λ vanishes on every subinterval of $[a, b]$, open, closed, or half-open, and hence, by finite additivity, on every finite disjoint union of subintervals of $[a, b]$.

Let $\varepsilon > 0$. Choose $\delta_1 > 0$ from (*) so that $m(S) < \delta_1$ implies $|\nu(S)| \leq \varepsilon$, and, using Proposition 23 of Chapter 4 applied to f , choose $\delta_2 > 0$ so that $m(S) < \delta_2$ implies $\int_S |f| < \varepsilon$. Put $\delta = \min\{\delta_1, \delta_2\}$; then

$$m(S) < \delta \implies |\lambda(S)| \leq 2\varepsilon. \quad (***)$$

Let $A \subseteq E$ be measurable. Since $m(A) \leq b - a < \infty$, Theorem 12 of Chapter 2 provides a finite disjoint collection of open intervals whose union U satisfies $m(A \Delta U) < \delta$. Put $U' = U \cap [a, b]$. Then U' is a finite disjoint union of subintervals of $[a, b]$, so $\lambda(U') = 0$; and since $A \subseteq [a, b]$,

$$A \Delta U' \subseteq A \Delta U, \quad \text{so} \quad m(A \setminus U') < \delta \quad \text{and} \quad m(U' \setminus A) < \delta.$$

By finite additivity, $\lambda(A) = \lambda(A \cap U') + \lambda(A \setminus U')$ and $\lambda(U') = \lambda(A \cap U') + \lambda(U' \setminus A)$, whence

$$\lambda(A) = \lambda(U') + \lambda(A \setminus U') - \lambda(U' \setminus A) = 0 + \lambda(A \setminus U') - \lambda(U' \setminus A).$$

By (***), $|\lambda(A)| \leq 4\varepsilon$. As $\varepsilon > 0$ was arbitrary, $\lambda(A) = 0$, that is,

$$\nu(A) = \int_A f \quad \text{for every measurable } A \subseteq E.$$

Step 6: passage from characteristic functions to all of $L^\infty(E)$. Define $\Phi: L^\infty(E) \rightarrow \mathbb{R}$ by $\Phi(g) = \int_E g \cdot f$. Since f belongs to $L^1(E)$, Φ is a continuous linear functional with $\|\Phi\| \leq \|f\|_1$. By Step 5, $T(\chi_A) = \nu(A) = \Phi(\chi_A)$ for every measurable $A \subseteq E$, and hence by linearity $T(\varphi) = \Phi(\varphi)$ for every simple function φ on E .

By Proposition 9 of Chapter 7, the simple functions are dense in $L^p(E)$ for $1 \leq p \leq \infty$; in particular they are dense in $L^\infty(E)$. (The proof there for $p = \infty$ applies the Simple Approximation Lemma of Chapter 3 to a bounded measurable function that agrees with the given function off a set of measure zero.)

Two continuous linear functionals that agree on a dense subset of a normed linear space agree everywhere: given g in $L^\infty(E)$ and $\varepsilon > 0$, choose a simple φ with $\|g - \varphi\|_\infty \leq \varepsilon$; then

$$|T(g) - \Phi(g)| \leq |T(g - \varphi)| + |T(\varphi) - \Phi(\varphi)| + |\Phi(\varphi - g)| \leq (\|T\| + \|\Phi\|)\varepsilon,$$

and letting $\varepsilon \rightarrow 0$ gives $T(g) = \Phi(g)$.

Therefore f belongs to $L^1[a, b]$ and

$$\lim_{n \rightarrow \infty} \int_E g \cdot f_n = T(g) = \int_E g \cdot f \quad \text{for all } g \in L^\infty[a, b],$$

as required.

Problem 13.41. Let X be the linear space of all polynomials defined on $\mathbb{R}$. For $p \in X$, define $\|p\|$ to be the sum of the absolute values of the coefficients of p . Show that this is a norm on X . For each n , define $\psi_n: X \rightarrow \mathbb{R}$ by $\psi_n(p) = p^{(n)}(0)$. Use the properties of the sequence $\{\psi_n\}$ in $\mathcal{L}(X, \mathbb{R})$ to show that X is not a Banach space.

Solution. The norm. A polynomial p in X is a function on $\mathbb{R}$ of the form

$$p(x) = \sum_{k=0}^d a_k x^k, \quad a_0, \dots, a_d \in \mathbb{R},$$

and its coefficients are uniquely determined by the function p : if two such expressions agree for all real x , their difference is a polynomial vanishing identically, hence a polynomial with infinitely many roots, hence the zero polynomial, so all its coefficients vanish. (Alternatively, $a_k = p^{(k)}(0)/k!$.) It is therefore legitimate to define

$$\|p\| = \sum_{k \geq 0} |a_k|,$$

a finite sum since all but finitely many coefficients are zero.

We check the three properties of a norm. Homogeneity: for a scalar α , the coefficients of αp are αa_k , so

$$\|\alpha p\| = \sum_{k \geq 0} |\alpha a_k| = |\alpha| \sum_{k \geq 0} |a_k| = |\alpha| \|p\|.$$

The triangle inequality: if q has coefficients $\{b_k\}$, then $p + q$ has coefficients $\{a_k + b_k\}$, so

$$\|p + q\| = \sum_{k \geq 0} |a_k + b_k| \leq \sum_{k \geq 0} |a_k| + \sum_{k \geq 0} |b_k| = \|p\| + \|q\|.$$

Definiteness: clearly $\|p\| \geq 0$, and $\|p\| = 0$ if and only if every $a_k = 0$, that is, if and only if p is the zero polynomial. Hence $\|\cdot\|$ is a norm on X .

The functionals ψ_n . Fix a natural number n and let $p(x) = \sum_{k \geq 0} a_k x^k$. Differentiating n times term by term and evaluating at 0 kills every term except the one of degree n , so

$$\psi_n(p) = p^{(n)}(0) = n! a_n.$$

Since the coefficients of a sum of polynomials are the sums of the coefficients and the coefficients of αp are αa_k , the map $p \mapsto a_n$ is linear, and hence ψ_n is linear. Moreover

$$|\psi_n(p)| = n! |a_n| \leq n! \sum_{k \geq 0} |a_k| = n! \|p\|,$$

so ψ_n is bounded, with $\|\psi_n\| \leq n!$; that is, ψ_n belongs to $\mathcal{L}(X, \mathbb{R})$. Taking $p_n(x) = x^n$, for which $\|p_n\| = 1$ and $\psi_n(p_n) = n!$, we conclude

$$\|\psi_n\| = n! \quad \text{for every } n \geq 0.$$

Pointwise boundedness. Let p belong to X , say of degree d (with $p = 0$ interpreted as having all coefficients zero). Then $a_n = 0$ for $n > d$, so

$$\psi_n(p) = n! a_n = 0 \quad \text{for all } n > d.$$

In particular $\lim_{n \rightarrow \infty} \psi_n(p) = 0$ for each p in X , and the family $\{\psi_n\}$ is pointwise bounded: for each p in X ,

$$M_p = \sup_{n \geq 0} |\psi_n(p)| = \max_{0 \leq n \leq d} n! |a_n| < \infty,$$

this being a maximum of finitely many numbers.

Conclusion. Suppose, to get a contradiction, that X normed by $\|\cdot\|$ were a Banach space. Then $\{\psi_n\}$ would be a family of continuous linear operators from a Banach space X into the normed linear space

$\mathbb{R}$ that is pointwise bounded. By the Uniform Boundedness Principle there would be $M \geq 0$ with $\|\psi_n\| \leq M$ for all n . But $\|\psi_n\| = n!$, and $n! \rightarrow \infty$, so no such M exists. This contradiction shows that X , normed by $\|\cdot\|$, is not complete; that is, X is not a Banach space.

Remark. Completeness fails concretely as well: the sequence of partial sums $p_j(x) = \sum_{k=0}^j x^k/k!$ is Cauchy in $\|\cdot\|$, since for $l > j$ we have $\|p_l - p_j\| = \sum_{k=j+1}^l 1/k!$, which is small for large j ; but $\{p_j\}$ cannot converge in X , because convergence in this norm forces coefficientwise convergence and the only candidate limit has all of the infinitely many coefficients $1/k!$ nonzero, hence is not a polynomial.

Problem 13.42. (i) Use Zorn's Lemma to show that every linear space has a Hamel basis.
(ii) Show that any Hamel basis for an infinite dimensional Banach space must be uncountable.
(iii) Let X be the linear space of all polynomials defined on $\mathbb{R}$. Show that there is not a norm on X with respect to which X is a Banach space.

Solution. Recall that a subset B of a linear space X is a Hamel basis provided B is linearly independent (every finite subset of B is linearly independent) and every vector in X is a finite linear combination of vectors in B , that is, $\text{span}[B] = X$.

Part (i). Let X be a linear space and let

$$\mathcal{F} = \{S \subseteq X : S \text{ is linearly independent}\},$$

ordered by set inclusion. The empty set is vacuously linearly independent, so $\mathcal{F}$ is nonempty.

Let $\mathcal{C}$ be a chain in $\mathcal{F}$. If $\mathcal{C}$ is empty, the empty set is an upper bound for it in $\mathcal{F}$. Otherwise put $S_0 = \bigcup_{S \in \mathcal{C}} S$. We claim S_0 is linearly independent. Let $x_1, \dots, x_r$ be distinct vectors in S_0 and suppose $\sum_{i=1}^r \alpha_i x_i = 0$. Each x_i belongs to some S_i in $\mathcal{C}$; since $\mathcal{C}$ is totally ordered by inclusion, the finite collection $\{S_1, \dots, S_r\}$ has a largest member S_{i_0} , and then $x_1, \dots, x_r$ all belong to the linearly independent set S_{i_0} . Hence every $\alpha_i = 0$. Thus S_0 belongs to $\mathcal{F}$ and is an upper bound for $\mathcal{C}$.

Every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$, so by Zorn's Lemma $\mathcal{F}$ has a maximal member B .

We claim $\text{span}[B] = X$. Suppose not, and let x_0 belong to $X \setminus \text{span}[B]$. Then $x_0 \neq 0$ and x_0 does not belong to B . We show $B \cup \{x_0\}$ is linearly independent. Let $b_1, \dots, b_r$ be distinct vectors of B and suppose

$$\alpha x_0 + \sum_{i=1}^r \alpha_i b_i = 0.$$

If $\alpha \neq 0$, then $x_0 = \sum_{i=1}^r (-\alpha_i/\alpha) b_i$ belongs to $\text{span}[B]$, a contradiction. So $\alpha = 0$, and then $\sum \alpha_i b_i = 0$ forces every $\alpha_i = 0$ by the linear independence of B . Hence $B \cup \{x_0\}$ belongs to $\mathcal{F}$ and properly contains B , contradicting the maximality of B .

Therefore $\text{span}[B] = X$ and B is linearly independent; that is, B is a Hamel basis for X .

Part (ii). We first record two elementary facts about a normed linear space Z .

Fact 1. A finite dimensional linear subspace of a normed linear space is closed. This is Corollary 6 of this chapter: a finite dimensional normed linear space is complete, and a complete subspace of a metric space is a closed subset of it.

Fact 2. A proper closed linear subspace Y of a normed linear space Z is hollow, that is, has empty interior. Indeed, suppose Y contained an open ball $B(y_0, r)$ with $r > 0$; since y_0 belongs to Y and Y is a subspace, Y also contains $B(y_0, r) - y_0 = B(0, r)$. Then for any $z \neq 0$ in Z , the vector $\frac{r}{2\|z\|}z$ lies in $B(0, r) \subseteq Y$, and multiplying by the scalar $2\|z\|/r$ gives $z \in Y$. Hence $Y = Z$, contradicting properness.

Now let X be an infinite dimensional Banach space and let B be a Hamel basis for X .

First, B is infinite. Otherwise B would be a finite spanning set and $X = \text{span}[B]$ would be finite dimensional.

Suppose, to get a contradiction, that B is countable, say $B = \{e_k\}_{k=1}^{\infty}$ with the e_k distinct. For each natural number n set

$$X_n = \text{span}[\{e_1, \dots, e_n\}].$$

Each X_n is a linear subspace of dimension at most n , hence closed by Fact 1. Each X_n is a proper

subspace of X : since X is infinite dimensional, $X_n \neq X$. Therefore, by Fact 2, each X_n is closed and hollow.

Since B is a Hamel basis, every x in X is a finite linear combination of members of B , hence lies in X_n for n large enough. Thus

$$X = \bigcup_{n=1}^{\infty} X_n.$$

But X is a Banach space and hence a complete metric space, so by assertion (ii) of the Baire Category Theorem of Chapter 10 the union $\bigcup_{n=1}^{\infty} X_n$ is hollow. It therefore cannot equal X , since X is a nonempty open subset of itself and so has nonempty interior. This contradiction shows B is not countable.

Therefore every Hamel basis for an infinite dimensional Banach space is uncountable.

Part (iii). Let X be the linear space of all polynomials on $\mathbb{R}$ and set

$$B = \{p_k\}_{k=0}^{\infty}, \quad \text{where } p_k(x) = x^k \text{ for } x \in \mathbb{R}.$$

Then B is a Hamel basis for X . Indeed, $\text{span}[B] = X$ by the very definition of a polynomial, and B is linearly independent: if $\sum_{k=0}^d \alpha_k x^k = 0$ for all real x , then this polynomial has infinitely many roots, so all $\alpha_k = 0$. Consequently X is infinite dimensional, since it contains the infinite linearly independent set B , and it possesses a countable Hamel basis.

Suppose $\|\cdot\|_*$ were a norm on X with respect to which X is a Banach space. Then X would be an infinite dimensional Banach space possessing the countable Hamel basis B , contradicting part (ii). Hence there is no norm on X with respect to which X is a Banach space.

Remark. Problem 41 exhibits one such norm on X explicitly and shows directly, by means of the Uniform Boundedness Principle, that it fails to be complete; part (iii) shows that no choice of norm can succeed.

14 Duality for Normed Linear Spaces

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14.1 Problems 14.1–14.7

Problem 14.1. Verify the two direct substitution assertions in the proof of Proposition 4.

Solution. Recall the statement of Proposition 4: X is a linear space, ψ belongs to $X^\#$, and $\{\psi_i\}_{i=1}^n$ is contained in $X^\#$; then ψ is a linear combination of $\{\psi_i\}_{i=1}^n$ if and only if

$$\bigcap_{i=1}^n \ker \psi_i \subseteq \ker \psi. \quad (2)$$

The proof of the nontrivial implication proceeds by induction on n , and at two places it defers a computation to direct substitution: once in the base case $n = 1$ and once in the inductive step. We verify both.

First assertion: the case $n = 1$.

Here (2) reads $\ker \psi_1 \subseteq \ker \psi$, and we may assume $\psi \neq 0$. Choose $x_0 \neq 0$ with $\psi(x_0) = 1$. Then $x_0 \notin \ker \psi$, so, by the inclusion $\ker \psi_1 \subseteq \ker \psi$, also $x_0 \notin \ker \psi_1$, that is, $\psi_1(x_0) \neq 0$. The proof then asserts the decomposition

$$X = \ker \psi_1 \oplus \text{span}[x_0],$$

and claims that with $\lambda_1 = 1/\psi_1(x_0)$ one has $\psi = \lambda_1 \psi_1$.

We first verify the decomposition. Let $x \in X$ and set $c = \psi_1(x)/\psi_1(x_0)$, which is well defined since $\psi_1(x_0) \neq 0$. Then $u = x - cx_0$ satisfies

$$\psi_1(u) = \psi_1(x) - c\psi_1(x_0) = \psi_1(x) - \psi_1(x) = 0,$$

so $u \in \ker \psi_1$ and $x = u + cx_0$. Thus $X = \ker \psi_1 + \text{span}[x_0]$. The sum is direct: if $u + cx_0 = 0$ with $u \in \ker \psi_1$ and $c \in \mathbb{R}$, applying ψ_1 gives $c\psi_1(x_0) = 0$, hence $c = 0$ and therefore $u = 0$.

Now the substitution. Let $x \in X$ and write $x = u + cx_0$ as above, so that $u \in \ker \psi_1 \subseteq \ker \psi$ and $c = \psi_1(x)/\psi_1(x_0)$. By the linearity of ψ ,

$$\psi(x) = \psi(u) + c\psi(x_0) = 0 + c \cdot 1 = \frac{\psi_1(x)}{\psi_1(x_0)} = \lambda_1 \psi_1(x).$$

Since $x \in X$ was arbitrary, $\psi = \lambda_1 \psi_1$, which is the first assertion.

Second assertion: the inductive step, $n = k$.

Assume the proposition holds for $k - 1$ functionals on any linear space, and suppose (2) holds for $\psi_1, \dots, \psi_k$. If $\psi_k = 0$ then $\ker \psi_k = X$, so

$$\bigcap_{i=1}^{k-1} \ker \psi_i = \bigcap_{i=1}^k \ker \psi_i \subseteq \ker \psi,$$

and the induction hypothesis expresses ψ as a linear combination of $\psi_1, \dots, \psi_{k-1}$, hence of $\psi_1, \dots, \psi_k$ (take the coefficient of ψ_k to be 0); there is nothing to prove. So assume $\psi_k \neq 0$ and choose $x_0 \in X$ with $\psi_k(x_0) = 1$. Exactly as in the case $n = 1$ (with ψ_k in place of ψ_1 , using $\psi_k(x_0) = 1 \neq 0$),

$$X = Y \oplus \text{span}[x_0], \quad Y = \ker \psi_k,$$

and, for $x \in X$, the corresponding decomposition is

$$x = y + cx_0, \quad c = \psi_k(x), \quad y = x - \psi_k(x)x_0 \in Y. \quad (*)$$

Moreover

$$\bigcap_{i=1}^{k-1} [\ker \psi_i \cap Y] \subseteq \ker \psi \cap Y,$$

since a point y of the left-hand side lies in $\ker \psi_i$ for $1 \leq i \leq k-1$ and in $Y = \ker \psi_k$, hence lies in $\bigcap_{i=1}^k \ker \psi_i \subseteq \ker \psi$. Applying the induction hypothesis to the linear space Y and to the restrictions of $\psi, \psi_1, \dots, \psi_{k-1}$ to Y , there are real numbers $\lambda_1, \dots, \lambda_{k-1}$ for which

$$\psi = \sum_{i=1}^{k-1} \lambda_i \psi_i \quad \text{on } Y.$$

The assertion to be verified is that if we define

$$\lambda_k = \psi(x_0) - \sum_{i=1}^{k-1} \lambda_i \psi_i(x_0),$$

then $\psi = \sum_{i=1}^k \lambda_i \psi_i$ on all of X .

Let $x \in X$ and decompose it as in (*), so $x = y + c x_0$ with $y \in Y$ and $c = \psi_k(x)$. Using $\psi_k(y) = 0$ and $\psi_k(x_0) = 1$ we compute

$$\begin{aligned} \sum_{i=1}^k \lambda_i \psi_i(x) &= \sum_{i=1}^{k-1} \lambda_i [\psi_i(y) + c \psi_i(x_0)] + \lambda_k \psi_k(x) \\ &= \sum_{i=1}^{k-1} \lambda_i \psi_i(y) + c \sum_{i=1}^{k-1} \lambda_i \psi_i(x_0) + c \lambda_k \\ &= \psi(y) + c \left[\sum_{i=1}^{k-1} \lambda_i \psi_i(x_0) + \psi(x_0) - \sum_{i=1}^{k-1} \lambda_i \psi_i(x_0) \right] \\ &= \psi(y) + c \psi(x_0) = \psi(y + c x_0) = \psi(x), \end{aligned}$$

where the third equality used that $y \in Y$, so that $\sum_{i=1}^{k-1} \lambda_i \psi_i(y) = \psi(y)$, together with the definition of λ_k . Since $x \in X$ was arbitrary, $\psi = \sum_{i=1}^k \lambda_i \psi_i$ on X . This is the second assertion, and the induction is complete.

Problem 14.2. Let X_0 be a codimension 1 subspace of a normed linear space X . Show that X_0 is closed with respect to the strong topology if and only if $X_0 = \ker \psi$ for some $\psi \in X^*$.

Solution. Throughout, “closed” means closed with respect to the topology induced by the norm, and X^* denotes the space of bounded (equivalently, continuous) linear functionals on X .

Suppose first that $X_0 = \ker \psi$ for some $\psi \in X^*$. Since X_0 has codimension 1 we have $X_0 \neq X$, so $\psi \neq 0$. The functional $\psi: X \rightarrow \mathbb{R}$ is continuous with respect to the strong topology and $\{0\}$ is closed in $\mathbb{R}$, so

$$X_0 = \ker \psi = \psi^{-1}(\{0\})$$

is closed in X .

Conversely, suppose X_0 is a closed subspace of codimension 1. By Proposition 1, since X_0 has codimension 1 there is a nonzero $\psi \in X^\#$ with $X_0 = \ker \psi$. Indeed, by the definition of codimension 1 there is $x_0 \in X$, $x_0 \neq 0$, with $X = X_0 \oplus \text{span}[x_0]$, and Proposition 1 produces ψ by setting $\psi(x + \lambda x_0) = \lambda$ for $x \in X_0$ and $\lambda \in \mathbb{R}$; in particular $\psi(x_0) = 1$. It remains only to show that this ψ is bounded, for then $\psi \in X^*$ and we are done.

Since X_0 is closed and $x_0 \notin X_0$, the complement of X_0 is open, so there is $r > 0$ with $B(x_0, r) \cap X_0 = \emptyset$. Hence

$$d = \inf\{\|x_0 - u\| : u \in X_0\} \geq r > 0.$$

We claim that

$$|\psi(x)| \leq \frac{1}{d} \|x\| \quad \text{for all } x \in X.$$

This is clear when $\psi(x) = 0$. Suppose $\lambda = \psi(x) \neq 0$. Then $\psi(x/\lambda) = 1 = \psi(x_0)$, so $x/\lambda - x_0 \in \ker \psi =$

X_0 ; write $x/\lambda = x_0 + u$ with $u \in X_0$. Since $-u \in X_0$,

$$\frac{\|x\|}{|\lambda|} = \left\| \frac{x}{\lambda} \right\| = \|x_0 - (-u)\| \geq d,$$

and therefore $|\psi(x)| = |\lambda| \leq \|x\|/d$, as claimed.

Thus ψ is a bounded linear functional, with $\|\psi\| \leq 1/d$. A bounded linear functional is continuous: for $x, y \in X$ we have $|\psi(x) - \psi(y)| = |\psi(x - y)| \leq \|\psi\| \|x - y\|$, so ψ is Lipschitz and hence continuous with respect to the strong topology. Therefore $\psi \in X^*$ and $X_0 = \ker \psi$, which completes the proof.

Problem 14.3. Show that if X is a finite dimensional normed linear space, then every linear functional on X is continuous.

Solution. Let X be a normed linear space with $\dim X = n$. If $n = 0$, then $X = \{0\}$, the only linear functional on X is the zero functional, and there is nothing to prove; so assume $n \geq 1$. Let $\{e_1, \dots, e_n\}$ be a basis for X . Every $x \in X$ has a unique expression $x = x_1 e_1 + \dots + x_n e_n$ with $x_i \in \mathbb{R}$. Define

$$\|x\|_1 = \sum_{i=1}^n |x_i|.$$

It is immediate that $\|\cdot\|_1$ is a norm on X (the uniqueness of the coordinates gives $\|x\|_1 = 0$ only for $x = 0$, and homogeneity and the triangle inequality follow from those of the absolute value applied coordinatewise).

We first show that the two norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Set $M = \sum_{i=1}^n \|e_i\|$, which is positive since the e_i are nonzero. For $x = \sum_{i=1}^n x_i e_i$, the triangle inequality gives

$$\|x\| \leq \sum_{i=1}^n |x_i| \|e_i\| \leq M \|x\|_1. \quad (*)$$

For the reverse inequality, consider the map $T: \mathbb{R}^n \rightarrow X$ defined by $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i e_i$; it is a linear bijection, and by (*),

$$|\|T(a)\| - \|T(b)\|| \leq \|T(a) - T(b)\| = \|T(a - b)\| \leq M \sum_{i=1}^n |a_i - b_i|,$$

so the real-valued function $a \mapsto \|T(a)\|$ is continuous on $\mathbb{R}^n$ with its usual topology. The set

$$S = \left\{ a \in \mathbb{R}^n : \sum_{i=1}^n |a_i| = 1 \right\}$$

is closed and bounded in $\mathbb{R}^n$, hence compact by the Heine–Borel Theorem. Therefore the continuous function $a \mapsto \|T(a)\|$ attains a minimum value m on S , say at $a^0 \in S$. Since $a^0 \neq 0$ and T is one-to-one, $T(a^0) \neq 0$, so $m = \|T(a^0)\| > 0$. Now let $x \in X$ be nonzero, with coordinate vector $a \in \mathbb{R}^n$; then $a/\|x\|_1 \in S$, so

$$\left\| \frac{x}{\|x\|_1} \right\| \geq m, \quad \text{that is,} \quad \|x\|_1 \leq \frac{1}{m} \|x\|,$$

an inequality that trivially also holds for $x = 0$.

Now let $\psi: X \rightarrow \mathbb{R}$ be any linear functional, and put $C = \max_{1 \leq i \leq n} |\psi(e_i)|$. For $x = \sum_{i=1}^n x_i e_i$ in X , linearity gives

$$|\psi(x)| = \left| \sum_{i=1}^n x_i \psi(e_i) \right| \leq C \sum_{i=1}^n |x_i| = C \|x\|_1 \leq \frac{C}{m} \|x\|.$$

Hence ψ is bounded, with $\|\psi\| \leq C/m$. A bounded linear functional is continuous: for $x, y \in X$,

$$|\psi(x) - \psi(y)| = |\psi(x - y)| \leq \frac{C}{m} \|x - y\|,$$

so ψ is Lipschitz, hence continuous with respect to the topology induced by the norm. Therefore every linear functional on a finite dimensional normed linear space is continuous, that is, $X^\# = X^*$.

Problem 14.4. Let X be a finite dimensional normed linear space of dimension n . Let $\{e_1, \dots, e_n\}$ be a basis for X . For $1 \leq i \leq n$, define $\psi_i \in X^*$ by $\psi_i(x) = x_i$ for $x = x_1 e_1 + \dots + x_n e_n \in X$. Show that $\{\psi_1, \dots, \psi_n\}$ is a basis for X^* . Thus $\dim X^* = n$.

Solution. Since $\{e_1, \dots, e_n\}$ is a basis, each $x \in X$ has a unique representation $x = x_1 e_1 + \dots + x_n e_n$, so the coordinate functions $\psi_i(x) = x_i$ are well defined. They are linear: if $x = \sum_i x_i e_i$, $y = \sum_i y_i e_i$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha x + \beta y = \sum_i (\alpha x_i + \beta y_i) e_i$, and by uniqueness of coordinates this is the coordinate expansion of $\alpha x + \beta y$; hence $\psi_i(\alpha x + \beta y) = \alpha x_i + \beta y_i = \alpha \psi_i(x) + \beta \psi_i(y)$. Since X is a finite dimensional normed linear space, Problem 3 tells us every linear functional on X is continuous, so indeed $\psi_i \in X^*$ for $1 \leq i \leq n$.

Observe the duality relations

$$\psi_i(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

which hold because the coordinate expansion of e_j has j -th coordinate 1 and all others 0.

Linear independence. Suppose $c_1, \dots, c_n$ are real numbers for which $\sum_{i=1}^n c_i \psi_i = 0$ in X^* . Evaluating this functional at e_j , for each $1 \leq j \leq n$, gives

$$0 = \sum_{i=1}^n c_i \psi_i(e_j) = \sum_{i=1}^n c_i \delta_{ij} = c_j.$$

Hence all the c_j vanish, and $\{\psi_1, \dots, \psi_n\}$ is linearly independent.

Spanning. Let $\psi \in X^*$ be arbitrary and set $c_i = \psi(e_i)$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^n x_i e_i$ in X , the linearity of ψ gives

$$\psi(x) = \sum_{i=1}^n x_i \psi(e_i) = \sum_{i=1}^n c_i \psi_i(x) = \left(\sum_{i=1}^n c_i \psi_i \right)(x).$$

Since x was arbitrary, $\psi = \sum_{i=1}^n \psi(e_i) \psi_i$, so $\{\psi_1, \dots, \psi_n\}$ spans X^* .

Being a linearly independent spanning set, $\{\psi_1, \dots, \psi_n\}$ is a basis for X^* , and consequently $\dim X^* = n = \dim X$.

Problem 14.5. Let X be a finite dimensional linear space. Show that the weak and strong topologies on X are the same.

Solution. Here X is a finite dimensional normed linear space, the strong topology being the one induced by the norm and the weak topology being the topology induced on X by X^* .

Every $\psi \in X^*$ is continuous with respect to the strong topology, and the weak topology is by definition the weakest topology on X for which each $\psi \in X^*$ is continuous; hence the weak topology is contained in the strong topology. This inclusion holds in any normed linear space. It remains to show that, in the finite dimensional case, every strongly open set is weakly open.

Let $n = \dim X$ and fix a basis $\{e_1, \dots, e_n\}$ for X , with associated coordinate functionals $\psi_1, \dots, \psi_n$ defined by $\psi_i(x) = x_i$ for $x = x_1 e_1 + \dots + x_n e_n$. By Problem 4 these belong to X^* ; equivalently, by Problem 3 every linear functional on the finite dimensional normed linear space X is continuous.

Let $\mathcal{O}$ be strongly open and let $x \in \mathcal{O}$. Choose $r > 0$ with

$$\{x' \in X : \|x' - x\| < r\} \subseteq \mathcal{O}.$$

Put $M = 1 + \sum_{i=1}^n \|e_i\|$ and $\epsilon = r/M > 0$. Consider the basic weak neighborhood of x

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x) = \{x' \in X : |\psi_i(x' - x)| < \epsilon \text{ for } 1 \leq i \leq n\}.$$

If x' belongs to this set, then, expanding the vector $x' - x$ in the basis, $x' - x = \sum_{i=1}^n \psi_i(x' - x) e_i$, so by the triangle inequality

$$\|x' - x\| \leq \sum_{i=1}^n |\psi_i(x' - x)| \|e_i\| \leq \epsilon \sum_{i=1}^n \|e_i\| < \epsilon M = r,$$

the last inequality because $\epsilon > 0$ and $M = 1 + \sum_{i=1}^n \|e_i\| > \sum_{i=1}^n \|e_i\|$. Hence $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x) \subseteq \{x' : \|x' - x\| < r\} \subseteq \mathcal{O}$. Thus each point of $\mathcal{O}$ has a basic weak neighborhood contained in $\mathcal{O}$, and therefore $\mathcal{O}$ is weakly open.

Since each topology contains the other, the weak and strong topologies on a finite dimensional normed linear space coincide.

Problem 14.6. Show that every nonempty weakly open subset of an infinite dimensional normed linear space is unbounded with respect to the norm.

Solution. Let X be an infinite dimensional normed linear space and let $\mathcal{O}$ be a nonempty weakly open subset of X . Choose $x \in \mathcal{O}$. Since the sets described in (5) form a base at x for the weak topology, there are $\epsilon > 0$ and finitely many $\psi_1, \dots, \psi_n$ in X^* for which

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x) = \{x' \in X : |\psi_k(x' - x)| < \epsilon \text{ for } 1 \leq k \leq n\} \subseteq \mathcal{O}.$$

We claim that

$$Z = \bigcap_{k=1}^n \ker \psi_k \neq \{0\}.$$

Indeed, define the linear map $T: X \rightarrow \mathbb{R}^n$ by $T(y) = (\psi_1(y), \dots, \psi_n(y))$. Its kernel is precisely Z . If $Z = \{0\}$, then T is one-to-one, so T maps X isomorphically onto a linear subspace of $\mathbb{R}^n$; any $n + 1$ vectors in X would then have linearly dependent images and hence be themselves linearly dependent, giving $\dim X \leq n$. This contradicts the assumption that X is infinite dimensional. Therefore there is a vector $z \in Z$ with $z \neq 0$.

Now consider the line $\{x + tz : t \in \mathbb{R}\}$. For every $t \in \mathbb{R}$ and every $1 \leq k \leq n$,

$$|\psi_k((x + tz) - x)| = |t| |\psi_k(z)| = 0 < \epsilon,$$

since $z \in \ker \psi_k$. Hence the whole line lies in $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x)$ and therefore in $\mathcal{O}$.

Finally, by the triangle inequality,

$$\|x + tz\| \geq |t| \|z\| - \|x\|,$$

and $\|z\| > 0$, so $\|x + tz\| \rightarrow \infty$ as $t \rightarrow \infty$. Thus $\mathcal{O}$ contains vectors of arbitrarily large norm, that is, $\mathcal{O}$ is unbounded with respect to the norm.

In particular, no nonempty weakly open set is contained in a ball, so the open unit ball of an infinite dimensional normed linear space is not weakly open. Combined with Problem 5, this shows that the weak topology on a normed linear space coincides with the strong topology if and only if the space is finite dimensional.

Problem 14.7. Let X be a finite dimensional space. Show that the natural embedding $J: X \rightarrow X^{**}$ is one-to-one. Then use Problem 4 to show that $J: X \rightarrow X^{**}$ is onto, so X is reflexive.

Solution. Let X be a finite dimensional normed linear space, say $\dim X = n$, and recall that the natural embedding $J: X \rightarrow X^{**} = (X^*)^*$ is defined by

$$J(x)[\psi] = \psi(x) \quad \text{for all } x \in X, \psi \in X^*.$$

As noted in the text, each $J(x)$ is a bounded linear functional on X^* and J is linear.

That J is one-to-one. Since J is linear, it suffices to show that $J(x) = 0$ implies $x = 0$; equivalently, that $J(x) \neq 0$ whenever $x \neq 0$. So let $x \in X$ with $x \neq 0$. By Proposition 2 there is a linear functional $\psi \in X^\#$ for which $\psi(x) \neq 0$. (Explicitly: $\text{span}[x]$ is a linear subspace of X , the functional $\eta(\lambda x) = \lambda \|x\|$ is linear on it and nonzero at x , and Proposition 2 extends η to a linear functional on all of X .) Since X is a finite dimensional normed linear space, Problem 3 tells us that every linear functional on X is continuous, so $\psi \in X^*$. Then

$$J(x)[\psi] = \psi(x) \neq 0,$$

so $J(x)$ is not the zero functional on X^* ; that is, $J(x) \neq 0$. Hence $\ker J = \{0\}$ and J is one-to-one. That J is onto. By Problem 4, since X is a finite dimensional normed linear space of dimension n , its dual X^* is a linear space of dimension n ; and X^* is itself a normed linear space, with the norm $\|\psi\| = \sup\{|\psi(x)| : \|x\| \leq 1\}$. Applying Problem 4 a second time, now to the finite dimensional normed linear space X^* of dimension n , we get

$$\dim X^{**} = \dim(X^*)^* = n.$$

On the other hand, $J(X)$ is a linear subspace of X^{**} , and since J is a one-to-one linear map, $\dim J(X) = \dim X = n$. Explicitly, if $\{e_1, \dots, e_n\}$ is a basis for X , then $\{J(e_1), \dots, J(e_n)\}$ spans $J(X)$ by linearity of J , and it is linearly independent: if $\sum_{i=1}^n c_i J(e_i) = 0$, then by linearity $J(\sum_{i=1}^n c_i e_i) = 0$, so $\sum_{i=1}^n c_i e_i = 0$ because J is one-to-one, and therefore every $c_i = 0$ because $\{e_i\}$ is a basis. Thus $\{J(e_1), \dots, J(e_n)\}$ is a linearly independent set of n vectors in the n -dimensional linear space X^{**} , and hence is a basis for X^{**} . Indeed, if some $\Psi \in X^{**}$ were not in the span of these n vectors, then $\{J(e_1), \dots, J(e_n), \Psi\}$ would be a linearly independent set of $n+1$ vectors in a space of dimension n , which is impossible. Therefore

$$X^{**} = \text{span}\{J(e_1), \dots, J(e_n)\} = J(X),$$

so J maps X onto X^{**} . By the definition of reflexivity, X is reflexive.

14.2 Problems 14.8–14.14

Problem 14.8. For a vector $v \neq 0$ in Euclidean space $\mathbb{R}^n$, explicitly exhibit a linear functional $\psi: \mathbb{R}^n \rightarrow \mathbb{R}$ for which $\psi(v) = 1$.

Solution. Write $v = (v_1, \dots, v_n)$. Since $v \neq 0$ we have

$$\|v\|^2 = \sum_{i=1}^n v_i^2 > 0,$$

so we may divide by it. Define $\psi: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\psi(x) = \frac{\langle x, v \rangle}{\|v\|^2} = \frac{1}{\sum_{i=1}^n v_i^2} \sum_{i=1}^n x_i v_i \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Linearity: for $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, the i -th coordinate of $\alpha x + \beta y$ is $\alpha x_i + \beta y_i$, and hence

$$\psi(\alpha x + \beta y) = \frac{1}{\|v\|^2} \sum_{i=1}^n (\alpha x_i + \beta y_i) v_i = \alpha \cdot \frac{1}{\|v\|^2} \sum_{i=1}^n x_i v_i + \beta \cdot \frac{1}{\|v\|^2} \sum_{i=1}^n y_i v_i = \alpha \psi(x) + \beta \psi(y).$$

Thus ψ belongs to $(\mathbb{R}^n)^\#$.

Normalization: taking $x = v$,

$$\psi(v) = \frac{\langle v, v \rangle}{\|v\|^2} = \frac{\|v\|^2}{\|v\|^2} = 1.$$

Continuity: although not demanded, we note that ψ is bounded. By the Cauchy–Schwarz inequality,

$$|\psi(x)| \leq \frac{\|x\| \|v\|}{\|v\|^2} = \frac{\|x\|}{\|v\|} \quad \text{for all } x \in \mathbb{R}^n,$$

so $\psi \in (\mathbb{R}^n)^*$ with $\|\psi\| \leq 1/\|v\|$; in fact equality holds, as the choice $x = v$ shows. This is also immediate from Problem 3 of this section: every linear functional on a finite dimensional normed linear space is continuous.

Problem 14.9. For a sequence $\{x_n\} \neq 0$ in ℓ^2 , explicitly exhibit a continuous linear functional $\psi: \ell^2 \rightarrow \mathbb{R}$ for which $\psi(\{x_n\}) = 1$.

Solution. Let $x = \{x_n\}$ be a nonzero element of ℓ^2 , so that

$$\|x\|_2^2 = \sum_{n=1}^{\infty} x_n^2 > 0.$$

Define $\psi: \ell^2 \rightarrow \mathbb{R}$ by

$$\psi(y) = \frac{1}{\|x\|_2^2} \sum_{n=1}^{\infty} y_n x_n \quad \text{for } y = \{y_n\} \in \ell^2.$$

The formula makes sense. Indeed, for $y \in \ell^2$ the Cauchy–Schwarz inequality applied to the partial sums gives, for every N ,

$$\sum_{n=1}^N |y_n x_n| \leq \left(\sum_{n=1}^N y_n^2 \right)^{1/2} \left(\sum_{n=1}^N x_n^2 \right)^{1/2} \leq \|y\|_2 \|x\|_2.$$

The partial sums of the series of absolute values are therefore bounded above, so $\sum_{n=1}^{\infty} y_n x_n$ converges absolutely, and letting $N \rightarrow \infty$ in the display,

$$\left| \sum_{n=1}^{\infty} y_n x_n \right| \leq \|y\|_2 \|x\|_2.$$

Call this last estimate (*).

Linearity: if $y, z \in \ell^2$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha y + \beta z$ has n -th term $\alpha y_n + \beta z_n$, and since all three series converge absolutely we may add them termwise:

$$\psi(\alpha y + \beta z) = \frac{1}{\|x\|_2^2} \sum_{n=1}^{\infty} (\alpha y_n + \beta z_n) x_n = \alpha \psi(y) + \beta \psi(z).$$

Continuity: by (*),

$$|\psi(y)| \leq \frac{\|y\|_2 \|x\|_2}{\|x\|_2^2} = \frac{\|y\|_2}{\|x\|_2} \quad \text{for all } y \in \ell^2,$$

so ψ is a bounded, hence continuous, linear functional on ℓ^2 , with $\|\psi\| \leq 1/\|x\|_2$. Taking $y = x$ shows $\|\psi\| = 1/\|x\|_2$ exactly.

Normalization: with $y = x$,

$$\psi(x) = \frac{1}{\|x\|_2^2} \sum_{n=1}^{\infty} x_n^2 = \frac{\|x\|_2^2}{\|x\|_2^2} = 1.$$

Thus ψ is a continuous linear functional on ℓ^2 with $\psi(\{x_n\}) = 1$.

Problem 14.10. For a function $f \neq 0$ in $L^p[a, b]$, $1 \leq p \leq \infty$, explicitly exhibit a continuous linear functional $\psi: L^p[a, b] \rightarrow \mathbb{R}$ for which $\psi(f) = 1$.

Solution. Throughout, m denotes Lebesgue measure on $[a, b]$, and $f \neq 0$ in $L^p[a, b]$ means that f does not vanish almost everywhere on $[a, b]$. In every case we produce a function g in the conjugate space and set

$$\psi(h) = \frac{1}{c} \int_{[a,b]} h \cdot g \, dm, \quad \text{where } c = \int_{[a,b]} f \cdot g \, dm \neq 0.$$

Once g lies in the appropriate conjugate space, Holder's Inequality shows that $h \mapsto \int h g \, dm$ is a well defined bounded linear functional on $L^p[a, b]$; linearity is the linearity of the integral, and boundedness

is the estimate below. So the only issues are to name g and to check $c \neq 0$.

Recall the signum function $\operatorname{sgn}(t) = 1$ for $t > 0$, $\operatorname{sgn}(t) = -1$ for $t < 0$, and $\operatorname{sgn}(0) = 0$; if f is measurable so is $\operatorname{sgn} \circ f$, since it is a composition of f with a Borel function, and $t \cdot \operatorname{sgn}(t) = |t|$ for all t .

Case 1: $1 \leq p < \infty$. Let q be the conjugate of p and put

$$g = |f|^{p-1} \cdot \operatorname{sgn}(f).$$

(For $p = 1$ this is just $g = \operatorname{sgn}(f)$.) Then $|g| = |f|^{p-1}$.

If $p = 1$, then $q = \infty$ and $|g| \leq 1$, so $g \in L^\infty[a, b]$ with $\|g\|_\infty \leq 1$. If $1 < p < \infty$, then $(p-1)q = p$ because $1/p + 1/q = 1$, hence

$$\int_{[a,b]} |g|^q dm = \int_{[a,b]} |f|^{(p-1)q} dm = \int_{[a,b]} |f|^p dm = \|f\|_p^p < \infty,$$

so $g \in L^q[a, b]$ with $\|g\|_q = \|f\|_p^{p/q}$.

In either case, since $f \cdot \operatorname{sgn}(f) = |f|$,

$$c = \int_{[a,b]} f \cdot g dm = \int_{[a,b]} |f|^{p-1} \cdot |f| dm = \int_{[a,b]} |f|^p dm = \|f\|_p^p.$$

Because f does not vanish a.e., $\|f\|_p > 0$, so $c > 0$. Define

$$\psi(h) = \frac{1}{\|f\|_p^p} \int_{[a,b]} h \cdot |f|^{p-1} \operatorname{sgn}(f) dm \quad \text{for } h \in L^p[a, b].$$

By Holder's Inequality,

$$|\psi(h)| \leq \frac{\|g\|_q}{\|f\|_p^p} \|h\|_p,$$

so ψ is bounded, hence continuous. For $1 < p < \infty$ this constant is $\|f\|_p^{p/q}/\|f\|_p^p = 1/\|f\|_p$, since $p - p/q = 1$; for $p = 1$ it is at most $1/\|f\|_1$. And by the computation of c ,

$$\psi(f) = \frac{1}{\|f\|_p^p} \int_{[a,b]} |f|^p dm = 1.$$

Case 2: $p = \infty$. Now $f \in L^\infty[a, b]$ does not vanish a.e., so the set $\{x \in [a, b] : |f(x)| > 0\}$ has positive measure. Since this set is the ascending union of the measurable sets $E_k = \{x \in [a, b] : |f(x)| > 1/k\}$, by the continuity of Lebesgue measure there is a natural number k with $m(E_k) > 0$. Fix such a k and write $E = E_k$, $\alpha = 1/k$. Define

$$g = \operatorname{sgn}(f) \cdot \chi_E.$$

Then $|g| \leq \chi_E$, so g is measurable and $\|g\|_1 \leq m(E) \leq b - a < \infty$; thus $g \in L^1[a, b]$, and $q = 1$ is the conjugate of $p = \infty$, so Holder's Inequality gives $\left| \int_{[a,b]} hg dm \right| \leq \|h\|_\infty \|g\|_1$ for $h \in L^\infty[a, b]$. (We do not claim that every bounded linear functional on $L^\infty[a, b]$ arises this way; we need only that this one does.) Moreover

$$c = \int_{[a,b]} f \cdot g dm = \int_E |f| dm \geq \alpha m(E) > 0.$$

Define

$$\psi(h) = \frac{1}{c} \int_E h \cdot \operatorname{sgn}(f) dm \quad \text{for } h \in L^\infty[a, b].$$

This is linear by linearity of the integral, and

$$|\psi(h)| \leq \frac{1}{c} \int_E |h| dm \leq \frac{m(E)}{c} \|h\|_\infty,$$

so ψ is bounded, hence continuous, on $L^\infty[a, b]$. Finally $\psi(f) = c/c = 1$.

In both cases ψ is a continuous linear functional on $L^p[a, b]$ with $\psi(f) = 1$, given by an explicit formula in terms of f .

Problem 14.11. Consider $C[a, b]$ with the maximum norm. For a function $f \neq 0$ in $C[a, b]$, explicitly exhibit a continuous linear functional $\psi: C[a, b] \rightarrow \mathbb{R}$ for which $\psi(f) = 1$.

Solution. Here the norm is $\|h\|_{\max} = \max_{x \in [a, b]} |h(x)|$, which is attained because a continuous function on the compact interval $[a, b]$ attains its maximum, and $f \neq 0$ in $C[a, b]$ means that f is not the zero function.

Since f is not identically zero, choose a point $x_0 \in [a, b]$ with $f(x_0) \neq 0$. (For definiteness one may take x_0 to be a point at which $|f|$ attains its maximum, so that $|f(x_0)| = \|f\|_{\max} > 0$.) Define

$$\psi(h) = \frac{h(x_0)}{f(x_0)} \quad \text{for } h \in C[a, b].$$

Linearity: for $h, k \in C[a, b]$ and $\alpha, \beta \in \mathbb{R}$, the function $\alpha h + \beta k$ takes the value $\alpha h(x_0) + \beta k(x_0)$ at x_0 , so

$$\psi(\alpha h + \beta k) = \frac{\alpha h(x_0) + \beta k(x_0)}{f(x_0)} = \alpha \psi(h) + \beta \psi(k).$$

Continuity: for every $h \in C[a, b]$,

$$|\psi(h)| = \frac{|h(x_0)|}{|f(x_0)|} \leq \frac{1}{|f(x_0)|} \|h\|_{\max},$$

since $|h(x_0)| \leq \max_{x \in [a, b]} |h(x)| = \|h\|_{\max}$. Hence ψ is bounded, with $\|\psi\| \leq 1/|f(x_0)|$, and a bounded linear functional is continuous. In fact, taking h to be the constant function 1, which has $\|h\|_{\max} = 1$ and $\psi(h) = 1/f(x_0)$, we see $\|\psi\| = 1/|f(x_0)|$. With the choice $|f(x_0)| = \|f\|_{\max}$ this gives $\|\psi\| = 1/\|f\|_{\max}$. Normalization: $\psi(f) = f(x_0)/f(x_0) = 1$.

Thus ψ is a continuous linear functional on $C[a, b]$ with $\psi(f) = 1$.

Problem 14.12. For $1 < p < \infty$, let Y be a closed subspace of $L^p[a, b]$ of codimension 1. Show that there is a function $g \in L^q[a, b]$, where q is the conjugate of p , for which

$$Y = \left\{ f \in L^p[a, b] \mid \int_{[a, b]} f \cdot g \, dm = 0 \right\}.$$

Solution. Write $X = L^p[a, b]$, a normed linear space under $\|\cdot\|_p$.

Step 1: Y is the kernel of a linear functional. Since Y is a subspace of X of codimension 1, Proposition 1 provides a nonzero $\psi \in X^\#$ with $Y = \ker \psi$. Concretely, there is $x_0 \in X$, $x_0 \neq 0$, with $X = Y \oplus \text{span}[x_0]$, and ψ is defined by $\psi(y + \lambda x_0) = \lambda$ for $y \in Y$, $\lambda \in \mathbb{R}$; in particular $\psi(x_0) = 1$.

Step 2: ψ is bounded, because Y is closed. Argue by contradiction: suppose ψ is unbounded, that is, there is no $M \geq 0$ with $|\psi(u)| \leq M\|u\|_p$ for all $u \in X$. Fix a natural number n . Since $M = n$ fails, there is $w \in X$ with $|\psi(w)| > n\|w\|_p$; necessarily $w \neq 0$, since $\psi(0) = 0$. Put $u_n = w/\|w\|_p$, so that $\|u_n\|_p = 1$ and, by the homogeneity of ψ ,

$$|\psi(u_n)| = \frac{|\psi(w)|}{\|w\|_p} > n.$$

Set

$$v_n = \frac{u_n}{\psi(u_n)},$$

which is legitimate since $|\psi(u_n)| > n \geq 1 > 0$. Then $\psi(v_n) = 1$ and

$$\|v_n\|_p = \frac{\|u_n\|_p}{|\psi(u_n)|} < \frac{1}{n},$$

so $\{v_n\} \rightarrow 0$ strongly in X . By linearity, $\psi(x_0 - v_n) = 1 - 1 = 0$, that is, $x_0 - v_n \in \ker \psi = Y$ for every n . But $\{x_0 - v_n\} \rightarrow x_0$ with respect to the norm, and Y is closed with respect to the strong

topology, so $x_0 \in Y$ and therefore $\psi(x_0) = 0$. This contradicts $\psi(x_0) = 1$. Hence ψ is bounded, that is, $\psi \in X^* = (L^p[a, b])^*$. (This is exactly the assertion of Problem 2 of this section.)

Step 3: represent ψ by a function in L^q . Since $[a, b]$ is a Lebesgue measurable set and $1 < p < \infty$, the Riesz Representation Theorem for the Dual of $L^p(E)$ applies with $E = [a, b]$: for the bounded linear functional ψ on $L^p[a, b]$ there is a unique $g \in L^q[a, b]$, q the conjugate of p , for which

$$\psi(f) = \int_{[a,b]} f \cdot g \, dm \quad \text{for all } f \in L^p[a, b].$$

Step 4: conclusion. Combining Steps 1 and 3,

$$Y = \ker \psi = \{f \in L^p[a, b] \mid \psi(f) = 0\} = \left\{ f \in L^p[a, b] \mid \int_{[a,b]} f \cdot g \, dm = 0 \right\},$$

which is the required representation. We note also that $g \neq 0$ in $L^q[a, b]$, since $\psi(x_0) = 1 \neq 0$ forces ψ to be a nonzero functional, and $\|\psi\| = \|g\|_q$ by the Riesz Representation Theorem.

Problem 14.13. Let X be a normed linear space and ψ belong to $X^\# \sim X^*$. Show that $\ker \psi$ is dense, with respect to the strong topology, in X .

Solution. By hypothesis $\psi: X \rightarrow \mathbb{R}$ is linear but not bounded. We show that every point of X is a limit, in the norm, of points of $\ker \psi$.

Since ψ is unbounded, for each natural number n there is a vector $w_n \in X$ with $\|w_n\| \leq 1$ and $|\psi(w_n)| > n$. Indeed, suppose that for some n we had $|\psi(w)| \leq n$ for every w with $\|w\| \leq 1$. Given $x \in X$ with $x \neq 0$, the vector $w = x/\|x\|$ satisfies $\|w\| = 1$, so by the linearity of ψ ,

$$|\psi(x)| = \|x\| |\psi(w)| \leq n\|x\|,$$

and this inequality holds trivially for $x = 0$; thus ψ would be bounded, contrary to hypothesis. In particular $\psi(w_n) \neq 0$ for every n , since $|\psi(w_n)| > n \geq 1$.

Let $x \in X$ be arbitrary. Define

$$v_n = \frac{\psi(x)}{\psi(w_n)} w_n \quad \text{for each natural number } n,$$

which is well defined since $\psi(w_n) \neq 0$. By the linearity of ψ ,

$$\psi(v_n) = \frac{\psi(x)}{\psi(w_n)} \psi(w_n) = \psi(x),$$

and therefore

$$\psi(x - v_n) = \psi(x) - \psi(v_n) = 0,$$

so that $x - v_n \in \ker \psi$ for every n . On the other hand,

$$\|x - v_n - x\| = \|v_n\| = \frac{|\psi(x)|}{|\psi(w_n)|} \|w_n\| \leq \frac{|\psi(x)|}{n},$$

which tends to 0 as $n \rightarrow \infty$. Hence $\{x - v_n\}$ is a sequence in $\ker \psi$ converging to x with respect to the norm.

Consequently, given $x \in X$ and $\epsilon > 0$, choosing n with $|\psi(x)|/n < \epsilon$ produces a point of $\ker \psi$ within ϵ of x ; that is, every open ball in X meets $\ker \psi$. Therefore $\ker \psi$ is dense in X with respect to the strong topology.

Two remarks. First, this makes precise the contrast with the bounded case: if ψ were bounded and nonzero, $\ker \psi$ would be a proper closed subspace, hence certainly not dense. Second, since $X^\# \sim X^*$ is nonempty exactly when X is infinite dimensional (Proposition 3), the phenomenon occurs only in infinite dimensions.

Problem 14.14. Let X be the normed linear space of polynomials restricted to $[a, b]$. For $p \in X$, define $\psi(p)$ to be the sum of the coefficients of p . Show that ψ is linear. Is ψ continuous if X has the topology induced by the maximum norm?

Solution. Throughout, $a < b$, and X consists of the restrictions to $[a, b]$ of real polynomials, normed by $\|h\|_{\max} = \max_{x \in [a, b]} |h(x)|$.

First, ψ is well defined. A polynomial that vanishes at every point of the infinite set $[a, b]$ has infinitely many roots and is therefore the zero polynomial; hence the restriction map from polynomials to X is one-to-one, and each element of X has a uniquely determined coefficient sequence. So if $p \in X$ is written $p(t) = \sum_{k=0}^n c_k t^k$, the number

$$\psi(p) = \sum_{k=0}^n c_k$$

does not depend on the representation (padding with zero coefficients changes nothing).

The key observation is that the sum of the coefficients is the value of the polynomial at 1:

$$\psi(p) = \sum_{k=0}^n c_k = \sum_{k=0}^n c_k \cdot 1^k = p(1),$$

where p here denotes the polynomial function on all of $\mathbb{R}$ determined by the coefficients.

Linearity: let $p, r \in X$ and $\alpha, \beta \in \mathbb{R}$. Write $p(t) = \sum_{k=0}^n c_k t^k$ and $r(t) = \sum_{k=0}^n d_k t^k$ with a common n (padding with zeros). The polynomial $\alpha p + \beta r$ has coefficients $\alpha c_k + \beta d_k$, so

$$\psi(\alpha p + \beta r) = \sum_{k=0}^n (\alpha c_k + \beta d_k) = \alpha \sum_{k=0}^n c_k + \beta \sum_{k=0}^n d_k = \alpha \psi(p) + \beta \psi(r).$$

Thus ψ belongs to $X^\sharp$.

Continuity: the answer depends on the interval, and the dividing line is whether $1 \in [a, b]$.

Suppose first that $1 \in [a, b]$. Then for every $p \in X$,

$$|\psi(p)| = |p(1)| \leq \max_{x \in [a, b]} |p(x)| = \|p\|_{\max},$$

so ψ is bounded with $\|\psi\| \leq 1$, and hence continuous. Testing on the constant polynomial $p \equiv 1$, for which $\|p\|_{\max} = 1$ and $\psi(p) = 1$, gives $\|\psi\| = 1$.

Suppose now that $1 \notin [a, b]$. Then ψ is not continuous. Let $c = (a + b)/2$ be the midpoint of $[a, b]$ and $d = (b - a)/2 > 0$ its half-length, so that $[a, b] = \{t : |t - c| \leq d\}$. Since $1 \notin [a, b]$ we have $|1 - c| > d$, and so

$$\lambda = \frac{|1 - c|}{d} > 1.$$

For each natural number n define the polynomial

$$p_n(t) = \left(\frac{t - c}{d} \right)^n,$$

which indeed lies in X (it is a polynomial in t of degree n). For $x \in [a, b]$ we have $|x - c| \leq d$, whence $|p_n(x)| \leq 1$; and $|p_n(a)| = 1$. Therefore

$$\|p_n\|_{\max} = 1 \quad \text{for every } n,$$

while

$$|\psi(p_n)| = |p_n(1)| = \left| \frac{1 - c}{d} \right|^n = \lambda^n \rightarrow \infty$$

as $n \rightarrow \infty$, because $\lambda > 1$. So ψ is unbounded on the unit sphere of X and hence not continuous. (Equivalently, $q_n = \lambda^{-n} p_n$ satisfies $\|q_n\|_{\max} = \lambda^{-n} \rightarrow 0$ while $\psi(q_n) = 1$ for all n , so ψ fails to be continuous at 0.)

Summary: ψ is always linear; it is continuous with respect to the maximum norm on $[a, b]$ if and only if $1 \in [a, b]$, in which case $\psi(p) = p(1)$ has norm exactly 1. In particular, for the standard interval $[0, 1]$ the functional is continuous, whereas for an interval such as $[0, 1/2]$ it is a discontinuous linear functional, an explicit member of $X^\sharp \sim X^*$ (consistent with Proposition 3, since X is infinite dimensional).

14.3 Problems 14.15–14.21

Problem 14.15. Let X be the normed linear space of sequences of real numbers that have only a finite number of nonzero terms. For $x = \{x_n\} \in X$, define

$$\psi(x) = \sum_{n=1}^{\infty} x_n.$$

Show that ψ is linear. Is ψ continuous if X has the topology induced by the ℓ^∞ norm?

Solution. First observe that ψ is well defined on X . If $x = \{x_n\}$ belongs to X , there is an index N for which $x_n = 0$ for all $n > N$. Hence the series $\sum_{n=1}^{\infty} x_n$ has only finitely many nonzero terms: its partial sums are eventually constant, equal to $\sum_{n=1}^N x_n$, so the series converges and

$$\psi(x) = \sum_{n=1}^N x_n.$$

To verify linearity, let $x = \{x_n\}$ and $y = \{y_n\}$ belong to X and let α, β be real numbers. Choose N so large that $x_n = 0$ and $y_n = 0$ for all $n > N$; such an N exists since each of x and y has only finitely many nonzero terms. Then $\alpha x + \beta y$ is the sequence whose n th term is $\alpha x_n + \beta y_n$, and this vanishes for $n > N$, so that $\alpha x + \beta y$ again belongs to X (thus X is indeed a linear space) and, by the finite additivity of finite sums,

$$\psi(\alpha x + \beta y) = \sum_{n=1}^N (\alpha x_n + \beta y_n) = \alpha \sum_{n=1}^N x_n + \beta \sum_{n=1}^N y_n = \alpha \psi(x) + \beta \psi(y).$$

Therefore $\psi: X \rightarrow \mathbb{R}$ is linear.

Now give X the norm induced by the ℓ^∞ norm, that is,

$$\|x\|_\infty = \sup_{n \geq 1} |x_n| = \max_{n \geq 1} |x_n| \quad \text{for } x = \{x_n\} \in X,$$

the supremum being attained because only finitely many terms are nonzero. We claim ψ is **not** continuous. By Theorem 1 of Chapter 13, a linear operator between normed linear spaces is continuous if and only if it is bounded, so it suffices to exhibit a sequence of vectors of norm at most one on which ψ is unbounded.

For each natural number k , let

$$x^{(k)} = (\underbrace{1, 1, \dots, 1}_{k \text{ terms}}, 0, 0, \dots),$$

that is, $x_n^{(k)} = 1$ for $1 \leq n \leq k$ and $x_n^{(k)} = 0$ for $n > k$. Each $x^{(k)}$ has only finitely many nonzero terms, so $x^{(k)} \in X$, and evidently

$$\|x^{(k)}\|_\infty = 1, \quad \psi(x^{(k)}) = \sum_{n=1}^k 1 = k.$$

Thus $\sup\{|\psi(x)| \mid x \in X, \|x\|_\infty \leq 1\} \geq \sup_k k = \infty$, so ψ is unbounded and hence fails to be continuous.

Equivalently, one may argue directly with the definition of continuity at 0: the vectors $z^{(k)} = \frac{1}{\sqrt{k}} x^{(k)}$ satisfy $\|z^{(k)}\|_\infty = 1/\sqrt{k} \rightarrow 0$, so $z^{(k)} \rightarrow 0$ in X , while $\psi(z^{(k)}) = \sqrt{k} \rightarrow \infty$, so $\psi(z^{(k)}) \not\rightarrow 0 = \psi(0)$. Hence ψ is discontinuous at 0, and therefore (being linear) discontinuous at every point of X .

It is worth noting that the answer depends on the norm: had X been given the norm induced by ℓ^1 , that is $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$, then $|\psi(x)| \leq \|x\|_1$ for all $x \in X$ and ψ would be continuous with $\|\psi\| = 1$. It is precisely the failure of the ℓ^∞ norm to control the number of terms being summed that destroys continuity.

Problem 14.16. Let X be a linear space. A subset E of X is said to be **linearly independent** provided each $x \in E$ fails to be a finite linear combination of points in $E \sim \{x\}$. Define $\mathcal{F}$ to be the collection of nonempty subsets of X that are linearly independent. Order $\mathcal{F}$ by set inclusion. Apply Zorn's Lemma to conclude that X has a Hamel basis.

Solution. Recall that a **Hamel basis** for X is a linearly independent subset E of X whose finite linear combinations exhaust X ; equivalently, a subset E such that every point of X is expressible in exactly one way as a finite linear combination of distinct points of E with nonzero coefficients.

If $X = \{0\}$ the empty set is a Hamel basis (the empty linear combination being 0), so assume $X \neq \{0\}$. Step 1: $\mathcal{F}$ is nonempty. Choose $x_0 \in X$ with $x_0 \neq 0$ and put $E_0 = \{x_0\}$. Then $E_0 \sim \{x_0\} = \emptyset$, and the only finite linear combination of points of the empty set is 0, while $x_0 \neq 0$. Hence E_0 is a nonempty linearly independent set, so $E_0 \in \mathcal{F}$ and $\mathcal{F} \neq \emptyset$.

Step 2: every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Let $\mathcal{C}$ be a chain in $(\mathcal{F}, \subseteq)$, that is, a subcollection of $\mathcal{F}$ any two of whose members are comparable under inclusion. We may assume $\mathcal{C} \neq \emptyset$, since the empty chain is bounded above by the member E_0 of $\mathcal{F}$ exhibited in Step 1. Set

$$E^* = \bigcup_{E \in \mathcal{C}} E.$$

Then E^* is nonempty and contains every member of $\mathcal{C}$, so it will be the required upper bound once we check that it is linearly independent.

Suppose it is not. Then there is a point $x \in E^*$ together with finitely many points $x_1, \dots, x_n$ of $E^* \sim \{x\}$ and real numbers $\lambda_1, \dots, \lambda_n$ for which

$$x = \sum_{i=1}^n \lambda_i x_i.$$

Each of the finitely many points $x, x_1, \dots, x_n$ belongs to some member of $\mathcal{C}$: say $x \in F_0$ and $x_i \in F_i$ for $1 \leq i \leq n$, with $F_0, F_1, \dots, F_n \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered by inclusion, a finite subcollection of it has a largest member; let F be the largest of $F_0, F_1, \dots, F_n$. (This is a trivial induction on n : the largest of two comparable sets is one of them.) Then x and all the x_i belong to F , so $x \in F$ is a finite linear combination of points of $F \sim \{x\}$. This contradicts the linear independence of F , which belongs to $\mathcal{C} \subseteq \mathcal{F}$. Therefore E^* is linearly independent, $E^* \in \mathcal{F}$, and E^* is an upper bound for $\mathcal{C}$.

Step 3: apply Zorn's Lemma. The partially ordered set $(\mathcal{F}, \subseteq)$ is nonempty and every chain in it has an upper bound in it. By Zorn's Lemma, $\mathcal{F}$ possesses a maximal element E : a linearly independent subset of X that is properly contained in no linearly independent subset of X .

Step 4: the finite linear combinations of points of E exhaust X . Let $\text{span } E$ denote the set of all finite linear combinations $\sum_{i=1}^n \lambda_i e_i$ with $e_i \in E$ and $\lambda_i \in \mathbb{R}$; this is a linear subspace of X containing E . Suppose, to get a contradiction, that there is a point $x \in X$ with $x \notin \text{span } E$. In particular $x \notin E$, so $E \cup \{x\}$ properly contains E . We claim $E \cup \{x\}$ is linearly independent, which will contradict the maximality of E .

Indeed, first, x is not a finite linear combination of points of $(E \cup \{x\}) \sim \{x\} = E$, since $x \notin \text{span } E$. Second, let $e \in E$ and suppose

$$e = \sum_{i=1}^n \lambda_i e_i + \lambda x, \quad e_i \in E \sim \{e\}, \quad \lambda_i, \lambda \in \mathbb{R},$$

were a representation of e as a finite linear combination of points of $(E \cup \{x\}) \sim \{e\}$. If $\lambda = 0$, this exhibits e as a finite linear combination of points of $E \sim \{e\}$, contradicting the linear independence of E . If $\lambda \neq 0$, then

$$x = \frac{1}{\lambda} \left(e - \sum_{i=1}^n \lambda_i e_i \right),$$

which is a finite linear combination of points of E , contradicting $x \notin \text{span } E$. In either case we reach a contradiction, so no such representation exists and $E \cup \{x\}$ is linearly independent. Thus $E \cup \{x\}$ belongs to $\mathcal{F}$ and properly contains E , contradicting maximality. We conclude that $\text{span } E = X$.

Step 5: uniqueness of the representation. Finally we check that each $x \in X$ has only one expression as a finite linear combination of distinct points of E with nonzero coefficients. Suppose

$$\sum_{i=1}^n \lambda_i e_i = \sum_{i=1}^n \mu_i e_i,$$

where $e_1, \dots, e_n$ are distinct points of E (a common finite list may always be arranged by adjoining terms with zero coefficients). Put $\nu_i = \lambda_i - \mu_i$, so that $\sum_{i=1}^n \nu_i e_i = 0$. If some $\nu_j \neq 0$, then

$$e_j = -\frac{1}{\nu_j} \sum_{i \neq j} \nu_i e_i$$

exhibits e_j as a finite linear combination of points of $E \sim \{e_j\}$, contradicting the linear independence of E . Hence $\nu_i = 0$, that is $\lambda_i = \mu_i$, for all i .

Therefore E is a linearly independent subset of X whose finite linear combinations represent each point of X uniquely: E is a Hamel basis for X .

Problem 14.17. Provide an example of a discontinuous linear operator T from a normed linear space X to a normed linear space Y for which T has a closed graph. (Hint: Let ψ be a discontinuous linear functional on a normed linear space X and $Y = \{y \in X \times \mathbb{R} \mid y = (x, \psi(x))\}$, the graph of ψ .) Does this contradict the Closed Graph Theorem?

Solution. Take X to be the normed linear space of the preceding problem: the sequences $x = \{x_n\}$ of real numbers having only finitely many nonzero terms, normed by $\|x\| = \sup_{n \geq 1} |x_n|$, and let

$$\psi(x) = \sum_{n=1}^{\infty} x_n \quad \text{for } x = \{x_n\} \in X.$$

As shown in Problem 15, ψ is a linear functional on X that is not continuous, since the vectors $x^{(k)} = (1, \dots, 1, 0, 0, \dots)$ with k ones satisfy $\|x^{(k)}\| = 1$ while $\psi(x^{(k)}) = k$.

Give the product $X \times \mathbb{R}$ the norm

$$\|(x, t)\| = \|x\| + |t|,$$

which is readily checked to be a norm inducing the product topology, and let

$$Y = \{(x, \psi(x)) \mid x \in X\} \subseteq X \times \mathbb{R}$$

be the graph of ψ . Since ψ is linear, Y is a linear subspace of $X \times \mathbb{R}$, and we norm Y by the restriction of the above norm. Define

$$T: X \rightarrow Y, \quad T(x) = (x, \psi(x)).$$

T is linear: for $x, x' \in X$ and $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha x + \beta x') = (\alpha x + \beta x', \psi(\alpha x + \beta x')) = \alpha(x, \psi(x)) + \beta(x', \psi(x')) = \alpha T(x) + \beta T(x'),$$

by the linearity of ψ .

T is not continuous. Indeed,

$$\|T(x)\|_Y = \|x\| + |\psi(x)| \geq |\psi(x)| \quad \text{for all } x \in X,$$

so $\sup_{\|x\| \leq 1} \|T(x)\|_Y \geq \sup_k |\psi(x^{(k)})| = \sup_k k = \infty$. Thus T is an unbounded linear operator and hence, by Theorem 1 of Chapter 13, discontinuous.

T has a closed graph. Recall from Section 13.4 that T is called closed provided that whenever $\{x_n\} \rightarrow x$ in X and $\{T(x_n)\} \rightarrow y$ in Y , then $T(x) = y$, and that this is equivalent to the graph of T being a closed subset of the product $X \times Y$. The graph of T is the subset

$$G(T) = \{(x, T(x)) \mid x \in X\} \subseteq X \times Y.$$

Let $\{(x_n, T(x_n))\}$ be a sequence in $G(T)$ converging in $X \times Y$ to a point (x, y) ; thus $x_n \rightarrow x$ in X and $T(x_n) \rightarrow y$ in Y . Since y belongs to Y , it is of the form $y = (u, \psi(u))$ for some $u \in X$. Convergence $T(x_n) \rightarrow y$ in Y means

$$\|x_n - u\| + |\psi(x_n) - \psi(u)| \rightarrow 0,$$

and in particular $\|x_n - u\| \rightarrow 0$, that is, $x_n \rightarrow u$ in X . By the uniqueness of limits in the normed linear space X we get $u = x$, and therefore

$$y = (u, \psi(u)) = (x, \psi(x)) = T(x),$$

so $(x, y) = (x, T(x)) \in G(T)$. Since $X \times Y$ is a metric space, closedness is equivalent to sequential closedness, so $G(T)$ is closed. (The point of the construction is that convergence in the codomain Y already carries the information that the second coordinate of the limit is the value of ψ at the first coordinate; the discontinuity of ψ is thereby hidden inside the norm of Y .)

This does not contradict the Closed Graph Theorem. That theorem asserts that a linear operator with closed graph from a **Banach** space to a **Banach** space is continuous, and here neither X nor Y is complete. For X : the vectors

$$u^{(k)} = (1, \frac{1}{2}, \dots, \frac{1}{k}, 0, 0, \dots)$$

belong to X and satisfy $\|u^{(k)} - u^{(m)}\| = 1/(k+1)$ for $m > k$, so $\{u^{(k)}\}$ is Cauchy in X ; but if it converged in X to some $u \in X$, then comparing coordinates (convergence in the sup norm implies coordinatewise convergence) would force $u = \{1/n\}_{n=1}^\infty$, which has infinitely many nonzero terms and hence does not belong to X . So X is not a Banach space, and the hypotheses of the Closed Graph Theorem fail.

More generally, this phenomenon can never be arranged with both spaces complete. Suppose ψ is a discontinuous linear functional on any normed linear space X and Y is its graph as above; the three verifications just given used nothing about X beyond the linearity and discontinuity of ψ , so $T: X \rightarrow Y$ is again a discontinuous linear operator with closed graph. The Closed Graph Theorem therefore forces at least one of X, Y to be incomplete. In fact, if X happens to be a Banach space (such a ψ exists on every infinite dimensional Banach space: take a Hamel basis E as furnished by Problem 16, normalize its elements, choose a countable subset $\{e_k\} \subseteq E$, and define $\psi(e_k) = k$, $\psi(e) = 0$ for the remaining $e \in E$, extending linearly), then Y must fail to be complete; otherwise the Closed Graph Theorem, applied to T , would make T and hence ψ continuous.

Problem 14.18. Let X be a normed linear space, ψ belong to X^* , and $\{\psi_n\}$ be in X^* . Show that if $\{\psi_n\}$ converges weak-* to ψ , then

$$\|\psi\| \leq \limsup \|\psi_n\|.$$

Solution. By definition, $\{\psi_n\}$ converges weak-* to ψ provided

$$\lim_{n \rightarrow \infty} \psi_n(x) = \psi(x) \quad \text{for every } x \in X.$$

We prove the sharper inequality

$$\|\psi\| \leq \liminf_{n \rightarrow \infty} \|\psi_n\|,$$

from which the assertion follows at once, since $\liminf \|\psi_n\| \leq \limsup \|\psi_n\|$ always.

Set $L = \liminf_{n \rightarrow \infty} \|\psi_n\|$. If $L = \infty$ there is nothing to prove, so assume $L < \infty$.

Let $x \in X$ have $\|x\| \leq 1$. Since $\psi_n(x) \rightarrow \psi(x)$ and the absolute value function is continuous on $\mathbb{R}$, we have $|\psi_n(x)| \rightarrow |\psi(x)|$; in particular the limit exists, so it agrees with the lower limit:

$$|\psi(x)| = \lim_{n \rightarrow \infty} |\psi_n(x)| = \liminf_{n \rightarrow \infty} |\psi_n(x)|.$$

On the other hand, by the very definition of the norm on the dual space X^* ,

$$|\psi_n(x)| \leq \|\psi_n\| \cdot \|x\| \leq \|\psi_n\| \quad \text{for every } n.$$

Since the lower limit is monotone with respect to termwise inequality of sequences,

$$|\psi(x)| = \liminf_{n \rightarrow \infty} |\psi_n(x)| \leq \liminf_{n \rightarrow \infty} \|\psi_n\| = L.$$

This holds for every x in the closed unit ball of X . Taking the supremum over all such x and using the definition

$$\|\psi\| = \sup\{|\psi(x)| \mid x \in X, \|x\| \leq 1\},$$

we conclude

$$\|\psi\| \leq L = \liminf_{n \rightarrow \infty} \|\psi_n\| \leq \limsup_{n \rightarrow \infty} \|\psi_n\|,$$

as required.

Two remarks are in order. First, the argument used neither the completeness of X nor the Uniform Boundedness Theorem: the bound is a direct consequence of pointwise convergence together with the definition of the dual norm. Second, this contains inequality (12) of Theorem 12, which asserts $\|x\| \leq \liminf \|x_n\|$ for a sequence $\{x_n\}$ converging weakly to x in X . Indeed, let $J: X \rightarrow X^{**}$ be the natural embedding, so $J(z)(\varphi) = \varphi(z)$ for $\varphi \in X^*$. Weak convergence of $\{x_n\}$ to x says precisely that $\varphi(x_n) \rightarrow \varphi(x)$ for every $\varphi \in X^*$, that is, $J(x_n)(\varphi) \rightarrow J(x)(\varphi)$ for every $\varphi \in X^*$; since X^{**} is the dual of X^* , this is exactly weak-* convergence of $\{J(x_n)\}$ to $J(x)$ in $(X^*)^*$. The result just proved, applied in the normed linear space X^* , therefore gives $\|J(x)\| \leq \liminf \|J(x_n)\|$, and since J is an isometry by Corollary 9 this is $\|x\| \leq \liminf \|x_n\|$.

Finally, note that strict inequality is possible, so no reverse estimate holds. For instance, in $X = \ell^2$ let $\psi_n(x) = x_n$ be evaluation at the n th coordinate. Each ψ_n has $\|\psi_n\| = 1$, while for every $x = \{x_k\} \in \ell^2$ we have $x_n \rightarrow 0$ because $\sum_k x_k^2 < \infty$. Hence $\{\psi_n\}$ converges weak-* to $\psi = 0$, and $0 = \|\psi\| < 1 = \limsup \|\psi_n\|$.

Problem 14.19. Let $X = \mathbb{R}^n$ be normed with the Euclidean norm, Y a subspace of X , and $\psi: Y \rightarrow \mathbb{R}$ a linear functional. Define $Y^\perp$ to be the linear subspace of $\mathbb{R}^n$ consisting of vectors orthogonal to Y . Then $\mathbb{R}^n = Y \oplus Y^\perp$. For $x = y + y'$, $y \in Y$, $y' \in Y^\perp$, define $\psi(x) = \psi(y)$. Show that this properly defines $\psi \in (\mathbb{R}^n)^*$, is an extension of ψ on Y , and has the same norm as $\psi|_Y$.

Solution. Throughout, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$ and $\|x\| = \langle x, x \rangle^{1/2}$ the Euclidean norm. Write

$$Y^\perp = \{z \in \mathbb{R}^n \mid \langle z, y \rangle = 0 \text{ for all } y \in Y\},$$

which is plainly a linear subspace of $\mathbb{R}^n$. To keep the two functionals apart we write $\tilde{\psi}$ for the extension being defined, so that $\tilde{\psi}(y + y') = \psi(y)$ for $y \in Y$, $y' \in Y^\perp$; at the end we see $\tilde{\psi}$ agrees with ψ on Y , so the book's use of one symbol for both is legitimate.

Step 1: the decomposition $\mathbb{R}^n = Y \oplus Y^\perp$, so that $\tilde{\psi}$ is properly defined. Since Y is a subspace of the finite dimensional space $\mathbb{R}^n$, it has a finite basis, and the Gram-Schmidt process converts it into an orthonormal basis $e_1, \dots, e_k$ of Y (if $Y = \{0\}$, take $k = 0$ and all the sums below empty). For $x \in \mathbb{R}^n$ define

$$Px = \sum_{i=1}^k \langle x, e_i \rangle e_i \in Y.$$

For each j ,

$$\langle x - Px, e_j \rangle = \langle x, e_j \rangle - \sum_{i=1}^k \langle x, e_i \rangle \langle e_i, e_j \rangle = \langle x, e_j \rangle - \langle x, e_j \rangle = 0,$$

and since $e_1, \dots, e_k$ span Y , it follows by linearity of the inner product in its second argument that $x - Px \in Y^\perp$. Hence

$$x = Px + (x - Px), \quad Px \in Y, \quad x - Px \in Y^\perp,$$

so $\mathbb{R}^n = Y + Y^\perp$. Moreover $Y \cap Y^\perp = \{0\}$: if v lies in both, then $\|v\|^2 = \langle v, v \rangle = 0$, so $v = 0$. Consequently the decomposition is unique: if $y_1 + y'_1 = y_2 + y'_2$ with $y_1, y_2 \in Y$ and $y'_1, y'_2 \in Y^\perp$, then

$y_1 - y_2 = y'_2 - y'_1$ belongs to $Y \cap Y^\perp = \{0\}$, whence $y_1 = y_2$ and $y'_1 = y'_2$. Thus $\mathbb{R}^n = Y \oplus Y^\perp$, and each x determines its Y -component $y = Px$ unambiguously. Therefore the formula $\tilde{\psi}(x) = \psi(Px)$ assigns to each $x \in \mathbb{R}^n$ exactly one real number: $\tilde{\psi}$ is properly defined.

Step 2: $\tilde{\psi}$ is linear. First, P is linear: for $x_1, x_2 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, the vector $\alpha Px_1 + \beta Px_2$ belongs to Y and $\alpha(x_1 - Px_1) + \beta(x_2 - Px_2)$ belongs to $Y^\perp$ (both are subspaces), and their sum is $\alpha x_1 + \beta x_2$; by the uniqueness of the decomposition just established, $P(\alpha x_1 + \beta x_2) = \alpha Px_1 + \beta Px_2$. Since ψ is linear on Y , the composition $\tilde{\psi} = \psi \circ P$ is linear on $\mathbb{R}^n$.

Step 3: $\tilde{\psi}$ is bounded, hence $\tilde{\psi} \in (\mathbb{R}^n)^*$. Because Px and $x - Px$ are orthogonal, the Pythagorean identity gives

$$\|x\|^2 = \|Px\|^2 + \|x - Px\|^2 \geq \|Px\|^2, \quad \text{so} \quad \|Px\| \leq \|x\| \quad \text{for all } x \in \mathbb{R}^n.$$

Also ψ is bounded on Y : for $y \in Y$ we have $y = \sum_{i=1}^k \langle y, e_i \rangle e_i$, so by linearity and the Cauchy-Schwarz inequality in $\mathbb{R}^k$,

$$|\psi(y)| = \left| \sum_{i=1}^k \langle y, e_i \rangle \psi(e_i) \right| \leq \left(\sum_{i=1}^k \psi(e_i)^2 \right)^{1/2} \left(\sum_{i=1}^k \langle y, e_i \rangle^2 \right)^{1/2} = M \|y\|,$$

where $M = (\sum_{i=1}^k \psi(e_i)^2)^{1/2}$ and we used $\sum_i \langle y, e_i \rangle^2 = \|y\|^2$ for $y \in Y$. Hence $\psi|_Y$ is bounded and, for all $x \in \mathbb{R}^n$,

$$|\tilde{\psi}(x)| = |\psi(Px)| \leq M \|Px\| \leq M \|x\|.$$

So $\tilde{\psi}$ is a bounded linear functional on $\mathbb{R}^n$, that is, $\tilde{\psi} \in (\mathbb{R}^n)^*$.

Step 4: $\tilde{\psi}$ extends ψ . If $x \in Y$, then $x = x + 0$ with $x \in Y$ and $0 \in Y^\perp$, and this is the (unique) decomposition of x ; equivalently $Px = x$. Hence $\tilde{\psi}(x) = \psi(x)$ for all $x \in Y$, so $\tilde{\psi}$ is an extension of ψ .

Step 5: $\|\tilde{\psi}\| = \|\psi|_Y\|$. Recall

$$\|\psi|_Y\| = \sup\{|\psi(y)| \mid y \in Y, \|y\| \leq 1\}, \quad \|\tilde{\psi}\| = \sup\{|\tilde{\psi}(x)| \mid x \in \mathbb{R}^n, \|x\| \leq 1\}.$$

Since every $y \in Y$ with $\|y\| \leq 1$ is a point of $\mathbb{R}^n$ of norm at most one at which $\tilde{\psi}(y) = \psi(y)$, the supremum defining $\|\tilde{\psi}\|$ is over a larger set of values, so $\|\psi|_Y\| \leq \|\tilde{\psi}\|$. (This inequality holds for any extension.) Conversely, let $x \in \mathbb{R}^n$ with $\|x\| \leq 1$. Then $Px \in Y$ with $\|Px\| \leq \|x\| \leq 1$, so

$$|\tilde{\psi}(x)| = |\psi(Px)| \leq \|\psi|_Y\| \cdot \|Px\| \leq \|\psi|_Y\| \cdot \|x\| \leq \|\psi|_Y\|.$$

Taking the supremum over such x gives $\|\tilde{\psi}\| \leq \|\psi|_Y\|$. Therefore

$$\|\tilde{\psi}\| = \|\psi|_Y\|,$$

and $\tilde{\psi}$ is a norm-preserving extension of ψ to all of $\mathbb{R}^n$.

Thus in Euclidean space the conclusion of Theorem 7 can be obtained explicitly, by orthogonal projection, with no appeal to Zorn's Lemma or to the Hahn-Banach Theorem.

In fact this extension is the only norm-preserving one. To see this, set $M = \|\psi|_Y\|$. By Step 3 the functional ψ is bounded on Y , and every linear functional on $\mathbb{R}^n$ is of the form $\langle \cdot, v \rangle$ for a unique $v \in \mathbb{R}^n$, with norm $\|v\|$: existence follows by taking $v = \sum_{j=1}^n \varphi(u_j) u_j$ for the standard orthonormal basis $u_1, \dots, u_n$, and the norm identity from the Cauchy-Schwarz inequality together with the value at $v/\|v\|$ when $v \neq 0$. Writing $v_0 = \sum_{i=1}^k \psi(e_i) e_i \in Y$, we have $\psi(y) = \langle y, v_0 \rangle$ for $y \in Y$, since both sides are linear on Y and agree on the basis $e_1, \dots, e_k$; and $\|v_0\| = M$, because $|\psi(y)| = |\langle y, v_0 \rangle| \leq \|v_0\| \|y\|$ on Y with equality at $y = v_0$. Now let $\varphi = \langle \cdot, v \rangle$ be any extension of ψ to $\mathbb{R}^n$ with $\|\varphi\| = M$. Decompose $v = Pv + (v - Pv)$. For $y \in Y$, $\langle y, v \rangle = \varphi(y) = \psi(y) = \langle y, v_0 \rangle$, so $v - v_0 \in Y^\perp$ and hence $Pv = v_0$. By the Pythagorean identity, $M^2 = \|v\|^2 = \|v_0\|^2 + \|v - v_0\|^2 = M^2 + \|v - v_0\|^2$, forcing $v = v_0$. Since $\tilde{\psi} = \langle \cdot, v_0 \rangle$, as one checks from $\tilde{\psi}(x) = \psi(Px) = \langle Px, v_0 \rangle = \langle x, v_0 \rangle$ (the last equality because $x - Px \in Y^\perp$ and $v_0 \in Y$), we conclude $\varphi = \tilde{\psi}$.

Problem 14.20. Let $X = L^p = L^p[a, b]$, $1 < p < \infty$ and m be Lebesgue measure. For $f \neq 0$ in L^p , define

$$\psi(h) = \frac{1}{\|f\|_p^{p-1}} \int_{[a,b]} \operatorname{sgn}(f) \cdot |f|^{p-1} \cdot h \, dm \quad \text{for all } h \in L^p.$$

Use Holder's Inequality to show that $\psi \in (L^p)^*$, $\|\psi\| = 1$ and $\psi(f) = \|f\|_p$.

Solution. Let $q = p/(p-1)$ be the conjugate of p , so that $1 < q < \infty$ and $1/p + 1/q = 1$. Here $\operatorname{sgn}(f)$ is the function equal to 1 where $f > 0$, to -1 where $f < 0$, and to 0 where $f = 0$. Since $f \neq 0$ in L^p , we have $\|f\|_p > 0$, so the normalizing constant $\|f\|_p^{p-1}$ is a positive real number and the formula makes sense. Put

$$g = \operatorname{sgn}(f) \cdot |f|^{p-1},$$

$$\text{so that } \psi(h) = \frac{1}{\|f\|_p^{p-1}} \int_{[a,b]} g h \, dm.$$

Step 1: g belongs to $L^q[a, b]$, with $\|g\|_q = \|f\|_p^{p-1}$. First, g is measurable. Indeed, fix a measurable representative of f . The function $\operatorname{sgn}(f)$ is measurable because it is the simple function $\chi_{\{f>0\}} - \chi_{\{f<0\}}$, and the sets $\{f > 0\}$ and $\{f < 0\}$ are measurable since f is; the function $|f|^{p-1}$ is measurable as the composition of the continuous function $t \mapsto t^{p-1}$ on $[0, \infty)$ with the measurable function $|f|$; and a product of measurable functions is measurable. Moreover g does not depend, up to equality almost everywhere, on the representative chosen: if $f_1 = f_2$ a.e. then $\operatorname{sgn}(f_1)|f_1|^{p-1} = \operatorname{sgn}(f_2)|f_2|^{p-1}$ a.e., so the integral defining $\psi(h)$ below is unchanged. Thus ψ depends only on the class of f in L^p . Next, since $(p-1)q = p$,

$$|g|^q = |\operatorname{sgn}(f)|^q \cdot |f|^{(p-1)q} \leq |f|^p,$$

with equality except on the set where $f = 0$, where both sides vanish; so in fact $|g|^q = |f|^p$ pointwise on $[a, b]$. As $f \in L^p$, $\int_{[a,b]} |f|^p \, dm < \infty$, and therefore

$$\int_{[a,b]} |g|^q \, dm = \int_{[a,b]} |f|^p \, dm = \|f\|_p^p < \infty.$$

Thus $g \in L^q[a, b]$ and

$$\|g\|_q = \left(\int_{[a,b]} |f|^p \, dm \right)^{1/q} = \|f\|_p^{p/q} = \|f\|_p^{p-1},$$

since $p/q = p(p-1)/p = p-1$.

Step 2: ψ is a well defined bounded linear functional on L^p with $\|\psi\| \leq 1$. Let $h \in L^p$. By Holder's Inequality applied to $g \in L^q$ and $h \in L^p$, the product gh is integrable over $[a, b]$ and

$$\left| \int_{[a,b]} g h \, dm \right| \leq \int_{[a,b]} |g h| \, dm \leq \|g\|_q \cdot \|h\|_p = \|f\|_p^{p-1} \cdot \|h\|_p.$$

Hence $\psi(h)$ is a well defined real number, and

$$|\psi(h)| = \frac{1}{\|f\|_p^{p-1}} \left| \int_{[a,b]} g h \, dm \right| \leq \|h\|_p \quad \text{for all } h \in L^p. \quad (*)$$

Furthermore, if $h_1, h_2 \in L^p$ and $\alpha, \beta \in \mathbb{R}$, then gh_1 and gh_2 are integrable, so by the linearity of integration for integrable functions,

$$\psi(\alpha h_1 + \beta h_2) = \frac{1}{\|f\|_p^{p-1}} \int_{[a,b]} (\alpha g h_1 + \beta g h_2) \, dm = \alpha \psi(h_1) + \beta \psi(h_2).$$

(Also, if $h_1 = h_2$ a.e. then $gh_1 = gh_2$ a.e. and the integrals agree, so ψ is well defined on equivalence classes, as it must be to be a functional on L^p .) Therefore ψ is a linear functional, and by $(*)$ it is bounded with

$$\|\psi\| = \sup\{|\psi(h)| \mid h \in L^p, \|h\|_p \leq 1\} \leq 1.$$

In particular $\psi \in (L^p)^*$.

Step 3: $\psi(f) = \|f\|_p$. At each point $x \in [a, b]$ we have $\operatorname{sgn}(f(x)) \cdot f(x) = |f(x)|$ (this holds also where $f(x) = 0$, both sides being 0), and therefore

$$g \cdot f = \operatorname{sgn}(f) \cdot |f|^{p-1} \cdot f = |f|^{p-1} \cdot |f| = |f|^p.$$

Consequently

$$\psi(f) = \frac{1}{\|f\|_p^{p-1}} \int_{[a,b]} |f|^p dm = \frac{\|f\|_p^p}{\|f\|_p^{p-1}} = \|f\|_p.$$

Step 4: $\|\psi\| = 1$. Since $f \neq 0$, we have $\|f\|_p > 0$, and the function $h_0 = f/\|f\|_p$ belongs to L^p with $\|h_0\|_p = 1$. By Step 3 and the linearity of ψ ,

$$\psi(h_0) = \frac{\psi(f)}{\|f\|_p} = \frac{\|f\|_p}{\|f\|_p} = 1.$$

Hence $\|\psi\| \geq |\psi(h_0)| = 1$. Combined with the estimate $\|\psi\| \leq 1$ of Step 2, this gives $\|\psi\| = 1$.

Thus ψ is a unit functional in $(L^p)^*$ that attains its norm at f , since $|\psi(f)| = \|f\|_p = \|\psi\| \cdot \|f\|_p$. This exhibits explicitly, for the spaces $L^p[a, b]$ with $1 < p < \infty$, the functional whose existence is asserted in general by Theorem 7 as a consequence of the Hahn-Banach Theorem: no Zorn's Lemma is needed, only Holder's Inequality and the observation that equality holds in Holder's Inequality for the pair $|f|^{p-1}, |f|$.

Problem 14.21. For each point x in a normed linear space X , show that

$$\|x\| = \sup \{ \psi(x) \mid \psi \in X^*, \|\psi\| \leq 1 \}.$$

Solution. Denote by B^* the closed unit ball of the dual space,

$$B^* = \{ \psi \in X^* \mid \|\psi\| \leq 1 \},$$

and set $S(x) = \sup \{ \psi(x) \mid \psi \in B^* \}$. Note that B^* is nonempty, since the zero functional belongs to it; hence the supremum is taken over a nonempty set of real numbers and $S(x) \geq 0$.

The inequality $S(x) \leq \|x\|$. Let $\psi \in B^*$. By the definition of the dual norm,

$$\psi(x) \leq |\psi(x)| \leq \|\psi\| \cdot \|x\| \leq \|x\|.$$

Thus $\|x\|$ is an upper bound for the set $\{ \psi(x) \mid \psi \in B^* \}$, and therefore $S(x) \leq \|x\|$.

The inequality $S(x) \geq \|x\|$. If $x = 0$, then $\psi(x) = 0$ for every $\psi \in X^*$, so $S(0) = 0 = \|0\|$ and there is nothing more to prove. Assume then that $x \neq 0$. According to Theorem 7, a consequence of the Hahn-Banach Theorem, there is a functional $\psi_x \in X^*$ for which

$$\|\psi_x\| = 1 \quad \text{and} \quad \psi_x(x) = \|x\|.$$

Since $\|\psi_x\| = 1 \leq 1$, the functional ψ_x belongs to B^* , and hence

$$S(x) \geq \psi_x(x) = \|x\|.$$

Combining the two inequalities gives $S(x) = \|x\|$, which is the asserted identity. Observe moreover that the supremum is attained, at ψ_x when $x \neq 0$ and at the zero functional when $x = 0$.

For completeness we recall the derivation of Theorem 7 from the Hahn-Banach Theorem, since it is the only nontrivial ingredient above. Let $x \neq 0$ and let $X_0 = \{ \lambda x \mid \lambda \in \mathbb{R} \}$ be the one dimensional subspace of X spanned by x . Define $\psi_0: X_0 \rightarrow \mathbb{R}$ by

$$\psi_0(\lambda x) = \lambda \|x\|,$$

which is properly defined because $x \neq 0$ forces the representation λx of a point of X_0 to have a unique coefficient λ , and which is clearly linear. Take $p(z) = \|z\|$ for $z \in X$; as the text observes, a norm is positively homogeneous and subadditive. For $\lambda \in \mathbb{R}$,

$$\psi_0(\lambda x) = \lambda \|x\| \leq |\lambda| \|x\| = \|\lambda x\| = p(\lambda x),$$

so $\psi_0 \leq p$ on X_0 . By the Hahn-Banach Theorem there is a linear functional $\psi_x: X \rightarrow \mathbb{R}$ extending ψ_0 with $\psi_x \leq p$ on all of X . Applying this to z and to $-z$ gives $\psi_x(z) \leq \|z\|$ and $-\psi_x(z) = \psi_x(-z) \leq \| -z \| = \|z\|$, so

$$|\psi_x(z)| \leq \|z\| \quad \text{for all } z \in X.$$

Hence ψ_x is bounded with $\|\psi_x\| \leq 1$. On the other hand $\psi_x(x) = \psi_0(x) = \|x\| > 0$, so $\|\psi_x\| \geq \psi_x(x)/\|x\| = 1$. Therefore $\|\psi_x\| = 1$ and $\psi_x(x) = \|x\|$, as required.

Two remarks. First, since $\psi \in B^*$ implies $-\psi \in B^*$ and $(-\psi)(x) = -\psi(x)$, the set $\{\psi(x) \mid \psi \in B^*\}$ is symmetric about 0, so the same value is obtained if $\psi(x)$ is replaced by $|\psi(x)|$:

$$\|x\| = \sup \{ |\psi(x)| \mid \psi \in X^*, \|\psi\| \leq 1 \}.$$

Second, this identity is precisely the statement that the natural embedding $J: X \rightarrow X^{**}$, defined by $J(x)(\psi) = \psi(x)$, is an isometry, since the right-hand side above is exactly $\|J(x)\|$; that is Corollary 9. It is the dual counterpart of the definition $\|\psi\| = \sup\{|\psi(x)| : \|x\| \leq 1\}$, and unlike that definition it is not a tautology: it depends on the Hahn-Banach Theorem for the existence of enough continuous linear functionals on X .

14.4 Problems 14.22–14.28

Problem 14.22. Let X be a normed linear space and Y a closed subspace of X . Show that for each $x_0 \in X \sim Y$, there is a $\psi \in X^*$ such that

$$\|\psi\| = 1, \quad \psi = 0 \text{ on } Y, \quad \text{and} \quad \psi(x_0) = d,$$

where $d = \text{dist}(x_0, Y) = \inf\{\|x_0 - y\| \mid y \in Y\}$.

Solution. Since Y is closed and $x_0 \notin Y$, the complement $X \sim Y$ is open, so there is an $r > 0$ with $\|x_0 - y\| \geq r$ for all $y \in Y$. Hence

$$d = \text{dist}(x_0, Y) \geq r > 0.$$

Step 1: a functional on $Z = Y \oplus [x_0]$. Let

$$Z = \{y + \lambda x_0 \mid y \in Y, \lambda \in \mathbb{R}\} = \text{span}[Y + x_0],$$

which is a linear subspace of X containing Y . The representation of a vector of Z in the form $y + \lambda x_0$ is unique: if $y_1 + \lambda_1 x_0 = y_2 + \lambda_2 x_0$ with $\lambda_1 \neq \lambda_2$, then

$$x_0 = \frac{y_2 - y_1}{\lambda_1 - \lambda_2} \in Y,$$

contradicting $x_0 \in X \sim Y$; therefore $\lambda_1 = \lambda_2$ and then $y_1 = y_2$. Consequently

$$\psi_0(y + \lambda x_0) = \lambda d \quad \text{for } y \in Y, \lambda \in \mathbb{R}$$

is a properly defined function on Z , and it is clearly linear. By construction $\psi_0 = 0$ on Y (take $\lambda = 0$) and $\psi_0(x_0) = d$ (take $y = 0, \lambda = 1$).

Step 2: $\|\psi_0\| \leq 1$. Let $y \in Y$ and $\lambda \in \mathbb{R}$. If $\lambda = 0$ then $\psi_0(y) = 0 \leq \|y\|$. If $\lambda \neq 0$, then $-y/\lambda$ belongs to Y , so by the definition of d as an infimum,

$$\|y + \lambda x_0\| = |\lambda| \cdot \left\| x_0 - \left(-\frac{y}{\lambda}\right) \right\| \geq |\lambda| d = |\psi_0(y + \lambda x_0)|.$$

Thus $|\psi_0(z)| \leq \|z\|$ for all $z \in Z$, that is, ψ_0 is bounded with $\|\psi_0\| \leq 1$.

Step 3: $\|\psi_0\| \geq 1$. By the definition of d as an infimum, choose a sequence $\{y_n\}$ in Y with

$$\lim_{n \rightarrow \infty} \|x_0 - y_n\| = d.$$

Put $z_n = x_0 - y_n \in Z$. Then $\psi_0(z_n) = d$ for every n , while $\|z_n\| \rightarrow d > 0$; in particular $z_n \neq 0$ for n large. For such n ,

$$\|\psi_0\| \geq \frac{|\psi_0(z_n)|}{\|z_n\|} = \frac{d}{\|z_n\|}.$$

Letting $n \rightarrow \infty$, the right-hand side tends to $d/d = 1$, so $\|\psi_0\| \geq 1$. Combining with Step 2, $\|\psi_0\| = 1$. Step 4: extension. By Theorem 7, the bounded linear functional ψ_0 on the subspace Z of X has an extension to a bounded linear functional ψ on all of X with the same norm. Then $\psi \in X^*$,

$$\|\psi\| = \|\psi_0\| = 1, \quad \psi = \psi_0 = 0 \text{ on } Y, \quad \psi(x_0) = \psi_0(x_0) = d,$$

which is precisely what was to be shown.

Problem 14.23. Let Y be a linear subspace of a normed linear space X and z be a vector in X . Show that

$$\text{dist}(z, Y) = \sup\{\psi(z) \mid \|\psi\| = 1, \psi = 0 \text{ on } Y\}.$$

Solution. Write $d = \text{dist}(z, Y) = \inf\{\|z - y\| \mid y \in Y\}$ and let

$$S = \{\psi \in X^* \mid \|\psi\| = 1, \psi = 0 \text{ on } Y\}.$$

Observe first that Y is not assumed closed; we relate the problem to $\bar{Y}$, which is a closed linear subspace of X (the closure of a linear subspace is a linear subspace).

Step 1: $\text{dist}(z, Y) = \text{dist}(z, \bar{Y})$. Since $Y \subseteq \bar{Y}$, the infimum over $\bar{Y}$ is no larger, so $\text{dist}(z, \bar{Y}) \leq d$. Conversely, let $u \in \bar{Y}$ and choose $y_n \in Y$ with $y_n \rightarrow u$. Then $\|z - y_n\| \rightarrow \|z - u\|$, and since $\|z - y_n\| \geq d$ for every n we get $\|z - u\| \geq d$. Taking the infimum over $u \in \bar{Y}$ gives $\text{dist}(z, \bar{Y}) \geq d$. Hence the two distances agree.

Step 2: the supremum is no greater than d . Let $\psi \in S$ and let $y \in Y$. Since $\psi(y) = 0$ and $\|\psi\| = 1$,

$$\psi(z) = \psi(z) - \psi(y) = \psi(z - y) \leq |\psi(z - y)| \leq \|\psi\| \cdot \|z - y\| = \|z - y\|.$$

Taking the infimum over $y \in Y$ yields $\psi(z) \leq d$. Hence $\sup\{\psi(z) \mid \psi \in S\} \leq d$.

Step 3: the supremum is attained when $z \notin \bar{Y}$. Suppose $z \in X \sim \bar{Y}$. Apply the preceding problem (Problem 22) to the closed subspace $\bar{Y}$ and the vector $z \notin \bar{Y}$: there is a $\psi \in X^*$ with

$$\|\psi\| = 1, \quad \psi = 0 \text{ on } \bar{Y}, \quad \psi(z) = \text{dist}(z, \bar{Y}).$$

Since $Y \subseteq \bar{Y}$, this ψ vanishes on Y , so $\psi \in S$; and by Step 1, $\psi(z) = d$. Together with Step 2 this shows the supremum equals d and is in fact attained.

Step 4: the case $z \in \bar{Y}$. Here $d = \text{dist}(z, \bar{Y}) = 0$ by Step 1. Every $\psi \in S$ is continuous and vanishes on Y , hence vanishes on $\bar{Y}$ and so $\psi(z) = 0$. Thus every element of the set on the right-hand side equals 0.

If $\bar{Y} \neq X$, the set S is nonempty (pick any $x_0 \in X \sim \bar{Y}$ and apply Problem 22 to $\bar{Y}$), so the supremum is a supremum of a nonempty set of zeros, namely $0 = d$, as required.

If $\bar{Y} = X$, then $d = 0$ for every $z \in X$, while $S = \emptyset$. Indeed, the linear span of the subspace Y is Y itself, so by Corollary 11 a functional $\psi \in X^*$ that vanishes on Y vanishes identically, hence has norm 0 and cannot belong to S . (One sees this directly as well: ψ is continuous and vanishes on a dense set, so $\psi \equiv 0$.) This single degenerate case – Y dense in X , which includes $Y = X$ – is the one place where the asserted identity requires a convention: the right-hand side is the supremum of the empty set, and the equality holds precisely under the convention $\sup \emptyset = 0$ appropriate to a set of distances. Every author who states this problem intends Y to be a proper subspace whose closure is not all of X , and in that case, by Steps 3 and 4, the identity holds literally and the supremum is attained whenever $z \notin \bar{Y}$.

Hence in all cases

$$\text{dist}(z, Y) = \sup\{\psi(z) \mid \|\psi\| = 1, \psi = 0 \text{ on } Y\},$$

the supremum being attained whenever $z \notin \bar{Y}$.

Finally, note that replacing ψ by $-\psi$ leaves S invariant, so the same formula holds with $|\psi(z)|$ in place of $\psi(z)$.

Problem 14.24. Let X be a vector space. A subset C of X is called a **cone** provided $x + y \in C$ and $\lambda x \in C$ whenever x, y belong to C and $\lambda > 0$. Define a partial order in X by defining $x \preceq y$ to mean $y - x \in C$. A linear functional f on X is said to be **positive** (with respect to the cone C) provided $f \geq 0$ on C . Let Y be any subspace of X with the property that for each $x \in X$ there is a $y \in Y$ with $x \preceq y$. Show that each positive linear functional on Y may be extended to a positive linear functional on X . [Hint: Adapt the Hahn-Banach Lemma and use Zorn's Lemma with respect to the relation $\preceq$ to find a maximal extension.]

Solution. The only properties of C we use are the two that define a cone: C is closed under addition and under multiplication by strictly positive scalars. (In particular nothing below requires $0 \in C$, so the argument is insensitive to whether one reads $\preceq$ as a reflexive order or as the strict relation $y - x \in C$.) We record two consequences used repeatedly:

- (i) transitivity: if $u \preceq v$ and $v \preceq w$, then $v - u \in C$ and $w - v \in C$, so $w - u = (w - v) + (v - u) \in C$, that is, $u \preceq w$;
- (ii) monotonicity: if Z is a subspace of X and g is a linear functional on Z with $g \geq 0$ on $C \cap Z$, and if $u, v \in Z$ satisfy $u \preceq v$, then $v - u \in C \cap Z$, so $g(v) - g(u) = g(v - u) \geq 0$.

Throughout, “positive linear functional on a subspace Z ” means a linear functional g on Z with $g \geq 0$ on $C \cap Z$.

Step 1: the one-step extension (the analogue of the Hahn-Banach Lemma). Let Z be a subspace of X with $Y \subseteq Z$, let g be a positive linear functional on Z , and let $w \in X \sim Z$. We claim g extends to a positive linear functional on $\text{span}[Z + w]$.

As in the proof of the Hahn-Banach Lemma, every vector of $\text{span}[Z + w]$ is written uniquely as $u + \lambda w$ with $u \in Z$ and $\lambda \in \mathbb{R}$ (uniqueness because $w \notin Z$). Hence, for each real number α , the formula $\tilde{g}(u + \lambda w) = g(u) + \lambda \alpha$ properly defines a linear functional $\tilde{g}$ on $\text{span}[Z + w]$ that agrees with g on Z (take $\lambda = 0$). So it suffices to choose α so that $\tilde{g}$ is nonnegative on $C \cap \text{span}[Z + w]$; that is, so that

$$g(u) + \lambda \alpha \geq 0 \quad \text{whenever } u \in Z, \lambda \in \mathbb{R}, u + \lambda w \in C. \quad (*)$$

Introduce the two sets of “lower” and “upper” elements of Z for w :

$$A = \{u \in Z \mid u \preceq w\}, \quad B = \{v \in Z \mid w \preceq v\}.$$

Both are nonempty. Indeed, by hypothesis applied to the vector $w \in X$ there is $v \in Y \subseteq Z$ with $w \preceq v$, so $v \in B$; and applying the hypothesis to the vector $-w$ there is $y_0 \in Y \subseteq Z$ with $-w \preceq y_0$, that is, $y_0 + w \in C$, that is, $-y_0 \preceq w$, so $-y_0 \in A$.

Moreover $g(u) \leq g(v)$ for all $u \in A, v \in B$: by (i), $u \preceq w \preceq v$ gives $u \preceq v$ with $u, v \in Z$, and (ii) gives $g(u) \leq g(v)$. Hence

$$-\infty < \sup_{u \in A} g(u) \leq \inf_{v \in B} g(v) < +\infty,$$

both being real numbers since A and B are nonempty. Choose any real α with

$$\sup_{u \in A} g(u) \leq \alpha \leq \inf_{v \in B} g(v). \quad (**)$$

We verify (*) for this α , distinguishing three cases for $u + \lambda w \in C$.

If $\lambda = 0$: then $u \in C \cap Z$, so $g(u) \geq 0$ since g is positive on Z .

If $\lambda > 0$: then $\lambda^{-1}(u + \lambda w) = w - (-u/\lambda) \in C$ because C is closed under multiplication by positive scalars. Thus $-u/\lambda \in A$, so by (**), $g(-u/\lambda) \leq \alpha$, i.e. $-g(u)/\lambda \leq \alpha$, i.e. $g(u) + \lambda \alpha \geq 0$.

If $\lambda < 0$: write $\lambda = -\mu$ with $\mu > 0$. Then $\mu^{-1}(u - \mu w) = (u/\mu) - w \in C$, so $u/\mu \in B$, whence by (**), $\alpha \leq g(u/\mu) = g(u)/\mu$, i.e. $g(u) - \mu \alpha \geq 0$, i.e. $g(u) + \lambda \alpha \geq 0$.

This proves (*). Thus $\tilde{g}$ is linear on $\text{span}[Z + w]$, agrees with g on Z , and is nonnegative on $C \cap \text{span}[Z + w]$; that is, $\tilde{g}$ is a positive linear functional extending g . Step 1 is proved.

Step 2: Zorn's Lemma. Let f be a positive linear functional on Y . Consider the family $\mathcal{F}$ of all linear functionals η defined on a subspace Z_η of X for which

$$Y \subseteq Z_\eta, \quad \eta = f \text{ on } Y, \quad \eta \geq 0 \text{ on } C \cap Z_\eta.$$

Then $\mathcal{F}$ is nonempty (f itself belongs to it) and is partially ordered by defining $\eta_1 \sqsubseteq \eta_2$ provided $Z_{\eta_1} \subseteq Z_{\eta_2}$ and $\eta_1 = \eta_2$ on Z_{η_1} . (This order on $\mathcal{F}$ is unrelated to the order $\preceq$ on X ; we use a different symbol to keep them apart.)

Let $\mathcal{F}_0$ be a totally ordered subfamily of $\mathcal{F}$. If $\mathcal{F}_0$ is empty, then f itself is an upper bound for it, so assume $\mathcal{F}_0 \neq \emptyset$, and let Z be the union of the domains of the functionals in $\mathcal{F}_0$. Since $\mathcal{F}_0$ is totally ordered, any two of these domains are contained in one of them, and therefore, since each domain is a linear subspace of X , so is Z . For $u \in Z$ choose $\eta \in \mathcal{F}_0$ with $u \in Z_\eta$ and define $\eta^*(u) = \eta(u)$; by the total ordering of $\mathcal{F}_0$ this is unambiguous, and η^* is a properly defined linear functional on Z . Since each functional in $\mathcal{F}_0$ contains Y in its domain, agrees with f on Y , and is nonnegative on the part of C in its domain, the same three properties hold for η^* : if $u \in C \cap Z$ then $u \in C \cap Z_\eta$ for some $\eta \in \mathcal{F}_0$ and $\eta^*(u) = \eta(u) \geq 0$. Hence $\eta^* \in \mathcal{F}$ and $\eta \sqsubseteq \eta^*$ for all $\eta \in \mathcal{F}_0$, so every totally ordered subfamily of $\mathcal{F}$ has an upper bound.

By Zorn's Lemma, $\mathcal{F}$ has a maximal member g_0 , with domain Z_0 . Suppose $Z_0 \neq X$ and choose $w \in X \sim Z_0$. Since $Y \subseteq Z_0$, the hypothesis on Y holds a fortiori for Z_0 : for each $x \in X$ there is $y \in Y \subseteq Z_0$ with $x \preceq y$. Therefore Step 1 applies with $Z = Z_0$ and produces a member of $\mathcal{F}$ that properly extends g_0 (its domain $\text{span}[Z_0 + w]$ properly contains Z_0 , since w belongs to it and not to Z_0). This contradicts the maximality of g_0 .

Hence $Z_0 = X$, and g_0 is a linear functional on all of X with $g_0 = f$ on Y and $g_0 \geq 0$ on C . That is, g_0 is a positive linear functional on X extending f .

Problem 14.25. Let X_0 be a subset of a metric space X . Use the Tietze Extension Theorem to show that every continuous real-valued function on X_0 has a continuous extension to all of X if and only if X_0 is closed. Does this contradict Theorem 7?

Solution. Denote the metric on X by ρ . By Proposition 7 of Chapter 11, every metric space is normal, so both Urysohn's Lemma and the Tietze Extension Theorem are available on X . Recall the form in which the book states the latter: if F is a closed subset of a normal topological space X and f is a continuous real-valued function on F taking values in a closed, bounded interval $[a, b]$, then f has a continuous extension to all of X that also takes values in $[a, b]$. The hypothesis of boundedness is the reason for the change of variable made below.

Sufficiency: if X_0 is closed, every continuous real-valued function on X_0 extends. Let $f: X_0 \rightarrow \mathbb{R}$ be continuous. Define the homeomorphism

$$\varphi: \mathbb{R} \rightarrow (-1, 1), \quad \varphi(t) = \frac{t}{1 + |t|},$$

whose inverse is $\varphi^{-1}(s) = s/(1 - |s|)$ on $(-1, 1)$. Then $g = \varphi \circ f$ is a continuous real-valued function on the closed set X_0 taking values in $[-1, 1]$ (indeed in $(-1, 1)$). By the Tietze Extension Theorem, g has a continuous extension $G: X \rightarrow [-1, 1]$.

The extension G may take the values ± 1 at points outside X_0 , so we correct it. Let

$$E = G^{-1}(\{-1, 1\}) = \{x \in X \mid |G(x)| = 1\},$$

a closed subset of X by continuity of $|G|$. Since $|G| = |g| < 1$ on X_0 , the sets E and X_0 are disjoint closed subsets of the normal space X . By Urysohn's Lemma there is a continuous $h: X \rightarrow [0, 1]$ with $h = 0$ on E and $h = 1$ on X_0 . Put $H = h \cdot G$, a continuous real-valued function on X . If $x \in E$, then $H(x) = 0$; if $x \notin E$, then $|H(x)| \leq |G(x)| < 1$. Thus H maps X into $(-1, 1)$, and $H = G = g$ on X_0 since $h = 1$ there. Therefore

$$F = \varphi^{-1} \circ H: X \rightarrow \mathbb{R}$$

is continuous and, on X_0 , $F = \varphi^{-1} \circ g = \varphi^{-1} \circ \varphi \circ f = f$. So F is a continuous extension of f to all of X .

Necessity: if every continuous real-valued function on X_0 extends, then X_0 is closed. Argue by contraposition. Suppose X_0 is not closed and choose $x^* \in \overline{X_0} \sim X_0$. Since $x^* \notin X_0$, we have $\rho(x, x^*) > 0$ for every $x \in X_0$, so

$$f(x) = \frac{1}{\rho(x, x^*)} \quad \text{for } x \in X_0$$

is a properly defined real-valued function on X_0 ; it is continuous, being the composition of the continuous function $x \mapsto \rho(x, x^*)$, which is nonvanishing on X_0 , with the function $t \mapsto 1/t$, continuous on $(0, \infty)$.

Suppose f had a continuous extension F to X . Since $x^* \in \overline{X_0}$, for each natural number n there is $x_n \in X_0$ with $\rho(x_n, x^*) < 1/n$; thus $x_n \rightarrow x^*$. By continuity of F at x^* , the sequence $\{F(x_n)\}$ converges to the real number $F(x^*)$ and is therefore bounded. But

$$F(x_n) = f(x_n) = \frac{1}{\rho(x_n, x^*)} > n \quad \text{for every } n,$$

so $\{F(x_n)\}$ is unbounded. This contradiction shows f has no continuous extension. Hence if every continuous real-valued function on X_0 extends continuously to X , then X_0 must be closed.

Does this contradict Theorem 7? No. The two statements are about different classes of functions, and the discrepancy is exactly the difference between arbitrary continuous functions and bounded linear ones.

Theorem 7 asserts that a **bounded linear** functional ψ on a **linear subspace** X_0 of a normed linear space X extends to a bounded linear functional on all of X of the same norm, and it imposes no closedness assumption on X_0 . There is no conflict, because a bounded linear functional never exhibits the pathology used above: if $\|\psi\| = M$, then for $u, v \in X_0$,

$$|\psi(u) - \psi(v)| = |\psi(u - v)| \leq M\|u - v\|,$$

so ψ is Lipschitz, hence uniformly continuous, hence remains bounded near any point of $\overline{X_0}$. Indeed such a ψ automatically extends continuously (and linearly, with the same norm) to $\overline{X_0}$ by taking limits along sequences, and Theorem 7 then carries it to all of X .

The result of this problem says that when X_0 is not closed there exists **some** continuous function on X_0 with no continuous extension; the function we produced, $x \mapsto 1/\rho(x, x^*)$, is unbounded near x^* and is not a bounded linear functional. Theorem 7 makes no claim about such functions. So the two results are entirely compatible.

Problem 14.26. Let (X, ρ) be a metric space that contains the closed set F . Show that a point $x \in X$ belongs to F if and only if every continuous functional on X that vanishes on F also vanishes at x . Can this be used to prove Theorem 10?

Solution. Here “continuous functional on X ” means a continuous real-valued function on X ; no linear structure on X is assumed.

Necessity. Suppose $x \in F$. If $f: X \rightarrow \mathbb{R}$ is continuous and $f = 0$ on F , then in particular $f(x) = 0$, since x is a point of F . (Continuity is not even needed here.)

Sufficiency. We prove the contrapositive: if $x \notin F$, we exhibit a continuous real-valued function on X that vanishes on F but not at x .

If $F = \emptyset$, take $f \equiv 1$. It is continuous, it vanishes on F vacuously, and $f(x) = 1 \neq 0$.

Assume $F \neq \emptyset$ and define the distance to F ,

$$f(u) = \rho(u, F) = \inf\{\rho(u, z) \mid z \in F\} \quad \text{for } u \in X.$$

This function is continuous; in fact it is Lipschitz with constant 1. Indeed, for $u, v \in X$ and any $z \in F$, the triangle inequality gives $\rho(u, z) \leq \rho(u, v) + \rho(v, z)$, so $f(u) \leq \rho(u, v) + \rho(v, z)$; taking the infimum over $z \in F$ gives $f(u) \leq \rho(u, v) + f(v)$, that is, $f(u) - f(v) \leq \rho(u, v)$. Interchanging u and v yields $|f(u) - f(v)| \leq \rho(u, v)$.

Clearly $f = 0$ on F , since $\rho(z, z) = 0$ for $z \in F$.

Finally $f(x) > 0$. Otherwise $\rho(x, F) = 0$, so for each natural number n there is $z_n \in F$ with $\rho(x, z_n) < 1/n$; then $z_n \rightarrow x$, and since F is closed, $x \in F$, contrary to assumption.

Thus f is a continuous functional on X vanishing on F with $f(x) \neq 0$, which completes the contrapositive and hence the proof of the equivalence.

Can this be used to prove Theorem 10? No.

Theorem 10 concerns a normed linear space X and a linear subspace X_0 , and it asserts that x belongs to $\overline{X_0}$ if and only if every ψ in the **dual space** X^* that vanishes on X_0 also vanishes at x . The elements of X^* are the continuous **linear** functionals, and the substance of Theorem 10 lies entirely in producing, for $x \notin \overline{X_0}$, a continuous functional that is linear and separates x from X_0 . The present problem produces only a continuous function. Applied to the closed set $F = \overline{X_0}$ in the metric space determined by the norm, it yields

$$f(u) = \text{dist}(u, \overline{X_0}) = \text{dist}(u, X_0),$$

which is continuous, vanishes exactly on $\overline{X_0}$, and is nonzero at x – but it is not linear. It cannot be: $f \geq 0$ on X , whereas a linear functional that is nonnegative everywhere on a linear space is identically zero (if $g(u) > 0$ then $g(-u) = -g(u) < 0$). So this argument supplies no member of X^* and proves nothing about X^* .

The linear separation in Theorem 10 rests on the Hahn-Banach Theorem, by way of Theorem 7, and cannot be obtained from the metric structure alone. That is more than a defect of the particular function chosen above: the present problem is a statement about the metric space (X, ρ) and nothing else, and its proof uses only ρ . Consequently it holds verbatim in any metric space, in particular in a metric linear space whose topology comes from a metric that is not induced by a norm. In such a space the conclusion of Theorem 10 can fail while the conclusion of this problem persists: if V is a metric linear space whose only continuous linear functional is the zero functional – the spaces $L^p[0, 1]$ with $0 < p < 1$ are the standard examples, a fact lying outside this book – then taking $X_0 = \{0\}$ and any $x \neq 0$, every continuous linear functional on V vanishes both on X_0 and at x , yet $x \notin \overline{X_0} = \{0\}$. So the answer stands: this problem yields a continuous function, never a continuous linear one, and only the latter proves Theorem 10. What the problem does supply is the easy half of Theorem 10, the observation that a continuous functional vanishing on X_0 vanishes on $\overline{X_0}$; the substantive half needs Theorem 7.

Problem 14.27. Let $[a, b]$ be a closed, bounded interval of real numbers and consider $L^\infty[a, b]$, now formally considered as the collection of equivalence classes for the relation of pointwise equality almost everywhere among essentially bounded functions. Let X be the subspace of $L^\infty[a, b]$ comprising those equivalence classes that contain a continuous function. Show that such an equivalence class contains exactly one continuous function. Thus X is linearly isomorphic to $C[a, b]$ and therefore, modulo this identification, we may consider $C[a, b]$ to be a linear subspace of $L^\infty[a, b]$. Show that $C[a, b]$ is a closed subspace of the Banach space $L^\infty[a, b]$.

Solution. Throughout, $a < b$, m denotes Lebesgue measure, and for a measurable function f we write $[f]$ for its equivalence class under equality almost everywhere on $[a, b]$. Recall that the norm on $L^\infty[a, b]$ is the essential supremum,

$$\|[f]\|_\infty = \text{ess sup}_{[a, b]} |f| = \inf\{M \geq 0 \mid |f| \leq M \text{ a.e. on } [a, b]\},$$

while $C[a, b]$ carries the maximum norm $\|f\|_{\max} = \max_{x \in [a, b]} |f(x)|$, the maximum being attained since a continuous function on a closed, bounded interval attains its maximum. Every continuous function on $[a, b]$ is bounded, hence essentially bounded, so $[f] \in L^\infty[a, b]$ for $f \in C[a, b]$.

Step 1: each class contains at most one continuous function. Suppose f and g are continuous on $[a, b]$ and $f = g$ a.e. on $[a, b]$. Put $h = f - g$, a continuous function that vanishes almost everywhere. Suppose $h(x_0) \neq 0$ for some $x_0 \in [a, b]$. By continuity of h at x_0 there is a $\delta > 0$ such that

$$|h(x)| > \frac{1}{2}|h(x_0)| > 0 \quad \text{for all } x \in I = (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

Since $a < b$ and $x_0 \in [a, b]$, the set I contains an interval of positive length, so $m(I) > 0$. Then $\{x \in [a, b] \mid h(x) \neq 0\} \supseteq I$ has positive measure, contradicting $h = 0$ a.e. Therefore $h \equiv 0$ on $[a, b]$, that is, $f = g$.

Consequently, an equivalence class belonging to X , which by definition contains at least one continuous function, contains exactly one.

Step 2: the identification is a linear isometry. Define

$$\Phi: C[a, b] \rightarrow X, \quad \Phi(f) = [f].$$

It is linear, since $[f + \lambda g] = [f] + \lambda[g]$. It is onto X by the definition of X , and one-to-one by Step 1. So Φ is a linear isomorphism of $C[a, b]$ onto X .

Moreover Φ is an isometry: we claim $\|[f]\|_\infty = \|f\|_{\max}$ for every $f \in C[a, b]$. Since $|f| \leq \|f\|_{\max}$ everywhere, hence a.e., we have $\|[f]\|_\infty \leq \|f\|_{\max}$. Conversely, let $M = \|f\|_{\max}$ and let $\epsilon > 0$. Choose $x_0 \in [a, b]$ with $|f(x_0)| = M$. If $M = 0$ the two norms are both 0; assume $M > 0$. By continuity there is $\delta > 0$ with $|f| > M - \epsilon$ on the set $I = (x_0 - \delta, x_0 + \delta) \cap [a, b]$, and $m(I) > 0$ as in Step 1. Hence the inequality $|f| \leq M - \epsilon$ fails on a set of positive measure, so $\|[f]\|_\infty \geq M - \epsilon$. As $\epsilon > 0$ was arbitrary, $\|[f]\|_\infty \geq M$. Therefore $\|[f]\|_\infty = \|f\|_{\max}$.

Under this identification we regard $C[a, b]$ as the linear subspace X of $L^\infty[a, b]$, and the two norms agree on it.

Step 3: $C[a, b]$ is closed in $L^\infty[a, b]$. Let $\{[f_n]\}$ be a sequence in X , with each $f_n \in C[a, b]$, converging in $L^\infty[a, b]$ to some $[g] \in L^\infty[a, b]$. A convergent sequence in a normed linear space is Cauchy, so given $\epsilon > 0$ there is N with $\|[f_n] - [f_k]\|_\infty < \epsilon$ for all $n, k \geq N$. Since $f_n - f_k$ is continuous, Step 2 gives

$$\|f_n - f_k\|_{\max} = \|[f_n] - [f_k]\|_\infty = \|[f_n] - [f_k]\|_\infty < \epsilon \quad \text{for } n, k \geq N.$$

Thus $\{f_n\}$ is a Cauchy sequence in the maximum norm, that is, it is uniformly Cauchy on $[a, b]$. For each fixed $x \in [a, b]$ the numerical sequence $\{f_n(x)\}$ is then Cauchy, so by the completeness of $\mathbb{R}$ it converges; call the limit $f(x)$. The convergence is uniform: given $\epsilon > 0$, take N as above, and for $n \geq N$ and any x , let $k \rightarrow \infty$ in $|f_n(x) - f_k(x)| < \epsilon$ to get $|f_n(x) - f(x)| \leq \epsilon$, a bound independent of x . Since a uniform limit of continuous functions is continuous, $f \in C[a, b]$.

Finally, uniform convergence means $\|f_n - f\|_{\max} \rightarrow 0$, so by Step 2 again $\|[f_n] - [f]\|_\infty \rightarrow 0$. Hence $[f_n] \rightarrow [f]$ in $L^\infty[a, b]$. Limits in a normed linear space are unique, so $[g] = [f] \in X$.

Therefore X contains the limit of every one of its convergent sequences, that is, $X = C[a, b]$ is a closed linear subspace of the Banach space $L^\infty[a, b]$.

(Equivalently: Step 2 exhibits X as isometrically isomorphic to $C[a, b]$ with the maximum norm, which is complete, so X is a complete subspace of a metric space and hence closed.)

Problem 14.28. Define $\psi: C[a, b] \rightarrow \mathbb{R}$ by $\psi(f) = f(a)$ for all $f \in C[a, b]$. Use Theorem 7 to extend ψ to a continuous linear functional on all of $L^\infty[a, b]$ (see the preceding problem). Show that there is no functional $h \in L^1[a, b]$ for which

$$\psi(f) = \int_a^b h \cdot f \quad \text{for all } f \in L^\infty[a, b].$$

Solution. Assume $a < b$. By the preceding problem, $C[a, b]$ is identified with a linear subspace of $L^\infty[a, b]$, and on this subspace the L^∞ norm coincides with the maximum norm.

Step 1: ψ is a bounded linear functional on the subspace $C[a, b]$. Evaluation at a is plainly linear: $(f + \lambda g)(a) = f(a) + \lambda g(a)$. It is bounded, since for $f \in C[a, b]$,

$$|\psi(f)| = |f(a)| \leq \max_{x \in [a, b]} |f(x)| = \|f\|_{\max} = \|[f]\|_\infty,$$

the last equality by the preceding problem. Hence $\|\psi\| \leq 1$; taking $f \equiv 1$, for which $\psi(f) = 1 = \|f\|_{\max}$, gives $\|\psi\| = 1$.

Step 2: extension. By Theorem 7, the bounded linear functional ψ on the linear subspace $C[a, b]$ of the normed linear space $L^\infty[a, b]$ has an extension to a bounded linear functional on all of $L^\infty[a, b]$ having the same norm. Denote this extension again by ψ ; thus $\psi \in (L^\infty[a, b])^*$, $\|\psi\| = 1$, and

$$\psi([f]) = f(a) \quad \text{for every } f \in C[a, b]. \quad (13)$$

Step 3: no $h \in L^1[a, b]$ represents ψ . Suppose, to get a contradiction, that there is an $h \in L^1[a, b]$ with

$$\psi(f) = \int_a^b h \cdot f \quad \text{for all } f \in L^\infty[a, b]. \quad (14)$$

For each natural number n with $1/n < b - a$, define $f_n: [a, b] \rightarrow \mathbb{R}$ by

$$f_n(x) = \max\{0, 1 - n(x - a)\} = \begin{cases} 1 - n(x - a), & a \leq x \leq a + 1/n, \\ 0, & a + 1/n \leq x \leq b. \end{cases}$$

Each f_n is continuous on $[a, b]$ (it is the maximum of two continuous functions), takes values in $[0, 1]$, and satisfies $f_n(a) = 1$.

By (13) and (14),

$$\int_a^b h \cdot f_n = \psi(f_n) = f_n(a) = 1 \quad \text{for every such } n. \quad (15)$$

On the other hand, consider the pointwise limit of $\{hf_n\}$. Fix $x \in (a, b]$. For all n with $n > 1/(x - a)$ we have $x > a + 1/n$, so $f_n(x) = 0$ and hence $h(x)f_n(x) = 0$. Therefore $hf_n \rightarrow 0$ pointwise on $(a, b]$, that is, almost everywhere on $[a, b]$ (the exceptional set $\{a\}$ has measure zero). Moreover, since $0 \leq f_n \leq 1$,

$$|h \cdot f_n| \leq |h| \quad \text{on } [a, b] \text{ for every } n,$$

and $|h|$ is integrable over $[a, b]$ because $h \in L^1[a, b]$. Each hf_n is measurable, being the product of the measurable function h with the continuous, hence measurable, function f_n . By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_a^b h \cdot f_n = \int_a^b 0 = 0.$$

This contradicts (15), which asserts that this sequence of integrals is constantly 1.

Hence no $h \in L^1[a, b]$ satisfies (14).

Remarks. The contradiction was reached using only continuous functions f_n , so in fact no $h \in L^1[a, b]$ satisfies $\int_a^b hf = f(a)$ even for all $f \in C[a, b]$ alone; the failure has nothing to do with the particular Hahn-Banach extension chosen in Step 2.

This exhibits a bounded linear functional on $L^\infty[a, b]$ that is not represented by integration against an L^1 function. Thus the natural isometric embedding of $L^1[a, b]$ into $(L^\infty[a, b])^*$, which sends h to the functional $f \mapsto \int_a^b hf$, is not onto: $(L^\infty[a, b])^*$ is strictly larger than $L^1[a, b]$. This is why the Riesz Representation Theorem for the dual of L^p is restricted to $1 \leq p < \infty$.

14.5 Problems 14.29–14.35

Problem 14.29. Show that a collection of bounded linear functions is equicontinuous if and only if it is uniformly bounded.

Solution. The equicontinuity of a collection is defined in Section 10.1 for collections of real-valued functions on a metric space, so the bounded linear functions of the statement are the bounded linear real-valued functions on a normed linear space X , that is, the members of X^* . Accordingly, let $\mathcal{F} \subseteq X^*$. (The argument below uses nothing about the target beyond the triangle inequality and homogeneity of the norm, so it applies verbatim to a collection of bounded linear operators from X to a normed linear space Y .) Here uniform boundedness means boundedness in the norm of the dual space, that is,

$$M := \sup\{\|\psi\| \mid \psi \in \mathcal{F}\} < \infty.$$

Recall the definition from Section 10.1: $\mathcal{F}$ is equicontinuous at $x \in X$ provided for each $\epsilon > 0$ there is a $\delta > 0$ such that for every $\psi \in \mathcal{F}$ and every $x' \in X$, if $\|x' - x\| < \delta$ then $|\psi(x') - \psi(x)| < \epsilon$; and $\mathcal{F}$ is equicontinuous provided it is equicontinuous at every point of X .

Suppose first that $\mathcal{F}$ is uniformly bounded, with M as above. Let $x \in X$ and $\epsilon > 0$. Define $\delta = \epsilon/(M+1) > 0$. If $\psi \in \mathcal{F}$ and $\|x' - x\| < \delta$, then by the linearity of ψ and the definition of the norm of a bounded linear functional,

$$|\psi(x') - \psi(x)| = |\psi(x' - x)| \leq \|\psi\| \cdot \|x' - x\| \leq M\delta < \epsilon.$$

Since δ depends on ϵ alone and not on the point x or on the member ψ of $\mathcal{F}$, the collection $\mathcal{F}$ is equicontinuous on X (indeed uniformly equicontinuous).

Conversely, suppose $\mathcal{F}$ is equicontinuous. In particular $\mathcal{F}$ is equicontinuous at the point $x = 0$. Take $\epsilon = 1$ in the definition of equicontinuity at 0: there is a $\delta > 0$ such that

$$\text{if } \|u\| < \delta, \text{ then } |\psi(u)| = |\psi(u) - \psi(0)| < 1 \text{ for all } \psi \in \mathcal{F},$$

where we used $\psi(0) = 0$, which holds by linearity. Now let $x \in X$ have $\|x\| \leq 1$ and let $\psi \in \mathcal{F}$. Set $u = (\delta/2)x$. Then $\|u\| = (\delta/2)\|x\| \leq \delta/2 < \delta$, so $|\psi(u)| < 1$. By homogeneity of ψ ,

$$|\psi(x)| = \frac{2}{\delta} |\psi(u)| < \frac{2}{\delta}.$$

Taking the supremum over all x in the closed unit ball of X gives $\|\psi\| \leq 2/\delta$. Since δ was chosen independently of ψ , we conclude

$$\sup\{\|\psi\| \mid \psi \in \mathcal{F}\} \leq \frac{2}{\delta} < \infty,$$

so $\mathcal{F}$ is uniformly bounded.

Therefore a collection of bounded linear functionals is equicontinuous if and only if it is uniformly bounded. Observe that the second implication used only equicontinuity at the single point 0, while the first delivered uniform equicontinuity; so for collections of linear maps, equicontinuity at one point, equicontinuity at every point, uniform equicontinuity, and uniform boundedness of the norms are all equivalent.

Problem 14.30. Let X be a separable normed linear space. Show that its closed unit sphere $S = \{x \in X \mid \|x\| = 1\}$ also is separable.

Solution. If $X = \{0\}$ then $S = \emptyset$, which is trivially separable. So assume $X \neq \{0\}$; then $S \neq \emptyset$, since for $x \neq 0$ the vector $x/\|x\|$ belongs to S .

Since X is separable, there is a countable set $D = \{d_n\}_{n=1}^{\infty}$ that is dense in X . Discard the zero vector if it occurs and define

$$D' = \left\{ \frac{d}{\|d\|} \mid d \in D, d \neq 0 \right\}.$$

Then D' is a countable subset of S , being the image of a subset of the countable set D under a map into S . We claim D' is dense in S ; since S is a metric space with the metric inherited from the norm, this exhibits a countable dense subset of S and hence proves S separable.

Let $x \in S$ and let $\epsilon > 0$. We may assume $\epsilon < 1$, for if the claim holds for all small ϵ it holds for all ϵ . By the density of D in X , choose $d \in D$ with

$$\|x - d\| < \frac{\epsilon}{3}.$$

First, $d \neq 0$: indeed, by the reverse triangle inequality,

$$|\|d\| - 1| = |\|d\| - \|x\|| \leq \|d - x\| < \frac{\epsilon}{3} < \frac{1}{3},$$

so $\|d\| > 2/3 > 0$. Thus $d/\|d\|$ belongs to D' . Now estimate, using homogeneity of the norm,

$$\left\| \frac{d}{\|d\|} - x \right\| \leq \left\| \frac{d}{\|d\|} - d \right\| + \|d - x\| = \left| \frac{1}{\|d\|} - 1 \right| \cdot \|d\| + \|d - x\| = |1 - \|d\|| + \|d - x\|.$$

By the two displayed estimates, the right-hand side is less than $\epsilon/3 + \epsilon/3 < \epsilon$.

Hence every point of S is within ϵ of a point of D' , for every $\epsilon > 0$. Therefore D' is a countable dense subset of S , and S is separable.

Problem 14.31. Find a compact metric space X for which $C(X)$, normed by the maximum norm, is reflexive.

Solution. Take X to be any nonempty finite metric space, for instance the one-point space $X = \{p\}$, or more generally $X = \{p_1, \dots, p_n\}$ with the discrete metric $\rho(p_i, p_j) = 1$ for $i \neq j$. Such an X is compact: it is a finite metric space, so every open cover has a finite subcover (choose, for each of the finitely many points, one member of the cover containing it). Every real-valued function on X is continuous, since with the discrete metric every subset is open; hence the map

$$f \mapsto (f(p_1), \dots, f(p_n))$$

is a linear isomorphism of $C(X)$ onto $\mathbb{R}^n$, and under the maximum norm on $C(X)$ it is an isometry onto $\mathbb{R}^n$ with the max norm. In particular $C(X)$ is a normed linear space of finite dimension n . We show that every finite-dimensional normed linear space is reflexive; this settles the problem, and for $n = 1$ it says simply that $C(\{p\}) = \mathbb{R}$ is reflexive.

So let Z be a normed linear space with $\dim Z = n < \infty$, and fix a basis $e_1, \dots, e_n$ of Z . For $z = \sum_{k=1}^n a_k e_k$ define $\|z\|_1 = \sum_{k=1}^n |a_k|$; this is a norm on Z . We first check that the two norms are equivalent.

On one hand, by the triangle inequality and homogeneity, with $c_1 = \max_{1 \leq k \leq n} \|e_k\|$,

$$\|z\| \leq \sum_{k=1}^n |a_k| \|e_k\| \leq c_1 \|z\|_1 \quad \text{for all } z \in Z.$$

Consequently $z \mapsto \|z\|$ is continuous on Z with respect to $\|\cdot\|_1$, since $|\|z\| - \|w\|| \leq \|z - w\| \leq c_1 \|z - w\|_1$. Identify Z with $\mathbb{R}^n$ through the coordinate map $z \mapsto (a_1, \dots, a_n)$; under this identification $\|\cdot\|_1$ is the usual ℓ^1 norm on $\mathbb{R}^n$, and the set

$$\Sigma = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_k |a_k| = 1\}$$

is closed and bounded in $\mathbb{R}^n$, hence compact by the Heine-Borel Theorem in $\mathbb{R}^n$. (The elementary inequalities $\max_k |a_k| \leq \sum_k |a_k| \leq n \max_k |a_k|$ show that the ℓ^1 metric and the Euclidean metric on $\mathbb{R}^n$ induce the same open sets, so it is unambiguous which topology is meant in the two preceding assertions.) The function $z \mapsto \|z\|$ is continuous on Σ with respect to this topology and strictly positive there (if $\|z\| = 0$ then $z = 0$, which is excluded since $\|z\|_1 = 1$). By the Extreme Value Theorem it attains a minimum $c_0 > 0$ on Σ . By homogeneity,

$$\|z\| \geq c_0 \|z\|_1 \quad \text{for all } z \in Z.$$

Now let ψ be any linear functional on Z (no continuity assumed). Then for $z = \sum_k a_k e_k$, with $c_2 = \max_{1 \leq k \leq n} |\psi(e_k)|$,

$$|\psi(z)| \leq \sum_{k=1}^n |a_k| |\psi(e_k)| \leq c_2 \|z\|_1 \leq \frac{c_2}{c_0} \|z\|.$$

So ψ is bounded. Therefore Z^* coincides with the algebraic dual of Z , and by linear algebra $\dim Z^* = n$. Applying the same reasoning to the finite-dimensional normed linear space Z^* , we get $\dim Z^{**} = n$ as well.

Finally, let $J: Z \rightarrow Z^{**}$ be the natural embedding, $J(z)[\psi] = \psi(z)$. It is linear, and it is an isometry: this is the content of Problem 21 of this chapter, which follows from Theorem 7 (a consequence of the Hahn-Banach Theorem), namely that for each $z \in Z$ there is $\psi \in Z^*$ with $\|\psi\| = 1$ and $\psi(z) = \|z\|$, so that

$$\|J(z)\| = \sup\{|\psi(z)| \mid \psi \in Z^*, \|\psi\| \leq 1\} = \|z\|.$$

In particular J is injective. Since J is an injective linear map between the vector spaces Z and Z^{**} , both of dimension $n < \infty$, it is onto: $\dim J(Z) = \dim Z = n = \dim Z^{**}$, and a subspace of Z^{**} of full dimension is all of Z^{**} . Thus J is an isomorphism of Z onto Z^{**} , that is, Z is reflexive.

Applying this with $Z = C(X)$ for X finite, we conclude that $C(X)$ is reflexive. In particular, for the one-point compact metric space $X = \{p\}$, $C(X)$ is isometrically $\mathbb{R}$ and is reflexive.

This is sharp: the remark preceding these problems records that for a compact Hausdorff space K , $C(K)$ with the maximum norm is reflexive if and only if K is a finite set (Problem 11 of Chapter 15). Consistently with this, Proposition 19 shows $C[a, b]$ is not reflexive.

Problem 14.32. Let c_0 be the subspace of ℓ^∞ consisting of sequences that converge to 0. Show that c_0 is a closed subspace of ℓ^∞ whose dual space is isomorphic to ℓ^1 . Conclude that c_0 is not reflexive and therefore neither is ℓ^∞ .

Solution. Throughout, $x = (x_1, x_2, \dots)$ denotes a real sequence, $\|x\|_\infty = \sup_k |x_k|$, $\|a\|_1 = \sum_k |a_k|$, and e_k is the sequence with 1 in the k th place and 0 elsewhere.

Step 1: c_0 is a closed linear subspace of ℓ^∞ . A sequence converging to 0 is bounded, so $c_0 \subseteq \ell^\infty$, and c_0 is clearly closed under addition and scalar multiplication, hence a linear subspace. To see it is closed, let $\{x^{(n)}\}$ be a sequence in c_0 converging in the ℓ^∞ norm to $x \in \ell^\infty$. Let $\epsilon > 0$. Choose n with $\|x^{(n)} - x\|_\infty < \epsilon/2$. Since $x^{(n)} \in c_0$, choose K so that $|x_k^{(n)}| < \epsilon/2$ for $k \geq K$. Then for $k \geq K$,

$$|x_k| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $x_k \rightarrow 0$, that is, $x \in c_0$. So c_0 is closed in ℓ^∞ ; since ℓ^∞ is a Banach space, c_0 is itself a Banach space.

Step 2: $(c_0)^*$ is isometrically isomorphic to ℓ^1 . For $a \in \ell^1$ define $T(a): c_0 \rightarrow \mathbb{R}$ by

$$T(a)[x] = \sum_{k=1}^{\infty} a_k x_k \quad \text{for all } x \in c_0.$$

The series converges absolutely, since $\sum_k |a_k x_k| \leq \|x\|_\infty \sum_k |a_k| = \|a\|_1 \|x\|_\infty$. Thus $T(a)$ is a well-defined linear functional on c_0 with $\|T(a)\| \leq \|a\|_1$, and $a \mapsto T(a)$ is linear from ℓ^1 into $(c_0)^*$.

T is an isometry. Fix $a \in \ell^1$ and $n \in \mathbb{N}$, and put $x = \sum_{k=1}^n \operatorname{sgn}(a_k) e_k$, where $\operatorname{sgn}(t)$ is 1, 0, or -1 according as $t > 0$, $t = 0$, or $t < 0$. Then x has only finitely many nonzero terms, so $x \in c_0$, and $\|x\|_\infty \leq 1$. Since $T(a)[x] = \sum_{k=1}^n |a_k|$,

$$\sum_{k=1}^n |a_k| \leq \|T(a)\|.$$

Letting $n \rightarrow \infty$ gives $\|a\|_1 \leq \|T(a)\|$, and with the reverse inequality above, $\|T(a)\| = \|a\|_1$.

T is onto. Let $\psi \in (c_0)^*$ and set $a_k = \psi(e_k)$. Taking $x = \sum_{k=1}^n \operatorname{sgn}(a_k) e_k$ as above, which lies in the closed unit ball of c_0 , we get

$$\sum_{k=1}^n |a_k| = \psi(x) \leq \|\psi\| \quad \text{for every } n,$$

so $a = (a_k) \in \ell^1$ with $\|a\|_1 \leq \|\psi\|$. Now let $x \in c_0$ and put $x^{(n)} = \sum_{k=1}^n x_k e_k$. Then

$$\|x - x^{(n)}\|_\infty = \sup_{k > n} |x_k| \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

precisely because $x_k \rightarrow 0$. Since ψ is linear and continuous,

$$\psi(x) = \lim_{n \rightarrow \infty} \psi(x^{(n)}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k a_k = \sum_{k=1}^{\infty} a_k x_k = T(a)[x].$$

Thus $\psi = T(a)$, and $T: \ell^1 \rightarrow (c_0)^*$ is a linear isometry onto $(c_0)^*$, that is, an isometric isomorphism.

Step 3: c_0 is not reflexive. Let $J: c_0 \rightarrow (c_0)^{**}$ be the natural embedding. Define $S: (c_0)^* \rightarrow \mathbb{R}$ by

$$S(\psi) = \sum_{k=1}^{\infty} \psi(e_k).$$

By Step 2, $\psi = T(a)$ with $a_k = \psi(e_k)$ and $a \in \ell^1$, so the series converges absolutely and $|S(\psi)| \leq \|a\|_1 = \|\psi\|$. Since $\psi \mapsto (\psi(e_k))_k$ is linear, S is a linear functional on $(c_0)^*$ with $\|S\| \leq 1$; thus $S \in (c_0)^{**}$. Suppose $S = J(x)$ for some $x \in c_0$. Then for every $\psi \in (c_0)^*$,

$$\sum_{k=1}^{\infty} \psi(e_k) = S(\psi) = J(x)[\psi] = \psi(x).$$

Apply this with $\psi = T(e_n)$, where e_n is now regarded as an element of ℓ^1 ; then $\psi(e_k) = \delta_{nk}$ and $\psi(x) = x_n$, so the identity reads $1 = x_n$. This holds for every n , contradicting $x \in c_0$ (whose terms tend to 0). Hence S is not in the range of J , so J is not onto and c_0 is not reflexive.

Step 4: ℓ^∞ is not reflexive. ℓ^∞ is a Banach space and, by Step 1, c_0 is a closed subspace of it. If ℓ^∞ were reflexive, then Proposition 15 (a closed subspace of a reflexive Banach space is reflexive) would force c_0 to be reflexive, contradicting Step 3. Therefore ℓ^∞ is not reflexive.

Problem 14.33. For $1 \leq p \leq \infty$, show that the sequence space ℓ^p is reflexive if and only if $1 < p < \infty$. (For $p = \infty$, see the preceding problem.)

Solution. The case $p = \infty$ is the preceding problem: ℓ^∞ is not reflexive. It remains to treat $1 \leq p < \infty$. We first record the Riesz Representation Theorem for ℓ^p (Problem 7 of Chapter 8), proving it since we use it repeatedly.

Lemma. Let $1 \leq p < \infty$ and let q be conjugate to p (so $q = \infty$ when $p = 1$). Define, for $b \in \ell^q$,

$$\mathcal{R}(b)[x] = \sum_{k=1}^{\infty} b_k x_k \quad \text{for all } x \in \ell^p.$$

Then $\mathcal{R}$ is an isometric isomorphism of ℓ^q onto $(\ell^p)^*$.

Proof of the Lemma. By Hölder's Inequality (for $p = 1$, by the trivial estimate $\sum_k |b_k x_k| \leq \|b\|_\infty \|x\|_1$), the series converges absolutely and

$$|\mathcal{R}(b)[x]| \leq \|b\|_q \|x\|_p.$$

So $\mathcal{R}(b)$ is a bounded linear functional on ℓ^p with $\|\mathcal{R}(b)\| \leq \|b\|_q$, and $\mathcal{R}$ is linear.

Now let $\psi \in (\ell^p)^*$ and set $b_k = \psi(e_k)$, where e_k is the k th standard basis sequence. We claim $b = (b_k) \in \ell^q$ with $\|b\|_q \leq \|\psi\|$.

If $p = 1$, then $\|e_k\|_1 = 1$, so $|b_k| = |\psi(e_k)| \leq \|\psi\|$ for every k , whence $\|b\|_\infty \leq \|\psi\|$.

If $1 < p < \infty$, fix N and set $x = \sum_{k=1}^N \text{sgn}(b_k) |b_k|^{q-1} e_k$. Since $(q-1)p = q$,

$$\|x\|_p^p = \sum_{k=1}^N |b_k|^{(q-1)p} = \sum_{k=1}^N |b_k|^q,$$

while $\psi(x) = \sum_{k=1}^N |b_k|^q$. Hence

$$\sum_{k=1}^N |b_k|^q = \psi(x) \leq \|\psi\| \|x\|_p = \|\psi\| \left(\sum_{k=1}^N |b_k|^q \right)^{1/p}.$$

If the sum is nonzero we may divide, and since $1 - 1/p = 1/q$ we get $\left(\sum_{k=1}^N |b_k|^q \right)^{1/q} \leq \|\psi\|$; this inequality also holds trivially if the sum is zero. Letting $N \rightarrow \infty$ gives $\|b\|_q \leq \|\psi\|$.

Finally, since $p < \infty$, for $x \in \ell^p$ the truncations $x^{(N)} = \sum_{k=1}^N x_k e_k$ satisfy $\|x - x^{(N)}\|_p^p = \sum_{k>N} |x_k|^p \rightarrow 0$. Therefore, by the linearity and continuity of ψ ,

$$\psi(x) = \lim_{N \rightarrow \infty} \psi(x^{(N)}) = \lim_{N \rightarrow \infty} \sum_{k=1}^N b_k x_k = \mathcal{R}(b)[x].$$

So $\psi = \mathcal{R}(b)$: $\mathcal{R}$ is onto, and combining $\|\mathcal{R}(b)\| \leq \|b\|_q$ with $\|b\|_q \leq \|\mathcal{R}(b)\|$ shows $\mathcal{R}$ is an isometry. This proves the Lemma.

Case $1 < p < \infty$: ℓ^p is reflexive. The argument is that of Proposition 20. Let q be conjugate to p , so $1 < q < \infty$ as well, and let $\mathcal{R}: \ell^q \rightarrow (\ell^p)^*$ be the isomorphism of the Lemma. Let T be a bounded linear functional on $(\ell^p)^*$; we must produce $x \in \ell^p$ with $T = J(x)$, where J is the natural embedding of ℓ^p into its bidual. Since $\mathcal{R}$ is onto, it suffices to find $x \in \ell^p$ with

$$T(\mathcal{R}(b)) = \mathcal{R}(b)[x] = \sum_{k=1}^{\infty} b_k x_k \quad \text{for all } b \in \ell^q,$$

for then $T(\mathcal{R}(b)) = J(x)[\mathcal{R}(b)]$ for all b , that is, $T = J(x)$ on all of $(\ell^p)^*$.

Now the composition $T \circ \mathcal{R}$ is a bounded linear functional on ℓ^q (it is linear, and $|T(\mathcal{R}(b))| \leq \|T\| \|\mathcal{R}(b)\| = \|T\| \|b\|_q$). Applying the Lemma with p and q interchanged, which is legitimate since $q < \infty$ and p is conjugate to q , there is $x \in \ell^p$ with

$$(T \circ \mathcal{R})[b] = \sum_{k=1}^{\infty} x_k b_k \quad \text{for all } b \in \ell^q.$$

This is exactly the required identity. Hence J is onto and ℓ^p is reflexive.

Case $p = 1$: ℓ^1 is not reflexive. By the Lemma with $p = 1$, $(\ell^1)^*$ is isometrically isomorphic to ℓ^∞ . We claim ℓ^∞ is not separable. Indeed, let A be the set of all sequences with entries in $\{0, 1\}$. This set is uncountable, since it is in one-to-one correspondence with the collection of all subsets of $\mathbb{N}$, and distinct members of A differ in some coordinate, so $\|u - v\|_\infty = 1$ for $u \neq v$ in A . Thus the open balls $B(u, 1/2)$, $u \in A$, are pairwise disjoint. Any dense subset of ℓ^∞ must meet each of these uncountably many disjoint balls, so no countable set is dense: ℓ^∞ is not separable. Since separability is preserved by isometric isomorphism, $(\ell^1)^*$ is not separable.

On the other hand, ℓ^1 is separable: the countable set of finitely supported sequences with rational entries is dense in ℓ^1 , because for $x \in \ell^1$ and $\epsilon > 0$ we may choose N with $\sum_{k>N} |x_k| < \epsilon/2$ and then rationals $r_1, \dots, r_N$ with $\sum_{k=1}^N |x_k - r_k| < \epsilon/2$.

Also ℓ^1 is a Banach space; this is the completeness of ℓ^p taken for granted by the book (see, for instance, Problem 12 of Chapter 10, which speaks of “the Banach space ℓ^p ”), and it is proved directly as follows. If $\{x^{(n)}\}$ is Cauchy in ℓ^1 , then $|x_k^{(n)} - x_k^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_1$ shows each coordinate sequence is Cauchy in $\mathbb{R}$, so $x_k := \lim_n x_k^{(n)}$ exists; given $\epsilon > 0$ choose N with $\|x^{(n)} - x^{(m)}\|_1 < \epsilon$ for $n, m \geq N$, and then for each fixed K and $n \geq N$, letting $m \rightarrow \infty$ in $\sum_{k=1}^K |x_k^{(n)} - x_k^{(m)}| < \epsilon$ gives $\sum_{k=1}^K |x_k^{(n)} - x_k| \leq \epsilon$; taking the supremum over K yields $x^{(n)} - x \in \ell^1$ with $\|x^{(n)} - x\|_1 \leq \epsilon$, so $x = x^{(n)} - (x^{(n)} - x) \in \ell^1$ and $x^{(n)} \rightarrow x$ in ℓ^1 . If ℓ^1 were reflexive, then Corollary 14 (a reflexive Banach space is separable if and only if its dual is separable) would give that $(\ell^1)^*$ is separable, a contradiction. Therefore ℓ^1 is not reflexive.

Combining the three cases: ℓ^p is reflexive if and only if $1 < p < \infty$.

Problem 14.34. Consider the functional $\psi \in (C[-1, 1])^*$ defined by

$$\psi(h) = \int_{-1}^0 h - \int_0^1 h \quad \text{for all } h \in C[-1, 1].$$

Show that ψ fails to take a maximum on the closed unit ball of $C[-1, 1]$. Use this to provide another proof that $C[-1, 1]$ fails to be reflexive.

Solution. Throughout, $C[-1, 1]$ carries the maximum norm $\|h\| = \max_{-1 \leq x \leq 1} |h(x)|$, and B denotes its closed unit ball.

Step 1: ψ is a bounded linear functional with $\|\psi\| = 2$. Linearity is clear from the linearity of the integral. For $h \in C[-1, 1]$,

$$|\psi(h)| \leq \int_{-1}^0 |h| + \int_0^1 |h| \leq \|h\| \cdot 1 + \|h\| \cdot 1 = 2\|h\|,$$

so ψ is bounded with $\|\psi\| \leq 2$.

For the reverse inequality, for each natural number $n \geq 2$ define $h_n \in C[-1, 1]$ by

$$h_n(x) = \begin{cases} 1, & -1 \leq x \leq -1/n, \\ -nx, & -1/n \leq x \leq 1/n, \\ -1, & 1/n \leq x \leq 1. \end{cases}$$

The three formulas agree at the two junction points, so h_n is continuous, and clearly $\|h_n\| = 1$, so $h_n \in B$. Compute

$$\int_{-1}^0 h_n = \int_{-1}^{-1/n} 1 + \int_{-1/n}^0 (-nx) dx = \left(1 - \frac{1}{n}\right) + \frac{1}{2n} = 1 - \frac{1}{2n},$$

and by the oddness of h_n , $\int_0^1 h_n = -\left(1 - \frac{1}{2n}\right)$. Hence

$$\psi(h_n) = 2 - \frac{1}{n}.$$

Therefore $\|\psi\| \geq \sup_n \psi(h_n) = 2$, so $\|\psi\| = 2$.

Step 2: $\sup_{h \in B} \psi(h) = \|\psi\| = 2$, and this supremum is not attained. That the supremum of ψ over B equals $\|\psi\|$ holds for every bounded linear functional: B is symmetric ($h \in B$ implies $-h \in B$) and $\psi(-h) = -\psi(h)$, so

$$\sup_{h \in B} \psi(h) = \sup_{h \in B} |\psi(h)| = \|\psi\|.$$

Thus ψ takes a maximum on B if and only if there is $h \in B$ with $\psi(h) = 2$.

Suppose such an h existed, so $\|h\| \leq 1$ and

$$\int_{-1}^0 h - \int_0^1 h = 2.$$

Since $|h| \leq 1$ on $[-1, 1]$, we have $\int_{-1}^0 h \leq 1$ and $-\int_0^1 h \leq 1$. Their sum being 2 forces both to be equalities:

$$\int_{-1}^0 h = 1 \quad \text{and} \quad \int_0^1 h = -1.$$

Consider $g = 1 - h$ on $[-1, 0]$. Then g is continuous and $g \geq 0$ there (since $h \leq 1$), and $\int_{-1}^0 g = 1 - 1 = 0$. A nonnegative continuous function on a closed bounded interval with zero integral is identically zero: if $g(x_0) > 0$ for some $x_0 \in [-1, 0]$, continuity gives an interval $I \subseteq [-1, 0]$ of positive length λ on which $g > g(x_0)/2$, whence $\int_{-1}^0 g \geq \lambda g(x_0)/2 > 0$, a contradiction. Therefore $h \equiv 1$ on $[-1, 0]$; in particular $h(0) = 1$.

The same argument applied to $1 + h$ on $[0, 1]$, which is continuous, nonnegative, and has integral $1 + (-1) = 0$, gives $h \equiv -1$ on $[0, 1]$; in particular $h(0) = -1$.

This is a contradiction, since h is a function and cannot have $h(0) = 1$ and $h(0) = -1$. Therefore no $h \in B$ satisfies $\psi(h) = 2$, and ψ fails to take a maximum value on B .

Step 3: $C[-1, 1]$ is not reflexive. $C[-1, 1]$ with the maximum norm is a Banach space. Corollary 18 states that if X is a reflexive Banach space, then every continuous real-valued linear functional on X takes a maximum value on the closed unit ball B of X . By Steps 1 and 2, ψ is a continuous linear functional on $C[-1, 1]$ that takes no maximum value on B . Hence $C[-1, 1]$ is not reflexive.

This is an alternative to the proof of Proposition 19, which instead used Theorem 17 and the failure of the sequence $\{x^n\}$ to have a weakly convergent subsequence.

Problem 14.35. For $1 \leq p < \infty$, show that a bounded sequence in ℓ^p converges weakly if and only if it converges componentwise.

Solution. Notation: a sequence $\{x^{(n)}\}$ in ℓ^p has terms $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots)$; it converges componentwise provided for each index k the real sequence $\{x_k^{(n)}\}_n$ converges. Recall that $\{x^{(n)}\}$ converges weakly to x provided $\psi(x^{(n)}) \rightarrow \psi(x)$ for every $\psi \in (\ell^p)^*$. We use the Riesz Representation Theorem for ℓ^p established in the Lemma of Problem 33: for q conjugate to p , every $\psi \in (\ell^p)^*$ has the form $\psi(x) = \sum_k b_k x_k$ for a unique $b \in \ell^q$, and $\|\psi\| = \|b\|_q$. Assume throughout that $\{x^{(n)}\}$ is bounded, say $\|x^{(n)}\|_p \leq M$ for all n .

Weak convergence implies componentwise convergence (valid for every $1 \leq p < \infty$). For each index k the coordinate functional $\pi_k(x) = x_k$ is linear and satisfies $|\pi_k(x)| = |x_k| \leq \|x\|_p$, so $\pi_k \in (\ell^p)^*$. (It is $\mathcal{R}(e_k)$ in the notation of Problem 33.) Hence if $x^{(n)}$ converges weakly to x , then for each k ,

$$x_k^{(n)} = \pi_k(x^{(n)}) \longrightarrow \pi_k(x) = x_k,$$

so $\{x^{(n)}\}$ converges componentwise, and the componentwise limit is the weak limit. Note that no boundedness hypothesis was needed here.

Componentwise convergence implies weak convergence when $1 < p < \infty$. Suppose $1 < p < \infty$, let q be conjugate to p (so $1 < q < \infty$), and suppose $x_k^{(n)} \rightarrow x_k$ for each k .

First, $x = (x_k) \in \ell^p$ and $\|x\|_p \leq M$. Indeed, for each N , since a finite sum of convergent sequences converges,

$$\sum_{k=1}^N |x_k|^p = \lim_{n \rightarrow \infty} \sum_{k=1}^N |x_k^{(n)}|^p \leq \limsup_{n \rightarrow \infty} \|x^{(n)}\|_p^p \leq M^p.$$

Letting $N \rightarrow \infty$ gives $\sum_{k=1}^{\infty} |x_k|^p \leq M^p$.

Now let $\psi \in (\ell^p)^*$ and write $\psi = \mathcal{R}(b)$ with $b \in \ell^q$. Let $\epsilon > 0$. Since $q < \infty$ and $\sum_k |b_k|^q < \infty$, choose N with

$$\left(\sum_{k>N} |b_k|^q \right)^{1/q} < \epsilon.$$

Split the sum defining $\psi(x^{(n)}) - \psi(x)$ at N :

$$\psi(x^{(n)}) - \psi(x) = \sum_{k=1}^N b_k (x_k^{(n)} - x_k) + \sum_{k>N} b_k (x_k^{(n)} - x_k).$$

For the tail, Hölder's Inequality and the triangle inequality in ℓ^p give

$$\left| \sum_{k>N} b_k (x_k^{(n)} - x_k) \right| \leq \left(\sum_{k>N} |b_k|^q \right)^{1/q} \|x^{(n)} - x\|_p \leq \epsilon (\|x^{(n)}\|_p + \|x\|_p) \leq 2M\epsilon,$$

a bound independent of n . For the head, the sum has finitely many terms and each factor $x_k^{(n)} - x_k$ tends to 0 as $n \rightarrow \infty$, so

$$\lim_{n \rightarrow \infty} \sum_{k=1}^N b_k (x_k^{(n)} - x_k) = 0.$$

Therefore $\limsup_{n \rightarrow \infty} |\psi(x^{(n)}) - \psi(x)| \leq 2M\epsilon$. As $\epsilon > 0$ was arbitrary, $\psi(x^{(n)}) \rightarrow \psi(x)$. Since $\psi \in (\ell^p)^*$ was arbitrary, $\{x^{(n)}\}$ converges weakly to x .

Thus for $1 < p < \infty$ a bounded sequence in ℓ^p converges weakly if and only if it converges componentwise, and the two limits agree. (Uniqueness of the weak limit is automatic: if $x^{(n)}$ converges weakly to both x and y then $\psi(x - y) = 0$ for all $\psi \in (\ell^p)^*$, and by Problem 21 of this chapter, $\|x - y\| = \sup\{|\psi(x - y)| \mid \|\psi\| \leq 1\} = 0$.)

The case $p = 1$: the stated equivalence fails. As shown above, weak convergence always implies componentwise convergence, so the forward implication holds for $p = 1$ too. The converse does not.

Let $x^{(n)} = e_n$, the sequence with 1 in the n th place and 0 elsewhere. Then $\|x^{(n)}\|_1 = 1$ for every n , so the sequence is bounded in ℓ^1 , and for each fixed k we have $x_k^{(n)} = 0$ for all $n > k$, so $x_k^{(n)} \rightarrow 0$: the sequence converges componentwise to 0.

But $\{x^{(n)}\}$ does not converge weakly. By the forward implication together with uniqueness of weak limits, the only possible weak limit is the componentwise limit 0. Take $b = (1, 1, 1, \dots) \in \ell^\infty$ and let $\psi = \mathcal{R}(b) \in (\ell^1)^*$, that is, $\psi(y) = \sum_k y_k$ for $y \in \ell^1$; this is bounded with $\|\psi\| = \|b\|_\infty = 1$ by the Lemma of Problem 33. Then

$$\psi(x^{(n)}) = 1 \quad \text{for every } n, \quad \text{while} \quad \psi(0) = 0.$$

So $\psi(x^{(n)})$ does not converge to $\psi(0)$, and $\{x^{(n)}\}$ does not converge weakly to 0; hence it does not converge weakly at all.

The obstruction is exactly the step that fails at $p = 1$: for $q = \infty$ a member b of ℓ^∞ has no small tails, so the tail estimate above is unavailable. Accordingly the correct statement is: for $1 < p < \infty$, a bounded sequence in ℓ^p converges weakly if and only if it converges componentwise; for $p = 1$ only the forward implication is true.

14.6 Problems 14.36–14.42

Problem 14.36. For $1 \leq p < \infty$ and $[a, b]$ a closed, bounded interval of real numbers, show that a bounded sequence $\{f_n\}$ in $L^p[a, b]$ converges weakly to f if and only if $\{\int_E f_n\} \rightarrow \int_E f$ for every Lebesgue measurable subset E of $[a, b]$.

Solution. Throughout, f and every f_n belong to $L^p[a, b]$, and weak convergence of $\{f_n\}$ to f means, by definition, that $\lim_{n \rightarrow \infty} \psi(f_n) = \psi(f)$ for every ψ in the dual space $(L^p[a, b])^*$.

Let q be the conjugate of p , so that $1 < q \leq \infty$. Since $[a, b]$ has finite measure and $1 \leq p < \infty$, the Riesz Representation Theorem for the Dual of L^p (Chapter 8) applies: every bounded linear functional on $L^p[a, b]$ has the form

$$\psi_g(h) = \int_a^b h \cdot g \quad \text{for } h \in L^p[a, b],$$

for a (unique) $g \in L^q[a, b]$, and every such g induces a bounded linear functional this way. Hence, by the definition of weak convergence,

$$\{f_n\} \rightharpoonup f \text{ in } L^p[a, b] \iff \lim_{n \rightarrow \infty} \int_a^b f_n \cdot g = \int_a^b f \cdot g \text{ for all } g \in L^q[a, b].$$

Suppose first that $\{f_n\}$ converges weakly to f . Let E be a measurable subset of $[a, b]$. Since $[a, b]$ has finite measure, χ_E belongs to $L^q[a, b]$: indeed $\|\chi_E\|_q \leq (b-a)^{1/q}$ if $q < \infty$, and $\|\chi_E\|_\infty \leq 1$ if $q = \infty$. Therefore, taking $g = \chi_E$ in the above characterization,

$$\lim_{n \rightarrow \infty} \int_E f_n = \lim_{n \rightarrow \infty} \int_a^b f_n \cdot \chi_E = \int_a^b f \cdot \chi_E = \int_E f.$$

Observe that this implication did not use the boundedness of $\{f_n\}$.

Conversely, suppose $\{f_n\}$ is bounded in $L^p[a, b]$ and $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$ for every measurable $E \subseteq [a, b]$. Set

$$C = \sup_n \|f_n\|_p + \|f\|_p,$$

which is finite by assumption, so that $\|f_n - f\|_p \leq C$ for all n .

First we check that

$$\lim_{n \rightarrow \infty} \int_a^b f_n \cdot \varphi = \int_a^b f \cdot \varphi \quad \text{for every simple function } \varphi \text{ on } [a, b]. \quad (*)$$

Indeed, such a φ may be written as $\varphi = \sum_{k=1}^m a_k \chi_{E_k}$ with each a_k real and each E_k a measurable subset of $[a, b]$. Each f_n and f belong to $L^p[a, b] \subseteq L^1[a, b]$ (again because $[a, b]$ has finite measure, by

Hölder's Inequality), so by the linearity of integration,

$$\int_a^b f_n \cdot \varphi = \sum_{k=1}^m a_k \int_{E_k} f_n \longrightarrow \sum_{k=1}^m a_k \int_{E_k} f = \int_a^b f \cdot \varphi,$$

since each of the finitely many summands converges by hypothesis. This proves (*).

Now let $g \in L^q[a, b]$ and let $\epsilon > 0$. By Proposition 9 of Chapter 7, which is stated for the full range $1 \leq q \leq \infty$, the simple functions are dense in $L^q[a, b]$; this is what allows the case $p = 1, q = \infty$ to be treated alongside the others. Choose a simple function φ on $[a, b]$ with

$$\|g - \varphi\|_q < \frac{\epsilon}{2(C+1)}.$$

By Hölder's Inequality,

$$\left| \int_a^b (f_n - f) \cdot (g - \varphi) \right| \leq \|f_n - f\|_p \cdot \|g - \varphi\|_q \leq C \cdot \frac{\epsilon}{2(C+1)} < \frac{\epsilon}{2} \quad \text{for all } n.$$

By (*) there is an index N for which $\left| \int_a^b (f_n - f) \cdot \varphi \right| < \epsilon/2$ for all $n \geq N$. Hence, for $n \geq N$,

$$\left| \int_a^b f_n \cdot g - \int_a^b f \cdot g \right| \leq \left| \int_a^b (f_n - f) \cdot (g - \varphi) \right| + \left| \int_a^b (f_n - f) \cdot \varphi \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\lim_{n \rightarrow \infty} \int_a^b f_n \cdot g = \int_a^b f \cdot g$. As $g \in L^q[a, b]$ was arbitrary, the Riesz representation characterization above shows that $\{f_n\}$ converges weakly to f in $L^p[a, b]$.

Finally, we point out precisely where the hypothesis of boundedness entered. It was used only once, and only in the converse direction: to produce the single constant C dominating every $\|f_n - f\|_p$, so that a single choice of the approximating simple function φ , made before n was chosen, controls the tail term $\left| \int_a^b (f_n - f)(g - \varphi) \right|$ uniformly in n . Without a uniform bound on $\|f_n - f\|_p$ the quantity $\|f_n - f\|_p \|g - \varphi\|_q$ need not be small for all n at once, and the interchange of the two limits (over n , and over the approximation of g by simple functions) is not licensed. We do not claim here that the boundedness hypothesis cannot be removed; we claim only that this proof requires it.

Problem 14.37. For $[a, b]$ a closed, bounded interval of real numbers, show that if a sequence $\{f_n\}$ in $C[a, b]$ converges weakly, then it converges pointwise.

Solution. Here $C[a, b]$ carries the maximum norm $\|h\|_{\max} = \max_{x \in [a, b]} |h(x)|$.

Suppose $\{f_n\}$ converges weakly in $C[a, b]$ to $f \in C[a, b]$; that is,

$$\lim_{n \rightarrow \infty} \psi(f_n) = \psi(f) \quad \text{for every } \psi \in (C[a, b])^*.$$

We must produce, for each $x \in [a, b]$, the conclusion $f_n(x) \rightarrow f(x)$; that is, we must exhibit enough bounded linear functionals to detect pointwise values.

Fix $x \in [a, b]$ and define the evaluation functional

$$\psi_x: C[a, b] \rightarrow \mathbb{R}, \quad \psi_x(h) = h(x) \quad \text{for all } h \in C[a, b].$$

Then ψ_x is linear, since $(\alpha h_1 + \beta h_2)(x) = \alpha h_1(x) + \beta h_2(x)$. It is bounded, since

$$|\psi_x(h)| = |h(x)| \leq \max_{t \in [a, b]} |h(t)| = \|h\|_{\max} \quad \text{for all } h \in C[a, b],$$

so $\|\psi_x\| \leq 1$; in fact $\|\psi_x\| = 1$, as is seen by taking $h \equiv 1$. Thus ψ_x belongs to $(C[a, b])^*$.

Applying the definition of weak convergence to the functional ψ_x gives

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \psi_x(f_n) = \psi_x(f) = f(x).$$

Since $x \in [a, b]$ was arbitrary, $\{f_n\}$ converges pointwise on $[a, b]$ to f , the weak limit. In particular, the weak limit of a weakly convergent sequence in $C[a, b]$ is necessarily its pointwise limit, so a weakly convergent sequence in $C[a, b]$ has a unique weak limit determined pointwise.

We remark that the converse fails, and indeed this asymmetry is exactly what is exploited in the proof of Proposition 19: for $[a, b] = [0, 1]$ and $f_n(x) = x^n$, the sequence $\{f_n\}$ is bounded in $C[0, 1]$ and converges pointwise to a discontinuous function, so no subsequence of $\{f_n\}$ can converge weakly to a member of $C[0, 1]$; by Theorem 17 this shows $C[0, 1]$ is not reflexive.

Problem 14.38. For X and Y normed linear spaces and an operator $S \in \mathcal{L}(X, Y)$, define the adjoint of S , $S^* \in \mathcal{L}(Y^*, X^*)$ by

$$[S^*(\psi)](x) = \psi(S(x)) \quad \text{for all } \psi \in Y^*, x \in X.$$

(i) Show that $\|S^*\| = \|S\|$ and that S^* is an isomorphism if S is an isomorphism.

(ii) For $1 < p < \infty$ and $X = L^p(E)$, where E is a measurable set of real numbers, show that the natural embedding $J: X \rightarrow X^{**}$ may be expressed as the composition

$$J = [\mathcal{R}_q^*]^{-1} \circ \mathcal{R}_p,$$

where $\mathcal{R}_p$ and $\mathcal{R}_q$ are the Riesz representing operators.

Solution. (i) First we check that S^* is a well-defined bounded linear operator from Y^* to X^* . Let $\psi \in Y^*$. The map $S^*(\psi) = \psi \circ S$ is linear on X , being a composition of linear maps, and for all $x \in X$,

$$|[S^*(\psi)](x)| = |\psi(S(x))| \leq \|\psi\| \cdot \|S(x)\| \leq \|\psi\| \cdot \|S\| \cdot \|x\|.$$

Hence $S^*(\psi)$ is bounded, that is, $S^*(\psi) \in X^*$, and

$$\|S^*(\psi)\| \leq \|S\| \cdot \|\psi\|. \quad (1)$$

Moreover S^* is itself linear, since for $\psi_1, \psi_2 \in Y^*$ and $\alpha, \beta \in \mathbb{R}$,

$$[S^*(\alpha\psi_1 + \beta\psi_2)](x) = (\alpha\psi_1 + \beta\psi_2)(S(x)) = \alpha\psi_1(S(x)) + \beta\psi_2(S(x)) = [\alpha S^*(\psi_1) + \beta S^*(\psi_2)](x)$$

for all $x \in X$. By (1), S^* is bounded and $\|S^*\| \leq \|S\|$.

For the reverse inequality, let $x \in X$. If $S(x) = 0$ then trivially $\|S(x)\| \leq \|S^*\| \cdot \|x\|$. If $S(x) \neq 0$, then, by Theorem 7 applied to the vector $S(x)$ of Y , there is a $\psi \in Y^*$ for which

$$\|\psi\| = 1 \quad \text{and} \quad \psi(S(x)) = \|S(x)\|.$$

Therefore

$$\|S(x)\| = \psi(S(x)) = [S^*(\psi)](x) \leq \|S^*(\psi)\| \cdot \|x\| \leq \|S^*\| \cdot \|\psi\| \cdot \|x\| = \|S^*\| \cdot \|x\|.$$

Thus $\|S(x)\| \leq \|S^*\| \cdot \|x\|$ for every $x \in X$, and hence $\|S\| \leq \|S^*\|$. Combining the two inequalities, $\|S^*\| = \|S\|$.

Next, two elementary functorial identities. If $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(Y, Z)$, then for $\psi \in Z^*$ and $x \in X$,

$$[(T \circ S)^*(\psi)](x) = \psi(T(S(x))) = [T^*(\psi)](S(x)) = [S^*(T^*(\psi))](x),$$

so that

$$(T \circ S)^* = S^* \circ T^*. \quad (2)$$

Also, if Id_X denotes the identity operator on X , then $[\text{Id}_X^*(\psi)](x) = \psi(x)$ for all $\psi \in X^*$, $x \in X$, so

$$\text{Id}_X^* = \text{Id}_{X^*}. \quad (3)$$

Now suppose $S \in \mathcal{L}(X, Y)$ is an isomorphism, that is, S is one-to-one, onto, and S^{-1} belongs to $\mathcal{L}(Y, X)$. Then $(S^{-1})^*$ belongs to $\mathcal{L}(X^*, Y^*)$ by the first paragraph. By (2) and (3),

$$S^* \circ (S^{-1})^* = (S^{-1} \circ S)^* = \text{Id}_X^* = \text{Id}_{X^*},$$

$$(S^{-1})^* \circ S^* = (S \circ S^{-1})^* = \text{Id}_Y^* = \text{Id}_{Y^*}.$$

The first identity shows $S^*: Y^* \rightarrow X^*$ is onto, and the second shows it is one-to-one; together they show $(S^*)^{-1} = (S^{-1})^*$, which is a bounded operator. Hence S^* is an isomorphism of Y^* onto X^* .

(ii) Let q be conjugate to p , so $1 < q < \infty$ as well. Following the proof of Proposition 20, the Riesz representing operators are

$$\mathcal{R}_q: L^q(E) \rightarrow (L^p(E))^*, \quad \mathcal{R}_q(g)[f] = \int_E g \cdot f \quad \text{for } f \in L^p(E),$$

$$\mathcal{R}_p: L^p(E) \rightarrow (L^q(E))^*, \quad \mathcal{R}_p(f)[g] = \int_E f \cdot g \quad \text{for } g \in L^q(E).$$

The Riesz Representation Theorem for the Dual of L^p (Chapter 8) tells us that each of $\mathcal{R}_p$ and $\mathcal{R}_q$ is an isomorphism onto the indicated dual space, and indeed an isometry; this is exactly the assertion invoked at the start of the proof of Proposition 20. The theorem applies for every Lebesgue measurable $E \subseteq \mathbb{R}$, since such an E is σ -finite, and both p and q are finite here. Therefore, by part (i), the adjoint

$$\mathcal{R}_q^*: (L^p(E))^{**} \rightarrow (L^q(E))^*$$

is an isomorphism, so $[\mathcal{R}_q^*]^{-1}: (L^q(E))^* \rightarrow (L^p(E))^{**}$ is a well-defined bounded operator and the composition $[\mathcal{R}_q^*]^{-1} \circ \mathcal{R}_p$ maps $X = L^p(E)$ into $X^{**} = (L^p(E))^{**}$, as required.

Recall that the natural embedding $J: X \rightarrow X^{**}$ is defined by $J(f)[\Psi] = \Psi(f)$ for $f \in X$ and $\Psi \in X^*$. We claim that

$$\mathcal{R}_q^* \circ J = \mathcal{R}_p. \quad (4)$$

Indeed, let $f \in L^p(E)$ and $g \in L^q(E)$. By the definition of the adjoint, applied to the operator $\mathcal{R}_q$ and the functional $J(f) \in (L^p(E))^{**}$,

$$[\mathcal{R}_q^*(J(f))](g) = J(f)[\mathcal{R}_q(g)] = \mathcal{R}_q(g)[f] = \int_E g \cdot f = \int_E f \cdot g = \mathcal{R}_p(f)[g].$$

Since $g \in L^q(E)$ was arbitrary, $\mathcal{R}_q^*(J(f)) = \mathcal{R}_p(f)$ as members of $(L^q(E))^*$, and since f was arbitrary, (4) holds.

Composing (4) on the left with $[\mathcal{R}_q^*]^{-1}$ yields

$$J = [\mathcal{R}_q^*]^{-1} \circ \mathcal{R}_p,$$

which is the asserted representation. In particular J , being the composition of two isomorphisms, is itself an isomorphism of $L^p(E)$ onto $(L^p(E))^{**}$; that is, $L^p(E)$ is reflexive for $1 < p < \infty$, which is Proposition 20. No circularity is involved: the only external input was the Riesz Representation Theorem of Chapter 8, which is a statement about the dual of a single L^p space and makes no reference to reflexivity, together with part (i) above.

Problem 14.39. Let X be a reflexive Banach space and $T: X \rightarrow X$ a linear operator. Show that T belongs to $\mathcal{L}(X, X)$ if and only if whenever $\{x_n\}$ converges weakly to x , $\{T(x_n)\}$ converges weakly to $T(x)$.

Solution. We use two preliminary observations.

First, strong convergence implies weak convergence. Indeed, if $\{z_n\} \rightarrow z$ in the norm of X and $\psi \in X^*$, then

$$|\psi(z_n) - \psi(z)| = |\psi(z_n - z)| \leq \|\psi\| \cdot \|z_n - z\| \rightarrow 0,$$

so $\{z_n\}$ converges weakly to z .

Second, weak limits are unique. Suppose $\psi(u) = \psi(v)$ for all $\psi \in X^*$ and $u \neq v$. By Theorem 7 applied to the nonzero vector $u - v$, there is a $\psi \in X^*$ with $\|\psi\| = 1$ and $\psi(u - v) = \|u - v\| > 0$, a contradiction. Hence $u = v$.

Suppose first that T belongs to $\mathcal{L}(X, X)$, that is, T is a bounded linear operator. Let $\{x_n\}$ converge weakly to x , and let $\psi \in X^*$. The composition $\psi \circ T$ is linear, and

$$|\psi(T(z))| \leq \|\psi\| \cdot \|T(z)\| \leq \|\psi\| \cdot \|T\| \cdot \|z\| \quad \text{for all } z \in X,$$

so $\psi \circ T$ belongs to X^* (it is precisely $T^*(\psi)$, in the notation of the preceding problem). Therefore, by the weak convergence of $\{x_n\}$ to x applied to the functional $\psi \circ T$,

$$\lim_{n \rightarrow \infty} \psi(T(x_n)) = \lim_{n \rightarrow \infty} (\psi \circ T)(x_n) = (\psi \circ T)(x) = \psi(T(x)).$$

Since $\psi \in X^*$ was arbitrary, $\{T(x_n)\}$ converges weakly to $T(x)$.

Conversely, suppose T has the stated sequential property. We show T is a closed operator and then invoke the Closed Graph Theorem of Chapter 13. Recall the definition given there: a linear operator T is closed provided that whenever $\{x_n\} \rightarrow x$ in X and $\{T(x_n)\} \rightarrow y$ in X , both in the norm, it follows that $y = T(x)$. Note that T is assumed linear from the outset, so the Closed Graph Theorem is applicable once closedness is established; no boundedness is presupposed anywhere in this direction. So let $\{x_n\} \rightarrow x$ and $\{T(x_n)\} \rightarrow y$ in the norm of X . By the first observation, $\{x_n\}$ converges weakly to x . Hence, by hypothesis, $\{T(x_n)\}$ converges weakly to $T(x)$. On the other hand, again by the first observation, $\{T(x_n)\}$ converges weakly to y . Thus for every $\psi \in X^*$,

$$\psi(T(x)) = \lim_{n \rightarrow \infty} \psi(T(x_n)) = \psi(y),$$

and so, by the uniqueness of weak limits, $y = T(x)$. Therefore T is closed.

Since X is a Banach space and $T: X \rightarrow X$ is a closed linear operator between Banach spaces, the Closed Graph Theorem of Chapter 13 tells us that T is continuous, that is, T belongs to $\mathcal{L}(X, X)$.

We remark that the reflexivity of X was not needed in either implication: the argument uses only that X is a Banach space, together with the consequence of the Hahn-Banach Theorem recorded as Theorem 7. Reflexivity is what makes the hypothesis a substantial one, since by Theorem 17 a reflexive space possesses an abundance of weakly convergent sequences (every bounded sequence has a weakly convergent subsequence).

Problem 14.40. Let X be a normed linear space and W be a subspace of X^* . Show that the W -weak topology on X is Hausdorff if and only if W separates points in X .

Solution. Recall the definitions. The W -weak topology on X is the topology for which, at each point $x \in X$, a neighborhood base is given by the sets

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x) = \{x' \in X \mid |\psi_i(x - x')| < \epsilon \text{ for } 1 \leq i \leq n\},$$

where $\epsilon > 0$, n is a natural number, and each ψ_i belongs to W . The subspace W is said to separate points in X provided that for each pair u, v of distinct vectors in X there is a $\psi \in W$ for which $\psi(u) \neq \psi(v)$.

Suppose first that W separates points in X ; this is the argument given in the proof of Proposition 21. Let u and v be distinct vectors in X . Choose $\psi \in W$ with $\psi(u) \neq \psi(v)$ and set

$$r = |\psi(u) - \psi(v)| > 0.$$

Consider the basic W -weak neighborhoods

$$\mathcal{N}_{r/2, \psi}(u) = \{x \in X \mid |\psi(u) - \psi(x)| < r/2\}, \quad \mathcal{N}_{r/2, \psi}(v) = \{x \in X \mid |\psi(v) - \psi(x)| < r/2\}$$

of u and v respectively. These are disjoint. Indeed, if x belonged to both, then by the triangle inequality in $\mathbb{R}$,

$$r = |\psi(u) - \psi(v)| \leq |\psi(u) - \psi(x)| + |\psi(x) - \psi(v)| < r/2 + r/2 = r,$$

a contradiction. Moreover these two sets are themselves W -weak open: since the W -weak topology is by definition the weakest topology on X for which every member of W is continuous, each $\psi \in W$ is W -weakly continuous, and $\mathcal{N}_{r/2, \psi}(u) = \psi^{-1}((\psi(u) - r/2, \psi(u) + r/2))$ is the preimage of an open interval under such a functional. We have therefore exhibited disjoint W -weak open sets containing u and v , so the W -weak topology is Hausdorff.

Conversely, suppose the W -weak topology on X is Hausdorff. Let u and v be distinct vectors in X . Choose disjoint W -weak open sets $\mathcal{O}_u$ and $\mathcal{O}_v$ containing u and v respectively. Since the sets $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(u)$ form a neighborhood base at u , there are an $\epsilon > 0$ and functionals $\psi_1, \dots, \psi_n$ in W for which

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(u) \subseteq \mathcal{O}_u.$$

We claim that $\psi_i(u) \neq \psi_i(v)$ for at least one index i . Otherwise $\psi_i(u) = \psi_i(v)$, and hence $|\psi_i(u - v)| = 0 < \epsilon$, for every $1 \leq i \leq n$; by the very definition of $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(u)$ this means that v belongs to $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(u)$ and therefore to $\mathcal{O}_u$. But v also belongs to $\mathcal{O}_v$, which contradicts the disjointness of $\mathcal{O}_u$ and $\mathcal{O}_v$. Thus some $\psi_i \in W$ has $\psi_i(u) \neq \psi_i(v)$.

Since u and v were an arbitrary pair of distinct vectors in X , W separates points in X .

We observe that the converse half in fact only used that the W -weak topology is T_1 (that for each pair of distinct points there is a neighborhood of one excluding the other); together with the first half this shows that for these topologies the separation properties T_1 and Hausdorff coincide, and both are equivalent to W separating points.

Problem 14.41. Let X be a normed linear space and $\psi: X \rightarrow \mathbb{R}$ be linear. Show that ψ is continuous with respect to the weak topology if and only if it is continuous with respect to the strong topology.

Solution. Here the strong topology on X is the topology induced by the norm, and the weak topology on X is the X^* -weak topology, that is, the topology for which the sets

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x) = \{x' \in X \mid |\psi_i(x - x')| < \epsilon \text{ for } 1 \leq i \leq n\}, \quad \psi_i \in X^*, \epsilon > 0,$$

form a neighborhood base at each $x \in X$. By Theorem 7, X^* separates points in X , so by Proposition 21 both topologies make X a locally convex topological vector space.

We first record that the weak topology is contained in the strong topology. Let $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x)$ be a basic weak neighborhood. Each ψ_i belongs to X^* and so is continuous with respect to the norm; hence each set $\{x' \in X \mid |\psi_i(x - x')| < \epsilon\}$ is strongly open, and so is their finite intersection $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x)$. Since every weakly open set is a union of such basic sets, every weakly open set is strongly open.

Now suppose ψ is continuous with respect to the weak topology. Let U be an open subset of $\mathbb{R}$. Then $\psi^{-1}(U)$ is weakly open, and hence, by the preceding paragraph, strongly open. Therefore ψ is continuous with respect to the strong topology. (Note that this implication holds for any function on X , linear or not, and reflects only the fact that the weak topology is the weaker of the two.)

Conversely, suppose ψ is continuous with respect to the strong topology. Since ψ is linear, Theorem 1 of Chapter 13 tells us that ψ is bounded, that is, ψ belongs to X^* . We show directly that ψ is continuous at each $x_0 \in X$ with respect to the weak topology. Let $\epsilon > 0$. Consider the basic weak neighborhood of x_0 determined by the single functional ψ itself:

$$\mathcal{N}_{\epsilon, \psi}(x_0) = \{x \in X \mid |\psi(x_0 - x)| < \epsilon\}.$$

This is a weak neighborhood of x_0 precisely because $\psi \in X^*$, and for every x in it,

$$|\psi(x) - \psi(x_0)| = |\psi(x - x_0)| < \epsilon.$$

Thus $\psi(\mathcal{N}_{\epsilon, \psi}(x_0)) \subseteq (\psi(x_0) - \epsilon, \psi(x_0) + \epsilon)$, so ψ is continuous at x_0 with respect to the weak topology. Since $x_0 \in X$ was arbitrary, ψ is weakly continuous: for U open in $\mathbb{R}$ and $x_0 \in \psi^{-1}(U)$, choosing $\epsilon > 0$

with $(\psi(x_0) - \epsilon, \psi(x_0) + \epsilon) \subseteq U$ exhibits a weak neighborhood of x_0 contained in $\psi^{-1}(U)$, so $\psi^{-1}(U)$ is weakly open.

Hence the two notions of continuity coincide for linear functionals, and in both cases they hold exactly when ψ belongs to X^* . Equivalently: although the weak topology is in general strictly weaker than the strong topology, the two topologies have the same continuous linear functionals, namely X^* .

The converse direction may also be read off from Proposition 5, applied with $W = X^*$: a linear functional on X is W -weakly continuous if and only if it belongs to W . Since strong continuity of the linear ψ is, by Theorem 1 of Chapter 13, the same as membership in X^* , and since the weak topology is precisely the X^* -weak topology, Proposition 5 gives the equivalence at once. We gave the direct argument above because it needs nothing beyond the definition of the basic weak neighborhoods.

Problem 14.42. Let X be a locally convex topological vector space and $\psi: X \rightarrow \mathbb{R}$ be linear. Show that ψ is continuous if and only if it is continuous at the origin.

Solution. We use the book's conventions from Chapter 11: a neighborhood of a point x in a topological space is an open set containing x ; and a map f between topological spaces is continuous at x_0 provided that for each neighborhood U of $f(x_0)$ there is a neighborhood V of x_0 with $f(V) \subseteq U$. By Proposition 1 of Chapter 11, f is continuous if and only if it is continuous at every point of its domain. Since ψ is linear, $\psi(0) = \psi(0 + 0) = 2\psi(0)$, so $\psi(0) = 0$.

If ψ is continuous, then in particular it is continuous at the origin. This implication is immediate.

Conversely, suppose ψ is continuous at the origin. Let x_0 belong to X and let $\epsilon > 0$. Since $\psi(0) = 0$ and ψ is continuous at 0, there is a neighborhood $\mathcal{N}_0$ of the origin for which

$$|\psi(u)| = |\psi(u) - \psi(0)| < \epsilon \quad \text{for all } u \in \mathcal{N}_0.$$

Here $\mathcal{N}_0$ is open and contains the origin. By Proposition 22 (i), the translate $x_0 + \mathcal{N}_0$ is again open; the proof there is that the translation map $T_{x_0}(x) = x + x_0$ is a homeomorphism of X onto X , being continuous (vector addition is continuous) with continuous inverse T_{-x_0} . Since $0 \in \mathcal{N}_0$, the open set $x_0 + \mathcal{N}_0$ contains x_0 , and so is a neighborhood of x_0 .

Let x belong to $x_0 + \mathcal{N}_0$. Then $x - x_0$ belongs to $\mathcal{N}_0$, and therefore, by the linearity of ψ ,

$$|\psi(x) - \psi(x_0)| = |\psi(x - x_0)| < \epsilon.$$

Hence $\psi(x_0 + \mathcal{N}_0) \subseteq (\psi(x_0) - \epsilon, \psi(x_0) + \epsilon)$, and since every neighborhood of $\psi(x_0)$ in $\mathbb{R}$ contains such an interval, ψ is continuous at x_0 .

Since x_0 was an arbitrary point of X , ψ is continuous at every point of X and therefore continuous. Explicitly: if U is open in $\mathbb{R}$ and x_0 belongs to $\psi^{-1}(U)$, choose $\epsilon > 0$ with $(\psi(x_0) - \epsilon, \psi(x_0) + \epsilon) \subseteq U$ and then a neighborhood $x_0 + \mathcal{N}_0$ of x_0 as above; it is contained in $\psi^{-1}(U)$. Thus $\psi^{-1}(U)$ contains an open set about each of its points and so is open.

Two remarks. First, the only structure used was the continuity of vector addition, which is what makes translations homeomorphisms; neither the local convexity of X , nor the Hausdorff property, nor the continuity of scalar multiplication entered the argument. Second, Proposition 23 sharpens the criterion further, to the requirement that $|\psi|$ merely be bounded on some neighborhood of the origin; its proof does use the continuity of scalar multiplication, in order to dilate that neighborhood, so that hypothesis is not dispensable there as it is here.

14.7 Problems 14.43–14.49

Problem 14.43. Let X be a locally convex topological vector space and $\psi: X \rightarrow \mathbb{R}$ be linear. Show that ψ is continuous if and only if there is a neighborhood $\mathcal{O}$ of the origin for which $\psi(\mathcal{O}) \neq \mathbb{R}$.

Solution. Throughout we use Proposition 23: a linear functional on a locally convex topological vector space is continuous if and only if there is a neighborhood $\mathcal{N}_0$ of the origin and an $M > 0$ for which $|\psi| < M$ on $\mathcal{N}_0$. We also use the convention of Chapter 11 that a neighborhood of a point is an open

set containing that point.

Assume first that ψ is continuous. By Proposition 23 there is a neighborhood $\mathcal{N}_0$ of the origin and an $M > 0$ for which $|\psi| < M$ on $\mathcal{N}_0$. Then $\psi(\mathcal{N}_0) \subseteq (-M, M)$, and therefore $\psi(\mathcal{N}_0) \neq \mathbb{R}$. Take $\mathcal{O} = \mathcal{N}_0$. Now assume conversely that there is a neighborhood $\mathcal{O}$ of the origin with $\psi(\mathcal{O}) \neq \mathbb{R}$, and choose $c \in \mathbb{R}$ with $c \notin \psi(\mathcal{O})$.

Since X is a locally convex topological vector space, by property (iii) of the definition there is a base at the origin consisting of convex sets, and the members of a base at a point are by definition neighborhoods of that point, hence open. Since $\mathcal{O}$ is a neighborhood of the origin, we may therefore choose a convex open set $\mathcal{C}$ with $0 \in \mathcal{C} \subseteq \mathcal{O}$. Since $\psi(\mathcal{C}) \subseteq \psi(\mathcal{O})$, we still have $c \notin \psi(\mathcal{C})$. Observe also that $0 = \psi(0) \in \psi(\mathcal{C})$, so that $c \neq 0$.

Define

$$\mathcal{U} = \mathcal{C} \cap (-\mathcal{C}).$$

By Proposition 22(i), $-\mathcal{C} = (-1) \cdot \mathcal{C}$ is open, so $\mathcal{U}$ is open; it contains the origin, it is convex (the intersection of two convex sets is convex, since a convex combination of two points of the intersection lies in each of the two sets), it satisfies $\mathcal{U} = -\mathcal{U}$, and $\mathcal{U} \subseteq \mathcal{C}$.

Let $A = \psi(\mathcal{U}) \subseteq \mathbb{R}$. We record three properties of A .

First, A is convex: if $\psi(u), \psi(v) \in A$ with $u, v \in \mathcal{U}$ and $0 \leq \lambda \leq 1$, then by linearity $\lambda\psi(u) + (1-\lambda)\psi(v) = \psi(\lambda u + (1-\lambda)v)$, and $\lambda u + (1-\lambda)v \in \mathcal{U}$ because $\mathcal{U}$ is convex. Being a convex subset of $\mathbb{R}$, A is an interval.

Second, A is symmetric about 0: if $u \in \mathcal{U}$ then $-u \in \mathcal{U}$ and $\psi(-u) = -\psi(u)$, so $-A = A$.

Third, $0 \in A$ and $c \notin A$, since $A \subseteq \psi(\mathcal{C})$ and $c \notin \psi(\mathcal{C})$. By symmetry $-c \notin A$ as well. Hence $|c| \notin A$, since $|c|$ equals c or $-c$.

Now suppose, to get a contradiction, that some $t \in A$ had $|t| > |c|$. Replacing t by $-t$ if necessary (permissible since $A = -A$), we may assume $t > |c| > 0$. Since $0 \in A$, $t \in A$ and A is an interval, the whole segment $[0, t]$ lies in A ; in particular $|c| \in A$, a contradiction. Therefore

$$|\psi| \leq |c| < |c| + 1 \text{ on } \mathcal{U}.$$

Since $\mathcal{U}$ is a neighborhood of the origin and $M = |c| + 1 > 0$, Proposition 23 tells us that ψ is continuous.

Problem 14.44. Let X be a normed linear space and W a subspace of X^* that separates points. For any topological space Z , show that a mapping $f : Z \rightarrow X$ is continuous, where X has the W -weak topology, if and only if $\psi \circ f : Z \rightarrow \mathbb{R}$ is continuous for all $\psi \in W$.

Solution. Recall that for a point $x \in X$, a neighborhood base of x with respect to the W -weak topology comprises the sets

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x) = \{x' \in X \mid |\psi_i(x - x')| < \epsilon \text{ for } 1 \leq i \leq n\},$$

where $\epsilon > 0$ and each ψ_i belongs to W .

First assume that f is continuous when X is given the W -weak topology. Fix $\psi \in W$. We claim $\psi \circ f : X \rightarrow \mathbb{R}$ is itself W -weakly continuous. Indeed, let $x_0 \in X$ and $\epsilon > 0$. The set $\mathcal{N}_{\epsilon, \psi}(x_0)$ is a W -weak neighborhood of x_0 , and for $x' \in \mathcal{N}_{\epsilon, \psi}(x_0)$ we have $|\psi(x_0) - \psi(x')| = |\psi(x_0 - x')| < \epsilon$. Thus ψ is continuous at x_0 , and since x_0 was arbitrary, ψ is W -weakly continuous. A composition of continuous mappings is continuous, so $\psi \circ f : Z \rightarrow \mathbb{R}$ is continuous.

Now assume conversely that $\psi \circ f$ is continuous for every $\psi \in W$. Let $z_0 \in Z$ and let $\mathcal{O}$ be a W -weakly open subset of X containing $f(z_0)$. Since the sets displayed above form a neighborhood base at $f(z_0)$, there are an $\epsilon > 0$ and functionals $\psi_1, \dots, \psi_n$ in W for which

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(f(z_0)) \subseteq \mathcal{O}.$$

For each index i with $1 \leq i \leq n$, the function $\psi_i \circ f$ is continuous at z_0 , so there is an open set $\mathcal{U}_i \subseteq Z$ containing z_0 for which

$$|\psi_i(f(z)) - \psi_i(f(z_0))| < \epsilon \text{ for all } z \in \mathcal{U}_i.$$

Set $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_i$. As a finite intersection of open sets containing z_0 , $\mathcal{U}$ is an open set containing z_0 . If $z \in \mathcal{U}$, then for each i , by the linearity of ψ_i ,

$$|\psi_i(f(z_0) - f(z))| = |\psi_i(f(z_0)) - \psi_i(f(z))| < \epsilon,$$

so that $f(z) \in \mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(f(z_0)) \subseteq \mathcal{O}$. Therefore $f(\mathcal{U}) \subseteq \mathcal{O}$, and f is continuous at z_0 . Since $z_0 \in Z$ was arbitrary, f is continuous.

Observe that the hypothesis that W separates points was used only to guarantee, by Proposition 21, that the W -weak topology is a Hausdorff topology making X a locally convex topological vector space; the equivalence itself required only the description of the W -weak neighborhood base.

Problem 14.45. Show that the topology on a finite dimensional locally convex topological vector space is induced by a norm.

Solution. Let X be a locally convex topological vector space with $\dim X = n$. If $n = 0$ then $X = \{0\}$ and the zero norm induces the only topology on X , so assume $n \geq 1$. Fix a basis $e_1, \dots, e_n$ of X and define the linear map

$$T : \mathbb{R}^n \rightarrow X, \quad T(a_1, \dots, a_n) = \sum_{i=1}^n a_i e_i,$$

where $\mathbb{R}^n$ carries the Euclidean norm $|\cdot|$ and its usual topology. Since $e_1, \dots, e_n$ is a basis, T is a linear bijection. We show T is a homeomorphism.

Step 1: T is continuous. For each fixed i , the map $a \mapsto a_i$ is continuous from $\mathbb{R}^n$ to $\mathbb{R}$, so the map $a \mapsto (a_i, e_i)$ is continuous from $\mathbb{R}^n$ into $\mathbb{R} \times X$; composing with the scalar multiplication map, which is continuous by property (ii) of the definition of a locally convex topological vector space, shows that $a \mapsto a_i e_i$ is continuous from $\mathbb{R}^n$ into X . Now vector addition is continuous by property (i), so a sum of two continuous X -valued maps is continuous, and by induction a sum of finitely many is. Hence T is continuous.

Step 2: T^{-1} is continuous. Let

$$S = \{a \in \mathbb{R}^n \mid |a| = 1\}$$

be the Euclidean unit sphere. S is compact, so by Step 1 and Proposition 20 of Chapter 11 the set $T(S)$ is a compact subset of X . Since T is one-to-one, $T(0) = 0$ and $0 \notin S$, we have $0 \notin T(S)$. The topology on X is Hausdorff, since that is part of the definition of a locally convex topological vector space, so by Proposition 16 of Chapter 11 the compact set $T(S)$ is closed; therefore $X \setminus T(S)$ is an open set containing the origin. By property (iii) of the definition there is a base at the origin consisting of convex sets, and members of a base at a point are neighborhoods, hence open; so we may choose a convex open set $\mathcal{C}$ with

$$0 \in \mathcal{C} \subseteq X \setminus T(S).$$

Put $U = T^{-1}(\mathcal{C}) \subseteq \mathbb{R}^n$. Since T is a linear bijection, U is convex and contains 0, and $U \cap S = \emptyset$ because $\mathcal{C} \cap T(S) = \emptyset$.

We claim $U \subseteq B := \{a \in \mathbb{R}^n \mid |a| < 1\}$. Otherwise there is $a \in U$ with $|a| \geq 1$. Set $\lambda = 1/|a|$, so $0 < \lambda \leq 1$. Then

$$\frac{a}{|a|} = \lambda a + (1 - \lambda) \cdot 0$$

is a convex combination of the two points a and 0 of U , hence lies in U ; but $|a/|a|| = 1$, so $a/|a| \in S$, contradicting $U \cap S = \emptyset$. This proves the claim.

Now let $x_0 \in X$ and let $\epsilon > 0$. By Proposition 22(i), $\epsilon \cdot \mathcal{C}$ is open and $x_0 + \epsilon \cdot \mathcal{C}$ is an open set containing x_0 . Since T^{-1} is linear,

$$T^{-1}(x_0 + \epsilon \cdot \mathcal{C}) = T^{-1}(x_0) + \epsilon \cdot U \subseteq T^{-1}(x_0) + \epsilon \cdot B,$$

which is precisely the Euclidean open ball of radius ϵ about $T^{-1}(x_0)$. Hence T^{-1} is continuous at x_0 , and since x_0 was arbitrary, T^{-1} is continuous.

Step 3: the norm. Define

$$\|x\| = |T^{-1}(x)| \text{ for all } x \in X.$$

Since T^{-1} is linear and one-to-one and $|\cdot|$ is a norm on $\mathbb{R}^n$, the function $\|\cdot\|$ is a norm on X : it is nonnegative and finite; $\|x\| = 0$ forces $T^{-1}(x) = 0$ and hence $x = 0$; $\|\lambda x\| = |T^{-1}(\lambda x)| = |\lambda T^{-1}(x)| = |\lambda| \|x\|$; and $\|x + y\| = |T^{-1}(x) + T^{-1}(y)| \leq \|x\| + \|y\|$.

By construction T carries the Euclidean ball of radius r about a onto the $\|\cdot\|$ -ball of radius r about $T(a)$, so T is a homeomorphism of $\mathbb{R}^n$ onto X equipped with the topology induced by $\|\cdot\|$. By Steps 1 and 2, T is also a homeomorphism of $\mathbb{R}^n$ onto X equipped with its given topology. Composing, the identity map on X is a homeomorphism from the $\|\cdot\|$ -topology onto the given topology; that is, the two topologies coincide. Thus the topology on X is induced by the norm $\|\cdot\|$.

Problem 14.46. Let X be a locally convex topological space. Show that the linear space X' of all linear continuous functionals $\psi : X \rightarrow \mathbb{R}$ also has a topology with respect to which it is a locally convex topological space on which, for each $x \in X$, the linear functional $\psi \mapsto \psi(x)$ is continuous.

Solution. First note that X' is a linear space under the pointwise operations: a sum of two continuous linear functionals is linear and continuous, as is a scalar multiple of one.

For $\psi_0 \in X'$, a number $\epsilon > 0$ and a finite list $x_1, \dots, x_n$ of points of X , define

$$\mathcal{N}_{\epsilon, x_1, \dots, x_n}(\psi_0) = \{\psi \in X' \mid |\psi(x_i) - \psi_0(x_i)| < \epsilon \text{ for } 1 \leq i \leq n\}.$$

Declare a subset $\mathcal{O}$ of X' to be open provided that for each $\psi_0 \in \mathcal{O}$ there are an $\epsilon > 0$ and points $x_1, \dots, x_n$ in X with $\mathcal{N}_{\epsilon, x_1, \dots, x_n}(\psi_0) \subseteq \mathcal{O}$. This is the topology on X' that we exhibit; it is the analogue for X' of the weak topology, with X playing the role of a separating family of linear functionals on X' . This collection of sets is a topology: $\emptyset$ and X' plainly qualify; the condition is inherited by arbitrary unions; and if $\mathcal{O}_1, \mathcal{O}_2$ are open and $\psi_0 \in \mathcal{O}_1 \cap \mathcal{O}_2$, choose $\epsilon_1, x_1, \dots, x_n$ for $\mathcal{O}_1$ and $\epsilon_2, y_1, \dots, y_m$ for $\mathcal{O}_2$; then with $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ the set $\mathcal{N}_{\epsilon, x_1, \dots, x_n, y_1, \dots, y_m}(\psi_0)$ is contained in $\mathcal{O}_1 \cap \mathcal{O}_2$.

Moreover each set $\mathcal{N}_{\epsilon, x_1, \dots, x_n}(\psi_0)$ is itself open: if ψ_1 belongs to it, put $\delta = \epsilon - \max_{1 \leq i \leq n} |\psi_1(x_i) - \psi_0(x_i)| > 0$; then for $\psi \in \mathcal{N}_{\delta, x_1, \dots, x_n}(\psi_1)$ and each i ,

$$|\psi(x_i) - \psi_0(x_i)| \leq |\psi(x_i) - \psi_1(x_i)| + |\psi_1(x_i) - \psi_0(x_i)| < \delta + (\epsilon - \delta) = \epsilon.$$

Consequently the sets $\mathcal{N}_{\epsilon, x_1, \dots, x_n}(\psi_0)$ form a neighborhood base at ψ_0 .

The topology is Hausdorff. Let ψ_1 and ψ_2 be distinct members of X' . Being distinct functions on X , there is a point $x \in X$ with $r = |\psi_1(x) - \psi_2(x)| > 0$. The sets $\mathcal{N}_{r/2, x}(\psi_1)$ and $\mathcal{N}_{r/2, x}(\psi_2)$ are open and disjoint, since a functional ψ in both would give $r = |\psi_1(x) - \psi_2(x)| \leq |\psi_1(x) - \psi(x)| + |\psi(x) - \psi_2(x)| < r$. Addition is continuous. Let $\psi_1, \psi_2 \in X'$ and let $\mathcal{N}_{\epsilon, x_1, \dots, x_n}(\psi_1 + \psi_2)$ be a basic neighborhood of their sum. If $\varphi_1 \in \mathcal{N}_{\epsilon/2, x_1, \dots, x_n}(\psi_1)$ and $\varphi_2 \in \mathcal{N}_{\epsilon/2, x_1, \dots, x_n}(\psi_2)$, then for each i

$$|(\varphi_1 + \varphi_2)(x_i) - (\psi_1 + \psi_2)(x_i)| \leq |\varphi_1(x_i) - \psi_1(x_i)| + |\varphi_2(x_i) - \psi_2(x_i)| < \epsilon,$$

so $\varphi_1 + \varphi_2$ lies in the given neighborhood. Thus addition is continuous at (ψ_1, ψ_2) .

Scalar multiplication is continuous. Let $(\lambda_0, \psi_0) \in \mathbb{R} \times X'$ and let $\mathcal{N}_{\epsilon, x_1, \dots, x_n}(\lambda_0 \psi_0)$ be a basic neighborhood of $\lambda_0 \psi_0$. Set $M = \max_{1 \leq i \leq n} |\psi_0(x_i)|$ and choose δ with $0 < \delta \leq 1$ and

$$(|\lambda_0| + 1)\delta + M\delta < \epsilon.$$

If $|\lambda - \lambda_0| < \delta$ and $\psi \in \mathcal{N}_{\delta, x_1, \dots, x_n}(\psi_0)$, then for each i ,

$$|\lambda \psi(x_i) - \lambda_0 \psi_0(x_i)| \leq |\lambda| |\psi(x_i) - \psi_0(x_i)| + |\lambda - \lambda_0| |\psi_0(x_i)| < (|\lambda_0| + 1)\delta + M\delta < \epsilon.$$

Hence $\lambda \psi \in \mathcal{N}_{\epsilon, x_1, \dots, x_n}(\lambda_0 \psi_0)$, and scalar multiplication is continuous at (λ_0, ψ_0) .

There is a base at the origin consisting of convex sets. For each $x \in X$ the evaluation $E_x : X' \rightarrow \mathbb{R}$, $E_x(\psi) = \psi(x)$, is a linear functional on X' , since $(\psi + \varphi)(x) = \psi(x) + \varphi(x)$ and $(\lambda \psi)(x) = \lambda \psi(x)$. A basic neighborhood of the origin of X' is

$$\mathcal{N}_{\epsilon, x_1, \dots, x_n}(0) = \{\psi \in X' \mid |E_{x_i}(\psi)| < \epsilon \text{ for } 1 \leq i \leq n\},$$

and this set is convex: if $|E_{x_i}(\psi)| < \epsilon$ and $|E_{x_i}(\varphi)| < \epsilon$ for all i and $0 \leq \lambda \leq 1$, then

$$|E_{x_i}(\lambda\psi + (1-\lambda)\varphi)| = |\lambda E_{x_i}(\psi) + (1-\lambda)E_{x_i}(\varphi)| \leq \lambda|E_{x_i}(\psi)| + (1-\lambda)|E_{x_i}(\varphi)| < \lambda\epsilon + (1-\lambda)\epsilon = \epsilon,$$

where the strict inequality holds because λ and $1-\lambda$ are nonnegative with sum 1, so at least one of them is positive and the corresponding bound is strict. Since these convex sets form a base at the origin, property (iii) holds. Thus X' , with this topology, is a locally convex topological vector space. Finally, fix $x \in X$ and consider the linear functional $E_x : \psi \mapsto \psi(x)$ on X' . Given $\psi_0 \in X'$ and $\epsilon > 0$, the set $\mathcal{N}_{\epsilon, x}(\psi_0)$ is a neighborhood of ψ_0 and, for ψ in it, $|E_x(\psi) - E_x(\psi_0)| = |\psi(x) - \psi_0(x)| < \epsilon$. Hence E_x is continuous at ψ_0 , and therefore continuous on X' .

Problem 14.47. Let X and Y be locally convex topological vector spaces and $T : X \rightarrow Y$ be linear, one-to-one, and onto. Show that T is a topological homeomorphism if and only if it maps a base at the origin for the topology on X to a base at the origin for the topology on Y .

Solution. Since T is linear, $T(0) = 0$, and since T is a linear bijection its inverse $T^{-1} : Y \rightarrow X$ is also linear. We prove: T is a homeomorphism if and only if there is a base $\mathcal{B}$ at the origin for the topology on X for which $T(\mathcal{B}) = \{T(B) \mid B \in \mathcal{B}\}$ is a base at the origin for the topology on Y . The forward implication will in fact show that when T is a homeomorphism, every base at the origin of X is carried onto a base at the origin of Y .

Assume first that T is a topological homeomorphism, and let $\mathcal{B}$ be any base at the origin for the topology on X . Let $B \in \mathcal{B}$. Then B is open in X and $0 \in B$, so $T(B) = (T^{-1})^{-1}(B)$ is open in Y (because T^{-1} is continuous) and $0 = T(0) \in T(B)$. Now let $\mathcal{O}$ be any open subset of Y containing the origin. Since T is continuous, $T^{-1}(\mathcal{O})$ is open in X and contains 0, so there is a $B \in \mathcal{B}$ with $B \subseteq T^{-1}(\mathcal{O})$, and therefore $T(B) \subseteq \mathcal{O}$. Hence $T(\mathcal{B})$ is a base at the origin for the topology on Y .

Assume conversely that $\mathcal{B}$ is a base at the origin for the topology on X and that $T(\mathcal{B})$ is a base at the origin for the topology on Y . In particular each $T(B)$, for $B \in \mathcal{B}$, is an open subset of Y containing the origin: by the definition in Chapter 11, a base at a point consists of neighborhoods of that point, and a neighborhood of a point is an open set containing it. For the same reason each $B \in \mathcal{B}$ is an open subset of X containing the origin.

We show T is continuous. Let $x_0 \in X$ and let $\mathcal{O}$ be an open subset of Y containing $T(x_0)$. By Proposition 22(i), applied in Y with the translation by $-T(x_0)$, the set $\mathcal{O} - T(x_0)$ is open in Y , and it contains the origin. Since $T(\mathcal{B})$ is a base at the origin, there is a $B \in \mathcal{B}$ with $T(B) \subseteq \mathcal{O} - T(x_0)$. By Proposition 22(i) applied in X , the set $x_0 + B$ is open in X and contains x_0 , and by linearity

$$T(x_0 + B) = T(x_0) + T(B) \subseteq T(x_0) + (\mathcal{O} - T(x_0)) = \mathcal{O}.$$

Hence T is continuous at x_0 ; as x_0 was arbitrary, T is continuous.

We show T^{-1} is continuous. Let $y_0 \in Y$ and let $\mathcal{U}$ be an open subset of X containing $T^{-1}(y_0)$. By Proposition 22(i) the set $\mathcal{U} - T^{-1}(y_0)$ is open in X and contains the origin, so there is a $B \in \mathcal{B}$ with $B \subseteq \mathcal{U} - T^{-1}(y_0)$. Now $T(B)$ is an open subset of Y containing the origin, so $y_0 + T(B)$ is an open subset of Y containing y_0 , again by Proposition 22(i). Since T^{-1} is linear and $T^{-1}(T(B)) = B$,

$$T^{-1}(y_0 + T(B)) = T^{-1}(y_0) + B \subseteq T^{-1}(y_0) + (\mathcal{U} - T^{-1}(y_0)) = \mathcal{U}.$$

Hence T^{-1} is continuous at y_0 ; as y_0 was arbitrary, T^{-1} is continuous.

Therefore T is a one-to-one mapping of X onto Y that is continuous and has continuous inverse, that is, T is a topological homeomorphism.

Problem 14.48. Let X be a linear space and the function $\sigma : X \rightarrow [0, \infty)$ have the following properties: for all $u, v \in X$, (i) $\sigma(u + v) \leq \sigma(u) + \sigma(v)$; (ii) $\sigma(u) = 0$ if and only if $u = 0$; (iii) $\sigma(u) = \sigma(-u)$. Define $\rho(u, v) = \sigma(u - v)$. Show that ρ is a metric on X .

Solution. Since σ takes values in $[0, \infty)$, the function ρ is defined on all of $X \times X$ and takes finite, nonnegative values.

Identity of indiscernibles. For $u, v \in X$ we have $\rho(u, v) = \sigma(u - v)$, and by property (ii), $\sigma(u - v) = 0$ if and only if $u - v = 0$, that is, if and only if $u = v$. In particular $\rho(u, u) = \sigma(0) = 0$, and $\rho(u, v) > 0$ whenever $u \neq v$.

Symmetry. For $u, v \in X$, since $v - u = -(u - v)$, property (iii) gives

$$\rho(v, u) = \sigma(v - u) = \sigma(-(u - v)) = \sigma(u - v) = \rho(u, v).$$

The triangle inequality. Let $u, v, w \in X$. Then $u - w = (u - v) + (v - w)$, so by property (i),

$$\rho(u, w) = \sigma((u - v) + (v - w)) \leq \sigma(u - v) + \sigma(v - w) = \rho(u, v) + \rho(v, w).$$

Thus ρ is a nonnegative, real-valued, symmetric function on $X \times X$ that vanishes exactly on the diagonal and satisfies the triangle inequality; that is, ρ is a metric on X .

Problem 14.49. (Nikodym) Let X be the linear space of all measurable real-valued functions on $[0, 1]$. Define

$$\sigma(f) = \int_{[0,1]} \frac{|f|}{1 + |f|} \text{ for all } f \in X.$$

- (i) Use Problem 48 to show that $\rho(u, v) = \sigma(u - v)$ defines a metric on X .
- (ii) Show that $\{f_n\} \rightarrow f$ with respect to the metric ρ if and only if $\{f_n\} \rightarrow f$ in measure.
- (iii) Show that (X, ρ) is a complete metric space.
- (iv) Show that the mapping $(f, g) \mapsto f + g$ is a continuous mapping of $X \times X$ into X .
- (v) Show that the mapping $(\lambda, f) \mapsto \lambda \cdot f$ is a continuous mapping of $\mathbb{R} \times X$ into X .
- (vi) Show that there are no nonzero continuous linear functionals ψ on X . [Hint: Let $\psi : X \rightarrow \mathbb{R}$ be linear and continuous. Show that there is an n such that $\psi(f) = 0$ whenever f is the characteristic function of an interval of length less than $1/n$. Hence $\psi(f) = 0$ for all step functions f .]

Solution. Integration is with respect to Lebesgue measure m on $[0, 1]$, and, as always in this setting, two functions in X that agree almost everywhere are identified; without this identification property (ii) of Problem 48 fails.

Throughout put

$$\varphi(t) = \frac{t}{1 + t} \text{ for } t \geq 0,$$

so that $\sigma(f) = \int_{[0,1]} \varphi(|f|)$. We record three elementary properties of φ .

First, φ is increasing on $[0, \infty)$, since $\varphi(t) = 1 - 1/(1 + t)$. Second, $0 = \varphi(0) \leq \varphi(t) < 1$ for all $t \geq 0$. Third, φ is subadditive: for $s, t \geq 0$,

$$\varphi(s + t) = \frac{s}{1 + s + t} + \frac{t}{1 + s + t} \leq \frac{s}{1 + s} + \frac{t}{1 + t} = \varphi(s) + \varphi(t).$$

If $f \in X$, then $\varphi(|f|) = \Phi \circ f$, where $\Phi(t) = t/(1 + t)$ is continuous on all of $\mathbb{R}$; by Proposition 7 of Chapter 3 the composition of such a Φ with the measurable function f is measurable, so $\varphi(|f|)$ is a measurable function, and $0 \leq \varphi(|f|) < 1$. Hence $\sigma(f)$ is defined and

$$0 \leq \sigma(f) \leq m([0, 1]) = 1,$$

so σ maps X into $[0, \infty)$.

(i) We verify the three hypotheses of Problem 48.

Subadditivity. Let $f, g \in X$. Pointwise $|f + g| \leq |f| + |g|$, so by the monotonicity and subadditivity of φ ,

$$\varphi(|f + g|) \leq \varphi(|f| + |g|) \leq \varphi(|f|) + \varphi(|g|) \text{ on } [0, 1].$$

Integrating and using the monotonicity and linearity of integration,

$$\sigma(f + g) \leq \sigma(f) + \sigma(g).$$

Vanishing. $\varphi(|f|)$ is a nonnegative measurable function, so by Proposition 9 of Chapter 4, $\sigma(f) = 0$ if and only if $\varphi(|f|) = 0$ a.e. on $[0, 1]$. Since $\varphi(t) = 0$ only for $t = 0$, this holds if and only if $f = 0$ a.e., that is, if and only if $f = 0$ as an element of X .

Evenness. $|-f| = |f|$, so $\sigma(-f) = \sigma(f)$.

By Problem 48, $\rho(u, v) = \sigma(u - v)$ is a metric on X .

(ii) Recall that $\{f_n\} \rightarrow f$ in measure on $[0, 1]$ means that for each $\eta > 0$,

$$\lim_{n \rightarrow \infty} m\{x \in [0, 1] \mid |f_n(x) - f(x)| > \eta\} = 0.$$

Write $h_n = |f_n - f|$ and, for $\eta > 0$, put $E_n(\eta) = \{x \in [0, 1] \mid h_n(x) > \eta\}$; each h_n is measurable and real-valued, so each $E_n(\eta)$ is a measurable subset of $[0, 1]$ and the measures below are defined.

We first establish two inequalities. On $E_n(\eta)$ the monotonicity of φ gives $\varphi(h_n) \geq \varphi(\eta) = \eta/(1 + \eta)$, so

$$\rho(f_n, f) = \int_{[0,1]} \varphi(h_n) \geq \int_{E_n(\eta)} \varphi(h_n) \geq \frac{\eta}{1 + \eta} m(E_n(\eta)),$$

that is,

$$m(E_n(\eta)) \leq \frac{1 + \eta}{\eta} \rho(f_n, f). \quad (\text{A})$$

On the other hand, splitting the integral and using $\varphi < 1$ on $E_n(\eta)$ and $\varphi(h_n) \leq \varphi(\eta) \leq \eta$ off $E_n(\eta)$,

$$\rho(f_n, f) = \int_{E_n(\eta)} \varphi(h_n) + \int_{[0,1] \setminus E_n(\eta)} \varphi(h_n) \leq 1 \cdot m(E_n(\eta)) + \eta \cdot m([0, 1] \setminus E_n(\eta)) \leq m(E_n(\eta)) + \eta. \quad (\text{B})$$

Both splittings are legitimate because $\varphi(h_n)$ is a bounded nonnegative measurable function on the finite measure space $[0, 1]$, hence integrable, and integration over $[0, 1]$ is the sum of the integrals over the two disjoint measurable pieces.

Now suppose $\rho(f_n, f) \rightarrow 0$ and let $\eta > 0$. By (A), $m(E_n(\eta)) \rightarrow 0$. Since $\eta > 0$ was arbitrary, $\{f_n\} \rightarrow f$ in measure.

Conversely suppose $\{f_n\} \rightarrow f$ in measure and let $\epsilon > 0$. Apply (B) with $\eta = \epsilon/2$: since $m(E_n(\epsilon/2)) \rightarrow 0$, there is an N such that $m(E_n(\epsilon/2)) < \epsilon/2$ for $n \geq N$, and then $\rho(f_n, f) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $\rho(f_n, f) \rightarrow 0$.

(iii) Let $\{f_n\}$ be Cauchy with respect to ρ . The inequality (A), applied with f replaced by f_k , reads

$$m\{x \in [0, 1] \mid |f_n(x) - f_k(x)| > \eta\} \leq \frac{1 + \eta}{\eta} \rho(f_n, f_k),$$

so $\{f_n\}$ is Cauchy in measure.

Choose inductively a strictly increasing sequence of indices $n_1 < n_2 < \dots$ such that

$$\rho(f_{n_{k+1}}, f_{n_k}) < \frac{2^{-k}}{1 + 2^{-k}} \cdot 2^{-k} \text{ for all } k.$$

Setting $\eta = 2^{-k}$ in the displayed inequality, the sets

$$E_k = \{x \in [0, 1] \mid |f_{n_{k+1}}(x) - f_{n_k}(x)| > 2^{-k}\}$$

satisfy $m(E_k) < 2^{-k}$, so $\sum_{k=1}^{\infty} m(E_k) < \infty$. By the Borel-Cantelli Lemma of Chapter 2, almost all $x \in [0, 1]$ belong to at most finitely many of the E_k . Let Z be the set of measure zero consisting of those x that belong to infinitely many E_k .

Fix $x \in [0, 1] \setminus Z$. There is a K with $|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq 2^{-k}$ for all $k \geq K$, so

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < \infty$$

and hence, the partial sums of a telescoping series being $f_{n_{j+1}}(x) - f_{n_1}(x)$, the sequence $\{f_{n_k}(x)\}$ is a Cauchy sequence of real numbers and so converges. Define $f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$ for $x \notin Z$ and

$f(x) = 0$ for $x \in Z$. Then f is real-valued and $\{f_{n_k}\} \rightarrow f$ pointwise a.e. on $[0, 1]$, so f is measurable by Proposition 9 of Chapter 3. Thus $f \in X$.

Since φ is continuous with $\varphi(0) = 0$, we have $\varphi(|f_{n_k} - f|) \rightarrow 0$ pointwise a.e. on $[0, 1]$, while $0 \leq \varphi(|f_{n_k} - f|) \leq 1$ and the constant function 1 is integrable over $[0, 1]$. By the Lebesgue Dominated Convergence Theorem,

$$\rho(f_{n_k}, f) = \int_{[0,1]} \varphi(|f_{n_k} - f|) \rightarrow 0.$$

Finally, a Cauchy sequence possessing a convergent subsequence converges to the same limit: given $\epsilon > 0$, choose N with $\rho(f_n, f_j) < \epsilon/2$ for $n, j \geq N$, then choose k with $n_k \geq N$ and $\rho(f_{n_k}, f) < \epsilon/2$; for $n \geq N$,

$$\rho(f_n, f) \leq \rho(f_n, f_{n_k}) + \rho(f_{n_k}, f) < \epsilon.$$

Therefore $\{f_n\} \rightarrow f$ in (X, ρ) , and (X, ρ) is complete.

(iv) Give $X \times X$ the product topology, which is induced by the metric $d((f, g), (f_0, g_0)) = \max\{\rho(f, f_0), \rho(g, g_0)\}$. Using the subadditivity of σ established in part (i),

$$\rho(f + g, f_0 + g_0) = \sigma((f - f_0) + (g - g_0)) \leq \sigma(f - f_0) + \sigma(g - g_0) = \rho(f, f_0) + \rho(g, g_0).$$

Hence, given $\epsilon > 0$, the choice $\delta = \epsilon/2$ gives $\rho(f + g, f_0 + g_0) < \epsilon$ whenever $d((f, g), (f_0, g_0)) < \delta$. So addition is continuous (indeed uniformly continuous) from $X \times X$ into X .

(v) We first record a homogeneity estimate. Let N be a natural number and $|\lambda| \leq N$. Pointwise $|\lambda f| \leq N|f|$, so by the monotonicity of φ and then the subadditivity of σ applied $N - 1$ times,

$$\sigma(\lambda f) \leq \sigma(Nf) = \sigma(\underbrace{f + \cdots + f}_N) \leq N\sigma(f). \quad (\text{C})$$

Second, for a fixed $f_0 \in X$ we have $\lim_{j \rightarrow \infty} \sigma(f_0/j) = 0$: indeed $\varphi(|f_0|/j) \rightarrow 0$ pointwise on $[0, 1]$ and $0 \leq \varphi(|f_0|/j) \leq 1$, so the Lebesgue Dominated Convergence Theorem applies.

Now fix $(\lambda_0, f_0) \in \mathbb{R} \times X$ and let $\epsilon > 0$. Choose a natural number $N \geq |\lambda_0| + 1$, and a natural number j with $\sigma(f_0/j) < \epsilon/2$. Put

$$\delta = \min\{1, 1/j, \epsilon/(2N)\} > 0.$$

Suppose $|\lambda - \lambda_0| < \delta$ and $\rho(f, f_0) < \delta$. By the subadditivity of σ ,

$$\rho(\lambda f, \lambda_0 f_0) = \sigma(\lambda f - \lambda_0 f_0) \leq \sigma(\lambda(f - f_0)) + \sigma((\lambda - \lambda_0)f_0).$$

For the first term, $|\lambda| < |\lambda_0| + 1 \leq N$, so by (C),

$$\sigma(\lambda(f - f_0)) \leq N\sigma(f - f_0) = N\rho(f, f_0) < N\delta \leq \epsilon/2.$$

For the second term, $|\lambda - \lambda_0| < \delta \leq 1/j$, so $|(\lambda - \lambda_0)f_0| \leq |f_0|/j$ pointwise, and the monotonicity of φ together with the monotonicity of integration gives

$$\sigma((\lambda - \lambda_0)f_0) \leq \sigma(f_0/j) < \epsilon/2.$$

Adding, $\rho(\lambda f, \lambda_0 f_0) < \epsilon$. Hence scalar multiplication is continuous at (λ_0, f_0) , and since this point was arbitrary, it is continuous on $\mathbb{R} \times X$.

(vi) Let $\psi : X \rightarrow \mathbb{R}$ be linear and continuous. Since $\psi(0) = 0$, continuity at the origin provides a $\delta > 0$ for which

$$g \in X \text{ and } \sigma(g) < \delta \implies |\psi(g)| < 1. \quad (\text{D})$$

The key observation is this: if E is a measurable subset of $[0, 1]$ with $m(E) < \delta$ and $g \in X$ is arbitrary, then $\psi(g\chi_E) = 0$. To see this, let $c \in \mathbb{R}$. Since $0 \leq \varphi < 1$,

$$\sigma(cg\chi_E) = \int_{[0,1]} \varphi(|cg\chi_E|) = \int_E \varphi(|cg|) \leq \int_E 1 = m(E) < \delta,$$

so by (D) and the linearity of ψ ,

$$|c| |\psi(g\chi_E)| = |\psi(cg\chi_E)| < 1 \text{ for every } c \in \mathbb{R}.$$

If $\psi(g\chi_E)$ were nonzero, the choice $c = 2/|\psi(g\chi_E)|$ would give $2 < 1$, which is false. Hence $\psi(g\chi_E) = 0$. Taking g to be the constant function 1 and choosing a natural number n with $1/n < \delta$, this yields the assertion of the hint: $\psi(\chi_I) = 0$ for every interval I contained in $[0, 1]$ of length less than $1/n$, and hence, by linearity and the fact that every interval is a finite disjoint union of such intervals, ψ vanishes on every step function.

In fact the observation gives the conclusion outright, with no appeal to approximation by step functions. Fix a natural number n with $1/n < \delta$ and partition $[0, 1]$ into the disjoint measurable sets

$$E_k = \left[\frac{k-1}{n}, \frac{k}{n} \right) \text{ for } 1 \leq k \leq n-1, \quad E_n = \left[\frac{n-1}{n}, 1 \right],$$

each of measure $1/n < \delta$. Let $f \in X$ be arbitrary. Each $f\chi_{E_k}$ is a measurable real-valued function on $[0, 1]$, hence belongs to X , and

$$f = \sum_{k=1}^n f\chi_{E_k}.$$

By the linearity of ψ and the key observation,

$$\psi(f) = \sum_{k=1}^n \psi(f\chi_{E_k}) = 0.$$

Since $f \in X$ was arbitrary, ψ is the zero functional. Therefore there are no nonzero continuous linear functionals on X .

14.8 Problems 14.50–14.56

Problem 14.50. (Day) For $0 < p < 1$, let X be the linear space of all measurable (with respect to the Lebesgue measure m) real-valued functions on $[0, 1]$ for which $|f|^p$ is integrable. Define

$$\sigma(f) = \int_{[0,1]} |f|^p dm \quad \text{for all } f \in X.$$

- (i) Use Problem 48 to show that $\rho(u, v) = \sigma(u - v)$ defines a metric on X .
- (ii) Show that the linear space X , with the topology determined by ρ , is a topological vector space.
- (iii) For a nonzero function f in X and natural number n , show that there is a partition $0 = x_0 < x_1 < \dots < x_n = 1$ of $[0, 1]$ for which $\int_{x_{k-1}}^{x_k} f = 1/n \cdot \int_0^1 f$, for all $1 \leq k \leq n$.
- (iv) For a nonzero function f in X and natural number n , show that there are functions $f_1, \dots, f_n$ for which $\rho(f_k, 0) < 1/n$ for $1 \leq k \leq n$ and

$$f = \sum_{k=1}^n 1/n \cdot f_k.$$

- (v) Show that there are no continuous nonzero linear functionals on X .

Solution. Throughout we follow the convention in force in this chapter and identify two functions in X that agree almost everywhere on $[0, 1]$; thus X is a linear space of equivalence classes.

Preliminary inequality. For $0 < p < 1$ and $a, b \geq 0$,

$$(a + b)^p \leq a^p + b^p.$$

Indeed, if $a + b = 0$ this is trivial. Otherwise put $t = a/(a + b)$ and $s = b/(a + b)$, so that $t, s \in [0, 1]$ and $t + s = 1$. Since $0 < p < 1$ and $0 \leq t \leq 1$ we have $t^p \geq t$, and likewise $s^p \geq s$. Hence $t^p + s^p \geq t + s = 1$, and multiplying through by $(a + b)^p$ gives $a^p + b^p \geq (a + b)^p$.

(i) First observe that X really is a linear space. If $f, g \in X$ and $\lambda \in \mathbb{R}$, then $|f + g|^p \leq (|f| + |g|)^p \leq |f|^p + |g|^p$ pointwise by the preliminary inequality, so $|f + g|^p$ is integrable by the integral comparison test, and $|\lambda f|^p = |\lambda|^p |f|^p$ is integrable. Thus X is closed under addition and scalar multiplication, and σ is a finite nonnegative function on X .

We verify the three properties required in Problem 48.

(a) Subadditivity: integrating the pointwise inequality $|u + v|^p \leq |u|^p + |v|^p$ over $[0, 1]$ and using linearity and monotonicity of the integral,

$$\sigma(u + v) = \int_{[0,1]} |u + v|^p dm \leq \int_{[0,1]} |u|^p dm + \int_{[0,1]} |v|^p dm = \sigma(u) + \sigma(v).$$

(b) $\sigma(u) = 0$ if and only if $u = 0$: clearly $\sigma(0) = 0$. Conversely, if $\sigma(u) = \int_{[0,1]} |u|^p dm = 0$, then since $|u|^p \geq 0$ is integrable with zero integral, $|u|^p = 0$ almost everywhere, hence $u = 0$ almost everywhere, that is, u is the zero element of X .

(c) Symmetry: $\sigma(-u) = \int_{[0,1]} |-u|^p dm = \int_{[0,1]} |u|^p dm = \sigma(u)$.

By Problem 48, $\rho(u, v) = \sigma(u - v)$ is a metric on X .

(ii) The topology determined by a metric is Hausdorff, so it remains to check that vector addition and scalar multiplication are continuous. Note first the two identities that will be used repeatedly: $\sigma(\lambda f) = |\lambda|^p \sigma(f)$ for $\lambda \in \mathbb{R}$, and σ is subadditive.

Addition. Let $(f_0, g_0) \in X \times X$ and $\epsilon > 0$. If $\rho(f, f_0) < \epsilon/2$ and $\rho(g, g_0) < \epsilon/2$, then by subadditivity

$$\rho(f + g, f_0 + g_0) = \sigma((f - f_0) + (g - g_0)) \leq \sigma(f - f_0) + \sigma(g - g_0) < \epsilon.$$

Hence $(f, g) \mapsto f + g$ is continuous from $X \times X$ (with the product topology) to X .

Scalar multiplication. Let $(\lambda_0, f_0) \in \mathbb{R} \times X$ and let $\epsilon > 0$. For any $\lambda \in \mathbb{R}$ and $f \in X$,

$$\sigma(\lambda f - \lambda_0 f_0) = \sigma((\lambda - \lambda_0)f + \lambda_0(f - f_0)) \leq |\lambda - \lambda_0|^p \sigma(f) + |\lambda_0|^p \sigma(f - f_0),$$

and $\sigma(f) \leq \sigma(f - f_0) + \sigma(f_0)$. So if $\delta \in (0, 1)$ and $|\lambda - \lambda_0| < \delta$, $\rho(f, f_0) = \sigma(f - f_0) < \delta$, then

$$\rho(\lambda f, \lambda_0 f_0) \leq \delta^p (\delta + \sigma(f_0)) + |\lambda_0|^p \delta.$$

The right-hand side tends to 0 as $\delta \rightarrow 0^+$ (here $\sigma(f_0)$ and λ_0 are fixed and $p > 0$), so choosing δ small enough makes it less than ϵ . Hence $(\lambda, f) \mapsto \lambda \cdot f$ is continuous. Therefore X , with the topology induced by ρ , is a topological vector space.

(iii) As printed, this part cannot be correct with the plain integral of f , for two independent reasons. First, a member of X need not be integrable at all, so the symbol $\int_0^1 f$ may be meaningless. Fix $0 < p < 1$ and let $f(x) = 1/x$ for $x \in (0, 1]$, $f(0) = 0$. Then $|f|^p = x^{-p}$ has $\int_{[0,1]} x^{-p} dm = 1/(1-p) < \infty$, so $f \in X$; but $\int_{[0,1]} |f| dm = \int_{[0,1]} x^{-1} dm = \infty$, so f is not integrable.

Second, even for an integrable f the assertion is false. Take $f = \chi_{[0,1/2)} - \chi_{[1/2,1]}$, which lies in X since $|f|^p = 1$, and put $G(x) = \int_0^x f$, so that $G(x) = x$ on $[0, 1/2]$ and $G(x) = 1 - x$ on $[1/2, 1]$. Here $\int_0^1 f = G(1) = 0$, so the asserted identity $\int_{x_{k-1}}^{x_k} f = (1/n) \int_0^1 f$ would read $G(x_k) - G(x_{k-1}) = 0$ for $1 \leq k \leq n$; since $G(x_0) = G(0) = 0$, induction would force $G(x_k) = 0$ for every k , in particular $G(x_1) = 0$ with $0 < x_1 < 1$. But $G > 0$ on $(0, 1)$. So no such partition exists.

(For an integrable f with $\int_0^1 f \neq 0$ the printed statement is in fact true, by repeated application of the Intermediate Value Theorem to the continuous function G on the successive intervals $[x_{k-1}, 1]$; the point is that this is not what the exercise needs.)

What is meant, and what part (iv) uses, is the splitting of the σ -mass of f into n equal pieces; that is, the integrand in question is $|f|^p$, which is integrable for every $f \in X$ by the very definition of X . We prove this version.

Let $f \in X$ be nonzero, so $\sigma(f) > 0$, and let n be a natural number. Define

$$F(x) = \int_{[0,x]} |f|^p dm \quad \text{for } x \in [0, 1].$$

Since $|f|^p$ is integrable, F is nondecreasing and continuous on $[0, 1]$; continuity is the continuity of the integral with respect to the domain (Proposition 23 of Chapter 4: for each $\epsilon > 0$ there is $\delta > 0$ such that $\int_E |f|^p dm < \epsilon$ whenever $m(E) < \delta$). Moreover $F(0) = 0$ and $F(1) = \sigma(f)$.

For $1 \leq k \leq n-1$ the number $(k/n)\sigma(f)$ lies strictly between $F(0)$ and $F(1)$, so by the Intermediate Value Theorem the set $\{x \in [0, 1] : F(x) = (k/n)\sigma(f)\}$ is nonempty; let x_k be its smallest element (the set is closed, since F is continuous). Put $x_0 = 0$ and $x_n = 1$. Since F is nondecreasing and $F(x_{k-1}) < F(x_k)$, we get $x_{k-1} < x_k$; thus $0 = x_0 < x_1 < \cdots < x_n = 1$ is a partition of $[0, 1]$, and for $1 \leq k \leq n$,

$$\int_{[x_{k-1}, x_k]} |f|^p dm = F(x_k) - F(x_{k-1}) = \frac{1}{n} \sigma(f).$$

(iv) Let $f \in X$ be nonzero. Fix a natural number n , take the partition of part (iii), and define

$$f_k = n \cdot f \cdot \chi_{[x_{k-1}, x_k]} \quad \text{for } 1 \leq k \leq n.$$

Each f_k belongs to X , and since the intervals $[x_{k-1}, x_k]$ overlap only in finitely many points (a set of measure zero),

$$\sum_{k=1}^n \frac{1}{n} f_k = \sum_{k=1}^n f \cdot \chi_{[x_{k-1}, x_k]} = f \quad \text{in } X.$$

Moreover, using $\sigma(\lambda g) = |\lambda|^p \sigma(g)$ and part (iii),

$$\rho(f_k, 0) = \sigma(f_k) = n^p \int_{[x_{k-1}, x_k]} |f|^p dm = n^p \cdot \frac{1}{n} \sigma(f) = n^{p-1} \sigma(f).$$

Since $0 < p < 1$ we have $p-1 < 0$, so $n^{p-1} \sigma(f) \rightarrow 0$ as $n \rightarrow \infty$. Consequently: given any $\epsilon > 0$, choosing n so large that $n^{p-1} \sigma(f) < \epsilon$ produces $f_1, \dots, f_n$ with $\rho(f_k, 0) < \epsilon$ for all k and $f = \sum_{k=1}^n (1/n) f_k$. This is the assertion needed, and is the correct form of (iv). (The literal bound $\rho(f_k, 0) < 1/n$ printed in the text cannot hold for every n : if $f = \sum_{k=1}^n (1/n) f_k$ then subadditivity and homogeneity give $\sigma(f) \leq n^{-p} \sum_{k=1}^n \sigma(f_k) < n^{-p}$, which fails once n is large, $\sigma(f)$ being a fixed positive number. The intended and useful statement is the one just proved: the ρ -size of the pieces can be made arbitrarily small.)

(v) Let $\psi : X \rightarrow \mathbb{R}$ be a linear functional that is continuous with respect to the topology induced by ρ , and suppose $\psi \neq 0$. Choose $g \in X$ with $\psi(g) \neq 0$; replacing g by $g/\psi(g)$ (permissible, since ψ is linear), we may assume $\psi(g) = 1$. In particular $g \neq 0$, so $\sigma(g) > 0$.

By continuity of ψ at the origin there is an $\epsilon > 0$ such that

$$|\psi(h)| < 1 \quad \text{whenever } \rho(h, 0) = \sigma(h) < \epsilon.$$

By part (iv), choose n so large that $n^{p-1} \sigma(g) < \epsilon$ and let $g_1, \dots, g_n$ be the corresponding functions: $\sigma(g_k) = n^{p-1} \sigma(g) < \epsilon$ for each k , and $g = \sum_{k=1}^n (1/n) g_k$. Then $|\psi(g_k)| < 1$ for every k , so by linearity

$$1 = |\psi(g)| = \left| \frac{1}{n} \sum_{k=1}^n \psi(g_k) \right| \leq \frac{1}{n} \sum_{k=1}^n |\psi(g_k)| < \frac{1}{n} \cdot n \cdot 1 = 1,$$

a contradiction. Therefore no nonzero continuous linear functional exists on X : the only continuous linear functional on X is the zero functional.

This justifies the assertion in the Remark preceding these problems: X is a topological vector space that is not locally convex, and on it the Hahn-Banach machinery is vacuous.

Problem 14.51. Let S be the space of all sequences of real numbers, and define

$$\sigma(x) = \sum \frac{|x_n|}{2^n [1 + |x_n|]} \quad \text{for all } x = \{x_n\} \in S.$$

Prove the analogues of (i), (iii), (iv), and (v) of the preceding problem. What is the most general

Solution. A word first on which four assertions are intended. Read literally, the preceding problem is Problem 50, but its part (iii) is a statement about partitions of $[0, 1]$ and has no meaning for a space of sequences, and its part (v) asserts that there are no nonzero continuous linear functionals, which is flatly incompatible with the closing question here. The parts referred to are therefore those of Problem 49 (Nikodym), whose σ is of the same shape as the present one, namely $\sigma(f) = \int_{[0,1]} |f|/(1+|f|) dm$. Those parts read, transcribed to S : (i) $\rho(u, v) = \sigma(u - v)$ is a metric on S ; (iii) (S, ρ) is a complete metric space; (iv) $(x, y) \mapsto x + y$ is continuous from $S \times S$ into S ; (v) $(\lambda, x) \mapsto \lambda \cdot x$ is continuous from $\mathbb{R} \times S$ into S . We prove these four assertions and then determine all continuous linear functionals. Throughout, write $\varphi(t) = t/(1+t)$ for $t \geq 0$, so that $\sigma(x) = \sum_{n=1}^{\infty} 2^{-n}\varphi(|x_n|)$. Since $0 \leq \varphi < 1$, the series converges and $0 \leq \sigma \leq 1$ on S ; in particular σ is a finite nonnegative function on the linear space S of all real sequences.

Two properties of φ are used constantly.

(1) φ is strictly increasing on $[0, \infty)$, with $\varphi(t) \rightarrow 0$ as $t \rightarrow 0^+$ and inverse $\varphi^{-1}(s) = s/(1-s)$ on $[0, 1)$. Indeed $\varphi(t) = 1 - 1/(1+t)$.

(2) φ is subadditive: for $a, b \geq 0$,

$$\varphi(a+b) = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b} = \varphi(a) + \varphi(b).$$

(3) $\varphi(t) \leq t$ for $t \geq 0$, since $1+t \geq 1$.

(i) We check the three hypotheses of Problem 48 for σ .

Subadditivity: for $x, y \in S$ and each n , $|x_n + y_n| \leq |x_n| + |y_n|$, so by (1) and (2),

$$\varphi(|x_n + y_n|) \leq \varphi(|x_n| + |y_n|) \leq \varphi(|x_n|) + \varphi(|y_n|).$$

Multiplying by 2^{-n} and summing over n gives $\sigma(x+y) \leq \sigma(x) + \sigma(y)$.

Nondegeneracy: $\sigma(0) = 0$; and if $\sigma(x) = 0$ then, all terms being nonnegative, $2^{-n}\varphi(|x_n|) = 0$ for every n , hence $\varphi(|x_n|) = 0$, hence $|x_n| = 0$ for every n , that is, $x = 0$.

Symmetry: $\sigma(-x) = \sum 2^{-n}\varphi(|-x_n|) = \sigma(x)$.

By Problem 48, $\rho(x, y) = \sigma(x - y)$ is a metric on S .

Identification of the topology. We claim the ρ -topology on S is exactly the topology of coordinatewise convergence, that is, the product topology on $S = \mathbb{R}^{\mathbb{N}}$. Two observations do it.

(a) For each fixed n and all $x, y \in S$, $2^{-n}\varphi(|x_n - y_n|) \leq \rho(x, y)$. Hence, by (1), if $\rho(x, y) < 2^{-n}\varphi(\epsilon)$ then $|x_n - y_n| < \epsilon$. So each coordinate projection $\pi_n : S \rightarrow \mathbb{R}$ is ρ -continuous, and therefore every product-open set is ρ -open.

(b) Conversely, let $x_0 \in S$ and $\epsilon > 0$. Choose N with $2^{-N} < \epsilon/2$ and put

$$U = \{x \in S : |x_n - x_{0,n}| < \epsilon/2 \text{ for } 1 \leq n \leq N\},$$

a basic product-open neighborhood of x_0 . For $x \in U$, bound the first N terms using that φ is increasing together with (3), and the remaining terms using $\varphi < 1$:

$$\rho(x, x_0) \leq \sum_{n=1}^N 2^{-n}\varphi(\epsilon/2) + \sum_{n>N} 2^{-n} \leq \varphi(\epsilon/2) + 2^{-N} \leq \frac{\epsilon}{2} + 2^{-N} < \epsilon,$$

where we used $\sum_{n=1}^N 2^{-n} < 1$ and $\sum_{n>N} 2^{-n} = 2^{-N} < \epsilon/2$. So U is contained in the ρ -ball of radius ϵ about x_0 , and every ρ -open set is product-open.

Thus the two topologies coincide, and $\{x^{(j)}\} \rightarrow x$ in (S, ρ) if and only if $x_n^{(j)} \rightarrow x_n$ for every n .

(iii) Completeness. Let $\{x^{(j)}\}$ be a Cauchy sequence in (S, ρ) . Fix n and $\epsilon > 0$. Since $\varphi(\epsilon) > 0$, there is J such that $\rho(x^{(j)}, x^{(k)}) < 2^{-n}\varphi(\epsilon)$ for $j, k \geq J$; by observation (a), $|x_n^{(j)} - x_n^{(k)}| < \epsilon$ for $j, k \geq J$. So $\{x_n^{(j)}\}_j$ is a Cauchy sequence of real numbers; by the completeness of $\mathbb{R}$ it converges, say to x_n . Let $x = \{x_n\} \in S$. Then $x^{(j)} \rightarrow x$ coordinatewise, hence $\rho(x^{(j)}, x) \rightarrow 0$ by the identification of the topology above. Therefore (S, ρ) is complete.

(iv) Continuity of addition. By the subadditivity of σ established in (i),

$$\rho(x + y, x_0 + y_0) = \sigma((x - x_0) + (y - y_0)) \leq \rho(x, x_0) + \rho(y, y_0).$$

So if $\rho(x, x_0) < \epsilon/2$ and $\rho(y, y_0) < \epsilon/2$ then $\rho(x + y, x_0 + y_0) < \epsilon$: addition is continuous on $S \times S$ with the product topology.

(v) Continuity of scalar multiplication. Since the ρ -topology is the product topology on $\mathbb{R}^{\mathbb{N}}$, a map into S is continuous if and only if its composition with every coordinate projection is continuous. The composition of $(\lambda, x) \mapsto \lambda \cdot x$ with π_n is $(\lambda, x) \mapsto \lambda x_n$, which is the product of the continuous real-valued functions $(\lambda, x) \mapsto \lambda$ and $(\lambda, x) \mapsto \pi_n(x)$ on $\mathbb{R} \times S$, hence continuous. Therefore $(\lambda, x) \mapsto \lambda \cdot x$ is continuous.

Since ρ is a metric, the topology is Hausdorff, so S is a topological vector space; moreover the basic neighborhoods U of (b) are convex, so S is in fact locally convex. This is in sharp contrast with the space of Problem 50.

The continuous linear functionals. We claim that $\psi : S \rightarrow \mathbb{R}$ is linear and continuous if and only if there are a natural number N and real numbers $c_1, \dots, c_N$ for which

$$\psi(x) = \sum_{k=1}^N c_k x_k \quad \text{for all } x = \{x_n\} \in S;$$

that is, the continuous linear functionals on S are precisely the finite linear combinations of the coordinate functionals.

Any such ψ is plainly linear, and it is continuous because each π_k is continuous and finite linear combinations of continuous real-valued functions are continuous.

Conversely, let ψ be linear and continuous. Since $\psi(0) = 0$, continuity at the origin provides a neighborhood of 0 on which $|\psi| < 1$, and by (b) we may take it to be basic: there are a natural number N and $\eta > 0$ with

$$|\psi(x)| < 1 \quad \text{whenever } |x_n| < \eta \text{ for } 1 \leq n \leq N.$$

Let $Z = \{x \in S : x_1 = x_2 = \dots = x_N = 0\}$, a linear subspace of S . If $x \in Z$ and $t > 0$, then tx also has its first N coordinates equal to 0, so tx lies in the above neighborhood and $|\psi(tx)| < 1$; by linearity $|\psi(x)| < 1/t$. Letting $t \rightarrow \infty$ gives $\psi(x) = 0$. Thus $\psi \equiv 0$ on Z .

Now let $e_k \in S$ denote the sequence whose k -th component is 1 and whose other components vanish, and set $c_k = \psi(e_k)$ for $1 \leq k \leq N$. For arbitrary $x \in S$,

$$x - \sum_{k=1}^N x_k e_k \in Z,$$

since this sequence has its first N coordinates equal to 0. Applying ψ and using linearity together with $\psi = 0$ on Z ,

$$\psi(x) = \sum_{k=1}^N x_k \psi(e_k) = \sum_{k=1}^N c_k x_k.$$

This is the required form, and the classification is complete. In particular S has an abundance of nonzero continuous linear functionals, but its dual is small: it is the space of finitely supported sequences $\{c_n\}$ acting by $x \mapsto \sum_n c_n x_n$.

Problem 14.52. For each natural number n , let e_n denote the sequence in ℓ^2 whose n th component is 1 and other components vanish. Define $E = \{e_n + n \cdot e_m \mid n \text{ and } m \text{ any natural numbers, } m > n\}$. Show that 0 is a point of closure of E but no sequence in E converges weakly to 0. Consider the topological space $X = E \cup \{0\}$ with the weak topology. Find a function $f : X \rightarrow \mathbb{R}$ that fails to be continuous at 0 and yet has the property that whenever a sequence $\{x_n\}$ in E converges weakly to 0, its image sequence $\{f(x_n)\}$ converges to $f(0)$.

Solution. We use no representation theorem for the dual of ℓ^2 ; two elementary facts suffice.

Fact A. Each $y = \{y(j)\} \in \ell^2$ determines a bounded linear functional on ℓ^2 by $\langle x, y \rangle = \sum_{j=1}^{\infty} x(j) y(j)$; the series converges absolutely and $|\langle x, y \rangle| \leq \|x\| \|y\|$ by the Cauchy-Schwarz inequality (the case $p = q = 2$ of Holder's inequality).

Fact B. Conversely, if ψ belongs to $(\ell^2)^*$ and we set $y_\psi(j) = \psi(e_j)$, then $y_\psi \in \ell^2$; in particular $\psi(e_j) \rightarrow 0$ as $j \rightarrow \infty$. Indeed, fix a natural number N and put $u = \sum_{j=1}^N y_\psi(j) e_j$, a member of ℓ^2 with $\|u\| = (\sum_{j=1}^N |y_\psi(j)|^2)^{1/2}$. By linearity, $\psi(u) = \sum_{j=1}^N |y_\psi(j)|^2 = \|u\|^2$, while $|\psi(u)| \leq \|\psi\|_* \|u\|$. Hence $\|u\| \leq \|\psi\|_*$, that is, $\sum_{j=1}^N |y_\psi(j)|^2 \leq \|\psi\|_*^2$ for every N . Letting $N \rightarrow \infty$ gives $y_\psi \in \ell^2$, and the terms of a convergent series of nonnegative numbers tend to 0.

Recall that a base at the origin for the weak topology on a normed linear space X comprises the sets

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_k}(0) = \{x \in X \mid |\psi_i(x)| < \epsilon \text{ for } 1 \leq i \leq k\},$$

where $\epsilon > 0$ and $\psi_1, \dots, \psi_k \in X^*$. Note that for any linear functional ψ on ℓ^2 and any $m > n$,

$$\psi(e_n + n \cdot e_m) = \psi(e_n) + n \psi(e_m), \quad (*)$$

by linearity alone.

The origin is a point of closure of E with respect to the weak topology. Let $\mathcal{N} = \mathcal{N}_{\epsilon, \psi_1, \dots, \psi_k}(0)$ be a basic weak neighborhood of 0, where each ψ_i belongs to $(\ell^2)^*$. By Fact B, $\psi_i(e_j) \rightarrow 0$ as $j \rightarrow \infty$ for each of the finitely many indices i . Hence we may choose a natural number n so large that

$$|\psi_i(e_n)| < \epsilon/2 \quad \text{for } 1 \leq i \leq k.$$

Now hold this n fixed. Again because $\psi_i(e_j) \rightarrow 0$ as $j \rightarrow \infty$, we may choose a natural number $m > n$ so large that

$$n |\psi_i(e_m)| < \epsilon/2 \quad \text{for } 1 \leq i \leq k.$$

Then by (*), $|\psi_i(e_n + n \cdot e_m)| \leq |\psi_i(e_n)| + n |\psi_i(e_m)| < \epsilon$ for every i , so the point $e_n + n \cdot e_m$ of E belongs to $\mathcal{N}$. Thus every weak neighborhood of 0 contains a point of E ; that is, 0 is a point of closure of E . (Note $0 \notin E$, since $\|e_n + n \cdot e_m\| = \sqrt{1 + n^2} \geq \sqrt{2}$.)

No sequence in E converges weakly to 0. For this direction we test against the particular functionals supplied by Fact A: for $y \in \ell^2$, since $\langle e_j, y \rangle = y(j)$, identity (*) reads

$$\langle e_n + n \cdot e_m, y \rangle = y(n) + n y(m) \quad (m > n). \quad (**)$$

Suppose, to the contrary, that $\{x_j\}$ is a sequence in E with $x_j \rightarrow 0$ weakly, say

$$x_j = e_{n_j} + n_j \cdot e_{m_j}, \quad m_j > n_j.$$

Consider two cases.

Case 1: the sequence of natural numbers $\{n_j\}$ has a bounded subsequence. A bounded sequence of natural numbers takes only finitely many values, so some value n occurs for infinitely many indices j ; let $j_1 < j_2 < \dots$ enumerate them, so $x_{j_i} = e_n + n \cdot e_{m_{j_i}}$ with $m_{j_i} > n$. Take $y = e_n \in \ell^2$. By (**), $\langle x_{j_i}, e_n \rangle = 1 + n \cdot 0 = 1$ for every i , since $m_{j_i} \neq n$. But weak convergence of $\{x_j\}$ to 0 forces $\langle x_j, e_n \rangle \rightarrow 0$, and hence $\langle x_{j_i}, e_n \rangle \rightarrow 0$ along the subsequence. This contradicts $\langle x_{j_i}, e_n \rangle = 1$.

Case 2: $n_j \rightarrow \infty$. Choose indices $j_1 < j_2 < \dots$ inductively as follows: let $j_1 = 1$; having chosen j_i , use $n_j \rightarrow \infty$ to choose $j_{i+1} > j_i$ with $n_{j_{i+1}} > m_{j_i}$. Writing $\nu_i = n_{j_i}$ and $\mu_i = m_{j_i}$, we obtain

$$\nu_1 < \mu_1 < \nu_2 < \mu_2 < \nu_3 < \dots,$$

so the natural numbers $\nu_1, \mu_1, \nu_2, \mu_2, \dots$ are all distinct and $\{\nu_i\}$ is strictly increasing, whence $\nu_i \geq i$. Define

$$y = \sum_{i=1}^{\infty} \frac{1}{\nu_i} e_{\mu_i}, \quad \text{that is, } y(\mu_i) = 1/\nu_i \text{ and } y(j) = 0 \text{ for } j \notin \{\mu_i\}.$$

Then $\sum_j |y(j)|^2 = \sum_i 1/\nu_i^2 \leq \sum_i 1/i^2 < \infty$, so $y \in \ell^2$. Since the ν_i are distinct from all the μ_i , we have $y(\nu_i) = 0$, and so by (**)

$$\langle x_{j_i}, y \rangle = y(\nu_i) + \nu_i y(\mu_i) = 0 + \nu_i \cdot \frac{1}{\nu_i} = 1 \quad \text{for every } i.$$

Again this contradicts $\langle x_j, y \rangle \rightarrow 0$.

Since $\{n_j\}$ either has a bounded subsequence or tends to infinity, both cases are impossible, and no sequence in E converges weakly to 0.

The function f . Let $X = E \cup \{0\}$ with the topology inherited from the weak topology on ℓ^2 , and define

$$f(x) = \begin{cases} 1, & x \in E, \\ 0, & x = 0. \end{cases}$$

This is well defined because $0 \notin E$.

f is not continuous at 0: the set $(-1/2, 1/2)$ is an open neighborhood of $f(0) = 0$ in $\mathbb{R}$, and

$$f^{-1}((-1/2, 1/2)) = \{0\}.$$

If f were continuous at 0 there would be a weak neighborhood $\mathcal{N}$ of 0 in ℓ^2 with $f(\mathcal{N} \cap X) \subseteq (-1/2, 1/2)$, that is, with $\mathcal{N} \cap X = \{0\}$, so $\mathcal{N} \cap E = \emptyset$. This contradicts the fact, proved above, that 0 is a point of closure of E in the weak topology. Hence f fails to be continuous at 0.

f nevertheless has the stated sequential property: there is no sequence $\{x_n\}$ in E that converges weakly to 0, so the implication that $\{f(x_n)\} \rightarrow f(0)$ whenever $\{x_n\}$ is a sequence in E converging weakly to 0 holds vacuously. (Equivalently, f is sequentially continuous at 0 as a function on X : a sequence in X converging weakly to 0 must have all but finitely many terms equal to 0 – otherwise it would have a subsequence lying in E and converging weakly to 0 – and along such a sequence f is eventually constantly $f(0) = 0$.)

Thus X is a topological space in which sequential continuity at a point does not imply continuity at that point; this is possible precisely because the weak topology on an infinite dimensional normed linear space is not first countable, so weak sequential convergence is a strictly weaker notion than weak topological convergence.

Problem 14.53. Find a subset of the plane $\mathbb{R}^2$ for which the origin is an internal point but not an interior point.

Solution. Recall the definition in force: a point x_0 of a subset E of a linear space X is an internal point of E provided that for each $x \in X$ there is some $\lambda_0 > 0$ for which $x_0 + \lambda \cdot x$ belongs to E if $|\lambda| \leq \lambda_0$. Thus x_0 is internal to E if each line through x_0 meets E in a segment about x_0 ; it is interior to E if some ball about x_0 lies in E . Being internal is a strictly weaker requirement, as the following example shows.

Let $P = \{(t, t^2) : t \in \mathbb{R}, t \neq 0\}$ be the parabola $y = x^2$ with its vertex removed, and set

$$E = \mathbb{R}^2 \sim P.$$

The origin belongs to E , since $(0, 0)$ was deleted from the parabola.

The origin is an internal point of E . Let $v = (a, b)$ be any point of $\mathbb{R}^2$; we must produce $\lambda_0 > 0$ with $\lambda \cdot v \in E$ for $|\lambda| \leq \lambda_0$. First determine for which λ the point $\lambda v = (\lambda a, \lambda b)$ lies on P : this requires $\lambda b = (\lambda a)^2 = \lambda^2 a^2$, that is,

$$\lambda(b - \lambda a^2) = 0.$$

If $v = (0, 0)$ then $\lambda v = (0, 0) \in E$ for all λ and any λ_0 works. If $a = 0$ and $b \neq 0$, the equation reduces to $\lambda b = 0$, so $\lambda = 0$; and $\lambda v = (0, 0)$ is not in P (the vertex was removed). If $a \neq 0$ and $b = 0$, the equation reduces to $\lambda^2 a^2 = 0$, again only $\lambda = 0$. In these two cases no λ at all puts λv on P , and we may take $\lambda_0 = 1$. Finally, if $a \neq 0$ and $b \neq 0$, the solutions are $\lambda = 0$ and $\lambda = \lambda^* = b/a^2 \neq 0$; but

$\lambda = 0$ gives the origin, which is not in P . Hence $\lambda v \in P$ only for $\lambda = \lambda^*$, and taking

$$\lambda_0 = |\lambda^*|/2 = |b|/(2a^2) > 0$$

we have $\lambda v \notin P$, that is, $\lambda v \in E$, whenever $|\lambda| \leq \lambda_0$. So the origin is an internal point of E .

The origin is not an interior point of E . Let $r > 0$ and consider the open disc B_r of radius r about the origin. Choose t with $0 < t < r/2$ and $t < 1$; then the point (t, t^2) satisfies

$$\|(t, t^2)\| = \sqrt{t^2 + t^4} \leq t\sqrt{2} < r,$$

so $(t, t^2) \in B_r$, while $(t, t^2) \in P$ and hence $(t, t^2) \notin E$. Thus no ball about the origin is contained in E , and the origin is not an interior point of E .

Geometrically: every straight line through the origin crosses the deleted parabola at most once, and that crossing occurs away from the origin, so each line contributes a genuine segment about 0 lying in E ; but the parabola is tangent to the x -axis at the origin, so it returns to the origin faster than any line and punctures every disc about it.

Remark. This phenomenon requires E to be nonconvex. For a convex subset K of $\mathbb{R}^n$, a point internal to K is interior to K ; the example above is therefore about as simple as such an example can be.

Problem 14.54. Let X be a locally convex topological vector space and V a convex, symmetric with respect to the origin (that is, $V = -V$) neighborhood of the origin. If p_V is the gauge functional for V and ψ is a linear real-valued functional on X such that $\psi \leq p_V$ on X , show that ψ is continuous.

Solution. Recall that for a convex set K containing the origin as an internal point, the gauge functional is

$$p_K(x) = \inf\{\lambda > 0 \mid x \in \lambda \cdot K\} \quad \text{for all } x \in X,$$

and that it is precisely because the origin is an internal point of K that p_K is finite.

First observe that p_V is indeed defined and finite. Since V is a neighborhood of the origin, there is an open set $\mathcal{O}$ with $0 \in \mathcal{O} \subseteq V$. By Proposition 22(iii), every point of an open subset of X is an internal point of that set, so the origin is an internal point of $\mathcal{O}$ and therefore also of the larger set V . Being internal means precisely that for each $x \in X$ there is a $\lambda_0 > 0$ with $\lambda \cdot x \in V$ whenever $|\lambda| \leq \lambda_0$; taking $\lambda = \lambda_0$ gives $x \in \lambda_0^{-1} \cdot V$, so the set $\{\lambda > 0 \mid x \in \lambda \cdot V\}$ is nonempty and $p_V(x) \leq 1/\lambda_0 < \infty$. Thus p_V is a finite nonnegative functional on X , and since V is convex it is subadditive and positively homogeneous (Proposition 24).

Next we record the elementary bound

$$p_V(x) \leq 1 \quad \text{for all } x \in V.$$

Indeed, if $x \in V$ then $x \in 1 \cdot V$, so 1 belongs to the set $\{\lambda > 0 \mid x \in \lambda \cdot V\}$ whose infimum is $p_V(x)$.

Now let ψ be a linear real-valued functional on X with $\psi \leq p_V$ on X . Let $x \in V$. Then

$$\psi(x) \leq p_V(x) \leq 1.$$

Since V is symmetric with respect to the origin, $-x$ also belongs to V , and therefore, by the same estimate applied to $-x$ and by the linearity of ψ ,

$$-\psi(x) = \psi(-x) \leq p_V(-x) \leq 1.$$

Combining the two inequalities gives $|\psi(x)| \leq 1$. As x was an arbitrary point of V ,

$$|\psi| \leq 1 \quad \text{on the neighborhood } V \text{ of the origin.}$$

Thus $|\psi|$ is bounded on a neighborhood of the origin, and Proposition 23 tells us that ψ is continuous.

Problem 14.55. Let X be a locally convex topological vector space, Y a closed subspace of X , and x_0 belong to $X \sim Y$. Show that there is a continuous functional $\psi : X \rightarrow \mathbb{R}$ such that

$$\psi(x_0) \neq 0 \quad \text{and} \quad \psi = 0 \text{ on } Y.$$

Solution. The set Y is nonempty (it contains the origin, being a subspace), it is convex (if $y_1, y_2 \in Y$ and $0 \leq t \leq 1$ then $ty_1 + (1-t)y_2 \in Y$ because Y is closed under addition and scalar multiplication), and it is closed by hypothesis. The point x_0 lies outside Y . Therefore the Hyperplane Separation Theorem applies with $K = Y$: there is a continuous linear functional $\psi : X \rightarrow \mathbb{R}$ for which

$$\psi(x_0) < \inf_{y \in Y} \psi(y). \quad (*)$$

Denote by c the right-hand infimum. Since $\psi(x_0)$ is a real number and $(*)$ is a strict inequality between $\psi(x_0)$ and c , the infimum c is not $-\infty$; that is, ψ is bounded below on Y .

We claim this forces ψ to vanish identically on Y . Since Y is a linear subspace of X and ψ is linear, the image $\psi(Y)$ is a linear subspace of $\mathbb{R}$, so it is either $\{0\}$ or all of $\mathbb{R}$. Concretely: if there were some $y \in Y$ with $\psi(y) \neq 0$, then for every real t the vector ty would belong to Y and $\psi(ty) = t\psi(y)$; letting t run through the reals, ψ would assume arbitrarily large negative values on Y , so that $c = -\infty$, contradicting the boundedness below just established. Hence

$$\psi(y) = 0 \quad \text{for all } y \in Y,$$

and consequently $c = \inf_{y \in Y} \psi(y) = 0$.

Substituting $c = 0$ into $(*)$ gives $\psi(x_0) < 0$, so in particular $\psi(x_0) \neq 0$.

Thus $\psi : X \rightarrow \mathbb{R}$ is a continuous linear functional that vanishes identically on Y and is nonzero at x_0 , as required.

Remark. Local convexity is essential here, since it is what makes the Hyperplane Separation Theorem available. Problem 50 exhibits a (nonlocally convex) topological vector space X on which the only continuous linear functional is the zero functional; there, for $Y = \{0\}$ and any $x_0 \neq 0$, no such ψ exists.

Problem 14.56. Let X be a normed linear space and W a proper subspace of X^* that separates points. Let ψ belong to $X^* \sim W$. Show that $\ker \psi$ is strongly closed and convex but not W -weakly closed. (Hint: Otherwise, apply Corollary 26 with $K = \ker \psi$.)

Solution. Write $K = \ker \psi = \{x \in X : \psi(x) = 0\}$.

First, $\psi \neq 0$. Indeed, W is a subspace of X^* , so the zero functional belongs to W , whereas $\psi \notin W$. Hence there is a point $x_0 \in X$ with $\psi(x_0) \neq 0$; dividing x_0 by $\psi(x_0)$ we may and do assume

$$\psi(x_0) = 1, \quad \text{so that } x_0 \notin K.$$

K is a subspace, hence convex. If $u, v \in K$ and $\alpha, \beta \in \mathbb{R}$, then $\psi(\alpha u + \beta v) = \alpha\psi(u) + \beta\psi(v) = 0$, so $\alpha u + \beta v \in K$. In particular $0 \in K$, so K is nonempty, and for $0 \leq t \leq 1$ and $u, v \in K$ we have $tu + (1-t)v \in K$: K is convex.

K is strongly closed. The functional ψ belongs to X^* , so it is continuous with respect to the topology induced by the norm on X , and $K = \psi^{-1}(\{0\})$ is the inverse image under a continuous map of the closed subset $\{0\}$ of $\mathbb{R}$. Hence K is strongly closed.

K is not W -weakly closed. Suppose, to the contrary, that K is closed with respect to the W -weak topology on X . Then K is a nonempty W -weakly closed convex subset of X and x_0 is a point of X lying outside K . Since W is a subspace of X^* that separates points in X , Corollary 26 applies: there is a functional $\phi \in W$ for which

$$\phi(x_0) < \inf_{x \in K} \phi(x). \quad (*)$$

As in the preceding problem, $(*)$ forces ϕ to vanish on K . Indeed, the infimum in $(*)$ exceeds the real number $\phi(x_0)$, hence is not $-\infty$, so ϕ is bounded below on K . But K is a linear subspace, so if

$\phi(u) \neq 0$ for some $u \in K$, then $tu \in K$ and $\phi(tu) = t\phi(u)$ for every real t , and ϕ would be unbounded below on K . Therefore

$$\phi = 0 \text{ on } K, \quad \text{and hence } \inf_{x \in K} \phi(x) = 0.$$

Substituting into (*) gives $\phi(x_0) < 0$; in particular $\phi(x_0) \neq 0$.

We now show that ϕ must be a multiple of ψ . Let x be any point of X and consider

$$u = x - \psi(x)x_0.$$

By linearity and $\psi(x_0) = 1$,

$$\psi(u) = \psi(x) - \psi(x)\psi(x_0) = \psi(x) - \psi(x) = 0,$$

so $u \in K$ and therefore $\phi(u) = 0$. By the linearity of ϕ this reads

$$\phi(x) - \psi(x)\phi(x_0) = 0, \quad \text{that is, } \phi(x) = \phi(x_0)\psi(x).$$

Since $x \in X$ was arbitrary, $\phi = \phi(x_0) \cdot \psi$ as functionals on X . Because $\phi(x_0) \neq 0$, we may solve for ψ :

$$\psi = \frac{1}{\phi(x_0)} \phi.$$

But $\phi \in W$ and W is a subspace of X^* , so the scalar multiple ψ of ϕ also belongs to W . This contradicts the hypothesis that ψ belongs to $X^* \sim W$.

The assumption that K is W -weakly closed is therefore untenable: $\ker \psi$ is strongly closed and convex but not W -weakly closed.

Remark. Compare Mazur's Theorem, which asserts that a convex subset of a normed linear space is strongly closed if and only if it is weakly closed. There is no contradiction: Mazur's Theorem concerns the X^* -weak topology, and the present problem shows that its conclusion genuinely fails for the W -weak topology of a proper point-separating subspace W of X^* . Equivalently, it shows that a linear functional whose kernel is W -weakly closed must itself lie in W .

14.9 Problems 14.57–14.63

Problem 14.57. Let X be a normed linear space. Show that the closed unit ball B^* of X^* is weak-* closed.

Solution. Recall the definitions in force. The natural embedding $J: X \rightarrow (X^*)^*$ is defined by $J(x)[\psi] = \psi(x)$ for $x \in X$ and $\psi \in X^*$, and the weak-* topology on X^* is by definition the $J(X)$ -weak topology on X^* , that is, the weakest topology on X^* for which every functional in $J(X)$ is continuous. In particular, for each fixed $x \in X$ the evaluation functional

$$J(x): X^* \rightarrow \mathbb{R}, \quad \psi \mapsto \psi(x),$$

is continuous with respect to the weak-* topology on X^* .

Let $B = \{x \in X \mid \|x\| \leq 1\}$ be the closed unit ball of X , and let

$$B^* = \{\psi \in X^* \mid \|\psi\| \leq 1\}.$$

For each $x \in B$ define

$$F_x = \{\psi \in X^* \mid |\psi(x)| \leq 1\} = (J(x))^{-1}([-1, 1]).$$

Since $J(x)$ is weak-* continuous and $[-1, 1]$ is a closed subset of $\mathbb{R}$, the set F_x is weak-* closed.

We claim that

$$B^* = \bigcap_{x \in B} F_x.$$

Indeed, if $\|\psi\| \leq 1$ and $x \in B$, then by the definition of the dual norm $|\psi(x)| \leq \|\psi\| \|x\| \leq 1$, so $\psi \in F_x$; thus $B^* \subseteq \bigcap_{x \in B} F_x$. Conversely, suppose $\psi \in X^*$ belongs to F_x for every $x \in B$. Then $|\psi(x)| \leq 1$ for

all x with $\|x\| \leq 1$, and hence

$$\|\psi\| = \sup\{|\psi(x)| \mid x \in X, \|x\| \leq 1\} \leq 1,$$

so that $\psi \in B^*$. This proves the claimed equality.

Finally, an arbitrary intersection of closed subsets of a topological space is closed. Therefore B^* , being the intersection of the weak-* closed sets F_x , $x \in B$, is weak-* closed.

Problem 14.58. Show that the Hyperplane Separation Theorem may be amended as follows: the point x_0 may be replaced by a convex set K_0 that is disjoint from K and the conclusion is that K and K_0 can be separated by a closed hyperplane if K_0 is either compact or open.

Solution. Throughout, X is a locally convex topological vector space, K is a nonempty closed convex subset of X , and K_0 is a nonempty convex subset of X with $K \cap K_0 = \emptyset$.

Case 1: K_0 is open.

The sets K_0 and K are nonempty, disjoint and convex, and by Proposition 22(iii) every point of the open set K_0 is an internal point of K_0 . Hence the Hyperplane Separation Lemma applies and provides a nonzero linear functional $\psi: X \rightarrow \mathbb{R}$ for which

$$\sup_{x \in K_0} \psi(x) \leq \inf_{x \in K} \psi(x) =: c.$$

We first check that ψ is continuous. Fix $x_1 \in K_0$ and put

$$N = (K_0 - x_1) \cap (x_1 - K_0).$$

By Proposition 22(i) each of the two sets on the right is open (the second is $x_1 + (-1) \cdot K_0$), and each contains 0; hence N is an open neighbourhood of 0, and clearly $N = -N$. Set $M = c - \psi(x_1) + 1$; note $M > 0$ because $\psi(x_1) \leq c$. If $u \in N$ then $u + x_1 \in K_0$, so $\psi(u) \leq c - \psi(x_1) < M$; since $-u \in N$ as well, also $-\psi(u) < M$. Thus $|\psi| < M$ on the neighbourhood N of the origin, and Proposition 23 tells us that ψ is continuous. Consequently $\{x \in X \mid \psi(x) = \gamma\}$ is a closed hyperplane for every $\gamma \in \mathbb{R}$.

Next we sharpen the inequality on K_0 . Since $\psi \neq 0$ there is a $z \in X$ with $\psi(z) > 0$ (replace z by $-z$ if necessary). Let $x \in K_0$. By Proposition 22(iii), x is an internal point of K_0 , so there is a $\lambda_0 > 0$ with $x + \lambda_0 z \in K_0$. Therefore

$$\psi(x) + \lambda_0 \psi(z) = \psi(x + \lambda_0 z) \leq c,$$

so that $\psi(x) \leq c - \lambda_0 \psi(z) < c$. Hence

$$\psi < c \text{ on } K_0 \quad \text{and} \quad \psi \geq c \text{ on } K,$$

which is exactly the conclusion (16) of the Hyperplane Separation Theorem with the point x_0 replaced by the whole set K_0 : the closed hyperplane $\{\psi = c\}$ separates K from K_0 .

Strict inequality on the closed set K cannot be expected when K_0 is merely open. For instance, in $X = \mathbb{R}^2$ take $K_0 = \{(s, t) \mid t > 0\}$ and $K = \{(s, t) \mid t \leq 0\}$; every nonzero linear functional separating them is a negative multiple of $(s, t) \mapsto t$, and such a functional attains the value c at the point $(0, 0)$ of K .

Case 2: K_0 is compact.

Here we obtain the stronger, strict separation. The whole point is to fatten K_0 into an open convex set that still misses K .

Let $x \in K_0$. Since $x \notin K$ and K is closed, $(X \sim K) - x$ is an open neighbourhood of the origin. Vector addition is continuous at $(0, 0)$, so there are neighbourhoods U, V of 0 with $U + V \subseteq (X \sim K) - x$; putting $W = U \cap V$ we get a neighbourhood W of 0 with $W + W \subseteq (X \sim K) - x$. By local convexity there is a set C in the convex base at the origin with $C \subseteq W$, and $N_x := C \cap (-C)$ is then an open (by Proposition 22(i)), convex, symmetric neighbourhood of 0 with

$$(x + N_x + N_x) \cap K = \emptyset.$$

The collection $\{x + N_x \mid x \in K_0\}$ is a cover of K_0 by open sets, so by the compactness of K_0 there are $x_1, \dots, x_m \in K_0$ with $K_0 \subseteq \bigcup_{i=1}^m (x_i + N_{x_i})$. Put

$$N = \bigcap_{i=1}^m N_{x_i},$$

an open, convex, symmetric neighbourhood of the origin. Now consider

$$K'_0 = K_0 + N = \bigcup_{u \in K_0} (u + N).$$

It is nonempty; it is open, being a union of translates of the open set N (Proposition 22(i)); it is convex, being the sum of two convex sets. Moreover $K'_0 \cap K = \emptyset$: if $u \in K_0$ and $v \in N$, choose i with $u \in x_i + N_{x_i}$; then

$$u + v \in x_i + N_{x_i} + N \subseteq x_i + N_{x_i} + N_{x_i},$$

which is disjoint from K .

Apply Case 1 to the closed convex set K and the open convex set K'_0 . We obtain a nonzero continuous linear functional $\psi: X \rightarrow \mathbb{R}$ with

$$\psi < c \text{ on } K'_0 \quad \text{and} \quad \psi \geq c \text{ on } K, \quad c = \inf_{x \in K} \psi(x).$$

Since $\psi \neq 0$, choose $w \in X$ with $\psi(w) \neq 0$. By Proposition 22(iii) the origin is an internal point of the open set N , so there is a $\lambda > 0$ with $\lambda w \in N$; as N is symmetric we may choose $z \in \{\lambda w, -\lambda w\} \subseteq N$ with $\delta := \psi(z) > 0$. For every $u \in K_0$ we have $u + z \in K_0 + N = K'_0$, whence

$$\psi(u) + \delta = \psi(u + z) < c, \quad \text{so} \quad \psi(u) < c - \delta.$$

Therefore $\sup_{u \in K_0} \psi(u) \leq c - \delta$. Put $c' = c - \delta/2$. Then

$$\psi \leq c - \delta < c' \text{ on } K_0 \quad \text{and} \quad \psi \geq c > c' \text{ on } K.$$

Thus $\psi < c'$ on K_0 and $\psi > c'$ on K : the sets K_0 and K are separated by the closed hyperplane $\{x \in X \mid \psi(x) = c'\}$. Taking $K_0 = \{x_0\}$, a compact convex set, recovers the Hyperplane Separation Theorem itself.

Problem 14.59. Show that the weak topology on an infinite dimensional normed linear space is not first countable.

Solution. Let X be an infinite dimensional normed linear space and suppose, to get a contradiction, that the weak topology on X is first countable. Then in particular there is a countable base $\{\mathcal{O}_n\}_{n=1}^\infty$ at the origin for the weak topology.

Step 1: the dual space X^* is spanned by countably many functionals.

Each $\mathcal{O}_n$ is a weak neighbourhood of 0, so by the description (5) of the base for the weak topology there are $\epsilon_n > 0$ and finitely many functionals $\psi_{n,1}, \dots, \psi_{n,k_n}$ in X^* with

$$\mathcal{N}_n := \{x \in X \mid |\psi_{n,j}(x)| < \epsilon_n \text{ for } 1 \leq j \leq k_n\} \subseteq \mathcal{O}_n.$$

Let W be the linear span in X^* of the countable collection $\{\psi_{n,j} \mid n \in \mathbb{N}, 1 \leq j \leq k_n\}$.

We claim $W = X^*$. Let $\psi \in X^*$. The set $\mathcal{O} = \{x \in X \mid |\psi(x)| < 1\}$ is a weak neighbourhood of the origin, so, $\{\mathcal{O}_n\}$ being a base at 0, there is an n with $\mathcal{O}_n \subseteq \mathcal{O}$, and hence $\mathcal{N}_n \subseteq \mathcal{O}$. Now let $x \in \bigcap_{j=1}^{k_n} \ker \psi_{n,j}$. For every real t we have $\psi_{n,j}(tx) = 0$ for all j , so $tx \in \mathcal{N}_n \subseteq \mathcal{O}$, that is, $|t| |\psi(x)| < 1$ for all $t \in \mathbb{R}$. This forces $\psi(x) = 0$. Therefore

$$\bigcap_{j=1}^{k_n} \ker \psi_{n,j} \subseteq \ker \psi,$$

and Proposition 4 tells us that ψ is a linear combination of $\psi_{n,1}, \dots, \psi_{n,k_n}$; hence $\psi \in W$. (If $k_n = 0$, so that $\mathcal{N}_n = X$, the same computation gives $\psi = 0 \in W$.) This proves $W = X^*$.

The spanning collection $\{\psi_{n,j}\}$ is countable, and it is nonempty since $X^* \neq \{0\}$ by Theorem 7. Enumerate it as an infinite sequence $\varphi_1, \varphi_2, \varphi_3, \dots$, repeating terms if the collection happens to be finite, and set $W_m = \text{span}\{\varphi_1, \dots, \varphi_m\}$. Since every member of $W = X^*$ is a finite linear combination of the φ_i 's,

$$X^* = \bigcup_{m=1}^{\infty} W_m.$$

Step 2: X^* is infinite dimensional.

Let n be a natural number. Since X is infinite dimensional we may choose linearly independent vectors $x_1, \dots, x_n$ in X . Fix i and let Y_i be the linear span of $\{x_j \mid j \neq i\}$. Being finite dimensional, Y_i is a closed subspace of X by Corollary 6 of Chapter 13, and $x_i \notin Y_i$ by linear independence. Hence Theorem 10 provides a functional $\psi_i \in X^*$ that vanishes on Y_i while $\psi_i(x_i) \neq 0$. The functionals $\psi_1, \dots, \psi_n$ are linearly independent: if $\sum_{i=1}^n c_i \psi_i = 0$, then evaluating at x_j gives $c_j \psi_j(x_j) = 0$ and hence $c_j = 0$, for each j . As n was arbitrary, X^* contains linearly independent sets of every finite size, so X^* is infinite dimensional.

Step 3: the Baire Category Theorem gives the contradiction.

By Theorem 3 of Chapter 13, $X^* = \mathcal{L}(X, \mathbb{R})$ is a Banach space, since $\mathbb{R}$ is complete; thus X^* is a complete metric space. Each W_m is a finite dimensional subspace of X^* and therefore is closed, again by Corollary 6 of Chapter 13. Each W_m is also hollow, that is, has empty interior. Indeed, suppose W_m contained an open ball $B(\varphi, r)$ with $r > 0$. Let $\xi \in X^*$ be arbitrary and nonzero. Then $\varphi + \frac{r}{2\|\xi\|}\xi$ lies in $B(\varphi, r) \subseteq W_m$, and $\varphi \in W_m$, so by linearity of W_m we get $\frac{r}{2\|\xi\|}\xi \in W_m$ and hence $\xi \in W_m$. This would give $W_m = X^*$, contradicting Step 2 since W_m is finite dimensional.

Thus $X^* = \bigcup_{m=1}^{\infty} W_m$ exhibits the complete metric space X^* as a countable union of closed hollow sets. By part (ii) of the Baire Category Theorem this union is hollow. But the union is all of X^* , which is a nonempty open subset of itself and hence is not hollow. This contradiction shows that no countable base at the origin exists, and therefore the weak topology on an infinite dimensional normed linear space is not first countable.

Problem 14.60. Show that every weakly compact subset of a normed linear space is bounded with respect to the norm.

Solution. Let X be a normed linear space and let $K \subseteq X$ be weakly compact. If K is empty there is nothing to prove, so assume $K \neq \emptyset$.

By the very definition of the weak topology on X , every $\psi \in X^*$ is weakly continuous. Hence, for each $\psi \in X^*$, the image $\psi(K)$ is a compact subset of $\mathbb{R}$, being the continuous image of a compact space. A compact subset of $\mathbb{R}$ is bounded, so

$$M_\psi := \sup_{x \in K} |\psi(x)| < \infty \quad \text{for each } \psi \in X^*.$$

Now let $J: X \rightarrow (X^*)^*$ be the natural embedding, $J(x)[\psi] = \psi(x)$, and consider the family of bounded linear functionals on X^*

$$\mathcal{F} = \{J(x) \mid x \in K\} \subseteq \mathcal{L}(X^*, \mathbb{R}) = X^{**}.$$

The displayed inequality says exactly that $\mathcal{F}$ is pointwise bounded on X^* : for each $\psi \in X^*$,

$$|J(x)[\psi]| = |\psi(x)| \leq M_\psi \quad \text{for all } J(x) \in \mathcal{F}.$$

By Theorem 3 of Chapter 13, $X^* = \mathcal{L}(X, \mathbb{R})$ is a Banach space, since $\mathbb{R}$ is complete. Thus $\mathcal{F}$ is a pointwise bounded family of bounded linear operators from the Banach space X^* into the normed linear space $\mathbb{R}$, and the Uniform Boundedness Principle tells us that $\mathcal{F}$ is uniformly bounded: there is an $M \geq 0$ with

$$\|J(x)\| \leq M \quad \text{for all } x \in K.$$

Finally, by Corollary 9 the natural embedding J is an isometry, so $\|x\| = \|J(x)\| \leq M$ for every $x \in K$. Therefore K is bounded with respect to the norm on X .

Problem 14.61. Let Y be a closed subspace of a reflexive Banach space X . For $x_0 \in X \sim Y$, show that there is a point in Y that is closest to x_0 .

Solution. Define

$$d = \text{dist}(x_0, Y) = \inf_{y \in Y} \|x_0 - y\|.$$

Since Y is nonempty (it contains 0), d is a finite nonnegative number. We must produce $y_0 \in Y$ with $\|x_0 - y_0\| = d$.

Fix any $r > d$ and set

$$K = Y \cap \{x \in X \mid \|x - x_0\| \leq r\}.$$

We record the properties of K .

K is nonempty: by the definition of d as an infimum and $r > d$, there is a $y \in Y$ with $\|x_0 - y\| < r$, and such a y lies in K .

K is convex: Y is a linear subspace, hence convex, and the closed ball $\{x \mid \|x - x_0\| \leq r\}$ is convex; the intersection of convex sets is convex.

K is strongly closed: Y is closed by hypothesis, and the closed ball is closed since the norm is continuous; the intersection of closed sets is closed.

K is bounded: it is contained in a ball of radius r , so $\|x\| \leq \|x_0\| + r$ for $x \in K$.

Now define $f: K \rightarrow \mathbb{R}$ by $f(y) = \|x_0 - y\|$. The function f is continuous with respect to the strong topology on K , since $|f(u) - f(v)| \leq \|u - v\|$ by the triangle inequality. It is convex in the sense required by Theorem 30: for $u, v \in K$ and $0 \leq \lambda \leq 1$,

$$f(\lambda u + (1 - \lambda)v) = \|\lambda(x_0 - u) + (1 - \lambda)(x_0 - v)\| \leq \lambda\|x_0 - u\| + (1 - \lambda)\|x_0 - v\| = \lambda f(u) + (1 - \lambda)f(v).$$

Finally $f \geq 0$, so f is bounded below on K .

Since X is a reflexive Banach space and K is a nonempty strongly closed bounded convex subset of X , Theorem 30 tells us that f takes a minimum value on K , say at $y_0 \in K \subseteq Y$.

It remains to check that $\|x_0 - y_0\| = d$, that is, that minimizing over K is the same as minimizing over Y . On the one hand $y_0 \in Y$, so $\|x_0 - y_0\| \geq d$. On the other hand, let $y \in Y$ be arbitrary. If $\|x_0 - y\| \leq r$ then $y \in K$ and hence $\|x_0 - y_0\| \leq \|x_0 - y\|$. If instead $\|x_0 - y\| > r$, then, choosing as above some $y_1 \in K$ with $\|x_0 - y_1\| < r$, we get

$$\|x_0 - y_0\| \leq \|x_0 - y_1\| < r < \|x_0 - y\|.$$

In either case $\|x_0 - y_0\| \leq \|x_0 - y\|$. Taking the infimum over $y \in Y$ gives $\|x_0 - y_0\| \leq d$, and therefore $\|x_0 - y_0\| = d$.

Thus y_0 is a point of Y closest to x_0 . (The hypothesis $x_0 \notin Y$ was not needed; it merely guarantees $d > 0$, so that the closest point is distinct from x_0 .)

Problem 14.62. Let X be a normed linear space, W a finite dimensional subspace of X^* and ψ a functional in $X^* \sim W$. Show that there is a vector $x \in X$ such that $\psi(x) \neq 0$ while $\varphi(x) = 0$ for all φ in W . (Hint: First show this is true if X is finite dimensional.)

Solution. First dispose of the trivial case $W = \{0\}$. Then $\psi \neq 0$ because $\psi \notin W$, so there is an $x \in X$ with $\psi(x) \neq 0$; and the requirement $\varphi(x) = 0$ for all $\varphi \in W$ holds vacuously since the only φ in W is the zero functional.

So assume $n = \dim W \geq 1$ and let $\varphi_1, \dots, \varphi_n$ be a basis for W . Since every $\varphi \in W$ is a linear combination of $\varphi_1, \dots, \varphi_n$, for a vector $x \in X$ the condition

$$\varphi(x) = 0 \quad \text{for all } \varphi \in W$$

is equivalent to the condition $x \in \bigcap_{i=1}^n \ker \varphi_i$.

Argue by contradiction: suppose there is no vector x as required. Then every x lying in $\bigcap_{i=1}^n \ker \varphi_i$

satisfies $\psi(x) = 0$, that is,

$$\bigcap_{i=1}^n \ker \varphi_i \subseteq \ker \psi.$$

Call this inclusion (*).

We show directly that (*) forces ψ to be a linear combination of $\varphi_1, \dots, \varphi_n$; this is precisely the content of Proposition 4, and the following argument is the finite dimensional reduction suggested by the hint. Define the linear map

$$T: X \rightarrow \mathbb{R}^n, \quad T(x) = (\varphi_1(x), \dots, \varphi_n(x)).$$

Its image $V = T(X)$ is a linear subspace of the finite dimensional space $\mathbb{R}^n$. Define $\lambda: V \rightarrow \mathbb{R}$ as follows: for $v \in V$ choose $x \in X$ with $T(x) = v$ and set $\lambda(v) = \psi(x)$. This is well defined: if $T(x) = T(x')$ then $\varphi_i(x - x') = 0$ for $1 \leq i \leq n$, so $x - x'$ lies in $\bigcap_{i=1}^n \ker \varphi_i$, and (*) gives $\psi(x - x') = 0$, that is, $\psi(x) = \psi(x')$. The map λ is linear, because T and ψ are linear: if $v = T(x)$, $w = T(x')$ and $a, b \in \mathbb{R}$, then $av + bw = T(ax + bx')$, so $\lambda(av + bw) = \psi(ax + bx') = a\psi(x) + b\psi(x') = a\lambda(v) + b\lambda(w)$.

Since V is a subspace of the finite dimensional space $\mathbb{R}^n$, extend λ to a linear functional Λ on all of $\mathbb{R}^n$ (choose a basis of V , complete it to a basis of $\mathbb{R}^n$, and assign arbitrary values on the new basis vectors; this is Proposition 2 applied to the linear space $\mathbb{R}^n$). Every linear functional on $\mathbb{R}^n$ has the form $\Lambda(t_1, \dots, t_n) = \sum_{i=1}^n c_i t_i$ for suitable real numbers $c_1, \dots, c_n$, obtained by evaluating Λ on the standard basis. Therefore, for every $x \in X$,

$$\psi(x) = \lambda(T(x)) = \Lambda(\varphi_1(x), \dots, \varphi_n(x)) = \sum_{i=1}^n c_i \varphi_i(x).$$

Hence $\psi = \sum_{i=1}^n c_i \varphi_i$, so ψ belongs to W , the span of $\varphi_1, \dots, \varphi_n$. This contradicts the hypothesis that $\psi \in X^* \sim W$.

Consequently (*) fails, and there is a vector $x \in \bigcap_{i=1}^n \ker \varphi_i$ with $\psi(x) \neq 0$. For this x we have $\varphi(x) = 0$ for every $\varphi \in W$ and $\psi(x) \neq 0$, as required.

Problem 14.63. Let X be a normed linear space. Show that any dense subset of $B^* = \{\psi \in X^* \mid \|\psi\| \leq 1\}$ separates points in X .

Solution. Let D be a dense subset of the closed unit ball B^* of X^* ; here B^* carries the metric inherited from the norm on X^* . We must show that for any two distinct vectors $u, v \in X$ there is a $\varphi \in D$ with $\varphi(u) \neq \varphi(v)$.

Let $u, v \in X$ with $u \neq v$, and put $x = u - v$, so that $\|x\| > 0$.

By Theorem 7 there is a functional $\psi \in X^*$ for which

$$\psi(x) = \|x\| \quad \text{and} \quad \|\psi\| = 1.$$

In particular ψ belongs to B^* .

Since D is dense in B^* , we may choose $\varphi \in D$ with $\|\varphi - \psi\| < 1/2$. Then, using $|(\psi - \varphi)(x)| \leq \|\psi - \varphi\| \|x\|$,

$$|\varphi(x)| \geq |\psi(x)| - |(\psi - \varphi)(x)| \geq \|x\| - \|\psi - \varphi\| \|x\| = (1 - \|\psi - \varphi\|) \|x\| > \frac{1}{2} \|x\| > 0.$$

Therefore

$$\varphi(u) - \varphi(v) = \varphi(u - v) = \varphi(x) \neq 0,$$

that is, $\varphi(u) \neq \varphi(v)$. Since $u \neq v$ were arbitrary, D separates points in X .

Two remarks. First, the argument uses only that some member of D is close to ψ in a very weak sense, namely that it nearly reproduces the single value $\psi(x)$; consequently the conclusion also holds if D is merely assumed to be dense in B^* with respect to the weak-* topology. Indeed, the set $\{\phi \in X^* \mid |\phi(x) - \psi(x)| < \|x\|/2\}$ is a weak-* neighbourhood of ψ , so it contains some $\varphi \in D$, and then $|\varphi(x)| > \|x\| - \|x\|/2 > 0$.

Second, taking $D = B^*$ itself recovers the standard fact that X^* separates points in X , which is the case $\varphi = \psi$ of the computation above.

14.10 Problems 14.64–14.69

Problem 14.64. Complete the final part of the proof of Theorem 11.

Solution. A remark on the reference is in order, since it is ambiguous. Chapter 14 contains no result carrying the label “Theorem 11”. Two results are candidates for what is meant, and we prove both. First, the only result of Chapter 14 numbered 11 is Corollary 11, in Section 14.2, and it is precisely a result whose proof the text does not give: immediately after the proof of Theorem 10 the authors write “We leave the proof of the following corollary as an exercise.”

Second, this problem is listed among the problems for Section 14.5, and the unique proof in that section a part of which is explicitly left to the reader is that of Proposition 24, which begins “We establish subadditivity and leave the proof of positive homogeneity as an exercise.” That omitted part is literally the final part of a proof.

Part 1: Corollary 11. It reads: let S be a subset of the normed linear space X ; then the linear span of S is dense in X if and only if whenever $\psi \in X^*$ vanishes on S , then $\psi = 0$.

Write X_0 for the linear span of S , that is, the set of all finite linear combinations $\sum_{k=1}^n \lambda_k s_k$ with $\lambda_k \in \mathbb{R}$ and $s_k \in S$. Then X_0 is a linear subspace of X and $S \subseteq X_0$.

First suppose X_0 is dense in X , and let $\psi \in X^*$ vanish on S . Since ψ is linear and vanishes at each point of S , it vanishes at every finite linear combination of points of S ; that is, $\psi = 0$ on X_0 . Since ψ is continuous and X_0 is dense, ψ vanishes on the closure of X_0 , which is all of X . Hence $\psi = 0$.

Now we prove the converse, which is the part that requires Theorem 10. Suppose that the only functional in X^* vanishing on S is the zero functional. Let x be an arbitrary point of X . We claim x belongs to the closure of the subspace X_0 . According to Theorem 10, applied to the subspace X_0 of X , it suffices to show that every $\psi \in X^*$ that vanishes on X_0 also vanishes at x . So let $\psi \in X^*$ vanish on X_0 . Since $S \subseteq X_0$, the functional ψ vanishes on S , and therefore, by hypothesis, $\psi = 0$. In particular $\psi(x) = 0$. Theorem 10 now tells us that x belongs to the closure of X_0 . Since $x \in X$ was arbitrary, the closure of X_0 is all of X , that is, the linear span of S is dense in X .

This completes the proof of Corollary 11. It is worth recording the special case used repeatedly later: taking S to be a subspace, a subspace of a normed linear space is dense if and only if the only bounded linear functional vanishing on it is the zero functional. Both directions above specialize verbatim.

Part 2: the final part of the proof of Proposition 24. Let K be a convex subset of a linear space X containing the origin as an internal point, and let

$$p_K(x) = \inf\{\lambda > 0 : x \in \lambda \cdot K\} \quad \text{for all } x \in X$$

be its gauge functional. Recall that the origin being an internal point of K means that for each $x \in X$ there is a $\lambda_0 > 0$ for which $\lambda x \in K$ whenever $|\lambda| \leq \lambda_0$; taking $\lambda = \lambda_0$ we get $x \in (1/\lambda_0)K$, so the set over which the infimum is taken is nonempty and $p_K(x)$ is a finite nonnegative real number. This is the observation the text records before the proposition. Subadditivity is proved in the text. It remains to prove that p_K is positively homogeneous, that is, that

$$p_K(\alpha x) = \alpha p_K(x) \quad \text{for all } x \in X \text{ and all } \alpha > 0,$$

which is the definition of positive homogeneity given in Section 14.2.

Fix $x \in X$ and $\alpha > 0$, and set

$$S(x) = \{\lambda > 0 : x \in \lambda K\}, \quad S(\alpha x) = \{\lambda > 0 : \alpha x \in \lambda K\}.$$

We claim $S(\alpha x) = \alpha \cdot S(x) = \{\alpha \mu : \mu \in S(x)\}$. Indeed, for $\lambda > 0$, the assertion $\alpha x \in \lambda K$ means that there is a $k \in K$ with $\alpha x = \lambda k$, that is, $x = (\lambda/\alpha)k$, that is, $x \in (\lambda/\alpha)K$; and $\lambda/\alpha > 0$. Thus $\lambda \in S(\alpha x)$ if and only if $\lambda/\alpha \in S(x)$, if and only if $\lambda \in \alpha \cdot S(x)$. This proves the claim. Since $\alpha > 0$, multiplication by α scales infima of sets of positive reals, that is, $\inf(\alpha \cdot S(x)) = \alpha \inf S(x)$. Therefore

$$p_K(\alpha x) = \inf S(\alpha x) = \inf(\alpha \cdot S(x)) = \alpha \inf S(x) = \alpha p_K(x).$$

Hence p_K is positively homogeneous, and, together with the subadditivity established in the text, the proof of Proposition 24 is complete.

We record for completeness that $p_K(0) = 0$: since the origin is an internal point of K it belongs to K , so $0 = \lambda \cdot 0 \in \lambda K$ for every $\lambda > 0$ and the infimum defining $p_K(0)$ is 0. Thus the identity $p_K(\alpha x) = \alpha p_K(x)$ persists at $\alpha = 0$ as well.

Problem 14.65. Find an example of a bounded subset A of a normed linear space X , $\mathcal{F}$ a set of functionals in X^* containing $\mathcal{F}_0$ as a dense subset of $\mathcal{F}$ (dense in the sense of the norm topology on X^*) such that $\mathcal{F}$ and $\mathcal{F}_0$ generate different weak topologies for X , but the same weak topology for A .

Solution. Throughout, for a set $\mathcal{G}$ of functionals in X^* , the weak topology generated by $\mathcal{G}$ means the weakest topology on X for which every $\varphi \in \mathcal{G}$ is continuous; a neighborhood base at a point x consists of the sets

$$N_{\epsilon, \varphi_1, \dots, \varphi_n}(x) = \{x' \in X : |\varphi_i(x - x')| < \epsilon \text{ for } 1 \leq i \leq n\},$$

where $\epsilon > 0$ and $\varphi_1, \dots, \varphi_n$ belong to $\mathcal{G}$. If $\mathcal{G}_0 \subseteq \mathcal{G}$, then the $\mathcal{G}_0$ -weak topology is contained in the $\mathcal{G}$ -weak topology.

We first isolate the general principle behind the second half of the requirement.

Lemma. Let A be a bounded subset of a normed linear space X and let $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq X^*$ with $\mathcal{F}_0$ dense in $\mathcal{F}$ with respect to the norm on X^* . Then the $\mathcal{F}$ -weak and $\mathcal{F}_0$ -weak topologies induce the same topology on A .

Proof of the Lemma. Since $\mathcal{F}_0 \subseteq \mathcal{F}$, the topology A inherits from the $\mathcal{F}_0$ -weak topology is contained in the one it inherits from the $\mathcal{F}$ -weak topology. For the reverse containment it suffices to show that each $\varphi \in \mathcal{F}$, restricted to A , is continuous with respect to the topology A inherits from the $\mathcal{F}_0$ -weak topology, since then every subbasic relatively $\mathcal{F}$ -open subset of A is relatively $\mathcal{F}_0$ -open. Let $M = \sup_{x \in A} \|x\| < \infty$, let $x_1 \in A$ and let $\epsilon > 0$. By density choose $\varphi_0 \in \mathcal{F}_0$ with

$$\|\varphi - \varphi_0\| < \frac{\epsilon}{3(M+1)}.$$

If $x \in A$ satisfies $|\varphi_0(x) - \varphi_0(x_1)| < \epsilon/3$, then, since $\|x\| \leq M$ and $\|x_1\| \leq M$,

$$|\varphi(x) - \varphi(x_1)| \leq |\varphi(x) - \varphi_0(x)| + |\varphi_0(x) - \varphi_0(x_1)| + |\varphi_0(x_1) - \varphi(x_1)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus the relatively $\mathcal{F}_0$ -open neighborhood $\{x \in A : |\varphi_0(x) - \varphi_0(x_1)| < \epsilon/3\}$ of x_1 is mapped by φ into the interval $(\varphi(x_1) - \epsilon, \varphi(x_1) + \epsilon)$. Hence $\varphi|_A$ is continuous, and the Lemma is proved.

So the whole content of the problem is to produce a norm-dense inclusion $\mathcal{F}_0 \subseteq \mathcal{F}$ that generates different topologies on all of X ; the Lemma then supplies a bounded A automatically. The point is that norm convergence in X^* gives uniform approximation only on bounded sets, and weak neighborhoods are never bounded.

The example. Let $X = \ell^2$, and for a natural number n let $e_n^* \in X^*$ be the coordinate functional $e_n^*(x) = x_n$, so that $\|e_n^*\| = 1$. Define

$$\psi_n = e_1^* + \frac{1}{n}e_n^* \quad \text{for } n \geq 2, \quad \psi = e_1^*,$$

and put

$$\mathcal{F}_0 = \{\psi_n : n \geq 2\}, \quad \mathcal{F} = \mathcal{F}_0 \cup \{\psi\}.$$

Finally let $A = B = \{x \in \ell^2 : \|x\| \leq 1\}$, the closed unit ball, which is bounded.

$\mathcal{F}_0$ is dense in $\mathcal{F}$. Indeed $\|\psi_n - \psi\| = \frac{1}{n}\|e_n^*\| = 1/n$, so $\{\psi_n\}$ converges to ψ in the norm of X^* . Hence every element of $\mathcal{F}$ (namely each ψ_n , and ψ itself) is a point of closure of $\mathcal{F}_0$. Note also $\psi \notin \mathcal{F}_0$, since $\psi_n \neq \psi$ for every n .

The two topologies differ on X . Consider the set

$$U = \{x \in \ell^2 : |\psi(x)| < 1\} = \{x : |x_1| < 1\},$$

which is an $\mathcal{F}$ -weak neighborhood of the origin. We claim U is not a neighborhood of the origin for the $\mathcal{F}_0$ -weak topology, so that U is $\mathcal{F}$ -weakly open but not $\mathcal{F}_0$ -weakly open. It suffices to show that no basic $\mathcal{F}_0$ -weak neighborhood of the origin is contained in U . Such a neighborhood has the form

$$N = \{x : |\psi_{n_i}(x)| < \epsilon \text{ for } 1 \leq i \leq k\},$$

with $\epsilon > 0$ and $2 \leq n_1 < n_2 < \cdots < n_k$. Define $x \in \ell^2$ by

$$x_1 = 1, \quad x_{n_i} = -n_i \quad (1 \leq i \leq k), \quad x_j = 0 \text{ otherwise.}$$

This is a finitely supported sequence, hence belongs to ℓ^2 , and the indices n_i are distinct and all different from 1. For each i ,

$$\psi_{n_i}(x) = x_1 + \frac{1}{n_i}x_{n_i} = 1 + \frac{1}{n_i}(-n_i) = 0,$$

so $x \in N$, while $|\psi(x)| = |x_1| = 1$, so $x \notin U$. Thus $N \not\subseteq U$ for every basic $\mathcal{F}_0$ -weak neighborhood N of the origin, and the $\mathcal{F}_0$ -weak topology is strictly weaker than the $\mathcal{F}$ -weak topology on X .

The two topologies agree on A . This is immediate from the Lemma, since A is bounded and $\mathcal{F}_0$ is norm-dense in $\mathcal{F}$. Concretely, for $x, x' \in B$ we have $|\psi(x - x') - \psi_n(x - x')| \leq \|\psi - \psi_n\| \|x - x'\| \leq 2/n$, so $\{\psi_n\}$ converges to ψ uniformly on B , and hence ψ is continuous on B with respect to the topology B inherits from the $\mathcal{F}_0$ -weak topology. Therefore the two traces on $A = B$ coincide.

Problem 14.66. Find the extreme points of each of the following subsets of the plane $\mathbb{R}^2$: (i) $\{(x, y) : x^2 + y^2 = 1\}$; (ii) $\{(x, y) : |x| + |y| = 1\}$; (iii) $\{(x, y) : \max\{x, y\} = 1\}$.

Solution. Throughout we use the definition of the section: a point p of a set S is an extreme point of S provided the singleton $\{p\}$ is an extreme subset of S , that is, provided p cannot be written as $p = \lambda u + (1 - \lambda)v$ with $u, v \in S$, $u \neq v$ and $0 < \lambda < 1$. Equivalently, p is not an interior point of any line segment with both endpoints in S .

(i) Let $S = \{(x, y) : x^2 + y^2 = 1\}$ be the unit circle. We claim every point of S is an extreme point. Indeed, S is contained in the closed unit disc $D = \{(x, y) : x^2 + y^2 \leq 1\}$, and $\|\cdot\|$ denoting the Euclidean norm, every point of S is a unit vector. Suppose $p \in S$ and $p = \lambda u + (1 - \lambda)v$ with $u, v \in S$ and $0 < \lambda < 1$. Then $\|u\| = \|v\| = 1$ and, by the triangle inequality,

$$1 = \|p\| \leq \lambda\|u\| + (1 - \lambda)\|v\| = 1,$$

so equality holds. Squaring, and writing $\langle \cdot, \cdot \rangle$ for the usual inner product,

$$1 = \|\lambda u + (1 - \lambda)v\|^2 = \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)\langle u, v \rangle.$$

Since $\lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda) = 1$ and $\lambda(1 - \lambda) > 0$, this forces $\langle u, v \rangle = 1$, and therefore

$$\|u - v\|^2 = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2 = 1 - 2 + 1 = 0,$$

so $u = v$. Hence no point of the circle is an interior point of a segment with endpoints on the circle, and every point of S is extreme.

(ii) Let $S = \{(x, y) : |x| + |y| = 1\}$, the boundary of the unit ball for the norm $\|(x, y)\|_1 = |x| + |y|$. We claim the extreme points of S are exactly the four points

$$(1, 0), \quad (-1, 0), \quad (0, 1), \quad (0, -1).$$

First, no other point of S is extreme. A point $p \in S$ other than these four has both coordinates nonzero, say $p = (x_0, y_0)$ with $x_0 \neq 0 \neq y_0$. Let $\sigma = \text{sgn}(x_0)$ and $\tau = \text{sgn}(y_0)$, and for $|t|$ small put $q_t = (x_0 + \sigma t, y_0 - \tau t)$. If $|t| < \min\{|x_0|, |y_0|\}$ then $x_0 + \sigma t$ has the sign of x_0 and $y_0 - \tau t$ has the sign of y_0 , so

$$|x_0 + \sigma t| + |y_0 - \tau t| = \sigma(x_0 + \sigma t) + \tau(y_0 - \tau t) = |x_0| + |y_0| = 1,$$

that is, $q_t \in S$. Fix such a $t > 0$. Then $q_t \neq q_{-t}$ and $p = \frac{1}{2}q_t + \frac{1}{2}q_{-t}$, so p is not extreme.

Second, each of the four listed points is extreme. By symmetry it suffices to treat $p = (1, 0)$. Note S is contained in the closed ball $B_1 = \{(x, y) : |x| + |y| \leq 1\}$, so it is enough to show $(1, 0)$ is an extreme point of B_1 . Suppose $(1, 0) = \lambda u + (1 - \lambda)v$ with $u = (u_1, u_2)$, $v = (v_1, v_2)$ in B_1 and $0 < \lambda < 1$. Then

$$1 = |\lambda u_1 + (1 - \lambda)v_1| \leq \lambda|u_1| + (1 - \lambda)|v_1| \leq \lambda(|u_1| + |u_2|) + (1 - \lambda)(|v_1| + |v_2|) \leq 1.$$

Hence all these inequalities are equalities. The second equality forces $\lambda|u_2| + (1 - \lambda)|v_2| = 0$, so $u_2 = v_2 = 0$; the third then gives $|u_1| = |v_1| = 1$; and $\lambda u_1 + (1 - \lambda)v_1 = 1$ with $u_1, v_1 \in \{-1, 1\}$ forces $u_1 = v_1 = 1$. Thus $u = v = (1, 0)$, and $(1, 0)$ is extreme.

(iii) Let $S = \{(x, y) : \max\{x, y\} = 1\}$. This is the union of the two rays

$$R_1 = \{(1, y) : y \leq 1\}, \quad R_2 = \{(x, 1) : x \leq 1\},$$

which meet at the corner $(1, 1)$. We claim the unique extreme point of S is $(1, 1)$.

No point of S other than $(1, 1)$ is extreme. Such a point lies on R_1 or on R_2 ; by the symmetry $(x, y) \mapsto (y, x)$, which carries S onto itself, we may assume it is $p = (1, y_0)$ with $y_0 < 1$. Choose $t > 0$ with $y_0 + t \leq 1$. Then $(1, y_0 + t)$ and $(1, y_0 - t)$ both lie in $R_1 \subseteq S$, they are distinct, and p is their midpoint. So p is not extreme.

The corner $(1, 1)$ is extreme. Every point $(x, y) \in S$ satisfies $\max\{x, y\} = 1$ and hence $x \leq 1$ and $y \leq 1$; thus S is contained in the convex set $C = \{(x, y) : x \leq 1, y \leq 1\}$. Suppose $(1, 1) = \lambda u + (1 - \lambda)v$ with $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $S \subseteq C$ and $0 < \lambda < 1$. From $\lambda u_1 + (1 - \lambda)v_1 = 1$ with $u_1 \leq 1$, $v_1 \leq 1$ and $\lambda, 1 - \lambda > 0$ we get

$$1 = \lambda u_1 + (1 - \lambda)v_1 \leq \lambda + (1 - \lambda) = 1,$$

with equality, which forces $u_1 = v_1 = 1$. The same argument in the second coordinate gives $u_2 = v_2 = 1$. Hence $u = v = (1, 1)$, and $(1, 1)$ is an extreme point.

Problem 14.67. In each of the following, B denotes the closed unit ball of a normed linear space X . (i) Show that the only possible extreme points of B have norm 1. (ii) If $X = L^p[a, b]$, $1 < p < \infty$, show that every unit vector in B is an extreme point of B . (iii) If $X = L^\infty[a, b]$, show that the extreme points of B are those functions $f \in B$ such that $|f| = 1$ a.e. on $[a, b]$. (iv) If $X = L^1[a, b]$, show that B fails to have any extreme points. (v) If $X = \ell^p$, $1 \leq p \leq \infty$, what are the extreme points of B ? (vi) If $X = C(K)$, where K is a compact Hausdorff topological space and X is normed by the maximum norm, what are the extreme points of B ?

Solution. Throughout, a point p of a convex set S is an extreme point provided $p = \lambda u + (1 - \lambda)v$ with $u, v \in S$ and $0 < \lambda < 1$ forces $u = v = p$. We assume $X \neq \{0\}$ (otherwise $B = \{0\}$ and the single point of B is trivially extreme).

(i) Let $f \in B$ with $\|f\| < 1$. Choose any $w \in X$ with $\|w\| = 1$ (possible since $X \neq \{0\}$) and put $\delta = 1 - \|f\| > 0$. Set

$$u = f + \delta w, \quad v = f - \delta w.$$

Then $\|u\| \leq \|f\| + \delta = 1$ and likewise $\|v\| \leq 1$, so $u, v \in B$; moreover $u - v = 2\delta w \neq 0$, so $u \neq v$; and $f = \frac{1}{2}u + \frac{1}{2}v$. Hence f is not an extreme point of B . Therefore every extreme point of B has norm exactly 1.

Before doing (ii) we record two elementary facts about real numbers, used repeatedly.

Fact A. For $p > 1$ the function $\varphi(t) = t^p$ is strictly convex on $[0, \infty)$: φ is twice differentiable on $(0, \infty)$ with $\varphi''(t) = p(p-1)t^{p-2} > 0$ there, and φ is continuous on $[0, \infty)$; hence for $0 \leq \alpha \neq \beta$ and $0 < \lambda < 1$,

$$(\lambda\alpha + (1 - \lambda)\beta)^p < \lambda\alpha^p + (1 - \lambda)\beta^p.$$

Fact B. For real numbers s, t and $0 < \lambda < 1$, one has $|\lambda s + (1 - \lambda)t| = \lambda|s| + (1 - \lambda)|t|$ if and only if s and t have the same sign in the weak sense $st \geq 0$. Consequently, if in addition $|s| = |t|$, then $s = t$.

(ii) Let $X = L^p[a, b]$ with $1 < p < \infty$ and let $f \in B$ with $\|f\|_p = 1$. Suppose

$$f = \lambda g + (1 - \lambda)h, \quad g, h \in B, \quad 0 < \lambda < 1.$$

Pointwise almost everywhere on $[a, b]$ we have $|f| \leq \lambda|g| + (1 - \lambda)|h|$, and since $t \mapsto t^p$ is increasing on $[0, \infty)$ and convex,

$$|f|^p \leq (\lambda|g| + (1 - \lambda)|h|)^p \leq \lambda|g|^p + (1 - \lambda)|h|^p \quad \text{a.e. on } [a, b].$$

Integrating over $[a, b]$ and using $\|g\|_p \leq 1$, $\|h\|_p \leq 1$,

$$1 = \int_a^b |f|^p \leq \int_a^b (\lambda|g| + (1 - \lambda)|h|)^p \leq \lambda \int_a^b |g|^p + (1 - \lambda) \int_a^b |h|^p \leq \lambda + (1 - \lambda) = 1.$$

Hence every inequality above is an equality. Since the difference of two integrable functions with $\varphi \leq \psi$ a.e. and $\int \varphi = \int \psi$ is a nonnegative function of zero integral, and a nonnegative measurable function with zero integral vanishes almost everywhere, we conclude that almost everywhere on $[a, b]$

$$(\lambda|g| + (1 - \lambda)|h|)^p = \lambda|g|^p + (1 - \lambda)|h|^p \quad \text{and} \quad |f| = \lambda|g| + (1 - \lambda)|h|.$$

By Fact A, the first equality forces $|g| = |h|$ a.e. Since $f = \lambda g + (1 - \lambda)h$, the second equality says $|\lambda g + (1 - \lambda)h| = \lambda|g| + (1 - \lambda)|h|$ a.e., so by Fact B, together with $|g| = |h|$ a.e., we get $g = h$ a.e., that is, $g = h$ in $L^p[a, b]$. Then also $f = g = h$. Therefore every unit vector of $L^p[a, b]$, $1 < p < \infty$, is an extreme point of B . Combined with (i), the extreme points of B are precisely the unit vectors.

(iii) Let $X = L^\infty[a, b]$.

Suppose first $f \in B$ and $|f| = 1$ a.e. Let $f = \lambda g + (1 - \lambda)h$ with $g, h \in B$ and $0 < \lambda < 1$. Each of the four conditions $|f(t)| = 1$, $|g(t)| \leq 1$, $|h(t)| \leq 1$ and $f(t) = \lambda g(t) + (1 - \lambda)h(t)$ holds outside a set of measure zero, so all four hold simultaneously for every t in a set T of full measure. For $t \in T$,

$$1 = |f(t)| = |\lambda g(t) + (1 - \lambda)h(t)| \leq \lambda|g(t)| + (1 - \lambda)|h(t)| \leq 1,$$

so $|g(t)| = |h(t)| = 1$ and equality holds in the triangle inequality. By Fact B, $g(t)$ and $h(t)$ have the same sign, and having the same absolute value, $g(t) = h(t)$. Thus $g = h$ a.e. and f is an extreme point of B .

Conversely, suppose $f \in B$ and it is not the case that $|f| = 1$ a.e. Fix a representative of f ; since $\|f\|_\infty \leq 1$, we have $|f| \leq 1$ outside a set of measure zero, and altering f on that null set we may assume $|f(t)| \leq 1$ for every t . The set $\{t : |f(t)| < 1\}$ then has positive measure, for otherwise $|f| = 1$ a.e. Since

$$\{t : |f(t)| < 1\} = \bigcup_{n=1}^{\infty} \{t : |f(t)| \leq 1 - 1/n\}$$

is a countable union, by the countable additivity of Lebesgue measure at least one of the sets in the union has positive measure: there is an n with $m(E) > 0$, where $E = \{t : |f(t)| \leq 1 - 1/n\}$. Put $\epsilon = 1/n$ and

$$g = f + \epsilon \chi_E, \quad h = f - \epsilon \chi_E.$$

Off E we have $|g| = |h| = |f| \leq 1$ a.e., and on E we have $|g| \leq |f| + \epsilon \leq 1$ and $|h| \leq 1$; so $g, h \in B$. Since $g - h = 2\epsilon \chi_E$ and $m(E) > 0$, $g \neq h$ in $L^\infty[a, b]$. As $f = \frac{1}{2}g + \frac{1}{2}h$, f is not extreme. Hence the extreme points of B are exactly the $f \in B$ with $|f| = 1$ a.e.

(iv) Let $X = L^1[a, b]$ and let $f \in B$. If $\|f\|_1 < 1$, part (i) shows f is not extreme. So assume $\|f\|_1 = 1$. Define

$$\Phi(t) = \int_a^t |f| \quad \text{for } t \in [a, b].$$

Since $|f|$ is integrable over $[a, b]$, Proposition 23 of Chapter 4 tells us that for each $\epsilon > 0$ there is a $\delta > 0$ such that $\int_A |f| < \epsilon$ whenever $A \subseteq [a, b]$ is measurable with $m(A) < \delta$. As $|\Phi(t) - \Phi(s)| = \int_s^t |f|$ for $a \leq s \leq t \leq b$, this shows Φ is (uniformly) continuous on $[a, b]$. Moreover $\Phi(a) = 0$ and $\Phi(b) = \|f\|_1 = 1$. By the Intermediate Value Theorem there is $c \in (a, b)$ with $\Phi(c) = 1/2$. Put $E = [a, c]$ and $F = (c, b]$, so that

$$\int_E |f| = \int_F |f| = \frac{1}{2}.$$

Define $g = 2f\chi_E$ and $h = 2f\chi_F$. Then $\|g\|_1 = 2\int_E |f| = 1$ and $\|h\|_1 = 1$, so $g, h \in B$; moreover

$$\|g - h\|_1 = \int_a^b |2f\chi_E - 2f\chi_F| = 2 \int_a^b |f| = 2 \neq 0,$$

so $g \neq h$. Finally $\frac{1}{2}g + \frac{1}{2}h = f\chi_E + f\chi_F = f$. Hence f is not an extreme point. Since no $f \in B$ is extreme, B has no extreme points.

(v) Let $X = \ell^p$. We treat the three cases; write e_n for the sequence whose n th component is 1 and whose other components are 0.

Case $1 < p < \infty$. The extreme points of B are precisely the unit vectors, that is, all x with $\|x\|_p = 1$. Indeed, ℓ^p is the L^p space of counting measure on the natural numbers, and the proof of (ii) applies verbatim with integrals replaced by sums: if $\|x\|_p = 1$ and $x = \lambda u + (1 - \lambda)v$ with $\|u\|_p, \|v\|_p \leq 1$ and $0 < \lambda < 1$, then

$$1 = \sum_n |x_n|^p \leq \sum_n (\lambda|u_n| + (1 - \lambda)|v_n|)^p \leq \lambda \sum_n |u_n|^p + (1 - \lambda) \sum_n |v_n|^p \leq 1,$$

so for every n we have $(\lambda|u_n| + (1 - \lambda)|v_n|)^p = \lambda|u_n|^p + (1 - \lambda)|v_n|^p$ and $|x_n| = \lambda|u_n| + (1 - \lambda)|v_n|$. By Facts A and B, $|u_n| = |v_n|$ and $u_n = v_n$ for every n ; thus $u = v$. Together with (i), the extreme points are exactly the unit vectors.

Case $p = \infty$. The extreme points of B are precisely the sequences x with $|x_n| = 1$ for every n , that is, the sequences all of whose entries are ± 1 . If $|x_n| = 1$ for all n and $x = \lambda u + (1 - \lambda)v$ with $u, v \in B$ and $0 < \lambda < 1$, then for each n ,

$$1 = |x_n| = |\lambda u_n + (1 - \lambda)v_n| \leq \lambda|u_n| + (1 - \lambda)|v_n| \leq 1,$$

so $|u_n| = |v_n| = 1$ and, by Fact B, $u_n = v_n$; hence $u = v$. Conversely, if $x \in B$ and $|x_N| < 1$ for some N , set $\epsilon = 1 - |x_N| > 0$ and $u = x + \epsilon e_N$, $v = x - \epsilon e_N$. Then $\|u\|_\infty \leq 1$ and $\|v\|_\infty \leq 1$, $u \neq v$, and $x = \frac{1}{2}u + \frac{1}{2}v$; so x is not extreme.

Case $p = 1$. The extreme points of B are precisely the vectors $\pm e_n$, n a natural number. First, e_n is extreme (and by symmetry so is $-e_n$): suppose $e_n = \lambda u + (1 - \lambda)v$ with $\|u\|_1, \|v\|_1 \leq 1$ and $0 < \lambda < 1$. Then $1 = \|e_n\|_1 \leq \lambda\|u\|_1 + (1 - \lambda)\|v\|_1 \leq 1$, so $\|u\|_1 = \|v\|_1 = 1$. Looking at the n th component, $\lambda u_n + (1 - \lambda)v_n = 1$ with $|u_n| \leq \|u\|_1 = 1$ and $|v_n| \leq 1$ forces $u_n = v_n = 1$; and then $\sum_{k \neq n} |u_k| = \|u\|_1 - |u_n| = 0$, so $u = e_n$, and likewise $v = e_n$.

Conversely, let $x \in B$ be extreme. By (i), $\|x\|_1 = 1$. Suppose x had two nonzero components, say $x_m \neq 0$ and $x_n \neq 0$ with $m \neq n$. Put $\alpha = |x_m|$, so $0 < \alpha < 1$ because $\|x\|_1 = 1$ and $|x_n| > 0$. Let

$$y = x_m e_m, \quad z = x - y,$$

so $\|y\|_1 = \alpha$ and $\|z\|_1 = 1 - \alpha > 0$. Set $u = y/\alpha$ and $v = z/(1 - \alpha)$. Then $\|u\|_1 = \|v\|_1 = 1$, so $u, v \in B$; $u \neq v$ because the n th component of u is 0 while that of v is $x_n/(1 - \alpha) \neq 0$; and

$$\alpha u + (1 - \alpha)v = y + z = x, \quad 0 < \alpha < 1,$$

contradicting extremeness. Hence x has exactly one nonzero component, of absolute value $\|x\|_1 = 1$; that is, $x = \pm e_n$ for some n .

(vi) Let $X = C(K)$ with K compact Hausdorff and $\|f\| = \max_{x \in K} |f(x)|$. The extreme points of B are precisely the $f \in C(K)$ with $|f(x)| = 1$ for every $x \in K$.

Suppose $|f| \equiv 1$ on K and $f = \lambda g + (1 - \lambda)h$ with $g, h \in B$ and $0 < \lambda < 1$. For each $x \in K$,

$$1 = |f(x)| = |\lambda g(x) + (1 - \lambda)h(x)| \leq \lambda|g(x)| + (1 - \lambda)|h(x)| \leq 1,$$

so $|g(x)| = |h(x)| = 1$ and equality holds in the triangle inequality; by Fact B, $g(x) = h(x)$. Hence $g = h$ and f is extreme.

Conversely, suppose $f \in B$ and $|f(x_0)| < 1$ for some $x_0 \in K$. Put $\epsilon = \frac{1}{2}(1 - |f(x_0)|) > 0$ and

$$U = \{x \in K : |f(x)| < 1 - \epsilon\},$$

an open set containing x_0 , since $|f|$ is continuous and $|f(x_0)| = 1 - 2\epsilon < 1 - \epsilon$. A compact Hausdorff space is normal, so by Urysohn's Lemma applied to the disjoint closed sets $\{x_0\}$ and $K \sim U$ there is a continuous $\varphi : K \rightarrow [0, 1]$ with $\varphi(x_0) = 1$ and $\varphi = 0$ on $K \sim U$. Define

$$g = f + \epsilon\varphi, \quad h = f - \epsilon\varphi.$$

Both are continuous. For $x \in U$, $|g(x)| \leq |f(x)| + \epsilon < 1$ and similarly $|h(x)| < 1$; for $x \in K \sim U$, $g(x) = h(x) = f(x)$, of absolute value at most 1. Hence $g, h \in B$. Also $g(x_0) - h(x_0) = 2\epsilon \neq 0$, so $g \neq h$, while $f = \frac{1}{2}g + \frac{1}{2}h$. Thus f is not extreme.

We remark that since $|f| \equiv 1$ and f is continuous, such an extreme f takes only the values ± 1 and $\{f = 1\}$ is a clopen subset of K ; so the extreme points of B are exactly the functions $\chi_A - \chi_{K \sim A}$ with A a clopen subset of K . In particular, if K is connected the only extreme points are the constants 1 and -1 .

Problem 14.68. A norm on a linear space is said to be strictly convex provided whenever u and v are distinct unit vectors and $0 < \lambda < 1$, then $\|\lambda u + (1 - \lambda)v\| < 1$. Show that the Euclidean norm on $\mathbb{R}^n$ and the usual norm on $L^p[a, b]$, $1 < p < \infty$, are strictly convex.

Solution. The Euclidean norm on $\mathbb{R}^n$. Write $\langle u, v \rangle = \sum_{k=1}^n u_k v_k$ for the usual inner product, so that $\|u\|^2 = \langle u, u \rangle$. Let u and v be distinct unit vectors and let $0 < \lambda < 1$. Expanding,

$$\|\lambda u + (1 - \lambda)v\|^2 = \lambda^2\|u\|^2 + 2\lambda(1 - \lambda)\langle u, v \rangle + (1 - \lambda)^2\|v\|^2 = \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)\langle u, v \rangle.$$

Now

$$\|u - v\|^2 = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2 = 2 - 2\langle u, v \rangle,$$

and $u \neq v$ gives $\|u - v\| > 0$, so $\langle u, v \rangle < 1$. Since $\lambda(1 - \lambda) > 0$,

$$\|\lambda u + (1 - \lambda)v\|^2 < \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda) = (\lambda + (1 - \lambda))^2 = 1,$$

so $\|\lambda u + (1 - \lambda)v\| < 1$. Hence the Euclidean norm is strictly convex. (The same computation works in any inner product space, in particular in ℓ^2 and $L^2[a, b]$.)

The norm on $L^p[a, b]$, $1 < p < \infty$. We use two elementary facts about real numbers.

Fact A. For $p > 1$ the function $\varphi(t) = t^p$ is strictly convex on $[0, \infty)$, since it is continuous there and $\varphi''(t) = p(p - 1)t^{p-2} > 0$ on $(0, \infty)$. Thus for $\alpha, \beta \geq 0$ with $\alpha \neq \beta$ and $0 < \lambda < 1$,

$$(\lambda\alpha + (1 - \lambda)\beta)^p < \lambda\alpha^p + (1 - \lambda)\beta^p,$$

while for $\alpha = \beta$ equality holds.

Fact B. For real s, t and $0 < \lambda < 1$, $|\lambda s + (1 - \lambda)t| = \lambda|s| + (1 - \lambda)|t|$ if and only if $st \geq 0$; and if moreover $|s| = |t|$, then $s = t$.

Now let $u, v \in L^p[a, b]$ be distinct unit vectors, so $\|u\|_p = \|v\|_p = 1$ and $u \neq v$ in $L^p[a, b]$ (that is, $u \neq v$ on a set of positive measure), and let $0 < \lambda < 1$. By Minkowski's Inequality,

$$\|\lambda u + (1 - \lambda)v\|_p \leq \lambda\|u\|_p + (1 - \lambda)\|v\|_p = 1.$$

We argue by contradiction and suppose $\|\lambda u + (1 - \lambda)v\|_p = 1$. Pointwise almost everywhere we have $|\lambda u + (1 - \lambda)v| \leq \lambda|u| + (1 - \lambda)|v|$, and since $t \mapsto t^p$ is increasing and convex on $[0, \infty)$,

$$|\lambda u + (1 - \lambda)v|^p \leq (\lambda|u| + (1 - \lambda)|v|)^p \leq \lambda|u|^p + (1 - \lambda)|v|^p \quad \text{a.e. on } [a, b].$$

All three functions are nonnegative and integrable, and integrating gives

$$1 = \int_a^b |\lambda u + (1 - \lambda)v|^p \leq \int_a^b (\lambda|u| + (1 - \lambda)|v|)^p \leq \lambda \int_a^b |u|^p + (1 - \lambda) \int_a^b |v|^p = \lambda + (1 - \lambda) = 1.$$

Hence both inequalities between the integrals are equalities. If two integrable functions satisfy $\phi \leq \psi$ almost everywhere and have equal integrals, then $\psi - \phi$ is a nonnegative integrable function of zero

integral and therefore vanishes almost everywhere. Applying this twice, we obtain, almost everywhere on $[a, b]$,

$$(\lambda|u| + (1 - \lambda)|v|)^p = \lambda|u|^p + (1 - \lambda)|v|^p$$

and

$$|\lambda u + (1 - \lambda)v| = \lambda|u| + (1 - \lambda)|v|.$$

By Fact A, the first equality forces $|u| = |v|$ almost everywhere. By Fact B, the second forces u and v to have the same sign almost everywhere, and combined with $|u| = |v|$ a.e. this gives $u = v$ almost everywhere on $[a, b]$, that is, $u = v$ in $L^p[a, b]$. This contradicts the assumption that u and v are distinct.

Therefore $\|\lambda u + (1 - \lambda)v\|_p < 1$, and the usual norm on $L^p[a, b]$, $1 < p < \infty$, is strictly convex.

Two remarks. The same argument, with integrals replaced by sums, shows that the norm on ℓ^p is strictly convex for $1 < p < \infty$. On the other hand the norms on $L^1[a, b]$ and $L^\infty[a, b]$ are not strictly convex. Let c be the midpoint of $[a, b]$. For $L^1[a, b]$ take

$$u = \frac{\chi_{[a, c]}}{c - a}, \quad v = \frac{\chi_{(c, b]}}{b - c}.$$

These are distinct, $\|u\|_1 = \|v\|_1 = 1$, and since u and v have disjoint supports,

$$\left\| \frac{1}{2}u + \frac{1}{2}v \right\|_1 = \frac{1}{2}\|u\|_1 + \frac{1}{2}\|v\|_1 = 1.$$

For $L^\infty[a, b]$ take $u \equiv 1$ and $v = \chi_{[a, c]} - \chi_{(c, b]}$; these are distinct unit vectors whose average is $\chi_{[a, c]}$, of L^∞ norm 1.

Problem 14.69. Let X be a reflexive Banach space with a strictly convex norm and K a nonempty closed convex subset of X . For $z \in X \sim K$, use the reflexivity of X to show that there is a point $x_0 \in K$ that is closest to z in the sense that

$$\|z - x_0\| \leq \|x - z\| \text{ for all } x \in K.$$

Then use the strict convexity of the norm to show that x_0 is unique and is an extreme point of K .

Solution. Existence of a closest point. Set

$$d = \text{dist}(z, K) = \inf_{x \in K} \|x - z\|.$$

Since K is closed and $z \notin K$, the complement of K is open and hence contains a ball about z ; therefore $d > 0$. Choose a sequence $\{x_n\}$ in K with $\lim_{n \rightarrow \infty} \|x_n - z\| = d$. Discarding finitely many terms, we may assume $\|x_n - z\| \leq d + 1$ for all n . Define

$$C = K \cap \{x \in X : \|x - z\| \leq d + 1\}.$$

As the intersection of two convex sets, C is convex; as the intersection of two strongly closed sets, it is strongly closed; it is bounded, being contained in a ball of radius $d + 1$; and it is nonempty, since it contains every x_n . Since X is a reflexive Banach space, Theorem 28 tells us that C is weakly sequentially compact. Hence there is a subsequence $\{x_{n_k}\}$ that converges weakly to some $x_0 \in C \subseteq K$. For each $\psi \in X^*$ we have $\psi(x_{n_k} - z) = \psi(x_{n_k}) - \psi(z) \rightarrow \psi(x_0) - \psi(z) = \psi(x_0 - z)$, so $\{x_{n_k} - z\}$ converges weakly to $x_0 - z$. Theorem 12 therefore gives

$$\|x_0 - z\| \leq \liminf_{k \rightarrow \infty} \|x_{n_k} - z\| = d.$$

On the other hand $x_0 \in K$, so $\|x_0 - z\| \geq d$ by the definition of d . Hence $\|z - x_0\| = d \leq \|x - z\|$ for all $x \in K$, and x_0 is a closest point of K to z .

Uniqueness. Suppose x_0 and x_1 both belong to K and $\|x_0 - z\| = \|x_1 - z\| = d$, and suppose $x_0 \neq x_1$. Since $d > 0$, the vectors

$$u = \frac{x_0 - z}{d}, \quad v = \frac{x_1 - z}{d}$$

are unit vectors, and they are distinct because $x_0 \neq x_1$. By convexity of K the midpoint $m = \frac{1}{2}x_0 + \frac{1}{2}x_1$ belongs to K , and

$$\|m - z\| = \left\| \frac{1}{2}(x_0 - z) + \frac{1}{2}(x_1 - z) \right\| = d \left\| \frac{1}{2}u + \frac{1}{2}v \right\| < d,$$

the strict inequality by strict convexity of the norm applied with $\lambda = \frac{1}{2}$. This contradicts the definition of d as the infimum of $\|x - z\|$ over $x \in K$. Hence the closest point is unique.

The final assertion. The last claim of the problem, that this unique closest point x_0 is an extreme point of K , is false as stated, and no argument can establish it. Here is a counterexample. Take $X = \mathbb{R}^2$ with the Euclidean norm, which is a reflexive Banach space and, by the preceding problem, has a strictly convex norm. Let

$$K = [-1, 1] \times [-1, 1], \quad z = (0, 2) \notin K.$$

K is nonempty, closed and convex. For $(x, y) \in K$,

$$\|(x, y) - z\|^2 = x^2 + (2 - y)^2 \geq 0 + (2 - 1)^2 = 1,$$

with equality if and only if $x = 0$ and $y = 1$. So $d = 1$ and the unique closest point is $x_0 = (0, 1)$. But

$$(0, 1) = \frac{1}{2}(-1, 1) + \frac{1}{2}(1, 1),$$

and $(-1, 1)$ and $(1, 1)$ are distinct points of K . Hence x_0 is not an extreme point of K : it is the midpoint of the top edge of the square.

What is true, and is presumably what was intended, is the following statement, which is exactly the uniqueness argument above restated: x_0 cannot be written as $x_0 = \lambda u + (1 - \lambda)v$ with $u, v \in K$ distinct, $0 < \lambda < 1$, and $\|u - z\| = \|v - z\| = d$. Equivalently, x_0 is an extreme point of the set of closest points $K \cap \{x : \|x - z\| = d\}$, which by uniqueness is the singleton $\{x_0\}$. Indeed if $u, v \in K$ were distinct with $\|u - z\| = \|v - z\| = d$, uniqueness would already be violated.

We can also record the correct weaker geometric conclusion, that x_0 is a boundary point of K . Suppose x_0 were an interior point of K , so that $\{x : \|x - x_0\| < r\} \subseteq K$ for some $r > 0$. Since $d = \|x_0 - z\| > 0$, choose t with $0 < t < \min\{1, r/d\}$ and put

$$x_t = x_0 + t(z - x_0).$$

Then $\|x_t - x_0\| = t\|z - x_0\| = td < r$, so $x_t \in K$, while

$$\|x_t - z\| = \|(1 - t)(x_0 - z)\| = (1 - t)d < d,$$

contradicting the minimality of d . Hence x_0 lies on the boundary of K . In the counterexample above $(0, 1)$ is indeed a boundary point of the square but not an extreme point of it, which shows that the boundary conclusion is the most that the hypotheses yield.

Finally, note that strict convexity of the norm does give the extreme-point conclusion when K is a closed ball, say $K = \{x : \|x - c\| \leq r\}$ with $r > 0$. Indeed, the closest point to an exterior z then lies on the sphere $\|x - c\| = r$, and every point x_0 of that sphere is an extreme point of K : if $x_0 = \lambda u + (1 - \lambda)v$ with $u, v \in K$ and $0 < \lambda < 1$, then

$$r = \|x_0 - c\| \leq \lambda\|u - c\| + (1 - \lambda)\|v - c\| \leq r,$$

so $\|u - c\| = \|v - c\| = r$; were $u \neq v$, the unit vectors $(u - c)/r$ and $(v - c)/r$ would be distinct and strict convexity of the norm would give $\|x_0 - c\| < r$, a contradiction. So the failure in the general case is a failure of the set K , not of the norm.

15 Compactness Regained: The Weak Topology

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15.1 Problems 15.1–15.7

Problem 15.1. For X a normed linear space with closed unit ball B , suppose the function $f: B \rightarrow [-1, 1]$ has the property that whenever $u, v, u + v$, and λu belong to B ,

$$f(u + v) = f(u) + f(v) \quad \text{and} \quad f(\lambda u) = \lambda f(u).$$

Show that f is the restriction to B of a linear functional on all of X which belongs to the closed unit ball of X^* .

Solution. The hypothesis is stated as a single implication: if $u, v, u + v$ and λu all belong to B , then both equalities hold. We first record that it may be used in the following two separate forms.

(A) If u, v and $u + v$ all lie in B , then $f(u + v) = f(u) + f(v)$. Indeed, apply the hypothesis to this u , this v and the scalar $\lambda = 0$; then $\lambda u = 0$ lies in B , so all four vectors $u, v, u + v, \lambda u$ belong to B and the first equality holds.

(B) If u and λu both lie in B , then $f(\lambda u) = \lambda f(u)$. Indeed, apply the hypothesis to this u , this λ and the vector $v = 0$; then $v = 0$ and $u + v = u$ lie in B , so all four vectors belong to B and the second equality holds.

Here we used that 0 belongs to B , which holds since $\|0\| = 0 \leq 1$.

Step 1: $f(0) = 0$. Take $u = v = 0$ in (A). Then u, v and $u + v = 0$ all belong to B , so $f(0) = f(0) + f(0)$ and therefore $f(0) = 0$.

Step 2: definition of the extension. Let x belong to X . If $t > 0$ and $s > 0$ are such that tx and sx both belong to B , we claim that

$$\frac{1}{t} f(tx) = \frac{1}{s} f(sx).$$

Indeed, we may assume $s \leq t$. Put $u = tx$ and $\lambda = s/t$, so that $0 < \lambda \leq 1$. Then $u = tx$ belongs to B and $\lambda u = sx$ belongs to B , so (B) gives $f(sx) = (s/t)f(tx)$, which is the claimed equality.

Since for $x \neq 0$ the vector tx belongs to B whenever $0 < t \leq 1/\|x\|$, we may therefore define unambiguously

$$\psi(x) = \frac{1}{t} f(tx) \quad \text{for any } t > 0 \text{ with } tx \in B,$$

and we set $\psi(0) = 0$; this is consistent with the displayed formula since $f(0) = 0$.

Step 3: ψ extends f . If x belongs to B we may take $t = 1$ in the definition, and hence $\psi(x) = f(x)$. Thus $\psi|_B = f$.

Step 4: ψ is positively homogeneous. Let $\lambda > 0$ and let x belong to X . Choose $t > 0$ so small that $t\lambda x$ belongs to B . Computing $\psi(\lambda x)$ with the parameter t and $\psi(x)$ with the parameter $t\lambda$ (both are legitimate, since $t(\lambda x) = (t\lambda)x$ belongs to B),

$$\psi(\lambda x) = \frac{1}{t} f(t\lambda x) = \lambda \cdot \frac{1}{t\lambda} f((t\lambda)x) = \lambda \psi(x).$$

Step 5: ψ is odd. Let x belong to X and choose $t > 0$ with $u = tx$ in B . Then $-u$ also belongs to B , since $\| -u \| = \|u\| \leq 1$; so (B) with $\lambda = -1$ gives $f(-u) = -f(u)$. Hence

$$\psi(-x) = \frac{1}{t} f(-tx) = -\frac{1}{t} f(tx) = -\psi(x).$$

Together with Step 4 and $\psi(0) = 0$ this gives $\psi(\lambda x) = \lambda \psi(x)$ for every real λ and every x in X .

Step 6: ψ is additive. Let x, y belong to X and set

$$t = \frac{1}{1 + \|x\| + \|y\|} > 0.$$

Then $\|tx\| \leq 1$, $\|ty\| \leq 1$ and $\|t(x + y)\| \leq 1$, so $u = tx$, $v = ty$ and $u + v = t(x + y)$ all belong to B . Statement (A) therefore applies and yields $f(t(x + y)) = f(tx) + f(ty)$. Dividing by t and using the definition of ψ three times (each time with the admissible parameter t),

$$\psi(x + y) = \frac{1}{t} f(t(x + y)) = \frac{1}{t} f(tx) + \frac{1}{t} f(ty) = \psi(x) + \psi(y).$$

By Steps 5 and 6, $\psi: X \rightarrow \mathbb{R}$ is linear.

Step 7: ψ belongs to the closed unit ball of X^* . If $\|x\| \leq 1$, then x belongs to B and hence $\psi(x) = f(x)$ lies in $[-1, 1]$; thus $|\psi(x)| \leq 1$ for all x in B . Consequently ψ is bounded and

$$\|\psi\| = \sup_{\|x\| \leq 1} |\psi(x)| \leq 1.$$

So ψ is a linear functional on all of X that belongs to the closed unit ball B^* of X^* and whose restriction to B is f . (It is the only such functional, since every x in X is a positive multiple of a vector in B and a linear functional is determined by its values on B .)

Problem 15.2. Let X be a normed linear space and K be a bounded convex weak-* closed subset of X^* . Show that K possesses an extreme point.

Solution. Of course K is assumed nonempty, since the empty set has no extreme point.

Denote by B^* the closed unit ball of X^* . We first observe that X^* , with the weak-* topology, is a locally convex topological vector space. Indeed, the weak-* topology on X^* is precisely the W -weak topology induced by the subspace $W = J(X)$ of X^{**} , where $J: X \rightarrow X^{**}$ is the natural embedding; and $J(X)$ separates the points of X^* , since if ψ and η are distinct members of X^* there is a point x in X with $\psi(x) \neq \eta(x)$, that is, $J(x)[\psi] \neq J(x)[\eta]$. Proposition 21 of Chapter 14 tells us that a normed linear space, here X^* , is a locally convex topological vector space with respect to the W -weak topology induced by any subspace W of its dual that separates points. Hence X^* with its weak-* topology is a locally convex topological vector space.

Next we show that K is weak-* compact. Since K is bounded, there is an $r > 0$ for which

$$K \subseteq rB^* = \{\psi \in X^* \mid \|\psi\| \leq r\}.$$

By Alaoglu's Theorem, B^* is weak-* compact. The mapping $M_r: X^* \rightarrow X^*$ defined by $M_r(\psi) = r\psi$ is a bijection which, together with its inverse $M_{1/r}$, is weak-* continuous: for each x in X the function $\psi \mapsto (r\psi)(x) = r\psi(x)$ is weak-* continuous, and the weak-* topology is the weakest topology for which all the functions $\psi \mapsto \psi(x)$, $x \in X$, are continuous. Thus M_r is a weak-* homeomorphism of X^* onto itself, and therefore $rB^* = M_r(B^*)$ is weak-* compact, being the continuous image of a compact space. Now K is a weak-* closed subset of the weak-* compact set rB^* . A closed subset of a compact topological space is compact. Therefore K is weak-* compact.

Finally, K is a nonempty, compact, convex subset of the locally convex topological vector space consisting of X^* with its weak-* topology. The Krein-Milman Lemma tells us that such a set has an extreme point. Therefore K possesses an extreme point.

(Corollary 2 is the special case $K = B^*$: the closed unit ball of X^* is convex, bounded, and weak-* closed, the latter because it is weak-* compact and the weak-* topology is Hausdorff.)

Problem 15.3. Show that any nonempty weakly open set in an infinite dimensional normed linear space is unbounded with respect to the norm.

Solution. Let X be an infinite dimensional normed linear space and let $\mathcal{O}$ be a nonempty weakly open subset of X . Choose x_0 in $\mathcal{O}$. By the description of the neighborhood base for the weak topology, there are an $\epsilon > 0$ and finitely many functionals $\psi_1, \dots, \psi_n$ in X^* for which

$$\mathcal{N} = \mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x_0) = \{x \in X \mid |\psi_i(x - x_0)| < \epsilon \text{ for } 1 \leq i \leq n\} \subseteq \mathcal{O}.$$

Consider the linear operator $T: X \rightarrow \mathbb{R}^n$ defined by

$$T(x) = (\psi_1(x), \dots, \psi_n(x)),$$

and let $Z = \ker T = \bigcap_{i=1}^n \ker \psi_i$, a linear subspace of X .

We claim $Z \neq \{0\}$. Indeed, suppose $Z = \{0\}$. Then T is one-to-one and hence T is a linear isomorphism of the linear space X onto the linear subspace $T(X)$ of $\mathbb{R}^n$. Since a linear subspace of $\mathbb{R}^n$ has dimension

at most n , and a one-to-one linear map carries linearly independent sets to linearly independent sets, the linear space X would have dimension at most n . This contradicts the assumption that X is infinite dimensional. Therefore there is a vector z in Z with $z \neq 0$.

Now for every real number t and every index i with $1 \leq i \leq n$,

$$|\psi_i((x_0 + tz) - x_0)| = |t| |\psi_i(z)| = 0 < \epsilon,$$

since z belongs to $\ker \psi_i$. Hence the entire line $\{x_0 + tz \mid t \in \mathbb{R}\}$ is contained in $\mathcal{N}$, and therefore in $\mathcal{O}$. But by the triangle inequality,

$$\|x_0 + tz\| \geq |t| \|z\| - \|x_0\| \rightarrow \infty \quad \text{as } |t| \rightarrow \infty,$$

because $\|z\| > 0$. Therefore $\mathcal{O}$ contains vectors of arbitrarily large norm, that is, $\mathcal{O}$ is unbounded with respect to the norm.

(Observe that the argument shows more: every nonempty weakly open subset of an infinite dimensional normed linear space contains a whole line.)

Problem 15.4. Use the Baire Category Theorem and the preceding problem to show that the weak topology on an infinite dimensional Banach space is not metrizable by a complete metric.

Solution. Let X be an infinite dimensional Banach space and argue by contradiction: assume there is a metric ρ on X which induces the weak topology on X and for which the metric space (X, ρ) is complete.

For each natural number n define

$$F_n = \{x \in X \mid \|x\| \leq n\},$$

the closed ball of radius n centered at the origin. Each F_n is convex, and each F_n is closed with respect to the topology induced by the norm, that is, strongly closed. Mazur's Theorem tells us that a convex subset of a normed linear space is strongly closed if and only if it is weakly closed. Therefore each F_n is weakly closed, hence closed in the metric space (X, ρ) .

Moreover

$$X = \bigcup_{n=1}^{\infty} F_n,$$

since each x in X has finite norm and so belongs to F_n for any $n \geq \|x\|$.

Now argue as follows. Suppose every F_n were hollow, that is, had empty interior in (X, ρ) . The metric space (X, ρ) is complete, so part (ii) of the Baire Category Theorem tells us that the union of the countable collection $\{F_n\}$ of closed hollow subsets is itself hollow. But that union is all of X , and X is a nonempty open subset of itself, so its interior is $X \neq \emptyset$. This contradiction shows that some F_N has nonempty interior with respect to the topology of ρ , which is the weak topology. Thus there is a nonempty weakly open set $\mathcal{O}$ with

$$\mathcal{O} \subseteq F_N.$$

But X is infinite dimensional, so by the preceding problem the nonempty weakly open set $\mathcal{O}$ is unbounded with respect to the norm, whereas every vector in $\mathcal{O} \subseteq F_N$ has norm at most N . This contradiction shows that no such complete metric ρ exists.

Therefore the weak topology on an infinite dimensional Banach space is not metrizable by a complete metric. In particular it differs from the strong topology, which is metrizable by the complete metric induced by the norm.

Two remarks on the hypotheses. The completeness of X in the norm was never used; what the argument needs is only that X be an infinite dimensional normed linear space, since the assumed metric ρ supplies the completeness required by the Baire Category Theorem. Also, the sets F_n are closed in the weak topology by Mazur's Theorem, and it is exactly here that convexity is essential: a strongly closed set that is not convex need not be weakly closed.

Problem 15.5. Is every Banach space isomorphic to the dual of a Banach space?

Solution. No. The space c_0 of real sequences converging to 0, normed by $\|x\|_\infty = \sup_k |x_k|$, is a Banach space which is not the dual of any normed linear space. That c_0 is a Banach space is immediate from its being a closed linear subspace of the Banach space ℓ^∞ : if $\{x^{(j)}\}$ lies in c_0 and converges in $\|\cdot\|_\infty$ to x , then given $\epsilon > 0$ choose j with $\|x - x^{(j)}\|_\infty < \epsilon/2$ and then K with $|x_k^{(j)}| < \epsilon/2$ for $k \geq K$, so $|x_k| < \epsilon$ for $k \geq K$.

The obstruction is supplied by Corollary 2: the closed unit ball of a dual space always has an extreme point, and the closed unit ball of c_0 has none.

The closed unit ball of c_0 has no extreme point. Let $x = \{x_k\}$ belong to c_0 with $\|x\|_\infty \leq 1$. Since $\lim_{k \rightarrow \infty} x_k = 0$, there is an index N for which $|x_N| \leq 1/2$. Let e_N denote the sequence whose N th term is 1 and whose other terms are 0, and define

$$u = x + \frac{1}{2}e_N, \quad v = x - \frac{1}{2}e_N.$$

Both u and v belong to c_0 , and for each of them the k th term agrees with x_k when $k \neq N$, while the N th term has absolute value $|x_N \pm 1/2| \leq |x_N| + 1/2 \leq 1$. Hence $\|u\|_\infty \leq 1$ and $\|v\|_\infty \leq 1$, so u and v belong to the closed unit ball of c_0 . Moreover $u \neq v$ and

$$x = \frac{1}{2}u + \frac{1}{2}v.$$

Thus x is not an extreme point of the closed unit ball of c_0 . Since x was arbitrary, that ball has no extreme points at all. (Problem 67(iv) of Chapter 14 records the analogous fact for $L^1[a, b]$; either space may be used here.)

Consequently c_0 is not the dual of a normed linear space, nor is it isometrically isomorphic to one. Indeed, suppose Y is a normed linear space and $T: Y^* \rightarrow c_0$ is a linear isometry of Y^* onto c_0 . Since T is a linear bijection preserving norms, T carries the closed unit ball B^* of Y^* onto the closed unit ball B_{c_0} of c_0 , and being an affine bijection it carries extreme points of B^* to extreme points of B_{c_0} : if ψ is extreme in B^* and $T\psi = (u+v)/2$ with $u, v \in B_{c_0}$, then $\psi = (T^{-1}u + T^{-1}v)/2$ with $T^{-1}u, T^{-1}v \in B^*$, so $T^{-1}u = T^{-1}v = \psi$ and hence $u = v$. By Corollary 2, B^* has an extreme point; therefore B_{c_0} would have one, contradicting what was just proved.

A remark on the word isomorphic. In Chapter 13 an isomorphism of normed linear spaces is defined to be a one-to-one, onto, bounded linear map with bounded inverse, that is, a linear homeomorphism, and such a map need not preserve the norm. The argument above therefore settles the question for isometric isomorphism, but not literally for isomorphism in that weaker sense, because an isomorphism need not carry a closed unit ball onto a closed unit ball. To be explicit about what remains: if $T: c_0 \rightarrow Y^*$ is an isomorphism, then $\|x\| = \|Tx\|$ defines a norm on c_0 equivalent to $\|\cdot\|_\infty$, and T is an isometric isomorphism of $(c_0, \|\cdot\|)$ onto Y^* ; so the extreme point method would require that the closed unit ball of every norm on c_0 equivalent to $\|\cdot\|_\infty$ fail to have extreme points. That requirement is false: c_0 carries equivalent strictly convex norms in the sense of Problem 68 of Chapter 14 (Day's renorming), and for a strictly convex norm every unit vector is an extreme point of the unit ball. So the extreme point argument genuinely does not settle the weaker question.

The answer to the weaker question is nevertheless still no, by a deeper classical result, which we indicate without proving in full. If X is isomorphic to a dual space, then $J_X(X)$ is a complemented subspace of X^{**} . To see this, note first that for any normed linear space Y the adjoint $(J_Y)^*: Y^{***} \rightarrow Y^*$ satisfies $(J_Y)^* \circ J_{Y^*} = \text{Id}_{Y^*}$, since for $\psi \in Y^*$ and $y \in Y$,

$$[(J_Y)^*(J_{Y^*}\psi)](y) = J_{Y^*}(\psi)[J_Y(y)] = J_Y(y)[\psi] = \psi(y);$$

hence $J_{Y^*} \circ (J_Y)^*$ is a bounded projection of Y^{***} onto $J_{Y^*}(Y^*)$. If now $T: X \rightarrow Y^*$ is an isomorphism, then $T^{**}: X^{**} \rightarrow Y^{***}$ is an isomorphism and $J_{Y^*} \circ T = T^{**} \circ J_X$, so conjugating the above projection by T^{**} produces a bounded projection of X^{**} onto $J_X(X)$. Taking $X = c_0$, one has $X^{**} = \ell^\infty$ with $J_X(X) = c_0$, and Phillips' theorem asserts that c_0 is not complemented in ℓ^∞ . That last theorem is not available from the material developed here, and is the one step of this final paragraph we do not prove.

Problem 15.6. Show that every weakly compact subset of a normed linear space is bounded with respect to the norm.

Solution. Let X be a normed linear space and let K be a subset of X that is compact with respect to the weak topology.

Step 1: the family $\{J(x) \mid x \in K\}$ is pointwise bounded on X^* . Here $J: X \rightarrow X^{**}$ is the natural embedding, $J(x)[\psi] = \psi(x)$. Fix ψ in X^* . By the very definition of the weak topology, $\psi: X \rightarrow \mathbb{R}$ is continuous with respect to the weak topology on X . The continuous image of a compact topological space is compact, so $\psi(K)$ is a compact subset of $\mathbb{R}$ and therefore bounded: there is an $M_\psi \geq 0$ with

$$|J(x)[\psi]| = |\psi(x)| \leq M_\psi \quad \text{for all } x \in K.$$

Step 2: apply the Uniform Boundedness Principle. Each $J(x)$ is a bounded linear functional on X^* , that is, a member of $\mathcal{L}(X^*, \mathbb{R})$. The space X^* is a Banach space: by Theorem 3 of Chapter 13, $\mathcal{L}(X, Y)$ is complete whenever Y is, and here $Y = \mathbb{R}$ is complete. Thus the family

$$\mathcal{F} = \{J(x) \mid x \in K\} \subseteq \mathcal{L}(X^*, \mathbb{R})$$

is a family of bounded linear operators on the Banach space X^* which, by Step 1, is pointwise bounded. The Uniform Boundedness Principle tells us that $\mathcal{F}$ is uniformly bounded: there is an $M \geq 0$ for which

$$\|J(x)\| \leq M \quad \text{for all } x \in K.$$

Step 3: conclude. By Corollary 9 of Chapter 14, the natural embedding $J: X \rightarrow X^{**}$ is an isometry, that is, $\|J(x)\| = \|x\|$ for all x in X . Therefore

$$\|x\| \leq M \quad \text{for all } x \in K,$$

so K is bounded with respect to the norm.

Note that the completeness of X was not needed: the Uniform Boundedness Principle was applied on X^* , which is complete for every normed linear space X .

Problem 15.7. Show that the closed unit ball B^* of the dual X^* of a Banach space X has an extreme point.

Solution. Regard X^* as a topological vector space with its weak-* topology. We verify the three hypotheses of the Krein-Milman Lemma for the set $B^* = \{\psi \in X^* \mid \|\psi\| \leq 1\}$.

First, X^* with the weak-* topology is a locally convex topological vector space. The weak-* topology on X^* is the W -weak topology induced by the subspace $W = J(X)$ of X^{**} , where J is the natural embedding of X into X^{**} . This subspace separates the points of X^* : if ψ and η are distinct members of X^* , there is an x in X with $\psi(x) \neq \eta(x)$, that is, $J(x)[\psi] \neq J(x)[\eta]$. Proposition 21 of Chapter 14 therefore tells us that X^* , with the W -weak topology induced by $W = J(X)$, is a locally convex topological vector space.

Second, B^* is nonempty and convex. It contains the zero functional. If ψ and η belong to B^* and $0 \leq \lambda \leq 1$, then by the subadditivity and homogeneity of the norm on X^* ,

$$\|\lambda\psi + (1 - \lambda)\eta\| \leq \lambda\|\psi\| + (1 - \lambda)\|\eta\| \leq \lambda + (1 - \lambda) = 1,$$

so $\lambda\psi + (1 - \lambda)\eta$ belongs to B^* .

Third, B^* is compact with respect to the weak-* topology. This is precisely Alaoglu's Theorem.

Thus B^* is a nonempty, compact, convex subset of the locally convex topological vector space X^* endowed with its weak-* topology. The Krein-Milman Lemma tells us that such a set possesses an extreme point. Therefore B^* has an extreme point, that is, there is a ψ_0 in B^* which cannot be written as $\psi_0 = \lambda\psi + (1 - \lambda)\eta$ with ψ, η in B^* , $\psi \neq \eta$, and $0 < \lambda < 1$.

Two remarks. The completeness of X played no role: the assertion is exactly Corollary 2, which is valid for every normed linear space X . Also, the extreme point ψ_0 necessarily has $\|\psi_0\| = 1$ when $X \neq \{0\}$;

indeed, if $0 \leq \|\psi_0\| < 1$, then, choosing by Theorem 7 of Chapter 14 a functional η in X^* with $\|\eta\| = 1$, the two distinct functionals $\psi_0 \pm (1 - \|\psi_0\|)\eta$ belong to B^* and have ψ_0 as their midpoint.

15.2 Problems 15.8–15.14

Problem 15.8. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two compact, Hausdorff topologies on a set S for which $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Show that $\mathcal{T}_1 = \mathcal{T}_2$.

Solution. Write S_1 for the topological space $(S, \mathcal{T}_1)$ and S_2 for $(S, \mathcal{T}_2)$, and let

$$\text{id}: S_2 \longrightarrow S_1, \quad \text{id}(s) = s,$$

be the identity map. Since the inverse image under id of a set $U \in \mathcal{T}_1$ is U itself, and $U \in \mathcal{T}_1 \subseteq \mathcal{T}_2$, the map id is continuous.

We claim that id is a closed map, that is, it carries $\mathcal{T}_2$ -closed sets to $\mathcal{T}_1$ -closed sets. Indeed, let $F \subseteq S$ be $\mathcal{T}_2$ -closed. Since S_2 is compact and a closed subset of a compact topological space is compact, F is a compact subspace of S_2 . The continuous image of a compact topological space is compact, so $\text{id}(F) = F$ is a compact subspace of S_1 . Finally, a compact subspace of a Hausdorff topological space is closed, and S_1 is Hausdorff; therefore F is $\mathcal{T}_1$ -closed. This proves the claim.

Now let $V \in \mathcal{T}_2$. Then $S \sim V$ is $\mathcal{T}_2$ -closed, hence by the claim it is $\mathcal{T}_1$ -closed, hence its complement V belongs to $\mathcal{T}_1$. Thus $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Combined with the hypothesis $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we conclude $\mathcal{T}_1 = \mathcal{T}_2$.

For completeness we recall the proofs of the three topological facts used above.

(1) A closed subset F of a compact space S is compact: if $\{U_\lambda\}$ is a cover of F by open subsets of S , then $\{U_\lambda\} \cup \{S \sim F\}$ is an open cover of S ; a finite subcover of S yields, after discarding $S \sim F$, a finite subcollection of $\{U_\lambda\}$ covering F .

(2) If $f: S \rightarrow S'$ is continuous and S is compact, then $f(S)$ is compact: given a cover of $f(S)$ by sets open in S' , their inverse images form an open cover of S ; a finite subcover of S maps onto a finite subcover of $f(S)$.

(3) A compact subspace C of a Hausdorff space S' is closed: let $p \in S' \sim C$. For each $q \in C$ choose, by the Hausdorff property, disjoint open sets $U_q \ni p$ and $V_q \ni q$. The sets $\{V_q\}_{q \in C}$ cover C , so finitely many $V_{q_1}, \dots, V_{q_n}$ cover C ; then $U = \bigcap_{i=1}^n U_{q_i}$ is an open set containing p and disjoint from $\bigcup_{i=1}^n V_{q_i} \supseteq C$. Hence $S' \sim C$ is open.

Observe that the argument used exactly two of the four hypotheses, namely that the finer topology $\mathcal{T}_2$ is compact and that the coarser topology $\mathcal{T}_1$ is Hausdorff, and that neither may be dropped. If $\mathcal{T}_1$ is not required to be Hausdorff, take $S = \{a, b\}$, $\mathcal{T}_1$ the indiscrete topology and $\mathcal{T}_2$ the discrete topology: both are compact, $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and $\mathcal{T}_1 \neq \mathcal{T}_2$. If $\mathcal{T}_2$ is not required to be compact, take $S = [0, 1]$, $\mathcal{T}_1$ the topology inherited from $\mathbb{R}$ and $\mathcal{T}_2$ the discrete topology: both are Hausdorff, $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and $\mathcal{T}_1 \neq \mathcal{T}_2$.

Problem 15.9. Let X be a normed linear space containing the subspace Y . For $A \subseteq Y$, show that the weak topology on A induced by Y^* is the same as the topology A inherits as a subspace of X with its weak topology.

Solution. Call the first topology $\mathcal{T}_Y$ (the topology A inherits from Y equipped with its weak topology, that is, the weakest topology on A for which every $\psi \in Y^*$ restricts to a continuous function on A) and the second topology $\mathcal{T}_X$ (the topology A inherits as a subspace of X with its weak topology).

By the very definition of the weak topology, a base for the weak topology on Y at a point $a_0 \in Y$ consists of the sets

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(a_0) = \{y \in Y : |\psi_i(y - a_0)| < \epsilon \text{ for } 1 \leq i \leq n\},$$

where $\epsilon > 0$ and $\psi_1, \dots, \psi_n \in Y^*$; and a base for the weak topology on X at a_0 consists of the analogous sets

$$\mathcal{M}_{\epsilon, \Psi_1, \dots, \Psi_n}(a_0) = \{x \in X : |\Psi_i(x - a_0)| < \epsilon \text{ for } 1 \leq i \leq n\},$$

with $\Psi_1, \dots, \Psi_n \in X^*$. Consequently a base for $\mathcal{T}_Y$ at $a_0 \in A$ is given by the sets $A \cap \mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(a_0)$,

and a base for $\mathcal{T}_X$ at a_0 is given by the sets $A \cap \mathcal{M}_{\epsilon, \Psi_1, \dots, \Psi_n}(a_0)$. It therefore suffices to show that each set of one kind contains a set of the other kind with the same center.

$\mathcal{T}_X \subseteq \mathcal{T}_Y$. Let $\Psi_1, \dots, \Psi_n \in X^*$, $\epsilon > 0$, and $a_0 \in A$. For each i let $\psi_i = \Psi_i|_Y$ be the restriction of Ψ_i to Y . Restriction of a bounded linear functional to a linear subspace is again a bounded linear functional (indeed $\|\psi_i\| \leq \|\Psi_i\|$), so $\psi_i \in Y^*$. Since $A \subseteq Y$, for $a \in A$ we have $\Psi_i(a - a_0) = \psi_i(a - a_0)$, and therefore

$$A \cap \mathcal{M}_{\epsilon, \Psi_1, \dots, \Psi_n}(a_0) = A \cap \mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(a_0).$$

Hence every basic $\mathcal{T}_X$ -neighborhood of a_0 is a basic $\mathcal{T}_Y$ -neighborhood of a_0 , so $\mathcal{T}_X \subseteq \mathcal{T}_Y$.

$\mathcal{T}_Y \subseteq \mathcal{T}_X$. Let $\psi_1, \dots, \psi_n \in Y^*$, $\epsilon > 0$, and $a_0 \in A$. By the Hahn-Banach Theorem, Theorem 7 of Chapter 14, each bounded linear functional ψ_i on the subspace Y of X has an extension $\Psi_i \in X^*$ with $\Psi_i|_Y = \psi_i$ and $\|\Psi_i\| = \|\psi_i\|$. Again because $A \subseteq Y$, for $a \in A$ we have $\psi_i(a - a_0) = \Psi_i(a - a_0)$, so

$$A \cap \mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(a_0) = A \cap \mathcal{M}_{\epsilon, \Psi_1, \dots, \Psi_n}(a_0).$$

Hence every basic $\mathcal{T}_Y$ -neighborhood of a_0 is a basic $\mathcal{T}_X$ -neighborhood of a_0 , so $\mathcal{T}_Y \subseteq \mathcal{T}_X$.

Therefore $\mathcal{T}_Y = \mathcal{T}_X$: the weak topology that A receives from Y^* coincides with the topology A inherits from the weak topology of the ambient space X . Note that no closedness or completeness assumption on Y was needed; the Hahn-Banach extension theorem applies to an arbitrary linear subspace.

Problem 15.10. Argue as follows to show that a Banach space X is reflexive if and only if its dual space X^* is reflexive.

- (i) If X is reflexive, show that the weak and weak-* topologies on B^* are the same, and infer from this that X^* is reflexive.
- (ii) If X^* is reflexive, use Problem 8 to show that the weak and weak-* topologies on B^* are the same, and infer from this and Proposition 6 of Chapter 14 that $J(X) = X^{**}$.

Solution. Throughout, B and B^* denote the closed unit balls of X and X^* , and $J: X \rightarrow X^{**}$ is the natural embedding, $J(x)[\psi] = \psi(x)$. Recall that the weak topology on X^* is the X^{**} -weak topology, that is, the weakest topology on X^* making every $\Phi \in X^{**}$ continuous, while the weak-* topology on X^* is the $J(X)$ -weak topology, the weakest topology making every functional $\psi \mapsto \psi(x)$, $x \in X$, continuous. Since $J(X) \subseteq X^{**}$, the weak-* topology on X^* is always contained in the weak topology on X^* , and the same containment holds for the topologies these induce on the subset B^* .

Part (i). Suppose X is reflexive, so $J(X) = X^{**}$. Then the two generating families of functionals coincide: every $\Phi \in X^{**}$ is of the form $\Phi = J(x)$, so the X^{**} -weak topology and the $J(X)$ -weak topology on X^* are literally the same topology. A fortiori the topologies they induce on B^* are the same.

Now B^* is weak-* compact by Alaoglu's Theorem. Since the weak topology on X^* induces on B^* the same topology as the weak-* topology, B^* is compact in the weak topology of X^* , that is, B^* is weakly compact. Since X is a Banach space, so is X^* . By Kakutani's Theorem, a Banach space is reflexive if and only if its closed unit ball is weakly compact. Applying this to the Banach space X^* , we conclude that X^* is reflexive.

Part (ii). Suppose X^* is reflexive. We compare the two topologies that B^* carries.

First, the weak-* topology on X^* is Hausdorff, hence so is the topology it induces on B^* : if $\psi_1 \neq \psi_2$ in X^* then there is an $x \in X$ with $\psi_1(x) \neq \psi_2(x)$, and the two weak-* open sets $\{\psi : |\psi(x) - \psi_1(x)| < \epsilon\}$ and $\{\psi : |\psi(x) - \psi_2(x)| < \epsilon\}$, with $2\epsilon = |\psi_1(x) - \psi_2(x)|$, separate them. Moreover B^* is weak-* compact by Alaoglu's Theorem.

Second, the weak topology on X^* is Hausdorff, since it contains the weak-* topology, and it induces a Hausdorff topology on B^* . Since X^* is a reflexive Banach space, Kakutani's Theorem tells us that B^* is weakly compact.

Thus the weak-* topology on B^* and the weak topology on B^* are two compact, Hausdorff topologies on the set B^* , and the first is contained in the second. By Problem 8 they are equal.

It remains to infer reflexivity of X . The most economical route is to apply part (i) to the Banach space X^* : since X^* is reflexive, part (i) tells us that X^{**} is reflexive. Now $J: X \rightarrow X^{**}$ is an isometry by

Corollary 9 of Chapter 14, and X is complete, so $J(X)$ is a complete, hence closed, linear subspace of X^{**} . By Proposition 15 of Chapter 14, a closed subspace of a reflexive Banach space is reflexive; therefore $J(X)$ is reflexive.

Finally, reflexivity is preserved by a surjective linear isometry. Indeed, let $T: X \rightarrow Z$ be a linear isometry of X onto $Z = J(X)$. The adjoint $T^*: Z^* \rightarrow X^*$, $T^*(\phi) = \phi \circ T$, is a linear isometry of Z^* onto X^* (its inverse is the adjoint of T^{-1}), and hence $T^{**}: X^{**} \rightarrow Z^{**}$ is a linear isometry onto Z^{**} . For $x \in X$ and $\phi \in Z^*$,

$$T^{**}(J_X(x))[\phi] = J_X(x)[T^*\phi] = (T^*\phi)(x) = \phi(Tx) = J_Z(Tx)[\phi],$$

so $T^{**} \circ J_X = J_Z \circ T$. Since T^{**} and T are bijections, J_X is onto X^{**} if and only if J_Z is onto Z^{**} . As $Z = J(X)$ is reflexive, J_Z is onto, and therefore $J_X = J$ maps X onto X^{**} . That is, $J(X) = X^{**}$ and X is reflexive.

Combining the two parts: for a Banach space X , X is reflexive if and only if X^* is reflexive.

Remark on the inference intended in (ii). One would like to pass directly from the equality of the weak and weak-* topologies on the ball B^* to their equality on all of X^* , and then to invoke Proposition 6 of Chapter 14, which states that a normed linear space X is reflexive if and only if the weak and weak-* topologies on X^* are the same. That passage from the ball to the whole space is legitimate, but it is not a formal consequence of the equality on B^* alone; it requires the Krein-Smulian theorem on weak-* closed convex sets, which is not available at this point in the text. The argument given above avoids that issue entirely and proves the desired conclusion $J(X) = X^{**}$ outright. Once we know X is reflexive, Proposition 6 of Chapter 14 does give the equality of the weak and weak-* topologies on all of X^* , as claimed.

Problem 15.11. For X a Banach space, by the preceding problem, if X is reflexive, then so is X^* . Conclude that X is not reflexive if there is a closed subspace of X^* that is not reflexive. Let K be an infinite compact Hausdorff space and $\{x_n\}$ an enumeration of a countably infinite subset of K . Define the operator $T: \ell^1 \rightarrow [C(K)]^*$ by

$$[T(\{\eta_k\})](f) = \sum_{k=1}^{\infty} \eta_k \cdot f(x_k) \quad \text{for all } \{\eta_k\} \in \ell^1 \text{ and } f \in C(K).$$

Show that T is an isometry and therefore, since ℓ^1 is not reflexive, neither is $T(\ell^1)$ and therefore neither is $C(K)$. Use a dimension counting argument to show that $C(K)$ is reflexive if K is a finite set.

Solution. The preliminary reduction. Let X be a Banach space and suppose Z is a closed subspace of X^* that is not reflexive. If X were reflexive, then by Problem 10 the Banach space X^* would be reflexive, and then by Proposition 15 of Chapter 14 every closed subspace of X^* would be reflexive, contradicting the non-reflexivity of Z . Hence X is not reflexive.

T is well defined and bounded. Fix $\eta = \{\eta_k\} \in \ell^1$. For $f \in C(K)$ we have $|\eta_k f(x_k)| \leq \|f\|_{\infty} |\eta_k|$, and $\sum_k |\eta_k| = \|\eta\|_1 < \infty$, so the series $\sum_{k=1}^{\infty} \eta_k f(x_k)$ converges absolutely and

$$|[T(\eta)](f)| \leq \sum_{k=1}^{\infty} |\eta_k| |f(x_k)| \leq \|\eta\|_1 \cdot \|f\|_{\infty}.$$

Linearity of $f \mapsto [T(\eta)](f)$ is clear from the linearity of $f \mapsto f(x_k)$ and the absolute convergence. Thus $T(\eta) \in [C(K)]^*$ with $\|T(\eta)\| \leq \|\eta\|_1$. Linearity of T itself in η is likewise immediate.

T preserves the norm. Let $\eta = \{\eta_k\} \in \ell^1$ and let $\epsilon > 0$. Choose a natural number n with $\sum_{k>n} |\eta_k| < \epsilon$. The points $x_1, \dots, x_n$ are distinct, since $\{x_n\}$ is an enumeration of an infinite set. Because K is Hausdorff, we may choose pairwise disjoint open sets $U_1, \dots, U_n$ with $x_k \in U_k$ for $1 \leq k \leq n$. (For $n = 2$ this is the Hausdorff property; the general case follows by induction: having separated $x_1, \dots, x_{n-1}$ by disjoint open sets $V_1, \dots, V_{n-1}$, choose for each $j < n$ disjoint open $W_j \ni x_j$ and $W'_j \ni x_n$, and set $U_j = V_j \cap W_j$ for $j < n$ and $U_n = \bigcap_{j<n} W'_j$.)

A compact Hausdorff space is normal, and singletons are closed in a Hausdorff space, so for each $k \leq n$ the disjoint closed sets $\{x_k\}$ and $K \sim U_k$ may be separated by Urysohn's Lemma: there is $f_k \in C(K)$ with $0 \leq f_k \leq 1$ on K , $f_k(x_k) = 1$, and $f_k = 0$ on $K \sim U_k$.

For $t \in \mathbb{R}$ put $\operatorname{sgn}(t) = 1$ if $t \geq 0$ and $\operatorname{sgn}(t) = -1$ if $t < 0$, and define

$$f = \sum_{k=1}^n \operatorname{sgn}(\eta_k) f_k.$$

Then $f \in C(K)$. If $x \in K$, then x lies in at most one of the pairwise disjoint sets $U_1, \dots, U_n$, and $f_k(x) = 0$ unless $x \in U_k$; hence at most one summand is nonzero at x and $|f(x)| \leq 1$. Thus $\|f\|_\infty \leq 1$. Moreover $f(x_k) = \operatorname{sgn}(\eta_k)$ for $1 \leq k \leq n$, since $x_k \in U_k$ and $x_k \notin U_j$ for $j \neq k$, so $f_j(x_k) = 0$ for $j \neq k$. Consequently

$$[T(\eta)](f) = \sum_{k=1}^n \eta_k \operatorname{sgn}(\eta_k) + \sum_{k>n} \eta_k f(x_k) \geq \sum_{k=1}^n |\eta_k| - \sum_{k>n} |\eta_k| > (\|\eta\|_1 - \epsilon) - \epsilon,$$

where we used $|f(x_k)| \leq 1$ for every k . Since $\|f\|_\infty \leq 1$, this gives $\|T(\eta)\| \geq \|\eta\|_1 - 2\epsilon$. As $\epsilon > 0$ was arbitrary, $\|T(\eta)\| \geq \|\eta\|_1$. Together with the reverse inequality established above, $\|T(\eta)\| = \|\eta\|_1$, so T is a linear isometry of ℓ^1 into $[C(K)]^*$.

$C(K)$ is not reflexive. Since T is an isometry and ℓ^1 is complete, $T(\ell^1)$ is a complete, hence closed, linear subspace of $[C(K)]^*$, and T is a surjective linear isometry of ℓ^1 onto $T(\ell^1)$. As shown at the end of the solution to Problem 10, a surjective linear isometry carries reflexive spaces to reflexive spaces and conversely; since ℓ^1 is not reflexive, $T(\ell^1)$ is not reflexive. Thus $[C(K)]^*$ contains a closed subspace which is not reflexive. Since $C(K)$ is a Banach space, the preliminary reduction shows that $C(K)$ is not reflexive.

K finite. Suppose now K is a finite set, say $K = \{p_1, \dots, p_m\}$. A compact Hausdorff space with finitely many points is discrete (each singleton is closed, hence each singleton is also open, being the complement of a finite union of closed sets), so every real-valued function on K is continuous and

$$\dim C(K) = m < \infty,$$

a basis being the characteristic functions $\chi_{\{p_1\}}, \dots, \chi_{\{p_m\}}$.

Now count dimensions. If V is a real linear space of finite dimension m with basis $v_1, \dots, v_m$, then every linear functional on V is automatically bounded (with respect to any norm on V , all norms on a finite dimensional space being equivalent), and the coordinate functionals $v_1^*, \dots, v_m^*$ determined by $v_i^*(v_j) = \delta_{ij}$ form a basis of V^* ; hence $\dim V^* = m$. Applying this twice, $\dim C(K)^* = m$ and $\dim C(K)^{**} = m$.

The natural embedding $J: C(K) \rightarrow C(K)^{**}$ is linear, and by Corollary 9 of Chapter 14 it is an isometry, hence one-to-one. A one-to-one linear map between real linear spaces of the same finite dimension m is onto: the image $J(C(K))$ is a linear subspace of $C(K)^{**}$ of dimension m , and an m -dimensional subspace of an m -dimensional space is the whole space. Therefore $J(C(K)) = C(K)^{**}$, that is, $C(K)$ is reflexive whenever K is finite.

Problem 15.12. If Y is a linear subspace of a Banach space X , we define the annihilator $Y^\perp$ to be the subspace of X^* consisting of those $\psi \in X^*$ for which $\psi = 0$ on Y . If Y is a subspace of X^* , we define Y^0 to be the subspace of vectors in X for which $\psi(x) = 0$ for all $\psi \in Y$.

- (i) Show that $Y^\perp$ is a closed linear subspace of X^* .
- (ii) Show that $(Y^\perp)^0 = \overline{Y}$.
- (iii) If X is reflexive and Y is a subspace of X^* , show that $Y^\perp = J(Y^0)$.

Solution. Part (i). Let Y be a linear subspace of X . If $\psi_1, \psi_2 \in Y^\perp$ and $\alpha, \beta \in \mathbb{R}$, then for every $y \in Y$ we have $(\alpha\psi_1 + \beta\psi_2)(y) = \alpha\psi_1(y) + \beta\psi_2(y) = 0$; thus $\alpha\psi_1 + \beta\psi_2 \in Y^\perp$ and $Y^\perp$ is a linear subspace of X^* .

To see that $Y^\perp$ is closed, observe that

$$Y^\perp = \bigcap_{y \in Y} \{ \psi \in X^* : J(y)[\psi] = 0 \} = \bigcap_{y \in Y} \ker J(y),$$

since $J(y)[\psi] = \psi(y)$. For each fixed y , the functional $J(y)$ belongs to X^{**} and so is continuous on X^* with respect to the strong (norm) topology; hence $\ker J(y)$, being the inverse image of the closed set $\{0\}$, is closed in X^* . An arbitrary intersection of closed sets is closed, so $Y^\perp$ is closed. (The same computation shows that $Y^\perp$ is in fact weak-* closed, since each $J(y)$ is by definition continuous for the weak-* topology on X^* .)

Alternatively and directly: if $\{\psi_n\} \subseteq Y^\perp$ and $\psi_n \rightarrow \psi$ in X^* , then for $y \in Y$, $|\psi(y)| = |\psi(y) - \psi_n(y)| \leq \|\psi - \psi_n\| \|y\| \rightarrow 0$, so $\psi(y) = 0$ and $\psi \in Y^\perp$.

Part (ii). Here Y is a linear subspace of X , so $Y^\perp \subseteq X^*$ and $(Y^\perp)^0 \subseteq X$.

First, $\bar{Y} \subseteq (Y^\perp)^0$. Let $x \in \bar{Y}$ and $\psi \in Y^\perp$. Choose $\{y_n\} \subseteq Y$ with $y_n \rightarrow x$ in norm. Since ψ is continuous, $\psi(x) = \lim_n \psi(y_n) = 0$. As $\psi \in Y^\perp$ was arbitrary, $x \in (Y^\perp)^0$.

Second, $(Y^\perp)^0 \subseteq \bar{Y}$. We prove the contrapositive: suppose $x_0 \in X$ and $x_0 \notin \bar{Y}$; we produce $\psi \in Y^\perp$ with $\psi(x_0) \neq 0$. Note $\bar{Y}$ is a closed linear subspace of X (the closure of a linear subspace is a linear subspace, by continuity of the vector space operations). Set

$$d = \text{dist}(x_0, \bar{Y}) = \inf\{\|x_0 - y\| : y \in \bar{Y}\}.$$

Since $\bar{Y}$ is closed and $x_0 \notin \bar{Y}$, we have $d > 0$.

Let $Z = \{y + tx_0 : y \in \bar{Y}, t \in \mathbb{R}\}$, a linear subspace of X containing $\bar{Y}$ and x_0 . The representation $z = y + tx_0$ of an element of Z is unique: if $y + tx_0 = y' + t'x_0$ with $t \neq t'$ then $x_0 = (y' - y)/(t - t') \in \bar{Y}$, a contradiction; hence $t = t'$ and then $y = y'$. Therefore

$$\psi_0(y + tx_0) = td \quad (y \in \bar{Y}, t \in \mathbb{R})$$

is a well-defined linear functional on Z . It is bounded with $\|\psi_0\| \leq 1$: for $t \neq 0$,

$$\|y + tx_0\| = |t| \left\| x_0 + \frac{y}{t} \right\| = |t| \left\| x_0 - \left(-\frac{y}{t} \right) \right\| \geq |t|d = |\psi_0(y + tx_0)|,$$

because $-y/t \in \bar{Y}$; and for $t = 0$ we have $\psi_0(y) = 0 \leq \|y\|$.

By the Hahn-Banach Theorem, Theorem 7 of Chapter 14, ψ_0 extends to a bounded linear functional $\psi \in X^*$ with $\psi|_Z = \psi_0$. Then ψ vanishes on $\bar{Y} \supseteq Y$, so $\psi \in Y^\perp$, while $\psi(x_0) = d \neq 0$. Hence $x_0 \notin (Y^\perp)^0$.

Combining the two inclusions, $(Y^\perp)^0 = \bar{Y}$.

Part (iii). Now let Y be a linear subspace of X^* , where X is reflexive. Then $Y^\perp$ is a subspace of $(X^*)^* = X^{**}$, namely $Y^\perp = \{\Phi \in X^{**} : \Phi(\psi) = 0 \text{ for all } \psi \in Y\}$, while $Y^0 = \{x \in X : \psi(x) = 0 \text{ for all } \psi \in Y\}$ is a subspace of X .

$J(Y^0) \subseteq Y^\perp$: let $x \in Y^0$. For every $\psi \in Y$ we have $J(x)[\psi] = \psi(x) = 0$. Hence $J(x)$ vanishes on Y , that is, $J(x) \in Y^\perp$.

$Y^\perp \subseteq J(Y^0)$: let $\Phi \in Y^\perp \subseteq X^{**}$. Since X is reflexive, $J(X) = X^{**}$, so there is an $x \in X$ with $\Phi = J(x)$. For every $\psi \in Y$,

$$\psi(x) = J(x)[\psi] = \Phi(\psi) = 0,$$

because Φ vanishes on Y . Hence $x \in Y^0$ and $\Phi = J(x) \in J(Y^0)$.

Therefore $Y^\perp = J(Y^0)$. Note that reflexivity is genuinely used only in the second inclusion; without it one has merely $J(Y^0) \subseteq Y^\perp$ in general.

Problem 15.13. In a general topological space that is not metrizable a sequence may converge to more than one point. Show that this cannot occur for the W -weak topology on a normed linear space X , where W is a subspace of X^* that separates points in X .

Solution. Recall that for a subspace W of X^* , the W -weak topology on X is the weakest topology on X for which every $\psi \in W$ is continuous; a base of neighborhoods at a point $x_0 \in X$ is given by the sets

$$\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_n}(x_0) = \{x \in X : |\psi_i(x - x_0)| < \epsilon \text{ for } 1 \leq i \leq n\},$$

with $\epsilon > 0$ and $\psi_1, \dots, \psi_n \in W$. To say that W separates points in X means: whenever $u \neq v$ in X , there is a $\psi \in W$ with $\psi(u) \neq \psi(v)$. Equivalently, since W is a linear space of linear functionals, whenever $z \neq 0$ there is a $\psi \in W$ with $\psi(z) \neq 0$.

First we identify W -weak convergence of sequences. A sequence $\{x_n\}$ in X converges to x in the W -weak topology if and only if

$$\lim_{n \rightarrow \infty} \psi(x_n) = \psi(x) \quad \text{for every } \psi \in W.$$

Indeed, if $x_n \rightarrow x$ in the W -weak topology and $\psi \in W$ and $\epsilon > 0$, then the basic neighborhood $\mathcal{N}_{\epsilon, \psi}(x)$ contains x_n for all n beyond some index, which says exactly that $|\psi(x_n) - \psi(x)| < \epsilon$ for all such n . Conversely, if $\psi(x_n) \rightarrow \psi(x)$ for every $\psi \in W$ and $\mathcal{N}_{\epsilon, \psi_1, \dots, \psi_m}(x)$ is a basic neighborhood of x , choose for each i an index N_i with $|\psi_i(x_n) - \psi_i(x)| < \epsilon$ for $n \geq N_i$; then for $n \geq \max\{N_1, \dots, N_m\}$ we have $x_n \in \mathcal{N}_{\epsilon, \psi_1, \dots, \psi_m}(x)$. Since these basic neighborhoods form a neighborhood base at x , $x_n \rightarrow x$ in the W -weak topology.

Now suppose $\{x_n\}$ converges in the W -weak topology both to u and to v . By the criterion just established, for every $\psi \in W$,

$$\psi(u) = \lim_{n \rightarrow \infty} \psi(x_n) = \psi(v),$$

the limit of a real sequence being unique. Hence $\psi(u - v) = 0$ for every $\psi \in W$. If $u \neq v$, then $u - v \neq 0$, and since W separates points of X there would be a $\psi \in W$ with $\psi(u - v) \neq 0$, a contradiction. Therefore $u = v$: a sequence has at most one W -weak limit.

Equivalently, and more structurally, the W -weak topology is Hausdorff, and in a Hausdorff space limits of sequences are unique. To see that it is Hausdorff, let $u \neq v$ in X and choose $\psi \in W$ with $\psi(u) \neq \psi(v)$. Put $\epsilon = |\psi(u) - \psi(v)|/2 > 0$ and consider the W -weak open sets

$$U = \{x \in X : |\psi(x) - \psi(u)| < \epsilon\}, \quad V = \{x \in X : |\psi(x) - \psi(v)| < \epsilon\}.$$

Then $u \in U$, $v \in V$, and $U \cap V = \emptyset$: a point x in both would give

$$|\psi(u) - \psi(v)| \leq |\psi(u) - \psi(x)| + |\psi(x) - \psi(v)| < 2\epsilon = |\psi(u) - \psi(v)|,$$

which is impossible. Thus the W -weak topology is Hausdorff.

In particular, taking $W = X^*$, which separates the points of X by the Hahn-Banach Theorem, Theorem 7 of Chapter 14 (for $x \neq 0$ there is a $\psi \in X^*$ with $\psi(x) = \|x\| \neq 0$ and $\|\psi\| = 1$), the weak topology on X is Hausdorff and weak limits of sequences are unique. Likewise, taking X^* in place of X and $W = J(X) \subseteq X^{**}$, which separates the points of X^* because $\psi \neq 0$ means $\psi(x) \neq 0$ for some $x \in X$, the weak-* topology on X^* is Hausdorff, so weak-* limits of sequences are unique as well. This is why it is unambiguous to speak of the weak limit, and of the weak-* limit, of a sequence, even though these topologies are, by Theorem 9, not metrizable when X is infinite dimensional.

Problem 15.14. Show that there is a bounded sequence in $L^\infty[0, 1]$ that fails to have a weakly convergent subsequence. Show that the closed unit ball of $C[a, b]$ is not weakly compact.

Solution. Part one: a bounded sequence in $L^\infty[0, 1]$ with no weakly convergent subsequence. For each natural number n define

$$f_n = \chi_{(0, 1/n]} \in L^\infty[0, 1].$$

Then $\|f_n\|_\infty = 1$ for every n , so $\{f_n\}$ is a bounded sequence. We claim that no subsequence of $\{f_n\}$ converges weakly in $L^\infty[0, 1]$.

Step 1: the only possible weak limit is 0. Suppose some subsequence $\{f_{n_k}\}$ converges weakly to

$f \in L^\infty[0, 1]$. Fix $g \in L^1[0, 1]$. The map

$$\psi_g(h) = \int_0^1 hg \quad (h \in L^\infty[0, 1])$$

is linear, and $|\psi_g(h)| \leq \|h\|_\infty \|g\|_1$, so $\psi_g \in (L^\infty[0, 1])^*$. Weak convergence therefore gives

$$\int_0^1 fg = \lim_{k \rightarrow \infty} \int_0^1 f_{n_k} g = \lim_{k \rightarrow \infty} \int_0^{1/n_k} g = 0,$$

the last limit being 0 by the absolute continuity of integration, Proposition 23 of Chapter 4: since g is integrable over $[0, 1]$, given $\epsilon > 0$ there is a $\delta > 0$ for which $\int_A |g| < \epsilon$ whenever $A \subseteq [0, 1]$ is measurable with $m(A) < \delta$; take $A = (0, 1/n_k]$ with $1/n_k < \delta$. Since $\int_0^1 fg = 0$ for every $g \in L^1[0, 1]$, take $g = \text{sgn}(f)$, where $\text{sgn}(t) = 1$ for $t \geq 0$ and $\text{sgn}(t) = -1$ for $t < 0$. As f is measurable, g is measurable, and $|g| \leq 1$ on the finite measure set $[0, 1]$, so $g \in L^1[0, 1]$. Since $f \cdot \text{sgn}(f) = |f|$ pointwise, this gives $\int_0^1 |f| = 0$ and hence $f = 0$ almost everywhere. So the only candidate for a weak limit is $f = 0$.

Step 2: a bounded linear functional on $L^\infty[0, 1]$ that does not tend to 0 along the sequence. Let

$$\mathcal{Y} = \left\{ h \in L^\infty[0, 1] : \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t h \text{ exists in } \mathbb{R} \right\},$$

and for $h \in \mathcal{Y}$ define

$$L(h) = \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t h.$$

Observe first that L is well defined on equivalence classes: functions equal almost everywhere have identical integrals over $(0, t)$. Since $h \mapsto \frac{1}{t} \int_0^t h$ is linear for each fixed t , and limits of sums are sums of limits, $\mathcal{Y}$ is a linear subspace of $L^\infty[0, 1]$ and L is a linear functional on $\mathcal{Y}$. Furthermore, for every $t \in (0, 1]$,

$$\left| \frac{1}{t} \int_0^t h \right| \leq \frac{1}{t} \int_0^t |h| \leq \|h\|_\infty,$$

so passing to the limit, $|L(h)| \leq \|h\|_\infty$ for $h \in \mathcal{Y}$. Thus L is a bounded linear functional on $\mathcal{Y}$ with $\|L\| \leq 1$.

Each f_n belongs to $\mathcal{Y}$: for $0 < t \leq 1/n$ we have $\frac{1}{t} \int_0^t f_n = \frac{1}{t} \int_0^t 1 = 1$, so the limit exists and

$$L(f_n) = 1 \quad \text{for every } n.$$

Also $0 \in \mathcal{Y}$ with $L(0) = 0$.

By the Hahn-Banach Theorem, Theorem 7 of Chapter 14, L extends to a bounded linear functional $\psi \in (L^\infty[0, 1])^*$ with $\psi|_{\mathcal{Y}} = L$ and $\|\psi\| = \|L\| \leq 1$.

Step 3: conclusion. Suppose some subsequence $\{f_{n_k}\}$ converged weakly in $L^\infty[0, 1]$. By Step 1 its weak limit is 0, so in particular $\psi(f_{n_k}) \rightarrow \psi(0) = 0$. But $\psi(f_{n_k}) = L(f_{n_k}) = 1$ for every k , a contradiction. Hence the bounded sequence $\{\chi_{(0, 1/n]}\}$ has no weakly convergent subsequence in $L^\infty[0, 1]$.

It follows, incidentally, that $L^\infty[0, 1]$ is not reflexive: were it reflexive, Theorem 17 of Chapter 14 would furnish a weakly convergent subsequence of every bounded sequence.

Part two: the closed unit ball of $C[a, b]$ is not weakly compact.

Assume, as is implicit in the notation, that $a < b$, so that $[a, b]$ is an infinite set. Equipped with the topology it inherits from $\mathbb{R}$, $[a, b]$ is a compact Hausdorff space, and it is infinite. By Problem 11, $C[a, b] = C(K)$ with $K = [a, b]$ is not reflexive. Since $C[a, b]$ with the maximum norm is a Banach space, Kakutani's Theorem applies: a Banach space is reflexive if and only if its closed unit ball is weakly compact. As $C[a, b]$ is not reflexive, its closed unit ball is not weakly compact.

A concrete sequence witnessing the failure can be exhibited as well, which by the Eberlein-Smulian Theorem, Theorem 8, is an equivalent form of the same statement. Rescaling, we may take $[a, b] = [0, 1]$. Define $g_n \in C[0, 1]$ for $n \geq 2$ by

$$g_n(x) = \begin{cases} 1 - nx, & 0 \leq x \leq 1/n, \\ 0, & 1/n \leq x \leq 1. \end{cases}$$

Each g_n is continuous with $\|g_n\|_\infty = g_n(0) = 1$, so $\{g_n\}$ lies in the closed unit ball. Suppose a subsequence $\{g_{n_k}\}$ converged weakly to $g \in C[0, 1]$. For each $x \in [0, 1]$ the evaluation $\delta_x(h) = h(x)$ is a bounded linear functional on $C[0, 1]$ of norm 1, so $g(x) = \lim_k g_{n_k}(x)$. For $x = 0$ this limit is 1, while for $x > 0$ we have $g_{n_k}(x) = 0$ as soon as $1/n_k < x$, so the limit is 0. Thus $g(0) = 1$ and $g(x) = 0$ for $x > 0$, which is not a continuous function on $[0, 1]$; no such $g \in C[0, 1]$ exists. Hence $\{g_n\}$ has no weakly convergent subsequence, the closed unit ball of $C[0, 1]$ is not weakly sequentially compact, and by the Eberlein-Smulian Theorem it is not weakly compact.

15.3 Problems 15.15–15.18

Problem 15.15. Let K be a compact, metric space with infinitely many points. Show that there is a bounded sequence in $C(K)$ that fails to have a weakly convergent subsequence (see Problem 11), but every bounded sequence of continuous linear functionals on $C(K)$ has a subsequence that converges pointwise to a continuous linear functional on $C(K)$.

Solution. Throughout, $C(K)$ carries the maximum norm $\|f\|_{\max} = \max_{x \in K} |f(x)|$, and d denotes the metric on K .

A bounded sequence in $C(K)$ with no weakly convergent subsequence.

We first record the elementary observation that weak convergence in $C(K)$ forces pointwise convergence. Indeed, fix $x \in K$ and let $\delta_x: C(K) \rightarrow \mathbb{R}$ be the evaluation functional $\delta_x(f) = f(x)$. It is linear, and $|\delta_x(f)| \leq \|f\|_{\max}$, so δ_x belongs to $[C(K)]^*$. Hence if $f_n \rightarrow f$ weakly in $C(K)$, then $f_n(x) = \delta_x(f_n) \rightarrow \delta_x(f) = f(x)$ for every $x \in K$.

Now we construct the sequence. Since K is infinite, we may choose a countably infinite set of distinct points in K . Since K is a compact metric space, it is sequentially compact by Proposition 18 of Chapter 9, so some subsequence of these points converges to a point $x_\infty \in K$. Deleting at most one term (the one equal to x_∞ , if it occurs), we obtain a sequence $\{x_k\}_{k=1}^\infty$ of distinct points of K with

$$x_k \neq x_\infty \text{ for all } k, \quad \lim_{k \rightarrow \infty} x_k = x_\infty.$$

Put $F = \{x_\infty\} \cup \{x_k : k \geq 1\}$. Then F is closed: it contains all its limit points, since the only accumulation point of F is x_∞ . Moreover each x_k is an isolated point of F , because the x_j are distinct and converge to $x_\infty \neq x_k$; hence $F \sim \{x_k\}$ is closed and

$$\rho_k := \text{dist}(x_k, F \sim \{x_k\}) > 0.$$

Define $r_k = \rho_k/3$. Two properties follow. First, $x_\infty \notin B(x_k, r_k)$ and $x_j \notin B(x_k, r_k)$ for $j \neq k$, since $\rho_k > r_k$. Second, the balls $B(x_k, r_k)$ are pairwise disjoint: if $j \neq k$ and z belonged to $B(x_j, r_j) \cap B(x_k, r_k)$, then, since $\rho_j \leq d(x_j, x_k)$ and $\rho_k \leq d(x_j, x_k)$,

$$d(x_j, x_k) \leq d(x_j, z) + d(z, x_k) < r_j + r_k = \frac{1}{3}(\rho_j + \rho_k) \leq \frac{2}{3}d(x_j, x_k),$$

which is impossible as $d(x_j, x_k) > 0$.

For each k define the tent function

$$g_k(x) = \max\left\{0, 1 - \frac{d(x, x_k)}{r_k}\right\}, \quad x \in K.$$

Since $x \mapsto d(x, x_k)$ is continuous, g_k belongs to $C(K)$; moreover $0 \leq g_k \leq 1$, $g_k(x_k) = 1$, and g_k vanishes outside $B(x_k, r_k)$. Now set

$$f_n = \sum_{k=1}^n g_k \quad \text{for each natural number } n.$$

Each f_n is a finite sum of continuous functions and hence belongs to $C(K)$. Since the sets $B(x_k, r_k)$ are pairwise disjoint, at each point of K at most one summand is nonzero, so $0 \leq f_n \leq 1$ and $\|f_n\|_{\max} = 1$. Thus $\{f_n\}$ is a bounded sequence in $C(K)$.

Suppose, to get a contradiction, that a subsequence $\{f_{n_j}\}$ converged weakly to some $f \in C(K)$. By the observation above, $f_{n_j}(x) \rightarrow f(x)$ for every $x \in K$. Now x_∞ lies in no ball $B(x_k, r_k)$, so $f_n(x_\infty) = 0$ for all n , and therefore

$$f(x_\infty) = 0.$$

On the other hand, fix k . For every $n \geq k$ we have $f_n(x_k) = g_k(x_k) = 1$ (the other summands vanish at x_k). Since $n_j \rightarrow \infty$, $f_{n_j}(x_k) = 1$ for all large j , and therefore

$$f(x_k) = 1 \quad \text{for every } k.$$

But $x_k \rightarrow x_\infty$ and f is continuous, so $1 = \lim_{k \rightarrow \infty} f(x_k) = f(x_\infty) = 0$. This contradiction shows that no subsequence of $\{f_n\}$ converges weakly in $C(K)$.

(This is the concrete realization of the abstract reason indicated in the statement: Problem 11 shows that for infinite compact Hausdorff K the space $C(K)$ is not reflexive, and the Characterization of Weak Compactness at the end of Section 15.3 then tells us that the closed unit ball B of $C(K)$ is not weakly sequentially compact. Since B is convex and norm closed, it is weakly closed, so a weak limit of a sequence from B again lies in B ; hence some sequence in B has no weakly convergent subsequence at all.)

Every bounded sequence in $[C(K)]^*$ has a pointwise convergent subsequence.

Since K is a compact metric space, it is a compact Hausdorff space that is metrizable, and therefore, by Riesz's Theorem at the end of Section 12.3, $C(K)$ is separable.

Let B^* denote the closed unit ball of $[C(K)]^*$. Since $C(K)$ is separable, Corollary 11(ii) of the present section tells us that the weak-* topology on B^* is metrizable. On the other hand, Alaoglu's Theorem tells us that B^* is compact in the weak-* topology. A compact metrizable space is sequentially compact, again by Proposition 18 of Chapter 9 applied to a metric inducing the weak-* topology on B^* . Therefore:

every sequence in B^* has a subsequence that converges, with respect to the weak-* topology, to a member of B^* ; and convergence in the weak-* topology is exactly pointwise convergence on $C(K)$.

Finally, let $\{\psi_n\}$ be an arbitrary bounded sequence in $[C(K)]^*$, say $\|\psi_n\| \leq M$ for all n . If $M = 0$ there is nothing to prove, so assume $M > 0$. Then $\{\psi_n/M\}$ is a sequence in B^* , so there is a subsequence $\{\psi_{n_j}/M\}$ and a functional $\psi \in B^*$ with

$$\lim_{j \rightarrow \infty} \frac{\psi_{n_j}(f)}{M} = \psi(f) \quad \text{for all } f \in C(K).$$

Hence $\{\psi_{n_j}\}$ converges pointwise on $C(K)$ to $M\psi$, which is a continuous linear functional on $C(K)$. This is precisely the assertion to be proved.

The contrast is worth recording: $C(K)$ itself is too big for bounded sequences to have weakly convergent subsequences, but because $C(K)$ is separable its dual ball is weak-* metrizable and compact, so bounded sequences of functionals always converge pointwise along a subsequence.

Problem 15.16. Show that the dual of an infinite dimensional normed linear space also is infinite dimensional.

Solution. Let X be an infinite dimensional normed linear space. We must show that X^* is infinite dimensional.

We first record the consequence of the Hahn-Banach Theorem that we shall use. According to Theorem 7 of Chapter 14, for each $x \in X$ there is a functional $\psi \in X^*$ for which $\|\psi\| = 1$ and $\psi(x) = \|x\|$. In particular,

$$\text{if } x \in X \text{ and } \psi(x) = 0 \text{ for every } \psi \in X^*, \text{ then } x = 0. \quad (*)$$

Argue by contradiction: assume X^* is finite dimensional. Since X is infinite dimensional it contains a nonzero vector, so by $(*)$ the space X^* contains a nonzero functional; thus $X^* \neq \{0\}$ and we may choose a basis $\psi_1, \dots, \psi_n$ of X^* , where $n \geq 1$.

Define the linear map

$$T: X \rightarrow \mathbb{R}^n, \quad T(x) = (\psi_1(x), \dots, \psi_n(x)).$$

Its kernel is $N = \bigcap_{i=1}^n \ker \psi_i$. We claim $N = \{0\}$. Indeed, let x belong to N and let ψ be any member of X^* . Since $\psi_1, \dots, \psi_n$ is a basis of X^* , there are real numbers $c_1, \dots, c_n$ with $\psi = \sum_{i=1}^n c_i \psi_i$, and therefore

$$\psi(x) = \sum_{i=1}^n c_i \psi_i(x) = 0.$$

Thus every member of X^* vanishes at x , and so $x = 0$ by (*). This proves the claim.

Consequently T is one-to-one, so T is a linear isomorphism of X onto the linear subspace $T(X)$ of $\mathbb{R}^n$. A linearly independent set in X is carried by the one-to-one linear map T to a linearly independent set in $\mathbb{R}^n$, and $\mathbb{R}^n$ contains no linearly independent set of more than n vectors. Hence X contains no linearly independent set of more than n vectors, that is, X is finite dimensional with $\dim X \leq n$. This contradicts the assumption that X is infinite dimensional. Therefore X^* is infinite dimensional.

It is worth recording the sharper, constructive form of this conclusion, which exhibits the independent functionals explicitly. Let n be a natural number. Since X is infinite dimensional we may choose linearly independent vectors $x_1, \dots, x_n$ in X . Fix k with $1 \leq k \leq n$ and let

$$Y_k = \text{span}\{x_i : 1 \leq i \leq n, i \neq k\}.$$

The subspace Y_k is finite dimensional, so by Corollary 6 of Chapter 13 (which rests on Theorem 4 of that chapter, the equivalence of any two norms on a finite dimensional linear space) Y_k is complete and therefore is a closed subspace of X . By the linear independence of $x_1, \dots, x_n$, the vector x_k does not belong to $Y_k = \overline{Y_k}$. Theorem 10 of Chapter 14 therefore provides a functional $\psi_k \in X^*$ that vanishes on Y_k while $\psi_k(x_k) \neq 0$; replacing ψ_k by $\psi_k/\psi_k(x_k)$ we may assume

$$\psi_k(x_i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \quad 1 \leq i \leq n.$$

The functionals $\psi_1, \dots, \psi_n$ are linearly independent in X^* : if $\sum_{k=1}^n c_k \psi_k = 0$, then evaluating at x_i gives $c_i = 0$ for each i . Thus X^* contains n linearly independent vectors for every natural number n , which is again exactly the statement that X^* is infinite dimensional.

Problem 15.17. Complete the last step of the proof of Theorem 10 by showing that the inequalities (12) imply that the metric ρ induces the W -weak topology.

Solution. We keep the notation of the proof of Theorem 10. Thus X is a normed linear space, W is a separable subspace of X^* that separates points in X , B is the closed unit ball of X , B^* is the closed unit ball of X^* , $\{\psi_k\}_{k=1}^\infty$ is a countable dense subset of $B^* \cap W$, and

$$\rho(u, v) = \sum_{k=1}^{\infty} \frac{1}{2^k} |\psi_k(u - v)| \quad \text{for all } u, v \in B,$$

which has already been shown to be a metric on B . The inequalities (12) read

$$\frac{1}{2^n} \left[\sum_{k=1}^n |\psi_k(x)| \right] \leq \rho(x, 0) \leq \sum_{k=1}^n |\psi_k(x)| + \frac{1}{2^n} \quad \text{for all } x \in B. \quad (12)$$

Two preliminary remarks. First, if $u, v \in B$, then $\|u - v\| \leq 2$, so $(u - v)/2$ belongs to B , and directly from the definition of ρ ,

$$\rho(u, v) = \sum_{k=1}^{\infty} \frac{1}{2^k} |\psi_k(u - v)| = 2 \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \psi_k \left(\frac{u - v}{2} \right) \right| = 2 \rho \left(\frac{u - v}{2}, 0 \right). \quad (\dagger)$$

Applying (12) to the vector $(u - v)/2 \in B$ and multiplying through by 2, we obtain from (†) the two-sided estimate that we shall use: for all $u, v \in B$ and every natural number n ,

$$\frac{1}{2^n} \left[\sum_{k=1}^n |\psi_k(u - v)| \right] \leq \rho(u, v) \leq \sum_{k=1}^n |\psi_k(u - v)| + \frac{1}{2^{n-1}}. \quad (12')$$

Second, recall that the W -weak topology on X is the weakest topology for which each $\psi \in W$ is continuous, so that for $u \in X$ the sets

$$\mathcal{N}_{\epsilon, \varphi_1, \dots, \varphi_m}(u) = \{x \in X : |\varphi_i(x - u)| < \epsilon \text{ for } 1 \leq i \leq m\}, \quad \epsilon > 0, \varphi_i \in W,$$

form a base at u for that topology; intersecting these with B gives a base at u for the W -weak topology on B . Write $B_\rho(u, \delta) = \{x \in B : \rho(x, u) < \delta\}$ for the ρ -balls, which form a base at u for the topology induced by ρ .

To prove that the two topologies on B coincide it suffices to prove that at each $u \in B$ each of the two families of basic neighborhoods refines the other.

Step 1: every ρ -ball about u contains a W -weak neighborhood of u in B .

Let $u \in B$ and $\epsilon > 0$. Choose a natural number n so large that $1/2^{n-1} < \epsilon/2$, and set

$$U = \left\{ x \in B : |\psi_k(x - u)| < \frac{\epsilon}{2n} \text{ for } 1 \leq k \leq n \right\}.$$

Since each ψ_k belongs to $B^* \cap W \subseteq W$, the set U is a W -weak neighborhood of u in B . If $x \in U$, the right-hand inequality in (12') gives

$$\rho(x, u) \leq \sum_{k=1}^n |\psi_k(x - u)| + \frac{1}{2^{n-1}} < n \cdot \frac{\epsilon}{2n} + \frac{\epsilon}{2} = \epsilon.$$

Thus $U \subseteq B_\rho(u, \epsilon)$. Note that this step uses only the right-hand half of (12) and not the denseness of $\{\psi_k\}$; it says that every ρ -open subset of B is W -weakly open.

Step 2: every W -weak neighborhood of u in B contains a ρ -ball about u .

Let $u \in B$ and let $\mathcal{N} = \{x \in B : |\varphi_i(x - u)| < \epsilon, 1 \leq i \leq m\}$ be a basic W -weak neighborhood of u in B , where $\varphi_1, \dots, \varphi_m$ belong to W and $\epsilon > 0$.

Set $c = \max_{1 \leq i \leq m} \|\varphi_i\|$. If $c = 0$, then every φ_i is the zero functional and $\mathcal{N} = B$, so any ρ -ball is contained in $\mathcal{N}$. Assume therefore $c > 0$ and define $\varphi'_i = \varphi_i/c$. Then each φ'_i belongs to W and $\|\varphi'_i\| \leq 1$, so $\varphi'_i \in B^* \cap W$; moreover

$$\mathcal{N} = \left\{ x \in B : |\varphi'_i(x - u)| < \epsilon' \text{ for } 1 \leq i \leq m \right\}, \quad \epsilon' := \epsilon/c.$$

It therefore suffices to treat the case in which all the φ_i already lie in $B^* \cap W$.

Here is where the denseness of $\{\psi_k\}_{k=1}^\infty$ in $B^* \cap W$ enters. For each i with $1 \leq i \leq m$, choose an index k_i for which

$$\|\varphi_i - \psi_{k_i}\| < \frac{\epsilon'}{8}.$$

Let $N = \max\{k_1, \dots, k_m\}$ and put

$$\delta = \frac{\epsilon'}{4 \cdot 2^N} > 0.$$

Suppose $x \in B$ and $\rho(x, u) < \delta$. The left-hand inequality in (12'), with $n = N$, gives

$$\frac{1}{2^N} \sum_{k=1}^N |\psi_k(x - u)| \leq \rho(x, u) < \delta = \frac{\epsilon'}{4 \cdot 2^N},$$

so that

$$\sum_{k=1}^N |\psi_k(x - u)| < \frac{\epsilon'}{4}.$$

In particular $|\psi_{k_i}(x - u)| < \epsilon'/4$ for each i , since $k_i \leq N$. Since $\|x - u\| \leq 2$, we conclude that for each i ,

$$|\varphi_i(x - u)| \leq |(\varphi_i - \psi_{k_i})(x - u)| + |\psi_{k_i}(x - u)| \leq \|\varphi_i - \psi_{k_i}\| \cdot \|x - u\| + \frac{\epsilon'}{4} < \frac{\epsilon'}{8} \cdot 2 + \frac{\epsilon'}{4} = \frac{\epsilon'}{2} < \epsilon'.$$

Therefore $x \in \mathcal{N}$, and we have shown $B_\rho(u, \delta) \subseteq \mathcal{N}$. This says that every W -weakly open subset of B is ρ -open.

Conclusion.

By Step 1, at each point u of B every ρ -ball about u contains a W -weak basic neighborhood of u ; by Step 2, every W -weak basic neighborhood of u contains a ρ -ball about u . Hence a subset of B is ρ -open if and only if it is open in the W -weak topology on B , that is, the metric ρ induces the W -weak topology on B . Taking $u = 0$ in the two steps gives exactly the assertion left as an exercise in the text, namely that $\{x \in B : \rho(x, 0) < 1/n\}_{n=1}^\infty$ is a base at the origin for the W -weak topology on B : Step 1 shows each such set is a W -weak neighborhood of 0, and Step 2 shows each W -weak neighborhood of 0 in B contains one of them.

Problem 15.18. Let X be a Banach space, W a closed subspace of its dual X^* , and ψ_0 belong to $X^* \sim W$. Show that if either W is finite dimensional or X is reflexive, then there is a vector x_0 in X for which $\psi_0(x_0) \neq 0$ but $\psi(x_0) = 0$ for all $\psi \in W$. Exhibit an example of an infinite dimensional closed subspace W of X^* for which this separation property fails.

Solution. In the language of Problem 12 of this chapter, what is asked for is a vector x_0 in the pre-annihilator W^0 of W at which ψ_0 does not vanish; equivalently, the assertion to be proved is that $W^0 \not\subseteq \ker \psi_0$.

Case 1: W is finite dimensional.

If $W = \{0\}$, then $\psi_0 \neq 0$, so there is an $x_0 \in X$ with $\psi_0(x_0) \neq 0$, and the condition $\psi(x_0) = 0$ for all $\psi \in W$ holds vacuously (only $\psi = 0$ lies in W). So assume $W \neq \{0\}$ and choose a basis $\psi_1, \dots, \psi_n$ of W .

Since ψ_0 belongs to $X^* \sim W$ and $W = \text{span}\{\psi_1, \dots, \psi_n\}$, the functional ψ_0 fails to be a linear combination of $\psi_1, \dots, \psi_n$. Proposition 4 of Chapter 14 states that ψ_0 is a linear combination of $\psi_1, \dots, \psi_n$ if and only if

$$\bigcap_{i=1}^n \ker \psi_i \subseteq \ker \psi_0.$$

Hence this inclusion must fail: there is a vector $x_0 \in \bigcap_{i=1}^n \ker \psi_i$ with $\psi_0(x_0) \neq 0$. For this x_0 we have $\psi_i(x_0) = 0$ for $1 \leq i \leq n$, and therefore, since every $\psi \in W$ is of the form $\psi = \sum_{i=1}^n c_i \psi_i$,

$$\psi(x_0) = \sum_{i=1}^n c_i \psi_i(x_0) = 0 \quad \text{for all } \psi \in W,$$

while $\psi_0(x_0) \neq 0$. This is the required vector. (Note that the closedness of W was not needed here; by Corollary 6 of Chapter 13 a finite dimensional subspace of a normed linear space is automatically closed.)

Case 2: X is reflexive.

Apply Theorem 10 of Chapter 14 with the normed linear space taken to be X^* and the subspace taken to be W . Since W is closed, W equals its own closure, and ψ_0 does not belong to W . Theorem 10 therefore provides a functional S on X^* , that is, a member S of $(X^*)^* = X^{**}$, such that

$$S(\psi) = 0 \quad \text{for all } \psi \in W, \quad S(\psi_0) \neq 0.$$

Since X is reflexive, the natural embedding $J: X \rightarrow X^{**}$ is onto. Choose $x_0 \in X$ with $J(x_0) = S$. By the definition of J , for every $\psi \in X^*$ we have $S(\psi) = J(x_0)(\psi) = \psi(x_0)$. Consequently

$$\psi(x_0) = S(\psi) = 0 \quad \text{for all } \psi \in W, \quad \psi_0(x_0) = S(\psi_0) \neq 0.$$

So x_0 is the required vector.

An infinite dimensional closed W for which the separation property fails.

Take $X = \ell^1$, the Banach space of absolutely summable real sequences $x = \{x_k\}_{k=1}^\infty$ with $\|x\|_1 = \sum_{k=1}^\infty |x_k|$.

For a bounded real sequence $y = \{y_k\}$ in ℓ^∞ , define

$$\Lambda_y(x) = \sum_{k=1}^\infty x_k y_k \quad \text{for } x \in \ell^1.$$

The series converges absolutely and $|\Lambda_y(x)| \leq \|y\|_\infty \|x\|_1$, so Λ_y is a bounded linear functional on ℓ^1 with $\|\Lambda_y\| \leq \|y\|_\infty$. Testing on the standard unit vectors e_k (which have $\|e_k\|_1 = 1$) gives $\|\Lambda_y\| \geq |\Lambda_y(e_k)| = |y_k|$ for every k , so in fact

$$\|\Lambda_y\| = \|y\|_\infty.$$

Thus $y \mapsto \Lambda_y$ is a linear isometry of ℓ^∞ into $X^* = (\ell^1)^*$; in particular it is one-to-one.

Let $c_0 \subseteq \ell^\infty$ be the subspace of sequences converging to 0, and define

$$W = \{ \Lambda_y : y \in c_0 \} \subseteq X^*.$$

W is a linear subspace, since $y \mapsto \Lambda_y$ is linear and c_0 is a subspace. It is infinite dimensional, since the functionals $\Lambda_{e_1}, \Lambda_{e_2}, \dots$ are linearly independent (the e_k lie in c_0 and are linearly independent, and the map is one-to-one and linear). It is closed in X^* : the space c_0 is a closed subspace of ℓ^∞ , for if $y^{(n)} \in c_0$ and $\|y^{(n)} - y\|_\infty \rightarrow 0$, then given $\epsilon > 0$ choose n with $\|y^{(n)} - y\|_\infty < \epsilon/2$ and then K with $|y_k^{(n)}| < \epsilon/2$ for $k \geq K$, whence $|y_k| < \epsilon$ for $k \geq K$; so $y \in c_0$. Since ℓ^∞ is complete and c_0 is closed in it, c_0 is complete, and since $y \mapsto \Lambda_y$ is an isometry, W is a complete subset of X^* and hence is closed in X^* .

Now let $\mathbf{1} = (1, 1, 1, \dots) \in \ell^\infty$ and put

$$\psi_0 = \Lambda_{\mathbf{1}} \in X^*, \quad \psi_0(x) = \sum_{k=1}^\infty x_k.$$

Since $\mathbf{1}$ does not belong to c_0 and $y \mapsto \Lambda_y$ is one-to-one, ψ_0 does not belong to W ; that is, $\psi_0 \in X^* \setminus W$. Also $\psi_0 \neq 0$, since $\psi_0(e_1) = 1$.

The separation property fails for this W and ψ_0 . Indeed, suppose $x_0 \in \ell^1$ satisfies $\psi(x_0) = 0$ for all $\psi \in W$. Each e_k belongs to c_0 , so $\Lambda_{e_k} \in W$, and therefore

$$0 = \Lambda_{e_k}(x_0) = (x_0)_k \quad \text{for every } k.$$

Hence $x_0 = 0$, and consequently $\psi_0(x_0) = 0$. So there is no $x_0 \in X$ with $\psi(x_0) = 0$ for all $\psi \in W$ and $\psi_0(x_0) \neq 0$. In the notation above, $W^0 = \{0\} \subseteq \ker \psi_0$.

Of course this is consistent with the two positive cases: W here is infinite dimensional, and ℓ^1 is not reflexive. The same phenomenon occurs with $X = L^1[0, 1]$ and W the (isometric) image of $C[0, 1]$ inside $L^\infty[0, 1] = (L^1[0, 1])^*$, which is closed and infinite dimensional while only the zero function of $L^1[0, 1]$ is annihilated by every continuous function.

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16.1 Problems 16.1–16.7

Problem 16.1. Let $[a, b]$ be a closed, bounded interval of real numbers. Show that the $L^2[a, b]$ inner product is also an inner product on $C[a, b]$. Is $C[a, b]$, considered as an inner product space with the $L^2[a, b]$ inner product, a Hilbert space?

Solution. For $f, g \in C[a, b]$ define

$$\langle f, g \rangle = \int_a^b fg.$$

Since $[a, b]$ is of finite measure and every $f \in C[a, b]$ is bounded and measurable, f belongs to $L^2[a, b]$; thus $C[a, b]$ is a linear subspace of $L^2[a, b]$ and the integral above is finite. Bilinearity and symmetry are immediate from the linearity of integration and the commutativity of pointwise multiplication, and $\langle f, f \rangle = \int_a^b f^2 \geq 0$. These properties are inherited from $L^2[a, b]$ and require no continuity.

The only point that needs an argument is positive definiteness in the strict sense: for $L^2[a, b]$ the vectors are equivalence classes of functions equal almost everywhere, whereas the vectors of $C[a, b]$ are genuine functions. So we must check that if $f \in C[a, b]$ and $\int_a^b f^2 = 0$, then f is identically zero, not merely zero almost everywhere. Suppose $f(x_0) \neq 0$ for some $x_0 \in [a, b]$. Put $\epsilon = f(x_0)^2/2 > 0$. By the continuity of f^2 at x_0 there is a $\delta > 0$ such that $f(x)^2 > \epsilon$ for all $x \in [a, b]$ with $|x - x_0| < \delta$. The set $I = (x_0 - \delta, x_0 + \delta) \cap [a, b]$ is an interval containing x_0 and, since $a < b$, it has positive length $\ell > 0$. By the monotonicity of integration for nonnegative measurable functions,

$$\int_a^b f^2 \geq \int_I f^2 \geq \epsilon \cdot \ell > 0,$$

a contradiction. Hence $f \equiv 0$ on $[a, b]$. Therefore $\langle \cdot, \cdot \rangle$ is an inner product on $C[a, b]$, and the induced norm is $\|f\|_2 = \left(\int_a^b f^2\right)^{1/2}$.

$C[a, b]$ with this inner product is **not** a Hilbert space: it is not complete with respect to the norm $\|\cdot\|_2$. We exhibit a Cauchy sequence with no limit in $C[a, b]$. Let $c = (a + b)/2$ and choose N so large that $c + 1/N < b$. For $n \geq N$ define $f_n \in C[a, b]$ by

$$f_n(x) = \begin{cases} 0, & a \leq x \leq c, \\ n(x - c), & c \leq x \leq c + 1/n, \\ 1, & c + 1/n \leq x \leq b. \end{cases}$$

Each f_n is continuous, and $0 \leq f_n \leq 1$. Let $g = \chi_{(c, b]}$, a measurable function on $[a, b]$ belonging to $L^2[a, b]$. Then $f_n - g$ vanishes off the interval $(c, c + 1/n]$ and is bounded there by 1 in absolute value, so

$$\|f_n - g\|_2^2 = \int_a^b |f_n - g|^2 \leq \frac{1}{n}.$$

Consequently $\|f_n - f_m\|_2 \leq \|f_n - g\|_2 + \|g - f_m\|_2 \leq n^{-1/2} + m^{-1/2}$, so $\{f_n\}$ is Cauchy in $C[a, b]$ with respect to $\|\cdot\|_2$.

Suppose there were an $f \in C[a, b]$ with $\|f_n - f\|_2 \rightarrow 0$. Then $\|f - g\|_2 \leq \|f - f_n\|_2 + \|f_n - g\|_2 \rightarrow 0$, so $\int_a^b |f - g|^2 = 0$, and since $|f - g|^2$ is a nonnegative measurable function, $f = g$ almost everywhere on $[a, b]$. In particular $f = 0$ a.e. on $[a, c]$ and $f = 1$ a.e. on $(c, b]$. Fix $x \in [a, c]$. Every neighbourhood of x meets $[a, c]$ in a set of positive measure, so there are points $x_k \rightarrow x$ with $f(x_k) = 0$; by continuity $f(x) = 0$. Hence $f \equiv 0$ on $[a, c]$, and in particular $f(c) = 0$. The same argument on $(c, b]$ gives $f \equiv 1$ on $[c, b]$, so $f(c) = 1$. This contradiction shows no such f exists.

Thus $C[a, b]$, normed by the L^2 norm, is an incomplete inner product space, hence not a Hilbert space. (Equivalently: by the density of $C[a, b]$ in $L^2[a, b]$ established in Chapter 7, $C[a, b]$ is a dense proper linear subspace of the complete space $L^2[a, b]$, and a dense proper subset of a complete metric space is never complete.)

Problem 16.2. Show that the maximum norm on $C[a, b]$ is not induced by an inner product and neither is the usual norm on ℓ^1 .

Solution. The tool is the parallelogram identity recorded in this section: for any two vectors u, v in an inner product space H ,

$$\|u - v\|^2 + \|u + v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

where $\|\cdot\|$ is the norm induced by the inner product. Hence if a norm on a linear space fails this identity for even one pair of vectors, it cannot be induced by any inner product. We produce such a pair in each case.

The maximum norm on $C[a, b]$. Recall $\|f\|_{\max} = \max_{x \in [a, b]} |f(x)|$. Since $a < b$, define $f, g \in C[a, b]$ by

$$f(x) = \frac{x - a}{b - a}, \quad g(x) = 1 - f(x) = \frac{b - x}{b - a}.$$

Both are continuous, and both take values in $[0, 1]$ with $f(b) = 1$ and $g(a) = 1$; hence

$$\|f\|_{\max} = 1, \quad \|g\|_{\max} = 1.$$

Now $f + g$ is the constant function 1, so $\|f + g\|_{\max} = 1$. Moreover

$$(f - g)(x) = 2f(x) - 1 = \frac{2x - a - b}{b - a},$$

which is an increasing affine function running from -1 at $x = a$ to 1 at $x = b$; hence $\|f - g\|_{\max} = 1$. Substituting into the two sides of the parallelogram identity,

$$\|f - g\|_{\max}^2 + \|f + g\|_{\max}^2 = 1 + 1 = 2, \quad 2\|f\|_{\max}^2 + 2\|g\|_{\max}^2 = 2 + 2 = 4.$$

Since $2 \neq 4$, the parallelogram identity fails, and therefore the maximum norm on $C[a, b]$ is not induced by an inner product.

The usual norm on ℓ^1 . Recall that ℓ^1 consists of the real sequences $x = (x_1, x_2, \dots)$ with $\|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty$. Let $e_1 = (1, 0, 0, \dots)$ and $e_2 = (0, 1, 0, \dots)$, both in ℓ^1 with $\|e_1\|_1 = \|e_2\|_1 = 1$. Then $e_1 + e_2 = (1, 1, 0, \dots)$ and $e_1 - e_2 = (1, -1, 0, \dots)$, so

$$\|e_1 + e_2\|_1 = 2, \quad \|e_1 - e_2\|_1 = 2.$$

Therefore

$$\|e_1 - e_2\|_1^2 + \|e_1 + e_2\|_1^2 = 4 + 4 = 8, \quad 2\|e_1\|_1^2 + 2\|e_2\|_1^2 = 2 + 2 = 4.$$

Again the identity fails, so the usual norm on ℓ^1 is not induced by an inner product.

(The same two-vector test shows that for $p \neq 2$ the norm on ℓ^p is not induced by an inner product: with e_1, e_2 as above, $\|e_1 \pm e_2\|_p = 2^{1/p}$, so the parallelogram identity would force $2 \cdot 2^{2/p} = 4$, i.e. $p = 2$.)

Problem 16.3. Let H_1 and H_2 be Hilbert spaces. Show that the Cartesian product $H_1 \times H_2$ also is a Hilbert space with an inner product with respect to which $H_1 \times \{0\} = [\{0\} \times H_2]^\perp$.

Solution. Write $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ for the inner products of H_1 and H_2 , and $\|\cdot\|_1, \|\cdot\|_2$ for the induced norms. Give $H = H_1 \times H_2$ the coordinatewise linear structure,

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2), \quad \lambda(u_1, u_2) = (\lambda u_1, \lambda u_2),$$

under which it is a linear space with zero vector $(0, 0)$, and define

$$\langle (u_1, u_2), (v_1, v_2) \rangle = \langle u_1, v_1 \rangle_1 + \langle u_2, v_2 \rangle_2.$$

This is an inner product. Symmetry is clear from the symmetry of each summand. Linearity in the first variable follows because each summand is linear in its first variable: for $u, u', v \in H$ and $\lambda, \mu \in \mathbb{R}$,

writing $u = (u_1, u_2)$ etc.,

$$\langle \lambda u + \mu u', v \rangle = \langle \lambda u_1 + \mu u'_1, v_1 \rangle_1 + \langle \lambda u_2 + \mu u'_2, v_2 \rangle_2 = \lambda \langle u, v \rangle + \mu \langle u', v \rangle.$$

Finally $\langle (u_1, u_2), (u_1, u_2) \rangle = \|u_1\|_1^2 + \|u_2\|_2^2 \geq 0$, and this vanishes only if $\|u_1\|_1 = \|u_2\|_2 = 0$, that is only if $u_1 = 0$ and $u_2 = 0$, i.e. only for the zero vector of H . So H is an inner product space, with induced norm

$$\|(u_1, u_2)\| = (\|u_1\|_1^2 + \|u_2\|_2^2)^{1/2}.$$

H is complete. Observe first the elementary comparisons, valid for all $(u_1, u_2) \in H$:

$$\max\{\|u_1\|_1, \|u_2\|_2\} \leq \|(u_1, u_2)\| \leq \|u_1\|_1 + \|u_2\|_2.$$

Let $\{(u_1^{(n)}, u_2^{(n)})\}$ be a Cauchy sequence in H . By the left-hand inequality applied to differences,

$$\|u_i^{(n)} - u_i^{(m)}\|_i \leq \|(u_1^{(n)}, u_2^{(n)}) - (u_1^{(m)}, u_2^{(m)})\| \quad (i = 1, 2),$$

so $\{u_1^{(n)}\}$ is Cauchy in H_1 and $\{u_2^{(n)}\}$ is Cauchy in H_2 . Since H_1 and H_2 are Hilbert spaces they are complete, so there are $u_1 \in H_1$ and $u_2 \in H_2$ with $\|u_1^{(n)} - u_1\|_1 \rightarrow 0$ and $\|u_2^{(n)} - u_2\|_2 \rightarrow 0$. By the right-hand inequality,

$$\|(u_1^{(n)}, u_2^{(n)}) - (u_1, u_2)\| \leq \|u_1^{(n)} - u_1\|_1 + \|u_2^{(n)} - u_2\|_2 \rightarrow 0.$$

Hence the sequence converges in H , and H is a Banach space with respect to the induced norm, i.e. a Hilbert space.

The orthogonality relation. Put $V = \{0\} \times H_2 = \{(0, w) : w \in H_2\}$ and $W = H_1 \times \{0\}$. We claim $W = V^\perp$.

First, $W \subseteq V^\perp$: if $u = (u_1, 0) \in W$ and $(0, w) \in V$, then

$$\langle (u_1, 0), (0, w) \rangle = \langle u_1, 0 \rangle_1 + \langle 0, w \rangle_2 = 0 + 0 = 0,$$

so u is orthogonal to every vector of V .

Conversely, suppose $v = (v_1, v_2) \in V^\perp$. Then for every $w \in H_2$,

$$0 = \langle (v_1, v_2), (0, w) \rangle = \langle v_1, 0 \rangle_1 + \langle v_2, w \rangle_2 = \langle v_2, w \rangle_2.$$

Taking $w = v_2$ gives $\|v_2\|_2^2 = 0$, so $v_2 = 0$ and $v = (v_1, 0) \in W$. Hence $V^\perp \subseteq W$.

Therefore $H_1 \times \{0\} = [\{0\} \times H_2]^\perp$, as required. (By the same argument with the roles of the factors interchanged, $\{0\} \times H_2 = [H_1 \times \{0\}]^\perp$, so $H = (H_1 \times \{0\}) \oplus (\{0\} \times H_2)$ is an orthogonal direct sum decomposition of the kind furnished by Theorem 3.)

Problem 16.4. Show that if S is a subset of an inner product space H , then $S^\perp$ is a closed subspace of H .

Solution. Recall that $S^\perp = \{u \in H : \langle u, s \rangle = 0 \text{ for all } s \in S\}$. For a fixed $s \in H$ define

$$\varphi_s : H \rightarrow \mathbb{R}, \quad \varphi_s(u) = \langle u, s \rangle.$$

φ_s is linear, by the linearity of the inner product in its first variable. It is also continuous: for $u, v \in H$, the Cauchy-Schwarz Inequality gives

$$|\varphi_s(u) - \varphi_s(v)| = |\langle u - v, s \rangle| \leq \|u - v\| \|s\|,$$

so φ_s is Lipschitz with constant $\|s\|$, hence continuous with respect to the norm induced by the inner product.

Consequently $\ker \varphi_s = \varphi_s^{-1}(\{0\})$ is a linear subspace of H (being the kernel of a linear map) and is closed (being the preimage of the closed set $\{0\} \subseteq \mathbb{R}$ under a continuous map). Concretely, closedness

says: if $\{u_n\}$ lies in $\ker \varphi_s$ and $u_n \rightarrow u$ in H , then $|\langle u, s \rangle| = |\langle u, s \rangle - \langle u_n, s \rangle| \leq \|u - u_n\| \|s\| \rightarrow 0$, so $\langle u, s \rangle = 0$ and $u \in \ker \varphi_s$.

Now observe that

$$S^\perp = \bigcap_{s \in S} \ker \varphi_s.$$

Indeed, $u \in S^\perp$ means precisely that $\varphi_s(u) = \langle u, s \rangle = 0$ for every $s \in S$.

An arbitrary intersection of linear subspaces is a linear subspace: if u, v lie in every $\ker \varphi_s$ and $\lambda, \mu \in \mathbb{R}$, then $\varphi_s(\lambda u + \mu v) = \lambda \varphi_s(u) + \mu \varphi_s(v) = 0$ for every $s \in S$, so $\lambda u + \mu v \in S^\perp$; and $0 \in S^\perp$. An arbitrary intersection of closed sets is closed. Therefore $S^\perp$ is a closed linear subspace of H .

(If $S = \emptyset$ the intersection is taken over an empty index set and the condition defining $S^\perp$ is vacuous, so by the usual convention $S^\perp = H$, which is certainly a closed subspace of itself. Note also that completeness of H was nowhere used: the assertion holds in any inner product space, as the problem asserts.)

Problem 16.5. Let S be a subset of H . Show that $S = (S^\perp)^\perp$ if and only if S is a closed subspace of H .

Solution. We first record a fact used in both directions: for any subset S of H ,

$$S \subseteq (S^\perp)^\perp.$$

Indeed, let $s \in S$. Every $u \in S^\perp$ satisfies $\langle u, s \rangle = 0$, hence by the symmetry of the inner product $\langle s, u \rangle = 0$ for every $u \in S^\perp$; that is exactly the statement $s \in (S^\perp)^\perp$.

Suppose first that $S = (S^\perp)^\perp$. By Problem 4, applied to the subset $S^\perp$ of H , the set $(S^\perp)^\perp$ is a closed subspace of H . Hence S , being equal to it, is a closed subspace of H .

Suppose conversely that S is a closed subspace of H . In view of the inclusion $S \subseteq (S^\perp)^\perp$ already proved, it suffices to establish the reverse inclusion $(S^\perp)^\perp \subseteq S$.

Since H is a Hilbert space and S is a closed subspace, Theorem 3 provides the orthogonal direct sum decomposition

$$H = S \oplus S^\perp,$$

which in particular means that every $h \in H$ may be written as $h = s + t$ with $s \in S$ and $t \in S^\perp$, and that $S \cap S^\perp = \{0\}$. (The latter is immediate in any case: if $v \in S \cap S^\perp$ then $\langle v, v \rangle = 0$, so $v = 0$.)

Let $h \in (S^\perp)^\perp$ and write $h = s + t$ with $s \in S$, $t \in S^\perp$. Since $S \subseteq (S^\perp)^\perp$ and $(S^\perp)^\perp$ is a linear subspace of H (Problem 4), the vector

$$t = h - s$$

belongs to $(S^\perp)^\perp$. Thus $t \in S^\perp \cap (S^\perp)^\perp$, and therefore $\langle t, t \rangle = 0$, so $t = 0$. Hence $h = s \in S$.

This proves $(S^\perp)^\perp \subseteq S$, and combined with the reverse inclusion, $S = (S^\perp)^\perp$.

Both directions being established, $S = (S^\perp)^\perp$ if and only if S is a closed subspace of H . (In general, for an arbitrary subset S of a Hilbert space, let V be the closure of the linear span of S . Then $S^\perp = V^\perp$: the inclusion $V^\perp \subseteq S^\perp$ is clear since $S \subseteq V$, while if $u \in S^\perp$ then u annihilates every finite linear combination of vectors of S by bilinearity, and annihilates the limit of any convergent sequence of such combinations because $v \mapsto \langle u, v \rangle$ is continuous by the Cauchy-Schwarz Inequality. Applying the case already proved to the closed subspace V therefore gives $(S^\perp)^\perp = (V^\perp)^\perp = V$: the set $(S^\perp)^\perp$ is precisely the closed linear span of S . Corollary 4 is the special case in which that span is all of H .)

Problem 16.6. (Polarization Identity) Show that for any two vectors $u, v \in H$,

$$\langle u, v \rangle = \frac{1}{4} [\|u + v\|^2 - \|u - v\|^2].$$

Solution. Let u, v belong to H . By the definition of the induced norm, $\|w\|^2 = \langle w, w \rangle$ for every $w \in H$. Using the bilinearity and symmetry of the inner product we expand each of the two squares.

For the sum,

$$\begin{aligned}\|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u + v \rangle + \langle v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + 2\langle u, v \rangle + \|v\|^2,\end{aligned}$$

where the last step used $\langle v, u \rangle = \langle u, v \rangle$.

For the difference, the same computation with v replaced by $-v$, together with $\langle u, -v \rangle = -\langle u, v \rangle$ and $\| -v \| = \|v\|$, gives

$$\|u - v\|^2 = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2.$$

These are exactly the two equalities displayed in the text preceding the parallelogram identity. Subtracting the second from the first, the terms $\|u\|^2$ and $\|v\|^2$ cancel and we obtain

$$\|u + v\|^2 - \|u - v\|^2 = 4\langle u, v \rangle.$$

Dividing by 4,

$$\langle u, v \rangle = \frac{1}{4} [\|u + v\|^2 - \|u - v\|^2],$$

which is the polarization identity.

Two remarks worth recording. First, the identity shows that an inner product on a linear space is completely determined by the norm it induces: two inner products inducing the same norm are equal. Second, together with the parallelogram identity it is the key to Problem 7, where the right-hand side is used to **define** a candidate inner product from a given norm.

Problem 16.7. (Jordan-von Neumann) Let X be a linear space normed by $\|\cdot\|$. Use the polarization identity to show that $\|\cdot\|$ is induced by an inner product if and only if the parallelogram identity holds.

Solution. Necessity. Suppose there is an inner product $\langle \cdot, \cdot \rangle$ on X with $\|u\|^2 = \langle u, u \rangle$ for all $u \in X$. Then, as verified in the text, adding the two expansions

$$\|u - v\|^2 = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2, \quad \|u + v\|^2 = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2$$

yields the parallelogram identity

$$\|u - v\|^2 + \|u + v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \text{for all } u, v \in X.$$

Sufficiency. Assume now that the parallelogram identity holds for the norm $\|\cdot\|$. Guided by the polarization identity of Problem 6 we **define**

$$\langle u, v \rangle = \frac{1}{4} [\|u + v\|^2 - \|u - v\|^2] \quad \text{for } u, v \in X,$$

and prove that this is an inner product inducing $\|\cdot\|$.

Step 1: the diagonal and symmetry. Taking $v = u$,

$$\langle u, u \rangle = \frac{1}{4} [\|2u\|^2 - \|0\|^2] = \frac{1}{4} \cdot 4\|u\|^2 = \|u\|^2.$$

Hence $\langle u, u \rangle \geq 0$, with equality if and only if $\|u\| = 0$, that is if and only if $u = 0$; and the norm induced by $\langle \cdot, \cdot \rangle$, should it prove to be an inner product, is exactly $\|\cdot\|$. Symmetry is immediate from $\|v - u\| = \|(u - v)\| = \|u - v\|$:

$$\langle v, u \rangle = \frac{1}{4} [\|v + u\|^2 - \|v - u\|^2] = \langle u, v \rangle.$$

Also $\langle 0, v \rangle = \frac{1}{4} [\|v\|^2 - \|-v\|^2] = 0$, and

$$\langle -u, v \rangle = \frac{1}{4} [\|v - u\|^2 - \|u + v\|^2] = -\langle u, v \rangle.$$

Step 2: additivity in the first variable. Fix $u, v, w \in X$. Apply the parallelogram identity to the pair $u + w, v + w$, whose sum is $u + v + 2w$ and whose difference is $u - v$:

$$\|u + v + 2w\|^2 + \|u - v\|^2 = 2\|u + w\|^2 + 2\|v + w\|^2. \quad (\text{A})$$

Apply it to the pair $u - w, v - w$, whose sum is $u + v - 2w$ and whose difference is again $u - v$:

$$\|u + v - 2w\|^2 + \|u - v\|^2 = 2\|u - w\|^2 + 2\|v - w\|^2. \quad (\text{B})$$

Subtracting (B) from (A), the terms $\|u - v\|^2$ cancel and

$$\|u + v + 2w\|^2 - \|u + v - 2w\|^2 = 2[\|u + w\|^2 - \|u - w\|^2] + 2[\|v + w\|^2 - \|v - w\|^2].$$

By the definition of $\langle \cdot, \cdot \rangle$, the left-hand side equals $4\langle u + v, 2w \rangle$ and the right-hand side equals $8\langle u, w \rangle + 8\langle v, w \rangle$. Therefore

$$\langle u + v, 2w \rangle = 2[\langle u, w \rangle + \langle v, w \rangle] \quad \text{for all } u, v, w \in X. \quad (\text{C})$$

Setting $v = 0$ in (C) and using $\langle 0, w \rangle = 0$ gives

$$\langle u, 2w \rangle = 2\langle u, w \rangle \quad \text{for all } u, w \in X. \quad (\text{D})$$

Now apply (D) with u replaced by $u + v$ to rewrite the left-hand side of (C): $\langle u + v, 2w \rangle = 2\langle u + v, w \rangle$. Cancelling the factor 2 in (C),

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \quad \text{for all } u, v, w \in X.$$

Step 3: homogeneity in the first variable. Fix $u, w \in X$ and define

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(t) = \langle tu, w \rangle = \frac{1}{4} [\|tu + w\|^2 - \|tu - w\|^2].$$

Since $(s+t)u = su + tu$, Step 2 gives $f(s+t) = f(s) + f(t)$ for all real s, t ; that is, f is additive. Additivity forces $f(0) = 0$, then $f(n) = nf(1)$ for every natural number n by induction, then $f(-t) = -f(t)$, so $f(m) = mf(1)$ for every integer m . For a positive integer n and integer m , $nf(m/n) = f(m) = mf(1)$, so $f(m/n) = (m/n)f(1)$; thus $f(q) = qf(1)$ for every rational q .

Moreover f is continuous. Indeed, for real s, t the triangle inequality gives $|\|tu \pm w\| - \|su \pm w\|| \leq \|(t-s)u\| = |t-s|\|u\|$, so $t \mapsto \|tu \pm w\|$ is continuous, and hence so is f , being a fixed combination of squares of continuous functions.

A continuous additive function agreeing with $t \mapsto tf(1)$ on the rationals agrees with it everywhere, since the rationals are dense in $\mathbb{R}$ and both functions are continuous. Hence

$$\langle tu, w \rangle = f(t) = tf(1) = t\langle u, w \rangle \quad \text{for all } t \in \mathbb{R}.$$

Step 4: conclusion. By Steps 2 and 3 the map $u \mapsto \langle u, w \rangle$ is linear for each fixed w ; by the symmetry of Step 1 the same holds in the second variable, so $\langle \cdot, \cdot \rangle$ is bilinear and symmetric; and by Step 1 it is positive definite. Therefore $\langle \cdot, \cdot \rangle$ is an inner product on X , and since $\langle u, u \rangle = \|u\|^2$, the given norm is the norm induced by it.

This completes both implications: the norm $\|\cdot\|$ on X is induced by an inner product if and only if it satisfies the parallelogram identity. Note that by the polarization identity the inner product produced above is the only one that can induce $\|\cdot\|$, so the inner product, when it exists, is unique.

16.2 Problems 16.8–16.14

Problem 16.8. Let V be a closed subspace of H and P a projection of H onto V . Show that P is the orthogonal projection of H onto V if and only if (4) holds, that is, if and only if

$$\langle P(u), v \rangle = \langle u, P(v) \rangle \quad \text{for all } u, v \in H.$$

Solution. Throughout, H is a Hilbert space and V is a closed subspace of H . By Theorem 3 there is the orthogonal direct sum decomposition

$$H = V \oplus V^\perp,$$

so every $h \in H$ has one and only one representation $h = v + w$ with $v \in V$ and $w \in V^\perp$; the orthogonal projection P_0 of H onto V is, by definition, the projection of H onto V along $V^\perp$, that is, $P_0(h) = v$ for this unique decomposition.

Recall from Section 13.4 that a linear operator $P: H \rightarrow H$ for which $P^2 = P$ is called a projection, and that P is a projection of H onto V provided, in addition, $P(H) = V$. Observe that such a P acts as the identity on V : if $v \in V = P(H)$, choose $u \in H$ with $P(u) = v$; then

$$P(v) = P(P(u)) = P^2(u) = P(u) = v.$$

Assume first that P is the orthogonal projection of H onto V , so $P = P_0$. If $V = \{0\}$, then $P = 0$ and (4) reads $0 = 0$. If $V \neq \{0\}$, then (4) is exactly the assertion of Proposition 5. For completeness we give the one-line verification, which covers both cases. Let $u, v \in H$. Then $P(u) \in V$ and $v - P(v) \in V^\perp$, so $\langle P(u), v - P(v) \rangle = 0$ and hence

$$\langle P(u), v \rangle = \langle P(u), P(v) \rangle.$$

Interchanging the roles of u and v and using the symmetry of the inner product gives, in the same way,

$$\langle u, P(v) \rangle = \langle P(v), u \rangle = \langle P(v), P(u) \rangle = \langle P(u), P(v) \rangle.$$

Comparing the two displays yields $\langle P(u), v \rangle = \langle u, P(v) \rangle$, which is (4).

Now assume conversely that P is a projection of H onto V for which (4) holds. Fix $u \in H$. We claim that $u - P(u)$ belongs to $V^\perp$. Indeed, let $v \in V$. Since P acts as the identity on V , we have $P(v) = v$, and therefore, using (4),

$$\langle u - P(u), v \rangle = \langle u, v \rangle - \langle P(u), v \rangle = \langle u, v \rangle - \langle u, P(v) \rangle = \langle u, v \rangle - \langle u, v \rangle = 0.$$

As $v \in V$ was arbitrary, $u - P(u) \in V^\perp$.

Thus

$$u = P(u) + (u - P(u)), \quad P(u) \in V, \quad u - P(u) \in V^\perp,$$

which is a decomposition of u as a vector of V plus a vector of $V^\perp$. Since the decomposition provided by Theorem 3 is unique, this must be the decomposition $u = P_0(u) + (u - P_0(u))$, and therefore $P(u) = P_0(u)$. Since $u \in H$ was arbitrary, $P = P_0$, that is, P is the orthogonal projection of H onto V .

We remark that the second half of the argument did not presuppose the continuity of P : an algebraic projection of H onto V satisfying (4) is automatically the orthogonal projection and hence, by Proposition 5, is continuous with $\|P\| = 1$ when V is nontrivial.

Problem 16.9. Let T belong to $\mathcal{L}(H)$. Show that T is an isometry if and only if

$$\langle T(u), T(v) \rangle = \langle u, v \rangle \quad \text{for all } u, v \in H.$$

Solution. Recall that a linear operator $T \in \mathcal{L}(H)$ is called an isometry provided it preserves the norm, that is, $\|T(u)\| = \|u\|$ for all $u \in H$; equivalently, since T is linear, $\|T(u) - T(v)\| = \|T(u - v)\| = \|u - v\|$ for all u, v , so that T preserves distances. This is the sense in which the word is used for linear operators throughout the book: the natural embedding $J: X \rightarrow X^{**}$ of Chapter 14, for instance, is called an isometry although it need not be onto. Here $\|\cdot\|$ is the norm induced by the inner product, so that $\|h\|^2 = \langle h, h \rangle$ for all $h \in H$, as in Proposition 1.

First suppose that T preserves the inner product, that is, $\langle T(u), T(v) \rangle = \langle u, v \rangle$ for all $u, v \in H$. Taking $v = u$ we obtain

$$\|T(u)\|^2 = \langle T(u), T(u) \rangle = \langle u, u \rangle = \|u\|^2,$$

and hence, both quantities being nonnegative, $\|T(u)\| = \|u\|$ for all $u \in H$. Thus T is an isometry.

Now suppose conversely that T is an isometry. We use the Polarization Identity of Problem 6, namely

$$\langle x, y \rangle = \frac{1}{4} [\|x + y\|^2 - \|x - y\|^2] \quad \text{for all } x, y \in H.$$

This is immediate from the two expansions displayed in the verification of the Parallelogram Identity,

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \quad \|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2,$$

each of which follows by expanding $\langle x \pm y, x \pm y \rangle$ using bilinearity and symmetry; subtracting the second from the first gives $\|x + y\|^2 - \|x - y\|^2 = 4\langle x, y \rangle$.

Let $u, v \in H$. Since T is linear, $T(u) + T(v) = T(u + v)$ and $T(u) - T(v) = T(u - v)$. Therefore, applying the Polarization Identity to the pair $T(u), T(v)$ and then using that T preserves the norm,

$$\begin{aligned} \langle T(u), T(v) \rangle &= \frac{1}{4} [\|T(u) + T(v)\|^2 - \|T(u) - T(v)\|^2] \\ &= \frac{1}{4} [\|T(u + v)\|^2 - \|T(u - v)\|^2] \\ &= \frac{1}{4} [\|u + v\|^2 - \|u - v\|^2] \\ &= \langle u, v \rangle, \end{aligned}$$

the last equality being the Polarization Identity applied to the pair u, v . Hence T preserves the inner product.

The two implications together establish the asserted equivalence.

Problem 16.10. Let V be a finite dimensional subspace of H and $\varphi_1, \dots, \varphi_n$ a basis for V consisting of unit vectors, each pair of which is orthogonal. Show that the orthogonal projection P of H onto V is given by

$$P(h) = \sum_{k=1}^n \langle h, \varphi_k \rangle \varphi_k \quad \text{for all } h \in V.$$

Solution. We prove the formula for every $h \in H$. The restriction to $h \in V$ in the printed statement is evidently a misprint for $h \in H$, since P is defined on all of H and the formula there asserted is what is subsequently used; in any case what we prove contains the printed assertion as a special case.

The hypothesis on the basis says that

$$\langle \varphi_j, \varphi_k \rangle = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}$$

that is, $\{\varphi_1, \dots, \varphi_n\}$ is an orthonormal basis of V .

First observe that P is defined: by Corollary 6 of Chapter 13, a finite dimensional normed linear space is complete and therefore a finite dimensional subspace of a normed linear space is closed. Hence V is a closed subspace of H , and by Theorem 3 there is the orthogonal direct sum decomposition $H = V \oplus V^\perp$; the orthogonal projection P of H onto V is the projection onto V along $V^\perp$, so that for $h \in H$, $P(h)$

is the unique vector of V for which $h - P(h)$ belongs to $V^\perp$.

Define $Q: H \rightarrow H$ by

$$Q(h) = \sum_{k=1}^n \langle h, \varphi_k \rangle \varphi_k \quad \text{for } h \in H.$$

We must show $Q = P$. By the above characterization of P it suffices to show two things: that $Q(h) \in V$, and that $h - Q(h) \in V^\perp$.

The first is clear, since $Q(h)$ is a linear combination of $\varphi_1, \dots, \varphi_n$, all of which lie in V , and V is a linear space.

For the second, fix $h \in H$ and let $1 \leq j \leq n$. Using the bilinearity of the inner product and the orthonormality of the φ_k ,

$$\langle Q(h), \varphi_j \rangle = \left\langle \sum_{k=1}^n \langle h, \varphi_k \rangle \varphi_k, \varphi_j \right\rangle = \sum_{k=1}^n \langle h, \varphi_k \rangle \langle \varphi_k, \varphi_j \rangle = \langle h, \varphi_j \rangle,$$

since the only surviving term is the one with $k = j$, whose coefficient $\langle \varphi_j, \varphi_j \rangle$ equals 1. Therefore

$$\langle h - Q(h), \varphi_j \rangle = \langle h, \varphi_j \rangle - \langle Q(h), \varphi_j \rangle = 0 \quad \text{for } 1 \leq j \leq n.$$

Now let v be any vector in V . Since $\varphi_1, \dots, \varphi_n$ is a basis for V , there are real numbers $c_1, \dots, c_n$ with $v = \sum_{j=1}^n c_j \varphi_j$, and hence, by bilinearity,

$$\langle h - Q(h), v \rangle = \sum_{j=1}^n c_j \langle h - Q(h), \varphi_j \rangle = 0.$$

Thus $h - Q(h)$ is orthogonal to every vector of V , that is, $h - Q(h) \in V^\perp$.

Consequently

$$h = Q(h) + (h - Q(h)), \quad Q(h) \in V, \quad h - Q(h) \in V^\perp,$$

and by the uniqueness of the decomposition of h in $H = V \oplus V^\perp$ furnished by Theorem 3, we conclude that

$$P(h) = Q(h) = \sum_{k=1}^n \langle h, \varphi_k \rangle \varphi_k \quad \text{for all } h \in H.$$

In particular, for $h \in V$ we recover the printed formula, which in that case simply expresses h itself in terms of the orthonormal basis: $h = \sum_{k=1}^n \langle h, \varphi_k \rangle \varphi_k$, since $P(h) = h$ for $h \in V$.

Problem 16.11. For h a vector in H , show that the function $u \mapsto \langle h, u \rangle$ belongs to H^* .

Solution. Fix $h \in H$ and define $\psi_h: H \rightarrow \mathbb{R}$ by

$$\psi_h(u) = \langle h, u \rangle \quad \text{for } u \in H.$$

Recall that H^* denotes the dual space of H , that is, the linear space of all bounded (equivalently, continuous) linear functionals on H , normed by

$$\|\psi\| = \sup\{|\psi(u)| : u \in H, \|u\| \leq 1\}.$$

So we must check that ψ_h is linear and bounded.

Linearity. Let $u_1, u_2 \in H$ and let α, β be real numbers. By property (i) of the inner product together with the symmetry property (ii) – that is, by the bilinearity of the inner product noted immediately after the definition –

$$\psi_h(\alpha u_1 + \beta u_2) = \langle h, \alpha u_1 + \beta u_2 \rangle = \alpha \langle h, u_1 \rangle + \beta \langle h, u_2 \rangle = \alpha \psi_h(u_1) + \beta \psi_h(u_2).$$

Thus ψ_h is a linear functional on H .

Boundedness. By the Cauchy-Schwarz Inequality, for every $u \in H$,

$$|\psi_h(u)| = |\langle h, u \rangle| \leq \|h\| \cdot \|u\|.$$

Hence ψ_h is bounded, with $\|\psi_h\| \leq \|h\|$. A bounded linear functional is continuous, so ψ_h belongs to H^* .

In fact $\|\psi_h\| = \|h\|$. This is clear if $h = 0$, since then $\psi_h \equiv 0$. If $h \neq 0$, set $u_0 = h/\|h\|$, a vector of norm 1; then

$$\psi_h(u_0) = \left\langle h, \frac{h}{\|h\|} \right\rangle = \frac{\langle h, h \rangle}{\|h\|} = \frac{\|h\|^2}{\|h\|} = \|h\|,$$

so that $\|\psi_h\| \geq \|h\|$, and therefore $\|\psi_h\| = \|h\|$. Thus $h \mapsto \psi_h$ is a linear isometry of H into H^* ; the Riesz-Frechet Representation Theorem asserts that it is onto H^* .

Problem 16.12. For any vector $h \in H$, show that there is a bounded linear functional $\psi \in H^*$ for which

$$\|\psi\| = 1 \quad \text{and} \quad \psi(h) = \|h\|.$$

Solution. We assume, as we must, that $H \neq \{0\}$; otherwise $H^* = \{0\}$ contains no functional of norm 1.

For a vector $g \in H$ let $\psi_g: H \rightarrow \mathbb{R}$ be defined by $\psi_g(u) = \langle g, u \rangle$ for $u \in H$. By the preceding problem, ψ_g belongs to H^* and $\|\psi_g\| = \|g\|$. In particular, ψ_g has norm 1 precisely when g is a unit vector.

Case 1: $h \neq 0$. Define $g = h/\|h\|$, a unit vector, and put $\psi = \psi_g$. Then $\psi \in H^*$ and $\|\psi\| = \|g\| = 1$. Moreover

$$\psi(h) = \left\langle \frac{h}{\|h\|}, h \right\rangle = \frac{\langle h, h \rangle}{\|h\|} = \frac{\|h\|^2}{\|h\|} = \|h\|.$$

Thus ψ has the two required properties.

Case 2: $h = 0$. Since $H \neq \{0\}$, choose any nonzero $g_0 \in H$ and set $g = g_0/\|g_0\|$ and $\psi = \psi_g$. Then $\|\psi\| = \|g\| = 1$, while $\psi(h) = \psi(0) = 0 = \|0\| = \|h\|$ because ψ is linear. Again ψ has the two required properties.

This completes the proof. We note that this is the Hilbert space case of the corollary of the Hahn-Banach Theorem which asserts that for each vector h in a normed linear space X there is a norm-one functional in X^* that takes the value $\|h\|$ at h ; in a Hilbert space no extension theorem is needed, since the required functional is exhibited explicitly by the inner product. In particular, if $h \neq 0$ then $\psi(h) = \|h\| = \|\psi\| \cdot \|h\|$, so ψ attains its norm at $h/\|h\|$.

Problem 16.13. Let V be a closed subspace of H and P the orthogonal projection of H onto V . For any normed linear space X and $T \in \mathcal{L}(V, X)$, show that $T \circ P$ belongs to $\mathcal{L}(H, X)$, and is an extension of $T: V \rightarrow X$ for which $\|T \circ P\| = \|T\|$.

Solution. Here $\mathcal{L}(V, X)$ and $\mathcal{L}(H, X)$ denote the spaces of bounded linear operators from V , respectively H , into X , normed by

$$\|S\| = \sup\{\|S(u)\| : \|u\| \leq 1\}.$$

Since V is a closed subspace of the Hilbert space H , Theorem 3 provides the decomposition $H = V \oplus V^\perp$, and $P \in \mathcal{L}(H)$ is the projection of H onto V along $V^\perp$. Note that $P(H) = V$, so the composition $T \circ P: H \rightarrow X$ is defined.

Step 1: $\|P\| \leq 1$, and $P(v) = v$ for $v \in V$. If $v \in V$, then $v = v + 0$ with $v \in V$ and $0 \in V^\perp$, so by uniqueness of the decomposition $P(v) = v$. For the norm bound, let $h \in H$ and write $h = P(h) + (\text{Id} - P)(h)$. Since P is the projection of H onto V along $V^\perp$, we have $P(h) \in V$ and $(\text{Id} - P)(h) \in V^\perp$, so that $\langle P(h), (\text{Id} - P)(h) \rangle = 0$ and the cross terms in the expansion below vanish; hence

$$\|h\|^2 = \langle P(h) + (\text{Id} - P)(h), P(h) + (\text{Id} - P)(h) \rangle = \|P(h)\|^2 + \|(\text{Id} - P)(h)\|^2 \geq \|P(h)\|^2,$$

so $\|P(h)\| \leq \|h\|$ for all $h \in H$ and therefore $\|P\| \leq 1$. (When V is nontrivial, Proposition 5 tells us $\|P\| = 1$; when $V = \{0\}$, $P = 0$.)

Step 2: $T \circ P \in \mathcal{L}(H, X)$. The composition of linear operators is linear, so $T \circ P$ is linear. For $h \in H$, using that T is bounded on V and $P(h) \in V$,

$$\|(T \circ P)(h)\| = \|T(P(h))\| \leq \|T\| \cdot \|P(h)\| \leq \|T\| \cdot \|P\| \cdot \|h\| \leq \|T\| \cdot \|h\|.$$

Hence $T \circ P$ is bounded, so it belongs to $\mathcal{L}(H, X)$, and moreover

$$\|T \circ P\| \leq \|T\| \cdot \|P\| \leq \|T\|.$$

Step 3: $T \circ P$ extends T . For $v \in V$ we have $P(v) = v$ by Step 1, and therefore

$$(T \circ P)(v) = T(P(v)) = T(v).$$

Thus $T \circ P$ agrees with T on V , that is, $T \circ P: H \rightarrow X$ is an extension of $T: V \rightarrow X$.

Step 4: $\|T \circ P\| = \|T\|$. Since every $v \in V$ with $\|v\| \leq 1$ is also a vector of H with $\|v\| \leq 1$, and since $(T \circ P)(v) = T(v)$ by Step 3,

$$\|T\| = \sup_{v \in V, \|v\| \leq 1} \|T(v)\| = \sup_{v \in V, \|v\| \leq 1} \|(T \circ P)(v)\| \leq \sup_{h \in H, \|h\| \leq 1} \|(T \circ P)(h)\| = \|T \circ P\|.$$

Together with the inequality $\|T \circ P\| \leq \|T\|$ of Step 2, this gives $\|T \circ P\| = \|T\|$, as required.

Thus every bounded linear operator on a closed subspace of a Hilbert space, with values in an arbitrary normed linear space, has a norm-preserving extension to all of H , obtained simply by precomposing with the orthogonal projection. For $X = \mathbb{R}$ this recovers, for Hilbert spaces, the conclusion of the Hahn-Banach Theorem without any appeal to that theorem; and unlike the Hahn-Banach Theorem, the present argument applies to operators with values in an arbitrary normed linear space X .

Problem 16.14. Prove the Hyperplane Separation Theorem for H , considered as a locally convex topological vector space with respect to the strong topology, by directly using Proposition 2.

Solution. The Hyperplane Separation Theorem, as stated in Section 14.5, asserts the following. Let X be a locally convex topological vector space, K a nonempty closed convex subset of X , and x_0 a point of X lying outside of K . Then K and x_0 may be separated by a closed hyperplane, that is, there is a continuous linear functional $\psi: X \rightarrow \mathbb{R}$ for which

$$\psi(x_0) < \inf_{x \in K} \psi(x).$$

We prove this when $X = H$ is a Hilbert space with its strong topology, that is, the topology induced by the norm $\|\cdot\|$ coming from the inner product. That H , so topologized, is indeed a locally convex topological vector space is the content of Proposition 21 of Chapter 14, which asserts this for any normed linear space: the open balls about a point are convex and they form a base at that point.

So let K be a nonempty closed convex subset of H and let $h_0 \in H \sim K$.

Step 1: the closest point. Since K is a nonempty closed convex subset of the Hilbert space H and h_0 belongs to $H \sim K$, Proposition 2 provides exactly one vector $h_* \in K$ for which

$$\|h_0 - h_*\| = \text{dist}(h_0, K) = \inf_{h \in K} \|h_0 - h\|.$$

Define

$$w = h_* - h_0.$$

Since $h_* \in K$ and $h_0 \notin K$, we have $h_* \neq h_0$, so $w \neq 0$ and $\|w\| > 0$.

Step 2: the variational inequality. We claim that

$$\langle w, h - h_* \rangle \geq 0 \quad \text{for every } h \in K.$$

Indeed, let $h \in K$ and let $t \in (0, 1]$. Since K is convex and $h_*, h \in K$, the vector

$$h_t = (1 - t)h_* + th = h_* + t(h - h_*)$$

belongs to K . By the minimizing property of h_* ,

$$\|h_0 - h_*\|^2 \leq \|h_0 - h_t\|^2 = \|(h_0 - h_*) - t(h - h_*)\|^2.$$

Expanding the right-hand side by bilinearity and symmetry of the inner product,

$$\|h_0 - h_t\|^2 = \|h_0 - h_*\|^2 - 2t \langle h_0 - h_*, h - h_* \rangle + t^2 \|h - h_*\|^2.$$

Combining the last two displays and cancelling $\|h_0 - h_*\|^2$ gives

$$2t \langle h_0 - h_*, h - h_* \rangle \leq t^2 \|h - h_*\|^2.$$

Dividing by $2t > 0$,

$$\langle h_0 - h_*, h - h_* \rangle \leq \frac{t}{2} \|h - h_*\|^2 \quad \text{for all } t \in (0, 1],$$

and letting $t \rightarrow 0^+$ we conclude that $\langle h_0 - h_*, h - h_* \rangle \leq 0$, that is,

$$\langle w, h - h_* \rangle = \langle h_* - h_0, h - h_* \rangle \geq 0.$$

This proves the claim.

Step 3: the separating functional. Define $\psi: H \rightarrow \mathbb{R}$ by

$$\psi(u) = \langle w, u \rangle \quad \text{for } u \in H.$$

By Problem 11, ψ belongs to H^* ; that is, ψ is linear, and by the Cauchy-Schwarz Inequality $|\psi(u)| \leq \|w\| \|u\|$, so ψ is bounded and hence continuous with respect to the strong topology on H . Also $\psi \neq 0$, since $\psi(w) = \|w\|^2 > 0$.

By Step 2, for every $h \in K$,

$$\psi(h) - \psi(h_*) = \langle w, h - h_* \rangle \geq 0,$$

so that

$$\psi(h_*) \leq \inf_{h \in K} \psi(h).$$

On the other hand,

$$\psi(h_*) - \psi(h_0) = \langle w, h_* - h_0 \rangle = \langle w, w \rangle = \|w\|^2 > 0.$$

Therefore

$$\psi(h_0) = \psi(h_*) - \|w\|^2 < \psi(h_*) \leq \inf_{h \in K} \psi(h),$$

which is exactly the required strict inequality

$$\psi(h_0) < \inf_{h \in K} \psi(h).$$

Thus K and h_0 are separated by the closed hyperplane $\{u \in H : \psi(u) = c\}$ for any real number c with $\psi(h_0) < c \leq \inf_{h \in K} \psi(h)$; for instance $c = \psi(h_*)$ works, and indeed the choice $c = \psi(h_0) + \frac{1}{2}\|w\|^2$ separates h_0 and K strictly, since then $\psi(h_0) < c < \psi(h_*) \leq \inf_K \psi$. Observe that neither the Hyperplane Separation Lemma nor the Hahn-Banach Theorem was used: the entire construction rests on Proposition 2 together with the Cauchy-Schwarz Inequality.

16.3 Problems 16.15–16.21

Problem 16.15. Use Proposition 2 to prove the Krein-Milman Lemma in a Hilbert space.

Solution. The Krein-Milman Lemma asserts that a nonempty, compact, convex subset K of a locally convex topological vector space X has an extreme point. We prove this when $X = H$ is a Hilbert space regarded, as in Problem 14 of this chapter, as a locally convex topological vector space with respect to its strong topology. The only tool used is Proposition 2: if K is a nonempty closed convex subset of H and h_0 belongs to $H \sim K$, then there is exactly one vector of K closest to h_0 . So let K be a nonempty, compact, convex subset of H .

Recall that a vector x in a convex set K is called an extreme point of K provided that whenever $x = \lambda u + (1 - \lambda)v$ with $u, v \in K$ and $0 < \lambda < 1$, then $u = v$ (and hence $u = v = x$).

Step 1: a point of maximal norm exists. The function $h \mapsto \|h\|$ is continuous on H and K is a nonempty compact subset of H . By the Extreme Value Theorem this function attains a maximum on K : there is a vector $h_* \in K$ with

$$\|h_*\| = M, \quad \text{where } M = \max_{h \in K} \|h\|.$$

If $M = 0$ then $K = \{0\}$ and 0 is trivially an extreme point of K . So assume $M > 0$.

Step 2: the key consequence of Proposition 2. Let

$$D = \{h \in H : \|h\| \leq M\}$$

be the closed ball of radius M about the origin. Then D is nonempty, convex (by the triangle inequality and homogeneity of the norm) and closed (by continuity of the norm). We claim:

If $u, v \in D$ satisfy $\|u\| = \|v\| = M$ and $\|u + v\| = 2M$, then $u = v$.

Indeed, put $h_0 = u + v$. Then $\|h_0\| = 2M > M$, so h_0 belongs to $H \sim D$. For every $h \in D$,

$$\|h_0 - h\| \geq \|h_0\| - \|h\| \geq 2M - M = M,$$

so $\text{dist}(h_0, D) \geq M$. On the other hand u and v belong to D and

$$\|h_0 - u\| = \|v\| = M, \quad \|h_0 - v\| = \|u\| = M.$$

Therefore $\text{dist}(h_0, D) = M$ and both u and v are vectors in D that are closest to h_0 . Proposition 2 asserts that there is exactly one such vector. Hence $u = v$, and the claim is proved.

Step 3: h_* is an extreme point of K . Suppose

$$h_* = \lambda u + (1 - \lambda)v, \quad u, v \in K, \quad 0 < \lambda < 1.$$

Define $g: [0, 1] \rightarrow \mathbb{R}$ by $g(t) = \|tu + (1 - t)v\|$. For $t \in [0, 1]$ the vector $tu + (1 - t)v$ belongs to K , since K is convex, and hence $g(t) \leq M$ by the maximality of M . Moreover g is a convex function: for $s, t \in [0, 1]$ and $0 \leq \theta \leq 1$,

$$\theta(tu + (1 - t)v) + (1 - \theta)(su + (1 - s)v) = (\theta t + (1 - \theta)s)u + (1 - \theta t - (1 - \theta)s)v,$$

so $g(\theta t + (1 - \theta)s) \leq \theta g(t) + (1 - \theta)g(s)$ by the triangle inequality and homogeneity of the norm.

Now $g(\lambda) = \|h_*\| = M$ and $0 < \lambda < 1$. We show $g \equiv M$ on $[0, 1]$. First, convexity gives

$$M = g(\lambda) \leq (1 - \lambda)g(0) + \lambda g(1) \leq (1 - \lambda)M + \lambda M = M,$$

and since $g(0) \leq M$ and $g(1) \leq M$, equality forces $g(0) = g(1) = M$; that is, $\|v\| = \|u\| = M$. Next, let $t \in (0, 1)$ be arbitrary. If $t \geq \lambda$, write $\lambda = (1 - s) \cdot 0 + s \cdot t$ with $s = \lambda/t \in (0, 1]$; convexity gives

$$M = g(\lambda) \leq (1 - s)g(0) + s g(t) = (1 - s)M + s g(t),$$

so $g(t) \geq M$ and hence $g(t) = M$. If $t \leq \lambda$, write $\lambda = (1 - s) \cdot 1 + s \cdot t$ with $s = (1 - \lambda)/(1 - t) \in (0, 1]$ and argue in the same way using $g(1) = M$. Thus $g \equiv M$ on $[0, 1]$.

In particular $g(1/2) = M$, that is $\|u + v\| = 2M$, while $\|u\| = \|v\| = M$ so that $u, v \in D$. By Step 2, $u = v$. Therefore h_* is an extreme point of K , and K has an extreme point.

Problem 16.16. Show that neither ℓ^1 , ℓ^∞ , $L^1[a, b]$ nor $L^\infty[a, b]$ is Hilbertable.

Solution. Recall the definition in the Remark of Section 16.1: a Banach space X is Hilbertable provided there is an inner product on X whose induced norm is equivalent to the given norm.

Step 1: two general observations.

(i) If X is Hilbertable, then every bounded sequence in X has a weakly convergent subsequence.

Indeed, let $\langle \cdot, \cdot \rangle$ be an inner product on X whose induced norm $\| \cdot \|'$ is equivalent to the given norm $\| \cdot \|$, so there are constants $c, C > 0$ with $c\|x\| \leq \|x\|' \leq C\|x\|$ for all $x \in X$. Since X is complete with respect to $\| \cdot \|$ and the two norms are equivalent, X is complete with respect to $\| \cdot \|'$; thus $H = (X, \langle \cdot, \cdot \rangle)$ is a Hilbert space. A linear functional on X is bounded with respect to $\| \cdot \|$ if and only if it is bounded with respect to $\| \cdot \|'$, so the two norms give X the same dual space and therefore exactly the same weakly convergent sequences; and a set is bounded for one norm if and only if it is bounded for the other. Now let $\{x_n\}$ be a bounded sequence in X . It is bounded in H , so by Theorem 6 it has a subsequence converging weakly in H , and hence converging weakly in X .

(ii) If X is Hilbertable and Y is a closed linear subspace of X , then Y is Hilbertable; and if Z is a normed linear space that is isometrically isomorphic (indeed, merely isomorphic) to Y , then Z is Hilbertable.

Indeed, with $\langle \cdot, \cdot \rangle$ as above, the restriction of $\langle \cdot, \cdot \rangle$ to $Y \times Y$ is an inner product on Y whose induced norm is the restriction of $\| \cdot \|'$, which is equivalent to the restriction of $\| \cdot \|$; and Y , being a closed subspace of a Banach space, is a Banach space. If $T: Z \rightarrow Y$ is a linear isomorphism of normed linear spaces, then $\langle z, w \rangle_Z := \langle Tz, Tw \rangle$ is an inner product on Z whose induced norm $z \mapsto \|Tz\|'$ is equivalent to $z \mapsto \|Tz\|$ and hence to $\|z\|_Z$.

Step 2: ℓ^1 is not Hilbertable. We do not need to know all of $(\ell^1)^*$; we only need a supply of bounded linear functionals, and these we exhibit directly. For $\psi = \{\psi_k\} \in \ell^\infty$ define $\Lambda_\psi(x) = \sum_{k=1}^\infty \psi_k x_k$ for $x = \{x_k\} \in \ell^1$. The series converges absolutely since $\sum_k |\psi_k x_k| \leq \|\psi\|_\infty \sum_k |x_k| = \|\psi\|_\infty \|x\|_1$, and the same estimate shows that the clearly linear functional Λ_ψ is bounded on ℓ^1 . Let e_n denote the sequence whose n th term is 1 and whose other terms are 0; then $\|e_n\|_1 = 1$, so $\{e_n\}$ is bounded. Suppose some subsequence $\{e_{n_j}\}$ converged weakly to $x = \{x_k\} \in \ell^1$. Testing against the coordinate functional Λ_ψ with $\psi = e_k$ (viewed as an element of ℓ^∞) gives $x_k = \lim_{j \rightarrow \infty} (e_{n_j})_k = 0$, since $(e_{n_j})_k = 0$ once $n_j > k$. As this holds for every k , $x = 0$. But testing against $\psi = (1, 1, 1, \dots) \in \ell^\infty$ gives $\Lambda_\psi(e_{n_j}) = 1$ for all j , while $\Lambda_\psi(0) = 0$. This contradiction shows that $\{e_n\}$ has no weakly convergent subsequence, so by Step 1(i) ℓ^1 is not Hilbertable.

Step 3: ℓ^∞ is not Hilbertable. For each n let $x^{(n)} \in \ell^\infty$ be the sequence whose first n terms equal 1 and whose remaining terms are 0; then $\|x^{(n)}\|_\infty = 1$. Suppose some subsequence $\{x^{(n_j)}\}$ converged weakly to $x = \{x_k\} \in \ell^\infty$. Each coordinate functional $\psi_k(y) = y_k$ is bounded on ℓ^∞ , and $(x^{(n_j)})_k = 1$ once $n_j \geq k$; hence $x_k = 1$ for every k , that is, $x = (1, 1, 1, \dots)$. Now let $c \subseteq \ell^\infty$ be the linear subspace of convergent sequences and define ψ_0 on c by $\psi_0(y) = \lim_{k \rightarrow \infty} y_k$. Then ψ_0 is linear and $|\psi_0(y)| \leq \|y\|_\infty$, so by the Hahn-Banach Theorem ψ_0 has an extension to a bounded linear functional ψ on ℓ^∞ . Each $x^{(n)}$ belongs to c with limit 0, so $\psi(x^{(n_j)}) = 0$ for all j , whereas $\psi(x) = \psi_0((1, 1, \dots)) = 1$. This contradicts weak convergence. Hence the bounded sequence $\{x^{(n)}\}$ has no weakly convergent subsequence, and by Step 1(i) ℓ^∞ is not Hilbertable.

Step 4: $L^1[a, b]$ is not Hilbertable. We may assume $a < b$. Again we need only a supply of bounded linear functionals: for $g \in L^\infty[a, b]$ the product fg is integrable over $[a, b]$ whenever $f \in L^1[a, b]$, with $\int_a^b |fg| \leq \|g\|_\infty \|f\|_1$ (this is Holder's Inequality for the conjugate pair $p = 1$, $q = \infty$), so $f \mapsto \int_a^b fg$ is a bounded linear functional on $L^1[a, b]$. For each natural number n with $1/n < b - a$ put

$$f_n = n \cdot \chi_{[a, a+1/n]},$$

so that $\|f_n\|_1 = 1$ and $\{f_n\}$ is bounded in $L^1[a, b]$. Suppose a subsequence $\{f_{n_j}\}$ converged weakly to $f \in L^1[a, b]$. Define $F(x) = \int_a^x f$ for $x \in [a, b]$; since f is integrable over $[a, b]$, F is continuous on $[a, b]$ (indeed absolutely continuous, by the continuity of integration established in Chapter 4), and $F(a) = 0$. Fix $0 < \delta < b - a$ and take $g = \chi_{[a+\delta, b]} \in L^\infty[a, b]$. For all j with $1/n_j < \delta$ we have

$\int_a^b f_{n_j} g = 0$, and therefore, by weak convergence,

$$F(b) - F(a + \delta) = \int_{a+\delta}^b f = \lim_{j \rightarrow \infty} \int_a^b f_{n_j} g = 0.$$

Thus $F(a + \delta) = F(b)$ for every $\delta \in (0, b - a)$; letting $\delta \rightarrow 0^+$ and using the continuity of F at a gives $F(b) = F(a) = 0$. On the other hand, taking $g \equiv 1 \in L^\infty[a, b]$ we get $\int_a^b f_{n_j} \cdot 1 = 1$ for every j , so $F(b) = \int_a^b f = 1$. This contradiction shows $\{f_n\}$ has no weakly convergent subsequence, so $L^1[a, b]$ is not Hilbertable.

Step 5: $L^\infty[a, b]$ is not Hilbertable. Partition $[a, b]$ into countably many pairwise disjoint measurable sets of positive measure, for example

$$E_j = \left(a + \frac{b-a}{j+1}, a + \frac{b-a}{j} \right] \quad \text{for } j \geq 1, \quad \text{together with } \{a\} \text{ adjoined to } E_1.$$

Define $T: \ell^\infty \rightarrow L^\infty[a, b]$ by

$$T(\{c_j\}) = \sum_{j=1}^{\infty} c_j \chi_{E_j}.$$

The right-hand side is a well-defined measurable function on $[a, b]$ (on each E_j it equals the constant c_j), and since each E_j has positive measure its essential supremum of absolute values is exactly $\sup_j |c_j|$. Thus T is linear and $\|T(c)\|_\infty = \|c\|_\infty$, so T is a linear isometry of ℓ^∞ onto its image $Y = T(\ell^\infty)$. Since ℓ^∞ is complete and T is an isometry, Y is a complete, hence closed, linear subspace of $L^\infty[a, b]$. If $L^\infty[a, b]$ were Hilbertable, then by Step 1(ii) Y would be Hilbertable and hence so would ℓ^∞ , contradicting Step 3. Therefore $L^\infty[a, b]$ is not Hilbertable.

Problem 16.17. Prove Proposition 7. [Proposition 7: Let $\{u_n\} \rightarrow u$ weakly in the Hilbert space H . Then $\{u_n\}$ is bounded and

$$\|u\| \leq \liminf \|u_n\|.$$

Moreover, if $\{v_n\} \rightarrow v$ strongly in H , then

$$\lim_{n \rightarrow \infty} \langle u_n, v_n \rangle = \langle u, v \rangle.$$

]

Solution. By the Riesz-Frechet Representation Theorem, weak convergence $\{u_n\} \rightarrow u$ in H means precisely that

$$\lim_{n \rightarrow \infty} \langle h, u_n \rangle = \langle h, u \rangle \quad \text{for all } h \in H.$$

Boundedness. For each natural number n let $\psi_n = T(u_n) \in H^*$ be the functional furnished by the Riesz-Frechet Representation Theorem, that is,

$$\psi_n(h) = \langle u_n, h \rangle \quad \text{for all } h \in H.$$

Fix $h \in H$. By weak convergence and the symmetry of the inner product, the sequence of real numbers $\{\psi_n(h)\} = \{\langle h, u_n \rangle\}$ converges, to $\langle h, u \rangle$; a convergent sequence of real numbers is bounded, so there is a constant $M_h \geq 0$ with $|\psi_n(h)| \leq M_h$ for all n . Thus the family $\{\psi_n\} \subseteq \mathcal{L}(H, \mathbb{R})$ is pointwise bounded on the Banach space H . By the Uniform Boundedness Principle there is a constant $M \geq 0$ with $\|\psi_n\| \leq M$ for all n . But the Riesz-Frechet Representation Theorem asserts that T is an isometry, so $\|u_n\| = \|\psi_n\| \leq M$ for all n . Hence $\{u_n\}$ is bounded.

The norm inequality. If $u = 0$ the inequality $\|u\| \leq \liminf \|u_n\|$ is trivial, since norms are nonnegative. So assume $u \neq 0$. Taking $h = u$ in the definition of weak convergence,

$$\|u\|^2 = \langle u, u \rangle = \lim_{n \rightarrow \infty} \langle u, u_n \rangle.$$

By the Cauchy-Schwarz Inequality, $\langle u, u_n \rangle \leq |\langle u, u_n \rangle| \leq \|u\| \cdot \|u_n\|$ for every n . Since the sequence $\{\langle u, u_n \rangle\}$ converges, its limit equals its limit inferior, and the limit inferior is monotone, so

$$\|u\|^2 = \liminf_{n \rightarrow \infty} \langle u, u_n \rangle \leq \liminf_{n \rightarrow \infty} (\|u\| \cdot \|u_n\|) = \|u\| \cdot \liminf_{n \rightarrow \infty} \|u_n\|,$$

where the last equality holds because $\|u\| > 0$ is a constant. Dividing by $\|u\| > 0$ gives $\|u\| \leq \liminf \|u_n\|$.

The limit of the inner products. Suppose $\{v_n\} \rightarrow v$ strongly in H , that is, $\lim_{n \rightarrow \infty} \|v_n - v\| = 0$. Let $M \geq 0$ be a bound for $\{\|u_n\|\}$, as provided above. For each n , by bilinearity,

$$\langle u_n, v_n \rangle - \langle u, v \rangle = \langle u_n, v_n - v \rangle + (\langle u_n, v \rangle - \langle u, v \rangle),$$

and therefore, by the triangle inequality and the Cauchy-Schwarz Inequality,

$$|\langle u_n, v_n \rangle - \langle u, v \rangle| \leq \|u_n\| \cdot \|v_n - v\| + |\langle v, u_n \rangle - \langle v, u \rangle| \leq M\|v_n - v\| + |\langle v, u_n \rangle - \langle v, u \rangle|.$$

The first term tends to 0 since $\{v_n\} \rightarrow v$ strongly and M is a constant; the second tends to 0 by the weak convergence of $\{u_n\}$ to u , applied with $h = v$. Hence

$$\lim_{n \rightarrow \infty} \langle u_n, v_n \rangle = \langle u, v \rangle.$$

Problem 16.18. Let H be an inner product space. Show that since H is a dense subset of a Banach space X whose norm restricts to the norm induced by the inner product on H , the inner product on H extends to X and induces the norm on X . Thus inner product spaces have Hilbert space completions.

Solution. Let H be an inner product space with inner product $\langle \cdot, \cdot \rangle_H$ and induced norm $\|\cdot\|_H$. Since $\|\cdot\|_H$ is a norm on the linear space H (Proposition 1), the normed linear space H has a completion: there is a Banach space X containing H as a dense linear subspace and whose norm $\|\cdot\|$ restricts on H to $\|\cdot\|_H$. We show that $\|\cdot\|$ is induced by an inner product on X that restricts on H to $\langle \cdot, \cdot \rangle_H$; then X , being a Banach space whose norm is induced by an inner product, is a Hilbert space, so H has a Hilbert space completion.

Step 1: the parallelogram identity holds on X . Define $F: X \times X \rightarrow \mathbb{R}$ by

$$F(x, y) = \|x - y\|^2 + \|x + y\|^2 - 2\|x\|^2 - 2\|y\|^2.$$

The maps $(x, y) \mapsto x + y$ and $(x, y) \mapsto x - y$ from $X \times X$ to X are continuous, since $\|(x' \pm y') - (x \pm y)\| \leq \|x' - x\| + \|y' - y\|$, and $z \mapsto \|z\|^2$ is continuous on X by the reverse triangle inequality. Hence F is continuous on $X \times X$.

If $u, v \in H$, then $\|u\| = \|u\|_H$ and likewise for $u \pm v$ (note $u \pm v \in H$, a linear subspace), so the Parallelogram Identity of Section 16.1, valid in the inner product space H , gives $F(u, v) = 0$. Thus F vanishes on the set $H \times H$, which is dense in $X \times X$ because H is dense in X . Since F is continuous and vanishes on a dense subset, F vanishes identically: for all $x, y \in X$,

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Step 2: the norm on X is induced by an inner product. By the Jordan-von Neumann Theorem (Problem 7 of this chapter), a norm on a linear space is induced by an inner product if and only if the parallelogram identity holds. By Step 1, therefore, there is an inner product $\langle \cdot, \cdot \rangle_X$ on X whose induced norm is $\|\cdot\|$.

Step 3: $\langle \cdot, \cdot \rangle_X$ extends $\langle \cdot, \cdot \rangle_H$. Apply the Polarization Identity (Problem 6 of this chapter) in the inner product space $(X, \langle \cdot, \cdot \rangle_X)$, whose induced norm is $\|\cdot\|$: for all $x, y \in X$,

$$\langle x, y \rangle_X = \frac{1}{4} [\|x + y\|^2 - \|x - y\|^2].$$

Now let $u, v \in H$. Then $u + v$ and $u - v$ belong to H and the norm of X restricts to $\|\cdot\|_H$ on H , so

$$\langle u, v \rangle_X = \frac{1}{4} [\|u + v\|_H^2 - \|u - v\|_H^2] = \langle u, v \rangle_H,$$

the last equality being the Polarization Identity applied in the inner product space H . Thus $\langle \cdot, \cdot \rangle_X$ restricts on $H \times H$ to $\langle \cdot, \cdot \rangle_H$, that is, the inner product on H extends to X , and by Step 2 the extension induces the norm on X .

Step 4: uniqueness of the extension, and the conclusion. If $\langle \cdot, \cdot \rangle'$ is any inner product on X that extends $\langle \cdot, \cdot \rangle_H$ and induces $\|\cdot\|$, then the Polarization Identity in $(X, \langle \cdot, \cdot \rangle')$ forces $\langle x, y \rangle' = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2] = \langle x, y \rangle_X$ for all $x, y \in X$. So the extension is unique. (Alternatively, an inner product is continuous in each variable by the Cauchy-Schwarz Inequality, so any two continuous extensions agreeing on the dense set $H \times H$ agree on $X \times X$.)

Finally, X is complete with respect to $\|\cdot\|$, and $\|\cdot\|$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle_X$; by the definition in Section 16.1, X is a Hilbert space. Since H sits isometrically as a dense subspace of X with its inner product preserved, every inner product space has a Hilbert space completion.

Problem 16.19. Show that an orthonormal subset of a separable Hilbert space H must be countable.

Solution. Let S be an orthonormal subset of H ; thus every vector in S is a unit vector and every two distinct vectors in S are orthogonal.

First observe that distinct vectors of S are far apart. Indeed, let $\varphi, \psi \in S$ with $\varphi \neq \psi$. Then $\langle \varphi, \psi \rangle = 0$ and $\|\varphi\| = \|\psi\| = 1$, so by bilinearity and symmetry

$$\|\varphi - \psi\|^2 = \langle \varphi - \psi, \varphi - \psi \rangle = \|\varphi\|^2 - 2\langle \varphi, \psi \rangle + \|\psi\|^2 = 1 - 0 + 1 = 2,$$

that is, $\|\varphi - \psi\| = \sqrt{2}$.

Consequently the open balls

$$B(\varphi) = \left\{ h \in H : \|h - \varphi\| < \frac{\sqrt{2}}{2} \right\}, \quad \varphi \in S,$$

are pairwise disjoint: if $h \in B(\varphi) \cap B(\psi)$ with $\varphi, \psi \in S$, then by the triangle inequality

$$\|\varphi - \psi\| \leq \|\varphi - h\| + \|h - \psi\| < \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2},$$

which by the preceding paragraph is impossible unless $\varphi = \psi$.

Now use separability. Since H is separable, there is a countable subset D of H that is dense in H . For each $\varphi \in S$ the ball $B(\varphi)$ is a nonempty open subset of H , so it contains a point of D ; using this, choose for each $\varphi \in S$ a point $d(\varphi) \in D \cap B(\varphi)$. The map $\varphi \mapsto d(\varphi)$ from S into D is one-to-one: if $d(\varphi) = d(\psi)$, then this common point lies in $B(\varphi) \cap B(\psi)$, and by the disjointness just established $\varphi = \psi$.

Thus S is in one-to-one correspondence with a subset of the countable set D , and a subset of a countable set is countable. Therefore S is countable.

Problem 16.20. Let $\{\varphi_k\}$ be an orthonormal sequence in a Hilbert space H . Show that $\{\varphi_k\}$ converges weakly to 0 in H .

Solution. By the Riesz-Frechet Representation Theorem, every bounded linear functional on H is of the form $u \mapsto \langle h, u \rangle$ for a (unique) $h \in H$, so that

$$\{\varphi_k\} \rightarrow 0 \text{ weakly in } H \quad \text{if and only if} \quad \lim_{k \rightarrow \infty} \langle h, \varphi_k \rangle = \langle h, 0 \rangle = 0 \quad \text{for all } h \in H.$$

So let h be an arbitrary vector in H . Since $\{\varphi_k\}$ is an orthonormal sequence in H , Bessel's Inequality gives

$$\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 \leq \|h\|^2 < \infty.$$

Thus the series of nonnegative terms $\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2$ converges, and therefore its terms tend to 0:

$$\lim_{k \rightarrow \infty} \langle \varphi_k, h \rangle^2 = 0, \quad \text{so} \quad \lim_{k \rightarrow \infty} \langle h, \varphi_k \rangle = 0,$$

where we used the symmetry of the inner product.

Since $h \in H$ was arbitrary, $\lim_{k \rightarrow \infty} \langle h, \varphi_k \rangle = 0$ for every $h \in H$, and hence $\{\varphi_k\} \rightarrow 0$ weakly in H .

Observe that $\{\varphi_k\}$ certainly does not converge strongly to 0, since $\|\varphi_k\| = 1$ for every k ; this exhibits the strict inequality that can occur in the estimate $\|u\| \leq \liminf \|u_n\|$ of Proposition 7.

Problem 16.21. Let $\{\varphi_k\}$ be an orthonormal basis for the separable Hilbert space H . Show that $\{u_n\} \rightharpoonup u$ in H if and only if for each k , $\lim_{n \rightarrow \infty} \langle u_n, \varphi_k \rangle = \langle u, \varphi_k \rangle$.

Solution. Throughout, $\{u_n\} \rightharpoonup u$ denotes weak convergence in H , and, as recorded in Section 16.2 as a consequence of the Riesz-Frechet Representation Theorem, $\{u_n\} \rightharpoonup u$ in H if and only if $\lim_{n \rightarrow \infty} \langle h, u_n \rangle = \langle h, u \rangle$ for every $h \in H$.

The forward implication. Assume $\{u_n\} \rightharpoonup u$ weakly in H . Fix a natural number k and take $h = \varphi_k$ in the definition of weak convergence. By the symmetry of the inner product,

$$\lim_{n \rightarrow \infty} \langle u_n, \varphi_k \rangle = \lim_{n \rightarrow \infty} \langle \varphi_k, u_n \rangle = \langle \varphi_k, u \rangle = \langle u, \varphi_k \rangle.$$

This holds for every k , as required. (No use of the basis property was needed here.)

The converse implication. As stated, the converse is false; one must also assume that $\{u_n\}$ is bounded, and a counterexample is given at the end. We therefore prove the converse for bounded sequences. This is the intended reading: by Proposition 7 a weakly convergent sequence is bounded, so the sharp statement is that, for a bounded sequence $\{u_n\}$, weak convergence to u is equivalent to convergence of every coordinate.

So assume $\{u_n\}$ is bounded, say $\|u_n\| \leq M$ for all n , and assume $\lim_{n \rightarrow \infty} \langle u_n, \varphi_k \rangle = \langle u, \varphi_k \rangle$ for each k . Let $h \in H$ and let $\epsilon > 0$. Since $\{\varphi_k\}$ is an orthonormal basis for H ,

$$h = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k,$$

the series converging strongly in H . Hence, setting $h_N = \sum_{k=1}^N \langle \varphi_k, h \rangle \varphi_k$, we may choose a natural number N for which $\|h - h_N\| < \epsilon$.

For every n , by bilinearity and the triangle inequality,

$$|\langle u_n - u, h \rangle| \leq |\langle u_n - u, h_N \rangle| + |\langle u_n - u, h - h_N \rangle|.$$

For the second term, the Cauchy-Schwarz Inequality gives

$$|\langle u_n - u, h - h_N \rangle| \leq \|u_n - u\| \cdot \|h - h_N\| \leq (M + \|u\|) \epsilon.$$

For the first term, expanding h_N by bilinearity,

$$|\langle u_n - u, h_N \rangle| = \left| \sum_{k=1}^N \langle \varphi_k, h \rangle (\langle u_n, \varphi_k \rangle - \langle u, \varphi_k \rangle) \right| \leq \sum_{k=1}^N |\langle \varphi_k, h \rangle| \cdot |\langle u_n, \varphi_k \rangle - \langle u, \varphi_k \rangle|.$$

This is a finite sum, of N terms, in which N and the numbers $|\langle \varphi_k, h \rangle|$ do not depend on n , and each factor $|\langle u_n, \varphi_k \rangle - \langle u, \varphi_k \rangle|$ tends to 0 as $n \rightarrow \infty$ by hypothesis. Hence

$$\lim_{n \rightarrow \infty} |\langle u_n - u, h_N \rangle| = 0.$$

Combining the two estimates,

$$\limsup_{n \rightarrow \infty} |\langle h, u_n \rangle - \langle h, u \rangle| = \limsup_{n \rightarrow \infty} |\langle u_n - u, h \rangle| \leq 0 + (M + \|u\|) \epsilon.$$

Since $\epsilon > 0$ was arbitrary and $M + \|u\|$ does not depend on ϵ , we conclude that $\lim_{n \rightarrow \infty} \langle h, u_n \rangle = \langle h, u \rangle$. As $h \in H$ was arbitrary, $\{u_n\} \rightharpoonup u$ weakly in H .

Why boundedness cannot be dropped. Take $u_n = n\varphi_n$ and $u = 0$. For a fixed k we have $\langle u_n, \varphi_k \rangle = n\langle \varphi_n, \varphi_k \rangle = 0$ whenever $n \neq k$, so $\langle u_n, \varphi_k \rangle = 0$ for all $n > k$ and therefore

$$\lim_{n \rightarrow \infty} \langle u_n, \varphi_k \rangle = 0 = \langle 0, \varphi_k \rangle \quad \text{for every } k.$$

Yet $\|u_n\| = n$, so $\{u_n\}$ is unbounded, and by Proposition 7 an unbounded sequence cannot converge weakly to any vector of H . Thus the coordinatewise condition alone does not imply weak convergence, and the boundedness hypothesis used above cannot be dropped. (Here we used that H is infinite dimensional, which is implicit in the existence of an orthonormal basis $\{\varphi_k\}$ indexed by the natural numbers. If H were finite dimensional the two conditions would indeed be equivalent without any boundedness assumption, since coordinate convergence in a finite orthonormal basis is strong convergence.)

16.4 Problems 16.22–16.28

Problem 16.22. Show that any two infinite dimensional separable Hilbert spaces are isometrically isomorphic and that any such isomorphism preserves the inner product.

Solution. Let H_1 and H_2 be infinite dimensional separable Hilbert spaces. By Theorem 11, H_1 possesses an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ and H_2 possesses an orthonormal basis $\{\psi_k\}_{k=1}^{\infty}$; since each space is infinite dimensional, each of these orthonormal sequences is genuinely infinite (were $\{\varphi_k\}$ finite, say $\varphi_1, \dots, \varphi_n$, then by (10) every $h \in H_1$ would be a linear combination of $\varphi_1, \dots, \varphi_n$ and H_1 would be finite dimensional).

Construction of the isomorphism. Let $h \in H_1$. By Bessel's Inequality,

$$\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 \leq \|h\|^2 < \infty.$$

For natural numbers n and p , the General Pythagorean Identity gives

$$\left\| \sum_{k=n+1}^{n+p} \langle \varphi_k, h \rangle \psi_k \right\|^2 = \sum_{k=n+1}^{n+p} \langle \varphi_k, h \rangle^2,$$

and the right-hand side is small for large n because the series above converges. Hence the partial sums of $\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \psi_k$ form a Cauchy sequence in H_2 , and H_2 is complete, so the series converges strongly in H_2 . Define

$$T(h) = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \psi_k \quad \text{for all } h \in H_1.$$

Since $h \mapsto \langle \varphi_k, h \rangle$ is linear for each k , and since strong limits respect linear combinations, T is linear. We check that T is an isometry. Fix $h \in H_1$ and put $T_n(h) = \sum_{k=1}^n \langle \varphi_k, h \rangle \psi_k$. By the General Pythagorean Identity, $\|T_n(h)\|^2 = \sum_{k=1}^n \langle \varphi_k, h \rangle^2$. Since $T_n(h) \rightarrow T(h)$ strongly and the norm is continuous,

$$\|T(h)\|^2 = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2.$$

On the other hand, the right-hand side equals $\|h\|^2$: indeed, setting $h_n = \sum_{k=1}^n \langle \varphi_k, h \rangle \varphi_k$, the General Pythagorean Identity gives $\|h_n\|^2 = \sum_{k=1}^n \langle \varphi_k, h \rangle^2$, while $\{h_n\} \rightarrow h$ strongly by (10) since $\{\varphi_k\}$ is an orthonormal basis for H_1 ; the norm is continuous (it is Lipschitz, by the triangle inequality), so $\|h_n\|^2 \rightarrow \|h\|^2$ and hence $\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 = \|h\|^2$. (This is Parseval's Identity, Problem 24.) Thus $\|T(h)\| = \|h\|$ for all $h \in H_1$; in particular T is one-to-one, since $T(h) = 0$ forces $\|h\| = 0$.

We check that T is onto. Let $g \in H_2$. By Bessel's Inequality in H_2 , $\sum_{k=1}^{\infty} \langle \psi_k, g \rangle^2 \leq \|g\|^2 < \infty$, so exactly the argument used above (with the roles of the two bases interchanged) shows that

$$h = \sum_{k=1}^{\infty} \langle \psi_k, g \rangle \varphi_k$$

converges strongly in H_1 . By the continuity of the inner product and the orthonormality of $\{\varphi_k\}$, $\langle \varphi_j, h \rangle = \langle \psi_j, g \rangle$ for every j . Therefore

$$T(h) = \sum_{k=1}^{\infty} \langle \psi_k, g \rangle \psi_k = g,$$

the last equality because $\{\psi_k\}$ is an orthonormal basis for H_2 . Hence T maps H_1 onto H_2 , and T is a linear isometry of H_1 onto H_2 : the two spaces are isometrically isomorphic.

Every such isomorphism preserves the inner product. Let S be any linear isometry of H_1 onto H_2 (indeed any linear isometry at all). By the Polarization Identity for the inner product (Problem 6), for all $u, v \in H_1$,

$$\langle S(u), S(v) \rangle = \frac{1}{4} [\|S(u) + S(v)\|^2 - \|S(u) - S(v)\|^2].$$

Since S is linear, $S(u) + S(v) = S(u + v)$ and $S(u) - S(v) = S(u - v)$, and since S is an isometry, $\|S(u + v)\| = \|u + v\|$ and $\|S(u - v)\| = \|u - v\|$. Therefore

$$\langle S(u), S(v) \rangle = \frac{1}{4} [\|u + v\|^2 - \|u - v\|^2] = \langle u, v \rangle,$$

again by the Polarization Identity, now in H_1 . Thus every isometric isomorphism between the two spaces preserves the inner product.

Problem 16.23. Let H be a Hilbert space and V a closed separable subspace of H for which $\{\varphi_k\}$ is an orthonormal basis. Show that the orthogonal projection of H onto V , P , is given by

$$P(h) = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k \quad \text{for all } h \in H.$$

Solution. First observe that V , being a closed subspace of the complete space H , is itself complete, and the inner product of H restricted to V makes V a Hilbert space. The hypothesis is that $\{\varphi_k\}$ is an orthonormal basis for this Hilbert space V , that is, by (10),

$$v = \sum_{k=1}^{\infty} \langle \varphi_k, v \rangle \varphi_k \quad \text{for all } v \in V,$$

the convergence being strong in V , hence also strong in H since the norms agree.

Let $h \in H$. By Theorem 3, $H = V \oplus V^\perp$, and by the definition of the orthogonal projection P of H onto V along $V^\perp$ we have the decomposition

$$h = P(h) + [h - P(h)], \quad P(h) \in V, \quad h - P(h) \in V^\perp.$$

Fix a natural number k . Since $\varphi_k \in V$ and $h - P(h) \in V^\perp$,

$$\langle \varphi_k, h - P(h) \rangle = 0,$$

and therefore, by the additivity of the inner product in its second argument,

$$\langle \varphi_k, h \rangle = \langle \varphi_k, P(h) \rangle.$$

Now apply the basis expansion displayed above to the vector $v = P(h) \in V$:

$$P(h) = \sum_{k=1}^{\infty} \langle \varphi_k, P(h) \rangle \varphi_k = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k,$$

where the second equality uses the identity $\langle \varphi_k, P(h) \rangle = \langle \varphi_k, h \rangle$ just established, term by term. The series therefore converges strongly in H (indeed in V) and its sum is $P(h)$. Since $h \in H$ was arbitrary, this is the asserted formula.

(One may also read the convergence off directly: by Bessel's Inequality $\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 \leq \|h\|^2 < \infty$, so by Proposition 9 the series $\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k$ converges strongly to a vector h_* , which lies in V because V is closed and each partial sum lies in V ; moreover $h - h_*$ is orthogonal to every φ_k by Proposition 9, hence, by the continuity of the inner product, to every vector in the closed linear span of $\{\varphi_k\}$, which is V . Thus $h = h_* + (h - h_*)$ with $h_* \in V$ and $h - h_* \in V^\perp$; by the uniqueness of the decomposition of h in $H = V \oplus V^\perp$, $h_* = P(h)$.)

Problem 16.24. (Parseval's Identities) Let $\{\varphi_k\}$ be an orthonormal basis for a Hilbert space H . Verify that

$$\|h\|^2 = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 \quad \text{for all } h \in H.$$

Also verify that

$$\langle u, v \rangle = \sum_{k=1}^{\infty} a_k \cdot b_k \quad \text{for all } u, v \in H,$$

where, for each natural number k , $a_k = \langle u, \varphi_k \rangle$ and $b_k = \langle v, \varphi_k \rangle$.

Solution. Throughout we use that $\{\varphi_k\}$ is an orthonormal basis, that is, by (10),

$$h = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k \quad \text{for all } h \in H,$$

the convergence being strong in H .

The first identity. Fix $h \in H$ and, for a natural number n , set

$$h_n = \sum_{k=1}^n \langle \varphi_k, h \rangle \varphi_k.$$

Since $\varphi_1, \dots, \varphi_n$ are orthonormal, the General Pythagorean Identity gives

$$\|h_n\|^2 = \sum_{k=1}^n \langle \varphi_k, h \rangle^2.$$

By (10), $\{h_n\} \rightarrow h$ strongly in H . The norm is a continuous function on H (it is Lipschitz, by the triangle inequality), so $\|h_n\| \rightarrow \|h\|$ and hence $\|h_n\|^2 \rightarrow \|h\|^2$. Therefore the partial sums $\sum_{k=1}^n \langle \varphi_k, h \rangle^2$ converge to $\|h\|^2$, that is,

$$\|h\|^2 = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2.$$

The second identity. Let $u, v \in H$ and put $a_k = \langle u, \varphi_k \rangle = \langle \varphi_k, u \rangle$ and $b_k = \langle v, \varphi_k \rangle = \langle \varphi_k, v \rangle$, the two expressions agreeing by the symmetry of the inner product. We first note that the series $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely: by the Cauchy-Schwarz Inequality in Euclidean space, for every n ,

$$\sum_{k=1}^n |a_k b_k| \leq \left(\sum_{k=1}^n a_k^2 \right)^{1/2} \left(\sum_{k=1}^n b_k^2 \right)^{1/2} \leq \|u\| \cdot \|v\|,$$

the last inequality by Bessel's Inequality; so the increasing sequence of partial sums of $\sum |a_k b_k|$ is bounded and hence convergent.

Now define $u_n = \sum_{k=1}^n a_k \varphi_k$ and $v_n = \sum_{k=1}^n b_k \varphi_k$. By orthonormality and the bilinearity of the inner product,

$$\langle u_n, v_n \rangle = \sum_{k=1}^n \sum_{j=1}^n a_k b_j \langle \varphi_k, \varphi_j \rangle = \sum_{k=1}^n a_k b_k.$$

By (10), $\{u_n\} \rightarrow u$ and $\{v_n\} \rightarrow v$ strongly in H ; in particular $\{\|v_n\|\}$ is a bounded sequence, say $\|v_n\| \leq M$ for all n . By the Cauchy-Schwarz Inequality,

$$|\langle u_n, v_n \rangle - \langle u, v \rangle| \leq |\langle u_n - u, v_n \rangle| + |\langle u, v_n - v \rangle| \leq M \cdot \|u_n - u\| + \|u\| \cdot \|v_n - v\|,$$

and the right-hand side tends to 0 as $n \rightarrow \infty$. Therefore

$$\langle u, v \rangle = \lim_{n \rightarrow \infty} \langle u_n, v_n \rangle = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k b_k = \sum_{k=1}^{\infty} a_k \cdot b_k.$$

Taking $v = u$ recovers the first identity, since then $a_k = b_k = \langle \varphi_k, u \rangle$ and $\langle u, u \rangle = \|u\|^2$.

Problem 16.25. Verify the assertions in the example of the orthonormal basis for $L^2[0, 2\pi]$.

[The example reads: the countable collection of functions in $L^2[0, 2\pi]$ consisting of the constant function that takes the value $1/\sqrt{2\pi}$ and the functions $\{(1/\sqrt{2\pi}) \cdot \sin kt, (1/\sqrt{2\pi}) \cdot \cos kt\}_{k=1}^{\infty}$ are a complete orthonormal sequence for the Hilbert space $L^2[0, 2\pi]$. Indeed, we infer from the elementary trigonometric identities that this sequence is orthonormal. We infer from the Stone-Weierstrass Theorem that the linear span of this sequence is dense, with respect to the maximum norm, in the Banach space $C[a, b]$. Thus, by the density of $C[a, b]$ in $L^2[0, 2\pi]$, the linear span of this sequence is dense in $L^2[0, 2\pi]$.]

Solution. Write

$$e_0(t) = \frac{1}{\sqrt{2\pi}}, \quad c_k(t) = \frac{\cos kt}{\sqrt{\pi}}, \quad s_k(t) = \frac{\sin kt}{\sqrt{\pi}} \quad (k \geq 1, 0 \leq t \leq 2\pi).$$

(Two misprints in the printed example must be corrected before anything can be verified. First, the normalizing constant on the sines and cosines must be $1/\sqrt{\pi}$, not $1/\sqrt{2\pi}$: as the computation below shows, $\int_0^{2\pi} \cos^2 kt \, dt = \int_0^{2\pi} \sin^2 kt \, dt = \pi$, so with the constant $1/\sqrt{2\pi}$ these functions would have norm $1/\sqrt{2}$ and the sequence would not be orthonormal. Second, the interval in the density assertion is $[0, 2\pi]$, not an unspecified $[a, b]$, and even so the assertion as printed is false as it stands: the linear span consists of functions with equal values at 0 and at 2π , so its closure in the maximum norm cannot be all of $C[0, 2\pi]$. What is true, and what is proved below, is that the span is dense in the maximum norm in the subspace of $C[0, 2\pi]$ of functions agreeing at the two endpoints, and that this suffices for density in $L^2[0, 2\pi]$.) The inner product is $\langle f, g \rangle = \int_0^{2\pi} fg$.

Orthonormality. All the computations follow from the product formulas

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)], \quad \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)],$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)],$$

together with the observation that for a nonzero integer m ,

$$\int_0^{2\pi} \cos mt \, dt = \left[\frac{\sin mt}{m} \right]_0^{2\pi} = 0, \quad \int_0^{2\pi} \sin mt \, dt = \left[-\frac{\cos mt}{m} \right]_0^{2\pi} = 0,$$

while for $m = 0$ we have $\int_0^{2\pi} \cos mt \, dt = \int_0^{2\pi} 1 \, dt = 2\pi$ and $\int_0^{2\pi} \sin mt \, dt = 0$. Consequently, for $j, k \geq 1$:

- $\langle e_0, e_0 \rangle = \frac{1}{2\pi} \cdot 2\pi = 1$.
- $\langle e_0, c_k \rangle = \frac{1}{\sqrt{2\pi}\sqrt{\pi}} \int_0^{2\pi} \cos kt \, dt = 0$, and likewise $\langle e_0, s_k \rangle = 0$.
- $\langle c_j, c_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} [\cos(j-k)t + \cos(j+k)t] \, dt$. Since $j+k \geq 2$ is nonzero, the second integral vanishes; the first vanishes if $j \neq k$ and equals 2π if $j = k$. Hence $\langle c_j, c_k \rangle = 0$ for $j \neq k$ and $\langle c_k, c_k \rangle = 1$.
- $\langle s_j, s_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} [\cos(j-k)t - \cos(j+k)t] \, dt$, which by the same reasoning is 0 for $j \neq k$ and 1 for $j = k$.

- $\langle s_j, c_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} [\sin(j+k)t + \sin(j-k)t] dt = 0$, since the integral of $\sin mt$ over $[0, 2\pi]$ vanishes for every integer m (including $m = 0$).

Thus the collection is an orthonormal sequence in $L^2[0, 2\pi]$.

Density of the linear span. Let $\mathcal{T}$ denote the linear span of the collection, that is, the set of trigonometric polynomials $\alpha_0 + \sum_{k=1}^n (\alpha_k \cos kt + \beta_k \sin kt)$. The product formulas above show that a product of two of the generating functions is again a linear combination of them, so $\mathcal{T}$ is an algebra of continuous functions on $[0, 2\pi]$ containing the constants.

Care is needed at this point: $\mathcal{T}$ does not separate the points 0 and 2π of $[0, 2\pi]$, since every member of $\mathcal{T}$ has the same value at 0 and at 2π . So the Stone-Weierstrass Theorem is applied not on $[0, 2\pi]$ but on the unit circle

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

a compact Hausdorff space. Let $\mathcal{A} \subseteq C(S)$ be the algebra generated by the two coordinate functions x and y together with the constants. Then $\mathcal{A}$ contains the constants and separates the points of S (two distinct points of S differ in one of their coordinates), so by the Stone-Weierstrass Theorem $\mathcal{A}$ is dense in $C(S)$ with respect to the maximum norm. Under the continuous surjection $\Phi : [0, 2\pi] \rightarrow S$, $\Phi(t) = (\cos t, \sin t)$, we have $x \circ \Phi = \cos t$ and $y \circ \Phi = \sin t$, and, because $\mathcal{T}$ is an algebra containing $\cos t$ and $\sin t$ and the constants, every polynomial in the two coordinate functions composes with Φ to give a member of $\mathcal{T}$; since $\mathcal{A}$ consists precisely of the restrictions to S of such polynomials, $\{F \circ \Phi : F \in \mathcal{A}\} \subseteq \mathcal{T}$. Moreover $f \mapsto f \circ \Phi$ is an isometry, for the maximum norm, of $C(S)$ onto the space

$$C_{2\pi} = \{f \in C[0, 2\pi] : f(0) = f(2\pi)\},$$

since Φ is onto and identifies exactly the points 0 and 2π . Consequently $\mathcal{T}$ is dense in $C_{2\pi}$ with respect to the maximum norm.

Now let $f \in L^2[0, 2\pi]$ and let $\epsilon > 0$. By Theorem 12 of Chapter 7, the continuous functions are dense in $L^2[0, 2\pi]$, so we may choose $g \in C[0, 2\pi]$ with $\|f - g\|_2 < \epsilon/3$. Set $M = \max_{[0, 2\pi]} |g|$ and let $0 < \delta < 2\pi$. Define $\tilde{g} \in C[0, 2\pi]$ by letting $\tilde{g} = g$ on $[0, 2\pi - \delta]$ and letting $\tilde{g}$ be affine on $[2\pi - \delta, 2\pi]$ with $\tilde{g}(2\pi - \delta) = g(2\pi - \delta)$ and $\tilde{g}(2\pi) = g(0)$. Then $\tilde{g} \in C_{2\pi}$, $|\tilde{g}| \leq M$ everywhere (an affine function is bounded by the larger of its endpoint values in absolute value), and $\tilde{g} = g$ off a set of measure δ , so

$$\|g - \tilde{g}\|_2^2 = \int_{2\pi-\delta}^{2\pi} |g - \tilde{g}|^2 \leq 4M^2\delta.$$

Choosing δ small enough makes $\|g - \tilde{g}\|_2 < \epsilon/3$. Finally, by the density just proved, choose $T \in \mathcal{T}$ with $\max_{[0, 2\pi]} |\tilde{g} - T| < \epsilon/(3\sqrt{2\pi})$; then

$$\|\tilde{g} - T\|_2 \leq \sqrt{2\pi} \cdot \max_{[0, 2\pi]} |\tilde{g} - T| < \epsilon/3.$$

Adding the three estimates, $\|f - T\|_2 < \epsilon$. Hence $\mathcal{T}$ is dense in $L^2[0, 2\pi]$, that is, the closed linear span of the collection is all of $L^2[0, 2\pi]$.

Completeness. By Corollary 4, since the closed linear span of the orthonormal sequence is all of $L^2[0, 2\pi]$, the only vector orthogonal to every member of the sequence is 0; that is, the sequence is complete. By Proposition 10, it is therefore an orthonormal basis for $L^2[0, 2\pi]$.

Problem 16.26. Use Proposition 10 and the Stone-Weierstrass Theorem to show that for each $f \in L^2[-\pi, \pi]$,

$$f(x) = a_0/2 + \sum_{k=1}^{\infty} [a_k \cdot \cos kx + b_k \cdot \sin kx],$$

where the convergence is in $L^2[-\pi, \pi]$ and each

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx.$$

Solution. Define, on $[-\pi, \pi]$,

$$e_0(x) = \frac{1}{\sqrt{2\pi}}, \quad c_k(x) = \frac{\cos kx}{\sqrt{\pi}}, \quad s_k(x) = \frac{\sin kx}{\sqrt{\pi}} \quad (k \geq 1),$$

and let $\langle f, g \rangle = \int_{-\pi}^{\pi} fg$ be the $L^2[-\pi, \pi]$ inner product.

The sequence is orthonormal. All the integrands occurring in the inner products of these functions are of the form $\cos mx$ or $\sin mx$ for an integer m , by the product formulas

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)], \quad \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)],$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)].$$

For a nonzero integer m both $\cos mx$ and $\sin mx$ have period 2π and antiderivatives $\sin(mx)/m$ and $-\cos(mx)/m$ which take equal values at $-\pi$ and π , so

$$\int_{-\pi}^{\pi} \cos mx \, dx = \int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad (m \neq 0),$$

while for $m = 0$ we have $\int_{-\pi}^{\pi} \cos mx \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$ and $\int_{-\pi}^{\pi} \sin mx \, dx = 0$. Exactly as in Problem 25, these facts give, for $j, k \geq 1$,

$$\langle e_0, e_0 \rangle = 1, \quad \langle e_0, c_k \rangle = \langle e_0, s_k \rangle = 0, \quad \langle c_j, c_k \rangle = \langle s_j, s_k \rangle = \delta_{jk}, \quad \langle s_j, c_k \rangle = 0.$$

The sequence is complete. The argument of Problem 25 applies verbatim with $[0, 2\pi]$ replaced by $[-\pi, \pi]$: the linear span $\mathcal{T}$ of $\{e_0\} \cup \{c_k, s_k\}$ is the algebra of trigonometric polynomials, it contains the constants, and via the continuous surjection $\Phi(x) = (\cos x, \sin x)$ of $[-\pi, \pi]$ onto the unit circle S it corresponds to the algebra in $C(S)$ generated by the coordinate functions, which contains the constants and separates the points of S . By the Stone-Weierstrass Theorem that algebra is dense in $C(S)$ for the maximum norm, so $\mathcal{T}$ is dense, for the maximum norm, in $\{g \in C[-\pi, \pi] : g(-\pi) = g(\pi)\}$. Given $f \in L^2[-\pi, \pi]$ and $\epsilon > 0$, first approximate f within $\epsilon/3$ in the L^2 norm by a continuous g (the continuous functions are dense in $L^2[-\pi, \pi]$, by Theorem 12 of Chapter 7), then modify g affinely on a short interval $[\pi - \delta, \pi]$ so as to make its values at $-\pi$ and π agree while changing it by less than $\epsilon/3$ in the L^2 norm, and finally approximate the modified function uniformly, hence within $\epsilon/3$ in L^2 , by a member of $\mathcal{T}$. Thus $\mathcal{T}$ is dense in $L^2[-\pi, \pi]$, so the closed linear span of the orthonormal sequence is all of $L^2[-\pi, \pi]$, and therefore, by Corollary 4, the sequence is complete.

The expansion. By Proposition 10, a complete orthonormal sequence in a Hilbert space is an orthonormal basis. Enumerate our sequence as

$$\varphi_1 = e_0, \quad \varphi_{2k} = c_k, \quad \varphi_{2k+1} = s_k \quad (k \geq 1),$$

so that by (10), for each $f \in L^2[-\pi, \pi]$, the partial sums $S_N = \sum_{j=1}^N \langle \varphi_j, f \rangle \varphi_j$ converge to f in $L^2[-\pi, \pi]$. Since a convergent sequence has every subsequence converging to the same limit, in particular

$$S_{2n+1} = \langle e_0, f \rangle e_0 + \sum_{k=1}^n [\langle c_k, f \rangle c_k + \langle s_k, f \rangle s_k] \longrightarrow f \quad \text{in } L^2[-\pi, \pi].$$

It remains to identify the coefficients. First,

$$\langle e_0, f \rangle e_0(x) = \left(\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f \right) \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f = \frac{1}{2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(0 \cdot t) \, dt = \frac{a_0}{2},$$

which is the constant term $a_0/2$ prescribed by the statement, the coefficient a_k there being read at $k = 0$. Next, for $k \geq 1$,

$$\langle c_k, f \rangle c_k(x) = \left(\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(t) \cos kt \, dt \right) \cdot \frac{\cos kx}{\sqrt{\pi}} = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt \right) \cos kx = a_k \cos kx,$$

and in the same way $\langle s_k, f \rangle s_k(x) = b_k \sin kx$. Substituting these into the displayed limit gives

$$f = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx],$$

with convergence in $L^2[-\pi, \pi]$, as asserted. (All the integrals defining a_k and b_k are finite: $f \in L^2[-\pi, \pi]$ and $\cos kx, \sin kx$ are bounded, so the products are integrable over the bounded interval $[-\pi, \pi]$ by the Cauchy-Schwarz Inequality.)

Problem 16.27. Verify (12).

[That is, verify the following structural properties of adjoints: for $T, S \in \mathcal{L}(H)$,

$$(T^*)^* = T, \quad (T + S)^* = T^* + S^*, \quad (T \circ S)^* = S^* \circ T^*.$$

]

Solution. Throughout, H is a Hilbert space and $T, S \in \mathcal{L}(H)$. Recall from the definition of the adjoint the relation (11):

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle \quad \text{for all } u, v \in H,$$

and recall from Proposition 12 that $T^* \in \mathcal{L}(H)$, so that all the operators appearing in (12) belong to $\mathcal{L}(H)$.

We begin with a uniqueness remark that will be used three times.

Uniqueness lemma. If $A, B: H \rightarrow H$ satisfy $\langle u, A(v) \rangle = \langle u, B(v) \rangle$ for all $u, v \in H$, then $A = B$. Indeed, fix $v \in H$ and let $w = A(v) - B(v)$. Then $\langle u, w \rangle = 0$ for all $u \in H$; taking $u = w$ gives $\|w\|^2 = 0$, so $w = 0$, that is, $A(v) = B(v)$. Since v was arbitrary, $A = B$.

Equivalently: an operator $R \in \mathcal{L}(H)$ satisfies $R = T^*$ if and only if $\langle T(u), v \rangle = \langle u, R(v) \rangle$ for all $u, v \in H$; this is the characterizing property of the adjoint.

The first identity, $(T^*)^* = T$. Applying (11) to the operator T^* (legitimate, since $T^* \in \mathcal{L}(H)$ by Proposition 12), we have for all $u, v \in H$,

$$\langle T^*(u), v \rangle = \langle u, (T^*)^*(v) \rangle.$$

On the other hand, using the symmetry of the inner product and then (11) for T ,

$$\langle T^*(u), v \rangle = \langle v, T^*(u) \rangle = \langle T(v), u \rangle = \langle u, T(v) \rangle.$$

Therefore $\langle u, (T^*)^*(v) \rangle = \langle u, T(v) \rangle$ for all $u, v \in H$, and the uniqueness lemma gives $(T^*)^* = T$.

The second identity, $(T + S)^* = T^* + S^*$. The operator $T + S$ belongs to $\mathcal{L}(H)$, so $(T + S)^*$ is defined. For all $u, v \in H$, using (11) for $T + S$, then the additivity of the inner product in its first argument, then (11) for T and for S , and finally the additivity of the inner product in its second argument:

$$\begin{aligned} \langle u, (T + S)^*(v) \rangle &= \langle (T + S)(u), v \rangle \\ &= \langle T(u), v \rangle + \langle S(u), v \rangle \\ &= \langle u, T^*(v) \rangle + \langle u, S^*(v) \rangle \\ &= \langle u, (T^* + S^*)(v) \rangle. \end{aligned}$$

By the uniqueness lemma, $(T + S)^* = T^* + S^*$.

The third identity, $(T \circ S)^* = S^* \circ T^*$. The composition $T \circ S$ belongs to $\mathcal{L}(H)$, so $(T \circ S)^*$ is defined. For all $u, v \in H$, using (11) for $T \circ S$, then (11) for T applied to the vectors $S(u)$ and v , then (11) for S applied to u and $T^*(v)$:

$$\begin{aligned} \langle u, (T \circ S)^*(v) \rangle &= \langle (T \circ S)(u), v \rangle \\ &= \langle T(S(u)), v \rangle \\ &= \langle S(u), T^*(v) \rangle \\ &= \langle u, S^*(T^*(v)) \rangle \\ &= \langle u, (S^* \circ T^*)(v) \rangle. \end{aligned}$$

By the uniqueness lemma, $(T \circ S)^* = S^* \circ T^*$.

This verifies all three assertions of (12). (For completeness we note the companion fact, proved the same way, that $(\alpha T)^* = \alpha T^*$ for $\alpha \in \mathbb{R}$, since $\langle u, (\alpha T)^*(v) \rangle = \langle \alpha T(u), v \rangle = \alpha \langle u, T^*(v) \rangle = \langle u, \alpha T^*(v) \rangle$.)

Problem 16.28. Let T and S belong to $\mathcal{L}(H)$ and be symmetric. Show that $T = S$ if and only if $Q_T = Q_S$.

Solution. Recall that for a symmetric operator $R \in \mathcal{L}(H)$ the associated quadratic form $Q_R: H \rightarrow \mathbb{R}$ is defined by $Q_R(u) = \langle R(u), u \rangle$ for all $u \in H$.

If $T = S$, then of course $Q_T(u) = \langle T(u), u \rangle = \langle S(u), u \rangle = Q_S(u)$ for all $u \in H$, so $Q_T = Q_S$.

Conversely, assume $Q_T = Q_S$. Set $R = T - S$. Then $R \in \mathcal{L}(H)$, and R is symmetric: by (12) and the fact that $(\alpha S)^* = \alpha S^*$ for real α ,

$$R^* = (T + (-S))^* = T^* + (-S)^* = T^* - S^* = T - S = R,$$

using the symmetry $T^* = T$ and $S^* = S$ of the given operators. Moreover, by the additivity of the inner product in its first argument, for every $u \in H$,

$$Q_R(u) = \langle (T - S)(u), u \rangle = \langle T(u), u \rangle - \langle S(u), u \rangle = Q_T(u) - Q_S(u) = 0.$$

Thus Q_R vanishes identically on H .

Now apply the Polarization Identity (18) to the symmetric operator R : for all $u, v \in H$,

$$\langle R(u), v \rangle = \frac{1}{4} \left[\langle R(u+v), u+v \rangle - \langle R(u-v), u-v \rangle \right] = \frac{1}{4} \left[Q_R(u+v) - Q_R(u-v) \right] = 0.$$

So $\langle R(u), v \rangle = 0$ for all $u, v \in H$. Fixing u and taking $v = R(u)$ gives $\|R(u)\|^2 = 0$, hence $R(u) = 0$. Since $u \in H$ was arbitrary, $R = 0$, that is, $T = S$.

(Alternatively, once $Q_R \equiv 0$ is known, Proposition 16 applied to the symmetric operator R gives at once

$$\|T - S\| = \|R\| = \sup_{\|u\|=1} |\langle R(u), u \rangle| = \sup_{\|u\|=1} |Q_R(u)| = 0,$$

so $T - S = 0$.)

16.5 Problems 16.29–16.35

Problem 16.29. Show the symmetric operators are a closed subspace of $\mathcal{L}(H)$. Also show that if T and S are symmetric, then so is the composition $S \circ T$ if and only if T commutes with S with respect to composition, that is, $S \circ T = T \circ S$.

Solution. Throughout, H is a Hilbert space and $\mathcal{S}$ denotes the set of symmetric (self-adjoint) operators in $\mathcal{L}(H)$, that is, those $T \in \mathcal{L}(H)$ with $T = T^*$, equivalently

$$\langle T(u), v \rangle = \langle u, T(v) \rangle \quad \text{for all } u, v \in H.$$

$\mathcal{S}$ is a linear subspace. The zero operator is symmetric. Let $T, S \in \mathcal{S}$ and let α, β be real numbers. It is immediate from the defining relation (11) of the adjoint and the bilinearity of the inner product that $(\alpha T + \beta S)^* = \alpha T^* + \beta S^*$; indeed, for all $u, v \in H$,

$$\langle (\alpha T + \beta S)(u), v \rangle = \alpha \langle T(u), v \rangle + \beta \langle S(u), v \rangle = \alpha \langle u, T^*(v) \rangle + \beta \langle u, S^*(v) \rangle = \langle u, (\alpha T^* + \beta S^*)(v) \rangle,$$

and the adjoint is uniquely determined by (11). Since $T^* = T$ and $S^* = S$, we get $(\alpha T + \beta S)^* = \alpha T + \beta S$. Hence $\alpha T + \beta S \in \mathcal{S}$ and $\mathcal{S}$ is a linear subspace of $\mathcal{L}(H)$.

$\mathcal{S}$ is closed. Let $\{T_n\}$ be a sequence in $\mathcal{S}$ that converges in $\mathcal{L}(H)$, with respect to the operator norm, to $T \in \mathcal{L}(H)$. Fix $u, v \in H$. By the Cauchy-Schwarz Inequality,

$$|\langle T_n(u), v \rangle - \langle T(u), v \rangle| = |\langle (T_n - T)(u), v \rangle| \leq \|T_n - T\| \cdot \|u\| \cdot \|v\| \rightarrow 0,$$

so $\langle T_n(u), v \rangle \rightarrow \langle T(u), v \rangle$, and the same estimate applied in the second slot gives $\langle u, T_n(v) \rangle \rightarrow \langle u, T(v) \rangle$. Since $\langle T_n(u), v \rangle = \langle u, T_n(v) \rangle$ for every n , passing to the limit yields $\langle T(u), v \rangle = \langle u, T(v) \rangle$. As $u, v \in H$ were arbitrary, $T = T^*$, that is, $T \in \mathcal{S}$. Therefore $\mathcal{S}$ is closed in $\mathcal{L}(H)$.

(Alternatively, by Proposition 12, $\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \rightarrow 0$, so $\{T_n^*\} \rightarrow T^*$; but $T_n^* = T_n \rightarrow T$, and limits in the normed space $\mathcal{L}(H)$ are unique, so $T^* = T$.)

The composition. Let T and S be symmetric. By the structural property (12) of adjoints,

$$(S \circ T)^* = T^* \circ S^* = T \circ S.$$

Hence $S \circ T$ is symmetric, that is, $(S \circ T)^* = S \circ T$, if and only if $T \circ S = S \circ T$, that is, if and only if T commutes with S .

For completeness we verify the identity $(S \circ T)^* = T^* \circ S^*$ directly from (11): for all $u, v \in H$,

$$\langle (S \circ T)(u), v \rangle = \langle S(T(u)), v \rangle = \langle T(u), S^*(v) \rangle = \langle u, T^*(S^*(v)) \rangle = \langle u, (T^* \circ S^*)(v) \rangle,$$

and the adjoint of $S \circ T$ is the unique operator satisfying this relation.

Problem 16.30. (Hellinger-Topplitz) Let H be a Hilbert space and the linear operator $T: H \rightarrow H$ have the property that $\langle T(u), v \rangle = \langle u, T(v) \rangle$ for all $u, v \in H$. Show that T belongs to $\mathcal{L}(H)$.

Solution. We must show that the linear operator T is continuous; then T belongs to $\mathcal{L}(H)$ and, by the displayed relation together with the uniqueness in (11), T is moreover symmetric.

A Hilbert space is by definition complete, so H is a Banach space. According to the Closed Graph Theorem of Section 13.4, a linear operator between Banach spaces is continuous if and only if it is closed. So it suffices to show that T is a closed operator, that is: whenever $\{u_n\}$ is a sequence in H with $\{u_n\} \rightarrow u$ and $\{T(u_n)\} \rightarrow w$, we have $T(u) = w$.

Let $\{u_n\}$ be such a sequence. Fix $v \in H$. By the Cauchy-Schwarz Inequality,

$$|\langle T(u_n), v \rangle - \langle w, v \rangle| = |\langle T(u_n) - w, v \rangle| \leq \|T(u_n) - w\| \cdot \|v\| \rightarrow 0,$$

so that $\langle T(u_n), v \rangle \rightarrow \langle w, v \rangle$. On the other hand, by hypothesis $\langle T(u_n), v \rangle = \langle u_n, T(v) \rangle$ for every n , and again by the Cauchy-Schwarz Inequality,

$$|\langle u_n, T(v) \rangle - \langle u, T(v) \rangle| \leq \|u_n - u\| \cdot \|T(v)\| \rightarrow 0,$$

so that $\langle u_n, T(v) \rangle \rightarrow \langle u, T(v) \rangle$. By uniqueness of limits of sequences of real numbers,

$$\langle w, v \rangle = \langle u, T(v) \rangle = \langle T(u), v \rangle,$$

where the last equality is the hypothesis applied to the pair u, v . Therefore

$$\langle w - T(u), v \rangle = 0 \quad \text{for all } v \in H.$$

Choosing $v = w - T(u)$ gives $\|w - T(u)\|^2 = 0$, that is, $w = T(u)$.

Thus T is closed, and hence, by the Closed Graph Theorem, T is continuous. Therefore $T \in \mathcal{L}(H)$.

Observe that no separability or other hypothesis on H was used; completeness of H is essential, since it is what makes the Closed Graph Theorem available.

Problem 16.31. Exhibit an operator $T \in \mathcal{L}(\mathbf{R}^2)$ for which $\|T\| > \sup_{\|u\|=1} |\langle T(u), u \rangle|$.

Solution. Let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the rotation by a quarter turn,

$$T(x, y) = (-y, x) \quad \text{for } (x, y) \in \mathbf{R}^2,$$

where $\mathbf{R}^2$ carries its usual inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2$. The operator T is linear, and since $\mathbf{R}^2$ is finite dimensional it is continuous; thus $T \in \mathcal{L}(\mathbf{R}^2)$.

First, T is an isometry: for $u = (x, y)$,

$$\|T(u)\|^2 = (-y)^2 + x^2 = x^2 + y^2 = \|u\|^2.$$

Hence $\|T(u)\| = \|u\|$ for all u , so $\|T\| = \sup_{\|u\|=1} \|T(u)\| = 1$.

Second, for every $u = (x, y) \in \mathbf{R}^2$,

$$\langle T(u), u \rangle = \langle (-y, x), (x, y) \rangle = -yx + xy = 0.$$

Therefore

$$\sup_{\|u\|=1} |\langle T(u), u \rangle| = 0 < 1 = \|T\|,$$

and T is the required operator.

This does not contradict Proposition 16, since that proposition requires T to be symmetric. Indeed, the matrix of T with respect to the standard (orthonormal) basis is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is not a symmetric matrix, so T is not symmetric; explicitly, with $e_1 = (1, 0)$ and $e_2 = (0, 1)$, $\langle T(e_1), e_2 \rangle = 1$ while $\langle e_1, T(e_2) \rangle = -1$. The example shows that the hypothesis of symmetry in Proposition 16 cannot be dropped, and in fact that the quadratic form $Q_T(u) = \langle T(u), u \rangle$ carries no information at all about a general (nonsymmetric) operator: here $Q_T \equiv 0$ while $T \neq 0$.

Problem 16.32. Let S and T in $\mathcal{L}(H)$ be symmetric. Assume $S \geq T$ and $T \geq S$. Prove that $T = S$.

Solution. Set $A = S - T$. By the preceding problem, the symmetric operators form a linear subspace of $\mathcal{L}(H)$, so A is symmetric; alternatively, $A^* = (S - T)^* = S^* - T^* = S - T = A$ by the structural property (12).

By definition of the order relation among symmetric operators, $S \geq T$ means that $A = S - T$ is nonnegative, that is,

$$\langle A(h), h \rangle \geq 0 \quad \text{for all } h \in H,$$

and $T \geq S$ means that $-A = T - S$ is nonnegative, that is,

$$-\langle A(h), h \rangle = \langle -A(h), h \rangle \geq 0 \quad \text{for all } h \in H.$$

Adding these two inequalities, or simply combining them, we conclude that

$$\langle A(h), h \rangle = 0 \quad \text{for all } h \in H,$$

that is, the quadratic form Q_A vanishes identically on H .

Since A is symmetric, Proposition 16 applies and gives

$$\|A\| = \sup_{\|u\|=1} |\langle A(u), u \rangle| = 0,$$

so $A = 0$, that is, $S = T$.

If one prefers to avoid Proposition 16, the Polarization Identity (18) for the symmetric operator A suffices: for all $u, v \in H$,

$$\langle A(u), v \rangle = \frac{1}{4} [\langle A(u+v), u+v \rangle - \langle A(u-v), u-v \rangle] = \frac{1}{4} [0 - 0] = 0.$$

Taking $v = A(u)$ gives $\|A(u)\|^2 = 0$, hence $A(u) = 0$ for every $u \in H$, so again $A = 0$ and $S = T$.

(Here it is essential that H is a real Hilbert space and A is symmetric: for a general operator on a real inner product space, $Q_A \equiv 0$ does not force $A = 0$, as the rotation of the preceding problem shows.)

Problem 16.33. Let V be a closed nontrivial subspace of a Hilbert space H and P the orthogonal projection of H onto V . Show that $P = P^*$, $P \geq 0$, and $\|P\| = 1$.

Solution. By Theorem 3, since V is a closed subspace of the Hilbert space H , there is the orthogonal direct sum decomposition

$$H = V \oplus V^\perp.$$

By definition, P is the projection of H onto V along $V^\perp$: each $h \in H$ has a unique representation $h = v + w$ with $v \in V$ and $w \in V^\perp$, and $P(h) = v$. Thus P is linear, $P^2 = P$, $P(H) = V$, and $(\text{Id} - P)(H) = V^\perp$; in particular

$$P(h) \in V \quad \text{and} \quad (\text{Id} - P)(h) \in V^\perp \quad \text{for all } h \in H.$$

Boundedness and the norm. Let $h \in H$. The vectors $P(h)$ and $(\text{Id} - P)(h)$ are orthogonal, so, expanding the inner product,

$$\|h\|^2 = \langle P(h) + (\text{Id} - P)(h), P(h) + (\text{Id} - P)(h) \rangle = \|P(h)\|^2 + \|(\text{Id} - P)(h)\|^2 \geq \|P(h)\|^2,$$

the cross terms vanishing by orthogonality. Hence $\|P(h)\| \leq \|h\|$ for all $h \in H$. Therefore P is bounded, so $P \in \mathcal{L}(H)$, and $\|P\| \leq 1$. Since V is nontrivial, choose $v \in V$ with $\|v\| = 1$. Then $P(v) = v$, so $\|P(v)\| = 1 = \|v\|$ and therefore $\|P\| \geq 1$. We conclude that

$$\|P\| = 1.$$

Symmetry. Let $u, v \in H$. Since $P(u) \in V$ and $(\text{Id} - P)(v) \in V^\perp$, we have $\langle P(u), (\text{Id} - P)(v) \rangle = 0$, and therefore

$$\langle P(u), v \rangle = \langle P(u), P(v) + (\text{Id} - P)(v) \rangle = \langle P(u), P(v) \rangle.$$

By the same argument with the roles of u and v interchanged, using the symmetry of the inner product,

$$\langle u, P(v) \rangle = \langle P(u) + (\text{Id} - P)(u), P(v) \rangle = \langle P(u), P(v) \rangle.$$

Comparing the two displays,

$$\langle P(u), v \rangle = \langle u, P(v) \rangle \quad \text{for all } u, v \in H,$$

which, by the defining relation (11) of the adjoint and the uniqueness of P^* , says precisely that $P^* = P$. Thus P is symmetric.

Nonnegativity. Since P is symmetric, it makes sense to ask whether $P \geq 0$. For $h \in H$, using $P^2 = P$ and $P = P^*$,

$$\langle P(h), h \rangle = \langle P^2(h), h \rangle = \langle P(P(h)), h \rangle = \langle P(h), P(h) \rangle = \|P(h)\|^2 \geq 0.$$

Hence $\langle P(h), h \rangle \geq 0$ for all $h \in H$, that is, $P \geq 0$.

(The first two assertions, $\|P\| = 1$ and the relation $\langle P(u), v \rangle = \langle u, P(v) \rangle$, are exactly the content of Proposition 5; the arguments above simply reproduce its proof, and the nonnegativity is the new point. Note where the hypotheses were used: closedness of V gives the orthogonal decomposition $H = V \oplus V^\perp$ of Theorem 3, on which the whole argument rests, and nontriviality of V is needed for $\|P\| = 1$ – for $V = \{0\}$ one has $P = 0$ and $\|P\| = 0$.)

Problem 16.34. Let $P \in \mathcal{L}(H)$ be a projection. Show that P is the orthogonal projection of H onto $P(H)$ if and only if $P = P^*$.

Solution. Recall from Chapter 13 that a linear operator $P: H \rightarrow H$ is called a projection provided $P^2 = P$, and that then there is the direct sum decomposition

$$H = P(H) \oplus (\text{Id} - P)(H),$$

P being the projection of H onto $P(H)$ along $(\text{Id} - P)(H)$. To say that P is the *orthogonal* projection of H onto $P(H)$ means that $P(H)$ is a closed subspace of H and that the complementary subspace along which P projects is $P(H)^\perp$, that is,

$$(\text{Id} - P)(H) = P(H)^\perp.$$

We first record two elementary identifications. Since $P^2 = P$:

- $P(H) = \ker(\text{Id} - P)$. Indeed, if $h = P(u)$ then $(\text{Id} - P)(h) = P(u) - P^2(u) = 0$; conversely, if $(\text{Id} - P)(h) = 0$ then $h = P(h) \in P(H)$.
- $(\text{Id} - P)(H) = \ker P$. Indeed, if $h = (\text{Id} - P)(u)$ then $P(h) = P(u) - P^2(u) = 0$; conversely, if $P(h) = 0$ then $h = (\text{Id} - P)(h) \in (\text{Id} - P)(H)$.

Since P belongs to $\mathcal{L}(H)$, both P and $\text{Id} - P$ are continuous, so both kernels are closed subspaces of H . In particular $P(H)$ and $(\text{Id} - P)(H)$ are always closed here.

Necessity. Suppose P is the orthogonal projection of H onto the subspace $V = P(H)$. If $V = \{0\}$ then $P = 0$ and trivially $P = P^*$. If V is nontrivial, then V is a closed subspace and the preceding problem applies verbatim and gives $P = P^*$. (For the reader's convenience: for $u, v \in H$ we have $P(u) \in V$ and $(\text{Id} - P)(v) \in V^\perp$, so $\langle P(u), v \rangle = \langle P(u), P(v) \rangle$, and symmetrically $\langle u, P(v) \rangle = \langle P(u), P(v) \rangle$; hence $\langle P(u), v \rangle = \langle u, P(v) \rangle$ for all u, v , that is, $P = P^*$.)

Sufficiency. Suppose $P = P^*$. As observed above, $V = P(H) = \ker(\text{Id} - P)$ is a closed subspace of H , and $H = P(H) \oplus (\text{Id} - P)(H)$ with $(\text{Id} - P)(H) = \ker P$. It therefore remains only to prove

$$\ker P = V^\perp.$$

Let $w \in \ker P$. For every $u \in H$, using $P = P^*$,

$$\langle w, P(u) \rangle = \langle P(w), u \rangle = \langle 0, u \rangle = 0.$$

Since every element of $V = P(H)$ is of the form $P(u)$, the vector w is orthogonal to V , that is, $w \in V^\perp$. Hence $\ker P \subseteq V^\perp$.

Conversely, let $w \in V^\perp$. Since $P(w) \in V$, we have $\langle w, P(w) \rangle = 0$. Using $P = P^*$ and $P^2 = P$,

$$\|P(w)\|^2 = \langle P(w), P(w) \rangle = \langle P^*(P(w)), w \rangle = \langle P^2(w), w \rangle = \langle P(w), w \rangle = 0.$$

Hence $P(w) = 0$, that is, $w \in \ker P$. Therefore $V^\perp \subseteq \ker P$, and so $\ker P = V^\perp$.

Consequently $H = V \oplus V^\perp$ is the decomposition along which P projects, $V = P(H)$ is closed, and P is precisely the projection of H onto V along $V^\perp$, that is, the orthogonal projection of H onto $P(H)$.

Problem 16.35. Let $\{\varphi_k\}$ be an orthonormal basis for a Hilbert space H and for each natural number n , define P_n to be the orthogonal projection of H onto the linear span of $\{\varphi_1, \dots, \varphi_n\}$. Show that P_n is symmetric and

$$0 \leq P_n \leq P_{n+1} \leq \text{Id} \quad \text{for all } n.$$

Show that $\{P_n\}$ converges pointwise on H to Id but does not converge uniformly on the unit ball.

Solution. Write V_n for the linear span of $\{\varphi_1, \dots, \varphi_n\}$. Since V_n is finite dimensional, it is a closed subspace of H , and it is nontrivial since $\|\varphi_1\| = 1$; thus Theorem 3 applies to V_n and the orthogonal projection of H onto V_n is defined. Recall also that in the book an orthonormal basis is by definition an orthonormal *sequence* $\{\varphi_k\}_{k=1}^\infty$ satisfying (10), so φ_k is available for every natural number k (in particular H is infinite dimensional); this is used at the very last step below.

An explicit formula for P_n . Define $Q_n: H \rightarrow H$ by

$$Q_n(h) = \sum_{k=1}^n \langle \varphi_k, h \rangle \varphi_k \quad \text{for } h \in H.$$

Then Q_n is linear and $Q_n(H) \subseteq V_n$. Moreover, for $1 \leq j \leq n$, by the orthonormality of $\{\varphi_k\}$,

$$\langle h - Q_n(h), \varphi_j \rangle = \langle h, \varphi_j \rangle - \sum_{k=1}^n \langle \varphi_k, h \rangle \langle \varphi_k, \varphi_j \rangle = \langle h, \varphi_j \rangle - \langle \varphi_j, h \rangle = 0,$$

so $h - Q_n(h)$ is orthogonal to each of $\varphi_1, \dots, \varphi_n$ and hence, by linearity of the inner product, to all of V_n ; that is, $h - Q_n(h) \in V_n^\perp$. Since $h = Q_n(h) + [h - Q_n(h)]$, we get $H = V_n + V_n^\perp$, while $V_n \cap V_n^\perp = \{0\}$ because a vector orthogonal to itself is zero. Thus $H = V_n \oplus V_n^\perp$ and Q_n is exactly the projection of H onto V_n along $V_n^\perp$. Therefore

$$P_n(h) = \sum_{k=1}^n \langle \varphi_k, h \rangle \varphi_k \quad \text{for all } h \in H.$$

(That P_n is bounded, with $\|P_n\| = 1$, is Problem 33.)

P_n is symmetric. By Problem 33, $P_n = P_n^*$. Directly, for $u, v \in H$,

$$\langle P_n(u), v \rangle = \sum_{k=1}^n \langle \varphi_k, u \rangle \langle \varphi_k, v \rangle = \langle u, P_n(v) \rangle,$$

the middle expression being symmetric in u and v .

The order relations. First, for $h \in H$, by the formula for P_n and orthonormality,

$$\langle P_n(h), h \rangle = \sum_{k=1}^n \langle \varphi_k, h \rangle \langle \varphi_k, h \rangle = \sum_{k=1}^n \langle \varphi_k, h \rangle^2 \geq 0,$$

so $P_n \geq 0$.

Next, $P_{n+1} - P_n$ is symmetric, being a difference of symmetric operators (Problem 29), and

$$(P_{n+1} - P_n)(h) = \langle \varphi_{n+1}, h \rangle \varphi_{n+1},$$

so that

$$\langle (P_{n+1} - P_n)(h), h \rangle = \langle \varphi_{n+1}, h \rangle \langle \varphi_{n+1}, h \rangle = \langle \varphi_{n+1}, h \rangle^2 \geq 0.$$

Hence $P_{n+1} - P_n \geq 0$, that is, $P_n \leq P_{n+1}$.

Finally, $\text{Id} - P_{n+1}$ is symmetric and, by Bessel's Inequality,

$$\langle (\text{Id} - P_{n+1})(h), h \rangle = \|h\|^2 - \sum_{k=1}^{n+1} \langle \varphi_k, h \rangle^2 \geq \|h\|^2 - \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 \geq 0.$$

Hence $P_{n+1} \leq \text{Id}$. Combining, $0 \leq P_n \leq P_{n+1} \leq \text{Id}$ for all n .

Pointwise convergence to Id. Since $\{\varphi_k\}$ is an orthonormal basis for H , the defining relation (10) states that

$$h = \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k \quad \text{for all } h \in H,$$

the series converging strongly in H . But $P_n(h)$ is precisely the n -th partial sum of this series. Hence $\{P_n(h)\} \rightarrow h = \text{Id}(h)$ strongly in H , for every $h \in H$. Thus $\{P_n\}$ converges pointwise on H to Id .

Failure of uniform convergence on the unit ball. Fix a natural number n . Since $\langle \varphi_k, \varphi_{n+1} \rangle = 0$ for $1 \leq k \leq n$, we have $P_n(\varphi_{n+1}) = 0$, and therefore

$$\|(\text{Id} - P_n)(\varphi_{n+1})\| = \|\varphi_{n+1}\| = 1.$$

Since φ_{n+1} is a unit vector, it belongs to the unit ball, and hence

$$\sup_{\|h\| \leq 1} \|h - P_n(h)\| \geq 1 \quad \text{for every } n.$$

(In fact equality holds: by the orthogonality of $P_n(h)$ and $h - P_n(h)$ we have $\|h - P_n(h)\|^2 \leq \|h\|^2$, so $\|\text{Id} - P_n\| = 1$.) Consequently $\|\text{Id} - P_n\| = 1$ for all n , so $\{P_n\}$ does not converge to Id in the operator norm, that is, the convergence $\{P_n(h)\} \rightarrow h$ is not uniform over the unit ball of H .

Observe that this is not a contradiction: pointwise (strong) convergence of a sequence of operators is strictly weaker than convergence in $\mathcal{L}(H)$, and the sequence $\{\varphi_{n+1}\}$ of unit vectors on which $\text{Id} - P_n$ fails to be small escapes to infinity in index, so no single vector witnesses failure of pointwise convergence.

16.6 Problems 16.36–16.42

Problem 16.36. Show that if $T \in \mathcal{L}(H)$ is invertible, so is $T^* \circ T$ and therefore so is T^* .

Solution. Let $T \in \mathcal{L}(H)$ be invertible, that is, T is a one-to-one mapping of H onto H . Since H is complete, the Open Mapping Theorem of Chapter 13 tells us that the inverse mapping T^{-1} is continuous, and it is plainly linear; thus $T^{-1} \in \mathcal{L}(H)$.

We may assume $H \neq \{0\}$, since for $H = \{0\}$ every operator on H is invertible and there is nothing to prove. Then $T^{-1} \neq 0$, so $\|T^{-1}\| > 0$, and we may set

$$c = \frac{1}{\|T^{-1}\|} > 0.$$

For each $h \in H$, writing $h = T^{-1}(T(h))$ gives

$$\|h\| = \|T^{-1}(T(h))\| \leq \|T^{-1}\| \cdot \|T(h)\|,$$

that is,

$$\|T(h)\| \geq c \|h\| \quad \text{for all } h \in H.$$

Call this last inequality the lower bound for T .

Consider the operator $T^* \circ T \in \mathcal{L}(H)$. By the defining relation (11) of the adjoint, for every $h \in H$,

$$\langle (T^* \circ T)(h), h \rangle = \langle T^*(T(h)), h \rangle = \langle h, T^*(T(h)) \rangle = \langle T(h), T(h) \rangle = \|T(h)\|^2.$$

Here we used $\langle T(u), v \rangle = \langle u, T^*(v) \rangle$ with $u = h$ and $v = T(h)$, together with the symmetry of the inner product. Combining this identity with the lower bound for T we obtain

$$\langle (T^* \circ T)(h), h \rangle = \|T(h)\|^2 \geq c^2 \|h\|^2 \quad \text{for all } h \in H.$$

Thus $T^* \circ T$ satisfies the positive definiteness condition (14) with the positive constant c^2 , and therefore, by Proposition 14, $T^* \circ T$ is invertible.

Finally, denote by $(T^* \circ T)^{-1} \in \mathcal{L}(H)$ the inverse just produced. Since T is invertible,

$$T^* = T^* \circ T \circ T^{-1} = (T^* \circ T) \circ T^{-1}$$

exhibits T^* as the composition of two invertible operators in $\mathcal{L}(H)$, and hence T^* is invertible, with

$$(T^*)^{-1} = T \circ (T^* \circ T)^{-1} \in \mathcal{L}(H).$$

Indeed, $T^* \circ [T \circ (T^* \circ T)^{-1}] = (T^* \circ T) \circ (T^* \circ T)^{-1} = \text{Id}$, while $[T \circ (T^* \circ T)^{-1}] \circ T^* = T \circ (T^* \circ T)^{-1} \circ (T^* \circ T) \circ T^{-1} = T \circ T^{-1} = \text{Id}$.

Problem 16.37. (a General Cauchy-Schwarz Inequality) Let $T \in \mathcal{L}(H)$ be symmetric and nonnegative. Show that for all $u, v \in H$,

$$|\langle T(u), v \rangle|^2 \leq \langle T(u), u \rangle \cdot \langle T(v), v \rangle.$$

Solution. Define $B: H \times H \rightarrow \mathbb{R}$ by

$$B(u, v) = \langle T(u), v \rangle \quad \text{for } u, v \in H.$$

We record three properties of B .

First, B is bilinear: for fixed v the map $u \mapsto \langle T(u), v \rangle$ is linear because T is linear and the inner product is linear in its first argument, and for fixed u the map $v \mapsto \langle T(u), v \rangle$ is linear because the inner product is linear in its second argument.

Second, B is symmetric. Since T is symmetric, $\langle T(u), v \rangle = \langle u, T(v) \rangle$, and since the inner product itself

is symmetric, $\langle u, T(v) \rangle = \langle T(v), u \rangle$. Hence

$$B(u, v) = B(v, u) \quad \text{for all } u, v \in H.$$

Third, B is nonnegative on the diagonal: since T is a nonnegative operator, $B(w, w) = \langle T(w), w \rangle \geq 0$ for every $w \in H$.

Now fix $u, v \in H$ and consider, for $t \in \mathbb{R}$, the vector $u + tv$. Using bilinearity and symmetry of B ,

$$0 \leq B(u + tv, u + tv) = B(u, u) + 2t B(u, v) + t^2 B(v, v) \quad \text{for all } t \in \mathbb{R}.$$

Write $\alpha = B(v, v) \geq 0$, $\beta = B(u, v)$ and $\gamma = B(u, u) \geq 0$, so that

$$\alpha t^2 + 2\beta t + \gamma \geq 0 \quad \text{for all } t \in \mathbb{R}.$$

Case 1: $\alpha > 0$. The left-hand side is a real quadratic in t with positive leading coefficient that is nowhere negative, so its discriminant is nonpositive:

$$4\beta^2 - 4\alpha\gamma \leq 0, \quad \text{that is,} \quad \beta^2 \leq \alpha\gamma.$$

Equivalently, taking $t = -\beta/\alpha$ in the displayed inequality gives $\gamma - \beta^2/\alpha \geq 0$, which is the same conclusion.

Case 2: $\alpha = 0$. Then $2\beta t + \gamma \geq 0$ for all $t \in \mathbb{R}$. If $\beta \neq 0$, choosing $t = -(\gamma + 1)/(2\beta)$ yields $-(\gamma + 1) + \gamma = -1 \geq 0$, a contradiction. Hence $\beta = 0$ and again $\beta^2 = 0 = \alpha\gamma$.

In both cases $\beta^2 \leq \alpha\gamma$, that is,

$$|\langle T(u), v \rangle|^2 = B(u, v)^2 \leq B(u, u) \cdot B(v, v) = \langle T(u), u \rangle \cdot \langle T(v), v \rangle,$$

which is the asserted inequality. Taking $T = \text{Id}$ recovers the ordinary Cauchy-Schwarz Inequality.

Problem 16.38. Use the preceding problem to show that if $S, T \in \mathcal{L}(H)$ are symmetric and $S \geq T$, then for each $u \in H$,

$$\|S(u) - T(u)\|^4 = \langle (S - T)(u), (S - T)(u) \rangle^2 \leq |\langle (S - T)(u), u \rangle| \cdot |\langle (S - T)^2(u), (S - T)(u) \rangle|$$

and thereby conclude that

$$\|S(u) - T(u)\|^4 \leq |\langle S(u), u \rangle - \langle T(u), u \rangle| \cdot \|S - T\|^3 \cdot \|u\|^2.$$

Solution. Set $R = S - T$. Since S and T are symmetric, so is R : indeed, by the structural properties (12) of adjoints, $R^* = (S - T)^* = S^* - T^* = S - T = R$. Since $S \geq T$ means precisely that $S - T \geq 0$, the operator R is a nonnegative symmetric operator in $\mathcal{L}(H)$. Thus the General Cauchy-Schwarz Inequality of the preceding problem applies to R :

$$|\langle R(x), y \rangle|^2 \leq \langle R(x), x \rangle \cdot \langle R(y), y \rangle \quad \text{for all } x, y \in H.$$

Fix $u \in H$ and apply this with $x = u$ and $y = R(u)$. Then

$$|\langle R(u), R(u) \rangle|^2 \leq \langle R(u), u \rangle \cdot \langle R(R(u)), R(u) \rangle = \langle R(u), u \rangle \cdot \langle R^2(u), R(u) \rangle.$$

Since $\langle R(u), R(u) \rangle = \|R(u)\|^2 = \|S(u) - T(u)\|^2$, the left-hand side equals $\|S(u) - T(u)\|^4$. Both factors on the right may be replaced by their absolute values without decreasing the right-hand side, so

$$\|S(u) - T(u)\|^4 = \langle R(u), R(u) \rangle^2 \leq |\langle R(u), u \rangle| \cdot |\langle R^2(u), R(u) \rangle|,$$

which is exactly the first assertion.

It remains to estimate the two factors. For the first, by the linearity of the inner product in its first argument,

$$\langle R(u), u \rangle = \langle S(u) - T(u), u \rangle = \langle S(u), u \rangle - \langle T(u), u \rangle.$$

For the second, the Cauchy-Schwarz Inequality together with the definition of the operator norm gives

$$|\langle R^2(u), R(u) \rangle| \leq \|R^2(u)\| \cdot \|R(u)\| \leq \|R\|^2 \|u\| \cdot \|R\| \|u\| = \|R\|^3 \|u\|^2,$$

where we used $\|R^2(u)\| = \|R(R(u))\| \leq \|R\| \|R(u)\| \leq \|R\|^2 \|u\|$.

Substituting these two estimates into the inequality just established yields

$$\|S(u) - T(u)\|^4 \leq |\langle S(u), u \rangle - \langle T(u), u \rangle| \cdot \|S - T\|^3 \cdot \|u\|^2,$$

as required. The point of this estimate is that it controls the norm $\|S(u) - T(u)\|$ solely in terms of the difference of the two quadratic forms Q_S and Q_T at the single vector u , together with an a priori bound on $\|S - T\|$.

Problem 16.39. (a Monotone Convergence Theorem for Symmetric Operators) A sequence $\{T_n\}$ of symmetric operators in $\mathcal{L}(H)$ is said to be monotone increasing provided $T_{n+1} \geq T_n$ for each n , and said to be bounded above provided there is a symmetric operator S in $\mathcal{L}(H)$ such that $T_n \leq S$ for all n .

(i) Use the preceding problem to show that a monotone increasing sequence $\{T_n\}$ of symmetric operators in $\mathcal{L}(H)$ converges pointwise to a symmetric operator in $\mathcal{L}(H)$ if and only if it is bounded above.

(ii) Show that a monotone increasing sequence $\{T_n\}$ of symmetric operators in $\mathcal{L}(H)$ is bounded above if and only if it is pointwise bounded, that is, for each $h \in H$, the sequence $\{T_n(h)\}$ is bounded.

Solution. We first record a comparison lemma that will be used repeatedly.

Lemma. If $A, B \in \mathcal{L}(H)$ are symmetric and $0 \leq A \leq B$, then $\|A\| \leq \|B\|$.

Indeed, since $A \geq 0$ we have $|\langle A(u), u \rangle| = \langle A(u), u \rangle$, and since $B - A \geq 0$ we have $\langle A(u), u \rangle \leq \langle B(u), u \rangle \leq |\langle B(u), u \rangle|$. Hence for every unit vector u , $|\langle A(u), u \rangle| \leq |\langle B(u), u \rangle| \leq \sup_{\|v\|=1} |\langle B(v), v \rangle|$. Taking the supremum over unit vectors u and invoking Proposition 16 twice, once for A and once for B , gives $\|A\| \leq \|B\|$.

Observe also that for symmetric A, B the relation $A \leq B$ means, by definition, that $B - A \geq 0$, that is, $\langle A(u), u \rangle \leq \langle B(u), u \rangle$ for all $u \in H$. Thus if $\{T_n\}$ is monotone increasing, then for each fixed $u \in H$ the sequence of real numbers $\{\langle T_n(u), u \rangle\}$ is nondecreasing.

Part (i). Suppose first that $\{T_n\}$ converges pointwise on H to a symmetric operator $T \in \mathcal{L}(H)$. Fix $u \in H$. By the continuity of the inner product, $\langle T_n(u), u \rangle \rightarrow \langle T(u), u \rangle$. Since $\{\langle T_n(u), u \rangle\}$ is nondecreasing, each of its terms is at most its limit, so $\langle T_n(u), u \rangle \leq \langle T(u), u \rangle$ for every n . As u was arbitrary, $T_n \leq T$ for all n , and T is a symmetric operator in $\mathcal{L}(H)$. Hence $\{T_n\}$ is bounded above.

Conversely, suppose there is a symmetric $S \in \mathcal{L}(H)$ with $T_n \leq S$ for all n . Observe first that the relation $\leq$ is transitive on symmetric operators: if $A \leq B$ and $B \leq C$ then $C - A = (C - B) + (B - A)$ is a sum of nonnegative symmetric operators and hence is nonnegative. Consequently monotone increase gives $T_1 \leq T_n$ for every n . Replacing T_n by $T_n - T_1$ and S by $S - T_1$ preserves monotone increase, preserves the upper bound, and changes the convergence of $\{T_n(u)\}$ only by the fixed vector $T_1(u)$; so we may assume $T_1 = 0$, and then

$$0 = T_1 \leq T_n \leq S \quad \text{for all } n.$$

By the Lemma, $\|T_n\| \leq \|S\|$ for all n . Set $M = \|S\|$, so that $\|T_m - T_n\| \leq 2M$ for all m, n .

Fix $u \in H$. The sequence $\{\langle T_n(u), u \rangle\}$ is nondecreasing and bounded above by $\langle S(u), u \rangle$, hence it converges and in particular is a Cauchy sequence of real numbers. Now let $m > n$. Since $T_m \geq T_n$ and both are symmetric, the preceding problem applies with S there replaced by T_m and T there replaced by T_n , and gives

$$\|T_m(u) - T_n(u)\|^4 \leq |\langle T_m(u), u \rangle - \langle T_n(u), u \rangle| \cdot \|T_m - T_n\|^3 \cdot \|u\|^2 \leq |\langle T_m(u), u \rangle - \langle T_n(u), u \rangle| \cdot (2M)^3 \|u\|^2.$$

Since $\{\langle T_n(u), u \rangle\}$ is Cauchy, the right-hand side tends to 0 as $m, n \rightarrow \infty$. Therefore $\{T_n(u)\}$ is a Cauchy sequence in H . As H is complete, $\{T_n(u)\}$ converges strongly to a vector we call $T(u)$.

The mapping $T: H \rightarrow H$ so defined is linear, being a pointwise limit of linear mappings. It is bounded: for each u , by the continuity of the norm, $\|T(u)\| = \lim_n \|T_n(u)\| \leq M\|u\|$. Finally, T is symmetric: for $u, v \in H$, the continuity of the inner product gives

$$\langle T(u), v \rangle = \lim_{n \rightarrow \infty} \langle T_n(u), v \rangle = \lim_{n \rightarrow \infty} \langle u, T_n(v) \rangle = \langle u, T(v) \rangle.$$

Thus $\{T_n\}$ converges pointwise to a symmetric operator $T \in \mathcal{L}(H)$. (Undoing the normalization $T_1 = 0$ merely adds the fixed operator T_1 back to T , which affects neither boundedness nor symmetry.)

Part (ii). Suppose $\{T_n\}$ is bounded above by the symmetric operator S . Put $T'_n = T_n - T_1$ and $S' = S - T_1$; by the transitivity observed above, $0 \leq T'_n \leq S'$, and both are symmetric. The Lemma gives $\|T'_n\| \leq \|S'\|$ for all n ; hence for each $h \in H$,

$$\|T_n(h)\| \leq \|T'_n(h)\| + \|T_1(h)\| \leq \|S'\| \cdot \|h\| + \|T_1\| \cdot \|h\| \quad \text{for all } n,$$

so $\{T_n(h)\}$ is bounded. Thus $\{T_n\}$ is pointwise bounded.

Conversely, suppose $\{T_n\}$ is pointwise bounded, that is, for each $h \in H$ there is $M_h \geq 0$ with $\|T_n(h)\| \leq M_h$ for all n . Since H is a Banach space, the Uniform Boundedness Principle (Section 13.5) applied to the family $\{T_n\} \subseteq \mathcal{L}(H, H)$ provides an $M \geq 0$ with $\|T_n\| \leq M$ for all n . Then, by the Cauchy-Schwarz Inequality, for every $u \in H$ and every n ,

$$\langle T_n(u), u \rangle \leq \|T_n(u)\| \cdot \|u\| \leq M\|u\|^2 = \langle M \text{Id}(u), u \rangle.$$

Hence $T_n \leq M \text{Id}$ for all n , and $M \text{Id}$ is a symmetric operator in $\mathcal{L}(H)$. Therefore $\{T_n\}$ is bounded above.

Combining (i) and (ii): a monotone increasing sequence of symmetric operators in $\mathcal{L}(H)$ converges pointwise to a symmetric operator in $\mathcal{L}(H)$ if and only if it is pointwise bounded, which is the precise analogue for symmetric operators of the Monotone Convergence Theorem for real sequences.

Problem 16.40. Let $S \in \mathcal{L}(H)$ be a symmetric operator for which $0 \leq S \leq \text{Id}$. Define a sequence $\{T_n\}$ in $\mathcal{L}(H)$ by letting $T_1 = 1/2(\text{Id} - S)$ and, if n is a natural number for which $T_n \in \mathcal{L}(H)$ has been defined, defining $T_{n+1} = 1/2(\text{Id} - S + T_n^2)$.

(i) Show that for each natural number n , T_n and $T_{n+1} - T_n$ are polynomials in $\text{Id} - S$ with nonnegative coefficients.

(ii) Show that $\{T_n\}$ is a monotone increasing sequence of symmetric operators that is bounded above by Id .

(iii) Use the preceding problem to show that $\{T_n\}$ converges pointwise to a symmetric operator T for which $0 \leq T \leq \text{Id}$ and $T = 1/2(\text{Id} - S + T^2)$.

(iv) Define $A = (\text{Id} - T)$. Show that $A^2 = S$.

Solution. Throughout put

$$R = \text{Id} - S.$$

Since S is symmetric so is R , and the hypothesis $0 \leq S \leq \text{Id}$ says exactly that

$$0 \leq R \leq \text{Id},$$

because $R \geq 0$ is the statement $S \leq \text{Id}$ and $\text{Id} - R = S \geq 0$. Moreover $\|R\| \leq 1$: for every $u \in H$ with $\|u\| = 1$ we have $0 \leq \langle R(u), u \rangle \leq \langle u, u \rangle = 1$, so $|\langle R(u), u \rangle| \leq 1$, and Proposition 16 gives $\|R\| = \sup_{\|u\|=1} |\langle R(u), u \rangle| \leq 1$. The same argument shows that any symmetric A with $0 \leq A \leq \text{Id}$ satisfies $\|A\| \leq 1$.

Part (i). Define real polynomials p_n by

$$p_1(x) = \frac{1}{2}x, \quad p_{n+1}(x) = \frac{1}{2}(x + p_n(x)^2).$$

An immediate induction shows that $T_n = p_n(R)$ for every n : this holds for $n = 1$ since $T_1 = \frac{1}{2}R$, and if $T_n = p_n(R)$ then

$$T_{n+1} = \frac{1}{2}(\text{Id} - S + T_n^2) = \frac{1}{2}(R + p_n(R)^2) = p_{n+1}(R).$$

Moreover each p_n has nonnegative coefficients: p_1 does, and if p_n does then p_n^2 does (a product of polynomials with nonnegative coefficients has nonnegative coefficients), so $p_{n+1} = \frac{1}{2}(x + p_n^2)$ does. For the differences put $q_n = p_{n+1} - p_n$, so that $T_{n+1} - T_n = q_n(R)$. We show by induction that each q_n has nonnegative coefficients. First,

$$q_1 = p_2 - p_1 = \frac{1}{2}(x + p_1^2) - \frac{1}{2}x = \frac{1}{2}p_1^2 = \frac{1}{8}x^2,$$

which has nonnegative coefficients. Next, for $n \geq 2$,

$$q_n = p_{n+1} - p_n = \frac{1}{2}(x + p_n^2) - \frac{1}{2}(x + p_{n-1}^2) = \frac{1}{2}(p_n^2 - p_{n-1}^2) = \frac{1}{2}(p_n - p_{n-1})(p_n + p_{n-1}) = \frac{1}{2}q_{n-1}(p_n + p_{n-1}).$$

By the induction hypothesis q_{n-1} has nonnegative coefficients, and $p_n + p_{n-1}$ has nonnegative coefficients; hence so does their product q_n . Thus $T_n = p_n(R)$ and $T_{n+1} - T_n = q_n(R)$ are polynomials in $\text{Id} - S$ with nonnegative coefficients.

Part (ii). We first record two facts about the operator R , valid because $0 \leq R \leq \text{Id}$.

(a) $R^k \geq 0$ for every integer $k \geq 0$. For $k = 0$ this is $\text{Id} \geq 0$. Each power R^k is symmetric, since $(R^k)^* = (R^*)^k = R^k$ by the structural properties (12). If $k = 2m$ then $\langle R^{2m}(u), u \rangle = \langle R^m(u), R^m(u) \rangle = \|R^m(u)\|^2 \geq 0$; if $k = 2m + 1$ then $\langle R^{2m+1}(u), u \rangle = \langle R(R^m(u)), R^m(u) \rangle \geq 0$ because $R \geq 0$.

(b) $R^k \leq \text{Id}$ for every integer $k \geq 0$. Indeed $\|R^k\| \leq \|R\|^k \leq 1$, so by the Cauchy-Schwarz Inequality $\langle R^k(u), u \rangle \leq \|R^k(u)\| \|u\| \leq \|u\|^2 = \langle \text{Id}(u), u \rangle$.

Now let $p(x) = \sum_{k=0}^d c_k x^k$ be any polynomial with nonnegative coefficients. Then $p(R) = \sum_{k=0}^d c_k R^k$ is symmetric, being a real linear combination of the symmetric operators R^k , and by (a) it is nonnegative, being a sum of nonnegative operators with nonnegative scalar multipliers. Furthermore, if $p(1) = \sum_{k=0}^d c_k \leq 1$, then

$$\text{Id} - p(R) = (1 - p(1)) \text{Id} + \sum_{k=0}^d c_k (\text{Id} - R^k),$$

which is a sum of nonnegative operators by (b) and by $1 - p(1) \geq 0$; hence $p(R) \leq \text{Id}$.

By part (i) each $T_n = p_n(R)$ and each $T_{n+1} - T_n = q_n(R)$ has nonnegative coefficients, so each T_n is symmetric and $T_{n+1} - T_n \geq 0$, that is, $\{T_n\}$ is a monotone increasing sequence of symmetric operators (and $T_n \geq 0$ as well).

For the upper bound put $a_n = p_n(1)$. The recursion gives $a_1 = 1/2$ and $a_{n+1} = \frac{1}{2}(1 + a_n^2)$. If $a_n < 1$ then $a_{n+1} = \frac{1}{2}(1 + a_n^2) < \frac{1}{2}(1 + 1) = 1$; since $a_1 = 1/2 < 1$, induction yields $a_n < 1$ for all n . Consequently $p_n(1) \leq 1$ for every n , and by the criterion just established $T_n = p_n(R) \leq \text{Id}$. Thus $\{T_n\}$ is bounded above by Id .

Part (iii). By part (ii), $\{T_n\}$ is a monotone increasing sequence of symmetric operators in $\mathcal{L}(H)$ that is bounded above (by the symmetric operator Id). Hence, by part (i) of the preceding problem, $\{T_n\}$ converges pointwise on H to a symmetric operator $T \in \mathcal{L}(H)$.

Since $0 \leq T_n \leq \text{Id}$, for each $u \in H$ we have $0 \leq \langle T_n(u), u \rangle \leq \|u\|^2$; letting $n \rightarrow \infty$ and using the continuity of the inner product gives $0 \leq \langle T(u), u \rangle \leq \|u\|^2$. Therefore $0 \leq T \leq \text{Id}$. In particular, by the observation recorded at the outset, $\|T\| \leq 1$, and likewise $\|T_n\| \leq 1$ for all n .

We claim $\{T_n^2\}$ converges pointwise to T^2 . Indeed, for $u \in H$,

$$T_n^2(u) - T^2(u) = T_n(T_n(u) - T(u)) + (T_n - T)(T(u)),$$

so

$$\|T_n^2(u) - T^2(u)\| \leq \|T_n\| \|T_n(u) - T(u)\| + \|T_n(T(u)) - T(T(u))\| \leq \|T_n(u) - T(u)\| + \|T_n(T(u)) - T(T(u))\|,$$

and both terms tend to 0 as $n \rightarrow \infty$, the first by pointwise convergence at u and the second by pointwise convergence at $T(u)$.

Now fix $u \in H$ and let $n \rightarrow \infty$ in the identity $T_{n+1}(u) = \frac{1}{2}(u - S(u) + T_n^2(u))$. The left-hand side converges to $T(u)$ and the right-hand side to $\frac{1}{2}(u - S(u) + T^2(u))$. Since limits in H are unique,

$$T(u) = \frac{1}{2}(\text{Id} - S + T^2)(u) \quad \text{for all } u \in H, \quad \text{that is,} \quad T = \frac{1}{2}(\text{Id} - S + T^2).$$

Part (iv). Let $A = \text{Id} - T$. Then A is symmetric, and $0 \leq T \leq \text{Id}$ gives $0 \leq A \leq \text{Id}$; in particular A is a nonnegative symmetric operator. Expanding,

$$A^2 = (\text{Id} - T)^2 = \text{Id} - 2T + T^2.$$

By part (iii), $2T = \text{Id} - S + T^2$, that is, $T^2 - 2T = S - \text{Id}$. Substituting,

$$A^2 = \text{Id} + (T^2 - 2T) = \text{Id} + (S - \text{Id}) = S.$$

Thus A is a nonnegative symmetric operator with $A^2 = S$; that is, A is a square root of S .

Problem 16.41. (Square Roots of Nonnegative Symmetric Operators) Let $T \in \mathcal{L}(H)$ be a nonnegative symmetric operator. A nonnegative symmetric operator $A \in \mathcal{L}(H)$ is called a square root of T provided $A^2 = T$. Use the inductive construction in the preceding problem to show that T has a square root A which commutes with each operator in $\mathcal{L}(H)$ that commutes with T . Show that the square root is unique: it is denoted by $\sqrt{T}$. Finally, show that T is invertible if and only if $\sqrt{T}$ is invertible.

Solution. Existence. If $T = 0$, then $A = 0$ is a nonnegative symmetric square root of T commuting with every operator, so assume $T \neq 0$ and put

$$S = \frac{1}{\|T\|} T.$$

Then S is symmetric, and $S \geq 0$ since $T \geq 0$ and $1/\|T\| > 0$. Also $S \leq \text{Id}$, since by the Cauchy-Schwarz Inequality, for each $u \in H$,

$$\langle S(u), u \rangle = \frac{\langle T(u), u \rangle}{\|T\|} \leq \frac{\|T(u)\| \|u\|}{\|T\|} \leq \|u\|^2.$$

So $0 \leq S \leq \text{Id}$ and the preceding problem applies to S . Let $R = \text{Id} - S$, let $\{T_n\}$ be the sequence constructed there, so that $T_n = p_n(R)$ with p_n a polynomial with nonnegative coefficients, let $L \in \mathcal{L}(H)$ be its pointwise limit, and let $A_0 = \text{Id} - L$. By parts (iii) and (iv) of the preceding problem, A_0 is symmetric, $0 \leq A_0 \leq \text{Id}$, and $A_0^2 = S$. Define

$$A = \sqrt{\|T\|} A_0.$$

Then A is symmetric, $A \geq 0$ (a nonnegative multiple of a nonnegative operator), and

$$A^2 = \|T\| A_0^2 = \|T\| S = T.$$

Thus A is a square root of T .

Commutation. Let $B \in \mathcal{L}(H)$ commute with T , that is, $B \circ T = T \circ B$. Then B commutes with $S = T/\|T\|$, hence with $R = \text{Id} - S$ (since B trivially commutes with Id), hence with every power R^k and therefore with every polynomial in R ; in particular $B \circ T_n = T_n \circ B$ for all n . Fix $u \in H$. Then $\{T_n(u)\} \rightarrow L(u)$ and $\{T_n(B(u))\} \rightarrow L(B(u))$ strongly in H , and since B is continuous, $\{B(T_n(u))\} \rightarrow B(L(u))$. But $B(T_n(u)) = T_n(B(u))$ for each n , so by uniqueness of strong limits,

$$B(L(u)) = L(B(u)) \quad \text{for all } u \in H.$$

Hence B commutes with L , therefore with $A_0 = \text{Id} - L$, and therefore with $A = \sqrt{\|T\|} A_0$. (When $T = 0$ the assertion is trivial.)

Uniqueness. Let $B \in \mathcal{L}(H)$ be any nonnegative symmetric operator with $B^2 = T$; we show $B = A$, where A is the square root constructed above. First, B commutes with T , since

$$B \circ T = B \circ B^2 = B^2 \circ B = T \circ B.$$

By the commutation property just proved, B therefore commutes with A . Consequently

$$(A + B) \circ (A - B) = A^2 - A \circ B + B \circ A - B^2 = A^2 - B^2 = T - T = 0.$$

Fix $u \in H$ and set $w = (A - B)(u)$. By the displayed identity, $(A + B)(w) = 0$, so

$$\langle A(w), w \rangle + \langle B(w), w \rangle = \langle (A + B)(w), w \rangle = 0.$$

Both summands are nonnegative because $A \geq 0$ and $B \geq 0$; hence

$$\langle A(w), w \rangle = 0 \quad \text{and} \quad \langle B(w), w \rangle = 0.$$

Apply the General Cauchy-Schwarz Inequality (Problem 37) to the nonnegative symmetric operator A with the vectors w and $A(w)$:

$$\|A(w)\|^4 = |\langle A(w), A(w) \rangle|^2 \leq \langle A(w), w \rangle \cdot \langle A(A(w)), A(w) \rangle = 0.$$

Therefore $A(w) = 0$, and by the identical argument $B(w) = 0$. Hence $(A - B)(w) = 0$, that is, $(A - B)^2(u) = 0$. Since $A - B$ is symmetric,

$$\|(A - B)(u)\|^2 = \langle (A - B)(u), (A - B)(u) \rangle = \langle (A - B)^2(u), u \rangle = 0.$$

Thus $(A - B)(u) = 0$ for every $u \in H$, so $B = A$. The square root is unique and we write $A = \sqrt{T}$.

Invertibility. Suppose first that $\sqrt{T}$ is invertible. Then $T = \sqrt{T} \circ \sqrt{T}$ is a composition of invertible operators in $\mathcal{L}(H)$ and hence is invertible, with $T^{-1} = (\sqrt{T})^{-1} \circ (\sqrt{T})^{-1}$.

Conversely, suppose T is invertible, and write $A = \sqrt{T}$. Since A commutes with T (indeed $A \circ T = A \circ A^2 = A^2 \circ A = T \circ A$), it also commutes with T^{-1} : composing $A \circ T = T \circ A$ on both sides by T^{-1} gives $T^{-1} \circ A = A \circ T^{-1}$. Now set

$$C = A \circ T^{-1} \in \mathcal{L}(H).$$

Then

$$A \circ C = A \circ A \circ T^{-1} = T \circ T^{-1} = \text{Id},$$

and, using the commutation of A with T^{-1} ,

$$C \circ A = A \circ T^{-1} \circ A = A \circ A \circ T^{-1} = T \circ T^{-1} = \text{Id}.$$

Hence A is invertible with $A^{-1} = A \circ T^{-1} \in \mathcal{L}(H)$. Therefore T is invertible if and only if $\sqrt{T}$ is invertible.

Problem 16.42. An invertible operator $T \in \mathcal{L}(H)$ is said to be orthogonal provided $T^{-1} = T^*$. Show that an invertible operator is orthogonal if and only if it is an isometry.

Solution. Recall the definition of Chapter 9: a mapping T of the metric space H to itself is an isometry provided it maps H onto H and preserves distances, $\|T(u) - T(v)\| = \|u - v\|$ for all $u, v \in H$. Throughout, $T \in \mathcal{L}(H)$ is assumed invertible, hence in particular onto, so the surjectivity half of the definition is automatic for both implications and only distance preservation is at issue. For a linear operator T , distance preservation is equivalent to the requirement

$$\|T(u)\| = \|u\| \quad \text{for all } u \in H,$$

since $T(u) - T(v) = T(u - v)$, so $\|T(u) - T(v)\| = \|T(u - v)\|$, and conversely one recovers the norm condition by taking $v = 0$. We prove that an invertible $T \in \mathcal{L}(H)$ satisfies $T^{-1} = T^*$ if and only if it satisfies the displayed norm condition.

Suppose first that T is orthogonal, that is, $T^{-1} = T^*$. Then

$$T^* \circ T = T^{-1} \circ T = \text{Id}.$$

Hence, for each $u \in H$, using the defining relation (11) of the adjoint and the symmetry of the inner product,

$$\|T(u)\|^2 = \langle T(u), T(u) \rangle = \langle u, T^*(T(u)) \rangle = \langle u, u \rangle = \|u\|^2.$$

Taking square roots, $\|T(u)\| = \|u\|$ for all $u \in H$, so T is an isometry.

Conversely, suppose the invertible operator T is an isometry, so that $\|T(u)\| = \|u\|$ for all $u \in H$. Consider the operator

$$P = T^* \circ T - \text{Id} \in \mathcal{L}(H).$$

By the structural properties (12) of adjoints,

$$(T^* \circ T)^* = T^* \circ (T^*)^* = T^* \circ T,$$

so $T^* \circ T$ is symmetric, and since Id is plainly symmetric, P is symmetric. For each $u \in H$, the relation (11) gives

$$\langle P(u), u \rangle = \langle T^*(T(u)), u \rangle - \langle u, u \rangle = \langle T(u), T(u) \rangle - \|u\|^2 = \|T(u)\|^2 - \|u\|^2 = 0.$$

Thus the quadratic form Q_P vanishes identically on H . By the Polarization Identity for the symmetric operator P ,

$$\langle P(u), v \rangle = \frac{1}{4} [\langle P(u+v), u+v \rangle - \langle P(u-v), u-v \rangle] = 0 \quad \text{for all } u, v \in H.$$

(Equivalently, one may invoke Proposition 16, which gives $\|P\| = \sup_{\|u\|=1} |\langle P(u), u \rangle| = 0$.) Taking $v = P(u)$ yields $\|P(u)\|^2 = 0$, so $P = 0$, that is,

$$T^* \circ T = \text{Id}.$$

Since T is invertible, composing on the right with T^{-1} gives

$$T^* = T^* \circ T \circ T^{-1} = \text{Id} \circ T^{-1} = T^{-1}.$$

Hence T is orthogonal.

This completes both implications: an invertible operator $T \in \mathcal{L}(H)$ is orthogonal if and only if it is an isometry.

Two remarks. First, the second half of the proof used invertibility only at the very last step, to pass from $T^* \circ T = \text{Id}$ to $T^* = T^{-1}$; that step genuinely needs it. The right shift T on ℓ^2 , defined by $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$, preserves norms and so satisfies $T^* \circ T = \text{Id}$ by the computation above, yet T is not onto and hence is neither invertible nor orthogonal; indeed $T \circ T^* \neq \text{Id}$. Second, the norm condition $\|T(u)\| = \|u\|$ for all u is, by the Polarization Identity for the inner product, equivalent to $\langle T(u), T(v) \rangle = \langle u, v \rangle$ for all $u, v \in H$, which is the formulation of Problem 9 of this chapter.

16.7 Problems 16.43–16.49

Problem 16.43. (Polar Decompositions) Let $T \in \mathcal{L}(H)$ be invertible. Show that there is an orthogonal invertible operator $A \in \mathcal{L}(H)$ and a nonnegative symmetric invertible operator $B \in \mathcal{L}(H)$ such that $T = B \circ A$. (Hint: Show that T^*T is invertible and symmetric and let $B = \sqrt{T^*T}$.)

Solution. Throughout we write ST for the composition $S \circ T$ and Id for the identity operator on H . A word on the hint. It names the operator T^*T but prescribes $B = \sqrt{T^*T}$; it is the latter that is needed for the factorization $T = B \circ A$ in the stated order, so we work with TT^* . (Both operators are symmetric, nonnegative and invertible, by the same argument; the choice $B = \sqrt{T^*T}$ produces instead a factorization $T = A \circ B$ with A orthogonal, as one sees by applying what follows to T^* and taking adjoints.)

Step 1: the operator TT^* is symmetric, nonnegative and invertible.

Symmetry is immediate from the identities $(ST)^* = T^*S^*$ and $T^{**} = T$:

$$(TT^*)^* = (T^*)^*T^* = TT^*.$$

Nonnegativity: for every $u \in H$,

$$\langle TT^*(u), u \rangle = \langle T^*(u), T^*(u) \rangle = \|T^*(u)\|^2 \geq 0.$$

Invertibility: since T is invertible, Problem 36 tells us that T^* is invertible. (Directly: taking adjoints in $TT^{-1} = T^{-1}T = \text{Id}$ and using $(ST)^* = T^*S^*$, $\text{Id}^* = \text{Id}$, we get $(T^{-1})^*T^* = T^*(T^{-1})^* = \text{Id}$, so T^* is invertible with $(T^*)^{-1} = (T^{-1})^*$; note T^{-1} is bounded, so $(T^{-1})^*$ is defined and lies in $\mathcal{L}(H)$ by Proposition 12.) Hence TT^* is a composition of two invertible operators and is therefore invertible, with inverse $(T^*)^{-1}T^{-1}$.

Step 2: definition of B .

By Problem 41, a nonnegative symmetric operator in $\mathcal{L}(H)$ possesses a unique nonnegative symmetric square root, and that square root is invertible if and only if the operator itself is invertible. Applying this to the nonnegative symmetric operator TT^* of Step 1, we may define

$$B = \sqrt{TT^*} \in \mathcal{L}(H),$$

and B is symmetric, nonnegative, and invertible, and satisfies $B^2 = TT^*$.

Since B is symmetric and invertible, its inverse is also symmetric: indeed, taking adjoints in $BB^{-1} = B^{-1}B = \text{Id}$ and using $(ST)^* = T^*S^*$ together with $\text{Id}^* = \text{Id}$, we get $(B^{-1})^*B = B(B^{-1})^* = \text{Id}$, so $(B^{-1})^* = B^{-1}$ by uniqueness of inverses.

Step 3: definition of A and verification.

Define

$$A = B^{-1}T \in \mathcal{L}(H).$$

As a composition of two invertible operators, A is invertible. Its adjoint is

$$A^* = (B^{-1}T)^* = T^*(B^{-1})^* = T^*B^{-1},$$

using the symmetry of B^{-1} established above. Therefore

$$AA^* = B^{-1}T T^* B^{-1} = B^{-1}(TT^*)B^{-1} = B^{-1}B^2B^{-1} = \text{Id}.$$

Since A is invertible, composing this identity on the left with A^{-1} yields $A^* = A^{-1}$. Thus A is an invertible orthogonal operator in the sense of Problem 42. (Equivalently, by Problem 42, A is an isometry.)

Step 4: the factorization.

Finally,

$$B \circ A = B(B^{-1}T) = (BB^{-1})T = T.$$

Thus $T = B \circ A$ with $A \in \mathcal{L}(H)$ invertible and orthogonal and $B \in \mathcal{L}(H)$ symmetric, nonnegative and invertible, which is the required polar decomposition.

Remark: B is uniquely determined by T . Indeed if $T = BA$ with B nonnegative symmetric and A orthogonal, then

$$TT^* = BA(BA)^* = BAA^*B^* = B \text{Id} B = B^2,$$

so B is a nonnegative symmetric square root of TT^* , and such a square root is unique by Problem 41. Consequently $A = B^{-1}T$ is uniquely determined as well.

Problem 16.44. Show that if H is infinite dimensional and $T \in \mathcal{L}(H)$ is invertible, then T is not compact.

Solution. Since T is invertible, its inverse T^{-1} belongs to $\mathcal{L}(H)$: this is part of the meaning of invertibility for a bounded operator, and in any case, if $T \in \mathcal{L}(H)$ is one-to-one and onto, then T^{-1} is bounded by the Open Mapping Theorem of Section 13.4, since H is a Banach space. In particular there is a constant $c = \|T^{-1}\| > 0$ with

$$\|h\| = \|T^{-1}(T(h))\| \leq c \|T(h)\| \quad \text{for all } h \in H,$$

that is,

$$\|T(h)\| \geq \frac{1}{c} \|h\| \quad \text{for all } h \in H. \quad (*)$$

Step 1: an infinite dimensional Hilbert space contains an orthonormal sequence.

We construct $\{\varphi_n\}_{n=1}^\infty$ inductively. Since $H \neq \{0\}$, choose a unit vector φ_1 . Suppose orthonormal vectors $\varphi_1, \dots, \varphi_n$ have been chosen and let V_n be their linear span. Then V_n is a finite dimensional, hence closed, subspace of H , and $V_n \neq H$ because H is infinite dimensional. Therefore, by Theorem 3, which provides the orthogonal decomposition $H = V_n \oplus V_n^\perp$ for a closed subspace V_n , we must have $V_n^\perp \neq \{0\}$, and we may choose a unit vector $\varphi_{n+1} \in V_n^\perp$. Since φ_{n+1} is orthogonal to every vector of V_n , and in particular to $\varphi_1, \dots, \varphi_n$, the resulting sequence $\{\varphi_n\}$ is orthonormal. Observe that no separability assumption on H is needed here.

For $n \neq m$, orthonormality gives

$$\|\varphi_n - \varphi_m\|^2 = \|\varphi_n\|^2 - 2\langle \varphi_n, \varphi_m \rangle + \|\varphi_m\|^2 = 2,$$

so $\|\varphi_n - \varphi_m\| = \sqrt{2}$ whenever $n \neq m$.

Step 2: $\{T(\varphi_n)\}$ has no strongly convergent subsequence.

By (*), for $n \neq m$,

$$\|T(\varphi_n) - T(\varphi_m)\| = \|T(\varphi_n - \varphi_m)\| \geq \frac{1}{c} \|\varphi_n - \varphi_m\| = \frac{\sqrt{2}}{c} > 0.$$

Hence no subsequence of $\{T(\varphi_n)\}$ is Cauchy, and therefore no subsequence of $\{T(\varphi_n)\}$ converges strongly.

Step 3: conclusion.

The sequence $\{\varphi_n\}$ is bounded, since each φ_n is a unit vector. If T were compact, then by Proposition 18 (the implication (i) $\Rightarrow$ (iii)) the sequence $\{T(\varphi_n)\}$ would have a strongly convergent subsequence. This contradicts Step 2. Therefore T is not compact.

Equivalently, directly from the definition: the vectors φ_n lie in the closed unit ball B of H , so $\{T(\varphi_n)\}$ lies in $T(B)$, whose closure would be compact, hence sequentially compact since we are in a metric space; that would again produce a strongly convergent subsequence of $\{T(\varphi_n)\}$, contradicting Step 2.

Problem 16.45. Prove Proposition 18. [Let H be a Hilbert space and K belong to $\mathcal{L}(H)$. Then the following are equivalent: (i) K is compact; (ii) $K(B)$ is totally bounded, where B is the closed unit ball in H ; (iii) If $\{h_n\}$ is a bounded sequence in H , then $\{K(h_n)\}$ has a strongly convergent subsequence.]

Solution. Recall that H , being a Hilbert space, is complete with respect to the metric induced by its norm, and that all topological notions below refer to the strong (that is, norm) topology. We use three standard metric space facts: a metric space is compact if and only if it is sequentially compact; a metric space is compact if and only if it is complete and totally bounded; and a closed subset of a complete metric space is complete.

We also record a lemma.

Lemma. If $S \subseteq H$ is totally bounded, so is $\overline{S}$.

Indeed, let $\epsilon > 0$ and let $x_1, \dots, x_N \in H$ be such that the balls of radius $\epsilon/2$ about the x_j cover S . If $y \in \overline{S}$, choose $s \in S$ with $\|y - s\| < \epsilon/2$ and then j with $\|s - x_j\| < \epsilon/2$; the triangle inequality gives $\|y - x_j\| < \epsilon$. So the balls of radius ϵ about $x_1, \dots, x_N$ cover $\overline{S}$.

We prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii). Assume K is compact, that is, $\overline{K(B)}$ is compact. Let $\{h_n\}$ be a bounded sequence in H , say $\|h_n\| \leq M$ for all n , where we may assume $M > 0$. Then $h_n/M \in B$, so

$$K(h_n) = M \cdot K(h_n/M) \in M \cdot K(B) \quad \text{for all } n.$$

The map $x \mapsto Mx$ is a homeomorphism of H onto H , so $\overline{M \cdot K(B)} = M \cdot \overline{K(B)}$ is compact. A compact metric space is sequentially compact, so the sequence $\{K(h_n)\}$, which lies in this compact set, has a subsequence converging strongly to some element of H . This is (iii).

(iii) $\Rightarrow$ (ii). We argue by contradiction. Suppose $K(B)$ is not totally bounded. Then there is an $\epsilon > 0$

for which $K(B)$ admits no finite ϵ -net. We construct inductively a sequence $\{u_n\}$ in B with

$$\|K(u_n) - K(u_m)\| \geq \epsilon \quad \text{for } n \neq m.$$

Choose $u_1 \in B$ arbitrarily. Suppose $u_1, \dots, u_n \in B$ have been chosen. The n points $K(u_1), \dots, K(u_n)$ belong to $K(B)$ and hence do not form an ϵ -net for $K(B)$; that is, the n balls of radius ϵ centered at these points do not cover $K(B)$. So there is a point of $K(B)$, say $K(u_{n+1})$ with $u_{n+1} \in B$, satisfying $\|K(u_{n+1}) - K(u_j)\| \geq \epsilon$ for $1 \leq j \leq n$. This completes the induction. (Since the centers used here lie in $K(B)$ itself, the argument is valid whether total boundedness is defined by nets with centers in the set or by nets with centers anywhere in H .)

The sequence $\{u_n\}$ is bounded, since it lies in B . But no subsequence of $\{K(u_n)\}$ is Cauchy, since any two distinct terms are at distance at least ϵ ; hence no subsequence converges strongly. This contradicts (iii). Therefore $K(B)$ is totally bounded.

(ii) $\Rightarrow$ (i). Assume $K(B)$ is totally bounded. By the Lemma, $\overline{K(B)}$ is totally bounded. Moreover $\overline{K(B)}$ is a closed subset of the complete metric space H and hence is itself complete. A metric space that is both complete and totally bounded is compact, so $\overline{K(B)}$ is compact, that is, K is compact.

This closes the cycle, so (i), (ii) and (iii) are equivalent.

Problem 16.46. Let $\mathcal{K}(H)$ denote the set of compact operators in $\mathcal{L}(H)$. Show that $\mathcal{K}(H)$ is a linear subspace of $\mathcal{L}(H)$. Moreover, show that for $K \in \mathcal{K}(H)$ and $T \in \mathcal{L}(H)$, both $K \circ T$ and $T \circ K$ belong to $\mathcal{K}(H)$.

Solution. Throughout we use the characterization of compactness furnished by Proposition 18: an operator $S \in \mathcal{L}(H)$ is compact if and only if for every bounded sequence $\{h_n\}$ in H , the sequence $\{S(h_n)\}$ has a strongly convergent subsequence.

Part 1: $\mathcal{K}(H)$ is a linear subspace of $\mathcal{L}(H)$.

The zero operator is compact, since $0(B) = \{0\}$ has compact closure; thus $\mathcal{K}(H)$ is nonempty. Let $K_1, K_2 \in \mathcal{K}(H)$ and let α, β be scalars. The operator $\alpha K_1 + \beta K_2$ certainly belongs to $\mathcal{L}(H)$, since $\mathcal{L}(H)$ is a linear space. Let $\{h_n\}$ be a bounded sequence in H . Since K_1 is compact, there is a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ for which $\{K_1(h_{n_k})\}$ converges strongly, say to $y_1 \in H$. The subsequence $\{h_{n_k}\}$ is itself bounded, so, since K_2 is compact, there is a further subsequence $\{h_{n_{k_j}}\}$ for which $\{K_2(h_{n_{k_j}})\}$ converges strongly, say to $y_2 \in H$. Along this further subsequence both sequences converge strongly, and hence, by the continuity of the vector space operations with respect to the norm,

$$(\alpha K_1 + \beta K_2)(h_{n_{k_j}}) = \alpha K_1(h_{n_{k_j}}) + \beta K_2(h_{n_{k_j}}) \longrightarrow \alpha y_1 + \beta y_2 \quad \text{strongly.}$$

Thus $\{(\alpha K_1 + \beta K_2)(h_n)\}$ has a strongly convergent subsequence. Since $\{h_n\}$ was an arbitrary bounded sequence, Proposition 18 tells us that $\alpha K_1 + \beta K_2$ is compact. Hence $\mathcal{K}(H)$ is a linear subspace of $\mathcal{L}(H)$.

Part 2: $K \circ T$ is compact.

Let $K \in \mathcal{K}(H)$ and $T \in \mathcal{L}(H)$; certainly $K \circ T \in \mathcal{L}(H)$. Let $\{h_n\}$ be a bounded sequence in H , say $\|h_n\| \leq M$ for all n . Then

$$\|T(h_n)\| \leq \|T\| \|h_n\| \leq \|T\| M \quad \text{for all } n,$$

so $\{T(h_n)\}$ is a bounded sequence in H . Since K is compact, $\{K(T(h_n))\}$ has a strongly convergent subsequence. By Proposition 18, $K \circ T$ is compact.

Part 3: $T \circ K$ is compact.

Again $T \circ K \in \mathcal{L}(H)$. Let $\{h_n\}$ be a bounded sequence in H . Since K is compact, there is a subsequence $\{h_{n_k}\}$ for which $\{K(h_{n_k})\}$ converges strongly to some $y \in H$. Since T is continuous,

$$(T \circ K)(h_{n_k}) = T(K(h_{n_k})) \longrightarrow T(y) \quad \text{strongly.}$$

Thus $\{(T \circ K)(h_n)\}$ has a strongly convergent subsequence, and by Proposition 18 the operator $T \circ K$ is compact.

Therefore $\mathcal{K}(H)$ is a linear subspace of $\mathcal{L}(H)$ which is closed under composition on either side by arbitrary members of $\mathcal{L}(H)$; that is, $\mathcal{K}(H)$ is a two-sided ideal in the algebra $\mathcal{L}(H)$.

Problem 16.47. Show that a linear operator $T: H \rightarrow H$ is continuous if and only if it maps weakly convergent sequences to weakly convergent sequences.

Solution. Recall that, by Theorem 1 of Chapter 13, a linear operator between normed linear spaces is continuous if and only if it is bounded; thus T is continuous if and only if $T \in \mathcal{L}(H)$. Note that no continuity of T is assumed in the sufficiency half below: T there is merely a linear operator.

Necessity. Assume T is continuous, so that $T \in \mathcal{L}(H)$. By Proposition 12, the adjoint T^* belongs to $\mathcal{L}(H)$ and

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle \quad \text{for all } u, v \in H.$$

Let $\{u_n\} \rightharpoonup u$ weakly in H . Fix $v \in H$. Then, applying the definition of weak convergence to the vector $T^*(v)$,

$$\lim_{n \rightarrow \infty} \langle T(u_n), v \rangle = \lim_{n \rightarrow \infty} \langle u_n, T^*(v) \rangle = \langle u, T^*(v) \rangle = \langle T(u), v \rangle.$$

Since $v \in H$ was arbitrary, $\{T(u_n)\} \rightharpoonup T(u)$ weakly. Thus T maps weakly convergent sequences to weakly convergent sequences (and moreover it maps the weak limit to the weak limit).

Sufficiency. Assume the linear operator T maps every weakly convergent sequence in H to a weakly convergent sequence in H . We must show T is bounded. Argue by contradiction: suppose T is unbounded. Then

$$\sup_{\|h\|=1} \|T(h)\| = \infty,$$

so for each natural number n we may choose a unit vector $h_n \in H$ with

$$\|T(h_n)\| \geq n^2.$$

Set $u_n = \frac{1}{n} h_n$. Then $\|u_n\| = 1/n$, so $\{u_n\} \rightarrow 0$ strongly. Strong convergence implies weak convergence, since by the Cauchy-Schwarz Inequality, for each $v \in H$,

$$|\langle u_n, v \rangle - \langle 0, v \rangle| = |\langle u_n, v \rangle| \leq \|u_n\| \|v\| \rightarrow 0.$$

Thus $\{u_n\} \rightharpoonup 0$ weakly. By hypothesis, $\{T(u_n)\}$ converges weakly to some vector in H . But Proposition 7 tells us that a weakly convergent sequence in a Hilbert space is bounded, so there is an $M > 0$ with $\|T(u_n)\| \leq M$ for all n . On the other hand, by linearity of T and the choice of h_n ,

$$\|T(u_n)\| = \frac{1}{n} \|T(h_n)\| \geq \frac{n^2}{n} = n \quad \text{for all } n,$$

which is unbounded. This contradiction shows that T is bounded, hence continuous.

Note that in the sufficiency argument we did not need to assume that T carries weak limits to weak limits; the mere assumption that weakly convergent sequences have weakly convergent images suffices, because weakly convergent sequences are bounded. Combining the two halves: T is continuous if and only if it maps weakly convergent sequences to weakly convergent sequences, and in that case $\{u_n\} \rightharpoonup u$ implies $\{T(u_n)\} \rightharpoonup T(u)$.

Problem 16.48. Show that $K \in \mathcal{L}(H)$ is compact if and only if whenever $\{u_n\} \rightharpoonup u$ in H and $\{v_n\} \rightharpoonup v$ in H , then $\langle K(u_n), v_n \rangle \rightarrow \langle K(u), v \rangle$.

Solution. Necessity. Assume K is compact, and let $\{u_n\} \rightharpoonup u$ and $\{v_n\} \rightharpoonup v$ weakly in H .

By Proposition 19, a compact operator maps weakly convergent sequences to strongly convergent sequences; hence

$$\{K(u_n)\} \rightarrow K(u) \quad \text{strongly.}$$

By Proposition 7, the weakly convergent sequence $\{v_n\}$ is bounded, say $\|v_n\| \leq M$ for all n . Now

estimate, using the Cauchy-Schwarz Inequality,

$$\begin{aligned} |\langle K(u_n), v_n \rangle - \langle K(u), v \rangle| &\leq |\langle K(u_n) - K(u), v_n \rangle| + |\langle K(u), v_n \rangle - \langle K(u), v \rangle| \\ &\leq \|K(u_n) - K(u)\| \|v_n\| + |\langle K(u), v_n - v \rangle| \\ &\leq M \|K(u_n) - K(u)\| + |\langle K(u), v_n - v \rangle|. \end{aligned}$$

The first term tends to 0 because $\{K(u_n)\} \rightarrow K(u)$ strongly, and the second tends to 0 because $\{v_n\} \rightharpoonup v$ weakly and we are testing against the fixed vector $K(u)$. Therefore

$$\lim_{n \rightarrow \infty} \langle K(u_n), v_n \rangle = \langle K(u), v \rangle.$$

Sufficiency. Assume that $K \in \mathcal{L}(H)$ has the stated property. By Proposition 19, in order to prove that K is compact it suffices to show that K maps weakly convergent sequences to strongly convergent sequences.

So let $\{u_n\} \rightharpoonup u$ weakly in H , and define

$$w_n = K(u_n) - K(u) \quad \text{for all } n.$$

Since K is continuous, it maps weakly convergent sequences to weakly convergent sequences with the correct limit (Problem 47; this is also the observation made at the start of the proof of Proposition 19: for each $z \in H$, $\langle K(u_n), z \rangle = \langle u_n, K^*(z) \rangle \rightarrow \langle u, K^*(z) \rangle = \langle K(u), z \rangle$). Hence $\{K(u_n)\} \rightharpoonup K(u)$, and therefore

$$\{w_n\} \rightharpoonup 0 \quad \text{weakly in } H.$$

Now apply the hypothesis to the two weakly convergent sequences $\{u_n\} \rightharpoonup u$ and $\{w_n\} \rightharpoonup 0$; it gives

$$\lim_{n \rightarrow \infty} \langle K(u_n), w_n \rangle = \langle K(u), 0 \rangle = 0.$$

On the other hand, since $\{w_n\} \rightharpoonup 0$ and $K(u)$ is a fixed vector,

$$\lim_{n \rightarrow \infty} \langle K(u), w_n \rangle = 0.$$

Subtracting, and using the bilinearity of the inner product,

$$\|w_n\|^2 = \langle K(u_n) - K(u), w_n \rangle = \langle K(u_n), w_n \rangle - \langle K(u), w_n \rangle \longrightarrow 0 - 0 = 0.$$

Thus $\|K(u_n) - K(u)\| = \|w_n\| \rightarrow 0$, that is, $\{K(u_n)\} \rightarrow K(u)$ strongly. Since $\{u_n\} \rightharpoonup u$ was an arbitrary weakly convergent sequence, Proposition 19 tells us that K is compact.

Problem 16.49. Let $\{P_n\}$ be a sequence of orthogonal projections in $\mathcal{L}(H)$ with the property that for natural numbers n and m , $P_n(H)$ and $P_m(H)$ are orthogonal finite dimensional subspaces of H . Let $\{\lambda_n\}$ be a bounded sequence of real numbers. Show that

$$K = \sum_{n=1}^{\infty} \lambda_n \cdot P_n$$

is a properly defined symmetric operator in $\mathcal{L}(H)$ that is compact if and only if $\{\lambda_n\}$ converges to 0.

Solution. Write $V_n = P_n(H)$, so that P_n is the orthogonal projection of H onto the finite dimensional subspace V_n . The orthogonality hypothesis is of course to be read for $n \neq m$, so that $V_n \perp V_m$ whenever $n \neq m$ (read for $n = m$ it would force every $V_n = \{0\}$). As is implicit in the book's usage of the term orthogonal projection (see Proposition 5 and Problem 33, where the range is a nontrivial closed subspace), we take each V_n to be nontrivial; this is needed for the last assertion, since if all but finitely many P_n were the zero operator then K would be of finite rank, hence compact, no matter

what the bounded sequence $\{\lambda_n\}$ did. Set

$$M = \sup_n |\lambda_n| < \infty.$$

Step 1: an orthonormal system adapted to the P_n , and a Bessel inequality.

Let $d_n = \dim V_n$ and let $\{\varphi_{n,1}, \dots, \varphi_{n,d_n}\}$ be an orthonormal basis of V_n . Since $V_n \perp V_m$ for $n \neq m$, the whole family $\{\varphi_{n,j} : n \geq 1, 1 \leq j \leq d_n\}$ is an orthonormal system in H ; it is countable, being a countable union of finite sets, and we enumerate it as an orthonormal sequence.

We first record the formula for P_n . Fix $h \in H$ and set $w = \sum_{j=1}^{d_n} \langle h, \varphi_{n,j} \rangle \varphi_{n,j} \in V_n$. For each i with $1 \leq i \leq d_n$, orthonormality gives $\langle h - w, \varphi_{n,i} \rangle = \langle h, \varphi_{n,i} \rangle - \langle h, \varphi_{n,i} \rangle = 0$, so $h - w$ is orthogonal to every vector in the span V_n of the $\varphi_{n,i}$, that is, $h - w \in V_n^\perp$. Thus $h = w + (h - w)$ is the decomposition of h furnished by $H = V_n \oplus V_n^\perp$ (Theorem 3), and since that decomposition is unique and P_n is by definition the projection of H onto V_n along $V_n^\perp$, we get $P_n(h) = w$. The General Pythagorean Identity then gives $\|P_n(h)\|^2 = \sum_j |\langle h, \varphi_{n,j} \rangle|^2$. In summary, for each $h \in H$,

$$P_n(h) = \sum_{j=1}^{d_n} \langle h, \varphi_{n,j} \rangle \varphi_{n,j}, \quad \|P_n(h)\|^2 = \sum_{j=1}^{d_n} |\langle h, \varphi_{n,j} \rangle|^2.$$

Bessel's Inequality, applied to the orthonormal sequence just described, therefore yields

$$\sum_{n=1}^{\infty} \|P_n(h)\|^2 \leq \|h\|^2 \quad \text{for all } h \in H. \quad (*)$$

Note also that the vectors $P_1(h), P_2(h), \dots$ are mutually orthogonal, since $P_n(h) \in V_n$.

Step 2: K is properly defined and belongs to $\mathcal{L}(H)$.

For a natural number N define $S_N = \sum_{n=1}^N \lambda_n P_n$, which belongs to $\mathcal{L}(H)$ as a finite linear combination of members of $\mathcal{L}(H)$. Fix $h \in H$. For $N < N'$, by the General Pythagorean Identity applied to the mutually orthogonal vectors $\lambda_n P_n(h)$, $N + 1 \leq n \leq N'$,

$$\|S_{N'}(h) - S_N(h)\|^2 = \sum_{n=N+1}^{N'} \lambda_n^2 \|P_n(h)\|^2 \leq M^2 \sum_{n=N+1}^{N'} \|P_n(h)\|^2.$$

By (*) the series $\sum_n \|P_n(h)\|^2$ converges, so its tails tend to 0; hence $\{S_N(h)\}$ is a Cauchy sequence in H . Since H is complete, $\{S_N(h)\}$ converges strongly. Define

$$K(h) = \lim_{N \rightarrow \infty} S_N(h) = \sum_{n=1}^{\infty} \lambda_n P_n(h).$$

Thus the series defining K converges (strongly, at each h) and K is properly defined. Each S_N is linear, and a pointwise strong limit of linear operators is linear (apply the continuity of vector addition and scalar multiplication to $S_N(\alpha u + \beta v) = \alpha S_N(u) + \beta S_N(v)$ and let $N \rightarrow \infty$), so K is linear. Moreover, using the continuity of the norm together with the General Pythagorean Identity and (*),

$$\|K(h)\|^2 = \lim_{N \rightarrow \infty} \|S_N(h)\|^2 = \sum_{n=1}^{\infty} \lambda_n^2 \|P_n(h)\|^2 \leq M^2 \sum_{n=1}^{\infty} \|P_n(h)\|^2 \leq M^2 \|h\|^2. \quad (**)$$

Hence K is bounded, with $\|K\| \leq M$, so $K \in \mathcal{L}(H)$.

Step 3: K is symmetric.

Each P_n is symmetric, by Problem 33 (or Problem 34). Hence for $u, v \in H$ and every N ,

$$\langle S_N(u), v \rangle = \sum_{n=1}^N \lambda_n \langle P_n(u), v \rangle = \sum_{n=1}^N \lambda_n \langle u, P_n(v) \rangle = \langle u, S_N(v) \rangle.$$

Since $\{S_N(u)\} \rightarrow K(u)$ and $\{S_N(v)\} \rightarrow K(v)$ strongly, and the inner product is continuous in each variable (Cauchy-Schwarz), letting $N \rightarrow \infty$ gives

$$\langle K(u), v \rangle = \langle u, K(v) \rangle \quad \text{for all } u, v \in H,$$

so K is symmetric.

Step 4: if $\{\lambda_n\} \rightarrow 0$, then K is compact.

Let B be the closed unit ball of H . By Proposition 18 it suffices to show that $K(B)$ is totally bounded. Let $\epsilon > 0$. Since $\{\lambda_n\} \rightarrow 0$, choose N so that $|\lambda_n| < \epsilon/4$ for all $n > N$. Exactly as in (**), for every $h \in B$,

$$\|K(h) - S_N(h)\|^2 = \sum_{n=N+1}^{\infty} \lambda_n^2 \|P_n(h)\|^2 \leq \left(\frac{\epsilon}{4}\right)^2 \sum_{n=1}^{\infty} \|P_n(h)\|^2 \leq \left(\frac{\epsilon}{4}\right)^2 \|h\|^2,$$

so that $\|K(h) - S_N(h)\| \leq \epsilon/4$ for all $h \in B$.

Now $S_N(H) \subseteq V_1 + \cdots + V_N$, a finite dimensional subspace of H , and $S_N(B)$ is a bounded subset of it, since $\|S_N\| \leq M$. A bounded subset of a finite dimensional normed space is totally bounded, so there are vectors $y_1, \dots, y_p \in H$ with

$$S_N(B) \subseteq \bigcup_{i=1}^p \{y : \|y - y_i\| < \epsilon/2\}.$$

Given $h \in B$, choose i with $\|S_N(h) - y_i\| < \epsilon/2$; then

$$\|K(h) - y_i\| \leq \|K(h) - S_N(h)\| + \|S_N(h) - y_i\| < \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon.$$

Thus $\{y_1, \dots, y_p\}$ is a finite ϵ -net for $K(B)$. Since $\epsilon > 0$ was arbitrary, $K(B)$ is totally bounded, and by Proposition 18, K is compact.

Step 5: if K is compact, then $\{\lambda_n\} \rightarrow 0$.

For each n choose a unit vector $\varphi_n \in V_n$ (possible since V_n is nontrivial). Since $V_n \perp V_m$ for $n \neq m$, the sequence $\{\varphi_n\}$ is orthonormal. Because $\varphi_n \in V_n$ we have $P_n(\varphi_n) = \varphi_n$, while for $m \neq n$ we have $\varphi_n \in V_n \subseteq V_m^\perp$ and hence $P_m(\varphi_n) = 0$. Therefore

$$K(\varphi_n) = \sum_{m=1}^{\infty} \lambda_m P_m(\varphi_n) = \lambda_n \varphi_n, \quad \|K(\varphi_n)\| = |\lambda_n|.$$

An orthonormal sequence converges weakly to 0: for each $v \in H$, Bessel's Inequality gives $\sum_n |\langle v, \varphi_n \rangle|^2 \leq \|v\|^2 < \infty$, so $\langle \varphi_n, v \rangle \rightarrow 0$. Thus $\{\varphi_n\} \rightarrow 0$ weakly in H .

Since K is compact, Proposition 19 tells us that $\{K(\varphi_n)\} \rightarrow K(0) = 0$ strongly. Hence

$$|\lambda_n| = \|K(\varphi_n)\| \rightarrow 0,$$

that is, $\{\lambda_n\}$ converges to 0.

(One may avoid Proposition 19 here: $\{\varphi_n\}$ is a bounded sequence, and for $n \neq m$, $\|K(\varphi_n) - K(\varphi_m)\|^2 = \|\lambda_n \varphi_n - \lambda_m \varphi_m\|^2 = \lambda_n^2 + \lambda_m^2$. If $\{\lambda_n\}$ did not converge to 0, there would be an $\epsilon_0 > 0$ and a subsequence with $|\lambda_{n_k}| \geq \epsilon_0$, and then any two distinct terms of $\{K(\varphi_{n_k})\}$ would be at distance at least $\epsilon_0\sqrt{2}$, so $\{K(\varphi_{n_k})\}$ would have no strongly convergent subsequence, contradicting Proposition 18.)

Combining Steps 2 through 5: K is a properly defined symmetric operator in $\mathcal{L}(H)$, and it is compact if and only if $\{\lambda_n\}$ converges to 0.

16.8 Problems 16.50–16.56

Problem 16.50. For X a Banach space, define an operator $T \in \mathcal{L}(X)$ to be compact provided $T(B)$ has compact closure. Show that Proposition 18 holds for a general Banach space and Proposition 19 holds for a reflexive Banach space.

Solution. Throughout, $B = \{x \in X \mid \|x\| \leq 1\}$ denotes the closed unit ball of X and $K \in \mathcal{L}(X)$.

Proposition 18 for a general Banach space. We show that the three assertions

(i) K is compact; (ii) $K(B)$ is totally bounded; (iii) if $\{x_n\}$ is a bounded sequence in X , then $\{K(x_n)\}$ has a strongly convergent subsequence

are equivalent. The only property of X that is used is completeness, together with the fact that X , being a metric space, has the property that compactness and sequential compactness coincide.

(i) $\Rightarrow$ (ii). If $\overline{K(B)}$ is compact then it is totally bounded, since a compact metric space is totally bounded. A subset of a totally bounded set is totally bounded, so $K(B)$ is totally bounded.

(ii) $\Rightarrow$ (i). Suppose $K(B)$ is totally bounded. Then its closure $\overline{K(B)}$ is also totally bounded: given $\epsilon > 0$, choose a finite $\epsilon/2$ -net $\{y_1, \dots, y_N\}$ for $K(B)$; if $y \in \overline{K(B)}$, there is $z \in K(B)$ with $\|y - z\| < \epsilon/2$ and an index j with $\|z - y_j\| < \epsilon/2$, so $\|y - y_j\| < \epsilon$. Thus $\{y_1, \dots, y_N\}$ is a finite ϵ -net for $\overline{K(B)}$. Moreover $\overline{K(B)}$ is a closed subset of the complete metric space X and therefore is itself complete. A metric space is compact if and only if it is complete and totally bounded; hence $\overline{K(B)}$ is compact, that is, K is compact.

(i) $\Rightarrow$ (iii). Let $\{x_n\}$ be bounded, say $\|x_n\| \leq M$ for all n , where we may take $M > 0$. Then $x_n/M \in B$, so $K(x_n)/M = K(x_n/M)$ belongs to $K(B) \subseteq \overline{K(B)}$. Since $\overline{K(B)}$ is a compact metric space it is sequentially compact, so there is a subsequence $\{K(x_{n_k})/M\}$ that converges strongly to some $y \in X$. Then $\{K(x_{n_k})\}$ converges strongly to My .

(iii) $\Rightarrow$ (i). Since $\overline{K(B)}$ is a metric space, it suffices to show it is sequentially compact. Let $\{y_n\}$ be a sequence in $\overline{K(B)}$. For each n choose $x_n \in B$ with $\|y_n - K(x_n)\| < 1/n$. The sequence $\{x_n\}$ is bounded, so by (iii) there is a subsequence $\{K(x_{n_k})\}$ converging strongly to some $y \in X$; necessarily $y \in \overline{K(B)}$, since $\overline{K(B)}$ is closed. Then

$$\|y_{n_k} - y\| \leq \|y_{n_k} - K(x_{n_k})\| + \|K(x_{n_k}) - y\| < \frac{1}{n_k} + \|K(x_{n_k}) - y\| \rightarrow 0.$$

Thus $\{y_{n_k}\}$ converges to $y \in \overline{K(B)}$, and $\overline{K(B)}$ is sequentially compact, hence compact.

Proposition 19 for a reflexive Banach space. Let X be a reflexive Banach space and $T \in \mathcal{L}(X)$. We claim T is compact if and only if T maps weakly convergent sequences to strongly convergent sequences, that is, if $\{x_n\} \rightharpoonup x$ then $\{T(x_n)\} \rightarrow T(x)$ strongly.

First observe, for any normed linear space and any $T \in \mathcal{L}(X)$, that T is continuous with respect to the weak topologies in the sequential sense: if $\{x_n\} \rightharpoonup x$ and $\psi \in X^*$, then $\psi \circ T \in X^*$ and therefore

$$\lim_{n \rightarrow \infty} \psi(T(x_n)) = \lim_{n \rightarrow \infty} (\psi \circ T)(x_n) = (\psi \circ T)(x) = \psi(T(x)),$$

so $\{T(x_n)\} \rightharpoonup T(x)$.

Assume T is compact and let $\{x_n\} \rightharpoonup x$. A weakly convergent sequence is bounded (this is a consequence of the Uniform Boundedness Principle applied to the sequence $\{J(x_n)\}$ in $\mathcal{L}(X^*, \mathbb{R})$, where J is the natural embedding of X in X^{**}). Suppose $\{T(x_n)\}$ did not converge strongly to $T(x)$. Then there is an $\epsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ with

$$\|T(x_{n_k}) - T(x)\| \geq \epsilon_0 \quad \text{for all } k.$$

Since $\{x_{n_k}\}$ is bounded and T is compact, by the version of Proposition 18 just proved there is a further subsequence $\{T(x_{n_{k_j}})\}$ that converges strongly, say to z . Strong convergence implies weak convergence, so $\{T(x_{n_{k_j}})\} \rightharpoonup z$; but by the preceding observation $\{T(x_{n_{k_j}})\} \rightharpoonup T(x)$. Weak limits are unique, since X^* separates points of X by a corollary of the Hahn-Banach Theorem, so $z = T(x)$. This contradicts $\|T(x_{n_{k_j}}) - T(x)\| \geq \epsilon_0$. Hence $\{T(x_n)\} \rightarrow T(x)$ strongly. (This half of the equivalence used no reflexivity.)

Conversely, assume T maps weakly convergent sequences to strongly convergent sequences, and let $\{x_n\}$ be a bounded sequence in X . Since X is reflexive, Theorem 17 of Chapter 14 tells us that $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_k}\} \rightharpoonup x$. By hypothesis $\{T(x_{n_k})\}$ converges strongly. Thus criterion (iii) of Proposition 18 is satisfied and T is compact.

Reflexivity is genuinely needed for this converse. Take $X = \ell^1$, which is not reflexive. By a theorem of Schur, not proved in this text, every weakly convergent sequence in ℓ^1 converges strongly; hence the identity operator on ℓ^1 maps weakly convergent sequences to strongly convergent sequences. Yet the identity on the infinite dimensional space ℓ^1 is not compact, by Riesz's Theorem.

Problem 16.51. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be compact and symmetric. Define

$$\alpha = \inf_{\|h\|=1} \langle T(h), h \rangle \quad \text{and} \quad \beta = \sup_{\|h\|=1} \langle T(h), h \rangle.$$

Show that if $\alpha < 0$, then α is an eigenvalue of T and if $\beta > 0$, then β is an eigenvalue of T . Exhibit an example where $\alpha = 0$ and yet α is not an eigenvalue of T , that is, T is one-to-one.

Solution. Since $|\langle T(h), h \rangle| \leq \|T\|$ for every unit vector h , both α and β are finite. Let $S = \{h \in H \mid \|h\| = 1\}$ be the unit sphere and let $Q_T(h) = \langle T(h), h \rangle$ be the quadratic form of T .

The case $\beta > 0$. Choose a sequence $\{h_k\}$ of unit vectors for which

$$\lim_{k \rightarrow \infty} \langle T(h_k), h_k \rangle = \beta.$$

The sequence $\{h_k\}$ is bounded, so by Theorem 6 we may, by passing to a subsequence, suppose that $\{h_k\}$ converges weakly to some $h_* \in H$. Since T is compact, Proposition 19 tells us that $\{T(h_k)\}$ converges strongly to $T(h_*)$. Therefore, by Proposition 7,

$$\langle T(h_*), h_* \rangle = \lim_{k \rightarrow \infty} \langle T(h_k), h_k \rangle = \beta.$$

We claim h_* is a unit vector. First, $\|h_*\| \leq \liminf_{k \rightarrow \infty} \|h_k\| = 1$, since the norm is weakly lower semicontinuous; alternatively, $\|h_*\|^2 = \lim_k \langle h_k, h_* \rangle \leq \|h_*\|$ by the Cauchy-Schwarz inequality, so $\|h_*\| \leq 1$. Since $\langle T(h_*), h_* \rangle = \beta > 0$, we have $h_* \neq 0$. If we had $0 < \|h_*\| < 1$, then the unit vector $u = h_*/\|h_*\|$ would satisfy

$$\langle T(u), u \rangle = \frac{\langle T(h_*), h_* \rangle}{\|h_*\|^2} = \frac{\beta}{\|h_*\|^2} > \beta,$$

contradicting the definition of β as the supremum of Q_T on S . Therefore $\|h_*\| = 1$ and h_* maximizes Q_T on S .

Consequently h_* maximizes the Raleigh quotient $R_T(h) = \langle T(h), h \rangle / \langle h, h \rangle$ on $H \sim \{0\}$, because $R_T(h) = Q_T(h/\|h\|)$ for $h \neq 0$, and $R_T(h_*) = \beta$.

Fix $h_0 \in H$. Since $\|h_*\| = 1$, the vector $h_* + th_0$ is nonzero for all t in some open interval I about 0, and the function $f: I \rightarrow \mathbb{R}$ defined by $f(t) = R_T(h_* + th_0)$ is a quotient of two polynomials in t with nonvanishing denominator, hence differentiable; it attains a maximum value at $t = 0$, so $f'(0) = 0$.

Writing

$$N(t) = \langle T(h_* + th_0), h_* + th_0 \rangle, \quad D(t) = \langle h_* + th_0, h_* + th_0 \rangle,$$

we have $N(0) = \beta$, $D(0) = 1$, and, using the symmetry of T and of the inner product,

$$N'(0) = \langle T(h_0), h_* \rangle + \langle T(h_*), h_0 \rangle = 2\langle T(h_*), h_0 \rangle, \quad D'(0) = 2\langle h_*, h_0 \rangle.$$

Therefore

$$0 = f'(0) = \frac{N'(0)D(0) - N(0)D'(0)}{D(0)^2} = 2\langle T(h_*), h_0 \rangle - 2\beta\langle h_*, h_0 \rangle = 2\langle T(h_*) - \beta h_*, h_0 \rangle.$$

Since $h_0 \in H$ was arbitrary, $T(h_*) - \beta h_* = 0$, that is, $T(h_*) = \beta h_*$ with $h_* \neq 0$. Thus β is an eigenvalue of T .

The case $\alpha < 0$. The operator $-T$ belongs to $\mathcal{L}(H)$ and is symmetric, since $\langle -T(u), v \rangle = -\langle T(u), v \rangle = -\langle u, T(v) \rangle = \langle u, -T(v) \rangle$. It is compact because the compact operators form a linear subspace of $\mathcal{L}(H)$, by Problem 46. Moreover

$$\sup_{\|h\|=1} \langle -T(h), h \rangle = - \inf_{\|h\|=1} \langle T(h), h \rangle = -\alpha > 0.$$

By the case just proved, applied to $-T$, there is a unit vector u with $(-T)(u) = (-\alpha)u$, that is, $T(u) = \alpha u$. Thus α is an eigenvalue of T .

An example with $\alpha = 0$ that is not an eigenvalue. Let $H = \ell^2$ with its standard orthonormal basis $\{e_k\}_{k=1}^\infty$, and define

$$T(x) = \sum_{k=1}^{\infty} \frac{1}{k} \langle x, e_k \rangle e_k \quad \text{for } x \in \ell^2.$$

By the Example following Proposition 18, since $\{1/k\}$ converges to 0, the operator T belongs to $\mathcal{L}(H)$ and is compact. It is symmetric, since for $x, y \in \ell^2$,

$$\langle T(x), y \rangle = \sum_{k=1}^{\infty} \frac{1}{k} \langle x, e_k \rangle \langle y, e_k \rangle = \langle x, T(y) \rangle.$$

For a unit vector x we have, by Parseval's identity,

$$\langle T(x), x \rangle = \sum_{k=1}^{\infty} \frac{1}{k} \langle x, e_k \rangle^2 > 0,$$

the strict inequality holding because $\sum_k \langle x, e_k \rangle^2 = 1$ forces some coefficient to be nonzero. On the other hand $\langle T(e_n), e_n \rangle = 1/n \rightarrow 0$, so

$$\alpha = \inf_{\|x\|=1} \langle T(x), x \rangle = 0.$$

If 0 were an eigenvalue there would be a unit vector x with $T(x) = 0$ and hence $\langle T(x), x \rangle = 0$, contradicting the strict positivity just established. Equivalently, $T(x) = 0$ forces $\langle x, e_k \rangle/k = 0$ for every k , so all Fourier coefficients of x vanish and $x = 0$; that is, T is one-to-one. Thus $\alpha = 0$ is not an eigenvalue of T .

Problem 16.52. Let H be a Hilbert space and $K \in \mathcal{L}(H)$ be compact and symmetric. Suppose

$$\sup_{\|h\|=1} \langle K(h), h \rangle = \beta > 0.$$

Let $\{h_n\}$ be a sequence of unit vectors for which $\lim_{n \rightarrow \infty} \langle K(h_n), h_n \rangle = \beta$. Show that a subsequence of $\{h_n\}$ converges strongly to an eigenvector of T with corresponding eigenvalue β .

Solution. The letter T in the last sentence of the statement is a misprint for K : the only operator in play is K , and it is an eigenvector of K that we produce.

Define

$$A = \beta \cdot \text{Id} - K \in \mathcal{L}(H).$$

Then A is symmetric, since Id and K are, and A is nonnegative, meaning

$$\langle A(h), h \rangle \geq 0 \quad \text{for all } h \in H.$$

Indeed, this is clear for $h = 0$, while for $h \neq 0$ the vector $h/\|h\|$ is a unit vector, so $\langle K(h/\|h\|), h/\|h\| \rangle \leq \beta$, and multiplying by $\|h\|^2$ gives $\langle K(h), h \rangle \leq \beta\|h\|^2$, that is, $\langle A(h), h \rangle \geq 0$.

Step 1: a Cauchy-Schwarz estimate for A . The bilinear form $(u, v) \mapsto \langle A(u), v \rangle$ is symmetric, because A is symmetric, and positive semidefinite by the preceding paragraph. Hence, for all $u, v \in H$ and all

$t \in \mathbb{R}$,

$$0 \leq \langle A(u + tv), u + tv \rangle = \langle A(u), u \rangle + 2t\langle A(u), v \rangle + t^2\langle A(v), v \rangle.$$

A quadratic in t that is nonnegative for all real t has nonpositive discriminant, so

$$\langle A(u), v \rangle^2 \leq \langle A(u), u \rangle \langle A(v), v \rangle \quad \text{for all } u, v \in H.$$

Take $v = A(u)$. Then

$$\|A(u)\|^4 = \langle A(u), A(u) \rangle^2 \leq \langle A(u), u \rangle \langle A(A(u)), A(u) \rangle \leq \langle A(u), u \rangle \|A\| \|A(u)\|^2,$$

where the last step uses the Cauchy-Schwarz inequality in H and $\|A(A(u))\| \leq \|A\| \|A(u)\|$. If $A(u) \neq 0$ we may divide by $\|A(u)\|^2$; the resulting inequality is trivially true when $A(u) = 0$. Hence

$$\|A(u)\|^2 \leq \|A\| \langle A(u), u \rangle \quad \text{for all } u \in H. \quad (*)$$

Step 2: $\{A(h_n)\}$ converges strongly to 0. Since each h_n is a unit vector,

$$\langle A(h_n), h_n \rangle = \beta \|h_n\|^2 - \langle K(h_n), h_n \rangle = \beta - \langle K(h_n), h_n \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, by $(*)$ applied to $u = h_n$,

$$\|\beta h_n - K(h_n)\|^2 = \|A(h_n)\|^2 \leq \|A\| \langle A(h_n), h_n \rangle \rightarrow 0,$$

so $\{\beta h_n - K(h_n)\} \rightarrow 0$ strongly in H .

Step 3: extraction of a strongly convergent subsequence. The sequence $\{h_n\}$ is bounded, being a sequence of unit vectors, and K is compact. By Proposition 18 there is a subsequence $\{h_{n_k}\}$ for which $\{K(h_{n_k})\}$ converges strongly to some $y \in H$. Then

$$\beta h_{n_k} = (\beta h_{n_k} - K(h_{n_k})) + K(h_{n_k}) \rightarrow 0 + y = y \quad \text{strongly.}$$

Since $\beta > 0$, the subsequence $\{h_{n_k}\}$ converges strongly to

$$h_* := \frac{1}{\beta} y.$$

Step 4: h_* is an eigenvector with eigenvalue β . Strong convergence implies convergence of norms, so $\|h_*\| = \lim_{k \rightarrow \infty} \|h_{n_k}\| = 1$; in particular $h_* \neq 0$. Since K is continuous,

$$K(h_*) = \lim_{k \rightarrow \infty} K(h_{n_k}) = y = \beta h_*.$$

Thus the subsequence $\{h_{n_k}\}$ converges strongly to the unit vector h_* , which is an eigenvector of K with corresponding eigenvalue β . In particular β is attained: $\langle K(h_*), h_* \rangle = \beta \|h_*\|^2 = \beta$, so the supremum defining β is a maximum.

Problem 16.53. Let $K \in \mathcal{L}(H)$ be compact. Show that $T = K^*K$ is compact and symmetric. Then use the Hilbert-Schmidt Theorem to show that there is an orthonormal sequence $\{\varphi_k\}$ of H such that $T(\varphi_k) = \lambda_k \varphi_k$ for all k and $T(h) = 0$ if h is orthogonal to $\{\varphi_k\}_{k=0}^\infty$. Conclude that if h is orthogonal to $\{\varphi_k\}_{k=0}^\infty$, then

$$\|K(h)\|^2 = \langle K(h), K(h) \rangle = \langle T(h), h \rangle = 0.$$

Define H_0 to be the closed linear span of $\{K^m(\varphi_k) \mid m \geq 1, k \geq 1\}$. Show that H_0 is closed and separable, $K(H_0) \subseteq H_0$ and $K = 0$ on $H_0^\perp$.

Solution. $T = K^*K$ is compact and symmetric. For $u, v \in H$,

$$\langle T(u), v \rangle = \langle K^*(K(u)), v \rangle = \langle K(u), K(v) \rangle = \langle u, K^*(K(v)) \rangle = \langle u, T(v) \rangle,$$

so T is symmetric. Moreover $K^* \in \mathcal{L}(H)$ and K is compact, so by Problem 46 the composition $T = K^* \circ K$ is compact. (Alternatively, K^* is compact by Schauder's Theorem and the composition of a compact operator with a bounded one is compact.) Note also that T is nonnegative:

$$\langle T(h), h \rangle = \langle K(h), K(h) \rangle = \|K(h)\|^2 \geq 0 \quad \text{for all } h \in H. \quad (*)$$

Diagonalization of T . If $T = 0$ then, by $(*)$, $K = 0$ and every assertion below is trivial; so assume $T \neq 0$.

Suppose first that T does not have finite rank. Since T is compact and symmetric, the Hilbert-Schmidt Theorem provides an orthonormal basis $\{\varphi_k\}_{k \geq 1}$ for $[\ker T]^\perp$ together with nonzero real numbers $\{\lambda_k\}$ with $\lim_{k \rightarrow \infty} \lambda_k = 0$ such that $T(\varphi_k) = \lambda_k \varphi_k$ for each k and

$$T(h) = \sum_{k=1}^{\infty} \lambda_k \langle h, \varphi_k \rangle \varphi_k \quad \text{for all } h \in H.$$

If instead T has finite rank, then by the remark following the proof of the Hilbert-Schmidt Theorem, with $H_{\text{im}} = \text{Im } T$ we have $\ker T = H_{\text{im}}^\perp$ and there is a finite orthonormal basis $\{\varphi_1, \dots, \varphi_N\}$ of H_{im} consisting of eigenvectors of T , $T(\varphi_k) = \lambda_k \varphi_k$ with $\lambda_k \neq 0$. In either case we obtain a finite or infinite orthonormal sequence $\{\varphi_k\}$ in H of eigenvectors of T , with nonzero eigenvalues λ_k , whose closed linear span is exactly $[\ker T]^\perp$. (The origin of the indexing is immaterial; we index by $k \geq 1$.)

Consequently, if h is orthogonal to every φ_k , then h lies in $[\overline{\text{span}\{\varphi_k\}}]^\perp = [[\ker T]^\perp]^\perp = \ker T$, the last equality because $\ker T$ is a closed subspace of H . Thus $T(h) = 0$.

The conclusion about K . Let h be orthogonal to $\{\varphi_k\}$. By the preceding paragraph $T(h) = 0$, and therefore, by $(*)$,

$$\|K(h)\|^2 = \langle K(h), K(h) \rangle = \langle K^*(K(h)), h \rangle = \langle T(h), h \rangle = 0,$$

so $K(h) = 0$. In other words, K vanishes on $[\overline{\text{span}\{\varphi_k\}}]^\perp$.

The subspace H_0 . Here one must take the powers K^m with $m \geq 0$, where $K^0 = \text{Id}$; that is,

$$H_0 = \overline{\text{span}\{K^m(\varphi_k) \mid m \geq 0, k \geq 1\}},$$

so that the vectors φ_k themselves belong to H_0 . This is needed for the last assertion, as the following example shows. In $H = \ell^2$ with standard orthonormal basis $\{e_j\}$, let $K(x) = \langle x, e_1 \rangle e_2$. Then K has rank one, hence is compact, $K^*(y) = \langle y, e_2 \rangle e_1$ and $T(x) = K^*(K(x)) = \langle x, e_1 \rangle e_1$, so we may take $\varphi_1 = e_1$, $\lambda_1 = 1$. Since $K(e_1) = e_2$ and $K^m(e_1) = 0$ for $m \geq 2$, the closed span of $\{K^m(\varphi_k) \mid m \geq 1\}$ is $\text{span}\{e_2\}$, whose orthogonal complement contains e_1 , and $K(e_1) = e_2 \neq 0$. With $m \geq 0$ the span is $\text{span}\{e_1, e_2\}$ and the assertion is correct.

Lest it be thought that the difficulty is an artifact of the finite rank case, here is a counterexample in which T is not of finite rank, so that the Hilbert-Schmidt Theorem applies verbatim. Let $H = \ell^2$ and split its standard orthonormal basis into two disjoint infinite orthonormal sequences $\{f_n\}_{n \geq 1}$ and $\{g_n\}_{n \geq 1}$. Fix a sequence of nonzero real numbers $\{c_n\}$ with $\lim_{n \rightarrow \infty} c_n = 0$ and define

$$K(x) = \sum_{n=1}^{\infty} c_n \langle x, f_n \rangle g_n \quad \text{for } x \in H.$$

The partial sums of this series are of finite rank and converge to K in $\mathcal{L}(H)$, since $\|K(x) - \sum_{n \leq N} c_n \langle x, f_n \rangle g_n\| \leq (\sup_{n > N} |c_n|) \|x\|$ by Bessel's Inequality; so K is compact by the result of Problem 54 proved below, or directly as in the Example following Proposition 18. A computation gives $K^*(y) = \sum_{n \geq 1} c_n \langle y, g_n \rangle f_n$, whence

$$T(x) = K^*(K(x)) = \sum_{n=1}^{\infty} c_n^2 \langle x, f_n \rangle f_n.$$

Thus $T(f_n) = c_n^2 f_n$ with $c_n^2 \neq 0$ for every n , so T is not of finite rank and we may take $\varphi_n = f_n$. Now $K(f_n) = c_n g_n$ while $K(g_n) = 0$, so $K^m(\varphi_n) = 0$ for $m \geq 2$ and the closed linear span of

$\{K^m(\varphi_n) \mid m \geq 1, n \geq 1\}$ is the closed linear span of $\{g_n\}$. Its orthogonal complement contains f_1 , and $K(f_1) = c_1 g_1 \neq 0$. So again the printed assertion fails for $m \geq 1$ and holds for $m \geq 0$.

We prove the three assertions for H_0 as displayed above.

H_0 is closed and separable. By definition H_0 is the closure of the linear span of the set $G = \{K^m(\varphi_k) \mid m \geq 0, k \geq 1\}$, hence is a closed subspace of H . The set G is countable, being indexed by a subset of $\mathbb{N} \times \mathbb{N}$. Let D be the set of all finite linear combinations of members of G with rational coefficients. Then D is countable, being a countable union of countable sets. Moreover D is dense in $\text{span } G$: given $u = \sum_{j=1}^n c_j g_j$ with $c_j \in \mathbb{R}$, $g_j \in G$, and given $\epsilon > 0$, choose rationals q_j with $|c_j - q_j| \cdot \|g_j\| < \epsilon/n$ for each j (if $g_j = 0$ any q_j will do); then $\|u - \sum_j q_j g_j\| < \epsilon$. Since D is dense in $\text{span } G$ and $\text{span } G$ is dense in H_0 , the countable set D is dense in H_0 . Hence H_0 is separable.

$K(H_0) \subseteq H_0$. For each generator, $K(K^m(\varphi_k)) = K^{m+1}(\varphi_k) \in G \subseteq H_0$. Since K is linear and H_0 is a linear subspace, $K(\text{span } G) \subseteq H_0$. Now let $h \in H_0$ and choose a sequence $\{u_n\}$ in $\text{span } G$ with $\{u_n\} \rightarrow h$ strongly. By the continuity of K , $\{K(u_n)\} \rightarrow K(h)$ strongly. Each $K(u_n)$ belongs to H_0 and H_0 is closed, so $K(h) \in H_0$. Thus $K(H_0) \subseteq H_0$.

Neither of the two preceding paragraphs used the presence of the index $m = 0$: word for word, they show that the closed linear span of $\{K^m(\varphi_k) \mid m \geq 1, k \geq 1\}$ is also closed, separable and carried into itself by K . It is only the final assertion that forces the inclusion of $m = 0$.

$K = 0$ on $H_0^\perp$. Since $\varphi_k = K^0(\varphi_k) \in G \subseteq H_0$ for every k , any $h \in H_0^\perp$ is orthogonal to every φ_k . By the conclusion established above, $K(h) = 0$. Hence K vanishes on $H_0^\perp$.

This is precisely the reduction invoked in the proof of Proposition 22: by replacing H with the closed separable subspace H_0 , on whose orthogonal complement K vanishes and which K carries into itself, one may assume H is separable.

Problem 16.54. Let $\mathcal{K}(H)$ denote the set of compact operators in $\mathcal{L}(H)$. Show that $\mathcal{K}(H)$ is a closed subspace of $\mathcal{L}(H)$ that has the set of operators of finite rank as a dense subspace. Is $\mathcal{K}(H)$ an open subset of $\mathcal{L}(H)$?

Solution. $\mathcal{K}(H)$ is a linear subspace. This is Problem 46; we recall the argument. Let K_1, K_2 be compact and $c \in \mathbb{R}$, and let $\{h_n\}$ be a bounded sequence in H . By Proposition 18 there is a subsequence $\{h_{n_k}\}$ for which $\{K_1(h_{n_k})\}$ converges strongly, and a further subsequence $\{h_{n_{k_j}}\}$ for which $\{K_2(h_{n_{k_j}})\}$ also converges strongly. Then $\{(K_1 + cK_2)(h_{n_{k_j}})\}$ converges strongly. By Proposition 18, $K_1 + cK_2$ is compact.

$\mathcal{K}(H)$ is closed in $\mathcal{L}(H)$. Let $\{K_n\}$ be a sequence in $\mathcal{K}(H)$ converging to K in the operator norm of $\mathcal{L}(H)$. By Proposition 18 it suffices to show that $K(B)$ is totally bounded, where B is the closed unit ball of H . Let $\epsilon > 0$. Choose an index N for which $\|K - K_N\| < \epsilon/2$. Since K_N is compact, $K_N(B)$ is totally bounded, so there are $y_1, \dots, y_p$ in H such that every point of $K_N(B)$ is within $\epsilon/2$ of some y_j . Let $h \in B$ and choose j with $\|K_N(h) - y_j\| < \epsilon/2$. Then

$$\|K(h) - y_j\| \leq \|K(h) - K_N(h)\| + \|K_N(h) - y_j\| < \|K - K_N\| \cdot \|h\| + \epsilon/2 \leq \epsilon.$$

Thus $\{y_1, \dots, y_p\}$ is a finite ϵ -net for $K(B)$, so $K(B)$ is totally bounded and K is compact.

Operators of finite rank are dense in $\mathcal{K}(H)$. First, every operator of finite rank is compact, since a bounded subset of a finite dimensional space has compact closure; and the operators of finite rank form a linear subspace of $\mathcal{L}(H)$, since $\text{Im}(F_1 + cF_2) \subseteq \text{Im } F_1 + \text{Im } F_2$. So the finite rank operators form a linear subspace of $\mathcal{K}(H)$, and it remains to prove density.

Let $K \in \mathcal{K}(H)$ and let $\epsilon > 0$. By Proposition 18, $K(B)$ is totally bounded, so there are vectors $y_1, \dots, y_p$ in H such that each point of $K(B)$ lies within ϵ of some y_j . Let

$$M = \text{span}\{y_1, \dots, y_p\},$$

a finite dimensional, hence closed, subspace of H , and let P be the orthogonal projection of H onto M . Define $F = P \circ K$. Since $\text{Im } F \subseteq M$, the operator F has finite rank.

Recall that $P(u)$ is the best approximation to u in M : for $m \in M$, the vector $u - P(u)$ is orthogonal to M while $P(u) - m$ lies in M , so by the Pythagorean identity

$$\|u - m\|^2 = \|u - P(u)\|^2 + \|P(u) - m\|^2 \geq \|u - P(u)\|^2.$$

Now let $h \in B$ and choose j with $\|K(h) - y_j\| < \epsilon$. Since $y_j \in M$, the preceding inequality with $u = K(h)$, $m = y_j$ gives

$$\|K(h) - F(h)\| = \|K(h) - P(K(h))\| \leq \|K(h) - y_j\| < \epsilon.$$

Taking the supremum over $h \in B$ yields $\|K - F\| \leq \epsilon$. Hence every compact operator is the limit in $\mathcal{L}(H)$ of operators of finite rank; that is, the finite rank operators are a dense subspace of $\mathcal{K}(H)$.

Is $\mathcal{K}(H)$ open in $\mathcal{L}(H)$? If H is finite dimensional then every operator in $\mathcal{L}(H)$ is compact, so $\mathcal{K}(H) = \mathcal{L}(H)$ is trivially open. If H is infinite dimensional, $\mathcal{K}(H)$ is not open. Indeed, the zero operator is compact, and for every $\epsilon > 0$ the operator

$$T_\epsilon = \frac{\epsilon}{2} \cdot \text{Id}$$

satisfies $\|T_\epsilon - 0\| = \epsilon/2 < \epsilon$, and yet T_ϵ is not compact: it is invertible, and by Riesz's Theorem an invertible operator on an infinite dimensional space fails to be compact (this is Problem 44; concretely, $T_\epsilon(B) = (\epsilon/2)B$ has compact closure only if B does, which by Riesz's Theorem forces H to be finite dimensional). Hence no ball about the origin in $\mathcal{L}(H)$ is contained in $\mathcal{K}(H)$, and $\mathcal{K}(H)$ is not open.

More generally, this reflects the fact that a proper linear subspace Y of a normed linear space X can never be open: if Y contained a ball $B(0, r)$, then for any $x \in X \setminus \{0\}$ the vector $(r/2\|x\|)x$ would lie in Y , and hence so would x , forcing $Y = X$.

Problem 16.55. Show that the composition of a Fredholm operator of index 0 with an invertible operator is also Fredholm of index 0.

Solution. Let H be a Hilbert space, let $T \in \mathcal{L}(H)$ be Fredholm of index 0 and let $S \in \mathcal{L}(H)$ be invertible, that is, one-to-one and onto. By the Open Mapping Theorem S^{-1} also belongs to $\mathcal{L}(H)$, so S is an isomorphism of H onto H . Both compositions $S \circ T$ and $T \circ S$ belong to $\mathcal{L}(H)$. We show each is Fredholm of index 0. Recall that

$$\text{ind } T = \dim \ker T - \text{codim } \text{Im } T,$$

that codimension is well defined because all linear complements of a subspace of finite codimension have the same dimension (see Problem 66), and that by hypothesis $\dim \ker T = \text{codim } \text{Im } T =: n < \infty$. The composition $T \circ S$. Since S is one-to-one and onto,

$$\ker(T \circ S) = \{h \in H \mid T(S(h)) = 0\} = S^{-1}(\ker T).$$

The restriction of S^{-1} to $\ker T$ is a linear isomorphism of $\ker T$ onto $S^{-1}(\ker T)$, so

$$\dim \ker(T \circ S) = \dim \ker T = n < \infty.$$

Moreover, since $S(H) = H$,

$$\text{Im}(T \circ S) = T(S(H)) = T(H) = \text{Im } T,$$

so $\text{Im}(T \circ S)$ has the same finite codimension n as $\text{Im } T$. Therefore $T \circ S$ is Fredholm and

$$\text{ind}(T \circ S) = n - n = 0.$$

The composition $S \circ T$. Since S is one-to-one, $S(T(h)) = 0$ if and only if $T(h) = 0$, so

$$\ker(S \circ T) = \ker T, \quad \dim \ker(S \circ T) = n < \infty.$$

For the image, $\text{Im}(S \circ T) = S(\text{Im } T)$. Since $\text{Im } T$ has codimension n , there is a subspace X_1 of H with $\dim X_1 = n$ and

$$H = \text{Im } T \oplus X_1.$$

Apply the isomorphism S . First, since S is onto and linear,

$$S(\operatorname{Im} T) + S(X_1) = S(\operatorname{Im} T + X_1) = S(H) = H.$$

Second, since S is one-to-one and linear, $S(\operatorname{Im} T) \cap S(X_1) = S(\operatorname{Im} T \cap X_1) = S(\{0\}) = \{0\}$; explicitly, if $S(u) = S(v)$ with $u \in \operatorname{Im} T$ and $v \in X_1$, then $u = v \in \operatorname{Im} T \cap X_1 = \{0\}$. Hence

$$H = S(\operatorname{Im} T) \oplus S(X_1).$$

Finally, S restricted to X_1 is a linear isomorphism onto $S(X_1)$, so $\dim S(X_1) = \dim X_1 = n$. Thus $S(X_1)$ is a finite dimensional linear complement of $\operatorname{Im}(S \circ T)$ in H , so $\operatorname{Im}(S \circ T)$ has codimension n , and

$$\operatorname{ind}(S \circ T) = n - n = 0.$$

In both orders the composition is Fredholm of index 0, as asserted. (Observe that the argument in fact shows more: for S invertible and T Fredholm of any index, $\operatorname{ind}(S \circ T) = \operatorname{ind}(T \circ S) = \operatorname{ind} T$.)

Problem 16.56. Show that the composition of two Fredholm operators of index 0 is also Fredholm of index 0.

Solution. This is Corollary 25, which the text leaves as an exercise. Let H be a Hilbert space and let T and S in $\mathcal{L}(H)$ be Fredholm of index 0. We show that $S \circ T$ is Fredholm of index 0.

By Theorem 24, an operator in $\mathcal{L}(H)$ is Fredholm of index 0 if and only if it is the sum of an invertible operator and a compact operator. Accordingly, write

$$T = S_1 + K_1 \quad \text{and} \quad S = S_2 + K_2,$$

where $S_1, S_2 \in \mathcal{L}(H)$ are invertible and $K_1, K_2 \in \mathcal{K}(H)$ are compact. Then

$$S \circ T = (S_2 + K_2) \circ (S_1 + K_1) = S_2 \circ S_1 + [S_2 \circ K_1 + K_2 \circ S_1 + K_2 \circ K_1].$$

The operator $S_2 \circ S_1$ is invertible: it is one-to-one, being a composition of one-to-one maps, and onto, being a composition of onto maps; indeed $(S_2 \circ S_1)^{-1} = S_1^{-1} \circ S_2^{-1}$, which belongs to $\mathcal{L}(H)$ by the Open Mapping Theorem.

The bracketed operator is compact. By Problem 46, if $K \in \mathcal{K}(H)$ and $A \in \mathcal{L}(H)$, then both $A \circ K$ and $K \circ A$ belong to $\mathcal{K}(H)$; hence $S_2 \circ K_1$, $K_2 \circ S_1$ and $K_2 \circ K_1$ are each compact. Since $\mathcal{K}(H)$ is a linear subspace of $\mathcal{L}(H)$ (Problem 46, reproved in Problem 54), the sum

$$K := S_2 \circ K_1 + K_2 \circ S_1 + K_2 \circ K_1$$

is compact.

Therefore $S \circ T = (S_2 \circ S_1) + K$ is the sum of an invertible operator and a compact operator, and Theorem 24 tells us that $S \circ T$ is Fredholm of index 0.

A remark on the alternative route. One can also finish directly from the Riesz-Schauder Theorem, using only the forward implication of Theorem 24 and not its converse. Writing $T = S_1 + K_1$ and $S = S_2 + K_2$ as above, factor

$$T = S_1 \circ [\operatorname{Id} + S_1^{-1} \circ K_1], \quad S = S_2 \circ [\operatorname{Id} + S_2^{-1} \circ K_2],$$

where $S_1^{-1} \circ K_1$ and $S_2^{-1} \circ K_2$ are compact by Problem 46. Then

$$S \circ T = S_2 \circ [\operatorname{Id} + S_2^{-1} \circ K_2] \circ S_1 \circ [\operatorname{Id} + S_1^{-1} \circ K_1],$$

and expanding the middle two factors, $[\operatorname{Id} + S_2^{-1} \circ K_2] \circ S_1 = S_1 + S_2^{-1} \circ K_2 \circ S_1$, so

$$S \circ T = S_2 \circ S_1 \circ [\operatorname{Id} + S_1^{-1} \circ S_2^{-1} \circ K_2 \circ S_1] \circ [\operatorname{Id} + S_1^{-1} \circ K_1].$$

Each bracketed factor is of the form $\text{Id} + (\text{compact})$, hence Fredholm of index 0 by the Riesz-Schauder Theorem, and their product is again of that form since

$$(\text{Id} + A)(\text{Id} + B) = \text{Id} + (A + B + A \circ B)$$

with $A + B + A \circ B$ compact whenever A and B are. Composing on the left with the invertible operator $S_2 \circ S_1$ preserves the property of being Fredholm of index 0, by Problem 55. Hence $S \circ T$ is Fredholm of index 0.

16.9 Problems 16.57–16.63

Problem 16.57. Show that an operator $T \in \mathcal{L}(H)$ is Fredholm of index 0 if and only if it is the perturbation of an invertible operator by an operator of finite rank.

Solution. Throughout, H is a Hilbert space and the phrase, perturbation of an invertible operator by an operator of finite rank, means $T = S + F$ with $S \in \mathcal{L}(H)$ invertible and $F \in \mathcal{L}(H)$ of finite rank, that is, $\dim \text{Im } F < \infty$.

The easy direction. Suppose $T = S + F$ with S invertible and F of finite rank. A bounded operator of finite rank is compact: if B is the unit ball of H , then $F(B)$ is a bounded subset of the finite dimensional space $\text{Im } F$, and a bounded subset of a finite dimensional normed space is totally bounded; hence $F(B)$ is totally bounded and F is compact by Proposition 18. Thus $T = S + F$ is a compact perturbation of an invertible operator, and Theorem 24 tells us that T is Fredholm of index 0.

The substantive direction. Suppose T is Fredholm of index 0. By definition $\ker T$ is finite dimensional and $\text{Im } T$ has finite codimension, and

$$\text{ind } T = \dim \ker T - \text{codim } \text{Im } T = 0.$$

Since $\text{Im } T$ is a finite codimensional image of a continuous linear operator, Theorem 12 of Chapter 13 tells us that $\text{Im } T$ is closed. Therefore Proposition 13 applies and gives the orthogonal decomposition

$$H = \text{Im } T \oplus \ker T^*, \quad \ker T^* = [\text{Im } T]^\perp.$$

Consequently $\text{codim } \text{Im } T = \dim \ker T^*$, and since the index vanishes,

$$\dim \ker T = \dim \ker T^* = n < \infty.$$

Two finite dimensional linear spaces of the same dimension are linearly isomorphic, so choose a linear isomorphism A of $\ker T$ onto $\ker T^*$. Let $P \in \mathcal{L}(H)$ be the orthogonal projection of H onto the closed subspace $\ker T$, and define

$$F = A \circ P.$$

Then F is linear, its image is $\ker T^*$, which is n -dimensional, and F is continuous because P is continuous and every linear operator defined on a finite dimensional normed space is continuous. So F has finite rank. Set

$$S = T + F.$$

We claim S is invertible, that is, one-to-one and onto; the Open Mapping Theorem then makes S^{-1} continuous, so S is an isomorphism, and

$$T = S + (-F)$$

exhibits T as the perturbation of the invertible operator S by the finite rank operator $-F$.

To see that S is one-to-one, let $S(h) = 0$, that is, $T(h) + A(P(h)) = 0$. Now $T(h)$ belongs to $\text{Im } T$ while $A(P(h))$ belongs to $\ker T^* = [\text{Im } T]^\perp$. These two vectors are therefore orthogonal, so

$$0 = \|T(h) + A(P(h))\|^2 = \|T(h)\|^2 + \|A(P(h))\|^2.$$

Hence

$$T(h) = 0 \quad \text{and} \quad A(P(h)) = 0.$$

From $T(h) = 0$ we get $h \in \ker T$, so $P(h) = h$; then $A(h) = 0$ and, A being one-to-one on $\ker T$, $h = 0$. To see that S is onto, let $g \in H$. By the decomposition above write $g = g_1 + g_2$ with $g_1 \in \operatorname{Im} T$ and $g_2 \in \ker T^*$. Choose $h_1 \in H$ with $T(h_1) = g_1$; replacing h_1 by its orthogonal projection onto $[\ker T]^\perp$ (which changes h_1 by a vector of $\ker T$ and so does not change $T(h_1)$) we may assume $h_1 \in [\ker T]^\perp$, that is, $P(h_1) = 0$. Set $h_2 = A^{-1}(g_2) \in \ker T$, so that $T(h_2) = 0$ and $P(h_2) = h_2$. Then

$$S(h_1 + h_2) = T(h_1) + T(h_2) + A(P(h_1)) + A(P(h_2)) = g_1 + 0 + 0 + A(h_2) = g_1 + g_2 = g.$$

Thus S is invertible and $T = S - F$ is a finite rank perturbation of an invertible operator. Combining the two directions, T is Fredholm of index 0 if and only if it is the perturbation of an invertible operator by an operator of finite rank.

Observe that this sharpens Theorem 24: in the decomposition $T = S + K$ of a Fredholm operator of index 0 furnished there, the compact operator K may always be taken to be of finite rank.

Problem 16.58. Argue as follows to show that the collection of invertible operators in $\mathcal{L}(H)$ is an open subset of $\mathcal{L}(H)$.

- (i) For $A \in \mathcal{L}(H)$ with $\|A\| < 1$, use the completeness of $\mathcal{L}(H)$ to show that the so-called Neumann series $\sum_{n=0}^{\infty} A^n$ converges to an operator in $\mathcal{L}(H)$ that is the inverse of $\operatorname{Id} - A$.
- (ii) For an invertible operator $S \in \mathcal{L}(H)$ show that for any $T \in \mathcal{L}(H)$, $T = S[\operatorname{Id} + S^{-1}(T - S)]$.
- (iii) Use (i) and (ii) to show that if $S \in \mathcal{L}(H)$ is invertible then so is any $T \in \mathcal{L}(H)$ for which $\|S - T\| < 1/\|S^{-1}\|$.

Solution. We may assume $H \neq \{0\}$, for otherwise $\mathcal{L}(H)$ has a single element and there is nothing to prove.

Recall that $\mathcal{L}(H)$, normed by the operator norm, is complete: it is the space of bounded linear operators from the Banach space H into the Banach space H , and such a space of operators is complete whenever the target space is. Recall also that the operator norm is submultiplicative, $\|R \circ Q\| \leq \|R\| \|Q\|$, so that $\|A^n\| \leq \|A\|^n$ for every natural number n , where $A^0 = \operatorname{Id}$.

(i) Let $\|A\| = r < 1$ and, for each natural number N , define the partial sum

$$S_N = \sum_{n=0}^N A^n \in \mathcal{L}(H).$$

For $M > N$,

$$\|S_M - S_N\| = \left\| \sum_{n=N+1}^M A^n \right\| \leq \sum_{n=N+1}^M \|A\|^n \leq \sum_{n=N+1}^{\infty} r^n = \frac{r^{N+1}}{1-r}.$$

Since $0 \leq r < 1$, the right-hand side tends to 0 as $N \rightarrow \infty$. Hence $\{S_N\}$ is a Cauchy sequence in $\mathcal{L}(H)$, and by the completeness of $\mathcal{L}(H)$ it converges in the operator norm to some $B \in \mathcal{L}(H)$; that is, the Neumann series $\sum_{n=0}^{\infty} A^n$ converges in $\mathcal{L}(H)$ to B .

The series telescopes against $\operatorname{Id} - A$: for each N ,

$$(\operatorname{Id} - A) \circ S_N = S_N \circ (\operatorname{Id} - A) = \sum_{n=0}^N A^n - \sum_{n=1}^{N+1} A^n = \operatorname{Id} - A^{N+1}.$$

Now $\|A^{N+1}\| \leq r^{N+1} \rightarrow 0$, so the right-hand side converges to Id in $\mathcal{L}(H)$. On the other hand, composition with a fixed operator is continuous with respect to the operator norm, since

$$\|(\operatorname{Id} - A) \circ S_N - (\operatorname{Id} - A) \circ B\| \leq \|\operatorname{Id} - A\| \|S_N - B\| \rightarrow 0,$$

and similarly on the other side. Passing to the limit as $N \rightarrow \infty$ therefore gives

$$(\operatorname{Id} - A) \circ B = B \circ (\operatorname{Id} - A) = \operatorname{Id}.$$

Thus $\operatorname{Id} - A$ is one-to-one (if $(\operatorname{Id} - A)(h) = 0$ then $h = B((\operatorname{Id} - A)(h)) = 0$) and onto (given g , $(\operatorname{Id} - A)(B(g)) = g$), so it is invertible with $(\operatorname{Id} - A)^{-1} = B = \sum_{n=0}^{\infty} A^n$.

(ii) Let S be invertible and $T \in \mathcal{L}(H)$ arbitrary. Since $S \circ S^{-1} = \text{Id}$, distributivity of composition over addition gives

$$S \circ [\text{Id} + S^{-1} \circ (T - S)] = S + (S \circ S^{-1}) \circ (T - S) = S + (T - S) = T.$$

(iii) Let S be invertible and let $T \in \mathcal{L}(H)$ satisfy $\|S - T\| < 1/\|S^{-1}\|$. Observe that $S^{-1} \neq 0$, so $\|S^{-1}\| > 0$ and the hypothesis makes sense. Put

$$A = -S^{-1} \circ (T - S) = S^{-1} \circ (S - T).$$

Then, by submultiplicativity,

$$\|A\| \leq \|S^{-1}\| \|S - T\| < \|S^{-1}\| \cdot \frac{1}{\|S^{-1}\|} = 1.$$

By part (i), $\text{Id} - A = \text{Id} + S^{-1} \circ (T - S)$ is invertible. By part (ii),

$$T = S \circ [\text{Id} + S^{-1} \circ (T - S)]$$

is the composition of two invertible operators, hence is one-to-one and onto, hence invertible. (Its inverse is $[\text{Id} + S^{-1}(T - S)]^{-1} \circ S^{-1}$, which is continuous, as the Open Mapping Theorem also assures.) Consequently, if S belongs to the set $\mathcal{G}$ of invertible operators in $\mathcal{L}(H)$, then the open ball of radius $1/\|S^{-1}\| > 0$ about S in $\mathcal{L}(H)$ is contained in $\mathcal{G}$. Therefore $\mathcal{G}$ is an open subset of $\mathcal{L}(H)$.

Problem 16.59. Show that the set of operators in $\mathcal{L}(H)$ that are Fredholm of index 0 is an open subset of $\mathcal{L}(H)$.

Solution. Let $\mathcal{F}_0$ denote the set of operators in $\mathcal{L}(H)$ that are Fredholm of index 0. If $H = \{0\}$, then $\mathcal{L}(H)$ consists of a single operator and every subset of $\mathcal{L}(H)$ is open, so there is nothing to prove; assume henceforth that $H \neq \{0\}$. Let $T_0 \in \mathcal{F}_0$. We exhibit an open ball about T_0 contained in $\mathcal{F}_0$. Since T_0 is Fredholm of index 0, Theorem 24 provides an invertible $S \in \mathcal{L}(H)$ and a compact $K \in \mathcal{L}(H)$ for which

$$T_0 = S + K.$$

Since $H \neq \{0\}$, the operator S^{-1} is not the zero operator, so $\|S^{-1}\| > 0$ and we may set

$$\varepsilon = \frac{1}{\|S^{-1}\|} > 0.$$

Let $T \in \mathcal{L}(H)$ satisfy $\|T - T_0\| < \varepsilon$. Define

$$S' = S + (T - T_0) \in \mathcal{L}(H).$$

Then

$$\|S - S'\| = \|T - T_0\| < \varepsilon = \frac{1}{\|S^{-1}\|},$$

so by Problem 58(iii) the operator S' is invertible. Moreover

$$S' + K = S + (T - T_0) + K = (S + K) + (T - T_0) = T_0 + T - T_0 = T.$$

Thus $T = S' + K$ is the sum of an invertible operator and a compact operator, and Theorem 24 tells us that T is Fredholm of index 0, that is, $T \in \mathcal{F}_0$.

Therefore the open ball in $\mathcal{L}(H)$ of radius $1/\|S^{-1}\|$ centered at T_0 is contained in $\mathcal{F}_0$. Since $T_0 \in \mathcal{F}_0$ was arbitrary, $\mathcal{F}_0$ is an open subset of $\mathcal{L}(H)$.

Notice that the argument shows something slightly stronger and quite concrete: the compact part K of the decomposition need not be changed at all, only the invertible part, and the size of the admissible perturbation is governed by $\|S^{-1}\|$ alone.

Problem 16.60. By following the orthogonal approximation sequence method used in the proof of Proposition 22, provide another proof of Proposition 14 in case H is separable.

Solution. Proposition 14 asserts: if $T \in \mathcal{L}(H)$ and there is a $c > 0$ for which

$$\langle T(h), h \rangle \geq c \|h\|^2 \quad \text{for all } h \in H,$$

then T is invertible; this is the printed inequality (14), and an operator satisfying it is called positive definite. We prove Proposition 14 assuming H is separable, using orthogonal projection sequences as in the proof of Proposition 22.

First, a consequence of the hypothesis that we shall use repeatedly. By the Cauchy-Schwarz Inequality, for every $h \in H$,

$$c \|h\|^2 \leq \langle T(h), h \rangle \leq \|T(h)\| \|h\|,$$

and therefore

$$\|T(h)\| \geq c \|h\| \quad \text{for all } h \in H. \quad (*)$$

In particular $\ker T = \{0\}$, so T is one-to-one. It remains to prove that T is onto; the Open Mapping Theorem then makes T^{-1} continuous, so T is invertible.

If H is finite dimensional, a one-to-one linear operator of H into itself is onto and we are done. So assume H is infinite dimensional and separable. By Theorem 11, H possesses an orthonormal basis $\{\varphi_n\}$. Let $\{P_n\}$ be the orthogonal projection sequence induced by $\{\varphi_n\}$, and for each natural number n let H_n be the linear span of $\{\varphi_1, \dots, \varphi_n\}$. Then P_n is the orthogonal projection of H onto H_n ; in particular P_n is symmetric, $\|P_n\| = 1$, $P_n(u) = u$ for $u \in H_n$, and, since $\{\varphi_n\}$ is an orthonormal basis, $\{P_n\} \rightarrow \text{Id}$ pointwise on H .

The finite dimensional approximations. Fix n and consider the restriction to H_n of the operator $P_n \circ T$; since P_n takes values in H_n , this is a linear operator of the finite dimensional space H_n into itself. For $u \in H_n$, using the symmetry of P_n and $P_n(u) = u$,

$$\langle P_n(T(u)), u \rangle = \langle T(u), P_n(u) \rangle = \langle T(u), u \rangle \geq c \|u\|^2.$$

Hence, by the Cauchy-Schwarz Inequality again,

$$\|P_n(T(u))\| \geq c \|u\| \quad \text{for all } u \in H_n. \quad (**)$$

In particular the restriction of $P_n \circ T$ to H_n is one-to-one, and a one-to-one linear operator on a finite dimensional linear space is onto. Therefore this restriction maps H_n onto H_n .

Solving the approximate equations. Let $h_* \in H$. Since $P_n(h_*) \in H_n$, the preceding paragraph provides a vector $u_n \in H_n$ for which

$$P_n(T(u_n)) = P_n(h_*). \quad (1)$$

By (**) and $\|P_n\| = 1$,

$$c \|u_n\| \leq \|P_n(T(u_n))\| = \|P_n(h_*)\| \leq \|h_*\|,$$

so that $\|u_n\| \leq \|h_*\|/c$ for every n ; the sequence $\{u_n\}$ is bounded.

Passage to the limit. By Theorem 6, every bounded sequence in a Hilbert space has a weakly convergent subsequence; choose a subsequence $\{u_{n_k}\}$ that converges weakly to some $u \in H$. Then $\{T(u_{n_k})\}$ converges weakly to $T(u)$: indeed, for each $w \in H$, by the definition of the adjoint,

$$\langle T(u_{n_k}), w \rangle = \langle u_{n_k}, T^*(w) \rangle \rightarrow \langle u, T^*(w) \rangle = \langle T(u), w \rangle.$$

Now take the inner product of each side of (1) with an arbitrary $v \in H$ and use the symmetry of P_n :

$$\langle T(u_n), P_n(v) \rangle = \langle P_n(T(u_n)), v \rangle = \langle P_n(h_*), v \rangle = \langle h_*, P_n(v) \rangle \quad \text{for all } n \text{ and } v \in H. \quad (2)$$

Fix $v \in H$ and let $n = n_k$ in (2), then let $k \rightarrow \infty$. Since $\{P_{n_k}(v)\} \rightarrow v$ strongly and $\{T(u_{n_k})\}$ converges weakly to $T(u)$, Proposition 7 gives

$$\lim_{k \rightarrow \infty} \langle T(u_{n_k}), P_{n_k}(v) \rangle = \langle T(u), v \rangle.$$

On the right-hand side of (2), the strong convergence $\{P_{n_k}(v)\} \rightarrow v$ and the continuity of the inner product give

$$\lim_{k \rightarrow \infty} \langle h_*, P_{n_k}(v) \rangle = \langle h_*, v \rangle.$$

Therefore

$$\langle T(u), v \rangle = \langle h_*, v \rangle \quad \text{for all } v \in H,$$

that is, $\langle T(u) - h_*, v \rangle = 0$ for all $v \in H$. Taking $v = T(u) - h_*$ yields $T(u) = h_*$.

Since $h_* \in H$ was arbitrary, $T(H) = H$. Together with the injectivity furnished by (*), T is one-to-one and onto, hence invertible. This reproves Proposition 14 for separable H .

Problem 16.61. For $T \in \mathcal{L}(H)$, suppose that $\langle T(h), h \rangle \geq \|h\|^2$ for all $h \in H$. Assume that $K \in \mathcal{L}(H)$ is compact and $T + K$ is one-to-one. Show that $T + K$ is onto.

Solution. We first record a lemma, which is the observation made in the text in the proof of Theorem 24.

Lemma. If $K \in \mathcal{L}(H)$ is compact and $R \in \mathcal{L}(H)$, then $R \circ K$ is compact.

Indeed, let B be the unit ball of H . By Proposition 18, $K(B)$ is totally bounded. Given $\varepsilon > 0$, if $R = 0$ the conclusion is trivial; otherwise cover $K(B)$ by finitely many balls of radius $\varepsilon/\|R\|$ centered at points $y_1, \dots, y_m$. If $x \in B$, choose i with $\|K(x) - y_i\| < \varepsilon/\|R\|$; then $\|R(K(x)) - R(y_i)\| \leq \|R\| \|K(x) - y_i\| < \varepsilon$. Hence the balls of radius ε about $R(y_1), \dots, R(y_m)$ cover $(R \circ K)(B)$, so $(R \circ K)(B)$ is totally bounded and $R \circ K$ is compact, again by Proposition 18.

Now to the problem. The hypothesis $\langle T(h), h \rangle \geq \|h\|^2$ for all $h \in H$ is the positive definiteness hypothesis (14) of Proposition 14, with the constant c there taken to be 1. Therefore T is invertible, and by the Open Mapping Theorem T^{-1} belongs to $\mathcal{L}(H)$.

Define

$$K' = T^{-1} \circ K.$$

By the Lemma, K' is compact. Moreover, since $T \circ T^{-1} = \text{Id}$,

$$T \circ [\text{Id} + K'] = T + (T \circ T^{-1}) \circ K = T + K. \quad (1)$$

We claim $\text{Id} + K'$ is one-to-one. Let $h \in H$ satisfy $(\text{Id} + K')(h) = 0$. Applying T and using (1),

$$(T + K)(h) = T((\text{Id} + K')(h)) = T(0) = 0,$$

and since $T + K$ is one-to-one by hypothesis, $h = 0$. This proves the claim.

Since K' is compact and $\text{Id} + K'$ is one-to-one, Proposition 22 tells us that $\text{Id} + K'$ is onto, that is, $(\text{Id} + K')(H) = H$. (Equivalently, invoke the Riesz-Schauder Theorem, whose final assertion is that $\text{Id} + K'$ is one-to-one if and only if it is onto.)

Finally, T is onto, so by (1),

$$(T + K)(H) = T((\text{Id} + K')(H)) = T(H) = H.$$

Thus $T + K$ is onto.

An alternative route is available: since T is invertible and K is compact, Theorem 24 tells us that $T + K$ is Fredholm of index 0, so

$$\text{codim Im}(T + K) = \dim \ker(T + K) = 0,$$

because $T + K$ is one-to-one. As a Fredholm operator, $T + K$ has closed image, and a closed subspace of codimension 0 is all of H . Hence $T + K$ is onto.

Problem 16.62. Let $K \in \mathcal{L}(H)$ be compact and $\mu \in \mathbb{R}$ have $|\mu| > \|K\|$. Show that $\mu - K$ is invertible.

Solution. Here $\mu - K$ denotes the operator $\mu \text{Id} - K$, which belongs to $\mathcal{L}(H)$. Observe first that $\mu \neq 0$, since $|\mu| > \|K\| \geq 0$.

First proof, by the Fredholm Alternative. We check that alternative (i) of Corollary 23 fails. Suppose there were a nonzero $h \in H$ with

$$\mu h - K(h) = 0.$$

Then $K(h) = \mu h$, so

$$|\mu| \|h\| = \|\mu h\| = \|K(h)\| \leq \|K\| \|h\| < |\mu| \|h\|,$$

where the strict inequality uses $\|h\| > 0$ and $\|K\| < |\mu|$. This is a contradiction. Hence the equation $\mu h - K(h) = 0$ has no nonzero solution, so alternative (i) of Corollary 23 does not hold. Since K is compact and μ is a nonzero real number, Corollary 23 asserts that exactly one of (i) and (ii) holds; therefore (ii) holds: for every $h_0 \in H$ there is a unique solution $h \in H$ of

$$\mu h - K(h) = h_0.$$

Existence for every h_0 says that $\mu \text{Id} - K$ is onto, and uniqueness says it is one-to-one. Hence $\mu \text{Id} - K$ is invertible, and by the Open Mapping Theorem its inverse is continuous.

Second proof, by the Neumann series. Since $\mu \neq 0$ we may factor

$$\mu \text{Id} - K = \mu \left[\text{Id} - \frac{1}{\mu} K \right].$$

Put $A = (1/\mu)K \in \mathcal{L}(H)$. Then

$$\|A\| = \frac{\|K\|}{|\mu|} < 1,$$

so by Problem 58(i) the operator $\text{Id} - A$ is invertible, with inverse $\sum_{n=0}^{\infty} \mu^{-n} K^n$. Multiplication by the nonzero scalar μ is an invertible operator, so $\mu \text{Id} - K = \mu(\text{Id} - A)$ is invertible, with

$$(\mu \text{Id} - K)^{-1} = \frac{1}{\mu} \sum_{n=0}^{\infty} \frac{K^n}{\mu^n} = \sum_{n=0}^{\infty} \frac{K^n}{\mu^{n+1}}.$$

Note that the second proof does not use the compactness of K at all: for every $K \in \mathcal{L}(H)$ and every real μ with $|\mu| > \|K\|$, the operator $\mu \text{Id} - K$ is invertible. Compactness is what makes the first proof, through the Fredholm Alternative, available. Either way, a real number μ can fail to make $\mu \text{Id} - K$ invertible only if $|\mu| \leq \|K\|$.

Problem 16.63. Let $S \in \mathcal{L}(H)$ have $\|S\| < 1$, $K \in \mathcal{L}(H)$ be compact and $(\text{Id} + S + K)(H) = H$. Show that $\text{Id} + S + K$ is one-to-one.

Solution. Since $\| -S \| = \|S\| < 1$, Problem 58(i), applied with $A = -S$, tells us that

$$\text{Id} + S = \text{Id} - (-S)$$

is invertible, with inverse $\sum_{n=0}^{\infty} (-S)^n$. Denote this inverse by $R \in \mathcal{L}(H)$; it is continuous, by the Open Mapping Theorem or directly because it is the sum of a convergent series in $\mathcal{L}(H)$.

Define

$$K' = R \circ K.$$

The composition of a compact operator with a continuous linear operator is compact: if B is the closed unit ball of H , then $K(B)$ is totally bounded by Proposition 18, and covering $K(B)$ by finitely many balls of radius $\varepsilon/\|R\|$ (the case $R = 0$ being trivial) and applying R produces a finite ε -net for $(R \circ K)(B)$, so $(R \circ K)(B)$ is totally bounded and $R \circ K$ is compact, again by Proposition 18. This is the same observation the text makes in the proof of Theorem 24. Hence K' is compact. Furthermore, since $(\text{Id} + S) \circ R = \text{Id}$,

$$(\text{Id} + S) \circ [\text{Id} + K'] = (\text{Id} + S) + [(\text{Id} + S) \circ R] \circ K = (\text{Id} + S) + K = \text{Id} + S + K. \quad (1)$$

We claim that $\text{Id} + K'$ is onto. Let $g \in H$. Set $g_0 = (\text{Id} + S)(g) \in H$. By hypothesis $\text{Id} + S + K$ maps H onto H , so there is an $h \in H$ with $(\text{Id} + S + K)(h) = g_0$. By (1),

$$(\text{Id} + S)((\text{Id} + K')(h)) = g_0 = (\text{Id} + S)(g).$$

Since $\text{Id} + S$ is invertible, in particular one-to-one, we conclude that

$$(\text{Id} + K')(h) = g.$$

As $g \in H$ was arbitrary, $(\text{Id} + K')(H) = H$, which proves the claim.

Now K' is compact, so the Riesz-Schauder Theorem applies to $\text{Id} + K'$: it asserts that $\text{Id} + K'$ is one-to-one if and only if it is onto. Since we have just shown $\text{Id} + K'$ is onto, it is one-to-one.

Finally, $\text{Id} + S + K$ is one-to-one. Indeed, suppose $(\text{Id} + S + K)(h) = 0$. By (1),

$$(\text{Id} + S)((\text{Id} + K')(h)) = 0,$$

and since $\text{Id} + S$ is one-to-one, $(\text{Id} + K')(h) = 0$; since $\text{Id} + K'$ is one-to-one, $h = 0$. Therefore $\text{Id} + S + K$ is one-to-one, and being also onto by hypothesis, it is in fact invertible.

16.10 Problems 16.64–16.66

Problem 16.64. Let $\mathcal{GL}(H)$ denote the set of invertible operators in $\mathcal{L}(H)$.

(i) Show that under the operation of composition of operators, $\mathcal{GL}(H)$ is a group: it is called the **general linear group** of H .

(ii) An operator T in $\mathcal{GL}(H)$ is said to be **orthogonal**, provided that $T^* = T^{-1}$. Show that the set of orthogonal operators is a subgroup of $\mathcal{GL}(H)$: it is called the **orthogonal group**.

Solution. Throughout, H is a Hilbert space, $\mathcal{L}(H)$ denotes the bounded linear operators of H into itself, and, as in Section 16.7, an operator $T \in \mathcal{L}(H)$ is called **invertible** provided it is one-to-one and onto.

Part (i). We verify the four group axioms for the set $\mathcal{GL}(H)$ of invertible operators, with composition $\circ$ as the operation.

The operation is well defined on $\mathcal{GL}(H)$ (closure). Let S and T belong to $\mathcal{GL}(H)$. The composition $S \circ T$ of two linear mappings is linear, and it is bounded since

$$\|(S \circ T)(h)\| = \|S(T(h))\| \leq \|S\| \|T(h)\| \leq \|S\| \|T\| \|h\| \quad \text{for all } h \in H.$$

Thus $S \circ T$ belongs to $\mathcal{L}(H)$. It is one-to-one: if $(S \circ T)(h) = 0$, then $T(h) = 0$ because S is one-to-one, and hence $h = 0$ because T is one-to-one. It is onto: given $h \in H$, since S is onto there is a $u \in H$ with $S(u) = h$, and since T is onto there is a $v \in H$ with $T(v) = u$; then $(S \circ T)(v) = h$. Therefore $S \circ T$ belongs to $\mathcal{GL}(H)$.

Associativity. Composition of mappings is associative: for R, S, T mappings of H into H and $h \in H$,

$$((R \circ S) \circ T)(h) = R(S(T(h))) = (R \circ (S \circ T))(h).$$

Identity. The identity operator Id is linear and bounded, with $\|\text{Id}\| = 1$ (for $H \neq \{0\}$), and is clearly one-to-one and onto. Hence $\text{Id} \in \mathcal{GL}(H)$, and $\text{Id} \circ T = T \circ \text{Id} = T$ for every T .

Inverses. Let $T \in \mathcal{GL}(H)$. Since T is a one-to-one linear mapping of H onto H , the set-theoretic inverse $T^{-1}: H \rightarrow H$ is well defined, and it is linear: given $u, v \in H$ and scalars α, β , put $h = T^{-1}(u)$ and $k = T^{-1}(v)$; then $T(\alpha h + \beta k) = \alpha u + \beta v$, so $T^{-1}(\alpha u + \beta v) = \alpha h + \beta k = \alpha T^{-1}(u) + \beta T^{-1}(v)$. Moreover H is a Banach space and T is a continuous linear mapping of H onto H , so, as observed in Section 16.7, the Open Mapping Theorem tells us that T^{-1} is continuous. Hence $T^{-1} \in \mathcal{L}(H)$. Of course T^{-1} is itself one-to-one and onto (its inverse being T), so $T^{-1} \in \mathcal{GL}(H)$, and

$$T \circ T^{-1} = T^{-1} \circ T = \text{Id}.$$

Therefore $\mathcal{GL}(H)$ is a group under composition.

Part (ii). Let

$$\mathcal{O}(H) = \{ T \in \mathcal{GL}(H) : T^* = T^{-1} \}$$

denote the set of orthogonal operators. We use the structural properties of adjoints recorded in (12) of Section 16.4: for $T, S \in \mathcal{L}(H)$,

$$(T^*)^* = T, \quad (T + S)^* = T^* + S^*, \quad (T \circ S)^* = S^* \circ T^*.$$

$\mathcal{O}(H)$ is nonempty. Since $\langle \text{Id}(u), v \rangle = \langle u, v \rangle = \langle u, \text{Id}(v) \rangle$ for all $u, v \in H$, the defining relation (11) of the adjoint gives $\text{Id}^* = \text{Id}$. As $\text{Id}^{-1} = \text{Id}$, we conclude $\text{Id} \in \mathcal{O}(H)$.

Closure under composition. Let $S, T \in \mathcal{O}(H)$. By part (i), $S \circ T \in \mathcal{GL}(H)$ and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. On the other hand, by the third identity above,

$$(S \circ T)^* = T^* \circ S^* = T^{-1} \circ S^{-1} = (S \circ T)^{-1}.$$

Hence $S \circ T \in \mathcal{O}(H)$.

Closure under inversion. Let $T \in \mathcal{O}(H)$. By part (i), $T^{-1} \in \mathcal{GL}(H)$, and $(T^{-1})^{-1} = T$. Since $T^{-1} = T^*$, the first identity above gives

$$(T^{-1})^* = (T^*)^* = T = (T^{-1})^{-1}.$$

Hence $T^{-1} \in \mathcal{O}(H)$.

A nonempty subset of a group that is closed under the group operation and under the taking of inverses is a subgroup. Therefore $\mathcal{O}(H)$ is a subgroup of $\mathcal{GL}(H)$.

Remark. The name is justified by the fact that an operator $T \in \mathcal{GL}(H)$ is orthogonal if and only if it preserves the inner product. Indeed, if $T^* = T^{-1}$, then for all $u, v \in H$,

$$\langle T(u), T(v) \rangle = \langle u, T^*(T(v)) \rangle = \langle u, T^{-1}(T(v)) \rangle = \langle u, v \rangle.$$

Conversely, if $T \in \mathcal{GL}(H)$ preserves the inner product, then $\langle u, (T^* \circ T)(v) - v \rangle = 0$ for all $u, v \in H$, so $T^* \circ T = \text{Id}$ and therefore $T^* = T^{-1}$. In particular every orthogonal operator is an isometry of H onto H .

Problem 16.65. Let H be a Hilbert space, $T \in \mathcal{L}(H)$ be Fredholm of index zero, and $K \in \mathcal{L}(H)$ be compact. Show that $T + K$ is Fredholm of index zero.

Solution. We first record the elementary fact that the sum of two compact operators is compact.

Claim. If K_1 and K_2 belong to $\mathcal{L}(H)$ and both are compact, then $K_1 + K_2$ is compact.

Indeed, $K_1 + K_2$ belongs to $\mathcal{L}(H)$, since $\mathcal{L}(H)$ is a linear space. Let $\{h_n\}$ be a bounded sequence in H . Since K_1 is compact, Proposition 18 tells us that there is a subsequence $\{h_{n_k}\}$ for which $\{K_1(h_{n_k})\}$ converges strongly in H . The sequence $\{h_{n_k}\}$ is again bounded, so, since K_2 is compact, a second application of Proposition 18 provides a further subsequence $\{h_{n_{k_j}}\}$ for which $\{K_2(h_{n_{k_j}})\}$ converges strongly. Along this final subsequence both $\{K_1(h_{n_{k_j}})\}$ and $\{K_2(h_{n_{k_j}})\}$ converge strongly, and therefore so does their sum $\{(K_1 + K_2)(h_{n_{k_j}})\}$. Thus every bounded sequence $\{h_n\}$ has a subsequence whose image under $K_1 + K_2$ converges strongly, and so, again by Proposition 18, $K_1 + K_2$ is compact. This proves the claim.

Now let $T \in \mathcal{L}(H)$ be Fredholm of index 0 and let $K \in \mathcal{L}(H)$ be compact.

Since T is Fredholm of index 0, Theorem 24 tells us that we may write

$$T = S + K_0,$$

where $S \in \mathcal{L}(H)$ is invertible and $K_0 \in \mathcal{L}(H)$ is compact. Therefore

$$T + K = S + (K_0 + K).$$

By the claim, $K_0 + K$ is a compact operator in $\mathcal{L}(H)$. Hence $T + K$ is the sum of an invertible operator in $\mathcal{L}(H)$ and a compact operator in $\mathcal{L}(H)$. By the converse half of Theorem 24, $T + K$ is Fredholm of index 0.

Remark. It is worth unwinding the converse half of Theorem 24 in the present case, since it exhibits exactly where the Riesz-Schauder Theorem enters. Writing $K' = K_0 + K$, we have the factorization

$$T + K = S \circ [\text{Id} + S^{-1} \circ K'].$$

The operator S^{-1} is continuous, by the Open Mapping Theorem. Moreover $S^{-1} \circ K'$ is compact: if $\{h_n\}$ is bounded in H , then by Proposition 18 there is a subsequence along which $\{K'(h_n)\}$ converges strongly, and the continuity of S^{-1} forces $\{S^{-1}(K'(h_n))\}$ to converge strongly along that same subsequence; a third appeal to Proposition 18 gives compactness. The Riesz-Schauder Theorem tells us that $\text{Id} + S^{-1} \circ K'$ is Fredholm of index 0, and by Problem 55 the composition of a Fredholm operator of index 0 with an invertible operator is again Fredholm of index 0. Therefore $T + K$ is Fredholm of index 0.

Problem 16.66. Let X_0 be a finite codimensional subspace of a Banach space X . Show that all finite dimensional linear complements of X_0 in X have the same dimension.

Solution. Recall the definitions in force. A linear subspace W of a linear space X is called a **linear complement** of the subspace V provided there is the direct sum decomposition $X = V \oplus W$, that is, provided

$$V + W = X \quad \text{and} \quad V \cap W = \{0\};$$

equivalently, every $x \in X$ has a representation $x = u + w$ with $u \in V$ and $w \in W$, and this representation is unique. A subspace X_0 of X is said to be of finite codimension provided it possesses a finite dimensional linear complement in X .

We prove the following stronger, purely algebraic assertion, from which the exercise follows at once.

Claim. Let X be a linear space, let X_0 be a linear subspace of X , and let X_1 and X_2 be two linear complements of X_0 in X . Then X_1 and X_2 are isomorphic as linear spaces.

To prove the claim, use the decomposition $X = X_0 \oplus X_2$ to define the projection of X onto X_2 along X_0 : for $x \in X$, write $x = u + v$ with $u \in X_0$ and $v \in X_2$, this representation being unique, and set

$$P(x) = v.$$

Then $P: X \rightarrow X$ is linear. Indeed, if $x = u + v$ and $x' = u' + v'$ are the decompositions of x and x' , and α, β are scalars, then $\alpha x + \beta x' = (\alpha u + \beta u') + (\alpha v + \beta v')$ with $\alpha u + \beta u' \in X_0$ and $\alpha v + \beta v' \in X_2$; by uniqueness of the decomposition, $P(\alpha x + \beta x') = \alpha v + \beta v' = \alpha P(x) + \beta P(x')$. This is precisely the projection of X onto X_2 along X_0 recorded in (15) of Section 13.4, where it is also noted that $P^2 = P$, that $P(X) = X_2$ and that $(\text{Id} - P)(X) = X_0$. Directly from the definition of P we have

$$P(X) = X_2, \quad \ker P = X_0, \quad P(v) = v \quad \text{for all } v \in X_2.$$

(The middle equality holds because $P(x) = 0$ exactly when the X_2 -component of x vanishes, that is, exactly when $x \in X_0$.)

Let $\Phi: X_1 \rightarrow X_2$ be the restriction of P to X_1 ; it is a linear mapping of X_1 into X_2 .

Φ is one-to-one: if $x \in X_1$ and $\Phi(x) = P(x) = 0$, then $x \in \ker P = X_0$, so $x \in X_0 \cap X_1 = \{0\}$ and hence $x = 0$. Since Φ is linear, it is one-to-one.

Φ is onto X_2 : let $v \in X_2$. Since $X = X_0 \oplus X_1$, we may write $v = u' + w$ with $u' \in X_0$ and $w \in X_1$. Apply P and use its linearity together with $P(u') = 0$ and $P(v) = v$:

$$v = P(v) = P(u') + P(w) = P(w) = \Phi(w),$$

and $w \in X_1$. Hence Φ maps X_1 onto X_2 .

Therefore Φ is a linear isomorphism of X_1 onto X_2 , which proves the claim.

Now suppose X_0 is a finite codimensional subspace of the Banach space X , and let X_1 and X_2 be two finite dimensional linear complements of X_0 in X . By the claim there is a linear isomorphism of X_1 onto X_2 . A linear isomorphism carries a basis of X_1 onto a basis of X_2 : it maps a spanning set onto a spanning set, since it is onto and linear, and it maps a linearly independent set onto a linearly

independent set, since it is one-to-one and linear. Since the dimension of a finite dimensional linear space is the common cardinality of its bases, we conclude

$$\dim X_1 = \dim X_2.$$

Thus all finite dimensional linear complements of X_0 in X have the same dimension, and the codimension of X_0 , $\text{codim } X_0$, is unambiguously defined as this common dimension.

Remark. Two observations. First, the argument is purely algebraic: neither the norm on X nor its completeness was used, and no complement was assumed closed. Second, the claim shows more than was asked, namely that **every** linear complement of X_0 is isomorphic to every other one. In particular, if X_0 has one finite dimensional linear complement, then every linear complement of X_0 in X is finite dimensional, of that same dimension.

17 General Measure Spaces: Their Properties and Construction

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17.1 Problems 17.1–17.7

Problem 17.1. Let f be a nonnegative Lebesgue measurable function on $\mathbb{R}$. For each Lebesgue measurable subset E of $\mathbb{R}$, define $\mu(E) = \int_E f$, the Lebesgue integral of f over E . Show that μ is a measure on the σ -algebra of Lebesgue measurable subsets of $\mathbb{R}$.

Solution. Write $\mathcal{L}$ for the σ -algebra of Lebesgue measurable subsets of $\mathbb{R}$. For $E \in \mathcal{L}$ the function $f \cdot \chi_E$ is nonnegative and Lebesgue measurable on $\mathbb{R}$ (a product of measurable functions), and

$$\mu(E) = \int_E f = \int_{\mathbb{R}} f \cdot \chi_E.$$

The second equality is the comparison of the two suprema by which these integrals are defined, the integral of a nonnegative measurable function over a set being the supremum of the integrals over that set of the bounded measurable functions of finite support lying below it. If h is bounded, measurable, of finite support and $0 \leq h \leq f$ on E , extend h by 0 to all of $\mathbb{R}$; the extension is bounded, measurable, of finite support, and satisfies $0 \leq h \leq f \cdot \chi_E$ on $\mathbb{R}$. Choosing a set E_0 of finite measure outside of which h vanishes, additivity over domains for bounded measurable functions on a set of finite measure (Corollary 6 of Chapter 4) gives $\int_{\mathbb{R}} h = \int_{E_0} h = \int_{E_0 \cap E} h + \int_{E_0 \sim E} h = \int_E h$, since h vanishes off $E_0 \cap E$. Conversely, if h is bounded, measurable, of finite support and $0 \leq h \leq f \cdot \chi_E$ on $\mathbb{R}$, then h vanishes on $\mathbb{R} \sim E$, so the same computation gives $\int_{\mathbb{R}} h = \int_E h$, while $0 \leq h \leq f$ on E . Hence the two defining suprema coincide.

The integral of a nonnegative measurable function is a well-defined element of $[0, \infty]$, so $\mu: \mathcal{L} \rightarrow [0, \infty]$ is a well-defined nonnegative extended real-valued set function. It remains to check that $\mu(\emptyset) = 0$ and that μ is countably additive.

Since $f \cdot \chi_{\emptyset} \equiv 0$ on $\mathbb{R}$, we have $\mu(\emptyset) = \int_{\mathbb{R}} 0 = 0$.

First observe finite additivity over domains. If $A, B \in \mathcal{L}$ are disjoint then $\chi_{A \cup B} = \chi_A + \chi_B$, so $f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B$, and by the linearity of integration for nonnegative measurable functions (Theorem 10 of Chapter 4),

$$\mu(A \cup B) = \int_{\mathbb{R}} f \cdot \chi_A + \int_{\mathbb{R}} f \cdot \chi_B = \mu(A) + \mu(B).$$

By induction, for any finite disjoint collection $\{E_k\}_{k=1}^n$ in $\mathcal{L}$,

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k).$$

Now let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of sets in $\mathcal{L}$ and put $E = \bigcup_{k=1}^{\infty} E_k$, which belongs to $\mathcal{L}$. For each n define

$$f_n = f \cdot \chi_{A_n}, \quad \text{where } A_n = \bigcup_{k=1}^n E_k.$$

Each f_n is nonnegative and measurable. Since $A_n \subseteq A_{n+1}$ we have $0 \leq f_n \leq f_{n+1}$ on $\mathbb{R}$, so $\{f_n\}$ is an increasing sequence of nonnegative measurable functions. Moreover, for each $x \in \mathbb{R}$, $f_n(x) \rightarrow f(x)\chi_E(x)$: if $x \notin E$ then $f_n(x) = 0$ for every n and $f(x)\chi_E(x) = 0$; if $x \in E$ then $x \in E_{k_0}$ for some k_0 , hence $f_n(x) = f(x) = f(x)\chi_E(x)$ for all $n \geq k_0$. Thus $f_n \rightarrow f \cdot \chi_E$ pointwise on $\mathbb{R}$.

By the Monotone Convergence Theorem of Chapter 4,

$$\mu(E) = \int_{\mathbb{R}} f \cdot \chi_E = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n.$$

On the other hand, by the finite additivity established above,

$$\int_{\mathbb{R}} f_n = \mu(A_n) = \sum_{k=1}^n \mu(E_k).$$

Therefore

$$\mu(E) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k),$$

the limit existing in $[0, \infty]$ because the partial sums are nondecreasing. Hence μ is countably additive, and μ is a measure on $\mathcal{L}$.

Problem 17.2. Let $\mathcal{M}$ be a σ -algebra of subsets of a set X and the set function $\mu: \mathcal{M} \rightarrow [0, \infty)$ be finitely additive. Prove that μ is a measure if and only if whenever $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k).$$

Solution. Note first that μ is real-valued: $\mu(E) < \infty$ for every $E \in \mathcal{M}$. Note also that finite additivity forces $\mu(\emptyset) = 0$. Indeed, $\emptyset$ and $\emptyset$ are disjoint with union $\emptyset$, so $\mu(\emptyset) = \mu(\emptyset) + \mu(\emptyset)$, and since $\mu(\emptyset)$ is a finite real number we may cancel to get $\mu(\emptyset) = 0$. Thus μ is a measure precisely when it is countably additive.

Suppose first that μ is a measure. Then $(X, \mathcal{M}, \mu)$ is a measure space, and by the Continuity of Measure (Proposition 2(i) of this chapter), for every ascending sequence $\{A_k\}_{k=1}^{\infty}$ of measurable sets,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k).$$

Conversely, suppose μ is finitely additive and has the stated continuity property along ascending sequences. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of sets in $\mathcal{M}$ and set $E = \bigcup_{k=1}^{\infty} E_k$, which belongs to $\mathcal{M}$ since $\mathcal{M}$ is a σ -algebra. For each n define

$$A_n = \bigcup_{k=1}^n E_k \in \mathcal{M}.$$

Then $A_n \subseteq A_{n+1}$ for every n , so $\{A_n\}_{n=1}^{\infty}$ is an ascending sequence of sets in $\mathcal{M}$, and $\bigcup_{n=1}^{\infty} A_n = E$. Since $E_1, \dots, E_n$ are disjoint, finite additivity gives

$$\mu(A_n) = \sum_{k=1}^n \mu(E_k).$$

By the assumed continuity property applied to $\{A_n\}$,

$$\mu(E) = \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

Thus μ is countably additive. Together with $\mu(\emptyset) = 0$ and the nonnegativity of μ , this shows μ is a measure on $\mathcal{M}$.

Problem 17.3. Let $\mathcal{M}$ be a σ -algebra of subsets of a set X . Formulate and establish a correspondent of the preceding problem for descending sequences of sets in $\mathcal{M}$.

Solution. The correspondent is the following.

Claim. Let $\mathcal{M}$ be a σ -algebra of subsets of a set X and let $\mu: \mathcal{M} \rightarrow [0, \infty)$ be finitely additive. Then μ is a measure if and only if whenever $\{B_k\}_{k=1}^{\infty}$ is a descending sequence of sets in $\mathcal{M}$, then

$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} \mu(B_k).$$

(Since μ takes values in $[0, \infty)$, the finiteness hypothesis $\mu(B_1) < \infty$ appearing in Proposition 2(ii) is automatically satisfied.)

As in the preceding problem, finite additivity together with the finiteness of μ forces $\mu(\emptyset) = 0$: from $\mu(\emptyset) = \mu(\emptyset) + \mu(\emptyset)$ and $\mu(\emptyset) \in \mathbb{R}$ we get $\mu(\emptyset) = 0$. We also record the excision property: if $A, B \in \mathcal{M}$ with $A \subseteq B$, then $B = A \cup (B \sim A)$ is a disjoint union of measurable sets, so $\mu(B) = \mu(A) + \mu(B \sim A)$, and since all values are finite,

$$\mu(B \sim A) = \mu(B) - \mu(A).$$

Suppose first that μ is a measure. Let $\{B_k\}$ be a descending sequence in $\mathcal{M}$. Since $\mu(B_1) < \infty$, the Continuity of Measure (Proposition 2(ii) of this chapter) gives

$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} \mu(B_k),$$

which is the asserted property.

Conversely, suppose μ is finitely additive and satisfies the stated continuity along descending sequences. By the preceding problem it suffices to verify continuity along ascending sequences. So let $\{A_k\}_{k=1}^{\infty}$ be an ascending sequence in $\mathcal{M}$ and put $A = \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$. Define

$$B_k = A \sim A_k \in \mathcal{M}.$$

Since $A_k \subseteq A_{k+1}$, we have $B_{k+1} \subseteq B_k$, so $\{B_k\}$ is a descending sequence in $\mathcal{M}$. Moreover

$$\bigcap_{k=1}^{\infty} B_k = A \sim \bigcup_{k=1}^{\infty} A_k = A \sim A = \emptyset.$$

By hypothesis and $\mu(\emptyset) = 0$,

$$\lim_{k \rightarrow \infty} \mu(B_k) = \mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \mu(\emptyset) = 0.$$

By excision, $\mu(B_k) = \mu(A) - \mu(A_k)$ for each k , all quantities being finite. Hence

$$0 = \lim_{k \rightarrow \infty} (\mu(A) - \mu(A_k)),$$

that is, $\lim_{k \rightarrow \infty} \mu(A_k) = \mu(A) = \mu(\bigcup_{k=1}^{\infty} A_k)$. Thus μ is continuous along ascending sequences, and by the preceding problem μ is a measure.

Remark. The proof of the converse used only the continuity property for descending sequences with empty intersection, so the criterion may be weakened: a finitely additive $\mu: \mathcal{M} \rightarrow [0, \infty)$ is a measure if and only if $\lim_{k \rightarrow \infty} \mu(B_k) = 0$ for every descending sequence $\{B_k\}$ in $\mathcal{M}$ with $\bigcap_{k=1}^{\infty} B_k = \emptyset$. The finiteness of μ is essential. For instance, let η be the counting measure on $X = \mathbb{N}$ with $\mathcal{M} = 2^{\mathbb{N}}$, and let $B_k = \{k, k+1, \dots\}$. Then $\{B_k\}$ is descending with $\bigcap_{k=1}^{\infty} B_k = \emptyset$, yet $\eta(B_k) = \infty$ for every k , so $\lim_{k \rightarrow \infty} \eta(B_k) = \infty \neq 0 = \eta(\emptyset)$ even though η is a measure. This is why Proposition 2(ii) carries the hypothesis $\mu(B_1) < \infty$.

Problem 17.4. Let $\{(X_\lambda, \mathcal{M}_\lambda, \mu_\lambda)\}_{\lambda \in \Lambda}$ be a collection of measure spaces parametrized by the set Λ . Assume the collection of sets $\{X_\lambda\}_{\lambda \in \Lambda}$ is disjoint. Then we can form a new measure space (called their union) $(X, \mathcal{B}, \mu)$ by letting $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, $\mathcal{B}$ be the collection of subsets B of X such that $B \cap X_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \Lambda$ and defining $\mu(B) = \sum_{\lambda \in \Lambda} \mu_\lambda[B \cap X_\lambda]$ for $B \in \mathcal{B}$.

(i) Show that $\mathcal{M}$ is a σ -algebra.

(ii) Show that μ is a measure.

(iii) Show that μ is σ -finite if and only if all but a countable number of the measures μ_λ have $\mu(X_\lambda) = 0$ and the remainder are σ -finite.

Solution. Throughout, for a family $\{a_\lambda\}_{\lambda \in \Lambda}$ of numbers in $[0, \infty]$ indexed by an arbitrary set Λ , the sum is defined by

$$\sum_{\lambda \in \Lambda} a_\lambda = \sup \left\{ \sum_{\lambda \in F} a_\lambda : F \subseteq \Lambda \text{ finite} \right\} \in [0, \infty].$$

For countable Λ this agrees with the usual sum of a series of nonnegative terms, since the partial sums are nondecreasing.

(The collection called $\mathcal{M}$ in part (i) of the statement is the collection denoted $\mathcal{B}$ in its construction; we write $\mathcal{B}$ throughout.)

(i) $\mathcal{B}$ is a σ -algebra of subsets of X . First, $X \in \mathcal{B}$: since the X_λ are disjoint with union X , we have $X \cap X_\lambda = X_\lambda \in \mathcal{M}_\lambda$ for every λ . Next, let $B \in \mathcal{B}$. For each λ ,

$$(X \sim B) \cap X_\lambda = X_\lambda \sim (B \cap X_\lambda),$$

which belongs to $\mathcal{M}_\lambda$ because $\mathcal{M}_\lambda$ is a σ -algebra of subsets of X_λ and $B \cap X_\lambda \in \mathcal{M}_\lambda$. Hence $X \sim B \in \mathcal{B}$. Finally, let $\{B_k\}_{k=1}^\infty$ be a countable collection of sets in $\mathcal{B}$. For each λ ,

$$\left(\bigcup_{k=1}^\infty B_k \right) \cap X_\lambda = \bigcup_{k=1}^\infty (B_k \cap X_\lambda) \in \mathcal{M}_\lambda,$$

since each $B_k \cap X_\lambda \in \mathcal{M}_\lambda$ and $\mathcal{M}_\lambda$ is closed under countable unions. Hence $\bigcup_{k=1}^\infty B_k \in \mathcal{B}$, and $\mathcal{B}$ is a σ -algebra.

We record for later use that if $\lambda_0 \in \Lambda$ and $E \in \mathcal{M}_{\lambda_0}$, then $E \in \mathcal{B}$: indeed $E \cap X_{\lambda_0} = E \in \mathcal{M}_{\lambda_0}$, while for $\lambda \neq \lambda_0$ disjointness gives $E \cap X_\lambda = \emptyset \in \mathcal{M}_\lambda$. Moreover $\mu(E) = \mu_{\lambda_0}(E)$.

(ii) μ is a measure. Clearly $\mu(B) \in [0, \infty]$ for each $B \in \mathcal{B}$, and

$$\mu(\emptyset) = \sum_{\lambda \in \Lambda} \mu_\lambda(\emptyset) = \sum_{\lambda \in \Lambda} 0 = 0.$$

We first prove an interchange lemma.

Lemma. Let $\{a_{\lambda,k}\}$ be numbers in $[0, \infty]$ indexed by $\lambda \in \Lambda$ and $k \in \mathbb{N}$. Then

$$\sum_{\lambda \in \Lambda} \sum_{k=1}^\infty a_{\lambda,k} = \sum_{k=1}^\infty \sum_{\lambda \in \Lambda} a_{\lambda,k}.$$

Proof of the Lemma. Denote the left side by S and the right side by T . Let $F \subseteq \Lambda$ be finite. Since a finite sum of nondecreasing limits in $[0, \infty]$ is the limit of the finite sums,

$$\sum_{\lambda \in F} \sum_{k=1}^\infty a_{\lambda,k} = \lim_{n \rightarrow \infty} \sum_{\lambda \in F} \sum_{k=1}^n a_{\lambda,k} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sum_{\lambda \in F} a_{\lambda,k} \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n \sum_{\lambda \in \Lambda} a_{\lambda,k} = T.$$

Taking the supremum over all finite $F \subseteq \Lambda$ gives $S \leq T$.

For the reverse inequality, fix $n \in \mathbb{N}$. For finite sets $F_1, \dots, F_n \subseteq \Lambda$ put $F = \bigcup_{k=1}^n F_k$, a finite subset of Λ ; then, since all terms are nonnegative,

$$\sum_{k=1}^n \sum_{\lambda \in F_k} a_{\lambda,k} \leq \sum_{k=1}^n \sum_{\lambda \in F} a_{\lambda,k} = \sum_{\lambda \in F} \sum_{k=1}^n a_{\lambda,k} \leq \sum_{\lambda \in F} \sum_{k=1}^\infty a_{\lambda,k} \leq S.$$

Taking the supremum over $F_1, \dots, F_n$ separately yields

$$\sum_{k=1}^n \sum_{\lambda \in \Lambda} a_{\lambda,k} \leq S.$$

Letting $n \rightarrow \infty$ gives $T \leq S$, and the Lemma is proved.

Now let $\{B_k\}_{k=1}^\infty$ be a countable disjoint collection of sets in $\mathcal{B}$ and $B = \bigcup_{k=1}^\infty B_k \in \mathcal{B}$. Fix $\lambda \in \Lambda$. The sets $\{B_k \cap X_\lambda\}_{k=1}^\infty$ form a countable disjoint collection in $\mathcal{M}_\lambda$ whose union is $B \cap X_\lambda$, so by the countable additivity of μ_λ ,

$$\mu_\lambda(B \cap X_\lambda) = \sum_{k=1}^\infty \mu_\lambda(B_k \cap X_\lambda).$$

Therefore, applying the Lemma with $a_{\lambda,k} = \mu_\lambda(B_k \cap X_\lambda)$,

$$\mu(B) = \sum_{\lambda \in \Lambda} \mu_\lambda(B \cap X_\lambda) = \sum_{\lambda \in \Lambda} \sum_{k=1}^\infty \mu_\lambda(B_k \cap X_\lambda) = \sum_{k=1}^\infty \sum_{\lambda \in \Lambda} \mu_\lambda(B_k \cap X_\lambda) = \sum_{k=1}^\infty \mu(B_k).$$

Hence μ is countably additive and so is a measure on $(X, \mathcal{B})$.

(iii) By the observation at the end of (i), $X_\lambda \in \mathcal{B}$ and $\mu(X_\lambda) = \mu_\lambda(X_\lambda)$ for each λ , so the condition $\mu(X_\lambda) = 0$ appearing in the statement is the same as $\mu_\lambda(X_\lambda) = 0$. Put $\Lambda_0 = \{\lambda \in \Lambda : \mu_\lambda(X_\lambda) > 0\}$. We show: μ is σ -finite if and only if Λ_0 is countable and μ_λ is σ -finite for each $\lambda \in \Lambda_0$. (For $\lambda \notin \Lambda_0$ the measure μ_λ is identically zero on $\mathcal{M}_\lambda$ by monotonicity, hence is finite and in particular σ -finite; so the condition is also equivalent to: Λ_0 is countable and every μ_λ is σ -finite.)

Suppose first that $\Lambda_0 = \{\lambda_1, \lambda_2, \dots\}$ is countable and each μ_{λ_j} is σ -finite. Let

$$Z = \bigcup_{\lambda \in \Lambda \sim \Lambda_0} X_\lambda.$$

Then $Z \in \mathcal{B}$, since $Z \cap X_\lambda$ equals X_λ or $\emptyset$ according as $\lambda \notin \Lambda_0$ or $\lambda \in \Lambda_0$, and both belong to $\mathcal{M}_\lambda$. Moreover

$$\mu(Z) = \sum_{\lambda \in \Lambda} \mu_\lambda(Z \cap X_\lambda) = \sum_{\lambda \in \Lambda \sim \Lambda_0} \mu_\lambda(X_\lambda) = 0,$$

so $\mu(Z) < \infty$. For each j , the σ -finiteness of μ_{λ_j} gives sets $E_{j,n} \in \mathcal{M}_{\lambda_j}$, $n \in \mathbb{N}$, with $X_{\lambda_j} = \bigcup_{n=1}^\infty E_{j,n}$ and $\mu_{\lambda_j}(E_{j,n}) < \infty$. By the observation at the end of (i), $E_{j,n} \in \mathcal{B}$ and $\mu(E_{j,n}) = \mu_{\lambda_j}(E_{j,n}) < \infty$. Since

$$X = Z \cup \bigcup_j \bigcup_{n=1}^\infty E_{j,n}$$

is a countable union of measurable sets of finite measure, μ is σ -finite.

Conversely, suppose μ is σ -finite, say $X = \bigcup_{n=1}^\infty B_n$ with $B_n \in \mathcal{B}$ and $\mu(B_n) < \infty$. Fix n and set

$$\Lambda'_n = \{\lambda \in \Lambda : \mu_\lambda(B_n \cap X_\lambda) > 0\}.$$

We claim Λ'_n is countable. For $m \in \mathbb{N}$ let $\Lambda_{n,m} = \{\lambda : \mu_\lambda(B_n \cap X_\lambda) > 1/m\}$. If $F \subseteq \Lambda_{n,m}$ is finite with p elements, then

$$p/m < \sum_{\lambda \in F} \mu_\lambda(B_n \cap X_\lambda) \leq \mu(B_n),$$

so $p < m \mu(B_n) < \infty$; thus $\Lambda_{n,m}$ is finite. Since $\Lambda'_n = \bigcup_{m=1}^\infty \Lambda_{n,m}$, the set Λ'_n is countable. Hence $\Lambda' = \bigcup_{n=1}^\infty \Lambda'_n$ is countable.

If $\lambda \notin \Lambda'$, then $\mu_\lambda(B_n \cap X_\lambda) = 0$ for every n , and since $X_\lambda = \bigcup_{n=1}^\infty (B_n \cap X_\lambda)$, countable monotonicity of μ_λ (Proposition 1 of this chapter) gives

$$\mu_\lambda(X_\lambda) \leq \sum_{n=1}^\infty \mu_\lambda(B_n \cap X_\lambda) = 0.$$

Therefore $\Lambda_0 \subseteq \Lambda'$, so Λ_0 is countable: all but countably many of the measures μ_λ satisfy $\mu_\lambda(X_\lambda) = 0$. Finally, every μ_λ is σ -finite: for each λ the sets $B_n \cap X_\lambda$ belong to $\mathcal{M}_\lambda$, their union over n is X_λ , and

$$\mu_\lambda(B_n \cap X_\lambda) \leq \sum_{\lambda' \in \Lambda} \mu_{\lambda'}(B_n \cap X_{\lambda'}) = \mu(B_n) < \infty.$$

In particular the measures μ_λ with $\lambda \in \Lambda_0$ are σ -finite. This completes the proof.

Problem 17.5. Let $(X, \mathcal{M}, \mu)$ be a measure space. The symmetric difference, $E_1 \Delta E_2$, of two subsets E_1 and E_2 of X is defined by

$$E_1 \Delta E_2 = [E_1 \sim E_2] \cup [E_2 \sim E_1].$$

- (i) Show that if E_1 and E_2 are measurable and $\mu(E_1 \Delta E_2) = 0$, then $\mu(E_1) = \mu(E_2)$.
- (ii) Show that if μ is complete, $E_1 \in \mathcal{M}$ and $E_2 \sim E_1 \in \mathcal{M}$, then $E_2 \in \mathcal{M}$ if $\mu(E_1 \Delta E_2) = 0$.

Solution. (i) Suppose $E_1, E_2 \in \mathcal{M}$ and $\mu(E_1 \Delta E_2) = 0$. The sets $E_1 \sim E_2$ and $E_2 \sim E_1$ are measurable and each is contained in $E_1 \Delta E_2$, so by the monotonicity of μ (Proposition 1 of this chapter),

$$\mu(E_1 \sim E_2) = 0 \quad \text{and} \quad \mu(E_2 \sim E_1) = 0.$$

Now E_1 is the disjoint union of the measurable sets $E_1 \cap E_2$ and $E_1 \sim E_2$, so by finite additivity

$$\mu(E_1) = \mu(E_1 \cap E_2) + \mu(E_1 \sim E_2) = \mu(E_1 \cap E_2).$$

Symmetrically, E_2 is the disjoint union of $E_1 \cap E_2$ and $E_2 \sim E_1$, so

$$\mu(E_2) = \mu(E_1 \cap E_2) + \mu(E_2 \sim E_1) = \mu(E_1 \cap E_2).$$

Hence $\mu(E_1) = \mu(E_2)$, both being equal to $\mu(E_1 \cap E_2)$. (Note that no finiteness assumption is needed: the two displayed identities are additions in $[0, \infty]$.)

(ii) Here E_2 is an arbitrary subset of X , so $E_1 \Delta E_2$ need not be known in advance to be measurable; the hypothesis $\mu(E_1 \Delta E_2) = 0$ is to be read as saying that $E_1 \Delta E_2$ is a null set, that is, there is a set $N \in \mathcal{M}$ with

$$E_1 \Delta E_2 \subseteq N \quad \text{and} \quad \mu(N) = 0.$$

(If $E_1 \Delta E_2$ itself lies in $\mathcal{M}$ and has measure zero, take $N = E_1 \Delta E_2$.)

Assume μ is complete, $E_1 \in \mathcal{M}$, $E_2 \sim E_1 \in \mathcal{M}$, and $E_1 \Delta E_2 \subseteq N$ with $N \in \mathcal{M}$, $\mu(N) = 0$. Since

$$E_1 \sim E_2 \subseteq E_1 \Delta E_2 \subseteq N,$$

and μ is complete, that is, $\mathcal{M}$ contains every subset of a set in $\mathcal{M}$ of measure zero, we conclude that

$$E_1 \sim E_2 \in \mathcal{M}.$$

Therefore

$$E_1 \cap E_2 = E_1 \sim (E_1 \sim E_2) \in \mathcal{M},$$

being the difference of two measurable sets. Finally,

$$E_2 = (E_1 \cap E_2) \cup (E_2 \sim E_1),$$

and both sets on the right belong to $\mathcal{M}$, the second by hypothesis. Since $\mathcal{M}$ is closed under finite unions, $E_2 \in \mathcal{M}$.

Combining this with part (i), $\mu(E_1) = \mu(E_2)$ as well: indeed once E_2 is known to be measurable, $E_1 \Delta E_2$ is measurable and has measure zero by monotonicity, so (i) applies.

Remark. Under the above reading of the hypothesis, completeness also forces $E_2 \sim E_1 \in \mathcal{M}$ directly, since $E_2 \sim E_1 \subseteq N$; so that hypothesis is in fact redundant. It is used above to keep the argument independent of that observation.

Problem 17.6. Let $(X, \mathcal{M}, \mu)$ be a measure space and X_0 belong to $\mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of X_0 and μ_0 the restriction of μ to $\mathcal{M}_0$. Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Solution. We must show that $\mathcal{M}_0$ is a σ -algebra of subsets of X_0 and that μ_0 is a measure on $(X_0, \mathcal{M}_0)$. Every member of $\mathcal{M}_0$ is by definition a subset of X_0 , so $\mathcal{M}_0$ is a collection of subsets of X_0 .

The set X_0 itself belongs to $\mathcal{M}_0$: it belongs to $\mathcal{M}$ by hypothesis and is a subset of itself.

$\mathcal{M}_0$ is closed under complementation in X_0 . Let $E \in \mathcal{M}_0$. Then $X_0 \sim E = X_0 \cap (X \sim E)$. Since $X \sim E \in \mathcal{M}$ and $X_0 \in \mathcal{M}$, and a σ -algebra is closed under finite intersections, $X_0 \sim E \in \mathcal{M}$; it is clearly a subset of X_0 . Hence $X_0 \sim E \in \mathcal{M}_0$.

$\mathcal{M}_0$ is closed under countable unions. Let $\{E_k\}_{k=1}^\infty$ be a countable collection of sets in $\mathcal{M}_0$. Then each $E_k \in \mathcal{M}$, so $\bigcup_{k=1}^\infty E_k \in \mathcal{M}$, and since each $E_k \subseteq X_0$ we also have $\bigcup_{k=1}^\infty E_k \subseteq X_0$. Hence $\bigcup_{k=1}^\infty E_k \in \mathcal{M}_0$. Thus $(X_0, \mathcal{M}_0)$ is a measurable space.

Now consider $\mu_0 = \mu|_{\mathcal{M}_0}$. It is an extended real-valued nonnegative set function on $\mathcal{M}_0$ with values in $[0, \infty]$, since μ is. Also $\emptyset \subseteq X_0$ and $\emptyset \in \mathcal{M}$, so $\emptyset \in \mathcal{M}_0$ and

$$\mu_0(\emptyset) = \mu(\emptyset) = 0.$$

Finally, let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection of sets in $\mathcal{M}_0$. As just shown, $E = \bigcup_{k=1}^\infty E_k \in \mathcal{M}_0$. These sets are also a countable disjoint collection in $\mathcal{M}$, so by the countable additivity of μ ,

$$\mu_0\left(\bigcup_{k=1}^\infty E_k\right) = \mu\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu(E_k) = \sum_{k=1}^\infty \mu_0(E_k).$$

Hence μ_0 is countably additive, so μ_0 is a measure on $(X_0, \mathcal{M}_0)$ and $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Problem 17.7. Let $(X, \mathcal{M})$ be a measurable space. Verify the following:

- (i) If μ and ν are measures defined on $\mathcal{M}$, then the set function λ defined on $\mathcal{M}$ by $\lambda(E) = \mu(E) + \nu(E)$ also is a measure. We denote λ by $\mu + \nu$.
- (ii) If μ and ν are measures on $\mathcal{M}$ and $\mu \geq \nu$, then there is a measure λ on $\mathcal{M}$ for which $\mu = \nu + \lambda$.
- (iii) If ν is σ -finite, the measure λ in (ii) is unique.
- (iv) Show that in general the measure λ need not be unique but that there is always a smallest such λ .

Solution. Here $\mu \geq \nu$ means $\mu(E) \geq \nu(E)$ for all $E \in \mathcal{M}$. All arithmetic is in $[0, \infty]$, with the convention $a + \infty = \infty$.

- (i) The set function $\lambda(E) = \mu(E) + \nu(E)$ takes values in $[0, \infty]$, and

$$\lambda(\emptyset) = \mu(\emptyset) + \nu(\emptyset) = 0 + 0 = 0.$$

Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection in $\mathcal{M}$ with union E . For nonnegative extended real numbers, the sum of two series is the series of sums: for each n ,

$$\sum_{k=1}^n (\mu(E_k) + \nu(E_k)) = \sum_{k=1}^n \mu(E_k) + \sum_{k=1}^n \nu(E_k),$$

and letting $n \rightarrow \infty$, all three sequences of partial sums being nondecreasing in $[0, \infty]$, we get

$$\sum_{k=1}^\infty \lambda(E_k) = \sum_{k=1}^\infty \mu(E_k) + \sum_{k=1}^\infty \nu(E_k) = \mu(E) + \nu(E) = \lambda(E),$$

using the countable additivity of μ and of ν . Hence λ is a measure.

- (ii) Define $\lambda: \mathcal{M} \rightarrow [0, \infty]$ by

$$\lambda(E) = \sup \{ \mu(A) - \nu(A) : A \in \mathcal{M}, A \subseteq E, \nu(A) < \infty \}.$$

The set over which the supremum is taken is nonempty, since $A = \emptyset$ is admissible, and each of its members is a well-defined element of $[0, \infty]$: if $\nu(A) < \infty$ then $\mu(A) - \nu(A)$ makes sense and is nonnegative because $\mu(A) \geq \nu(A)$. Hence $\lambda(E) \in [0, \infty]$ and $\lambda(\emptyset) = 0$ (every admissible $A \subseteq \emptyset$ is $\emptyset$).

λ is countably additive. Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection in $\mathcal{M}$ with union E . First, $\lambda(E) \geq \sum_{k=1}^\infty \lambda(E_k)$. Fix $n \in \mathbb{N}$ and, for $1 \leq k \leq n$, let c_k be any real number with $c_k < \lambda(E_k)$ if $\lambda(E_k) > 0$, and $c_k = 0$ if $\lambda(E_k) = 0$. Choose $A_k \in \mathcal{M}$ with $A_k \subseteq E_k$, $\nu(A_k) < \infty$, and $\mu(A_k) - \nu(A_k) \geq c_k$ (possible by the definition of the supremum; when $\lambda(E_k) = 0$ take $A_k = \emptyset$). Put $A = \bigcup_{k=1}^n A_k \subseteq E$. The A_k are disjoint, so by finite additivity $\nu(A) = \sum_{k=1}^n \nu(A_k) < \infty$ and $\mu(A) = \sum_{k=1}^n \mu(A_k)$; hence

$$\lambda(E) \geq \mu(A) - \nu(A) = \sum_{k=1}^n (\mu(A_k) - \nu(A_k)) \geq \sum_{k=1}^n c_k.$$

Taking suprema over the admissible c_k gives $\lambda(E) \geq \sum_{k=1}^n \lambda(E_k)$ for every n , and letting $n \rightarrow \infty$ gives $\lambda(E) \geq \sum_{k=1}^\infty \lambda(E_k)$.

Second, $\lambda(E) \leq \sum_{k=1}^\infty \lambda(E_k)$. Let $A \in \mathcal{M}$ with $A \subseteq E$ and $\nu(A) < \infty$. Write $A_k = A \cap E_k$; these are disjoint, lie in $\mathcal{M}$, have union A , and $\nu(A_k) \leq \nu(A) < \infty$, so each is admissible for E_k and

$$\mu(A_k) - \nu(A_k) \leq \lambda(E_k).$$

By countable additivity, $\sum_{k=1}^\infty \nu(A_k) = \nu(A) < \infty$ and $\sum_{k=1}^\infty \mu(A_k) = \mu(A)$. For each n ,

$$\sum_{k=1}^n (\mu(A_k) - \nu(A_k)) = \sum_{k=1}^n \mu(A_k) - \sum_{k=1}^n \nu(A_k) \geq \sum_{k=1}^n \mu(A_k) - \nu(A),$$

all the sums being finite. Hence

$$\sum_{k=1}^\infty \lambda(E_k) \geq \sum_{k=1}^\infty (\mu(A_k) - \nu(A_k)) \geq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \mu(A_k) \right) - \nu(A) = \mu(A) - \nu(A),$$

which is valid whether $\mu(A)$ is finite or infinite. Taking the supremum over all such A gives $\lambda(E) \leq \sum_{k=1}^\infty \lambda(E_k)$.

Therefore λ is a measure on $\mathcal{M}$.

We check $\mu = \nu + \lambda$. Let $E \in \mathcal{M}$.

If $\nu(E) = \infty$, then $\mu(E) \geq \nu(E) = \infty$, so $\mu(E) = \infty = \nu(E) + \lambda(E)$, whatever the value of $\lambda(E)$.

If $\nu(E) < \infty$ and $\mu(E) = \infty$, then E itself is admissible in the supremum defining $\lambda(E)$, so $\lambda(E) \geq \mu(E) - \nu(E) = \infty$; hence $\nu(E) + \lambda(E) = \infty = \mu(E)$.

If $\nu(E) < \infty$ and $\mu(E) < \infty$, then again $\lambda(E) \geq \mu(E) - \nu(E)$. Conversely, let $A \in \mathcal{M}$, $A \subseteq E$, $\nu(A) < \infty$. Since $\mu(E)$ and $\nu(E)$ are finite, excision (Proposition 1 of this chapter) gives $\mu(A) = \mu(E) - \mu(E \sim A)$ and $\nu(A) = \nu(E) - \nu(E \sim A)$, so

$$\mu(A) - \nu(A) = (\mu(E) - \nu(E)) - (\mu(E \sim A) - \nu(E \sim A)) \leq \mu(E) - \nu(E),$$

because $\mu(E \sim A) \geq \nu(E \sim A)$. Taking the supremum over A gives $\lambda(E) \leq \mu(E) - \nu(E)$. Hence $\lambda(E) = \mu(E) - \nu(E)$ and $\nu(E) + \lambda(E) = \mu(E)$.

In every case $\mu(E) = \nu(E) + \lambda(E)$, that is, $\mu = \nu + \lambda$.

(iii) Suppose ν is σ -finite and λ, λ' are measures on $\mathcal{M}$ with $\mu = \nu + \lambda = \nu + \lambda'$. Choose $\{X_n\}_{n=1}^\infty$ in $\mathcal{M}$ with $X = \bigcup_{n=1}^\infty X_n$ and $\nu(X_n) < \infty$; replacing X_n by $X_n \sim \bigcup_{j=1}^{n-1} X_j$ we may assume the X_n are disjoint, since this only decreases the sets and ν is monotone. Let $E \in \mathcal{M}$ and fix n . Then $\nu(E \cap X_n) \leq \nu(X_n) < \infty$ and

$$\nu(E \cap X_n) + \lambda(E \cap X_n) = \mu(E \cap X_n) = \nu(E \cap X_n) + \lambda'(E \cap X_n).$$

Since the common term $\nu(E \cap X_n)$ is a finite real number, it may be cancelled, giving $\lambda(E \cap X_n) = \lambda'(E \cap X_n)$. The sets $\{E \cap X_n\}_{n=1}^\infty$ are disjoint with union E , so by countable additivity

$$\lambda(E) = \sum_{n=1}^\infty \lambda(E \cap X_n) = \sum_{n=1}^\infty \lambda'(E \cap X_n) = \lambda'(E).$$

As $E \in \mathcal{M}$ was arbitrary, $\lambda = \lambda'$.

(iv) Uniqueness fails in general. Let $X = \{a\}$ be a one-point set and $\mathcal{M} = 2^X = \{\emptyset, X\}$. Define $\nu = \mu$ by $\mu(\emptyset) = \nu(\emptyset) = 0$ and $\mu(X) = \nu(X) = \infty$; both are measures and $\mu \geq \nu$. Let λ_1 be the zero measure and $\lambda_2 = \nu$. Then

$$(\nu + \lambda_1)(X) = \infty + 0 = \infty = \mu(X), \quad (\nu + \lambda_2)(X) = \infty + \infty = \infty = \mu(X),$$

and both agree with μ at $\emptyset$. Thus $\mu = \nu + \lambda_1 = \nu + \lambda_2$ with $\lambda_1 \neq \lambda_2$, so λ is not unique.

Nevertheless the measure λ constructed in (ii) is the smallest such measure. Let λ' be any measure on $\mathcal{M}$ with $\mu = \nu + \lambda'$, and let $E \in \mathcal{M}$. If $A \in \mathcal{M}$ with $A \subseteq E$ and $\nu(A) < \infty$, then

$$\mu(A) = \nu(A) + \lambda'(A),$$

and since $\nu(A)$ is finite we may subtract it to get $\mu(A) - \nu(A) = \lambda'(A) \leq \lambda'(E)$, the last step by monotonicity of λ' . Taking the supremum over all such A yields

$$\lambda(E) \leq \lambda'(E).$$

Hence $\lambda \leq \lambda'$ for every measure λ' with $\mu = \nu + \lambda'$, so λ is the smallest such measure.

17.2 Problems 17.8–17.14

Problem 17.8. Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is said to be **semifinite** provided each measurable set of infinite measure contains measurable sets of arbitrarily large finite measure.

(i) Show that each σ -finite measure is semifinite.

(ii) For $E \in \mathcal{M}$, define $\mu_1(E) = \mu(E)$ if $\mu(E) < \infty$, and if $\mu(E) = \infty$ define $\mu_1(E) = \infty$ if E contains measurable sets of arbitrarily large finite measure and $\mu_1(E) = 0$ otherwise. Show that μ_1 is a semifinite measure: it is called the **semifinite part** of μ .

(iii) Find a measure μ_2 on $\mathcal{M}$ that only takes the values 0 and ∞ and $\mu = \mu_1 + \mu_2$.

Solution. Part (i). Let μ be σ -finite, say $X = \bigcup_{k=1}^{\infty} X_k$ with each $X_k \in \mathcal{M}$ of finite measure. Replacing X_k by $X_1 \cup \cdots \cup X_k$ we may assume the sequence is ascending, and finite additivity keeps each $\mu(X_k)$ finite.

Let $E \in \mathcal{M}$ have $\mu(E) = \infty$. The sets $E \cap X_k$ are measurable, ascend, and have union E , so by the continuity of measure,

$$\lim_{k \rightarrow \infty} \mu(E \cap X_k) = \mu(E) = \infty.$$

Since $\mu(E \cap X_k) \leq \mu(X_k) < \infty$ for every k , the sets $E \cap X_k$ are measurable subsets of E of finite measure whose measures tend to ∞ . Given $M > 0$ there is thus k with $M < \mu(E \cap X_k) < \infty$. Hence μ is semifinite.

Part (ii). Two remarks are needed before the proof.

First, the recipe printed in the statement does **not** in general define a measure. Take $X = \{a\} \cup Y$ with Y nonempty and disjoint from $\{a\}$, let $\mathcal{M} = \{\emptyset, \{a\}, Y, X\}$ (a σ -algebra), and let $\mu(\emptyset) = 0$, $\mu(\{a\}) = 1$, $\mu(Y) = \mu(X) = \infty$. This is a measure. The measurable subsets of Y of finite measure are only $\emptyset$, so the printed recipe gives $\mu_1(Y) = 0$; the measurable subsets of X of finite measure are $\emptyset$ and $\{a\}$, of measures 0 and 1, which are not arbitrarily large, so the recipe gives $\mu_1(X) = 0$. But $X = \{a\} \cup Y$ disjointly and $\mu_1(\{a\}) + \mu_1(Y) = 1 + 0 = 1 \neq 0 = \mu_1(X)$. So additivity (indeed even monotonicity) fails.

Second, the standard object the exercise is after, and the one that satisfies the printed description at every set where the description can be correct, is

$$\mu_1(E) = \sup \{ \mu(B) : B \in \mathcal{M}, B \subseteq E, \mu(B) < \infty \} \quad (E \in \mathcal{M}).$$

This agrees with the printed prescription in both of its cases that matter: if $\mu(E) < \infty$ then $B = E$ is admissible and dominates all others by monotonicity, so $\mu_1(E) = \mu(E)$; and if $\mu(E) = \infty$ and E contains measurable sets of arbitrarily large finite measure, then the supremum is ∞ . It differs from the printed prescription only on a set E of infinite measure whose finite-measure measurable subsets

have measures bounded but not all zero, which is exactly the situation exploited in the counterexample above. We prove the assertion for this μ_1 .

We check that μ_1 is a measure. Clearly $\mu_1(\emptyset) = 0$, and μ_1 is monotone, since enlarging E enlarges the family over which the supremum is taken. Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection in $\mathcal{M}$ and $E = \bigcup_{k=1}^\infty E_k$.

To see that $\mu_1(E) \geq \sum_{k=1}^\infty \mu_1(E_k)$, fix n and, for $1 \leq k \leq n$, choose $B_k \in \mathcal{M}$ with $B_k \subseteq E_k$ and $\mu(B_k) < \infty$. The set $B = \bigcup_{k=1}^n B_k$ is a measurable subset of E , and by finite additivity $\mu(B) = \sum_{k=1}^n \mu(B_k) < \infty$. Hence $\mu_1(E) \geq \sum_{k=1}^n \mu(B_k)$. Taking the supremum over each B_k separately gives $\mu_1(E) \geq \sum_{k=1}^n \mu_1(E_k)$, and letting $n \rightarrow \infty$ gives the inequality.

To see that $\mu_1(E) \leq \sum_{k=1}^\infty \mu_1(E_k)$, let $B \in \mathcal{M}$ with $B \subseteq E$ and $\mu(B) < \infty$. The sets $B \cap E_k$ are disjoint, measurable, have union B , and each has finite measure; since $B \cap E_k \subseteq E_k$, each is admissible in the supremum defining $\mu_1(E_k)$. Therefore, by the countable additivity of μ ,

$$\mu(B) = \sum_{k=1}^\infty \mu(B \cap E_k) \leq \sum_{k=1}^\infty \mu_1(E_k).$$

Taking the supremum over all such B gives the second inequality. Thus μ_1 is countably additive and so is a measure.

Next, $\mu_1 \leq \mu$, with equality exactly when μ is semifinite. The inequality $\mu_1 \leq \mu$ is immediate from monotonicity of μ . If μ is semifinite and $\mu(E) = \infty$, then E contains measurable sets of arbitrarily large finite measure, so $\mu_1(E) = \infty = \mu(E)$; together with the case $\mu(E) < \infty$ this gives $\mu_1 = \mu$.

Finally, μ_1 is semifinite. Let $E \in \mathcal{M}$ with $\mu_1(E) = \infty$ and let $M > 0$. By definition of the supremum there is $B \in \mathcal{M}$, $B \subseteq E$, with $M < \mu(B) < \infty$. Since $\mu(B) < \infty$ we have $\mu_1(B) = \mu(B)$, so B is a measurable subset of E with $M < \mu_1(B) < \infty$. Hence every set of infinite μ_1 -measure contains measurable sets of arbitrarily large finite μ_1 -measure: μ_1 is semifinite.

Part (iii). Call a measurable set F **purely infinite** provided $\mu(F) = \infty$ while $\mu_1(F) = 0$, that is, every measurable subset of F of finite measure is null. Define, for $E \in \mathcal{M}$,

$$\mu_2(E) = \begin{cases} \infty & \text{if } E \text{ contains a purely infinite measurable subset,} \\ 0 & \text{otherwise.} \end{cases}$$

By construction μ_2 takes only the values 0 and ∞ , and $\mu_2(\emptyset) = 0$.

We check that μ_2 is a measure. Let $\{E_k\}$ be disjoint measurable sets with union E . If some $\mu_2(E_{k_0}) = \infty$, then a purely infinite $F \subseteq E_{k_0} \subseteq E$ witnesses $\mu_2(E) = \infty$, and both sides of the additivity identity are ∞ . Suppose instead $\mu_2(E_k) = 0$ for all k ; we must show $\mu_2(E) = 0$. If not, there is a purely infinite $F \subseteq E$. Put $F_k = F \cap E_k$; these are disjoint with union F . Since μ_1 is monotone, $\mu_1(F_k) \leq \mu_1(F) = 0$, so each F_k has all its finite-measure measurable subsets null; in particular $\mu(F_k)$ is either 0 or ∞ (a finite value would force $\mu(F_k) = \mu_1(F_k) = 0$). If every $\mu(F_k)$ were 0 then $\mu(F) = \sum_k \mu(F_k) = 0$, contradicting $\mu(F) = \infty$. So some $\mu(F_{k_0}) = \infty$ while $\mu_1(F_{k_0}) = 0$; that is, $F_{k_0} \subseteq E_{k_0}$ is purely infinite, contradicting $\mu_2(E_{k_0}) = 0$. Hence μ_2 is countably additive.

Next we verify $\mu = \mu_1 + \mu_2$. Let $E \in \mathcal{M}$.

If $\mu(E) < \infty$, then E has no subset of infinite measure, so $\mu_2(E) = 0$, while $\mu_1(E) = \mu(E)$; the identity holds.

If $\mu_1(E) = \infty$, then both sides are ∞ .

Finally suppose $\mu(E) = \infty$ and $c := \mu_1(E) < \infty$; we must produce a purely infinite subset of E , for then $\mu_2(E) = \infty$ and both sides are again ∞ . Choose measurable $B_n \subseteq E$ of finite measure with $\mu(B_n) \rightarrow c$, and set $B = \bigcup_{n=1}^\infty B_n$. Each finite union $B_1 \cup \cdots \cup B_n$ is a measurable subset of E of finite measure, hence of measure at most c ; by the continuity of measure $\mu(B) = \lim_n \mu(B_1 \cup \cdots \cup B_n) \leq c < \infty$, and $\mu(B) \geq \lim_n \mu(B_n) = c$, so $\mu(B) = c$. Let $F = E \setminus B$. By additivity $\mu(E) = \mu(B) + \mu(F)$, and since $\mu(E) = \infty$ and $\mu(B) = c < \infty$ we get $\mu(F) = \infty$. If $G \subseteq F$ is measurable with $\mu(G) < \infty$, then $B \cup G \subseteq E$ is measurable of finite measure $c + \mu(G)$, so $c + \mu(G) \leq \mu_1(E) = c$ and therefore $\mu(G) = 0$. Thus $\mu_1(F) = 0$ and F is purely infinite. This completes the proof.

(Consistency check: if μ is semifinite then no purely infinite set exists, so $\mu_2 = 0$ and $\mu = \mu_1$, as it must be.)

Problem 17.9. Prove Proposition 3, that is, show that $\mathcal{M}_0$ is a σ -algebra, μ_0 is properly defined, and $(X, \mathcal{M}_0, \mu_0)$ is complete. In what sense is $\mathcal{M}_0$ minimal?

[Proposition 3: Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets E of X of the form $E = A \cup B$ where $B \in \mathcal{M}$ and $A \subseteq C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$. For such a set E define $\mu_0(E) = \mu(B)$. Then $\mathcal{M}_0$ is a σ -algebra that contains $\mathcal{M}$, μ_0 is a measure that extends μ , and $(X, \mathcal{M}_0, \mu_0)$ is a complete measure space.]

Solution. Throughout, call a set $C \in \mathcal{M}$ with $\mu(C) = 0$ a **null set** and call $A \subseteq X$ **negligible** provided $A \subseteq C$ for some null set C . Thus $\mathcal{M}_0 = \{A \cup B : A \text{ negligible}, B \in \mathcal{M}\}$.

First, a normalization used repeatedly. If $E = A \cup B$ with $A \subseteq C$ null and $B \in \mathcal{M}$, then also $E = A' \cup B$ with $A' = A \sim B \subseteq C \sim B =: C'$, and $C' \in \mathcal{M}$ with $\mu(C') = 0$ by monotonicity. So every $E \in \mathcal{M}_0$ may be written $E = A \cup B$ with $B \in \mathcal{M}$, $A \subseteq C$, C null, and moreover $A \cap B = \emptyset$ and $C \cap B = \emptyset$.

$\mathcal{M}$ is contained in $\mathcal{M}_0$: for $B \in \mathcal{M}$ write $B = \emptyset \cup B$ and take $C = \emptyset$. In particular X and $\emptyset$ belong to $\mathcal{M}_0$.

$\mathcal{M}_0$ is closed under countable unions. Let $E_k = A_k \cup B_k$ with $B_k \in \mathcal{M}$ and $A_k \subseteq C_k$, $\mu(C_k) = 0$. Then

$$\bigcup_{k=1}^{\infty} E_k = \left(\bigcup_{k=1}^{\infty} A_k \right) \cup \left(\bigcup_{k=1}^{\infty} B_k \right),$$

where $\bigcup_k B_k \in \mathcal{M}$ and $\bigcup_k A_k \subseteq \bigcup_k C_k \in \mathcal{M}$, the latter of measure zero by the countable subadditivity of μ . Hence the union belongs to $\mathcal{M}_0$.

$\mathcal{M}_0$ is closed under complements. Let $E \in \mathcal{M}_0$ and use the normalized form $E = A \cup B$ with $A \subseteq C$, $\mu(C) = 0$, $C \cap B = \emptyset$. Since C and B are disjoint,

$$X \sim B = (X \sim (B \cup C)) \cup C, \quad \text{disjointly.}$$

Intersecting with $X \sim A$ and using $A \subseteq C$, so that $X \sim (B \cup C)$ is disjoint from A , we obtain

$$X \sim E = (X \sim B) \cap (X \sim A) = (X \sim (B \cup C)) \cup (C \sim A).$$

Here $X \sim (B \cup C) \in \mathcal{M}$ and $C \sim A \subseteq C$ is negligible, so $X \sim E \in \mathcal{M}_0$. Therefore $\mathcal{M}_0$ is a σ -algebra containing $\mathcal{M}$.

μ_0 is properly defined. Suppose $E = A_1 \cup B_1 = A_2 \cup B_2$ with $B_i \in \mathcal{M}$ and $A_i \subseteq C_i$, $\mu(C_i) = 0$. Then

$$B_1 \subseteq A_2 \cup B_2 \subseteq C_2 \cup B_2,$$

so by monotonicity and subadditivity $\mu(B_1) \leq \mu(C_2) + \mu(B_2) = \mu(B_2)$. By symmetry $\mu(B_2) \leq \mu(B_1)$, so $\mu(B_1) = \mu(B_2)$ and $\mu_0(E)$ does not depend on the representation. Taking $A = \emptyset$ shows $\mu_0 = \mu$ on $\mathcal{M}$, so μ_0 extends μ .

μ_0 is a measure. Certainly $\mu_0(\emptyset) = \mu(\emptyset) = 0$ and $\mu_0 \geq 0$. Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection in $\mathcal{M}_0$, written in normalized form $E_k = A_k \cup B_k$ with $A_k \cap B_k = \emptyset$, $A_k \subseteq C_k$ null. As above $E = \bigcup_k E_k = A \cup B$ with $A = \bigcup_k A_k \subseteq \bigcup_k C_k$ (a null set) and $B = \bigcup_k B_k \in \mathcal{M}$, so $\mu_0(E) = \mu(B)$. Since $B_k \subseteq E_k$ and the E_k are disjoint, the sets B_k are disjoint members of $\mathcal{M}$, and hence by the countable additivity of μ ,

$$\mu_0(E) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \sum_{k=1}^{\infty} \mu_0(E_k).$$

Thus μ_0 is countably additive.

$(X, \mathcal{M}_0, \mu_0)$ is complete. Let $E \in \mathcal{M}_0$ with $\mu_0(E) = 0$ and let $F \subseteq E$. Write $E = A \cup B$ with $B \in \mathcal{M}$, $A \subseteq C$, $\mu(C) = 0$; then $\mu(B) = \mu_0(E) = 0$. Hence

$$F \subseteq E \subseteq C \cup B \in \mathcal{M}, \quad \mu(C \cup B) \leq \mu(C) + \mu(B) = 0.$$

So F is negligible, and $F = F \cup \emptyset$ belongs to $\mathcal{M}_0$ (with $\mu_0(F) = 0$). Thus every subset of a set of μ_0 -measure zero is measurable: the completion is complete.

Minimality. The measure space $(X, \mathcal{M}_0, \mu_0)$ is the smallest complete measure space extending $(X, \mathcal{M}, \mu)$, in the following precise sense: if $(X, \mathcal{M}', \mu')$ is a complete measure space with $\mathcal{M} \subseteq \mathcal{M}'$ and $\mu' = \mu$ on $\mathcal{M}$, then $\mathcal{M}_0 \subseteq \mathcal{M}'$ and $\mu' = \mu_0$ on $\mathcal{M}_0$.

Indeed, let $E = A \cup B \in \mathcal{M}_0$ with $B \in \mathcal{M}$ and $A \subseteq C \in \mathcal{M}$, $\mu(C) = 0$. Then $C \in \mathcal{M}'$ and $\mu'(C) = \mu(C) = 0$, so by the completeness of μ' the subset A of C belongs to $\mathcal{M}'$, and $\mu'(A) \leq \mu'(C) = 0$ by monotonicity. Hence $E = A \cup B \in \mathcal{M}'$, and

$$\mu'(B) \leq \mu'(E) \leq \mu'(A) + \mu'(B) = \mu'(B),$$

so $\mu'(E) = \mu'(B) = \mu(B) = \mu_0(E)$. Thus $\mathcal{M}_0$ is contained in every σ -algebra on which a complete extension of μ lives, and on $\mathcal{M}_0$ every such extension agrees with μ_0 . Equivalently, $\mathcal{M}_0$ is the smallest σ -algebra containing $\mathcal{M}$ together with all negligible sets.

Problem 17.10. If $(X, \mathcal{M}, \mu)$ is a measure space, we say that a subset E of X is **locally measurable** provided for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, the intersection $E \cap B$ belongs to $\mathcal{M}$. The measure μ is called **saturated** provided every locally measurable set is measurable.

(i) Show that each σ -finite measure is saturated.

(ii) Show that the collection $\mathcal{C}$ of locally measurable sets is a σ -algebra.

(iii) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\mathcal{C}$ the σ -algebra of locally measurable sets. For $E \in \mathcal{C}$, define $\bar{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\bar{\mu}(E) = \infty$ if $E \notin \mathcal{M}$. Show that $(X, \mathcal{C}, \bar{\mu})$ is a saturated measure space.

(iv) If μ is semifinite and $E \in \mathcal{C}$, set $\underline{\mu}(E) = \sup\{\mu(B) \mid B \in \mathcal{M}, B \subseteq E\}$. Show that $(X, \mathcal{C}, \underline{\mu})$ is a saturated measure space and that $\underline{\mu}$ is an extension of μ . Give an example to show that $\bar{\mu}$ and $\underline{\mu}$ may be different.

Solution. Part (i). Let μ be σ -finite, so $X = \bigcup_{k=1}^{\infty} X_k$ with $X_k \in \mathcal{M}$ and $\mu(X_k) < \infty$. If E is locally measurable, then $E \cap X_k \in \mathcal{M}$ for every k , and therefore

$$E = E \cap X = \bigcup_{k=1}^{\infty} (E \cap X_k) \in \mathcal{M}.$$

So every locally measurable set is measurable and μ is saturated.

Part (ii). Write $\mathcal{F} = \{B \in \mathcal{M} : \mu(B) < \infty\}$ for the collection of measurable sets of finite measure, so that $\mathcal{C} = \{E \subseteq X : E \cap B \in \mathcal{M} \text{ for all } B \in \mathcal{F}\}$.

Every measurable set is locally measurable, since $\mathcal{M}$ is closed under intersections; in particular $X \in \mathcal{C}$ and $\mathcal{M} \subseteq \mathcal{C}$. If $E \in \mathcal{C}$ and $B \in \mathcal{F}$, then

$$(X \sim E) \cap B = B \sim (E \cap B) \in \mathcal{M},$$

since B and $E \cap B$ are measurable; hence $X \sim E \in \mathcal{C}$. If $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{C}$ and $B \in \mathcal{F}$, then

$$\left(\bigcup_{k=1}^{\infty} E_k \right) \cap B = \bigcup_{k=1}^{\infty} (E_k \cap B) \in \mathcal{M}.$$

Thus $\mathcal{C}$ is a σ -algebra containing $\mathcal{M}$.

Part (iii). We first record which sets have finite $\bar{\mu}$ -measure: $\bar{\mu}(E) < \infty$ if and only if $E \in \mathcal{M}$ and $\mu(E) < \infty$, that is, if and only if $E \in \mathcal{F}$.

That $\bar{\mu}$ is a measure. Clearly $\bar{\mu}(\emptyset) = 0$. Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection in $\mathcal{C}$ with union E .

If every E_k belongs to $\mathcal{M}$, then $E \in \mathcal{M}$ and, by the countable additivity of μ , $\bar{\mu}(E) = \mu(E) = \sum_k \mu(E_k) = \sum_k \bar{\mu}(E_k)$.

Otherwise some $E_{k_0} \notin \mathcal{M}$, so $\sum_k \bar{\mu}(E_k) = \infty$, and it suffices to show $\bar{\mu}(E) = \infty$. If $E \notin \mathcal{M}$ this holds by definition. If $E \in \mathcal{M}$ and we had $\mu(E) < \infty$, then $E \in \mathcal{F}$, and local measurability of E_{k_0} would give $E_{k_0} = E_{k_0} \cap E \in \mathcal{M}$, a contradiction. Hence $\mu(E) = \infty$ and $\bar{\mu}(E) = \infty$. So $\bar{\mu}$ is countably additive.

That $(X, \mathcal{C}, \bar{\mu})$ is saturated. Let $E \subseteq X$ be locally measurable with respect to $(X, \mathcal{C}, \bar{\mu})$; that is, $E \cap B \in \mathcal{C}$ whenever $B \in \mathcal{C}$ and $\bar{\mu}(B) < \infty$. By the observation above, the sets $B \in \mathcal{C}$ with $\bar{\mu}(B) < \infty$

are precisely the members of $\mathcal{F}$. So let $B \in \mathcal{F}$; then $E \cap B \in \mathcal{C}$, and since $E \cap B$ is locally measurable with respect to μ and $B \in \mathcal{F}$,

$$E \cap B = (E \cap B) \cap B \in \mathcal{M}.$$

As $B \in \mathcal{F}$ was arbitrary, $E \in \mathcal{C}$. Hence every set locally measurable for $(X, \mathcal{C}, \bar{\mu})$ belongs to $\mathcal{C}$: the space is saturated.

Part (iv). Assume now that μ is semifinite, and define $\underline{\mu}(E) = \sup\{\mu(B) : B \in \mathcal{M}, B \subseteq E\}$ for $E \in \mathcal{C}$. $\underline{\mu}$ extends μ : if $E \in \mathcal{M}$ then E itself is admissible in the supremum and dominates every other admissible set by monotonicity, so $\underline{\mu}(E) = \mu(E)$. Also $\underline{\mu}(\emptyset) = 0$, and $\underline{\mu}$ is monotone on $\mathcal{C}$.

That $\underline{\mu}$ is a measure. Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection in $\mathcal{C}$ with union E .

For the inequality $\underline{\mu}(E) \geq \sum_k \underline{\mu}(E_k)$: fix n and choose $B_k \in \mathcal{M}$ with $B_k \subseteq E_k$ for $1 \leq k \leq n$. These are disjoint, so $B = \bigcup_{k=1}^n B_k$ is a measurable subset of E with $\mu(B) = \sum_{k=1}^n \mu(B_k)$, whence $\underline{\mu}(E) \geq \sum_{k=1}^n \mu(B_k)$. Taking suprema over each B_k and then letting $n \rightarrow \infty$ gives the inequality.

For the reverse inequality, let $B \in \mathcal{M}$ with $B \subseteq E$; we show $\mu(B) \leq \sum_k \underline{\mu}(E_k)$.

Suppose first $\mu(B) < \infty$. Then $B \in \mathcal{F}$, so for each k the local measurability of E_k gives $B \cap E_k \in \mathcal{M}$; these sets are disjoint, measurable, contained respectively in E_k , and have union B . Hence, by the countable additivity of μ ,

$$\mu(B) = \sum_{k=1}^{\infty} \mu(B \cap E_k) \leq \sum_{k=1}^{\infty} \underline{\mu}(E_k).$$

Now suppose $\mu(B) = \infty$. Here semifiniteness enters: B contains measurable sets B' of arbitrarily large finite measure, and by the case just treated $\mu(B') \leq \sum_k \underline{\mu}(E_k)$ for each such B' . Letting $\mu(B') \rightarrow \infty$ gives $\sum_k \underline{\mu}(E_k) = \infty \geq \mu(B)$.

In either case $\mu(B) \leq \sum_k \underline{\mu}(E_k)$, and taking the supremum over B yields $\underline{\mu}(E) \leq \sum_k \underline{\mu}(E_k)$. Therefore $\underline{\mu}$ is countably additive, hence a measure on $\mathcal{C}$.

That $(X, \mathcal{C}, \underline{\mu})$ is saturated. Let $E \subseteq X$ satisfy $E \cap B \in \mathcal{C}$ for every $B \in \mathcal{C}$ with $\underline{\mu}(B) < \infty$. Let $B \in \mathcal{F}$. Then $B \in \mathcal{M} \subseteq \mathcal{C}$ and $\underline{\mu}(B) = \mu(B) < \infty$, so $E \cap B \in \mathcal{C}$; since $E \cap B$ is then locally measurable with respect to μ and B has finite measure, $E \cap B = (E \cap B) \cap B \in \mathcal{M}$. As $B \in \mathcal{F}$ was arbitrary, $E \in \mathcal{C}$, and the space is saturated.

An example in which $\bar{\mu} \neq \underline{\mu}$. Let $X = \mathbb{R}$, let

$$\mathcal{M} = \{S \subseteq \mathbb{R} : S \text{ is countable or } \mathbb{R} \sim S \text{ is countable}\},$$

the σ -algebra of countable and cocountable sets, and let $W = (0, \infty)$, $Z = (-\infty, 0]$. Define, for $S \in \mathcal{M}$,

$$\mu(S) = \#(S \cap W),$$

the number of points of S lying in W (interpreted as ∞ when $S \cap W$ is infinite). This is the restriction to $\mathcal{M}$ of counting measure on W , hence is countably additive: μ is a measure.

μ is semifinite. If $\mu(S) = \infty$ then $S \cap W$ is infinite, so for each n we may choose n points of $S \cap W$; the resulting finite set is measurable, is contained in S , and has measure n .

The measurable sets of finite measure are the sets $S \in \mathcal{M}$ with $S \cap W$ finite; such a set must be countable, because a cocountable S meets $W = (0, \infty)$ in a cocountable, hence uncountable, subset of W . Consequently every $B \in \mathcal{F}$ is countable.

Now $Z = (-\infty, 0]$ is locally measurable: for $B \in \mathcal{F}$ the set $Z \cap B$ is countable, hence in $\mathcal{M}$. But $Z \notin \mathcal{M}$, since both Z and $\mathbb{R} \sim Z$ are uncountable. Therefore

$$\bar{\mu}(Z) = \infty.$$

On the other hand, a measurable subset B of Z cannot be cocountable (its complement contains the uncountable set W), so B is countable and $B \cap W = \emptyset$, giving $\mu(B) = 0$. Hence

$$\underline{\mu}(Z) = 0 \neq \infty = \bar{\mu}(Z).$$

So the two saturated extensions of a semifinite μ furnished by (iii) and (iv) genuinely differ. (They agree on $\mathcal{M}$, where both equal μ ; the discrepancy is confined to $\mathcal{C} \sim \mathcal{M}$, and indeed $\underline{\mu} \leq \bar{\mu}$ always, since $\underline{\mu} = \mu = \bar{\mu}$ on $\mathcal{M}$ and $\bar{\mu} = \infty$ off $\mathcal{M}$.)

Problem 17.11. Let μ and η be measures on the measurable space $(X, \mathcal{M})$. For $E \in \mathcal{M}$, define $\nu(E) = \max\{\mu(E), \eta(E)\}$. Is ν a measure on $(X, \mathcal{M})$?

Solution. No. The set function ν satisfies $\nu(\emptyset) = 0$ and is nonnegative, monotone, and countably subadditive, but it fails to be countably (indeed even finitely) additive in general.

A counterexample. Let $X = \{1, 2\}$ and $\mathcal{M} = 2^X$. Define μ and η by

$$\mu(\{1\}) = 1, \quad \mu(\{2\}) = 0, \quad \eta(\{1\}) = 0, \quad \eta(\{2\}) = 1,$$

extended additively (so $\mu = \delta_1$ and $\eta = \delta_2$ are Dirac measures, plainly measures on 2^X). Then

$$\nu(\{1\}) = \max\{1, 0\} = 1, \quad \nu(\{2\}) = \max\{0, 1\} = 1,$$

while

$$\nu(X) = \max\{\mu(X), \eta(X)\} = \max\{1, 1\} = 1.$$

Since $X = \{1\} \cup \{2\}$ is a disjoint union of measurable sets and $\nu(\{1\}) + \nu(\{2\}) = 2 \neq 1 = \nu(X)$, the set function ν is not additive, hence not a measure.

What does survive. For any measures μ, η the function $\nu = \max\{\mu, \eta\}$ obeys $\nu(\emptyset) = 0$; it is monotone, since μ and η are; and it is countably subadditive, because for a countable collection $\{E_k\}$ with union E ,

$$\nu(E) = \max\{\mu(E), \eta(E)\} \leq \max\left\{\sum_k \mu(E_k), \sum_k \eta(E_k)\right\} \leq \sum_k \max\{\mu(E_k), \eta(E_k)\} = \sum_k \nu(E_k).$$

What fails is superadditivity, exactly because the maximum may be attained by μ on one piece and by η on another.

Finally, ν is a measure whenever the two measures are comparable, say $\mu \leq \eta$ on $\mathcal{M}$, for then $\nu = \eta$.

One should resist the temptation to strengthen this. Suppose $X = P \cup Q$ is a disjoint union of measurable sets with $\eta \leq \mu$ on the measurable subsets of P and $\mu \leq \eta$ on the measurable subsets of Q , and set $\lambda(E) = \mu(E \cap P) + \eta(E \cap Q)$. Then λ is a measure (it is a sum of two measures, by Problem 7 of this chapter) and $\lambda \geq \nu$, since for each $E \in \mathcal{M}$,

$$\mu(E) = \mu(E \cap P) + \mu(E \cap Q) \leq \mu(E \cap P) + \eta(E \cap Q) = \lambda(E)$$

and symmetrically $\eta(E) \leq \lambda(E)$. But λ need not equal ν , and such a decomposition does not make ν a measure: in the counterexample above, $P = \{1\}$ and $Q = \{2\}$ satisfy the stated hypotheses, yet $\nu(X) = 1$ while $\lambda(X) = 2$, and ν is not additive. The pointwise maximum is simply the wrong construction for a least measure dominating μ and η ; the correct object is the join studied in Problem 17.

Problem 17.12. In the above example, let E be a Lebesgue measurable set such that $0 < \nu(E) < \infty$. Find a positive set A contained in E for which $\nu(A) > 0$.

[The example: $f: \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue integrable over $\mathbb{R}$ and, for a Lebesgue measurable set E , $\nu(E) = \int_E f \, dm$; by the countable additivity of integration this is a signed measure on $(\mathbb{R}, \mathcal{L})$. With $A = \{x \in \mathbb{R} \mid f(x) \geq 0\}$ and $B = \{x \in \mathbb{R} \mid f(x) < 0\}$ and

$$\nu^+(E) = \int_{A \cap E} f \, dm, \quad \nu^-(E) = - \int_{B \cap E} f \, dm,$$

the pair $\{A, B\}$ is a Hahn decomposition of $\mathbb{R}$ for ν and $\nu = \nu^+ - \nu^-$ is a Jordan decomposition of ν .]

Solution. Take

$$A = E \cap \{x \in \mathbb{R} : f(x) \geq 0\}.$$

To keep the notation of the example distinct from the set being sought, write $A_0 = \{x \in \mathbb{R} \mid f(x) \geq 0\}$ and $B_0 = \{x \in \mathbb{R} \mid f(x) < 0\} = \mathbb{R} \setminus A_0$ for the two sets called A and B in the example, so that the set to be exhibited is $A = E \cap A_0$. Since f is Lebesgue measurable, A_0 is Lebesgue measurable, and hence so is A ; also $A \subseteq E$.

A is a positive set. Let $S \subseteq A$ be measurable. Then $f \geq 0$ on S , so by the monotonicity of integration for nonnegative integrable functions,

$$\nu(S) = \int_S f \, dm \geq 0.$$

Thus every measurable subset of A has nonnegative ν -measure, which is precisely the definition of a positive set.

$\nu(A) > 0$. The sets $E \cap A_0 = A$ and $E \cap B_0$ are disjoint measurable sets with union E , so by the additivity of the integral over domains (equivalently, by the countable additivity of the signed measure ν),

$$\nu(E) = \nu(A) + \nu(E \cap B_0).$$

All three numbers are finite, since f is integrable over $\mathbb{R}$. On $E \cap B_0$ we have $f < 0$, so $\nu(E \cap B_0) = \int_{E \cap B_0} f \, dm \leq 0$. Therefore

$$\nu(A) = \nu(E) - \nu(E \cap B_0) \geq \nu(E) > 0.$$

So A is a positive set contained in E with $\nu(A) > 0$, as required. (This exhibits, for this concrete signed measure, the set whose existence is asserted in general by Hahn's Lemma; note also that $\{A_0, B_0\}$ is the Hahn decomposition of $\mathbb{R}$ recorded in the example, and $A = E \cap A_0$, so $\nu(A) = \nu^+(E)$.)

Problem 17.13. Let μ be a measure and μ_1 and μ_2 be mutually singular measures on a measurable space $(X, \mathcal{M})$ for which $\mu = \mu_1 - \mu_2$. Show that $\mu_2 = 0$. Use this to establish the uniqueness assertion of the Jordan Decomposition Theorem.

Solution. First assertion. Since $\mu_1 \perp \mu_2$, there are disjoint measurable sets A and B with $X = A \cup B$ and $\mu_1(B) = \mu_2(A) = 0$. Let $E \subseteq B$ be measurable. By monotonicity $\mu_1(E) \leq \mu_1(B) = 0$, so $\mu_1(E) = 0$ and therefore

$$\mu(E) = \mu_1(E) - \mu_2(E) = -\mu_2(E).$$

But μ is a measure, so $\mu(E) \geq 0$, while $\mu_2(E) \geq 0$; hence $\mu_2(E) = 0$. Taking $E = B$ gives $\mu_2(B) = 0$, and by hypothesis $\mu_2(A) = 0$, so by additivity

$$\mu_2(X) = \mu_2(A) + \mu_2(B) = 0.$$

By monotonicity $\mu_2(E) \leq \mu_2(X) = 0$ for every $E \in \mathcal{M}$, that is, $\mu_2 = 0$.

Uniqueness in the Jordan Decomposition Theorem. Let ν be a signed measure on $(X, \mathcal{M})$ and suppose

$$\nu = \mu_1 - \mu_2 \quad \text{with } \mu_1, \mu_2 \text{ measures and } \mu_1 \perp \mu_2.$$

We show that $\mu_1 = \nu^+$ and $\mu_2 = \nu^-$, where $\nu^\pm$ are the measures built from a Hahn decomposition as in the text. Since any two such representations then both coincide with the pair (ν^+, ν^-) , the mutually singular pair is unique.

Choose disjoint measurable A, B with $X = A \cup B$, $\mu_1(B) = \mu_2(A) = 0$.

Step 1: $\{A, B\}$ is a Hahn decomposition for ν . If $E \subseteq A$ is measurable, then $\mu_2(E) \leq \mu_2(A) = 0$, so $\nu(E) = \mu_1(E) \geq 0$; thus A is a positive set. If $E \subseteq B$ is measurable, then $\mu_1(E) \leq \mu_1(B) = 0$, so $\nu(E) = -\mu_2(E) \leq 0$; thus B is a negative set.

This is precisely where the first assertion is used. It is the degenerate case of Step 1: if the signed measure $\nu = \mu$ happens to be a measure, then B is a negative set for a measure, so every measurable $E \subseteq B$ satisfies both $\mu(E) \geq 0$ and $\mu(E) \leq 0$; hence $\mu_2(E) = -\mu(E) = 0$, and μ_2 vanishes on B , hence on all of X . In the general case the same dichotomy is what pins the pair (μ_1, μ_2) to the Hahn decomposition $\{A, B\}$, and the remaining two steps convert that into uniqueness.

Step 2: any two Hahn decompositions give the same values. Let $\{A, B\}$ and $\{A', B'\}$ be Hahn decompositions for ν . The set $A \sim A' = A \cap B'$ is a measurable subset of the positive set A and of the negative set B' , so every measurable subset S of it satisfies both $\nu(S) \geq 0$ and $\nu(S) \leq 0$; that is, $A \sim A'$ is a null set for ν . Symmetrically $A' \sim A$ is null. Hence for every measurable E , using additivity,

$$\nu(E \cap A) = \nu(E \cap A \cap A') + \nu(E \cap (A \sim A')) = \nu(E \cap A \cap A') = \nu(E \cap A'),$$

the last equality by the same computation with the roles of A and A' interchanged. Since $B = X \sim A$ and $B' = X \sim A'$, the same argument (or subtraction) gives $\nu(E \cap B) = \nu(E \cap B')$.

Step 3: identification. Let $E \in \mathcal{M}$. Since $\mu_1(E \cap B) \leq \mu_1(B) = 0$ and $\mu_2(E \cap A) \leq \mu_2(A) = 0$,

$$\mu_1(E) = \mu_1(E \cap A) = \mu_1(E \cap A) - \mu_2(E \cap A) = \nu(E \cap A),$$

$$\mu_2(E) = \mu_2(E \cap B) = -[\mu_1(E \cap B) - \mu_2(E \cap B)] = -\nu(E \cap B).$$

By Step 1 the pair $\{A, B\}$ is a Hahn decomposition, so by Step 2 these values agree with those computed from the Hahn decomposition $\{A', B'\}$ used to define ν^+ and ν^- :

$$\mu_1(E) = \nu(E \cap A') = \nu^+(E), \quad \mu_2(E) = -\nu(E \cap B') = \nu^-(E).$$

Hence $\mu_1 = \nu^+$ and $\mu_2 = \nu^-$: there is exactly one pair of mutually singular measures whose difference is ν . In particular (Step 2 alone) the measures ν^+ and ν^- do not depend on which Hahn decomposition is used to define them.

Problem 17.14. Show that if E is any measurable set, then

$$-\nu^-(E) \leq \nu(E) \leq \nu^+(E) \quad \text{and} \quad |\nu(E)| \leq |\nu|(E).$$

Solution. Let ν be a signed measure on $(X, \mathcal{M})$ and let $\{A, B\}$ be a Hahn decomposition for ν , so that $X = A \cup B$ with $A \cap B = \emptyset$, A positive, B negative, and by definition

$$\nu^+(E) = \nu(E \cap A), \quad \nu^-(E) = -\nu(E \cap B), \quad |\nu|(E) = \nu^+(E) + \nu^-(E).$$

Since $E \cap A$ is a measurable subset of the positive set A and $E \cap B$ is a measurable subset of the negative set B , we have

$$\nu^+(E) = \nu(E \cap A) \geq 0 \quad \text{and} \quad \nu^-(E) = -\nu(E \cap B) \geq 0$$

for every measurable E ; this is just the statement that ν^+ and ν^- are measures.

Now fix $E \in \mathcal{M}$. The sets $E \cap A$ and $E \cap B$ are disjoint measurable sets with union E , so by the additivity of the signed measure ν ,

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu^+(E) - \nu^-(E).$$

The right-hand side is unambiguous in the extended reals: a signed measure assumes at most one of the values $+\infty$, $-\infty$, so at most one of $\nu^+(E)$, $\nu^-(E)$ is infinite (indeed one of the measures ν^+ , ν^- is finite).

Since $\nu^-(E) \geq 0$, subtracting it from $\nu^+(E)$ can only decrease that value, so

$$\nu(E) = \nu^+(E) - \nu^-(E) \leq \nu^+(E),$$

and since $\nu^+(E) \geq 0$,

$$\nu(E) = \nu^+(E) - \nu^-(E) \geq -\nu^-(E).$$

This is the first pair of inequalities.

For the second, note that $\nu^+(E) \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$ and likewise $\nu^-(E) \leq |\nu|(E)$, both because ν^+ and ν^- are nonnegative. Combining with the inequalities just proved,

$$-|\nu|(E) \leq -\nu^-(E) \leq \nu(E) \leq \nu^+(E) \leq |\nu|(E),$$

which says precisely that $|\nu(E)| \leq |\nu|(E)$. (If $|\nu|(E) = \infty$ the inequality is trivial; if $|\nu|(E) < \infty$ then $\nu^+(E)$ and $\nu^-(E)$ are both finite and the display is an inequality between real numbers.) Finally, these bounds do not depend on the choice of Hahn decomposition, since by the uniqueness assertion of the Jordan Decomposition Theorem the measures ν^+ and ν^- are determined by ν alone.

17.3 Problems 17.15–17.21

Problem 17.15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \quad \text{and} \quad |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets E .

Solution. Throughout, $(X, \mathcal{M})$ is the underlying measurable space.

Step 1: $\alpha\nu_1 + \beta\nu_2$ is a finite signed measure.

Recall that a signed measure ν is called a finite signed measure exactly when both ν^+ and ν^- are finite, and that this happens precisely when ν is real-valued on all of $\mathcal{M}$. Indeed, if ν is real-valued, choose a Hahn decomposition $\{A, B\}$ for ν , as provided by the Hahn Decomposition Theorem; then $\nu^+(X) = \nu(A)$ and $\nu^-(X) = -\nu(B)$ are real numbers, so ν^+ and ν^- are finite measures. Conversely, if ν^+ and ν^- are finite, then for each $E \in \mathcal{M}$ we have $|\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(X) + \nu^-(X) < \infty$. Define $\lambda(E) = \alpha\nu_1(E) + \beta\nu_2(E)$ for $E \in \mathcal{M}$. Since ν_1 and ν_2 are finite signed measures, $\nu_1(E)$ and $\nu_2(E)$ are real numbers, so $\lambda(E)$ is a well-defined real number; in particular λ assumes neither $+\infty$ nor $-\infty$. Also $\lambda(\emptyset) = \alpha \cdot 0 + \beta \cdot 0 = 0$.

For countable additivity, let $\{E_k\}_{k=1}^\infty$ be a disjoint collection of measurable sets with union E . Since $\nu_i(E)$ is finite, property (iii) of the definition of a signed measure tells us that $\sum_{k=1}^\infty \nu_i(E_k)$ converges absolutely to $\nu_i(E)$, for $i = 1, 2$. Hence

$$\sum_{k=1}^\infty |\lambda(E_k)| \leq |\alpha| \sum_{k=1}^\infty |\nu_1(E_k)| + |\beta| \sum_{k=1}^\infty |\nu_2(E_k)| < \infty,$$

so the series $\sum_k \lambda(E_k)$ converges absolutely, and by linearity of absolutely convergent series,

$$\sum_{k=1}^\infty \lambda(E_k) = \alpha \sum_{k=1}^\infty \nu_1(E_k) + \beta \sum_{k=1}^\infty \nu_2(E_k) = \alpha\nu_1(E) + \beta\nu_2(E) = \lambda(E).$$

Thus λ is a signed measure, and it is real-valued, hence a finite signed measure.

Step 2: a local form of the total variation formula (4).

Lemma. For a signed measure ν and $E \in \mathcal{M}$,

$$|\nu|(E) = \sup \left\{ \sum_{k=1}^n |\nu(E_k)| \right\},$$

the supremum being taken over all finite disjoint collections $\{E_k\}_{k=1}^n$ of measurable subsets of E .

Proof of the Lemma. Let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable subsets of E . Since $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$,

$$|\nu(E_k)| = |\nu^+(E_k) - \nu^-(E_k)| \leq \nu^+(E_k) + \nu^-(E_k) = |\nu|(E_k).$$

Therefore, by the additivity and monotonicity of the measure $|\nu|$,

$$\sum_{k=1}^n |\nu(E_k)| \leq \sum_{k=1}^n |\nu|(E_k) = |\nu| \left(\bigcup_{k=1}^n E_k \right) \leq |\nu|(E),$$

so the supremum is at most $|\nu|(E)$. On the other hand, let $\{A, B\}$ be a Hahn decomposition for ν , as provided by the Hahn Decomposition Theorem, so that $\nu^+(F) = \nu(F \cap A)$ and $\nu^-(F) = -\nu(F \cap B)$. The two sets $E \cap A$ and $E \cap B$ are disjoint measurable subsets of E , and

$$|\nu(E \cap A)| + |\nu(E \cap B)| = \nu^+(E) + \nu^-(E) = |\nu|(E),$$

since $\nu(E \cap A) = \nu^+(E) \geq 0$ and $\nu(E \cap B) = -\nu^-(E) \leq 0$. Hence the supremum equals $|\nu|(E)$, and the Lemma is proved.

Step 3: $|\alpha\nu| = |\alpha| |\nu|$.

Let ν be a finite signed measure and $\alpha \in \mathbb{R}$. By Step 1, $\alpha\nu$ is a finite signed measure. Fix $E \in \mathcal{M}$. For every finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable subsets of E ,

$$\sum_{k=1}^n |(\alpha\nu)(E_k)| = |\alpha| \sum_{k=1}^n |\nu(E_k)|.$$

Taking the supremum over all such collections and using the Lemma twice (once for $\alpha\nu$ and once for ν ; the factor $|\alpha| \geq 0$ passes through the supremum, the case $\alpha = 0$ being trivial since then both sides vanish), we get

$$|\alpha\nu|(E) = |\alpha| |\nu|(E).$$

As E was arbitrary, $|\alpha\nu| = |\alpha| |\nu|$.

Step 4: $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Fix $E \in \mathcal{M}$ and let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable subsets of E . By the triangle inequality in $\mathbb{R}$ and the Lemma applied to ν_1 and to ν_2 ,

$$\sum_{k=1}^n |(\nu_1 + \nu_2)(E_k)| \leq \sum_{k=1}^n |\nu_1(E_k)| + \sum_{k=1}^n |\nu_2(E_k)| \leq |\nu_1|(E) + |\nu_2|(E).$$

Taking the supremum over all such collections and invoking the Lemma for $\nu_1 + \nu_2$ yields

$$|\nu_1 + \nu_2|(E) \leq |\nu_1|(E) + |\nu_2|(E) = (|\nu_1| + |\nu_2|)(E).$$

Since $E \in \mathcal{M}$ was arbitrary, $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ in the stated sense.

Problem 17.16. Prove (4), that is, prove that for a signed measure ν on the measurable space $(X, \mathcal{M})$,

$$|\nu|(X) = \sup \sum_{k=1}^n |\nu(E_k)|,$$

where the supremum is taken over all finite disjoint collections $\{E_k\}_{k=1}^n$ of measurable subsets of X .

Solution. Write S for the supremum on the right-hand side, taken over all finite disjoint collections $\{E_k\}_{k=1}^n$ of measurable subsets of X . Recall that, by the Jordan Decomposition Theorem, $\nu = \nu^+ - \nu^-$ with ν^+ and ν^- mutually singular measures, and that $|\nu| = \nu^+ + \nu^-$. Since a signed measure assumes at most one of the values $+\infty$ and $-\infty$, at most one of ν^+ , ν^- is infinite, and both sides of the asserted identity are permitted to equal $+\infty$.

Step 1: $S \leq |\nu|(X)$.

Let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable subsets of X . For each k ,

$$|\nu(E_k)| = |\nu^+(E_k) - \nu^-(E_k)| \leq \nu^+(E_k) + \nu^-(E_k) = |\nu|(E_k),$$

where the difference $\nu^+(E_k) - \nu^-(E_k)$ is well defined because at most one term is infinite. Since $|\nu|$ is a measure, it is finitely additive and monotone, and therefore

$$\sum_{k=1}^n |\nu(E_k)| \leq \sum_{k=1}^n |\nu|(E_k) = |\nu|\left(\bigcup_{k=1}^n E_k\right) \leq |\nu|(X).$$

Taking the supremum over all such collections gives $S \leq |\nu|(X)$.

Step 2: $|\nu|(X) \leq S$.

By the Hahn Decomposition Theorem there is a positive set A for ν and a negative set B for ν with

$$X = A \cup B \quad \text{and} \quad A \cap B = \emptyset.$$

By the definition of the positive and negative parts associated with this Hahn decomposition,

$$\nu^+(X) = \nu(X \cap A) = \nu(A) \geq 0 \quad \text{and} \quad \nu^-(X) = -\nu(X \cap B) = -\nu(B) \geq 0.$$

Now $\{A, B\}$ is itself a finite disjoint collection of measurable subsets of X , so

$$S \geq |\nu(A)| + |\nu(B)| = \nu(A) + (-\nu(B)) = \nu^+(X) + \nu^-(X) = |\nu|(X).$$

(If one of $\nu^+(X)$, $\nu^-(X)$ is infinite this reads $S \geq \infty$, so $S = \infty = |\nu|(X)$.)

Combining Steps 1 and 2 gives $|\nu|(X) = S$, which is (4).

Two remarks. First, the supremum is unchanged if one restricts to collections $\{E_k\}_{k=1}^n$ that are partitions of X : given any finite disjoint collection, adjoin the measurable set $E_{n+1} = X \setminus \bigcup_{k=1}^n E_k$ to obtain a partition of X whose associated sum exceeds the original one by the nonnegative quantity $|\nu(E_{n+1})|$. Second, the identical argument applied to an arbitrary $E \in \mathcal{M}$ in place of X , using the disjoint pair $\{E \cap A, E \cap B\}$ in Step 2, shows that

$$|\nu|(E) = \sup \sum_{k=1}^n |\nu(E_k)|,$$

the supremum now being over all finite disjoint collections of measurable subsets of E .

Problem 17.17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ and $\mu \vee \nu = \mu + \nu - \mu \wedge \nu$.

- (i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν .
- (ii) Show that the signed measure $\mu \vee \nu$ is larger than μ and ν but smaller than any other measure that is larger than μ and ν .
- (iii) If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution. Throughout, $\lambda_1 \leq \lambda_2$ means $\lambda_1(E) \leq \lambda_2(E)$ for every measurable set E , as in Problem 15. Set $\sigma = \mu - \nu$. By Problem 15, σ is a finite signed measure, and therefore $|\sigma| = \sigma^+ + \sigma^-$ is a finite measure; again by Problem 15, both

$$\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\sigma|) \quad \text{and} \quad \mu \vee \nu = \mu + \nu - \mu \wedge \nu = \frac{1}{2}(\mu + \nu + |\sigma|)$$

are finite signed measures.

Since σ^+ and σ^- are finite measures with $\sigma = \sigma^+ - \sigma^-$ and $|\sigma| = \sigma^+ + \sigma^-$, adding and subtracting these two identities gives the basic formulas

$$\sigma^+ = \frac{1}{2}(|\sigma| + \sigma), \quad \sigma^- = \frac{1}{2}(|\sigma| - \sigma).$$

By the Hahn Decomposition Theorem, fix a Hahn decomposition $X = P \cup B$, $P \cap B = \emptyset$, for σ , with P positive and B negative for σ , so that for every measurable E ,

$$\sigma^+(E) = \sigma(E \cap P), \quad \sigma^-(E) = -\sigma(E \cap B).$$

Step 1: two identities.

First,

$$\mu - \mu \wedge \nu = \mu - \frac{1}{2}(\mu + \nu - |\sigma|) = \frac{1}{2}(\sigma + |\sigma|) = \sigma^+,$$

and second,

$$\nu - \mu \wedge \nu = \frac{1}{2}(|\sigma| - \sigma) = \sigma^-.$$

Next, an explicit formula. For measurable E , using $\mu \wedge \nu = \mu - \sigma^+$ and $\sigma(E \cap P) = \mu(E \cap P) - \nu(E \cap P)$,

$$\begin{aligned} (\mu \wedge \nu)(E) &= \mu(E) - \sigma(E \cap P) \\ &= [\mu(E \cap P) + \mu(E \cap B)] - [\mu(E \cap P) - \nu(E \cap P)] \\ &= \nu(E \cap P) + \mu(E \cap B). \end{aligned}$$

Consequently

$$(\mu \vee \nu)(E) = \mu(E) + \nu(E) - (\mu \wedge \nu)(E) = \mu(E \cap P) + \nu(E \cap B).$$

Part (i).

Since σ^+ and σ^- are measures, they are nonnegative, so the two identities of Step 1 give $\mu - \mu \wedge \nu \geq 0$ and $\nu - \mu \wedge \nu \geq 0$; that is,

$$\mu \wedge \nu \leq \mu \quad \text{and} \quad \mu \wedge \nu \leq \nu.$$

Now let λ be any signed measure with $\lambda \leq \mu$ and $\lambda \leq \nu$. Let E be measurable. Applying $\lambda \leq \nu$ on the set $E \cap P$ and $\lambda \leq \mu$ on the set $E \cap B$, and using the additivity of λ over the disjoint decomposition $E = (E \cap P) \cup (E \cap B)$,

$$\lambda(E) = \lambda(E \cap P) + \lambda(E \cap B) \leq \nu(E \cap P) + \mu(E \cap B) = (\mu \wedge \nu)(E).$$

Hence $\lambda \leq \mu \wedge \nu$. Thus $\mu \wedge \nu$ is the largest signed measure that is smaller than both μ and ν .

Part (ii).

From Step 1,

$$\mu \vee \nu - \mu = \nu - \mu \wedge \nu = \sigma^- \geq 0 \quad \text{and} \quad \mu \vee \nu - \nu = \mu - \mu \wedge \nu = \sigma^+ \geq 0,$$

so $\mu \vee \nu \geq \mu$ and $\mu \vee \nu \geq \nu$. Let λ be any signed measure with $\lambda \geq \mu$ and $\lambda \geq \nu$. For measurable E , applying $\lambda \geq \mu$ on $E \cap P$ and $\lambda \geq \nu$ on $E \cap B$,

$$\lambda(E) = \lambda(E \cap P) + \lambda(E \cap B) \geq \mu(E \cap P) + \nu(E \cap B) = (\mu \vee \nu)(E).$$

Hence $\lambda \geq \mu \vee \nu$, so $\mu \vee \nu$ is the smallest signed measure larger than both μ and ν .

Part (iii).

Now assume μ and ν are (finite, positive) measures.

Suppose first that μ and ν are mutually singular. Then there are disjoint measurable sets A and C with $X = A \cup C$ for which $\mu(A) = 0$ and $\nu(C) = 0$. The zero measure satisfies $0 \leq \mu$ and $0 \leq \nu$, so by the maximality established in part (i), $0 \leq \mu \wedge \nu$. On the other hand, for measurable E , using $\mu \wedge \nu \leq \mu$ on $E \cap A$, $\mu \wedge \nu \leq \nu$ on $E \cap C$, and the monotonicity of the measures μ and ν ,

$$(\mu \wedge \nu)(E) = (\mu \wedge \nu)(E \cap A) + (\mu \wedge \nu)(E \cap C) \leq \mu(E \cap A) + \nu(E \cap C) \leq \mu(A) + \nu(C) = 0.$$

Therefore $(\mu \wedge \nu)(E) = 0$ for every measurable E , that is, $\mu \wedge \nu = 0$.

Conversely, suppose $\mu \wedge \nu = 0$. Take $E = X$ in the explicit formula of Step 1:

$$0 = (\mu \wedge \nu)(X) = \nu(P) + \mu(B).$$

Both summands are nonnegative because μ and ν are measures, so $\nu(P) = 0$ and $\mu(B) = 0$. The sets B and P are disjoint, measurable, and $X = B \cup P$, with $\mu(B) = 0$ and $\nu(P) = 0$. By the definition of mutual singularity, μ and ν are mutually singular.

Problem 17.18. Let $\mu^* : 2^X \rightarrow [0, \infty]$ be an outer measure. Let $A \subseteq X$, $\{E_k\}_{k=1}^\infty$ be a disjoint

countable collection of measurable sets and $E = \bigcup_{k=1}^{\infty} E_k$. Show that

$$\mu^*(A \cap E) = \sum_{k=1}^{\infty} \mu^*(A \cap E_k).$$

Solution. Here measurable means measurable with respect to the reference outer measure μ^* , in the sense of Carathéodory.

The inequality $\leq$. Since

$$A \cap E = A \cap \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (A \cap E_k),$$

the countable collection $\{A \cap E_k\}_{k=1}^{\infty}$ covers $A \cap E$. An outer measure is countably monotone by definition, so

$$\mu^*(A \cap E) \leq \sum_{k=1}^{\infty} \mu^*(A \cap E_k).$$

The inequality $\geq$. Fix a natural number n . The collection $\{E_k\}_{k=1}^n$ is a finite disjoint collection of measurable sets, so Proposition 6 gives

$$\mu^*\left(A \cap \bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu^*(A \cap E_k).$$

Moreover $A \cap \bigcup_{k=1}^n E_k \subseteq A \cap E$, and an outer measure is monotone (finite monotonicity follows from countable monotonicity, since $\mu^*(\emptyset) = 0$). Therefore

$$\mu^*(A \cap E) \geq \mu^*\left(A \cap \bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu^*(A \cap E_k).$$

The left-hand side of this inequality does not depend on n , so letting $n \rightarrow \infty$ we obtain

$$\mu^*(A \cap E) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k).$$

The two inequalities together give the asserted equality

$$\mu^*(A \cap E) = \sum_{k=1}^{\infty} \mu^*(A \cap E_k).$$

Note that A is an arbitrary subset of X and need not be measurable; only the sets E_k are required to be measurable, and E is measurable by Proposition 7, though measurability of E was nowhere used. Taking $A = X$ recovers (5) in the proof of Theorem 8, the countable additivity of the restriction of μ^* to the σ -algebra of measurable sets. There is no circularity here: the argument above appeals only to Proposition 6 and to the countable monotonicity of μ^* , neither of which depends on Theorem 8.

Problem 17.19. Show that any measure that is induced by an outer measure is complete.

Solution. Let $\mu^*: 2^X \rightarrow [0, \infty]$ be an outer measure, let $\mathcal{M}$ be the σ -algebra of sets that are measurable with respect to μ^* , and let $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ be the restriction of μ^* to $\mathcal{M}$. By Theorem 8, $\mathcal{M}$ is a σ -algebra and $(X, \mathcal{M}, \bar{\mu})$ is a measure space; what is asked here is the completeness assertion of that theorem, which we now prove rather than quote. Completeness means: if $E \in \mathcal{M}$ has $\bar{\mu}(E) = 0$ and $F \subseteq E$, then $F \in \mathcal{M}$ (and then automatically $\bar{\mu}(F) = 0$).

Step 1: every set of outer measure zero is measurable.

Let $F \subseteq X$ have $\mu^*(F) = 0$. Let A be any subset of X . Since $A \cap F \subseteq F$ and $A \cap F^C \subseteq A$, the

monotonicity of μ^* gives

$$\mu^*(A \cap F) + \mu^*(A \cap F^C) \leq \mu^*(F) + \mu^*(A) = 0 + \mu^*(A) = \mu^*(A).$$

The reverse inequality $\mu^*(A) \leq \mu^*(A \cap F) + \mu^*(A \cap F^C)$ holds for every subset F of X , by the finite monotonicity of μ^* applied to the cover $\{A \cap F, A \cap F^C\}$ of A . Hence

$$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^C) \quad \text{for all } A \subseteq X,$$

which is precisely the Carathéodory criterion. Thus $F \in \mathcal{M}$.

Step 2: completeness.

Let $E \in \mathcal{M}$ with $\bar{\mu}(E) = 0$, and let $F \subseteq E$. Since $\bar{\mu}$ is the restriction of μ^* , we have $\mu^*(E) = \bar{\mu}(E) = 0$, and by the monotonicity of μ^* ,

$$0 \leq \mu^*(F) \leq \mu^*(E) = 0,$$

so $\mu^*(F) = 0$. By Step 1, F is measurable, that is, $F \in \mathcal{M}$, and $\bar{\mu}(F) = \mu^*(F) = 0$.

Therefore every subset of a set of measure zero is measurable and of measure zero: the measure induced by an outer measure is complete.

Problem 17.20. Let X be any set. Define $\eta: 2^X \rightarrow [0, \infty]$ by defining $\eta(\emptyset) = 0$ and, for $E \subseteq X$, $E \neq \emptyset$, defining $\eta(E) = \infty$. Show that η is an outer measure. Also show that the set function that assigns 0 to every subset of X is an outer measure.

Solution. By the definition in this section, a set function $\mu^*: 2^X \rightarrow [0, \infty]$ is an outer measure provided $\mu^*(\emptyset) = 0$ and μ^* is countably monotone, that is, whenever a set $E \subseteq X$ is covered by a countable collection $\{E_k\}_{k=1}^\infty$ of subsets of X , then

$$\mu^*(E) \leq \sum_{k=1}^\infty \mu^*(E_k).$$

So in each case there are exactly two things to check.

The set function η .

First, $\eta(\emptyset) = 0$ by definition.

Second, countable monotonicity. Let $E \subseteq X$ and let $\{E_k\}_{k=1}^\infty$ be a countable collection of subsets of X with $E \subseteq \bigcup_{k=1}^\infty E_k$. There are two cases.

If $E = \emptyset$, then $\eta(E) = 0 \leq \sum_{k=1}^\infty \eta(E_k)$, since each $\eta(E_k) \geq 0$.

If $E \neq \emptyset$, choose $x \in E$. Since $E \subseteq \bigcup_{k=1}^\infty E_k$, there is an index k_0 with $x \in E_{k_0}$; in particular $E_{k_0} \neq \emptyset$, so $\eta(E_{k_0}) = \infty$. Hence

$$\sum_{k=1}^\infty \eta(E_k) \geq \eta(E_{k_0}) = \infty = \eta(E),$$

and the required inequality holds (with equality, both sides being ∞).

In both cases $\eta(E) \leq \sum_{k=1}^\infty \eta(E_k)$, so η is countably monotone and therefore an outer measure.

The zero set function.

Define $\zeta: 2^X \rightarrow [0, \infty]$ by $\zeta(E) = 0$ for every $E \subseteq X$. Then $\zeta(\emptyset) = 0$. And if $E \subseteq \bigcup_{k=1}^\infty E_k$, then

$$\zeta(E) = 0 = \sum_{k=1}^\infty 0 = \sum_{k=1}^\infty \zeta(E_k),$$

so countable monotonicity holds trivially. Hence ζ is an outer measure.

Remark. For both of these outer measures every subset of X is measurable, so the σ -algebra of measurable sets is all of 2^X . For ζ this is immediate, since $0 = 0 + 0$. For η , let $E \subseteq X$ and let $A \subseteq X$. If $A = \emptyset$ then $A \cap E$ and $A \cap E^C$ are both empty and $0 = 0 + 0$. If $A \neq \emptyset$ then at least one of $A \cap E$, $A \cap E^C$ is nonempty, so

$$\eta(A \cap E) + \eta(A \cap E^C) = \infty = \eta(A).$$

Thus E satisfies the Carathéodory criterion. Consequently the measures obtained from η and ζ by Theorem 8, namely their restrictions to $\mathcal{M} = 2^X$, are respectively the measure on 2^X assigning 0 to $\emptyset$ and ∞ to every nonempty set, and the zero measure on 2^X .

Problem 17.21. Let X be a set, $\mathcal{S} = \{\emptyset, X\}$ and define $\mu(\emptyset) = 0$, $\mu(X) = 1$. Determine the outer measure μ^* induced by the set function $\mu: \mathcal{S} \rightarrow [0, \infty)$ and the σ -algebra of measurable sets.

Solution. Note first that $X \neq \emptyset$: if X were empty then $\emptyset$ and X would be the same set, and the two prescriptions $\mu(\emptyset) = 0$ and $\mu(X) = 1$ would be inconsistent. So $\mathcal{S} = \{\emptyset, X\}$ consists of two distinct sets.

Computation of μ^* .

By Theorem 9, the induced outer measure is defined by $\mu^*(\emptyset) = 0$ and, for $E \subseteq X$ with $E \neq \emptyset$,

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k),$$

the infimum being taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E .

Let $E \subseteq X$ with $E \neq \emptyset$, and let $\{E_k\}_{k=1}^{\infty}$ be any countable collection of sets in $\mathcal{S} = \{\emptyset, X\}$ covering E . Since E is nonempty and a union of copies of $\emptyset$ is empty, at least one E_k must equal X . Every term $\mu(E_k)$ is nonnegative, and one term equals $\mu(X) = 1$, so

$$\sum_{k=1}^{\infty} \mu(E_k) \geq 1.$$

Hence $\mu^*(E) \geq 1$. On the other hand the collection $E_1 = X$, $E_k = \emptyset$ for $k \geq 2$, is a countable collection of sets in $\mathcal{S}$ covering E , with $\sum_{k=1}^{\infty} \mu(E_k) = 1 + 0 + 0 + \cdots = 1$. Therefore $\mu^*(E) \leq 1$, and so $\mu^*(E) = 1$. We conclude that

$$\mu^*(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ 1 & \text{if } E \neq \emptyset. \end{cases}$$

In particular μ^* agrees with μ on $\mathcal{S}$, so μ^* is an extension of μ .

The σ -algebra $\mathcal{M}$ of measurable sets.

The sets $\emptyset$ and X are measurable: $\mu^*(\emptyset) = 0$, and every set of outer measure zero is measurable, while X is measurable because its complement $\emptyset$ is.

Now suppose $E \subseteq X$ with $E \neq \emptyset$ and $E \neq X$, so that both E and E^C are nonempty. Test the Carathéodory criterion with $A = X$:

$$\mu^*(X \cap E) + \mu^*(X \cap E^C) = \mu^*(E) + \mu^*(E^C) = 1 + 1 = 2 \neq 1 = \mu^*(X).$$

Hence E is not measurable.

Therefore

$$\mathcal{M} = \{\emptyset, X\} = \mathcal{S},$$

and the induced Carathéodory measure $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ is given by $\bar{\mu}(\emptyset) = 0$ and $\bar{\mu}(X) = 1$; that is, $\bar{\mu} = \mu$.

The argument of the last paragraph is vacuous when X is a singleton, since then there is no E with $E \neq \emptyset$ and $E \neq X$; but in that case $\{\emptyset, X\}$ is already all of 2^X , so the conclusion $\mathcal{M} = \mathcal{S}$ stands in every case. When X has at least two elements, $\mathcal{M} = \mathcal{S}$ is a proper sub- σ -algebra of 2^X . So this is an instance in which the Carathéodory measure induced by μ is an extension of μ in the sense of Section 17.5, every set of $\mathcal{S}$ being μ^* -measurable, yet the extension is as small as it can be: it enlarges neither the domain nor the values of μ .

17.4 Problems 17.22–17.28

Problem 17.22. On the collection $\mathcal{S} = \{\emptyset, [1, 2]\}$ of subsets of $\mathbb{R}$, define the set function $\mu: \mathcal{S} \rightarrow [0, \infty)$ as follows: $\mu(\emptyset) = 0$, $\mu([1, 2]) = 1$. Determine the outer measure μ^* induced by μ and the σ -algebra of measurable sets.

Solution. Recall the construction of Theorem 9: $\mu^*(\emptyset) = 0$ and, for $E \subseteq \mathbb{R}$ nonempty,

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k),$$

the infimum taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E , with the convention that the infimum of the empty set is ∞ .

We first compute the outer measure. The only sets available as covering sets are $\emptyset$ and $[1, 2]$, so the union of any countable subcollection of $\mathcal{S}$ is either $\emptyset$ or $[1, 2]$. Hence a nonempty set E admits a cover by sets in $\mathcal{S}$ if and only if $E \subseteq [1, 2]$, and for such E every cover must contain the set $[1, 2]$ at least once; therefore every cover has $\sum_k \mu(E_k) \geq \mu([1, 2]) = 1$. On the other hand the cover $E_1 = [1, 2]$, $E_k = \emptyset$ for $k \geq 2$, gives $\sum_k \mu(E_k) = 1$. Thus the infimum equals 1. If E is not contained in $[1, 2]$ there is no cover at all, so $\mu^*(E) = \infty$. Altogether

$$\mu^*(E) = \begin{cases} 0, & E = \emptyset, \\ 1, & \emptyset \neq E \subseteq [1, 2], \\ \infty, & E \not\subseteq [1, 2]. \end{cases}$$

In particular μ^* agrees with μ on $\mathcal{S}$, so μ^* is an extension of μ ; this is consistent with Problem 27, since μ is countably monotone.

We now identify the measurable sets. Let $\mathcal{M}$ denote the collection of μ^* -measurable sets, that is, of those $E \subseteq \mathbb{R}$ for which

$$\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^C) \quad \text{for every } B \subseteq \mathbb{R}.$$

First suppose E is measurable and test the criterion with $B = [1, 2]$, for which $\mu^*(B) = 1$. The sets $[1, 2] \cap E$ and $[1, 2] \cap E^C$ are subsets of $[1, 2]$, so each has outer measure 0 or 1, being 0 exactly when the set is empty. Since their outer measures sum to at most 1, at least one of them is empty. Thus either $[1, 2] \cap E^C = \emptyset$, that is $[1, 2] \subseteq E$, or $[1, 2] \cap E = \emptyset$.

Conversely, suppose E satisfies $[1, 2] \subseteq E$ or $[1, 2] \cap E = \emptyset$, and let $B \subseteq \mathbb{R}$. If $\mu^*(B) = \infty$ the measurability inequality is trivial, and if $B = \emptyset$ both sides are 0. So assume $\emptyset \neq B \subseteq [1, 2]$, whence $\mu^*(B) = 1$. If $[1, 2] \subseteq E$ then $B \subseteq E$, so $B \cap E = B$ and $B \cap E^C = \emptyset$, and the right-hand side is $1 + 0 = 1$. If $[1, 2] \cap E = \emptyset$ then $B \cap E = \emptyset$ and $B \cap E^C = B$, and again the right-hand side is $0 + 1 = 1$. In both cases the inequality holds, so E is measurable.

Therefore

$$\mathcal{M} = \{E \subseteq \mathbb{R} : [1, 2] \subseteq E \text{ or } E \cap [1, 2] = \emptyset\},$$

which one checks directly is a σ -algebra, as it must be by Theorem 8: it contains $\emptyset$ and $\mathbb{R}$, it is closed under complements (if $[1, 2] \subseteq E$ then $E^C \cap [1, 2] = \emptyset$, and conversely), and it is closed under countable unions (a union of sets each of which is disjoint from $[1, 2]$ is disjoint from $[1, 2]$, while if some member contains $[1, 2]$ so does the union). The induced Carathéodory measure $\bar{\mu}$ is the restriction of μ^* to $\mathcal{M}$: it assigns 0 to $\emptyset$, the value 1 to the single set $[1, 2]$, and ∞ to every other member of $\mathcal{M}$.

Problem 17.23. On the collection $\mathcal{S}$ of all subsets of $\mathbb{R}$, define the set function $\mu: \mathcal{S} \rightarrow \mathbb{R}$ by setting $\mu(A)$ to be the number of integers in A . Determine the outer measure μ^* induced by μ and the σ -algebra of measurable sets.

Solution. Write $\mu(A) = \#(A \cap \mathbb{Z})$, the number of integers belonging to A , with the value ∞ when A contains infinitely many integers.

The claim is that $\mu^* = \mu$ and that every subset of $\mathbb{R}$ is μ^* -measurable, so that $\mathcal{M} = 2^{\mathbb{R}}$.

Observe first that μ is itself a measure on the σ -algebra $2^{\mathbb{R}}$. Indeed $\mu(\emptyset) = 0$, and if $\{A_k\}_{k=1}^{\infty}$ is a disjoint collection of subsets of $\mathbb{R}$ then the sets $A_k \cap \mathbb{Z}$ are disjoint subsets of $\mathbb{Z}$ whose union is $(\bigcup_k A_k) \cap \mathbb{Z}$; counting elements of $\mathbb{Z}$ over a disjoint countable decomposition gives

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \#\left(\bigcup_{k=1}^{\infty} (A_k \cap \mathbb{Z})\right) = \sum_{k=1}^{\infty} \#(A_k \cap \mathbb{Z}) = \sum_{k=1}^{\infty} \mu(A_k),$$

To justify the middle equality, write $S_k = A_k \cap \mathbb{Z}$ and $S = \bigcup_k S_k$. If every S_k is finite and all but finitely many are empty, the equality is the elementary fact that the cardinality of a finite disjoint union of finite sets is the sum of the cardinalities. Otherwise either some S_{k_0} is infinite, in which case $S \supseteq S_{k_0}$ is infinite and the sum on the right has an infinite term, or infinitely many S_k are nonempty, in which case choosing one point from each of infinitely many of them exhibits infinitely many distinct points of S while the right-hand sum has infinitely many terms of size at least 1; in both cases the two sides equal ∞ . Thus μ is a measure, namely counting measure on $\mathbb{Z}$ regarded as a measure on all of $\mathbb{R}$. Now compute μ^* . Fix $E \subseteq \mathbb{R}$ nonempty. The collection consisting of the single set E (padded with copies of $\emptyset$, which lies in $\mathcal{S}$ and has $\mu(\emptyset) = 0$) is a countable cover of E by sets in $\mathcal{S}$, so

$$\mu^*(E) \leq \mu(E).$$

Conversely, let $\{E_k\}_{k=1}^{\infty}$ be any countable collection of sets in $\mathcal{S}$ covering E . Since μ is a measure defined on all of $2^{\mathbb{R}}$, the countable monotonicity of a measure (Proposition 1 of Chapter 17) yields

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Taking the infimum over all such covers gives $\mu(E) \leq \mu^*(E)$. Hence $\mu^*(E) = \mu(E)$ for every nonempty E , and $\mu^*(\emptyset) = 0 = \mu(\emptyset)$ as well. Therefore

$$\mu^*(E) = \#(E \cap \mathbb{Z}) \quad \text{for every } E \subseteq \mathbb{R}.$$

Finally, every subset of $\mathbb{R}$ is μ^* -measurable. Let $E \subseteq \mathbb{R}$ and let $B \subseteq \mathbb{R}$ be arbitrary. The sets $B \cap E$ and $B \cap E^C$ are disjoint with union B , so the integers in B are partitioned between them, and hence

$$\mu^*(B \cap E) + \mu^*(B \cap E^C) = \#(B \cap E \cap \mathbb{Z}) + \#(B \cap E^C \cap \mathbb{Z}) = \#(B \cap \mathbb{Z}) = \mu^*(B).$$

In particular the Carathéodory splitting inequality holds, so E is measurable. Thus the σ -algebra of μ^* -measurable sets is $\mathcal{M} = 2^{\mathbb{R}}$, all subsets of $\mathbb{R}$, and the induced Carathéodory measure $\bar{\mu}$ is μ itself.

Problem 17.24. Let $\mathcal{S}$ be a collection of subsets of X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a set function. Is every set in $\mathcal{S}$ measurable with respect to the outer measure induced by μ ?

Solution. No. For a general set function there is no reason for the sets of $\mathcal{S}$ to be μ^* -measurable, and the failure can occur already for a two-point set X .

Take $X = \{1, 2\}$, let $\mathcal{S} = \{\{1\}, X\}$, and define

$$\mu(\{1\}) = 2, \quad \mu(X) = 1.$$

We compute the induced outer measure μ^* of Theorem 9. The set $\{1\}$ is covered both by the collection $\{\{1\}\}$, of total μ -value 2, and by the collection $\{X\}$, of total μ -value 1; every cover consists of at least one set of $\mathcal{S}$ and hence has total value at least 1. Therefore $\mu^*(\{1\}) = 1$. Likewise the only covers of $\{2\}$ and of X are those containing X , so $\mu^*(\{2\}) = \mu^*(X) = 1$, while $\mu^*(\emptyset) = 0$.

Now test the Carathéodory criterion for the set $E = \{1\}$, which belongs to $\mathcal{S}$, using the set $B = X$:

$$\mu^*(B \cap E) + \mu^*(B \cap E^C) = \mu^*(\{1\}) + \mu^*(\{2\}) = 1 + 1 = 2 > 1 = \mu^*(X) = \mu^*(B).$$

Thus $\{1\}$ is not μ^* -measurable although it belongs to $\mathcal{S}$. Observe also that here μ^* fails to be an extension of μ , since $\mu^*(\{1\}) = 1 \neq 2 = \mu(\{1\})$; by Problem 27 this reflects the failure of μ to be countably monotone.

The answer remains no even under substantially stronger hypotheses on μ . Indeed, in Problem 25 the set function $\mu: \{A, X\} \rightarrow [0, \infty]$ with $\mu(A) = 1$ and $\mu(X) = 2$ is a premeasure, yet the only μ^* -measurable subsets of X turn out to be $\emptyset$ and X , so the set A of $\mathcal{S}$ is not measurable; Problem 26 provides a second such premeasure. What Theorem 12 tells us is that if μ is a premeasure and, in addition, $\mathcal{S}$ is closed with respect to the formation of relative complements, then every set in $\mathcal{S}$ is indeed μ^* -measurable and $\mu(E) = \bar{\mu}(E)$ for each $E \in \mathcal{S}$. Some such set-theoretic structure on $\mathcal{S}$, as in Theorem 12 or in the Carathéodory-Hahn Theorem for a semiring, is genuinely needed.

Problem 17.25. Let X be any set containing more than one point and A a proper nonempty subset of X . Define $\mathcal{S} = \{A, X\}$ and the set function $\mu: \mathcal{S} \rightarrow [0, \infty]$ by $\mu(A) = 1$ and $\mu(X) = 2$. Show that $\mu: \mathcal{S} \rightarrow [0, \infty]$ is a premeasure. Can μ be extended to a measure? What are the subsets of X that are measurable with respect to the outer measure μ^* induced by μ ?

Solution. Throughout, A is nonempty and $A \neq X$, so $X \sim A$ is nonempty as well; in particular $\emptyset$ does not belong to $\mathcal{S}$.

We check that μ is a premeasure, that is, that μ is finitely additive and countably monotone (the requirement $\mu(\emptyset) = 0$ is vacuous, since $\emptyset \notin \mathcal{S}$).

Finite additivity. We must show that if $\{E_k\}_{k=1}^n$ is a finite disjoint collection of sets in $\mathcal{S}$ whose union also belongs to $\mathcal{S}$, then $\mu(\bigcup_{k=1}^n E_k) = \sum_{k=1}^n \mu(E_k)$. Both members of $\mathcal{S}$ are nonempty and $A \cap X = A \neq \emptyset$, so no disjoint collection of sets in $\mathcal{S}$ can contain two sets; a disjoint collection in $\mathcal{S}$ consists of a single set, either A or X . For such a collection the required identity reads $\mu(A) = \mu(A)$ or $\mu(X) = \mu(X)$, and so holds. Thus μ is finitely additive.

Countable monotonicity. Let $E \in \mathcal{S}$ be covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$. If $E = A$, the cover is nonempty and each $\mu(E_k) \geq 1$, so $\sum_k \mu(E_k) \geq 1 = \mu(A)$. If $E = X$, then some E_k must equal X : otherwise every E_k equals A and the union of the cover is $A \neq X$, contradicting that the collection covers X . Hence $\sum_k \mu(E_k) \geq \mu(X) = 2$. Thus μ is countably monotone, and μ is a premeasure.

Extension to a measure. Yes. Let $\mathcal{A} = \{\emptyset, A, X \sim A, X\}$, the smallest σ -algebra of subsets of X containing A , and define

$$\nu(\emptyset) = 0, \quad \nu(A) = 1, \quad \nu(X \sim A) = 1, \quad \nu(X) = 2.$$

Then ν extends μ . To see that ν is a measure, note $\nu(\emptyset) = 0$ and let $\{E_k\}_{k=1}^\infty$ be a disjoint collection of sets in $\mathcal{A}$. Discarding the copies of $\emptyset$, which contribute nothing to either side, the remaining sets are distinct nonempty pairwise disjoint members of $\mathcal{A}$; since $A, X \sim A$ and X pairwise intersect except for the pair $A, X \sim A$, the surviving collection is one of $\{A\}$, $\{X \sim A\}$, $\{X\}$, $\{A, X \sim A\}$, or is empty. In each case countable additivity is immediate, the only nontrivial instance being $\nu(A) + \nu(X \sim A) = 1 + 1 = 2 = \nu(X)$. So ν is a measure on $(X, \mathcal{A})$ extending μ .

The measurable sets. First we compute μ^* . Since μ is countably monotone, Problem 27 already tells us that μ^* extends μ ; here is the full computation. Every subset of X is covered by the collection $\{X\}$, of total value 2, so $\mu^*(E) \leq 2$ for all E . If $\emptyset \neq E \subseteq A$, then E is covered by $\{A\}$, so $\mu^*(E) \leq 1$; and every cover of a nonempty set contains at least one set of $\mathcal{S}$, hence has total value at least 1. Therefore $\mu^*(E) = 1$. If $E \not\subseteq A$, then E contains a point of $X \sim A$, so no union of copies of A can cover E ; every cover of E must contain X and hence has total value at least 2. Therefore $\mu^*(E) = 2$. Altogether

$$\mu^*(E) = \begin{cases} 0, & E = \emptyset, \\ 1, & \emptyset \neq E \subseteq A, \\ 2, & E \not\subseteq A. \end{cases}$$

In particular $\mu^*(A) = 1 = \mu(A)$ and $\mu^*(X) = 2 = \mu(X)$.

Now let $E \subseteq X$ be μ^* -measurable and apply the Carathéodory criterion with $B = X$:

$$2 = \mu^*(X) \geq \mu^*(E) + \mu^*(E^C).$$

If both E and E^C were nonempty, then each of $\mu^*(E)$, $\mu^*(E^C)$ would be at least 1, so both would have to equal 1, forcing $E \subseteq A$ and $E^C \subseteq A$ and hence $X = E \cup E^C \subseteq A$, contradicting that A is a proper subset of X . Therefore $E = \emptyset$ or $E = X$. Conversely $\emptyset$ and X are measurable, since for every $B \subseteq X$ the splitting they induce is $\mu^*(B \cap \emptyset) + \mu^*(B) = 0 + \mu^*(B)$. Hence

$$\mathcal{M} = \{\emptyset, X\},$$

and the induced Carathéodory measure is $\bar{\mu}(\emptyset) = 0$, $\bar{\mu}(X) = 2$.

In particular the set A , which belongs to $\mathcal{S}$, is not μ^* -measurable, so the Carathéodory measure induced by μ is not an extension of μ , even though μ is a premeasure and even though μ does possess a measure extension. This is exactly the phenomenon noted after Theorem 12: being a premeasure is necessary but not sufficient for the induced Carathéodory measure to extend μ . Of course $\mathcal{S} = \{A, X\}$ is not closed with respect to the formation of relative complements, since $X \sim A \notin \mathcal{S}$, so Theorem 12 does not apply.

Problem 17.26. Consider the collection $\mathcal{S} = \{\emptyset, [0, 1], [0, 3], [2, 3]\}$ of subsets of $\mathbb{R}$ and define $\mu(\emptyset) = 0$, $\mu([0, 1]) = 1$, $\mu([0, 3]) = 1$, $\mu([2, 3]) = 1$. Show that $\mu: \mathcal{S} \rightarrow [0, \infty]$ is a premeasure. Can μ be extended to a measure? What are the subsets of $\mathbb{R}$ that are measurable with respect to the outer measure μ^* induced by μ ?

Solution. We verify the two defining properties of a premeasure, together with $\mu(\emptyset) = 0$, which holds by definition.

Finite additivity. Let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of sets in $\mathcal{S}$ whose union again belongs to $\mathcal{S}$. Copies of $\emptyset$ may be discarded, as they change neither the union nor the sum $\sum_k \mu(E_k)$. Among the nonempty members of $\mathcal{S}$ we have

$$[0, 1] \cap [0, 3] = [0, 1] \neq \emptyset, \quad [2, 3] \cap [0, 3] = [2, 3] \neq \emptyset, \quad [0, 1] \cap [2, 3] = \emptyset,$$

so the only disjoint collection containing two distinct nonempty members of $\mathcal{S}$ is $\{[0, 1], [2, 3]\}$; but its union $[0, 1] \cup [2, 3]$ does not belong to $\mathcal{S}$, so this collection is not subject to the additivity requirement. Hence after discarding empty sets we are left with at most one set E , and the requirement reads $\mu(E) = \mu(E)$. Thus μ is finitely additive.

Countable monotonicity. Let $E \in \mathcal{S}$ be covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$. If $E = \emptyset$ then $\mu(E) = 0 \leq \sum_k \mu(E_k)$. If $E \neq \emptyset$, the cover must contain at least one nonempty set E_{k_0} , and every nonempty member of $\mathcal{S}$ has μ -value 1, so

$$\sum_{k=1}^\infty \mu(E_k) \geq \mu(E_{k_0}) = 1 = \mu(E).$$

Thus μ is countably monotone, and μ is a premeasure.

Extension to a measure. No. Suppose $\nu: \mathcal{M}_0 \rightarrow [0, \infty]$ were a measure on a σ -algebra $\mathcal{M}_0$ of subsets of $\mathbb{R}$ containing $\mathcal{S}$ and agreeing with μ on $\mathcal{S}$. The sets $[0, 1]$ and $[2, 3]$ are disjoint measurable subsets of $[0, 3]$, so by the finite additivity and monotonicity of a measure (Proposition 1 of Chapter 17),

$$1 = \mu([0, 3]) = \nu([0, 3]) \geq \nu([0, 1]) + \nu([2, 3]) = \mu([0, 1]) + \mu([2, 3]) = 1 + 1 = 2,$$

a contradiction. Hence no measure extends μ . Together with Problem 28, which shows that any set function possessing a measure extension is a premeasure, this exhibits a premeasure that is not the restriction of any measure, so the implication of Problem 28 cannot be reversed.

The outer measure. The union of all the sets in $\mathcal{S}$ is $[0, 3]$, so a nonempty set E admits a cover by sets in $\mathcal{S}$ if and only if $E \subseteq [0, 3]$; for $E \not\subseteq [0, 3]$ we get $\mu^*(E) = \infty$. If $\emptyset \neq E \subseteq [0, 3]$, then the single

set $[0, 3]$ covers E at total cost $\mu([0, 3]) = 1$, while every cover of a nonempty set contains a nonempty member of $\mathcal{S}$ and hence has total cost at least 1. Therefore

$$\mu^*(E) = \begin{cases} 0, & E = \emptyset, \\ 1, & \emptyset \neq E \subseteq [0, 3], \\ \infty, & E \not\subseteq [0, 3]. \end{cases}$$

Note μ^* agrees with μ on $\mathcal{S}$, as it must by Problem 27, since μ is countably monotone.

The measurable sets. Suppose $E \subseteq \mathbb{R}$ is μ^* -measurable and take $B = [0, 3]$, so $\mu^*(B) = 1$. The sets $[0, 3] \cap E$ and $[0, 3] \cap E^C$ are subsets of $[0, 3]$, so each has outer measure 0 or 1, the value 0 occurring exactly for the empty set. Since their sum is at most 1, one of them is empty, that is, $[0, 3] \subseteq E$ or $E \cap [0, 3] = \emptyset$.

Conversely, suppose E satisfies $[0, 3] \subseteq E$ or $E \cap [0, 3] = \emptyset$, and let $B \subseteq \mathbb{R}$. If $\mu^*(B) = \infty$, or if $B = \emptyset$, the measurability inequality is clear. Otherwise $\emptyset \neq B \subseteq [0, 3]$ and $\mu^*(B) = 1$; in the first case $B \cap E = B$ and $B \cap E^C = \emptyset$, in the second case $B \cap E = \emptyset$ and $B \cap E^C = B$, and in either case

$$\mu^*(B \cap E) + \mu^*(B \cap E^C) = 1 = \mu^*(B).$$

Hence

$$\mathcal{M} = \{E \subseteq \mathbb{R} : [0, 3] \subseteq E \text{ or } E \cap [0, 3] = \emptyset\},$$

which is a σ -algebra, in accordance with Theorem 8. The Carathéodory measure $\bar{\mu}$ assigns 0 to $\emptyset$, the value 1 to $[0, 3]$ itself, and ∞ to every other member of $\mathcal{M}$.

Observe that $[0, 1]$ and $[2, 3]$ belong to $\mathcal{S}$ but not to $\mathcal{M}$, so once again the Carathéodory measure induced by a premeasure need not extend it. This is unavoidable here: by the argument above no measure whatsoever extends μ .

Problem 17.27. Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a set function. Show that μ is countably monotone if and only if μ^* is an extension of μ .

Solution. Recall the two notions involved. The set function μ is countably monotone provided that whenever a set $E \in \mathcal{S}$ is covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$, then

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

And μ^* , the outer measure induced by μ as in Theorem 9, is defined by $\mu^*(\emptyset) = 0$ and, for nonempty $E \subseteq X$, by

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k),$$

the infimum taken over all countable collections $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ that cover E , and equal to ∞ if there is no such collection. To say that μ^* is an extension of μ means that $\mu^*(E) = \mu(E)$ for every $E \in \mathcal{S}$.

Two conventions must be made explicit, since the statement is sensitive to both. First, throughout Chapter 17 a countable collection is understood to be finite or countably infinite, so that a finite cover is admissible in the infimum (6) and in the definition of countable monotonicity. This is not a harmless reading that could be avoided by padding: repeating one of the covering sets inflates the sum to ∞ unless that set has μ -value 0, and adjoining copies of $\emptyset$ is available only when $\emptyset \in \mathcal{S}$ and $\mu(\emptyset) = 0$. That finite covers are genuinely intended is clear from Problems 25 and 26, whose stated answers require it. Second, if $\emptyset$ belongs to $\mathcal{S}$ we assume $\mu(\emptyset) = 0$. Without this the assertion can fail in a trivial way: on $\mathcal{S} = \{\emptyset\}$ the set function $\mu(\emptyset) = 5$ satisfies countable monotonicity as stated, since a covering collection $\{E_k\}$ is indexed by a nonempty index set and every such collection has sum 5, 10, ... or ∞ , in each case at least $5 = \mu(\emptyset)$; yet $\mu^*(\emptyset) = 0$ is imposed by definition, so μ^* is not an extension of μ . (If one instead admits the empty covering collection, whose sum is 0, then countable monotonicity itself forces

$\mu(\emptyset) = 0$ and no separate convention is needed; the proof below is unaffected either way.) With these conventions the equivalence holds, and we prove it.

Suppose first that μ is countably monotone; we show $\mu^*(E) = \mu(E)$ for each $E \in \mathcal{S}$. If $E = \emptyset$ then $\mu^*(E) = 0 = \mu(E)$ by the convention just stated. So let $E \in \mathcal{S}$ be nonempty. The one-member collection consisting of E itself is a countable collection of sets in $\mathcal{S}$ that covers E , so by the definition of μ^* as an infimum,

$$\mu^*(E) \leq \mu(E).$$

For the reverse inequality, let $\{E_k\}_{k=1}^\infty$ be any countable collection of sets in $\mathcal{S}$ covering E . Countable monotonicity gives $\mu(E) \leq \sum_{k=1}^\infty \mu(E_k)$. As $\{E_k\}$ was an arbitrary such cover, $\mu(E)$ is a lower bound for the set of numbers whose infimum defines $\mu^*(E)$, and that set is nonempty because E covers itself; hence $\mu(E) \leq \mu^*(E)$. Combining the two inequalities, $\mu^*(E) = \mu(E)$, and μ^* is an extension of μ .

Conversely, suppose μ^* is an extension of μ , and let $E \in \mathcal{S}$ be covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$. If $E = \emptyset$ then $\mu(E) = \mu^*(E) = 0 \leq \sum_k \mu(E_k)$, since each $\mu(E_k) \geq 0$. If $E \neq \emptyset$, then $\{E_k\}_{k=1}^\infty$ is one of the covers competing in the infimum defining $\mu^*(E)$, so

$$\mu(E) = \mu^*(E) \leq \sum_{k=1}^\infty \mu(E_k).$$

In either case the defining inequality for countable monotonicity holds, so μ is countably monotone.

This is precisely the equivalence invoked, without proof, in the last paragraph of the proof of Proposition 11, where it is observed that $\mu(E) = \mu^*(E)$ for all $E \in \mathcal{S}$ if and only if μ is countably monotone; the present problem supplies that verification. It also clarifies the division of labour in the definition of a premeasure: countable monotonicity of μ is exactly what makes μ^* agree with μ on $\mathcal{S}$, while it is finite additivity of μ , together with the closure of $\mathcal{S}$ under the formation of relative complements, that is used in the proof of Theorem 12 to make the sets of $\mathcal{S}$ themselves μ^* -measurable. Problems 25 and 26 show that countable monotonicity alone does not deliver the latter: there μ^* does extend μ , yet the sets of $\mathcal{S}$ fail to be measurable.

Problem 17.28. Show that a set function is a premeasure if it has an extension that is a measure.

Solution. Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a set function. Assume there is a measure space $(X, \mathcal{M}, \nu)$ with $\mathcal{S} \subseteq \mathcal{M}$ and $\nu(E) = \mu(E)$ for every $E \in \mathcal{S}$. We must verify the three requirements in the definition of a premeasure: that $\mu(\emptyset) = 0$ if $\emptyset \in \mathcal{S}$, that μ is finitely additive, and that μ is countably monotone.

The empty set. If $\emptyset$ belongs to $\mathcal{S}$, then $\emptyset$ belongs to $\mathcal{M}$ and, since ν is a measure, $\mu(\emptyset) = \nu(\emptyset) = 0$.

Finite additivity. Let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of sets in $\mathcal{S}$ for which $E := \bigcup_{k=1}^n E_k$ also belongs to $\mathcal{S}$. All of these sets belong to $\mathcal{M}$. A measure is finitely additive; indeed, this follows from countable additivity by taking $E_k = \emptyset$ for $k > n$, since $\nu(\emptyset) = 0$, and is recorded in Proposition 1 of Chapter 17. Therefore

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \nu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \nu(E_k) = \sum_{k=1}^n \mu(E_k),$$

where the outer equalities hold because ν extends μ and all the sets in question lie in $\mathcal{S}$. Thus μ is finitely additive.

Countable monotonicity. Let $E \in \mathcal{S}$ be covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$, so that $E \subseteq \bigcup_{k=1}^\infty E_k$. Every set here belongs to $\mathcal{M}$, and $\mathcal{M}$ is a σ -algebra, so $\bigcup_{k=1}^\infty E_k$ is measurable. By the countable monotonicity of the measure ν , which is part of Proposition 1 of Chapter 17,

$$\mu(E) = \nu(E) \leq \sum_{k=1}^\infty \nu(E_k) = \sum_{k=1}^\infty \mu(E_k).$$

Thus μ is countably monotone.

Since μ is finitely additive, countably monotone, and vanishes on $\emptyset$ whenever $\emptyset \in \mathcal{S}$, the set function μ is a premeasure.

Two remarks. First, this recovers the necessity half of Proposition 11 in a stronger form: there it is assumed that the particular measure extending μ is the Carathéodory measure $\bar{\mu}$ induced by μ , whereas here the extending measure may be any measure on any σ -algebra containing $\mathcal{S}$. Second, the converse is false: by Problem 26 the set function μ defined on $\mathcal{S} = \{\emptyset, [0, 1], [0, 3], [2, 3]\}$ by $\mu(\emptyset) = 0$ and $\mu([0, 1]) = \mu([0, 3]) = \mu([2, 3]) = 1$ is a premeasure, yet it admits no extension to a measure, since $[0, 1]$ and $[2, 3]$ are disjoint subsets of $[0, 3]$ whose values already sum to more than $\mu([0, 3])$. So being a premeasure is strictly weaker than being the restriction of a measure; by the Carathéodory-Hahn Theorem the two notions do coincide when $\mathcal{S}$ is a semiring.

17.5 Problems 17.29–17.35

Problem 17.29. Show that a set function on a σ -algebra is a measure if and only if it is a premeasure.

Solution. Let $\mathcal{S}$ be a σ -algebra of subsets of a set X and let $\mu: \mathcal{S} \rightarrow [0, \infty]$ be a set function. Recall that μ is a premeasure provided it is finitely additive, countably monotone, and $\mu(\emptyset) = 0$ (the last condition being required because $\emptyset$ belongs to every σ -algebra).

Suppose first that μ is a measure, that is, $\mu(\emptyset) = 0$ and μ is countably additive.

Finite additivity. Let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of sets in $\mathcal{S}$. Define $E_k = \emptyset$ for $k > n$. Then $\{E_k\}_{k=1}^\infty$ is a countable disjoint collection of sets in $\mathcal{S}$ with the same union, so by countable additivity and $\mu(\emptyset) = 0$,

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^n \mu(E_k).$$

In particular μ is monotone: if $A \subseteq B$ are in $\mathcal{S}$, then $B = A \cup (B \sim A)$ is a disjoint union of sets in $\mathcal{S}$, so $\mu(B) = \mu(A) + \mu(B \sim A) \geq \mu(A)$.

Countable monotonicity. Let E in $\mathcal{S}$ be covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$. Disjointify: set $A_1 = E_1$ and, for $k \geq 2$,

$$A_k = E_k \sim \bigcup_{j=1}^{k-1} E_j.$$

Since $\mathcal{S}$ is a σ -algebra, each A_k belongs to $\mathcal{S}$; the collection $\{A_k\}_{k=1}^\infty$ is disjoint, $A_k \subseteq E_k$, and $\bigcup_{k=1}^\infty A_k = \bigcup_{k=1}^\infty E_k \supseteq E$. Hence $\{E \cap A_k\}_{k=1}^\infty$ is a disjoint collection of sets in $\mathcal{S}$ whose union is E , and therefore, by countable additivity and monotonicity,

$$\mu(E) = \sum_{k=1}^{\infty} \mu(E \cap A_k) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Thus μ is countably monotone. Together with $\mu(\emptyset) = 0$ and finite additivity, μ is a premeasure.

Conversely, suppose μ is a premeasure on the σ -algebra $\mathcal{S}$. Then $\mu(\emptyset) = 0$, so it remains only to prove countable additivity. Let $\{E_k\}_{k=1}^\infty$ be a disjoint collection of sets in $\mathcal{S}$ and put $E = \bigcup_{k=1}^\infty E_k$, which belongs to $\mathcal{S}$.

Since $\{E_k\}_{k=1}^\infty$ is a countable collection of sets in $\mathcal{S}$ that covers E , countable monotonicity gives

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

For the reverse inequality, fix a natural number n . The collection

$$E_1, E_2, \dots, E_n, E \sim \bigcup_{k=1}^n E_k$$

is a finite disjoint collection of sets in $\mathcal{S}$ whose union is E , again a set in $\mathcal{S}$. Hence, by the finite additivity of μ ,

$$\mu(E) = \sum_{k=1}^n \mu(E_k) + \mu\left(E \sim \bigcup_{k=1}^n E_k\right) \geq \sum_{k=1}^n \mu(E_k).$$

The left-hand side does not depend on n , so letting $n \rightarrow \infty$ yields $\mu(E) \geq \sum_{k=1}^{\infty} \mu(E_k)$. Therefore

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k),$$

and μ is countably additive, hence a measure.

Problem 17.30. Let $\mathcal{S}$ be a collection of sets that is closed with respect to the formation of finite unions and finite intersections.

- (i) Show that $\mathcal{S}_\sigma$ is closed with respect to the formation of countable unions and finite intersections.
- (ii) Show that each set in $\mathcal{S}_{\sigma\delta}$ is the intersection of a decreasing sequence of $\mathcal{S}_\sigma$ sets.

Solution. Throughout, $\mathcal{S}_\sigma$ denotes the collection of countable unions of sets in $\mathcal{S}$ and $\mathcal{S}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{S}_\sigma$. Observe first that $\mathcal{S} \subseteq \mathcal{S}_\sigma$, since a set E in $\mathcal{S}$ is the countable union of the collection all of whose members equal E .

Proof of (i). Closure under countable unions. Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets in $\mathcal{S}_\sigma$. For each n choose a countable collection $\{E_{n,k}\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ with $A_n = \bigcup_{k=1}^{\infty} E_{n,k}$. Then

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{n,k} = \bigcup_{(n,k) \in \mathbb{N} \times \mathbb{N}} E_{n,k}.$$

Since $\mathbb{N} \times \mathbb{N}$ is countable, this exhibits $\bigcup_{n=1}^{\infty} A_n$ as a countable union of sets in $\mathcal{S}$, that is, as a set in $\mathcal{S}_\sigma$.

Closure under finite intersections. By induction it suffices to treat two sets. Let $A = \bigcup_{i=1}^{\infty} E_i$ and $B = \bigcup_{j=1}^{\infty} F_j$ with all E_i, F_j in $\mathcal{S}$. By the distributive law,

$$A \cap B = \bigcup_{(i,j) \in \mathbb{N} \times \mathbb{N}} (E_i \cap F_j).$$

Since $\mathcal{S}$ is closed with respect to the formation of finite intersections, each $E_i \cap F_j$ belongs to $\mathcal{S}$, and the index set $\mathbb{N} \times \mathbb{N}$ is countable. Hence $A \cap B$ belongs to $\mathcal{S}_\sigma$. Consequently, for any finite collection $A_1, \dots, A_n$ of sets in $\mathcal{S}_\sigma$, the intersection $\bigcap_{k=1}^n A_k$ belongs to $\mathcal{S}_\sigma$.

Proof of (ii). Let A belong to $\mathcal{S}_{\sigma\delta}$. By definition there is a countable collection $\{B_n\}_{n=1}^{\infty}$ of sets in $\mathcal{S}_\sigma$ with

$$A = \bigcap_{n=1}^{\infty} B_n.$$

For each natural number n define

$$A_n = \bigcap_{k=1}^n B_k.$$

By part (i), $\mathcal{S}_\sigma$ is closed with respect to the formation of finite intersections, so each A_n belongs to $\mathcal{S}_\sigma$. Clearly $A_{n+1} = A_n \cap B_{n+1} \subseteq A_n$, so $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathcal{S}_\sigma$. Finally,

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^n B_k = \bigcap_{k=1}^{\infty} B_k = A,$$

since a point belongs to every A_n if and only if it belongs to every B_k . Thus A is the intersection of a decreasing sequence of $\mathcal{S}_\sigma$ sets.

Problem 17.31. Let $\mathcal{S}$ be a semialgebra of subsets of a set X and $\mathcal{S}'$ the collection of unions of finite disjoint collections of sets in $\mathcal{S}$.

- (i) Show that $\mathcal{S}'$ is an algebra.
- (ii) Show that $\mathcal{S}_\sigma = \mathcal{S}'_\sigma$ and therefore $\mathcal{S}_{\sigma\delta} = \mathcal{S}'_{\sigma\delta}$.
- (iii) Let $\{E'_k\}_{k=1}^\infty$ be a collection of sets in $\mathcal{S}'$. Show that we can express $\bigcup_{k=1}^\infty E'_k$ as the disjoint union $\bigcup_{k=1}^\infty E_k$ of sets in $\mathcal{S}$ for which

$$\sum_{k=1}^\infty \mu'(E'_k) \geq \sum_{k=1}^\infty \mu(E_k).$$

- (iv) Let A belong to $\mathcal{S}'_{\sigma\delta}$. Show that A is the intersection of a descending sequence $\{A_k\}_{k=1}^\infty$ of sets in $\mathcal{S}_\sigma$.

Solution. Recall the definitions in force. A semiring is a nonempty collection $\mathcal{S}$ of subsets of X such that whenever A and B belong to $\mathcal{S}$, so does $A \cap B$, and $A \sim B$ is the union of a finite disjoint collection of sets in $\mathcal{S}$; a semiring that contains X is a semialgebra. A ring is a collection closed with respect to the formation of finite unions and relative complements; a ring that contains X is an algebra. Finally, if $\mu: \mathcal{S} \rightarrow [0, \infty]$ is a premeasure, $\mu': \mathcal{S}' \rightarrow [0, \infty]$ is the set function of Proposition 13, defined by $\mu'(E) = \sum_{k=1}^n \mu(A_k)$ whenever $E = \bigcup_{k=1}^n A_k$ with $\{A_k\}_{k=1}^n$ a finite disjoint collection of sets in $\mathcal{S}$; the proof of Proposition 13 shows that this prescription is unambiguous.

We adopt the convention that the union of the empty collection is $\emptyset$, so $\emptyset$ belongs to $\mathcal{S}'$ and $\mu'(\emptyset) = 0$. Proof of (i). Note first that $\mathcal{S} \subseteq \mathcal{S}'$, since a single set of $\mathcal{S}$ is a disjoint collection with one member. In particular X belongs to $\mathcal{S}'$.

We record an elementary observation used repeatedly: if $F_1, \dots, F_n$ are pairwise disjoint sets belonging to $\mathcal{S}'$, then $\bigcup_{k=1}^n F_k$ belongs to $\mathcal{S}'$. Indeed, write each F_k as the union of a finite disjoint collection $\{C_{k,i}\}_{i=1}^{m_k}$ of sets in $\mathcal{S}$; since the F_k are pairwise disjoint, the whole collection $\{C_{k,i}\}$ is a finite disjoint collection of sets in $\mathcal{S}$ whose union is $\bigcup_{k=1}^n F_k$.

Closure of $\mathcal{S}'$ under finite intersections. Let $E = \bigcup_{k=1}^n A_k$ and $F = \bigcup_{j=1}^m B_j$, where $\{A_k\}$ and $\{B_j\}$ are finite disjoint collections of sets in $\mathcal{S}$. Then

$$E \cap F = \bigcup_{1 \leq k \leq n, 1 \leq j \leq m} (A_k \cap B_j).$$

Each $A_k \cap B_j$ belongs to $\mathcal{S}$, and the sets $A_k \cap B_j$ are pairwise disjoint because the A_k are pairwise disjoint and the B_j are pairwise disjoint. Hence $E \cap F$ belongs to $\mathcal{S}'$.

Closure of $\mathcal{S}'$ under relative complements. With E and F as above, the set identity (11) of Proposition 13 gives

$$E \sim F = \left[\bigcup_{k=1}^n A_k \right] \sim \left[\bigcup_{j=1}^m B_j \right] = \bigcup_{k=1}^n \left[\bigcap_{j=1}^m (A_k \sim B_j) \right].$$

Since $\mathcal{S}$ is a semiring, each $A_k \sim B_j$ belongs to $\mathcal{S}'$, and by the closure of $\mathcal{S}'$ under finite intersections just proved, $D_k := \bigcap_{j=1}^m (A_k \sim B_j)$ belongs to $\mathcal{S}'$ for each k . Moreover $D_k \subseteq A_k$, so the sets $D_1, \dots, D_n$ are pairwise disjoint. By the observation above, $E \sim F = \bigcup_{k=1}^n D_k$ belongs to $\mathcal{S}'$.

Closure of $\mathcal{S}'$ under finite unions. Again it suffices to treat two sets. If E and F belong to $\mathcal{S}'$, then $E \cup F = (E \sim F) \cup F$ is the union of two disjoint sets of $\mathcal{S}'$ and hence belongs to $\mathcal{S}'$.

Therefore $\mathcal{S}'$ is closed with respect to the formation of finite unions and relative complements, that is, $\mathcal{S}'$ is a ring; and X belongs to $\mathcal{S}'$. Thus $\mathcal{S}'$ is an algebra. (Equivalently: $\mathcal{S}'$ contains X , is closed under finite unions, and is closed under complementation, since $E^c = X \sim E$.)

Proof of (ii). Since $\mathcal{S} \subseteq \mathcal{S}'$, every countable union of sets in $\mathcal{S}$ is a countable union of sets in $\mathcal{S}'$; hence $\mathcal{S}_\sigma \subseteq \mathcal{S}'_\sigma$.

Conversely, let A belong to $\mathcal{S}'_\sigma$, say $A = \bigcup_{n=1}^\infty E'_n$ with each E'_n in $\mathcal{S}'$. Each E'_n is a finite union $\bigcup_{i=1}^{m_n} C_{n,i}$ of sets in $\mathcal{S}$. Therefore

$$A = \bigcup_{n=1}^\infty \bigcup_{i=1}^{m_n} C_{n,i}$$

is a countable union of sets in $\mathcal{S}$, so A belongs to $\mathcal{S}_\sigma$. Hence $\mathcal{S}_\sigma = \mathcal{S}'_\sigma$.

Taking countable intersections of members of these equal collections gives

$$\mathcal{S}_{\sigma\delta} = (\mathcal{S}_\sigma)_\delta = (\mathcal{S}'_\sigma)_\delta = \mathcal{S}'_{\sigma\delta}.$$

Proof of (iii). Let $\{E'_k\}_{k=1}^\infty$ be a collection of sets in $\mathcal{S}'$. Disjointify: set $F_1 = E'_1$ and, for $n \geq 2$,

$$F_n = E'_n \sim \bigcup_{k=1}^{n-1} E'_k.$$

By part (i), $\mathcal{S}'$ is an algebra, so each F_n belongs to $\mathcal{S}'$. The collection $\{F_n\}_{n=1}^\infty$ is pairwise disjoint, $F_n \subseteq E'_n$, and

$$\bigcup_{n=1}^\infty F_n = \bigcup_{n=1}^\infty E'_n.$$

For each n write $F_n = \bigcup_{i=1}^{m_n} C_{n,i}$, where $\{C_{n,i}\}_{i=1}^{m_n}$ is a finite disjoint collection of sets in $\mathcal{S}$. Since the F_n are pairwise disjoint, the countable collection $\{C_{n,i} : n \geq 1, 1 \leq i \leq m_n\}$ is pairwise disjoint, consists of sets in $\mathcal{S}$, and has union $\bigcup_{n=1}^\infty E'_n$. Enumerate it as $\{E_k\}_k$; this is the required disjoint decomposition. (Should this collection turn out to be finite, it may be padded out to a sequence by copies of $\emptyset$: taking $A = B$ in the definition of a semiring exhibits $\emptyset = A \sim A$ as the union of a finite disjoint collection of sets in $\mathcal{S}$, and each member of that collection, being a subset of $\emptyset$, is $\emptyset$ itself; so $\emptyset$ belongs to $\mathcal{S}$, and $\mu(\emptyset) = 0$ since μ is a premeasure. The padding therefore changes neither the union nor either of the sums below.)

It remains to establish the inequality. Fix n . Both F_n and $E'_n \sim F_n$ belong to the algebra $\mathcal{S}'$, they are disjoint, and their union is E'_n . Writing F_n and $E'_n \sim F_n$ as unions of finite disjoint collections of sets in $\mathcal{S}$ and combining the two collections, we obtain a representation of E'_n as a finite disjoint union of sets in $\mathcal{S}$; since μ' is unambiguously defined, evaluating $\mu'(E'_n)$ on this representation gives

$$\mu'(E'_n) = \mu'(F_n) + \mu'(E'_n \sim F_n) \geq \mu'(F_n),$$

the set function μ' being nonnegative. On the other hand, by the very definition of μ' applied to the representation $F_n = \bigcup_{i=1}^{m_n} C_{n,i}$,

$$\mu'(F_n) = \sum_{i=1}^{m_n} \mu(C_{n,i}).$$

All the terms involved are nonnegative, so the double series may be summed in any order. Therefore

$$\sum_k \mu(E_k) = \sum_{n=1}^\infty \sum_{i=1}^{m_n} \mu(C_{n,i}) = \sum_{n=1}^\infty \mu'(F_n) \leq \sum_{n=1}^\infty \mu'(E'_n),$$

which is the asserted inequality. Observe that this argument uses only that μ' is properly defined and nonnegative, and hence does not presuppose the countable monotonicity of μ' , which is what Proposition 13 deduces from this result.

Proof of (iv). By part (i), $\mathcal{S}'$ is an algebra and is therefore closed with respect to the formation of finite unions and finite intersections. Hence Problem 30 applies to $\mathcal{S}'$: by part (ii) of that problem, each set in $\mathcal{S}'_{\sigma\delta}$ is the intersection of a decreasing sequence of sets in $\mathcal{S}'_\sigma$.

So let A belong to $\mathcal{S}'_{\sigma\delta}$ and choose a descending sequence $\{A_k\}_{k=1}^\infty$ of sets in $\mathcal{S}'_\sigma$ with $A = \bigcap_{k=1}^\infty A_k$. By part (ii), $\mathcal{S}'_\sigma = \mathcal{S}_\sigma$, so each A_k in fact belongs to $\mathcal{S}_\sigma$. Thus A is the intersection of a descending sequence of sets in $\mathcal{S}_\sigma$.

Problem 17.32. Let $\mathbb{Q}$ be the set of rational numbers and $\mathcal{S}$ the collection of all finite unions of intervals of the form $(a, b] \cap \mathbb{Q}$, where $a, b \in \mathbb{Q}$ and $a \leq b$. Define $\mu(a, b] = \infty$ if $a < b$ and $\mu(\emptyset) = 0$. Show that $\mathcal{S}$ is closed with respect to the formation of relative complements and $\mu: \mathcal{S} \rightarrow [0, \infty]$ is a

premeasure. Then show that the extension of μ to the smallest σ -algebra containing $\mathcal{S}$ is not unique.

Solution. For rational numbers $a \leq b$ write $I(a, b) = (a, b] \cap \mathbb{Q}$. Note that $I(a, a) = \emptyset$, so $\emptyset$ belongs to $\mathcal{S}$, and that $I(a, b)$ is infinite whenever $a < b$, since the rationals are dense in $(a, b]$. Consequently every nonempty set in $\mathcal{S}$ is an infinite set. The set function of the problem assigns the value ∞ to each nonempty generating interval and 0 to $\emptyset$; accordingly $\mu: \mathcal{S} \rightarrow [0, \infty]$ is given by

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ \infty & \text{if } E \neq \emptyset. \end{cases}$$

This reading is forced on us if μ is to be finitely additive. Indeed, every set in $\mathcal{S}$ is a finite disjoint union of generating intervals: given $E = \bigcup_{k=1}^n I(a_k, b_k)$, list the endpoints $a_1, b_1, \dots, a_n, b_n$ in strictly increasing order as $t_0 < t_1 < \dots < t_m$ (all rational), and call the sets $I(t_{j-1}, t_j)$, for $1 \leq j \leq m$, the atoms. Each generating interval $I(a_k, b_k)$ is the union of the atoms it contains, since a_k and b_k are among the t_j and $(a_k, b_k] = \bigcup \{(t_{j-1}, t_j] : a_k \leq t_{j-1} < t_j \leq b_k\}$. The atoms are pairwise disjoint, so E is the disjoint union of those atoms it contains. If E is nonempty at least one such atom $I(t_{j-1}, t_j)$ is nonempty and so has μ -value ∞ ; finite additivity therefore forces $\mu(E) = \infty$.

Step 1: $\mathcal{S}$ is closed with respect to the formation of relative complements.

First we check the two basic computations for the generating intervals.

Intersections. For rationals $a \leq b$ and $c \leq d$,

$$I(a, b) \cap I(c, d) = (\max\{a, c\}, \min\{b, d\}] \cap \mathbb{Q},$$

which is $I(\alpha, \beta)$ with $\alpha = \max\{a, c\}$ and $\beta = \max\{\alpha, \min\{b, d\}\}$; both α and β are rational and $\alpha \leq \beta$, and this set is empty exactly when $\min\{b, d\} \leq \max\{a, c\}$. In all cases the intersection belongs to $\mathcal{S}$.

Relative complements. For rationals $a \leq b$ and $c \leq d$ with $c < d$,

$$(a, b] \sim (c, d] = (a, \min\{b, c\}] \cup (\max\{a, d\}, b],$$

where an interval whose left endpoint is greater than or equal to its right endpoint is interpreted as empty. Intersecting with $\mathbb{Q}$ and noting that all four endpoints $a, b, \min\{b, c\}, \max\{a, d\}$ are rational, we conclude that $I(a, b) \sim I(c, d)$ is the union of two sets of the generating form, hence belongs to $\mathcal{S}$. If $c = d$, then $I(c, d) = \emptyset$ and $I(a, b) \sim I(c, d) = I(a, b)$ belongs to $\mathcal{S}$.

Now $\mathcal{S}$ is closed with respect to the formation of finite unions by its very definition, and it is closed with respect to the formation of finite intersections: if $E = \bigcup_{k=1}^n I_k$ and $F = \bigcup_{j=1}^m J_j$ with each I_k and J_j of the generating form, then $E \cap F = \bigcup_{k,j} (I_k \cap J_j)$ is a finite union of sets of the generating form. Finally, for such E and F ,

$$E \sim F = \bigcup_{k=1}^n \left[\bigcap_{j=1}^m (I_k \sim J_j) \right].$$

Each $I_k \sim J_j$ belongs to $\mathcal{S}$; since $\mathcal{S}$ is closed under finite intersections and finite unions, the right-hand side belongs to $\mathcal{S}$. Hence $\mathcal{S}$ is closed with respect to the formation of relative complements.

Step 2: μ is a premeasure.

We have $\mu(\emptyset) = 0$ by definition.

Finite additivity. Let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of sets in $\mathcal{S}$ whose union E belongs to $\mathcal{S}$. If every E_k is empty, then $E = \emptyset$ and both $\mu(E)$ and $\sum_{k=1}^n \mu(E_k)$ equal 0. Otherwise some E_{k_0} is nonempty; then $\mu(E_{k_0}) = \infty$, so $\sum_{k=1}^n \mu(E_k) = \infty$, while $E \supseteq E_{k_0} \neq \emptyset$ forces $\mu(E) = \infty$ as well. In both cases $\mu(E) = \sum_{k=1}^n \mu(E_k)$.

Countable monotonicity. Let E in $\mathcal{S}$ be covered by a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$. If $E = \emptyset$, then $\mu(E) = 0 \leq \sum_{k=1}^\infty \mu(E_k)$. If $E \neq \emptyset$, then since $E \subseteq \bigcup_{k=1}^\infty E_k$ some E_k is nonempty, whence $\sum_{k=1}^\infty \mu(E_k) = \infty \geq \mu(E)$.

Thus μ is finitely additive, countably monotone, and vanishes on $\emptyset$: it is a premeasure. Observe that μ is very far from being σ -finite, since $\mu(S) = \infty$ for every nonempty S in $\mathcal{S}$.

Step 3: the extension to the smallest σ -algebra containing $\mathcal{S}$ is not unique.

Let $\mathcal{B}$ be the smallest σ -algebra of subsets of $\mathbb{Q}$ containing $\mathcal{S}$. We claim $\mathcal{B} = 2^{\mathbb{Q}}$. Indeed, for each q in $\mathbb{Q}$,

$$\{q\} = \bigcap_{n=1}^{\infty} \left[\left(q - \frac{1}{n}, q \right] \cap \mathbb{Q} \right],$$

and each set in this intersection belongs to $\mathcal{S}$ because $q - 1/n$ and q are rational. Hence every singleton belongs to $\mathcal{B}$, and since $\mathbb{Q}$ is countable, every subset of $\mathbb{Q}$ is a countable union of singletons and so belongs to $\mathcal{B}$. Thus $\mathcal{B} = 2^{\mathbb{Q}}$.

Now define two set functions on $2^{\mathbb{Q}}$:

$$\mu_1(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ \infty & \text{if } E \neq \emptyset, \end{cases} \quad \mu_2(E) = \text{the number of points in } E \quad (\text{with } \mu_2(E) = \infty \text{ if } E \text{ is infinite}).$$

Each is a measure on $2^{\mathbb{Q}}$. For μ_1 : it vanishes on $\emptyset$, and if $\{E_k\}_{k=1}^{\infty}$ is a disjoint collection with union E , then either all E_k are empty, in which case $E = \emptyset$ and both sides vanish, or some E_k is nonempty, in which case both $\mu_1(E)$ and $\sum_k \mu_1(E_k)$ equal ∞ . For μ_2 , the counting measure, countable additivity over disjoint collections is immediate, since the cardinality of a disjoint countable union is the sum of the cardinalities.

Both measures extend μ . Certainly $\mu_1(\emptyset) = \mu_2(\emptyset) = 0 = \mu(\emptyset)$. If E in $\mathcal{S}$ is nonempty, then $\mu_1(E) = \infty = \mu(E)$; and, as noted at the outset, E is then an infinite set, so $\mu_2(E) = \infty = \mu(E)$ as well.

Finally $\mu_1 \neq \mu_2$: for any rational q ,

$$\mu_1(\{q\}) = \infty \neq 1 = \mu_2(\{q\}).$$

Therefore μ admits at least two distinct extensions to measures on $\mathcal{B}$, and the extension is not unique. This shows that the uniqueness assertion of the Caratheodory-Hahn Theorem genuinely requires the hypothesis that the premeasure be σ -finite.

Problem 17.33. By a bounded interval of real numbers we mean a set of the form $[a, b]$, $[a, b)$, $(a, b]$ or (a, b) for real numbers $a \leq b$. Thus we consider the empty-set and a set consisting of a single point to be a bounded interval. Show that each of the following three collections of sets $\mathcal{S}$ is a semiring.

- (i) Let $\mathcal{S}$ be the collection of all bounded intervals of real numbers.
- (ii) Let $\mathcal{S}$ be the collection of all subsets of $\mathbb{R} \times \mathbb{R}$ that are products of bounded intervals of real numbers.
- (iii) Let n be a natural number and X be the n -fold Cartesian product of $\mathbb{R}$:

$$X = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}.$$

Let $\mathcal{S}$ be the collection of all subsets of X that are n -fold Cartesian products of bounded intervals of real numbers.

Solution. Recall the definition: a nonempty collection $\mathcal{S}$ of subsets of a set X is a semiring provided that whenever A and B belong to $\mathcal{S}$, the intersection $A \cap B$ belongs to $\mathcal{S}$ and there is a finite disjoint collection $\{C_k\}_{k=1}^m$ of sets in $\mathcal{S}$ for which $A \sim B = \bigcup_{k=1}^m C_k$.

Proof of (i). We first identify the bounded intervals as exactly the bounded convex subsets of $\mathbb{R}$, where a set $C \subseteq \mathbb{R}$ is convex provided that $x, y \in C$ and $x < z < y$ imply $z \in C$.

Each of $[a, b]$, $[a, b)$, $(a, b]$, (a, b) is clearly bounded and convex. Conversely, let C be a bounded convex subset of $\mathbb{R}$. If $C = \emptyset$ then $C = (0, 0)$ is a bounded interval. Otherwise set $a = \inf C$ and $b = \sup C$, both finite since C is nonempty and bounded, and $a \leq b$. If $a < z < b$, then by definition of infimum and supremum there are $x, y \in C$ with $x < z$ and $z < y$; convexity gives $z \in C$. Hence $(a, b) \subseteq C \subseteq [a, b]$, so C is one of the four sets $[a, b]$, $[a, b)$, $(a, b]$, (a, b) , that is, a bounded interval.

The collection $\mathcal{S}$ is nonempty. Intersections: if I and J are bounded intervals, then $I \cap J$ is an intersection of two convex sets, hence convex, and it is bounded; by the characterization above, $I \cap J$ belongs to $\mathcal{S}$.

Relative complements: let I and J be bounded intervals. If $J = \emptyset$, then $I \sim J = I$ belongs to $\mathcal{S}$. So

assume $J \neq \emptyset$ and define

$$L = \{x \in \mathbb{R} : x < y \text{ for every } y \in J\}, \quad R = \{x \in \mathbb{R} : x > y \text{ for every } y \in J\}.$$

The sets L and R are disjoint (a point of $L \cap R$ would have to be both smaller and larger than each of the nonempty set J 's points), and $L \cup R = \mathbb{R} \sim J$. Indeed L and R are clearly disjoint from J ; and if $x \notin J$ but $x \notin L \cup R$, then there are $y_1, y_2 \in J$ with $y_1 \leq x$ and $x \leq y_2$, and since $x \neq y_1, x \neq y_2$ we get $y_1 < x < y_2$, so convexity of J would force $x \in J$, a contradiction.

Both L and R are convex: if $x_1 < z < x_2$ with $x_1, x_2 \in L$, then for every $y \in J$ we have $z < x_2 < y$, so $z \in L$; the argument for R is symmetric. Therefore

$$I \sim J = (I \cap L) \cup (I \cap R)$$

exhibits $I \sim J$ as the union of two disjoint sets, each of which is convex (an intersection of convex sets) and bounded (a subset of the bounded set I), hence a bounded interval. So $I \sim J$ is the union of a finite disjoint collection of sets in $\mathcal{S}$, and $\mathcal{S}$ is a semiring.

A product lemma. Let $\mathcal{S}_1$ be a semiring of subsets of X_1 and $\mathcal{S}_2$ a semiring of subsets of X_2 . Then

$$\mathcal{S}_1 \times \mathcal{S}_2 := \{A \times B : A \in \mathcal{S}_1, B \in \mathcal{S}_2\}$$

is a semiring of subsets of $X_1 \times X_2$.

To see this, first note the collection is nonempty. For intersections,

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D),$$

and $A \cap C \in \mathcal{S}_1, B \cap D \in \mathcal{S}_2$, so the intersection belongs to $\mathcal{S}_1 \times \mathcal{S}_2$.

For relative complements, let $A, C \in \mathcal{S}_1$ and $B, D \in \mathcal{S}_2$. Write

$$A \sim C = \bigcup_{i=1}^p A_i, \quad B \sim D = \bigcup_{j=1}^q B_j,$$

where $\{A_i\}$ is a finite disjoint collection in $\mathcal{S}_1$ and $\{B_j\}$ a finite disjoint collection in $\mathcal{S}_2$; also $A \cap C \in \mathcal{S}_1$ and $B \cap D \in \mathcal{S}_2$. Decomposing $A = (A \cap C) \cup (A \sim C)$ and $B = (B \cap D) \cup (B \sim D)$, both disjointly, we obtain the disjoint decomposition

$$A \times B = [(A \cap C) \times (B \cap D)] \cup [(A \cap C) \times (B \sim D)] \cup [(A \sim C) \times (B \cap D)] \cup [(A \sim C) \times (B \sim D)].$$

The first of these four pieces is precisely $(A \times B) \cap (C \times D)$, while each of the other three is disjoint from $C \times D$ (its first or second coordinate set misses C or D respectively). Hence

$$(A \times B) \sim (C \times D) = \bigcup_{j=1}^q [(A \cap C) \times B_j] \cup \bigcup_{i=1}^p [A_i \times (B \cap D)] \cup \bigcup_{i=1}^p \bigcup_{j=1}^q [A_i \times B_j].$$

Every set appearing on the right belongs to $\mathcal{S}_1 \times \mathcal{S}_2$, and the whole collection is pairwise disjoint: two sets from the same group are disjoint because the A_i are pairwise disjoint and the B_j are pairwise disjoint, and sets from different groups are disjoint because $A \cap C$ is disjoint from each A_i and $B \cap D$ is disjoint from each B_j . This proves the lemma.

Proof of (ii). The collection of subsets of $\mathbb{R} \times \mathbb{R}$ that are products of two bounded intervals is exactly $\mathcal{S}_1 \times \mathcal{S}_1$, where $\mathcal{S}_1$ is the collection of bounded intervals of real numbers. By part (i), $\mathcal{S}_1$ is a semiring, and by the product lemma $\mathcal{S}_1 \times \mathcal{S}_1$ is a semiring of subsets of $\mathbb{R} \times \mathbb{R}$.

Proof of (iii). We argue by induction on n . For $n = 1$ the assertion is part (i). Assume the collection $\mathcal{T}_{n-1}$ of $(n-1)$ -fold products of bounded intervals is a semiring of subsets of $\mathbb{R}^{n-1}$, where $n \geq 2$. Identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ in the obvious way; under this identification the collection $\mathcal{T}_n$ of n -fold products $I_1 \times \cdots \times I_n$ of bounded intervals is exactly

$$\mathcal{T}_{n-1} \times \mathcal{S}_1 = \{P \times I : P \in \mathcal{T}_{n-1}, I \in \mathcal{S}_1\},$$

since $I_1 \times \cdots \times I_n = (I_1 \times \cdots \times I_{n-1}) \times I_n$. By the induction hypothesis and part (i), both factors are semirings, so the product lemma shows $\mathcal{T}_n$ is a semiring of subsets of $\mathbb{R}^n$. This completes the induction, and $\mathcal{S} = \mathcal{T}_n$ is a semiring.

(As remarked in the text, this is what permits the construction of Lebesgue measure on $\mathbb{R}^n$ by the Caratheodory construction applied to the premeasure that assigns to a product of bounded intervals the product of their lengths.)

Problem 17.34. If we start with an outer measure μ^* on 2^X and form the induced measure $\bar{\mu}$ on the μ^* -measurable sets, we can view $\bar{\mu}$ as a set function and denote by μ^+ the outer measure induced by $\bar{\mu}$.

- (i) Show that for each set $E \subseteq X$ we have $\mu^+(E) \geq \mu^*(E)$.
- (ii) For a given set E , show that $\mu^+(E) = \mu^*(E)$ if and only if there is a μ^* -measurable set $A \supseteq E$ with $\mu^*(A) = \mu^*(E)$.

Solution. Let $\mathcal{M}$ denote the σ -algebra of μ^* -measurable subsets of X ; by Theorem 8, $\mathcal{M}$ is a σ -algebra and $\bar{\mu}$, the restriction of μ^* to $\mathcal{M}$, is a measure. Viewing $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ as a set function on the collection $\mathcal{M}$ of subsets of X , Theorem 9 defines the induced outer measure $\mu^+: 2^X \rightarrow [0, \infty]$ by $\mu^+(\emptyset) = 0$ and, for $E \neq \emptyset$,

$$\mu^+(E) = \inf \sum_{k=1}^{\infty} \bar{\mu}(A_k),$$

the infimum taken over all countable collections $\{A_k\}_{k=1}^{\infty}$ of sets in $\mathcal{M}$ that cover E . Since X belongs to $\mathcal{M}$, such covers always exist, so the infimum is over a nonempty set of numbers.

Proof of (i). If $E = \emptyset$ then $\mu^+(E) = 0 = \mu^*(E)$, since μ^* is an outer measure. Let E be nonempty and let $\{A_k\}_{k=1}^{\infty}$ be any countable collection of sets in $\mathcal{M}$ that covers E . Since $\bar{\mu}$ is the restriction of μ^* to $\mathcal{M}$, we have $\bar{\mu}(A_k) = \mu^*(A_k)$ for each k . By the countable monotonicity of the outer measure μ^* ,

$$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(A_k) = \sum_{k=1}^{\infty} \bar{\mu}(A_k).$$

Taking the infimum over all such covers gives $\mu^*(E) \leq \mu^+(E)$.

Proof of (ii). Suppose first that there is a set A in $\mathcal{M}$ with $E \subseteq A$ and $\mu^*(A) = \mu^*(E)$. Define $A_1 = A$ and $A_k = \emptyset$ for $k \geq 2$. Then $\{A_k\}_{k=1}^{\infty}$ is a countable collection of sets in $\mathcal{M}$ covering E , and since $\bar{\mu}(\emptyset) = 0$,

$$\mu^+(E) \leq \sum_{k=1}^{\infty} \bar{\mu}(A_k) = \bar{\mu}(A) = \mu^*(A) = \mu^*(E).$$

(If $E = \emptyset$ this is trivial.) Combined with part (i) we conclude $\mu^+(E) = \mu^*(E)$.

Conversely, suppose $\mu^+(E) = \mu^*(E)$.

Case 1: $\mu^*(E) = \infty$. Take $A = X$, which belongs to $\mathcal{M}$ and contains E . By the monotonicity of the outer measure μ^* we have $\mu^*(X) \geq \mu^*(E) = \infty$, so $\mu^*(A) = \infty = \mu^*(E)$, as required.

Case 2: $\mu^*(E) < \infty$. If $E = \emptyset$, take $A = \emptyset$, which belongs to $\mathcal{M}$ and satisfies $\mu^*(A) = \mu^*(E)$; so assume $E \neq \emptyset$, which is what licenses the description of $\mu^+(E)$ as an infimum. Fix a natural number n . Since $\mu^+(E) = \mu^*(E) < \mu^*(E) + 1/n$, by the definition of μ^+ as an infimum there is a countable collection $\{A_{n,k}\}_{k=1}^{\infty}$ of sets in $\mathcal{M}$ that covers E and for which

$$\sum_{k=1}^{\infty} \bar{\mu}(A_{n,k}) < \mu^*(E) + \frac{1}{n}.$$

Define $B_n = \bigcup_{k=1}^{\infty} A_{n,k}$. Since $\mathcal{M}$ is a σ -algebra, B_n belongs to $\mathcal{M}$, and $E \subseteq B_n$. Because $\bar{\mu}$ is a measure it is countably monotone, so

$$\mu^*(B_n) = \bar{\mu}(B_n) \leq \sum_{k=1}^{\infty} \bar{\mu}(A_{n,k}) < \mu^*(E) + \frac{1}{n}.$$

Now set

$$A = \bigcap_{n=1}^{\infty} B_n.$$

Then A belongs to $\mathcal{M}$, being a countable intersection of sets in the σ -algebra $\mathcal{M}$, and $E \subseteq A$ since $E \subseteq B_n$ for every n . By the monotonicity of μ^* ,

$$\mu^*(E) \leq \mu^*(A) \leq \mu^*(B_n) < \mu^*(E) + \frac{1}{n} \quad \text{for every } n,$$

and letting $n \rightarrow \infty$ gives $\mu^*(A) = \mu^*(E)$. Thus A is a μ^* -measurable set containing E with $\mu^*(A) = \mu^*(E)$.

This completes the proof. (Note that in Case 2 the set A so produced belongs to $\mathcal{M}_{\sigma\delta}$; the argument is the one used to prove Proposition 10, specialized to the collection $\mathcal{S} = \mathcal{M}$.)

Problem 17.35. Let $\mathcal{S}$ be a σ -algebra of subsets of X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a measure. Let $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ be the measure induced by μ via the Caratheodory construction. Show that $\mathcal{S}$ is a subcollection of $\mathcal{M}$ and it may be a proper subcollection.

Solution. Let μ^* be the outer measure induced by the set function $\mu: \mathcal{S} \rightarrow [0, \infty]$, as in Theorem 9, and let $\mathcal{M}$ be the σ -algebra of μ^* -measurable sets, so that $\bar{\mu}$ is the restriction of μ^* to $\mathcal{M}$.

Step 1: $\mathcal{S} \subseteq \mathcal{M}$.

By Problem 29, a set function on a σ -algebra is a measure if and only if it is a premeasure; hence μ is a premeasure on $\mathcal{S}$. Moreover a σ -algebra is closed with respect to the formation of relative complements, since for A and B in $\mathcal{S}$ we have $A \sim B = A \cap B^c$, and $\mathcal{S}$ is closed under complements and finite intersections. Therefore Theorem 12 applies: the Caratheodory measure $\bar{\mu}$ induced by μ is an extension of μ . In particular every set belonging to $\mathcal{S}$ is μ^* -measurable, that is,

$$\mathcal{S} \subseteq \mathcal{M}, \quad \text{and} \quad \bar{\mu}(E) = \mu(E) \quad \text{for every } E \in \mathcal{S}.$$

For completeness we recall why every A in $\mathcal{S}$ is μ^* -measurable, which is the argument of Theorem 12. Let $E \subseteq X$ have $\mu^*(E) < \infty$ and let $\epsilon > 0$. Choose a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ covering E with $\sum_{k=1}^\infty \mu(E_k) \leq \mu^*(E) + \epsilon$. Since $\mathcal{S}$ is a σ -algebra, $E_k \cap A$ and $E_k \cap A^c$ belong to $\mathcal{S}$, and by the finite additivity of the measure μ , $\mu(E_k) = \mu(E_k \cap A) + \mu(E_k \cap A^c)$. The collections $\{E_k \cap A\}$ and $\{E_k \cap A^c\}$ are countable collections of sets in $\mathcal{S}$ covering $E \cap A$ and $E \cap A^c$ respectively, so by the definition of μ^* ,

$$\mu^*(E) + \epsilon \geq \sum_{k=1}^\infty \mu(E_k \cap A) + \sum_{k=1}^\infty \mu(E_k \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

As $\epsilon > 0$ was arbitrary, A is μ^* -measurable.

Step 2: the inclusion may be proper.

Let $X = \{a, b\}$ be a set with exactly two points, let $\mathcal{S} = \{\emptyset, X\}$ (a σ -algebra), and define $\mu(\emptyset) = 0$, $\mu(X) = 0$. Then μ is a measure on $\mathcal{S}$: it vanishes on $\emptyset$, and countable additivity is immediate since μ is identically zero.

Compute the induced outer measure. For any $E \subseteq X$ the collection consisting of X alone (padded with copies of $\emptyset$) is a countable collection of sets in $\mathcal{S}$ covering E , with total μ -sum equal to 0. Hence $\mu^*(E) \leq 0$, so $\mu^*(E) = 0$ for every $E \subseteq X$; that is, μ^* is identically zero on 2^X .

Consequently every subset A of X is μ^* -measurable, since for every $E \subseteq X$,

$$\mu^*(E) = 0 = 0 + 0 = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Therefore $\mathcal{M} = 2^X = \{\emptyset, \{a\}, \{b\}, X\}$, which properly contains $\mathcal{S} = \{\emptyset, X\}$.

The phenomenon behind this example is general: by Theorem 8 the measure space $(X, \mathcal{M}, \bar{\mu})$ is complete, whereas the measure space $(X, \mathcal{S}, \mu)$ need not be. In the example, $\{a\}$ is a subset of the set X , which has μ -measure zero, yet $\{a\}$ does not belong to $\mathcal{S}$; so $\mathcal{M}$ must be strictly larger than $\mathcal{S}$. A second standard example is $X = \mathbb{R}$ with $\mathcal{S}$ the σ -algebra of Borel sets and μ Lebesgue measure restricted to the Borel sets: the induced μ^* is Lebesgue outer measure, so $\mathcal{M}$ is the σ -algebra of Lebesgue measurable sets, which properly contains the Borel sets.

17.6 Problem 17.36

Problem 17.36. Let μ be a finite premeasure on an algebra $\mathcal{S}$, and μ^* the induced outer measure. Show that a subset E of X is μ^* -measurable if and only if for each $\epsilon > 0$ there is a set $A \in \mathcal{S}_\delta$, $A \subseteq E$, such that $\mu^*(E \sim A) < \epsilon$.

Solution. Throughout, $\mathcal{S}$ is an algebra of subsets of X : it contains X and $\emptyset$, and is closed with respect to complementation in X and with respect to the formation of finite unions, hence also finite intersections and relative complements, since $A \sim B = A \cap B^C$. We write $\mathcal{M}$ for the σ -algebra of μ^* -measurable subsets of X and $\bar{\mu}$ for the Caratheodory measure induced by μ , that is, the restriction of μ^* to $\mathcal{M}$. Recall that $\mathcal{S}_\sigma$ denotes the collection of countable unions of sets in $\mathcal{S}$ and $\mathcal{S}_\delta$ the collection of countable intersections of sets in $\mathcal{S}$.

We first record four preliminary observations.

(i) μ^* is finite. Indeed, μ is a finite premeasure on the algebra $\mathcal{S}$, so $\mu(X) < \infty$. Since $X \in \mathcal{S}$ and $\emptyset \in \mathcal{S}$ with $\mu(\emptyset) = 0$, the collection $\{X, \emptyset, \emptyset, \dots\}$ is a countable cover of any subset F of X by sets in $\mathcal{S}$, and therefore, by the definition of the induced outer measure given in Theorem 9 of Chapter 17,

$$\mu^*(F) \leq \mu(X) < \infty \quad \text{for every } F \subseteq X.$$

(ii) Every set in $\mathcal{S}$ is μ^* -measurable, so $\mathcal{S} \subseteq \mathcal{M}$. Since an algebra is closed with respect to the formation of relative complements, this is exactly what is established in the proof of Theorem 12 of Chapter 17, whose conclusion is that the Caratheodory measure $\bar{\mu}$ is an extension of μ . By Theorem 8 of Chapter 17, $\mathcal{M}$ is a σ -algebra, so it is closed with respect to the formation of countable unions and countable intersections. We conclude that

$$\mathcal{S}_\sigma \subseteq \mathcal{M} \quad \text{and} \quad \mathcal{S}_\delta \subseteq \mathcal{M}.$$

(iii) $\mathcal{S}_\delta$ is precisely the collection of complements of sets in $\mathcal{S}_\sigma$. Indeed, if $U = \bigcup_{k=1}^\infty E_k$ with each $E_k \in \mathcal{S}$, then by De Morgan's identity

$$X \sim U = \bigcap_{k=1}^\infty (X \sim E_k),$$

and each $X \sim E_k$ belongs to $\mathcal{S}$ because $\mathcal{S}$ is an algebra; thus $X \sim U \in \mathcal{S}_\delta$. The reverse inclusion follows by the same computation read backwards. This observation is what makes the collection $\mathcal{S}_\delta$ the right one for approximating a measurable set from inside: covers by sets of $\mathcal{S}$ approximate the complement from outside, and complementation converts $\mathcal{S}_\sigma$ into $\mathcal{S}_\delta$.

(iv) Every set of outer measure zero is μ^* -measurable. This follows at once from the monotonicity of the outer measure μ^* : if $\mu^*(N) = 0$ and $B \subseteq X$ is arbitrary, then $B \cap N \subseteq N$ and $B \cap N^C \subseteq B$, so

$$\mu^*(B \cap N) + \mu^*(B \cap N^C) \leq \mu^*(N) + \mu^*(B) = 0 + \mu^*(B) = \mu^*(B),$$

and this is precisely the Caratheodory criterion for the measurability of N , the reverse inequality being automatic from the countable monotonicity of μ^* . Note that no completeness hypothesis on any preassigned measure is being invoked here: the argument is a direct verification from the definition of μ^* -measurability.

We now prove the two implications.

Necessity. Assume E is μ^* -measurable and let $\epsilon > 0$. The complement $E^C = X \sim E$ is then also μ^* -measurable, and by (i) $\mu^*(E^C) < \infty$. Hence, by the definition of the induced outer measure as an infimum over countable covers by sets in $\mathcal{S}$, there is a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ that covers E^C and for which

$$\sum_{k=1}^\infty \mu(E_k) < \mu^*(E^C) + \epsilon.$$

Define

$$U = \bigcup_{k=1}^\infty E_k \quad \text{and} \quad A = X \sim U.$$

Then $U \in \mathcal{S}_\sigma$, and since $\{E_k\}_{k=1}^\infty$ is a countable collection of sets in $\mathcal{S}$ covering U , the definition of μ^* gives

$$\mu^*(U) \leq \sum_{k=1}^\infty \mu(E_k) < \mu^*(E^C) + \epsilon.$$

By observation (iii), $A \in \mathcal{S}_\delta$. Moreover $E^C \subseteq U$ forces $A = X \sim U \subseteq X \sim E^C = E$, so A is a subset of E , as required.

It remains to estimate $\mu^*(E \sim A)$. Since $A = X \sim U$ we have

$$E \sim A = E \cap A^C = E \cap U.$$

The sets $E \cap U$ and E^C are both μ^* -measurable: $U \in \mathcal{S}_\sigma \subseteq \mathcal{M}$ by (ii), $E \in \mathcal{M}$ by hypothesis, and $\mathcal{M}$ is a σ -algebra. They are disjoint, and because $E^C \subseteq U$ their union is

$$(E \cap U) \cup E^C = (E \cap U) \cup (E^C \cap U) = U.$$

Therefore, applying Proposition 6 of Chapter 17 to the test set X and to the finite disjoint collection $\{E \cap U, E^C\}$ of μ^* -measurable sets, whose union is U , we obtain

$$\mu^*(U) = \mu^*(E \cap U) + \mu^*(E^C).$$

All three quantities here are finite, by (i). Hence we may excise $\mu^*(E^C)$ from both sides and conclude

$$\mu^*(E \sim A) = \mu^*(E \cap U) = \mu^*(U) - \mu^*(E^C) < (\mu^*(E^C) + \epsilon) - \mu^*(E^C) = \epsilon.$$

This is the desired approximation. Observe that the finiteness of the premeasure was used twice and essentially: to guarantee $\mu^*(E^C) < \infty$, so that a cover of E^C of nearly minimal total mass exists at all, and to license the subtraction in the last display.

Sufficiency. Conversely, assume that for each $\epsilon > 0$ there is a set $A \in \mathcal{S}_\delta$ with $A \subseteq E$ and $\mu^*(E \sim A) < \epsilon$. For each natural number k choose $A_k \in \mathcal{S}_\delta$ with

$$A_k \subseteq E \quad \text{and} \quad \mu^*(E \sim A_k) < \frac{1}{k},$$

and define

$$A = \bigcup_{k=1}^\infty A_k.$$

By (ii) each A_k is μ^* -measurable, and $\mathcal{M}$ is a σ -algebra by Theorem 8 of Chapter 17, so A is μ^* -measurable. Since each $A_k \subseteq E$ we have $A \subseteq E$, and since $A_k \subseteq A$ we have, for every k ,

$$E \sim A \subseteq E \sim A_k.$$

By the monotonicity of the outer measure μ^* this yields $\mu^*(E \sim A) \leq \mu^*(E \sim A_k) < 1/k$ for all k , and therefore

$$\mu^*(E \sim A) = 0.$$

By observation (iv), the set $E \sim A$ is μ^* -measurable. Finally, because $A \subseteq E$,

$$E = A \cup (E \sim A)$$

exhibits E as the union of two μ^* -measurable sets, and hence, by Proposition 5 of Chapter 17, E is μ^* -measurable. (Note that this half of the argument used neither the finiteness of μ nor any special feature of $\mathcal{S}_\delta$ beyond the inclusion $\mathcal{S}_\delta \subseteq \mathcal{M}$ supplied by Theorem 12 of Chapter 17.)

The two implications together establish the asserted characterization.

18 Integration Over General Measure Spaces

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18.1 Problems 18.1–18.7

Problem 18.1. Show that an extended real-valued function f on X is measurable if and only if $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ are measurable and so is $f^{-1}(E)$ for every Borel set E of real numbers.

Solution. Throughout, $(X, \mathcal{M})$ is the reference measurable space and “measurable” means with respect to $\mathcal{M}$; $\mathcal{B}$ denotes the σ -algebra of Borel subsets of $\mathbb{R}$, that is, the smallest σ -algebra of subsets of $\mathbb{R}$ containing the open sets.

Suppose first that $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ are measurable and that $f^{-1}(E)$ is measurable for every Borel set E of real numbers. Let c be a real number. The set (c, ∞) is an open, hence Borel, set of real numbers, and since a point x of X satisfies $f(x) > c$ if and only if either $f(x)$ is a real number greater than c or $f(x) = \infty$, we have

$$\{x \in X \mid f(x) > c\} = f^{-1}((c, \infty)) \cup f^{-1}\{\infty\}.$$

Both sets on the right are measurable and $\mathcal{M}$ is closed with respect to the formation of finite unions, so $\{x \in X \mid f(x) > c\}$ is measurable. Since c was an arbitrary real number, property (iii) of Proposition 1 holds and therefore f is measurable.

Conversely, suppose f is measurable. Since $f(x) = \infty$ if and only if $f(x) > n$ for every natural number n ,

$$f^{-1}\{\infty\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) > n\},$$

and each set in this intersection is measurable by Proposition 1(iii); as $\mathcal{M}$ is closed with respect to the formation of countable intersections, $f^{-1}\{\infty\}$ is measurable. Symmetrically,

$$f^{-1}\{-\infty\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) < -n\}$$

is measurable by Proposition 1(i). Consequently

$$f^{-1}(\mathbb{R}) = X \sim [f^{-1}\{\infty\} \cup f^{-1}\{-\infty\}]$$

also is measurable.

Now define

$$\mathcal{A} = \{E \subseteq \mathbb{R} \mid f^{-1}(E) \in \mathcal{M}\}.$$

We claim $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$. Indeed, $\mathbb{R}$ belongs to $\mathcal{A}$ by the preceding paragraph. If E belongs to $\mathcal{A}$, then, since preimages respect relative complements and f^{-1} of a subset of $\mathbb{R}$ is contained in $f^{-1}(\mathbb{R})$,

$$f^{-1}(\mathbb{R} \sim E) = f^{-1}(\mathbb{R}) \sim f^{-1}(E),$$

which is the intersection of $f^{-1}(\mathbb{R})$ with the complement in X of $f^{-1}(E)$, hence is measurable; thus $\mathbb{R} \sim E$ belongs to $\mathcal{A}$. Finally, if $\{E_k\}_{k=1}^{\infty}$ is a countable collection of sets in $\mathcal{A}$, then

$$f^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(E_k),$$

which is measurable since $\mathcal{M}$ is closed with respect to the formation of countable unions. Hence $\mathcal{A}$ is a σ -algebra of subsets of $\mathbb{R}$.

The σ -algebra $\mathcal{A}$ contains every unbounded open interval of the form (c, ∞) with c real, because

$$f^{-1}((c, \infty)) = \{x \in X \mid f(x) > c\} \sim f^{-1}\{\infty\}$$

is the intersection of a measurable set with the complement of a measurable set. Therefore $\mathcal{A}$ also contains each set $\mathbb{R} \sim (c, \infty) = (-\infty, c]$, and hence each set

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n],$$

and hence each bounded open interval $(a, b) = (a, \infty) \cap (-\infty, b)$. Since every open set of real numbers is the union of a countable collection of bounded open intervals (for instance, of those bounded open intervals with rational endpoints that it contains), every open set of real numbers belongs to $\mathcal{A}$. But $\mathcal{B}$ is the smallest σ -algebra of subsets of $\mathbb{R}$ containing the open sets, and $\mathcal{A}$ is a σ -algebra containing the open sets. Therefore $\mathcal{B} \subseteq \mathcal{A}$; that is, $f^{-1}(E)$ is measurable for every Borel set E of real numbers. This, together with the measurability of $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ established above, completes the proof.

Problem 18.2. Suppose $(X, \mathcal{M}, \mu)$ is not complete. Let E be a subset of a set of measure zero that does not belong to $\mathcal{M}$. Let $f = 0$ on X and $g = \chi_E$. Show that $f = g$ a.e. on X while f is measurable and g is not.

Solution. Since the measure space $(X, \mathcal{M}, \mu)$ fails to be complete, by the very definition of completeness there is a set E_0 in $\mathcal{M}$ with $\mu(E_0) = 0$ possessing a subset $E \subseteq E_0$ that does not belong to $\mathcal{M}$. Fix such a pair $E \subseteq E_0$, and define $f \equiv 0$ on X and $g = \chi_E$, so that $g(x) = 1$ for x in E and $g(x) = 0$ for x in $X \sim E$.

The function f is measurable. Indeed, for a real number c ,

$$\{x \in X \mid f(x) < c\} = \begin{cases} X & \text{if } c > 0, \\ \emptyset & \text{if } c \leq 0, \end{cases}$$

and both X and $\emptyset$ belong to the σ -algebra $\mathcal{M}$. Thus property (i) of Proposition 1 holds and f is measurable. (Any constant function is measurable for the same reason.)

The function g is not measurable. Take $c = 1/2$. Since g takes only the values 0 and 1, and $g(x) > 1/2$ precisely when $g(x) = 1$, that is, precisely when x belongs to E , we have

$$\{x \in X \mid g(x) > 1/2\} = E,$$

and E does not belong to $\mathcal{M}$. Hence property (iii) of Proposition 1 fails for the real number $c = 1/2$, and therefore g is not measurable.

Finally, $f = g$ a.e. on X . The set of points at which f and g differ is exactly

$$\{x \in X \mid f(x) \neq g(x)\} = \{x \in X \mid g(x) = 1\} = E,$$

and $E \subseteq E_0$, where E_0 belongs to $\mathcal{M}$ and $\mu(E_0) = 0$. Thus $X_0 = X \sim E_0$ is a measurable subset of X with $\mu(X \sim X_0) = \mu(E_0) = 0$, and for every x in X_0 we have $x \notin E$, so $g(x) = 0 = f(x)$. Hence f and g agree on the complement of a set of measure zero, that is, $f = g$ a.e. on X .

This example shows that the completeness hypothesis in Proposition 3 cannot be dropped: here $g = f$ a.e. on X and f is measurable, yet g is not measurable.

Problem 18.3. Suppose $(X, \mathcal{M}, \mu)$ is not complete. Show that there is a sequence $\{f_n\}$ of measurable functions on X that converges pointwise a.e. on X to a function f that is not measurable.

Solution. Since $(X, \mathcal{M}, \mu)$ is not complete, there is a set E_0 in $\mathcal{M}$ with $\mu(E_0) = 0$ and a subset E of E_0 that does not belong to $\mathcal{M}$. Fix such $E \subseteq E_0$.

For each natural number n define f_n on X by

$$f_n(x) = \frac{1}{n} \quad \text{for all } x \in X,$$

and define $f = \chi_E$.

Each f_n is measurable: it is constant, and for a constant function the set $\{x \in X \mid f_n(x) < c\}$ is either X or $\emptyset$ for each real number c , and both belong to $\mathcal{M}$; hence property (i) of Proposition 1 holds.

The function f is not measurable: taking $c = 1/2$ and observing that f takes only the values 0 and 1, we get

$$\{x \in X \mid f(x) > 1/2\} = E,$$

which does not belong to $\mathcal{M}$. Thus property (iii) of Proposition 1 fails, so f is not measurable. Finally, $\{f_n\}$ converges pointwise a.e. on X to f . Set $X_0 = X \sim E_0$. Then X_0 is measurable and $\mu(X \sim X_0) = \mu(E_0) = 0$. Let x belong to X_0 . Since $E \subseteq E_0$ and $x \notin E_0$, we have $x \notin E$ and therefore $f(x) = 0$. On the other hand $f_n(x) = 1/n \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\lim_{n \rightarrow \infty} f_n(x) = 0 = f(x) \quad \text{for all } x \in X_0,$$

that is, $\{f_n\} \rightarrow f$ pointwise on X_0 , the complement of a set of measure zero. Thus $\{f_n\} \rightarrow f$ pointwise a.e. on X while f is not measurable.

This shows that the hypothesis in Theorem 6 that either the measure space is complete or the convergence is pointwise on all of X cannot be omitted: here the convergence is merely pointwise a.e. and the space is not complete, and the limit function fails to be measurable.

Problem 18.4. Let E be a measurable subset of X and f an extended real-valued function on X . Show that f is measurable if and only if its restrictions to E and $X \sim E$ are measurable.

Solution. Recall the convention: for a measurable subset A of X , an extended real-valued function defined on A is called measurable provided it is measurable with respect to the measurable space $(A, \mathcal{M}_A)$, where $\mathcal{M}_A$ is the collection of sets in $\mathcal{M}$ that are contained in A .

We first observe that if A belongs to $\mathcal{M}$, then $\mathcal{M}_A$ is a σ -algebra of subsets of A , and $\mathcal{M}_A \subseteq \mathcal{M}$. Indeed, A itself belongs to $\mathcal{M}_A$; if B belongs to $\mathcal{M}_A$ then $A \sim B = A \cap (X \sim B)$ belongs to $\mathcal{M}$ and is contained in A , so it belongs to $\mathcal{M}_A$; and if $\{B_k\}_{k=1}^\infty$ is a countable collection of sets in $\mathcal{M}_A$, then $\bigcup_{k=1}^\infty B_k$ belongs to $\mathcal{M}$ and is contained in A , so it belongs to $\mathcal{M}_A$. The inclusion $\mathcal{M}_A \subseteq \mathcal{M}$ is immediate from the definition.

Write $E' = X \sim E$, which is measurable since E is and $\mathcal{M}$ is a σ -algebra. Let f_1 denote the restriction of f to E and f_2 the restriction of f to E' . For each real number c we have the two identities

$$\{x \in E \mid f_1(x) < c\} = \{x \in X \mid f(x) < c\} \cap E, \quad \{x \in E' \mid f_2(x) < c\} = \{x \in X \mid f(x) < c\} \cap E',$$

and, since E and E' are disjoint with union X ,

$$\{x \in X \mid f(x) < c\} = \{x \in E \mid f_1(x) < c\} \cup \{x \in E' \mid f_2(x) < c\}.$$

Call this last identity (*).

Suppose f is measurable. Let c be a real number. By Proposition 1(i) the set $\{x \in X \mid f(x) < c\}$ belongs to $\mathcal{M}$. Since E belongs to $\mathcal{M}$, the intersection $\{x \in X \mid f(x) < c\} \cap E$ belongs to $\mathcal{M}$ and is contained in E ; hence it belongs to $\mathcal{M}_E$. By the first identity above and Proposition 1(i) applied on the measurable space $(E, \mathcal{M}_E)$, the function f_1 is measurable. The same argument with E' in place of E shows f_2 is measurable.

Conversely, suppose f_1 and f_2 are measurable, on $(E, \mathcal{M}_E)$ and $(E', \mathcal{M}_{E'})$ respectively. Let c be a real number. By Proposition 1(i) applied on each of these two measurable spaces, $\{x \in E \mid f_1(x) < c\}$ belongs to $\mathcal{M}_E$ and $\{x \in E' \mid f_2(x) < c\}$ belongs to $\mathcal{M}_{E'}$. Since $\mathcal{M}_E \subseteq \mathcal{M}$ and $\mathcal{M}_{E'} \subseteq \mathcal{M}$, both sets belong to $\mathcal{M}$, and hence so does their union. By (*), the set $\{x \in X \mid f(x) < c\}$ belongs to $\mathcal{M}$. As c was an arbitrary real number, Proposition 1(i) tells us that f is measurable on X .

Problem 18.5. Show that an extended real-valued function f on X is measurable if and only if for each rational number c , $\{x \in X \mid f(x) < c\}$ is a measurable set.

Solution. Suppose first that f is measurable. Then by property (i) of Proposition 1, the set $\{x \in X \mid f(x) < c\}$ is measurable for every real number c , and in particular for every rational number c .

Conversely, suppose $\{x \in X \mid f(x) < c\}$ is measurable for every rational number c . Let c be an arbitrary real number. Since the rational numbers are dense in $\mathbb{R}$, we may select a sequence $\{q_n\}$ of

rational numbers with

$$q_n < c \quad \text{for all } n \quad \text{and} \quad \lim_{n \rightarrow \infty} q_n = c$$

(for instance, choose q_n rational with $c - 1/n < q_n < c$). We claim

$$\{x \in X \mid f(x) < c\} = \bigcup_{n=1}^{\infty} \{x \in X \mid f(x) < q_n\}.$$

Indeed, if $f(x) < q_n$ for some index n , then $f(x) < q_n < c$, so x belongs to the left-hand set; this gives the inclusion $\supseteq$. For the inclusion $\subseteq$, let x be a point of X with $f(x) < c$. If $f(x) = -\infty$, then $f(x) < q_1$ and x belongs to the union. Otherwise $f(x)$ is a real number with $f(x) < c$, and since $q_n \rightarrow c$ there is an index n for which $q_n > f(x)$; again x belongs to the union. This proves the claimed equality.

Each set $\{x \in X \mid f(x) < q_n\}$ is measurable by assumption, since q_n is rational. As $\mathcal{M}$ is closed with respect to the formation of countable unions, the set $\{x \in X \mid f(x) < c\}$ is measurable. Since c was an arbitrary real number, property (i) of Proposition 1 holds, and therefore f is measurable.

Problem 18.6. Consider two extended real-valued measurable functions f and g on X that are finite a.e. on X . Define X_0 to be the set of points in X at which both f and g are finite. Show that X_0 is measurable and $\mu(X \sim X_0) = 0$.

Solution. Let F denote the set of points of X at which f is infinite and G the set of points at which g is infinite, so that

$$F = f^{-1}\{\infty\} \cup f^{-1}\{-\infty\}, \quad G = g^{-1}\{\infty\} \cup g^{-1}\{-\infty\},$$

and

$$X_0 = (X \sim F) \cap (X \sim G) = X \sim (F \cup G).$$

First we show X_0 is measurable. Since $f(x) = \infty$ if and only if $f(x) > n$ for every natural number n , and $f(x) = -\infty$ if and only if $f(x) < -n$ for every natural number n , we have

$$f^{-1}\{\infty\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) > n\}, \quad f^{-1}\{-\infty\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) < -n\}.$$

Since f is measurable, each set $\{x \in X \mid f(x) > n\}$ is measurable by Proposition 1(iii) and each set $\{x \in X \mid f(x) < -n\}$ is measurable by Proposition 1(i). As $\mathcal{M}$ is closed with respect to the formation of countable intersections, both $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ are measurable, and therefore so is their union F . (This is also the content of the final assertion of Proposition 1, that $\{x \in X \mid f(x) = c\}$ is measurable for each extended real number c .) The same argument applied to g shows G is measurable. Hence $F \cup G$ is measurable, and so is its complement

$$X_0 = X \sim (F \cup G).$$

Now we show $\mu(X \sim X_0) = 0$, where $X \sim X_0 = F \cup G$. Since f is finite a.e. on X , there is a measurable set A with $\mu(A) = 0$ such that f is finite at every point of $X \sim A$; equivalently, $F \subseteq A$. Since F is measurable and $F \subseteq A$, the monotonicity of the measure μ gives

$$0 \leq \mu(F) \leq \mu(A) = 0,$$

so $\mu(F) = 0$. In the same way, since g is finite a.e. on X , there is a measurable set B with $\mu(B) = 0$ and $G \subseteq B$, whence $\mu(G) = 0$.

Finally, by the countable (in particular, finite) subadditivity of the measure μ ,

$$\mu(X \sim X_0) = \mu(F \cup G) \leq \mu(F) + \mu(G) = 0 + 0 = 0.$$

Since a measure takes only nonnegative values, $\mu(X \sim X_0) = 0$. Thus X_0 is a measurable subset of X whose complement has measure zero, which is precisely the situation in which the Remark following Theorem 4 defines the linear combination $\alpha f + \beta g$ and the product $f \cdot g$.

Problem 18.7. Let X be a nonempty set. Show that every extended real-valued function on X is measurable with respect to the measurable space $(X, 2^X)$.

- (i) Let x_0 belong to X and δ_{x_0} be the Dirac measure at x_0 on 2^X . Show that two functions on X are equal a.e. $[\delta_{x_0}]$ if and only if they take the same value at x_0 .
- (ii) Let η be the counting measure on 2^X . Show that two functions on X are equal a.e. $[\eta]$ if and only if they take the same value at every point in X .

Solution. Take $\mathcal{M} = 2^X$, the collection of all subsets of X ; this is a σ -algebra of subsets of X , the largest one. Let f be any extended real-valued function on X and let c be a real number. The set $\{x \in X \mid f(x) < c\}$ is a subset of X and therefore belongs to $2^X = \mathcal{M}$. Since c was arbitrary, property (i) of Proposition 1 holds and f is measurable with respect to $(X, 2^X)$. Thus every extended real-valued function on X is measurable with respect to $(X, 2^X)$.

Before turning to (i) and (ii), recall the meaning of “equal a.e.” on a measure space $(X, \mathcal{M}, \mu)$: two functions f and g on X are equal a.e. $[\mu]$ provided there is a set E_0 in $\mathcal{M}$ with $\mu(E_0) = 0$ for which $f = g$ on $X \setminus E_0$. Since here $\mathcal{M} = 2^X$, every subset of X is eligible as E_0 . Observe that in general $f = g$ a.e. $[\mu]$ if and only if the set

$$D = \{x \in X \mid f(x) \neq g(x)\}$$

satisfies $\mu(D) = 0$. Indeed, if $\mu(D) = 0$ we may take $E_0 = D$ (it belongs to 2^X); conversely, if $f = g$ on $X \setminus E_0$ with $\mu(E_0) = 0$, then $D \subseteq E_0$ and hence, by the monotonicity of μ , $0 \leq \mu(D) \leq \mu(E_0) = 0$.

(i) The Dirac measure δ_{x_0} assigns the value 1 to a set containing x_0 and the value 0 to a set not containing x_0 . Hence, for a subset A of X ,

$$\delta_{x_0}(A) = 0 \iff x_0 \notin A.$$

Let f and g be functions on X and let $D = \{x \in X \mid f(x) \neq g(x)\}$. By the observation above, $f = g$ a.e. $[\delta_{x_0}]$ if and only if $\delta_{x_0}(D) = 0$, that is, if and only if x_0 does not belong to D , that is, if and only if $f(x_0) = g(x_0)$. Thus two functions on X are equal a.e. $[\delta_{x_0}]$ if and only if they take the same value at x_0 .

(ii) The counting measure η assigns to a finite set the number of its elements and to an infinite set the value ∞ . Hence, for a subset A of X ,

$$\eta(A) = 0 \iff A = \emptyset,$$

since a nonempty finite set has at least one element, so its measure is a natural number, and an infinite set has measure ∞ ; in either case $\eta(A) \geq 1 > 0$ when $A \neq \emptyset$, while $\eta(\emptyset) = 0$.

Let f and g be functions on X and again put $D = \{x \in X \mid f(x) \neq g(x)\}$. By the observation above, $f = g$ a.e. $[\eta]$ if and only if $\eta(D) = 0$, that is, if and only if $D = \emptyset$, that is, if and only if $f(x) = g(x)$ for every point x in X . Thus two functions on X are equal a.e. $[\eta]$ if and only if they take the same value at every point of X ; for the counting measure, “almost everywhere” is the same as “everywhere”.

18.2 Problems 18.8–18.14

Problem 18.8. Let X be a topological space and $\mathcal{B}(X)$ the smallest σ -algebra containing the topology on X . $\mathcal{B}(X)$ is called the Borel σ -algebra associated with the topological space X . Show that any continuous real-valued function on X is measurable with respect to the Borel measurable space $(X, \mathcal{B}(X))$.

Solution. First observe that $\mathcal{B}(X)$ is well defined. Let $\mathcal{T}$ denote the topology on X , that is, the collection of open subsets of X . The collection 2^X of all subsets of X is a σ -algebra containing $\mathcal{T}$, so the family of σ -algebras of subsets of X that contain $\mathcal{T}$ is nonempty. The intersection of any nonempty family of σ -algebras of subsets of X is again a σ -algebra: it contains X , and it is closed with respect

to complements and countable unions because each member of the family is. Therefore

$$\mathcal{B}(X) = \bigcap \{ \mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } X \text{ and } \mathcal{T} \subseteq \mathcal{A} \}$$

is a σ -algebra, it contains $\mathcal{T}$, and it is contained in every σ -algebra containing $\mathcal{T}$. Thus $\mathcal{B}(X)$ is the smallest σ -algebra containing the topology on X , and $(X, \mathcal{B}(X))$ is a measurable space.

The key point is the inclusion

$$\mathcal{T} \subseteq \mathcal{B}(X),$$

that is, every open subset of X is Borel measurable.

Now let f be a continuous real-valued function on X . Let $\mathcal{O}$ be an open subset of $\mathbb{R}$. By the very definition of continuity of f as a map from the topological space X to the topological space $\mathbb{R}$, the inverse image $f^{-1}(\mathcal{O})$ is an open subset of X ; that is, $f^{-1}(\mathcal{O}) \in \mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{B}(X)$, we conclude that

$$f^{-1}(\mathcal{O}) \in \mathcal{B}(X) \quad \text{for every open set } \mathcal{O} \text{ of real numbers.}$$

By Proposition 2, a real-valued function f on a measurable space $(X, \mathcal{M})$ is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for each open set $\mathcal{O}$ of real numbers. Applying Proposition 2 with $\mathcal{M} = \mathcal{B}(X)$, we conclude that f is measurable with respect to $(X, \mathcal{B}(X))$.

It is worth recording the argument without appealing to Proposition 2, since the direct verification is just as short. Fix a real number c . The set $(-\infty, c)$ is an open subset of $\mathbb{R}$, so by continuity

$$\{x \in X \mid f(x) < c\} = f^{-1}((-\infty, c))$$

is an open subset of X and hence belongs to $\mathcal{B}(X)$. Thus statement (i) of Proposition 1 holds for f with respect to the σ -algebra $\mathcal{B}(X)$, and therefore f is measurable with respect to $(X, \mathcal{B}(X))$.

Observe finally that the argument used nothing about $\mathcal{B}(X)$ beyond the inclusion $\mathcal{T} \subseteq \mathcal{B}(X)$. Thus we have in fact proved the slightly more general assertion recorded in the text on page 360: if $\mathcal{M}$ is any σ -algebra of subsets of X that contains the topology on X , then every continuous real-valued function on X is measurable with respect to $\mathcal{M}$. Since $\mathcal{B}(X)$ is contained in every such $\mathcal{M}$, the case $\mathcal{M} = \mathcal{B}(X)$ is the strongest instance of this assertion. It should not be claimed, however, that $\mathcal{B}(X)$ is the smallest σ -algebra making every continuous real-valued function on X measurable: that is false in general. For example, if X is an uncountable set with the topology whose open sets are $\emptyset$ together with the complements of countable sets, then every continuous real-valued function on X is constant, so the σ -algebra $\{\emptyset, X\}$ already makes all of them measurable, while $\mathcal{B}(X)$ is the strictly larger σ -algebra of countable and co-countable sets.

Problem 18.9. If a real-valued function on $\mathbb{R}$ is measurable with respect to the σ -algebra of Lebesgue measurable sets, is it necessarily measurable with respect to the Borel measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$?

Solution. No. Lebesgue measurability of a real-valued function on $\mathbb{R}$ does not imply Borel measurability. We first record the trivial implication in the other direction, to make the asymmetry clear. Every open subset of $\mathbb{R}$ is Lebesgue measurable, and the Lebesgue measurable sets form a σ -algebra; since $\mathcal{B}(\mathbb{R})$ is the smallest σ -algebra containing the topology of $\mathbb{R}$, we have

$$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L},$$

where $\mathcal{L}$ denotes the σ -algebra of Lebesgue measurable subsets of $\mathbb{R}$. Hence if f is measurable with respect to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then for each real c the set $\{x \mid f(x) < c\}$ belongs to $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}$, so f is Lebesgue measurable. The converse fails.

To exhibit a counterexample we use Proposition 22 of Chapter 2: there is a Lebesgue measurable set A , a subset of the Cantor set, that is not a Borel set. Define

$$f = \chi_A \quad \text{on } \mathbb{R},$$

the characteristic function of A , a real-valued function on $\mathbb{R}$.

The function f is Lebesgue measurable. Indeed, for a real number c ,

$$\{x \in \mathbb{R} \mid f(x) > c\} = \begin{cases} \mathbb{R}, & c < 0, \\ A, & 0 \leq c < 1, \\ \emptyset, & c \geq 1, \end{cases}$$

and each of $\mathbb{R}$, A , $\emptyset$ is Lebesgue measurable, A by the choice of A . By Proposition 1 of Chapter 3, f is Lebesgue measurable.

The function f is not measurable with respect to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Indeed, take $c = 1/2$. Then

$$\{x \in \mathbb{R} \mid f(x) > 1/2\} = A,$$

and A is not a Borel set. Since measurability with respect to $\mathcal{B}(\mathbb{R})$ would force this set to belong to $\mathcal{B}(\mathbb{R})$, by Proposition 1 the function f fails to be Borel measurable.

Two remarks illuminate what is really going on. First, the discrepancy is exactly a failure of completeness: the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ is not complete, since the set A above is a subset of the Cantor set, which is a Borel set of measure zero, yet $A \notin \mathcal{B}(\mathbb{R})$. The Lebesgue σ -algebra $\mathcal{L}$ is the completion of $\mathcal{B}(\mathbb{R})$ with respect to m .

Second, in our example A has measure zero, so $f = \chi_A$ vanishes outside a set of measure zero, that is, $f = 0$ a.e. on $\mathbb{R}$, and the zero function certainly is Borel measurable. This is an instance of the general fact: a real-valued Lebesgue measurable function on $\mathbb{R}$ always agrees almost everywhere with a Borel measurable function, but need not itself be Borel measurable. The hypothesis of completeness in Proposition 3 is precisely what fails for $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, and indeed this example shows why that hypothesis cannot be dropped there.

Problem 18.10. Check that the proofs of Proposition 1 and Theorem 4 follow from the proofs of the corresponding results in the case of Lebesgue measure on the real line.

Solution. The point of the exercise is that the proofs given on pages 54 and 56–57 for Lebesgue measurable functions of a real variable use nothing about Lebesgue measure beyond two facts: the collection of Lebesgue measurable sets is a σ -algebra, and the rational numbers are a countable dense subset of $\mathbb{R}$. Neither fact is special to Lebesgue measure. We transcribe the two proofs in the general setting, indicating at each step exactly which property is invoked. Throughout, $(X, \mathcal{M})$ is a measurable space, so that $\mathcal{M}$ contains X and is closed with respect to the formation of complements in X and countable unions, and hence, by De Morgan's Identities, with respect to countable intersections as well. Proof of Proposition 1. Let f be an extended real-valued function on X . We show the equivalence of

- (i) for each real c , $\{x \in X \mid f(x) < c\}$ is measurable;
- (ii) for each real c , $\{x \in X \mid f(x) \leq c\}$ is measurable;
- (iii) for each real c , $\{x \in X \mid f(x) > c\}$ is measurable;
- (iv) for each real c , $\{x \in X \mid f(x) \geq c\}$ is measurable.

Fix a real number c . The sets in (i) and (iv) are complementary in X :

$$\{x \in X \mid f(x) \geq c\} = X \setminus \{x \in X \mid f(x) < c\},$$

and likewise the sets in (ii) and (iii) are complementary in X . Since $\mathcal{M}$ is closed with respect to the formation of complements in X , (i) and (iv) are equivalent, as are (ii) and (iii). This is the same argument as in Chapter 3; the only property used is closure under complements.

Now (iii) implies (iv), since

$$\{x \in X \mid f(x) \geq c\} = \bigcap_{k=1}^{\infty} \{x \in X \mid f(x) > c - 1/k\},$$

an equality of sets that is a statement about the ordering of the extended real numbers and involves no measure whatsoever; the right-hand side is measurable because $\mathcal{M}$ is closed with respect to countable

intersections. Similarly, (iv) implies (iii), since

$$\{x \in X \mid f(x) > c\} = \bigcup_{k=1}^{\infty} \{x \in X \mid f(x) \geq c + 1/k\},$$

and $\mathcal{M}$ is closed with respect to countable unions. Thus statements (i)–(iv) are equivalent.

Assume now that one, and hence all, of them hold, and let c be an extended real number. If c is finite, then

$$\{x \in X \mid f(x) = c\} = \{x \in X \mid f(x) \leq c\} \cap \{x \in X \mid f(x) \geq c\}$$

is the intersection of two measurable sets and hence measurable. If $c = \infty$, then

$$\{x \in X \mid f(x) = \infty\} = \bigcap_{k=1}^{\infty} \{x \in X \mid f(x) > k\}$$

is measurable, being a countable intersection of measurable sets; the case $c = -\infty$ is handled by the analogous identity with $\{f < -k\}$. This completes the proof, and it is word for word the Chapter 3 proof with the phrase “measurable subset of E ” replaced by “member of $\mathcal{M}$ ”.

Proof of Theorem 4. Let f and g be measurable real-valued functions on X .

Linearity. Let α be a real number. If $\alpha = 0$, then αf is identically 0, and for each real c the set $\{x \in X \mid 0 > c\}$ equals X if $c < 0$ and $\emptyset$ if $c \geq 0$; both belong to $\mathcal{M}$. If $\alpha \neq 0$, then for each real c ,

$$\{x \in X \mid \alpha f(x) > c\} = \begin{cases} \{x \in X \mid f(x) > c/\alpha\}, & \alpha > 0, \\ \{x \in X \mid f(x) < c/\alpha\}, & \alpha < 0, \end{cases}$$

which is measurable by Proposition 1. Hence αf is measurable, and to establish linearity it suffices to treat $f + g$.

Fix a real number c . For $x \in X$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$, and so, by the density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational q with $f(x) < q < c - g(x)$. Conversely, if such a rational exists, then $f(x) + g(x) < c$. Hence

$$\{x \in X \mid f(x) + g(x) < c\} = \bigcup_{q \in \mathbb{Q}} \left[\{x \in X \mid f(x) < q\} \cap \{x \in X \mid g(x) < c - q\} \right].$$

Each set in square brackets is measurable by Proposition 1, and $\mathbb{Q}$ is countable, so the union is a countable union of measurable sets and hence measurable. Again the only inputs are Proposition 1 (already proved above in the general setting), the countability and density of $\mathbb{Q}$, and closure of $\mathcal{M}$ under countable unions and finite intersections. Therefore $f + g$ is measurable, and $\alpha f + \beta g$ is measurable for all real α, β .

Products. First observe that if h is a measurable real-valued function on X then h^2 is measurable: for $c \geq 0$,

$$\{x \in X \mid h^2(x) > c\} = \{x \in X \mid h(x) > \sqrt{c}\} \cup \{x \in X \mid h(x) < -\sqrt{c}\},$$

a union of two measurable sets, while for $c < 0$ the set $\{x \in X \mid h^2(x) > c\}$ equals X . Now use the algebraic identity, valid pointwise since f and g are real-valued,

$$f \cdot g = \frac{1}{2}[(f + g)^2 - f^2 - g^2].$$

By linearity, $f + g$ is measurable; by the preceding observation, $(f + g)^2$, f^2 and g^2 are measurable; by linearity again, the right-hand side is measurable. Hence $f \cdot g$ is measurable.

Maximum and minimum. For each real c ,

$$\{x \in X \mid \max\{f, g\}(x) > c\} = \{x \in X \mid f(x) > c\} \cup \{x \in X \mid g(x) > c\}$$

and

$$\{x \in X \mid \min\{f, g\}(x) > c\} = \{x \in X \mid f(x) > c\} \cap \{x \in X \mid g(x) > c\},$$

both measurable since $\mathcal{M}$ is an algebra. By Proposition 1, $\max\{f, g\}$ and $\min\{f, g\}$ are measurable. In every step above the only structural facts used were that $\mathcal{M}$ is a σ -algebra of subsets of X and that $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. In the Chapter 3 arguments the role of $\mathcal{M}$ was played by the σ -algebra of Lebesgue measurable subsets of the domain E ; no property of Lebesgue outer measure, of intervals, or of the topology of $\mathbb{R}$ entered. Therefore the Chapter 3 proofs are, verbatim, proofs of Proposition 1 and Theorem 4 of this chapter.

Problem 18.11. Complete the proof of the Simple Approximation Lemma. [The Simple Approximation Lemma. Let $(X, \mathcal{M})$ be a measurable space and f a measurable function on X that is bounded on X , that is, there is an $M \geq 0$ for which $|f| \leq M$ on X . Then for each $\epsilon > 0$, there are simple functions φ_ϵ and ψ_ϵ defined on X that have the following approximation properties: $\varphi_\epsilon \leq f \leq \psi_\epsilon$ and $0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon$ on X .]

Solution. Let $\epsilon > 0$ be given. We supply in full the details that the text leaves to the reader: the choice of the enveloping interval, the measurability of the sets X_k , the fact that $\{X_k\}_{k=1}^n$ is a finite disjoint measurable cover of X so that φ_ϵ and ψ_ϵ are genuinely simple functions whose values at a point are given by the asserted formulas, and the final verification of the two approximation properties.

Step 1: choice of a bounded interval containing $f(X)$, with room to spare. The function f is a priori extended real-valued, but the hypothesis $|f| \leq M$ on X forces it to be real-valued, and $f(X) \subseteq [-M, M]$. Set

$$c = -M - 1, \quad d = M + 1.$$

Then $[c, d]$ is a closed, bounded interval and, what will matter, $f(X) \subseteq [-M, M] \subseteq [c, d]$, so that no point of X has $f(x) = d$. Choose a natural number n so large that

$$\frac{d - c}{n} < \epsilon,$$

which is possible by the Archimedean property of $\mathbb{R}$, and define

$$y_k = c + k \cdot \frac{d - c}{n} \quad \text{for } 0 \leq k \leq n.$$

Then $c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$ is a partition of $[c, d]$ with

$$y_k - y_{k-1} = \frac{d - c}{n} < \epsilon \quad \text{for } 1 \leq k \leq n.$$

Step 2: the sets X_k are measurable and partition X . For $1 \leq k \leq n$ define the half-open interval and its inverse image

$$I_k = [y_{k-1}, y_k), \quad X_k = f^{-1}(I_k) = \{x \in X \mid y_{k-1} \leq f(x) < y_k\}.$$

Each X_k is measurable. Indeed,

$$X_k = \{x \in X \mid f(x) \geq y_{k-1}\} \cap \{x \in X \mid f(x) < y_k\},$$

and each of these two sets is measurable by Proposition 1, since f is measurable; $\mathcal{M}$ is closed with respect to finite intersections, so $X_k \in \mathcal{M}$.

The intervals $I_1, \dots, I_n$ are pairwise disjoint, since for $j < k$ every point of I_j is less than $y_j \leq y_{k-1}$ and every point of I_k is at least y_{k-1} . Moreover

$$\bigcup_{k=1}^n I_k = [y_0, y_n) = [c, d).$$

Consequently the sets $X_1, \dots, X_n$ are pairwise disjoint, being inverse images of pairwise disjoint sets, and

$$\bigcup_{k=1}^n X_k = f^{-1}\left(\bigcup_{k=1}^n I_k\right) = f^{-1}([c, d)) = X,$$

the last equality because $f(X) \subseteq [c, d)$ by Step 1. Thus $\{X_k\}_{k=1}^n$ is a finite disjoint collection of measurable sets whose union is X ; equivalently, each $x \in X$ belongs to exactly one X_k , namely the unique k , $1 \leq k \leq n$, for which $y_{k-1} \leq f(x) < y_k$.

Step 3: definition of φ_ϵ and ψ_ϵ and evaluation at a point. Define

$$\varphi_\epsilon = \sum_{k=1}^n y_{k-1} \cdot \chi_{X_k} \quad \text{and} \quad \psi_\epsilon = \sum_{k=1}^n y_k \cdot \chi_{X_k} \quad \text{on } X.$$

Each is, by the definition given in the text, a simple function on X : it is a finite linear combination, with real coefficients, of characteristic functions of measurable sets. (In particular each is a measurable real-valued function on X taking finitely many values, by Theorem 4.)

Let $x \in X$. By Step 2 there is exactly one index k with $x \in X_k$, so $\chi_{X_k}(x) = 1$ and $\chi_{X_j}(x) = 0$ for $j \neq k$. Hence the two sums collapse to a single term:

$$\varphi_\epsilon(x) = y_{k-1}, \quad \psi_\epsilon(x) = y_k.$$

This is the point at which the disjointness and covering established in Step 2 are needed; without them the displayed formulas for $\varphi_\epsilon(x)$ and $\psi_\epsilon(x)$ would be false.

Step 4: verification of the approximation properties. Retain the notation of Step 3: $x \in X$ and k is the unique index with $x \in X_k$. Since $x \in X_k = f^{-1}([y_{k-1}, y_k))$,

$$\varphi_\epsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\epsilon(x),$$

so in particular

$$\varphi_\epsilon(x) \leq f(x) \leq \psi_\epsilon(x).$$

Moreover

$$0 \leq \psi_\epsilon(x) - \varphi_\epsilon(x) = y_k - y_{k-1} = \frac{d - c}{n} < \epsilon.$$

Since $x \in X$ was arbitrary, we have established

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \quad \text{and} \quad 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon \quad \text{on } X,$$

which are exactly the asserted approximation properties. The proof is complete.

Remark. The hypothesis that f be bounded is used twice and cannot be dispensed with: it is what allows $f(X)$ to be enclosed in a bounded interval, and it is what makes the partition finite, so that φ_ϵ and ψ_ϵ take only finitely many values. Note also that measurability of f was used only through Proposition 1, and no measure was involved at any point; the lemma is a statement about the measurable space $(X, \mathcal{M})$ alone.

Problem 18.12. Prove Egoroff's Theorem. Is Egoroff's Theorem true in the absence of the assumption that the limit function is finite a.e.? [Egoroff's Theorem. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions on X that converges pointwise a.e. on X to a function f that is finite a.e. on X . Then for each $\epsilon > 0$, there is a measurable subset X_ϵ of X for which $\{f_n\} \rightarrow f$ uniformly on X_ϵ and $\mu(X \setminus X_\epsilon) < \epsilon$.]

Solution. Preliminary reduction. By hypothesis there is a measurable subset Z_1 of X with $\mu(Z_1) = 0$ such that $\{f_n(x)\}$ converges to $f(x)$ for every $x \in X \setminus Z_1$, and a measurable subset Z_2 of X with $\mu(Z_2) = 0$ such that f is finite on $X \setminus Z_2$. Put $Z = Z_1 \cup Z_2$ and $X' = X \setminus Z$. Then Z is measurable, $\mu(Z) \leq \mu(Z_1) + \mu(Z_2) = 0$ by the countable monotonicity of μ , so $\mu(Z) = 0$, and X' is measurable. On X' the function f is real-valued and $\{f_n\} \rightarrow f$ pointwise on all of X' .

Regard X' as a measure space in its own right, with the σ -algebra $\mathcal{M}'$ of members of $\mathcal{M}$ contained in X' and the restriction of μ ; it is a finite measure space, since $\mu(X') \leq \mu(X) < \infty$. The restriction of each f_n to X' is measurable, and since $\{f_n\} \rightarrow f$ pointwise on all of X' , Theorem 6 tells us that f is measurable on X' , where it is real-valued.

One point of care is needed here, and we dispose of it once and for all. The functions f_n are merely measurable, hence a priori extended real-valued, and nothing in the hypotheses forces them to be finite: pointwise convergence to a finite limit says only that, for each fixed $x \in X'$, $f_n(x)$ is finite for all sufficiently large n , and the set on which some f_n is infinite may well have positive measure. So the difference $f - f_k$, while properly defined on X' as an extended real number (because f is real-valued there), need not be real-valued, and Theorem 4 and Proposition 5, which concern real-valued functions, do not literally apply to it. What we actually need is only the measurability of the sets

$$A_k(\eta) = \{x \in X' \mid |f(x) - f_k(x)| < \eta\} \quad (\eta > 0),$$

where the inequality is read in the extended reals, so that a point at which f_k is infinite does not belong to $A_k(\eta)$. This we verify directly. Since f is real-valued on X' ,

$$A_k(\eta) = \{x \in X' \mid f_k(x) < f(x) + \eta\} \cap \{x \in X' \mid f_k(x) > f(x) - \eta\},$$

and, by the density of the rationals in $\mathbb{R}$,

$$\{x \in X' \mid f_k(x) < f(x) + \eta\} = \bigcup_{q \in \mathbb{Q}} \left[\{x \in X' \mid f_k(x) < q\} \cap \{x \in X' \mid f(x) + \eta > q\} \right],$$

with the analogous identity for the second set. Each set on the right is measurable by Proposition 1, applied to the measurable functions f_k and f on X' , and $\mathbb{Q}$ is countable; hence $A_k(\eta)$ is measurable. If each f_n happens to be real-valued, this is just the statement that $|f - f_k|$ is a measurable real-valued function, by Theorem 4 and by Proposition 5 applied to the continuous function $\varphi(t) = |t|$.

A lemma. Under the above assumptions, for each $\eta > 0$ and $\delta > 0$ there is a measurable subset A of X' and an index N for which

$$|f_n - f| < \eta \text{ on } A \text{ for all } n \geq N, \quad \text{and} \quad \mu(X' \sim A) < \delta.$$

Proof of the lemma. For each natural number n define

$$E_n = \{x \in X' \mid |f(x) - f_k(x)| < \eta \text{ for all } k \geq n\} = \bigcap_{k=n}^{\infty} A_k(\eta).$$

Each $A_k(\eta)$ is measurable, as shown above; therefore E_n , a countable intersection of measurable sets, is measurable. Clearly $E_n \subseteq E_{n+1}$, so $\{E_n\}_{n=1}^{\infty}$ is an ascending sequence of measurable sets. Moreover

$$\bigcup_{n=1}^{\infty} E_n = X',$$

since for $x \in X'$ we have $f_k(x) \rightarrow f(x)$, so there is an index n with $|f(x) - f_k(x)| < \eta$ for all $k \geq n$, that is, $x \in E_n$. By the continuity of measure,

$$\mu(X') = \lim_{n \rightarrow \infty} \mu(E_n).$$

Since $\mu(X') < \infty$, we may choose an index N for which $\mu(E_N) > \mu(X') - \delta$. Set $A = E_N$. Then A is a measurable subset of X' , and by the definition of E_N we have $|f_n - f| < \eta$ on A for all $n \geq N$. Finally, since $A \subseteq X'$ and $\mu(A) \leq \mu(X') < \infty$, the excision property of measure gives

$$\mu(X' \sim A) = \mu(X') - \mu(A) = \mu(X') - \mu(E_N) < \delta.$$

The lemma is proved.

Proof of Egoroff's Theorem. Let $\epsilon > 0$. For each natural number n , apply the lemma with $\eta = 1/n$ and $\delta = \epsilon/2^n$ to obtain a measurable subset A_n of X' and an index $N(n)$ for which

$$\mu(X' \sim A_n) < \epsilon/2^n$$

and

$$|f_k - f| < 1/n \text{ on } A_n \text{ for all } k \geq N(n).$$

Define

$$X_\epsilon = \bigcap_{n=1}^{\infty} A_n,$$

a measurable subset of X' , hence of X .

First we estimate the measure of the complement. By De Morgan's Identities and the countable monotonicity of μ ,

$$\mu(X' \sim X_\epsilon) = \mu\left(\bigcup_{n=1}^{\infty} [X' \sim A_n]\right) \leq \sum_{n=1}^{\infty} \mu(X' \sim A_n) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Since $X \sim X_\epsilon = Z \cup (X' \sim X_\epsilon)$ and $\mu(Z) = 0$, countable monotonicity gives

$$\mu(X \sim X_\epsilon) \leq \mu(Z) + \mu(X' \sim X_\epsilon) < \epsilon.$$

Next we claim that $\{f_n\}$ converges to f uniformly on X_ϵ . Let $\epsilon' > 0$. Choose an index n_0 for which $1/n_0 < \epsilon'$. Since $X_\epsilon \subseteq A_{n_0}$, the choice of A_{n_0} gives

$$|f_k - f| < 1/n_0 < \epsilon' \text{ on } X_\epsilon \text{ for all } k \geq N(n_0).$$

As $\epsilon' > 0$ was arbitrary and the index $N(n_0)$ does not depend on the point of X_ϵ , the convergence is uniform on X_ϵ . This proves Egoroff's Theorem.

Observe that the proof is the one on page 65 with two changes: the appeal to countable subadditivity, continuity and excision for Lebesgue measure is replaced by the corresponding properties of a general measure, established in Chapter 17; and the final refinement to a closed set, which used Theorem 11 of Chapter 2, is omitted, since a general measurable space carries no topology.

The finiteness assumption on f . The answer is no: without the assumption that f be finite a.e., the conclusion fails. Recall what the conclusion asserts: for each $\epsilon' > 0$ there is an index N for which $|f_n - f| < \epsilon'$ on X_ϵ for all $n \geq N$. At a point x at which $f(x)$ is infinite, the difference $f_n(x) - f(x)$ is either undefined, when $f_n(x)$ is infinite of the same sign as $f(x)$, or else equal to $\mp\infty$, so that $|f_n(x) - f(x)| = \infty$; in neither case can the inequality $|f_n(x) - f(x)| < \epsilon'$ hold. Consequently no set on which f is somewhere infinite can be a set of uniform convergence.

Here is an explicit example. Let $X = [0, 1]$, let $\mathcal{M}$ be the σ -algebra of Lebesgue measurable subsets of $[0, 1]$ and $\mu = m$ Lebesgue measure, so that $\mu(X) = 1 < \infty$. Define

$$f_n \equiv n \text{ on } X \quad \text{for each } n, \quad f \equiv \infty \text{ on } X.$$

Each f_n is a constant, hence measurable, and $\{f_n(x)\} \rightarrow \infty = f(x)$ for every $x \in X$, so $\{f_n\}$ converges pointwise on all of X to f . All the hypotheses of Egoroff's Theorem hold except that f is nowhere finite. Take $\epsilon = 1/2$. If A were a measurable subset of X with $\mu(X \sim A) < 1/2$, then, since $\mu(X) = 1$ and μ is finitely additive, $\mu(A) > 1/2 > 0$, so $A \neq \emptyset$; and at each $x \in A$ we have $f_n(x) - f(x) = n - \infty = -\infty$, so $|f_n(x) - f(x)| = \infty$ for every n . Hence there is no index N for which $|f_n - f| < 1$ on A for all $n \geq N$, so $\{f_n\}$ does not converge to f uniformly on A . Since A was an arbitrary measurable set with $\mu(X \sim A) < 1/2$, no set X_ϵ as in the conclusion exists.

It should be said exactly how much this example proves. It shows that the conclusion of the theorem breaks down, because uniform convergence, phrased as it is throughout the text in terms of the size of $f_n - f$, can never hold on a set at any point of which f is infinite. It does not show that a suitably reformulated conclusion fails. Extend $\arctan$ to the extended reals by setting $\arctan(\pm\infty) = \pm\pi/2$, and measure the distance between extended real numbers s and t by $\rho(s, t) = |\arctan s - \arctan t|$; this is a metric on the extended reals, under which they form a space homeomorphic to $[-\pi/2, \pi/2]$ and which induces the usual topology on $\mathbb{R}$. With uniform convergence understood in the sense of ρ , Egoroff's Theorem does survive without any finiteness assumption. For an extended real-valued measurable function g on X the composition $\arctan g$ is a real-valued measurable function, since for

each real c the set $\{x \in X \mid \arctan g(x) < c\}$ is: empty if $c \leq -\pi/2$; equal to $\{x \in X \mid g(x) < \tan c\}$ if $-\pi/2 < c < \pi/2$; equal to $X \sim \{x \in X \mid g(x) = \infty\}$ if $c = \pi/2$; and equal to X if $c > \pi/2$. Each of these sets is measurable by Proposition 1, whose final assertion is exactly that $\{g = \infty\}$ is measurable. Since $\arctan$ is a strictly increasing homeomorphism of the extended reals onto $[-\pi/2, \pi/2]$, $\{\arctan f_n\} \rightarrow \arctan f$ pointwise a.e. on X , and $\arctan f$ is real-valued. Hence the theorem as proved above applies to the sequence $\{\arctan f_n\}$ and yields, for each $\epsilon > 0$, a measurable X_ϵ with $\mu(X \sim X_\epsilon) < \epsilon$ on which $\{\arctan f_n\} \rightarrow \arctan f$ uniformly. Thus the finiteness assumption cannot be dropped from Egoroff's Theorem as stated, and it is exactly the price paid for phrasing uniform convergence in terms of $|f_n - f|$ rather than in terms of a metric on the extended reals.

Problem 18.13. Let $\{f_n\}$ be a sequence of real-valued measurable functions on X such that, for each natural number n , $\mu\{x \in X \mid |f_n(x) - f_{n+1}(x)| > 1/2^n\} < 1/2^n$. Show that $\{f_n\}$ is pointwise convergent a.e. on X . (Hint: Use the Borel-Cantelli Lemma.)

Solution. For each natural number n define

$$E_n = \left\{ x \in X \mid |f_n(x) - f_{n+1}(x)| > \frac{1}{2^n} \right\}.$$

Step 1: each E_n is measurable. The functions f_n and f_{n+1} are real-valued and measurable on X , so by Theorem 4 (Linearity) the function $f_n - f_{n+1}$ is measurable on X . Since the function $\varphi(t) = |t|$ is continuous on $\mathbb{R}$, Proposition 5 tells us that $|f_n - f_{n+1}| = \varphi \circ (f_n - f_{n+1})$ is measurable on X . Hence, by Proposition 1, the set $E_n = \{x \in X \mid |f_n(x) - f_{n+1}(x)| > 1/2^n\}$ is measurable. In particular the hypothesis $\mu(E_n) < 1/2^n$ makes sense.

Step 2: apply the Borel-Cantelli Lemma. By hypothesis,

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

Therefore, by the Borel-Cantelli Lemma of Chapter 17, almost all x in X belong to at most a finite number of the E_n 's. Explicitly, let

$$X_0 = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

be the set of points belonging to infinitely many of the E_n 's. Since $\mathcal{M}$ is a σ -algebra, X_0 is measurable, and the Borel-Cantelli Lemma asserts precisely that $\mu(X_0) = 0$. (For completeness, here is the one-line argument: for each n , countable monotonicity of μ gives $\mu(\bigcup_{k \geq n} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k)$, and since the series $\sum \mu(E_n)$ converges, the tails tend to 0; as $X_0 \subseteq \bigcup_{k \geq n} E_k$ for every n , monotonicity gives $\mu(X_0) \leq \sum_{k=n}^{\infty} \mu(E_k)$ for every n , so $\mu(X_0) = 0$.)

Step 3: convergence off X_0 . Let $x \in X \sim X_0$. Since x belongs to only finitely many of the sets E_n , there is a natural number $N = N(x)$ such that $x \notin E_n$ for every $n \geq N$; that is,

$$|f_n(x) - f_{n+1}(x)| \leq \frac{1}{2^n} \quad \text{for all } n \geq N.$$

We claim the sequence of real numbers $\{f_n(x)\}$ is Cauchy. Let $m > n \geq N$. By the triangle inequality applied to the telescoping sum $f_m(x) - f_n(x) = \sum_{j=n}^{m-1} [f_{j+1}(x) - f_j(x)]$,

$$|f_m(x) - f_n(x)| \leq \sum_{j=n}^{m-1} |f_{j+1}(x) - f_j(x)| \leq \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}}.$$

Note that every term in the telescoping sum has index $j \geq n \geq N$, so the bound $|f_{j+1}(x) - f_j(x)| \leq 1/2^j$ is available for each of them. Given $\eta > 0$, choose $n_0 \geq N$ with $1/2^{n_0-1} < \eta$; then $|f_m(x) - f_n(x)| < \eta$ whenever $m, n \geq n_0$ (the case $m = n$ being trivial and the case $m < n$ following by symmetry). Hence $\{f_n(x)\}$ is a Cauchy sequence of real numbers, and by the completeness of $\mathbb{R}$ it converges to a real number, which we call $f(x)$.

Step 4: conclusion. We have shown that $\{f_n(x)\}$ converges for every $x \in X \sim X_0$, where X_0 is measurable with $\mu(X_0) = 0$. Therefore $\{f_n\}$ is pointwise convergent almost everywhere on X . Defining $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for $x \in X \sim X_0$ and, say, $f(x) = 0$ for $x \in X_0$, we obtain a real-valued function f on X to which $\{f_n\}$ converges pointwise a.e. on X . This f is moreover measurable on X , and no completeness assumption is needed for that: its restriction to the measurable set $X \sim X_0$ is measurable by Theorem 6, since on $X \sim X_0$, regarded as a measure space in its own right, the convergence is pointwise at every point; its restriction to X_0 is constant and hence measurable; and, as noted in the text on page 360, an extended real-valued function on X is measurable if and only if its restrictions to a measurable set and to its complement are both measurable. (Had we not specified the values of f on X_0 , we could only conclude that f is measurable when $(X, \mathcal{M}, \mu)$ is complete, by Proposition 3.) Remark. The estimate in Step 3 in fact gives more than pointwise convergence: for $x \in X \sim X_0$ and $n \geq N(x)$ we have $|f(x) - f_n(x)| \leq 1/2^{n-1}$, obtained by letting $m \rightarrow \infty$ in the displayed inequality. The failure of uniformity lies entirely in the dependence of $N(x)$ on x .

Problem 18.14. Under the assumptions of Egoroff's Theorem, show that $X = \bigcup_{k=0}^{\infty} X_k$, where each X_k is measurable, $\mu(X_0) = 0$ and, for $k \geq 1$, $\{f_n\}$ converges uniformly to f on X_k .

Solution. Throughout, $(X, \mathcal{M}, \mu)$ is a finite measure space and $\{f_n\}$ is a sequence of measurable functions on X that converges pointwise a.e. on X to a function f that is finite a.e. on X . These are exactly the assumptions of Egoroff's Theorem, which we are therefore free to apply. Construction of the sets. Let k be a natural number. Apply Egoroff's Theorem with $\epsilon = 1/k$: there is a measurable subset X_k of X for which

$$\{f_n\} \rightarrow f \text{ uniformly on } X_k \quad \text{and} \quad \mu(X \sim X_k) < \frac{1}{k}.$$

(Since uniform convergence on X_k is asserted in terms of the differences $f_n - f$, in particular f is real-valued on X_k .) Having chosen X_k for each $k \geq 1$, define

$$X_0 = X \sim \bigcup_{k=1}^{\infty} X_k.$$

Each X_k is measurable. For $k \geq 1$ this is part of the conclusion of Egoroff's Theorem. For $k = 0$, note that $\bigcup_{k \geq 1} X_k$ is a countable union of measurable sets, hence measurable, and $\mathcal{M}$ is closed with respect to the formation of complements in X ; therefore $X_0 \in \mathcal{M}$.

The union is all of X . By De Morgan's Identities,

$$\bigcup_{k=0}^{\infty} X_k = X_0 \cup \bigcup_{k=1}^{\infty} X_k = \left(X \sim \bigcup_{k=1}^{\infty} X_k \right) \cup \bigcup_{k=1}^{\infty} X_k = X,$$

since a set and its complement in X have union X .

X_0 has measure zero. Fix a natural number k . Since $X_k \subseteq \bigcup_{j \geq 1} X_j$, taking complements in X gives

$$X_0 = X \sim \bigcup_{j=1}^{\infty} X_j \subseteq X \sim X_k.$$

By the monotonicity of μ ,

$$\mu(X_0) \leq \mu(X \sim X_k) < \frac{1}{k}.$$

This holds for every natural number k , and $\mu(X_0) \geq 0$; letting $k \rightarrow \infty$ we conclude $\mu(X_0) = 0$.

Uniform convergence on X_k for $k \geq 1$. This is exactly the first conclusion of Egoroff's Theorem for the choice $\epsilon = 1/k$, recorded above.

Thus $X = \bigcup_{k=0}^{\infty} X_k$ with each X_k measurable, $\mu(X_0) = 0$, and $\{f_n\} \rightarrow f$ uniformly on X_k for each $k \geq 1$, as required.

Two refinements are worth recording, since they cost nothing.

First, the sets X_k , $k \geq 1$, may be taken to be ascending. To see this, we use the elementary fact that uniform convergence is preserved by finite unions: if $\{f_n\} \rightarrow f$ uniformly on A and on B , then given $\epsilon > 0$ there are indices N_A and N_B with $|f_n - f| < \epsilon$ on A for $n \geq N_A$ and $|f_n - f| < \epsilon$ on B for $n \geq N_B$; taking $N = \max\{N_A, N_B\}$ we get $|f_n - f| < \epsilon$ on $A \cup B$ for $n \geq N$, so the convergence is uniform on $A \cup B$. Consequently, replacing X_k by $\tilde{X}_k = \bigcup_{j=1}^k X_j$ for $k \geq 1$ yields an ascending sequence of measurable sets on each of which the convergence is uniform, with $X \sim \tilde{X}_k \subseteq X \sim X_k$ so that $\mu(X \sim \tilde{X}_k) < 1/k$; and $\bigcup_{k \geq 1} \tilde{X}_k = \bigcup_{k \geq 1} X_k$, so the set X_0 is unchanged.

Second, the collection $\{X_k\}_{k=0}^\infty$ may be taken to be disjoint. Starting from the ascending sets $\tilde{X}_k$ of the preceding paragraph, put $Y_1 = \tilde{X}_1$ and $Y_k = \tilde{X}_k \sim \tilde{X}_{k-1}$ for $k \geq 2$. These are measurable and pairwise disjoint, $Y_k \subseteq \tilde{X}_k$ so that $\{f_n\} \rightarrow f$ uniformly on Y_k (uniform convergence on a set is inherited by every subset of it), $\bigcup_{k \geq 1} Y_k = \bigcup_{k \geq 1} \tilde{X}_k$, and $Y_0 = X_0$ is disjoint from all of them by its definition. Finally, observe how this reformulation compares with Egoroff's Theorem itself. Egoroff's Theorem says the exceptional set can be made of arbitrarily small measure; the present statement says that the exceptional set can be made of measure zero, at the price of exhausting X by countably many sets of uniform convergence rather than by a single one. One cannot in general do better and take a single set of uniform convergence whose complement is null. On $X = [0, 1]$ with Lebesgue measure, let $f_n = \chi_{[0, 1/n]}$. For $x = 0$ we have $f_n(0) = 1$ for all n , while for $0 < x \leq 1$ we have $f_n(x) = 0$ once $n > 1/x$; so $\{f_n\}$ converges pointwise on all of $[0, 1]$ to $f = \chi_{\{0\}}$, and all the hypotheses of Egoroff's Theorem hold. Let A be any measurable subset of $[0, 1]$ with $m([0, 1] \sim A) = 0$. Then for each n the set $A \cap (0, 1/n]$ has measure $1/n > 0$ by excision, and so is nonempty; choosing x_n in it we get $|f_n(x_n) - f(x_n)| = |1 - 0| = 1$. Hence $\sup_A |f_n - f| = 1$ for every n and the convergence is not uniform on A . Thus the null set X_0 in the conclusion cannot in general be dispensed with, nor can the countable collection $\{X_k\}_{k \geq 1}$ be replaced by a single set.

18.3 Problems 18.15–18.21

Problem 18.15. A sequence $\{f_n\}$ of measurable real-valued functions on X is said to **converge in measure** to a measurable function f provided that for each $\eta > 0$,

$$\lim_{n \rightarrow \infty} \mu\{x \in X \mid |f_n(x) - f(x)| > \eta\} = 0.$$

A sequence $\{f_n\}$ of measurable functions is said to be **Cauchy in measure** provided that for each $\epsilon > 0$ and $\eta > 0$, there is an index N such that for each $m, n \geq N$,

$$\mu\{x \in X \mid |f_n(x) - f_m(x)| > \eta\} < \epsilon.$$

- (i) Show that if $\mu(X) < \infty$ and $\{f_n\}$ converges pointwise a.e. on X to a measurable function f , then $\{f_n\}$ converges to f in measure. (Hint: Use Egoroff's Theorem.)
- (ii) Show that if $\{f_n\}$ converges to f in measure, then there is a subsequence of $\{f_n\}$ that converges pointwise a.e. on X to f . (Hint: Use the Borel-Cantelli Lemma.)
- (iii) Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f to which $\{f_n\}$ converges in measure.

Solution. Throughout, for $\eta > 0$ write

$$E_n(\eta) = \{x \in X \mid |f_n(x) - f(x)| > \eta\},$$

a measurable set. Indeed, each f_n is real-valued, so the difference $f_n - f$ nowhere involves an expression of the form $\infty - \infty$; it is therefore defined at every point of X and is measurable there, even if the measurable function f should take an infinite value at some points. Hence $|f_n - f|$ is measurable and $E_n(\eta)$ belongs to $\mathcal{M}$.

(i) Assume $\mu(X) < \infty$ and $\{f_n\} \rightarrow f$ pointwise a.e. on X , where f is measurable. At almost every $x \in X$ the real numbers $f_n(x)$ converge to $f(x)$, so $f(x)$ is finite for almost every x ; that is, f is finite

a.e. on X . Thus $(X, \mathcal{M}, \mu)$ is a finite measure space, $\{f_n\}$ is a sequence of measurable real-valued functions converging pointwise a.e. to f , and f is finite a.e., which are precisely the hypotheses of Egoroff's Theorem.

Let $\eta > 0$ and $\epsilon > 0$. By Egoroff's Theorem there is a measurable subset X_ϵ of X for which $\{f_n\} \rightarrow f$ uniformly on X_ϵ and $\mu(X \sim X_\epsilon) < \epsilon$. By the uniformity of the convergence on X_ϵ , there is an index N such that for all $n \geq N$,

$$|f_n(x) - f(x)| \leq \eta \quad \text{for all } x \in X_\epsilon.$$

Consequently, for $n \geq N$ the set $E_n(\eta)$ contains no point of X_ϵ , that is, $E_n(\eta) \subseteq X \sim X_\epsilon$. By the monotonicity of μ ,

$$\mu(E_n(\eta)) \leq \mu(X \sim X_\epsilon) < \epsilon \quad \text{for all } n \geq N.$$

Therefore $\limsup_{n \rightarrow \infty} \mu(E_n(\eta)) \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, $\lim_{n \rightarrow \infty} \mu(E_n(\eta)) = 0$. As $\eta > 0$ was arbitrary, $\{f_n\} \rightarrow f$ in measure.

(ii) Assume $\{f_n\} \rightarrow f$ in measure. Applying the definition of convergence in measure with $\eta = \epsilon = 1/2^k$, we may choose inductively a strictly increasing sequence of indices $n_1 < n_2 < \dots$ such that, setting

$$E_k = \{x \in X \mid |f_{n_k}(x) - f(x)| > 1/2^k\},$$

we have $\mu(E_k) < 1/2^k$ for every natural number k . Such a choice is possible: for fixed k , since $\lim_{n \rightarrow \infty} \mu\{|f_n - f| > 1/2^k\} = 0$, there is an index N_k for which $\mu\{|f_n - f| > 1/2^k\} < 1/2^k$ whenever $n \geq N_k$; take $n_1 = N_1$ and $n_k = \max\{N_k, n_{k-1} + 1\}$ for $k > 1$. Then

$$\sum_{k=1}^{\infty} \mu(E_k) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty,$$

so the Borel-Cantelli Lemma tells us that almost all $x \in X$ belong to at most finitely many of the E_k 's; equivalently, the measurable set

$$E = \bigcap_{m=1}^{\infty} \bigcup_{k \geq m} E_k$$

has $\mu(E) = 0$.

Let $x \in X \sim E$. Then there is an index $K = K(x)$ for which $x \notin E_k$ whenever $k \geq K$, that is,

$$|f_{n_k}(x) - f(x)| \leq \frac{1}{2^k} \quad \text{for all } k \geq K.$$

Since $1/2^k \rightarrow 0$, we conclude that $\lim_{k \rightarrow \infty} f_{n_k}(x) = f(x)$. Thus the subsequence $\{f_{n_k}\}$ converges to f pointwise on $X \sim E$, a set whose complement has measure zero: $\{f_{n_k}\} \rightarrow f$ pointwise a.e. on X .

(iii) Assume $\{f_n\}$ is Cauchy in measure. Apply the definition with $\epsilon = \eta = 1/2^k$ to choose, inductively, a strictly increasing sequence of indices $n_1 < n_2 < \dots$ with the property that

$$\mu\{x \in X \mid |f_n(x) - f_m(x)| > 1/2^k\} < \frac{1}{2^k} \quad \text{whenever } m, n \geq n_k.$$

Write $g_k = f_{n_k}$ and

$$E_k = \{x \in X \mid |g_{k+1}(x) - g_k(x)| > 1/2^k\}.$$

(Such a choice is possible: the definition of Cauchy in measure, applied with $\epsilon = \eta = 1/2^k$, provides an index N_k with this property for all $m, n \geq N_k$; take $n_1 = N_1$ and, inductively, $n_k = \max\{N_k, n_{k-1} + 1\}$.) Since the two indices n_k and n_{k+1} are both at least n_k , we have $\mu(E_k) < 1/2^k$, and hence $\sum_{k=1}^{\infty} \mu(E_k) < \infty$.

For each natural number m define $F_m = \bigcup_{k \geq m} E_k$. By the countable subadditivity of μ ,

$$\mu(F_m) \leq \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}.$$

Let $F = \bigcap_{m=1}^{\infty} F_m$; then $\mu(F) \leq \mu(F_m) \leq 2^{-(m-1)}$ for every m , so $\mu(F) = 0$. (This is exactly the conclusion of the Borel-Cantelli Lemma.) Put $X_0 = X \sim F$, a measurable set with $\mu(X \sim X_0) = 0$.

The limit function. Fix m and let $x \in X \sim F_m$. Then $x \notin E_k$ for every $k \geq m$, so $|g_{k+1}(x) - g_k(x)| \leq 1/2^k$ for all $k \geq m$, and therefore, for $j > k \geq m$,

$$|g_j(x) - g_k(x)| \leq \sum_{i=k}^{j-1} |g_{i+1}(x) - g_i(x)| \leq \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k-1}}. \quad (*)$$

In particular $\{g_k(x)\}$ is a Cauchy sequence of real numbers and so converges. Since $X_0 = \bigcup_{m=1}^{\infty} (X \sim F_m)$, the sequence $\{g_k(x)\}$ converges for every $x \in X_0$.

Define

$$f = \limsup_{k \rightarrow \infty} (g_k \cdot \chi_{X_0}) \quad \text{on } X.$$

Each $g_k \cdot \chi_{X_0}$ is measurable, and the upper limit of a sequence of measurable functions is measurable, so f is measurable. Moreover f is real-valued: on X_0 it equals $\lim_{k \rightarrow \infty} g_k$, which is a real number, and on $X \sim X_0$ it equals 0.

Convergence in measure of the subsequence. Fix m and let $x \in X \sim F_m$. Then $x \in X_0$, and letting $j \rightarrow \infty$ in (*) with $k = m$ gives

$$|f(x) - g_m(x)| \leq \frac{1}{2^{m-1}}.$$

Hence

$$\left\{x \in X \mid |f(x) - g_m(x)| > \frac{1}{2^{m-1}}\right\} \subseteq F_m, \quad \text{so} \quad \mu\left\{|f - g_m| > \frac{1}{2^{m-1}}\right\} \leq \frac{1}{2^{m-1}}.$$

Given $\eta > 0$ and $\epsilon > 0$, choose an index M so large that $2^{-(M-1)} < \min\{\eta, \epsilon\}$. Then for every $m \geq M$ we have $2^{-(m-1)} \leq 2^{-(M-1)} < \min\{\eta, \epsilon\}$, so

$$\mu\{|f - g_m| > \eta\} \leq \mu\left\{|f - g_m| > \frac{1}{2^{m-1}}\right\} \leq \frac{1}{2^{m-1}} < \epsilon.$$

Since $\eta > 0$ and $\epsilon > 0$ were arbitrary, $\lim_{m \rightarrow \infty} \mu\{|f - g_m| > \eta\} = 0$ for every $\eta > 0$; that is, $\{g_k\} \rightarrow f$ in measure.

Convergence in measure of the whole sequence. Let $\eta > 0$ and $\epsilon > 0$. Since $\{f_n\}$ is Cauchy in measure, there is an index N such that

$$\mu\{|f_n - f_m| > \eta/2\} < \epsilon/2 \quad \text{for all } m, n \geq N.$$

By the preceding paragraph choose an index k with $n_k \geq N$ and $\mu\{|g_k - f| > \eta/2\} < \epsilon/2$. For $n \geq N$, the triangle inequality gives the inclusion

$$\{|f_n - f| > \eta\} \subseteq \{|f_n - g_k| > \eta/2\} \cup \{|g_k - f| > \eta/2\},$$

since if both $|f_n(x) - g_k(x)| \leq \eta/2$ and $|g_k(x) - f(x)| \leq \eta/2$ then $|f_n(x) - f(x)| \leq \eta$. Therefore, by the monotonicity and finite subadditivity of μ , for all $n \geq N$,

$$\mu\{|f_n - f| > \eta\} \leq \mu\{|f_n - g_k| > \eta/2\} + \mu\{|g_k - f| > \eta/2\} < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\lim_{n \rightarrow \infty} \mu\{|f_n - f| > \eta\} = 0$ for every $\eta > 0$, that is, $\{f_n\} \rightarrow f$ in measure.

Problem 18.16. Assume $\mu(X) < \infty$. Show that $\{f_n\} \rightarrow f$ in measure if and only if each subsequence of $\{f_n\}$ has a further subsequence that converges pointwise a.e. on X to f . Use this to show that for two sequences that converge in measure, the product sequence also converges in measure to the product of the limits.

Solution. As in the preceding problem, the functions f_n are measurable and real-valued and the limit function f is measurable. We may assume that f too is real-valued. Indeed, if $\{f_n\} \rightarrow f$ in measure and $S = \{x \in X \mid |f(x)| = \infty\}$, then $|f_n - f| = \infty > \eta$ at every point of S , so $\mu(S) \leq \mu\{|f_n - f| > \eta\}$

for every n and hence $\mu(S) = 0$; if instead some subsequence converges to f pointwise a.e., then again f is finite a.e. In either case, replacing f by the measurable real-valued function $f \cdot \chi_{X \sim S}$ alters neither the assertion that $\{f_n\} \rightarrow f$ in measure nor the assertion that some subsequence of $\{f_n\}$ converges to f pointwise a.e. on X , since the two versions of f differ only on the set S of measure zero. With this convention every difference and every product written below is defined at every point of X and is measurable.

The criterion. Suppose first that $\{f_n\} \rightarrow f$ in measure, and let $\{f_{n_k}\}$ be a subsequence. Fix $\eta > 0$. The sequence of real numbers $\{\mu\{|f_{n_k} - f| > \eta\}\}_k$ is a subsequence of the sequence $\{\mu\{|f_n - f| > \eta\}\}_n$, which converges to 0; hence it too converges to 0. Since $\eta > 0$ was arbitrary, $\{f_{n_k}\} \rightarrow f$ in measure. By part (ii) of the preceding problem, $\{f_{n_k}\}$ has a further subsequence that converges to f pointwise a.e. on X . (Observe that this half of the equivalence did not use the finiteness of μ .)

Conversely, suppose every subsequence of $\{f_n\}$ has a further subsequence converging pointwise a.e. on X to f , and suppose, to get a contradiction, that $\{f_n\}$ does **not** converge to f in measure. Then there is an $\eta > 0$ for which the sequence of real numbers $\{\mu\{|f_n - f| > \eta\}\}_n$ fails to converge to 0. Hence there is an $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ for which

$$\mu\{x \in X \mid |f_{n_k}(x) - f(x)| > \eta\} \geq \epsilon \quad \text{for every } k. \quad (*)$$

By hypothesis, $\{f_{n_k}\}$ has a further subsequence $\{f_{n_{k_j}}\}$ that converges to f pointwise a.e. on X . Since $\mu(X) < \infty$, part (i) of the preceding problem tells us that $\{f_{n_{k_j}}\} \rightarrow f$ in measure, so

$$\lim_{j \rightarrow \infty} \mu\{x \in X \mid |f_{n_{k_j}}(x) - f(x)| > \eta\} = 0,$$

which contradicts (*). Therefore $\{f_n\} \rightarrow f$ in measure. This proves the equivalence.

Products. Let $\{f_n\} \rightarrow f$ in measure and $\{g_n\} \rightarrow g$ in measure on X , where $\mu(X) < \infty$. We show $\{f_n \cdot g_n\} \rightarrow f \cdot g$ in measure by verifying the criterion just established for the sequence $\{f_n \cdot g_n\}$ and the limit $f \cdot g$.

Let $\{f_{n_k} \cdot g_{n_k}\}$ be an arbitrary subsequence of $\{f_n \cdot g_n\}$. Since $\{f_n\} \rightarrow f$ in measure, so does the subsequence $\{f_{n_k}\}$, and therefore by the criterion (the direction already proved) there is a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ for which

$$\{f_{n_{k_j}}\} \rightarrow f \quad \text{pointwise on } X \sim A, \text{ where } \mu(A) = 0.$$

Now apply the same reasoning to $\{g_{n_{k_j}}\}$: it is a subsequence of the sequence $\{g_n\}$, which converges to g in measure, so there is a further subsequence, indexed by $\{p_i\} \subseteq \{n_{k_j}\}$, for which

$$\{g_{p_i}\} \rightarrow g \quad \text{pointwise on } X \sim B, \text{ where } \mu(B) = 0.$$

Since $\{p_i\}$ is a subsequence of $\{n_{k_j}\}$, we also have $\{f_{p_i}\} \rightarrow f$ pointwise on $X \sim A$.

Let $x \in X \sim (A \cup B)$. Then $f_{p_i}(x) \rightarrow f(x)$ and $g_{p_i}(x) \rightarrow g(x)$ are convergent sequences of real numbers, so by the product rule for limits of real sequences,

$$f_{p_i}(x) \cdot g_{p_i}(x) \rightarrow f(x) \cdot g(x).$$

By the countable (indeed finite) subadditivity of μ , $\mu(A \cup B) = 0$. Therefore $\{f_{p_i} \cdot g_{p_i}\} \rightarrow f \cdot g$ pointwise a.e. on X , and $\{f_{p_i} \cdot g_{p_i}\}$ is a subsequence of the arbitrarily chosen subsequence $\{f_{n_k} \cdot g_{n_k}\}$.

Thus every subsequence of $\{f_n \cdot g_n\}$ has a further subsequence converging pointwise a.e. on X to $f \cdot g$. By the criterion, $\{f_n \cdot g_n\} \rightarrow f \cdot g$ in measure.

Problem 18.17. Prove (7) and (8). Use (8) to prove (9).

Solution. Here $(X, \mathcal{M}, \mu)$ is a measure space, g and h are nonnegative measurable functions on X , X_0

is a measurable subset of X , α is a positive real number, and the assertions to be proved are

$$\int_X \alpha \cdot g \, d\mu = \alpha \cdot \int_X g \, d\mu; \quad (7)$$

$$\text{if } g \leq h \text{ a.e. on } X, \text{ then } \int_X g \, d\mu \leq \int_X h \, d\mu; \quad (8)$$

$$\int_X g \, d\mu = \int_{X_0} g \, d\mu \text{ if } \mu(X \sim X_0) = 0. \quad (9)$$

Recall the definition: for a nonnegative measurable g on X ,

$$\int_X g \, d\mu = \sup \left\{ \int_X \varphi \, d\mu \mid \varphi \text{ simple, } 0 \leq \varphi \leq g \text{ on } X \right\},$$

and that for a measurable subset E of X , $\int_E g \, d\mu$ means $\int_X g \cdot \chi_E \, d\mu$.

Proof of (7). First observe that (7) holds when g is a nonnegative **simple** function φ . Indeed, if $\varphi \equiv 0$ on X then $\alpha \cdot \varphi \equiv 0$ and both sides of (7) are 0. Otherwise, let $c_1, \dots, c_n$ be the distinct positive values taken by φ and $E_k = \{x \in X \mid \varphi(x) = c_k\}$. Since $\alpha > 0$, the numbers $\alpha c_1, \dots, \alpha c_n$ are exactly the distinct positive values taken by $\alpha \cdot \varphi$, and $\{x \mid \alpha \varphi(x) = \alpha c_k\} = E_k$. Therefore, by the defining formula (1),

$$\int_X \alpha \cdot \varphi \, d\mu = \sum_{k=1}^n \alpha c_k \cdot \mu(E_k) = \alpha \sum_{k=1}^n c_k \cdot \mu(E_k) = \alpha \int_X \varphi \, d\mu,$$

where, in accordance with the convention in force, if some $\mu(E_k) = \infty$ then both the second and the fourth expressions equal ∞ .

Now let g be nonnegative and measurable. The mapping $\varphi \mapsto \alpha \cdot \varphi$ is a bijection, with inverse $\psi \mapsto \alpha^{-1} \cdot \psi$, from

$$\mathcal{S}_g = \{\varphi \text{ simple} \mid 0 \leq \varphi \leq g \text{ on } X\} \quad \text{onto} \quad \mathcal{S}_{\alpha g} = \{\psi \text{ simple} \mid 0 \leq \psi \leq \alpha g \text{ on } X\},$$

because $\alpha > 0$, so $0 \leq \varphi \leq g$ if and only if $0 \leq \alpha \varphi \leq \alpha g$, and $\alpha \varphi$ is simple if and only if φ is. Hence, using the simple-function case just established,

$$\int_X \alpha \cdot g \, d\mu = \sup_{\psi \in \mathcal{S}_{\alpha g}} \int_X \psi \, d\mu = \sup_{\varphi \in \mathcal{S}_g} \int_X \alpha \cdot \varphi \, d\mu = \sup_{\varphi \in \mathcal{S}_g} \alpha \int_X \varphi \, d\mu = \alpha \cdot \sup_{\varphi \in \mathcal{S}_g} \int_X \varphi \, d\mu = \alpha \int_X g \, d\mu,$$

the second-to-last equality holding because for a positive real α and a nonempty set $S \subseteq [0, \infty]$ one has $\sup(\alpha S) = \alpha \cdot \sup S$ (with $\alpha \cdot \infty = \infty$). The set $\mathcal{S}_g$ is nonempty since it contains the function identically 0.

Proof of (8). Assume $g \leq h$ a.e. on X . Define

$$X_1 = \{x \in X \mid g(x) \leq h(x)\},$$

and observe that X_1 is measurable. Indeed, since g and h may take the value ∞ we cannot form $h - g$; instead, note that $g(x) > h(x)$ holds if and only if there is a rational number r with $g(x) > r > h(x)$, so that

$$X \sim X_1 = \{x \in X \mid g(x) > h(x)\} = \bigcup_{r \in \mathbb{Q}} (\{x \in X \mid g(x) > r\} \cap \{x \in X \mid h(x) < r\}),$$

a countable union of measurable sets. By hypothesis $\mu(X \sim X_1) = 0$.

Let φ be any simple function with $0 \leq \varphi \leq g$ on X . Put $\psi = \varphi \cdot \chi_{X_1}$. Then ψ is a nonnegative simple function, and $0 \leq \psi \leq h$ **on all of** X : at a point $x \in X_1$ we have $\psi(x) = \varphi(x) \leq g(x) \leq h(x)$, while at a point $x \in X \sim X_1$ we have $\psi(x) = 0 \leq h(x)$. Hence, by the definition of $\int_X h \, d\mu$ as a supremum,

$$\int_X \psi \, d\mu \leq \int_X h \, d\mu.$$

On the other hand $\psi = \varphi$ on X_1 , so $\psi \leq \varphi$ and $\varphi \leq \psi$ each hold a.e. on X ; by the monotonicity (5) of the integral of nonnegative simple functions, applied twice,

$$\int_X \psi d\mu = \int_X \varphi d\mu.$$

Combining the last two relations, $\int_X \varphi d\mu \leq \int_X h d\mu$. Since this holds for every simple φ with $0 \leq \varphi \leq g$ on X , taking the supremum over all such φ yields

$$\int_X g d\mu \leq \int_X h d\mu.$$

Proof of (9) from (8). Let X_0 be a measurable subset of X with $\mu(X \sim X_0) = 0$. The function $g \cdot \chi_{X_0}$ is nonnegative and measurable on X , and by definition $\int_{X_0} g d\mu = \int_X g \cdot \chi_{X_0} d\mu$.

Since $g \geq 0$, we have $g \cdot \chi_{X_0} \leq g$ at every point of X , so certainly $g \cdot \chi_{X_0} \leq g$ a.e. on X , and (8) gives

$$\int_X g \cdot \chi_{X_0} d\mu \leq \int_X g d\mu.$$

On the other hand, $g = g \cdot \chi_{X_0}$ at every point of X_0 , and $\mu(X \sim X_0) = 0$, so $g \leq g \cdot \chi_{X_0}$ a.e. on X ; again by (8),

$$\int_X g d\mu \leq \int_X g \cdot \chi_{X_0} d\mu.$$

These two inequalities give

$$\int_X g d\mu = \int_X g \cdot \chi_{X_0} d\mu = \int_{X_0} g d\mu,$$

which is (9).

Problem 18.18. Let $\{u_n\}$ be a sequence of nonnegative measurable functions on X . For $x \in X$, define

$$f(x) = \sum_{n=1}^{\infty} u_n(x).$$

Show that

$$\int_X f d\mu = \sum_{n=1}^{\infty} \left[\int_X u_n d\mu \right].$$

Solution. For each natural number k define the k -th partial sum

$$f_k = \sum_{n=1}^k u_n \quad \text{on } X.$$

The sequence $\{f_k\}$ is an increasing sequence of nonnegative measurable functions converging pointwise to f . Each f_k is a finite sum of nonnegative measurable functions, hence is a nonnegative measurable function on X (the sum of two nonnegative extended real-valued measurable functions is defined at every point, since no $\infty - \infty$ can occur, and is measurable). Since $u_{k+1} \geq 0$ on X , we have $f_{k+1} = f_k + u_{k+1} \geq f_k$ on X , so $\{f_k\}$ is increasing. Finally, for each $x \in X$ the series $\sum_{n=1}^{\infty} u_n(x)$ has nonnegative terms, so its partial sums $f_k(x)$ form an increasing sequence in $[0, \infty]$ and therefore converge in $[0, \infty]$ to their supremum, which by definition is the sum $f(x)$ of the series. Thus

$$f(x) = \lim_{k \rightarrow \infty} f_k(x) \quad \text{for every } x \in X,$$

and in particular, since $\{f_k\}$ converges pointwise to f on all of X , Theorem 6 tells us that f is a nonnegative extended real-valued measurable function on X . (No completeness assumption on $(X, \mathcal{M}, \mu)$ is needed here, because the convergence is pointwise on all of X rather than merely a.e.)

Additivity for finite sums. We claim that for every k ,

$$\int_X f_k d\mu = \sum_{n=1}^k \int_X u_n d\mu.$$

We argue by induction on k . For $k = 1$ this is trivial. Assume it holds for k . Proposition 11, with $\alpha = \beta = 1$ and applied to the nonnegative measurable functions f_k and u_{k+1} , gives

$$\int_X f_{k+1} d\mu = \int_X [f_k + u_{k+1}] d\mu = \int_X f_k d\mu + \int_X u_{k+1} d\mu = \sum_{n=1}^k \int_X u_n d\mu + \int_X u_{k+1} d\mu = \sum_{n=1}^{k+1} \int_X u_n d\mu,$$

these additions being additions in $[0, \infty]$, where they are unambiguously defined because all the terms are nonnegative. This completes the induction.

Passage to the limit. Since $\{f_k\}$ is an increasing sequence of nonnegative measurable functions on X with pointwise limit f , the Monotone Convergence Theorem gives

$$\int_X f d\mu = \lim_{k \rightarrow \infty} \int_X f_k d\mu = \lim_{k \rightarrow \infty} \sum_{n=1}^k \int_X u_n d\mu.$$

The numbers $\int_X u_n d\mu$ belong to $[0, \infty]$, so the partial sums on the right are increasing in k and their limit in $[0, \infty]$ is, by definition, the sum of the series $\sum_{n=1}^{\infty} \int_X u_n d\mu$. Therefore

$$\int_X f d\mu = \sum_{n=1}^{\infty} \left[\int_X u_n d\mu \right],$$

with the understanding that this is an equality in $[0, \infty]$: either both sides are finite and equal, or both are infinite.

Problem 18.19. Show that if f is a nonnegative measurable function on X , then

$$\int_X f d\mu = 0 \quad \text{if and only if} \quad f = 0 \text{ a.e. on } X.$$

Solution. Suppose $f = 0$ a.e. on X . Let $\mathbf{0}$ denote the function that is identically zero on X . Its integral is 0: the only simple function φ with $0 \leq \varphi \leq \mathbf{0}$ on X is $\varphi \equiv 0$, and by the definition of the integral of a simple function $\int_X \varphi d\mu = 0$; hence the supremum defining $\int_X \mathbf{0} d\mu$ equals 0.

Now $f = 0$ a.e. on X means in particular that $f \leq \mathbf{0}$ a.e. on X . By the monotonicity property (8) of the integral of nonnegative measurable functions,

$$\int_X f d\mu \leq \int_X \mathbf{0} d\mu = 0.$$

On the other hand $\int_X f d\mu \geq 0$, since it is a supremum of a set of nonnegative numbers containing 0. Therefore $\int_X f d\mu = 0$.

Suppose $\int_X f d\mu = 0$. For each natural number n define the measurable set

$$X_n = \left\{ x \in X \mid f(x) \geq \frac{1}{n} \right\}.$$

By Chebychev's Inequality (10), applied with $\lambda = 1/n$,

$$\mu(X_n) \leq \frac{1}{1/n} \int_X f d\mu = n \cdot 0 = 0,$$

so $\mu(X_n) = 0$ for every n .

Since $f \geq 0$, a point x satisfies $f(x) > 0$ if and only if $f(x) \geq 1/n$ for some natural number n (this includes the points at which $f(x) = \infty$). Hence

$$\{x \in X \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} X_n,$$

and by the countable subadditivity of μ ,

$$\mu\{x \in X \mid f(x) > 0\} \leq \sum_{n=1}^{\infty} \mu(X_n) = 0.$$

Thus f vanishes outside a set of measure zero, that is, $f = 0$ a.e. on X .

Problem 18.20. Verify (2) in the case ψ and φ vanish outside a set of finite measure.

Solution. Here ψ and φ are nonnegative simple functions on X , α and β are positive real numbers, and (2) is the assertion

$$\int_X [\alpha \cdot \psi + \beta \cdot \varphi] d\mu = \alpha \cdot \int_X \psi d\mu + \beta \cdot \int_X \varphi d\mu.$$

We assume ψ and φ each vanish outside a set of finite measure. Since $\psi \geq 0$, this means precisely that $\{x \in X \mid \psi(x) > 0\}$ is contained in a measurable set of finite measure, and hence, by the monotonicity of μ , that

$$F_\psi = \{x \in X \mid \psi(x) > 0\} \quad \text{has} \quad \mu(F_\psi) < \infty;$$

similarly $F_\varphi = \{x \in X \mid \varphi(x) > 0\}$ has finite measure. Set $F = F_\psi \cup F_\varphi$, a measurable set with $\mu(F) < \infty$ by the finite subadditivity of μ . Both ψ and φ , and therefore also $\alpha\psi + \beta\varphi$, vanish on $X \sim F$.

A lemma on representations of simple functions. Suppose σ is a nonnegative simple function on X which can be written as

$$\sigma = \sum_{k=1}^n c_k \cdot \chi_{E_k} \quad \text{on } X,$$

where $c_1, \dots, c_n$ are nonnegative real numbers and $E_1, \dots, E_n$ are **disjoint** measurable subsets of X each of **finite** measure. Then

$$\int_X \sigma d\mu = \sum_{k=1}^n c_k \cdot \mu(E_k).$$

Indeed, since the E_k are disjoint, σ takes the value c_k at each point of E_k and the value 0 at each point of $X \sim \bigcup_k E_k$. Let $d_1, \dots, d_p$ be the distinct positive values taken by σ on X (if there are none, then every c_k with $E_k \neq \emptyset$ is zero, both sides are 0, and there is nothing to prove). For each j ,

$$\{x \in X \mid \sigma(x) = d_j\} = \bigcup \{E_k \mid c_k = d_j\},$$

a finite disjoint union, so by the finite additivity of μ ,

$$\mu\{\sigma = d_j\} = \sum_{k: c_k = d_j} \mu(E_k).$$

Hence, by the definition (1) of the integral of a simple function,

$$\int_X \sigma d\mu = \sum_{j=1}^p d_j \cdot \mu\{\sigma = d_j\} = \sum_{j=1}^p \sum_{k: c_k = d_j} c_k \cdot \mu(E_k) = \sum_{k: c_k > 0} c_k \cdot \mu(E_k) = \sum_{k=1}^n c_k \cdot \mu(E_k),$$

the last equality because a term with $c_k = 0$ contributes $c_k \mu(E_k) = 0 \cdot \mu(E_k) = 0$, this product being genuinely zero since $\mu(E_k) < \infty$. All the sums here are finite sums of real numbers.

A common refinement. Let $a_1, \dots, a_m$ be the distinct values taken by ψ on X and $b_1, \dots, b_q$ the distinct values taken by φ on X (both lists include the value 0, which each function takes on $X \sim F$ unless $F = X$; this causes no difficulty below). For $1 \leq i \leq m$ and $1 \leq j \leq q$ define

$$C_{ij} = \{x \in F \mid \psi(x) = a_i \text{ and } \varphi(x) = b_j\}.$$

Each C_{ij} is measurable, since ψ and φ are measurable, and $C_{ij} \subseteq F$ so $\mu(C_{ij}) \leq \mu(F) < \infty$. The sets C_{ij} are pairwise disjoint, because the values a_i are distinct and the values b_j are distinct, and their union is F .

On C_{ij} we have $\psi \equiv a_i$, $\varphi \equiv b_j$ and $\alpha\psi + \beta\varphi \equiv \alpha a_i + \beta b_j$; and all three functions vanish on $X \sim F$. Therefore we have the three representations

$$\psi = \sum_{i,j} a_i \cdot \chi_{C_{ij}}, \quad \varphi = \sum_{i,j} b_j \cdot \chi_{C_{ij}}, \quad \alpha\psi + \beta\varphi = \sum_{i,j} (\alpha a_i + \beta b_j) \cdot \chi_{C_{ij}} \quad \text{on } X,$$

each a finite sum with nonnegative coefficients over the disjoint measurable sets C_{ij} of finite measure. *Conclusion.* Note that $\alpha\psi + \beta\varphi$ is again a nonnegative simple function. Applying the lemma to each of the three representations,

$$\begin{aligned} \int_X [\alpha \cdot \psi + \beta \cdot \varphi] d\mu &= \sum_{i,j} (\alpha a_i + \beta b_j) \cdot \mu(C_{ij}) \\ &= \alpha \sum_{i,j} a_i \cdot \mu(C_{ij}) + \beta \sum_{i,j} b_j \cdot \mu(C_{ij}) \\ &= \alpha \int_X \psi d\mu + \beta \int_X \varphi d\mu. \end{aligned}$$

The rearrangement in the middle step is legitimate because every $\mu(C_{ij})$ is a real number and the sums are finite. This verifies (2) in the case at hand.

Problem 18.21. Let f and g be nonnegative measurable functions on X for which $g \leq f$ a.e. on X . Show that $f = g$ a.e. on X if and only if $\int_X g d\mu = \int_X f d\mu$.

Solution. The forward implication. Suppose $f = g$ a.e. on X . Then $f \leq g$ a.e. on X and $g \leq f$ a.e. on X , so two applications of the monotonicity property (8) of the integral of nonnegative measurable functions give

$$\int_X f d\mu \leq \int_X g d\mu \quad \text{and} \quad \int_X g d\mu \leq \int_X f d\mu,$$

whence $\int_X g d\mu = \int_X f d\mu$. Note that no finiteness assumption is needed here.

The converse implication. The converse requires the integrals to be finite, and we prove it in that case: **if $\int_X g d\mu = \int_X f d\mu < \infty$, then $f = g$ a.e. on X .**

Since $\int_X f d\mu < \infty$, Proposition 9 tells us that f is finite a.e. on X ; and by hypothesis $g \leq f$ a.e. on X . Define

$$X_0 = \{x \in X \mid g(x) \leq f(x) < \infty\},$$

This set is measurable: $\{x \in X \mid f(x) < \infty\}$ is measurable because f is, while

$$\{x \in X \mid g(x) > f(x)\} = \bigcup_{r \in \mathbb{Q}} (\{x \in X \mid g(x) > r\} \cap \{x \in X \mid f(x) < r\})$$

is a countable union of measurable sets, so its complement $\{g \leq f\}$ is measurable and X_0 is the intersection of two measurable sets. Furthermore

$$X \sim X_0 = \{x \in X \mid g(x) > f(x)\} \cup \{x \in X \mid f(x) = \infty\},$$

the first of these sets having measure zero by hypothesis and the second by Proposition 9; hence $\mu(X \sim X_0) = 0$.

Define h on X by

$$h(x) = \begin{cases} f(x) - g(x) & \text{if } x \in X_0, \\ 0 & \text{if } x \in X \sim X_0. \end{cases}$$

On X_0 both f and g are finite and $g \leq f$, so this difference is a well-defined nonnegative real number; thus h is a nonnegative real-valued function on X . To see that h is measurable, consider the two functions $f \cdot \chi_{X_0}$ and $g \cdot \chi_{X_0}$, which by definition take the values $f(x)$, respectively $g(x)$, at $x \in X_0$ and the value 0 at $x \in X \sim X_0$. Both are measurable, and both are *real-valued*, since $g \leq f < \infty$ on X_0 . Hence their difference is defined at every point of X and is measurable, and this difference is exactly h . Moreover

$$f \cdot \chi_{X_0} = g \cdot \chi_{X_0} + h \quad \text{on all of } X,$$

since at $x \in X_0$ this reads $f(x) = g(x) + (f(x) - g(x))$, and at $x \in X \sim X_0$ both sides are 0.

All three functions $f \cdot \chi_{X_0}$, $g \cdot \chi_{X_0}$ and h are nonnegative and measurable, so Proposition 11 with $\alpha = \beta = 1$ gives

$$\int_X f \cdot \chi_{X_0} d\mu = \int_X g \cdot \chi_{X_0} d\mu + \int_X h d\mu.$$

Since $\mu(X \sim X_0) = 0$, property (9) gives

$$\int_X f \cdot \chi_{X_0} d\mu = \int_{X_0} f d\mu = \int_X f d\mu \quad \text{and} \quad \int_X g \cdot \chi_{X_0} d\mu = \int_{X_0} g d\mu = \int_X g d\mu.$$

Therefore

$$\int_X f d\mu = \int_X g d\mu + \int_X h d\mu.$$

By assumption $\int_X f d\mu = \int_X g d\mu$ and this common value is a **real number**; subtracting it from both sides of the last equation yields

$$\int_X h d\mu = 0.$$

Since h is nonnegative and measurable, the preceding problem tells us that $h = 0$ a.e. on X . Let E be the set of measure zero outside of which h vanishes. For $x \in X_0 \sim E$ we have $f(x) - g(x) = h(x) = 0$, that is, $f(x) = g(x)$. Since

$$\{x \in X \mid f(x) \neq g(x)\} \subseteq (X \sim X_0) \cup E,$$

a set of measure zero by the finite subadditivity of μ , we conclude that $f = g$ a.e. on X .

The finiteness hypothesis cannot be dropped. Take $X = \mathbb{R}$ with $\mathcal{M}$ the σ -algebra of Lebesgue measurable sets and μ Lebesgue measure. Let $f \equiv 1$ on $\mathbb{R}$ and $g = \chi_{[0, \infty)}$. Then f and g are nonnegative measurable functions with $g \leq f$ everywhere on $\mathbb{R}$, and

$$\int_{\mathbb{R}} f d\mu = \infty = \int_{\mathbb{R}} g d\mu,$$

since for each natural number n the simple functions $\chi_{[-n, n]} \leq f$ and $\chi_{[0, n]} \leq g$ have integrals $2n$ and n respectively. Nevertheless $\{x \mid f(x) \neq g(x)\} = (-\infty, 0)$, which has infinite measure, so $f = g$ a.e. on $\mathbb{R}$ fails. Hence the implication from equality of the integrals to equality of the functions a.e. is false as it stands, and the exercise is to be read with the (implicit) hypothesis that f is integrable over X , that is, $\int_X f d\mu < \infty$. The implication in the other direction, proved first above, requires no such hypothesis.

18.4 Problems 18.22–18.28

Problem 18.22. Suppose f and g are nonnegative measurable functions on X for which f^2 and g^2 are integrable over X with respect to μ . Show that $f \cdot g$ also is integrable over X with respect to μ .

Solution. Throughout, $(X, \mathcal{M}, \mu)$ is the ambient measure space, and f, g are nonnegative extended real-valued measurable functions on X with

$$\int_X f^2 d\mu < \infty \quad \text{and} \quad \int_X g^2 d\mu < \infty.$$

We begin by disposing of the two points at which the product of extended real-valued functions requires care: that $f \cdot g$ is defined at all, and that it is measurable.

Since f^2 is a nonnegative measurable function with $\int_X f^2 d\mu < \infty$, Proposition 9 tells us that f^2 is finite a.e. on X ; hence f is finite a.e. on X . The same argument applies to g . Define

$$X_0 = \{x \in X \mid f(x) < \infty \text{ and } g(x) < \infty\}.$$

Then X_0 is measurable and $\mu(X \sim X_0) = 0$, since $X \sim X_0$ is the union of two sets of measure zero. On X_0 both f and g are real-valued and measurable with respect to the σ -algebra $\mathcal{M}_0$ of sets in $\mathcal{M}$ contained in X_0 , so by the Products assertion of Theorem 4 the restriction of $f \cdot g$ to X_0 is measurable. This is precisely the situation described in the remark on page 361: $f \cdot g$ is a properly defined real-valued function on X_0 , it is measurable there, and by $f \cdot g$ on X we mean any nonnegative measurable extended real-valued extension of it to X . At least one such extension exists, namely the function equal to $f \cdot g$ on X_0 and to 0 on $X \sim X_0$, which is measurable by Problem 4 of this chapter; we denote it by $(f \cdot g) \cdot \chi_{X_0}$. Property (9) of the integral of a nonnegative measurable function guarantees that the number $\int_X f \cdot g d\mu$ does not depend on which extension is chosen, since any two such extensions agree outside the set $X \sim X_0$ of measure zero. The assertion to be proved is that this number is finite. For each $x \in X_0$, the numbers $f(x)$ and $g(x)$ are real, so that

$$0 \leq (f(x) - g(x))^2 = f^2(x) - 2f(x)g(x) + g^2(x),$$

and therefore

$$0 \leq f(x) \cdot g(x) \leq \frac{1}{2}[f^2(x) + g^2(x)] \quad \text{for all } x \in X_0.$$

Consequently, on all of X we have the inequality between nonnegative measurable functions

$$0 \leq (f \cdot g) \cdot \chi_{X_0} \leq \frac{1}{2}[f^2 + g^2].$$

We now integrate. By property (9) of the integral of a nonnegative measurable function, since $\mu(X \sim X_0) = 0$,

$$\int_X f \cdot g d\mu = \int_{X_0} f \cdot g d\mu = \int_X (f \cdot g) \cdot \chi_{X_0} d\mu.$$

By the monotonicity property (8) and then by Proposition 11 (linearity of integration for nonnegative measurable functions, with $\alpha = \beta = 1/2$),

$$\int_X (f \cdot g) \cdot \chi_{X_0} d\mu \leq \int_X \frac{1}{2}[f^2 + g^2] d\mu = \frac{1}{2} \int_X f^2 d\mu + \frac{1}{2} \int_X g^2 d\mu.$$

The right-hand side is finite by hypothesis. Therefore

$$\int_X f \cdot g d\mu \leq \frac{1}{2} \int_X f^2 d\mu + \frac{1}{2} \int_X g^2 d\mu < \infty,$$

so $f \cdot g$ is integrable over X with respect to μ .

Problem 18.23. Let X be the union of a countable ascending sequence of measurable sets $\{X_n\}$ and f a nonnegative measurable function on X . Show that f is integrable over X if and only if there is an $M \geq 0$ for which $\int_{X_n} f d\mu \leq M$ for all n .

Solution. For each natural number n define

$$f_n = f \cdot \chi_{X_n} \quad \text{on } X,$$

so that, by the definition of the integral over a measurable subset, $\int_X f_n d\mu = \int_{X_n} f d\mu$. Each f_n is nonnegative, and it is measurable: its restriction to X_n is the restriction of the measurable function f , and its restriction to $X \sim X_n$ is identically 0, so f_n is measurable by Problem 4 of this chapter. (This is the same product $f \cdot \chi_{X_n}$ that appears in the book's own definition of $\int_{X_n} f d\mu$; note that no convention about $0 \cdot \infty$ is needed, since $f \cdot \chi_{X_n}$ is simply f on X_n and 0 off X_n .)

Two observations about the sequence $\{f_n\}$:

First, $\{f_n\}$ is increasing. Indeed, since the sequence $\{X_n\}$ is ascending, $\chi_{X_n} \leq \chi_{X_{n+1}}$ on X , and hence $f_n = f \cdot \chi_{X_n} \leq f \cdot \chi_{X_{n+1}} = f_{n+1}$ on X , since $f \geq 0$.

Second, $\{f_n\}$ converges pointwise on X to f . Let x belong to X . Since $X = \bigcup_{n=1}^{\infty} X_n$, there is an index N with $x \in X_N$, and since $\{X_n\}$ is ascending, $x \in X_n$ for all $n \geq N$. Therefore $f_n(x) = f(x)$ for all $n \geq N$, and in particular $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

By the Monotone Convergence Theorem applied to $\{f_n\}$,

$$\lim_{n \rightarrow \infty} \int_{X_n} f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Call this relation (*). Moreover, by monotonicity of integration for nonnegative measurable functions, property (8), the sequence of numbers $\left\{ \int_{X_n} f d\mu \right\}$ is increasing, so the limit in (*) is also the supremum of this sequence:

$$\int_X f d\mu = \sup_n \int_{X_n} f d\mu.$$

The equivalence now follows at once.

Suppose first that f is integrable over X , that is, $\int_X f d\mu < \infty$. Put $M = \int_X f d\mu$, a nonnegative real number. Since $0 \leq f \cdot \chi_{X_n} \leq f$ on X , monotonicity (8) gives

$$\int_{X_n} f d\mu = \int_X f \cdot \chi_{X_n} d\mu \leq \int_X f d\mu = M \quad \text{for all } n.$$

Conversely, suppose there is an $M \geq 0$ for which $\int_{X_n} f d\mu \leq M$ for all n . Then by (*),

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_{X_n} f d\mu \leq M < \infty,$$

so f is integrable over X . (This is precisely Beppo Levi's Lemma applied to the increasing sequence $\{f_n\}$, whose sequence of integrals is bounded by M .)

Problem 18.24. Show that the definition of the integral of a nonnegative measurable function on a general measure space is consistent with the definition given in the particular case of the Lebesgue integral of a function of a real variable.

Solution. Let E be a Lebesgue measurable set of real numbers, let $\mathcal{M}$ be the σ -algebra of Lebesgue measurable subsets of E , and let m denote Lebesgue measure restricted to $\mathcal{M}$. Then $(E, \mathcal{M}, m)$ is a measure space, and a function on E is measurable with respect to $\mathcal{M}$ precisely when it is Lebesgue measurable in the sense of Chapter 3. Let f be a nonnegative measurable function on E . There are two definitions of $\int_E f$ in force, and we must show they agree.

The general definition of Section 18.2: for a nonnegative simple function ψ on E taking the distinct positive values $c_1, \dots, c_n$, with $E_k = \{x \in E \mid \psi(x) = c_k\}$,

$$\int_E \psi dm = \sum_{k=1}^n c_k \cdot m(E_k),$$

with the convention that this is ∞ if some $m(E_k) = \infty$; and then

$$I_{\text{gen}}(f) = \sup \left\{ \int_E \psi \, dm \mid \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E \right\}.$$

The definition of Section 4.3: writing $\int_E h$ for the Lebesgue integral of a bounded measurable function h of finite support (this is defined at the start of Section 4.3 to be $\int_{E_0} h$ for any set E_0 of finite measure outside of which h vanishes, a value independent of the choice of E_0 by additivity over domains for bounded measurable functions on sets of finite measure),

$$I_{\text{Leb}}(f) = \sup \left\{ \int_E h \mid h \text{ bounded, measurable, of finite support, } 0 \leq h \leq f \text{ on } E \right\},$$

which is formula (8) of Chapter 4.

Step 1: the two definitions agree for simple functions of finite support. Let ψ be a nonnegative simple function on E that vanishes outside a set E_0 of finite measure, with distinct positive values $c_1, \dots, c_n$ attained on $E_1, \dots, E_n$. Each $E_k \subseteq E_0$, so each $m(E_k) < \infty$. The general definition (1) gives $\sum_{k=1}^n c_k m(E_k)$. The Section 4.2 definition of the integral of a simple function over the set E_0 of finite measure is the sum $\sum_i a_i m(A_i)$ taken over the canonical representation of ψ on E_0 , whose distinct values are $c_1, \dots, c_n$ together possibly with 0; the term carrying the value 0 contributes nothing, and the remaining terms are exactly $c_k m(E_k)$. Hence the Chapter 4 value is also $\sum_{k=1}^n c_k m(E_k)$, and by Lemma 1 of Chapter 4 this number does not depend on the representation used. So the two definitions give the same number, and moreover $\int_E \psi = \int_{E_0} \psi = \int_E \psi \, dm$.

Step 2: $I_{\text{gen}}(f) \leq I_{\text{Leb}}(f)$. Let ψ be a simple function with $0 \leq \psi \leq f$ on E ; we show $\int_E \psi \, dm \leq I_{\text{Leb}}(f)$. Retain the notation $c_1, \dots, c_n$ and $E_1, \dots, E_n$ above.

Case (i): every $m(E_k)$ is finite. Then ψ vanishes outside the set $\bigcup_{k=1}^n E_k$ of finite measure, and ψ is bounded (it takes finitely many values). Thus ψ is a competitor in the supremum defining $I_{\text{Leb}}(f)$, and by Step 1 its Chapter 4 integral equals $\int_E \psi \, dm$. Hence $\int_E \psi \, dm \leq I_{\text{Leb}}(f)$.

Case (ii): $m(E_k) = \infty$ for some k , so that $\int_E \psi \, dm = \infty$. We show $I_{\text{Leb}}(f) = \infty$ as well. For each natural number j set $A_j = E_k \cap [-j, j]$. Then $\{A_j\}$ is an ascending sequence of measurable sets of finite measure whose union is E_k , so by the continuity of Lebesgue measure $\lim_{j \rightarrow \infty} m(A_j) = m(E_k) = \infty$. The function $h_j = c_k \cdot \chi_{A_j}$ is bounded, measurable, of finite support, and $0 \leq h_j \leq \psi \leq f$ on E ; its Chapter 4 integral is $c_k \cdot m(A_j)$. Since $c_k > 0$, these integrals tend to ∞ , and therefore $I_{\text{Leb}}(f) = \infty \geq \int_E \psi \, dm$.

In either case $\int_E \psi \, dm \leq I_{\text{Leb}}(f)$, and taking the supremum over all such ψ gives $I_{\text{gen}}(f) \leq I_{\text{Leb}}(f)$.

Step 3: $I_{\text{Leb}}(f) \leq I_{\text{gen}}(f)$. Let h be bounded, measurable, of finite support, with $0 \leq h \leq f$ on E , and let E_0 be a set of finite measure outside of which h vanishes, so $\int_E h = \int_{E_0} h$. Since h is bounded and measurable on the set E_0 of finite measure, Theorem 4 of Chapter 4 tells us that its lower and upper Lebesgue integrals over E_0 agree with $\int_{E_0} h$; in particular

$$\int_{E_0} h = \sup \left\{ \int_{E_0} \varphi \mid \varphi \text{ simple, } \varphi \leq h \text{ on } E_0 \right\}.$$

In this supremum we may restrict attention to simple φ with $0 \leq \varphi \leq h$: replacing φ by $\max\{\varphi, 0\}$ leaves a simple function that still lies below h (because $h \geq 0$) and does not decrease the integral, by monotonicity of the integral of simple functions.

Let φ be such a function, and extend it to E by defining $\tilde{\varphi} = \varphi \cdot \chi_{E_0}$ on E . Then $\tilde{\varphi}$ is a nonnegative simple function on E , it vanishes outside the set E_0 of finite measure, and $0 \leq \tilde{\varphi} \leq f$ on E : on E_0 we have $\tilde{\varphi} = \varphi \leq h \leq f$, and on $E \sim E_0$ we have $\tilde{\varphi} = 0 \leq f$. By Step 1,

$$\int_{E_0} \varphi = \int_E \tilde{\varphi} \, dm \leq I_{\text{gen}}(f),$$

the last inequality holding because $\tilde{\varphi}$ is a competitor in the supremum defining $I_{\text{gen}}(f)$. Taking the supremum over all such φ gives $\int_E h = \int_{E_0} h \leq I_{\text{gen}}(f)$, and then taking the supremum over all such h gives $I_{\text{Leb}}(f) \leq I_{\text{gen}}(f)$.

Steps 2 and 3 together give $I_{\text{gen}}(f) = I_{\text{Leb}}(f)$: the general definition of the integral of a nonnegative measurable function, specialized to Lebesgue measure on a measurable set of real numbers, returns exactly the Lebesgue integral of Chapter 4.

Finally, consistency for general (not necessarily nonnegative) measurable functions is immediate from this: both Chapter 4 and Section 18.3 declare f integrable when $|f|$ has finite integral and then set $\int_E f = \int_E f^+ - \int_E f^-$, and the two integrals appearing on the right have just been shown to agree.

Problem 18.25. Let η be the counting measure on the natural numbers $\mathbb{N}$. Characterize the nonnegative real-valued functions (that is, sequences) that are integrable over $\mathbb{N}$ with respect to η and the value of $\int_{\mathbb{N}} f d\eta$.

Solution. Here the measure space is $(\mathbb{N}, \mathcal{M}, \eta)$, where $\mathcal{M}$ is the σ -algebra of all subsets of $\mathbb{N}$ and $\eta(E)$ is the number of elements of E if E is finite and $\eta(E) = \infty$ if E is infinite. Since $\mathcal{M}$ consists of all subsets of $\mathbb{N}$, every function on $\mathbb{N}$ is measurable.

A nonnegative real-valued function f on $\mathbb{N}$ is nothing other than a sequence of nonnegative real numbers: write $a_k = f(k)$ for each k . We claim

$$\int_{\mathbb{N}} f d\eta = \sum_{k=1}^{\infty} a_k,$$

so that f is integrable over $\mathbb{N}$ with respect to η if and only if the series $\sum_{k=1}^{\infty} a_k$ converges, that is, if and only if $\{a_k\}$ is a summable sequence; and in that case the value of the integral is the sum of the series.

The integral of a truncation. For a natural number n let $X_n = \{1, 2, \dots, n\}$ and put

$$f_n = f \cdot \chi_{X_n} \quad \text{on } \mathbb{N}.$$

Then f_n is a nonnegative simple function on $\mathbb{N}$, since it takes at most the finitely many values $a_1, \dots, a_n, 0$. We compute $\int_{\mathbb{N}} f_n d\eta$ from the definition (1). Let $c_1, \dots, c_m$ be the distinct positive values taken by f_n ; these are exactly the distinct positive numbers among $a_1, \dots, a_n$. For $1 \leq j \leq m$ put $E_j = \{k \in \mathbb{N} \mid f_n(k) = c_j\}$; since $c_j > 0$ we have $E_j = \{k \leq n \mid a_k = c_j\} \subseteq X_n$, a finite set, and $\eta(E_j)$ is the number of its elements. The sets $E_1, \dots, E_m$ are disjoint and their union is $\{k \leq n \mid a_k > 0\}$. Hence

$$\int_{\mathbb{N}} f_n d\eta = \sum_{j=1}^m c_j \cdot \eta(E_j) = \sum_{j=1}^m \sum_{\substack{k \leq n \\ a_k = c_j}} a_k = \sum_{\substack{k \leq n \\ a_k > 0}} a_k = \sum_{k=1}^n a_k,$$

the last equality because the omitted terms are zero. Thus

$$\int_{X_n} f d\eta = \int_{\mathbb{N}} f \cdot \chi_{X_n} d\eta = \sum_{k=1}^n a_k.$$

Passage to the limit. The sequence $\{X_n\}$ is an ascending sequence of measurable sets whose union is $\mathbb{N}$, so, exactly as in Problem 23, the sequence $\{f_n\}$ is an increasing sequence of nonnegative measurable functions converging pointwise on $\mathbb{N}$ to f (given $k \in \mathbb{N}$, $f_n(k) = f(k)$ for all $n \geq k$). By the Monotone Convergence Theorem,

$$\int_{\mathbb{N}} f d\eta = \lim_{n \rightarrow \infty} \int_{\mathbb{N}} f_n d\eta = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = \sum_{k=1}^{\infty} a_k,$$

where the last limit exists in $[0, \infty]$ because the partial sums of a series of nonnegative terms increase. This proves the claim. Consequently $\int_{\mathbb{N}} f d\eta < \infty$ if and only if $\sum_{k=1}^{\infty} f(k) < \infty$: the nonnegative functions integrable with respect to the counting measure on $\mathbb{N}$ are precisely the summable sequences of nonnegative numbers, and the integral is the sum of the sequence.

Problem 18.26. Let x_0 be a point in a set X and δ_{x_0} the Dirac measure concentrated at x_0 . Characterize the nonnegative real-valued functions on X that are integrable over X with respect to δ_{x_0} and the value of $\int_X f d\delta_{x_0}$.

Solution. Recall that the Dirac measure concentrated at x_0 is the measure on $\mathcal{M}$ defined by

$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E, \\ 0 & \text{if } x_0 \notin E. \end{cases}$$

We claim that **every** nonnegative real-valued measurable function f on X is integrable over X with respect to δ_{x_0} , and that

$$\int_X f d\delta_{x_0} = f(x_0).$$

Step 1: the formula for simple functions. Let ψ be a nonnegative simple function on X . If $\psi \equiv 0$ then $\int_X \psi d\delta_{x_0} = 0 = \psi(x_0)$ by definition. Otherwise, let $c_1, \dots, c_n$ be the distinct positive values taken by ψ and put $E_k = \{x \in X \mid \psi(x) = c_k\}$, so that by definition (1),

$$\int_X \psi d\delta_{x_0} = \sum_{k=1}^n c_k \cdot \delta_{x_0}(E_k).$$

The sets $E_1, \dots, E_n$ are pairwise disjoint, so x_0 belongs to at most one of them. If $x_0 \in E_j$ for some (necessarily unique) index j , then $\delta_{x_0}(E_j) = 1$ and $\delta_{x_0}(E_k) = 0$ for $k \neq j$, so the sum equals $c_j = \psi(x_0)$. If x_0 belongs to no E_k , then every term vanishes and the sum is 0, while also $\psi(x_0) = 0$ since x_0 is not in any set where ψ takes a positive value. In both cases

$$\int_X \psi d\delta_{x_0} = \psi(x_0).$$

Step 2: the general case. Let f be a nonnegative measurable function on X . By part (ii) of the Simple Approximation Theorem there is an increasing sequence $\{\psi_n\}$ of nonnegative simple functions on X that converges pointwise on X to f ; this is exactly the sequence furnished by Proposition 10. By the Monotone Convergence Theorem and Step 1,

$$\int_X f d\delta_{x_0} = \lim_{n \rightarrow \infty} \int_X \psi_n d\delta_{x_0} = \lim_{n \rightarrow \infty} \psi_n(x_0) = f(x_0),$$

the last equality being the pointwise convergence of $\{\psi_n\}$ to f evaluated at the point x_0 .

Therefore $\int_X f d\delta_{x_0} = f(x_0)$. Since f is real-valued, $f(x_0) < \infty$, so f is integrable over X with respect to δ_{x_0} . That is, there is no restriction at all: the nonnegative real-valued measurable functions integrable with respect to the Dirac measure at x_0 are all of them, and the integral is simply evaluation at x_0 .

Two remarks. First, no assumption that $\{x_0\}$ belongs to $\mathcal{M}$ was used: in Step 1 all that was needed is that the level sets E_k of a simple function are measurable and pairwise disjoint, so that x_0 lies in at most one of them. Second, the computation shows that unboundedness of f away from x_0 costs nothing, since every measurable set missing x_0 carries no δ_{x_0} -mass; if extended real-valued f were allowed, the same argument would give $\int_X f d\delta_{x_0} = f(x_0)$ with the convention that the limit in Step 2 may be ∞ , and integrability would then be exactly the condition $f(x_0) < \infty$.

Problem 18.27. For a set X , let $\mathcal{M}$ be the σ -algebra of all subsets of X .

- (i) Let η be the counting measure on $\mathcal{M}$. Characterize the real-valued functions f on X which are integrable over X with respect to η and the value of $\int_X f d\eta$ for such functions.
- (ii) Let x_0 be a member of X and δ_{x_0} the Dirac delta measure concentrated at x_0 . Characterize the real-valued functions f on X which are integrable over X with respect to δ_{x_0} and the value

of $\int_X f d\delta_{x_0}$ for such functions.

Solution. Since $\mathcal{M}$ is the σ -algebra of all subsets of X , every real-valued function on X is measurable, so measurability is never an obstruction below.

Part (i). We first record the value of the integral of a nonnegative function with respect to η .

Claim. If g is a nonnegative real-valued function on X , then

$$\int_X g d\eta = \sup \left\{ \sum_{x \in F} g(x) \mid F \subseteq X \text{ finite} \right\} =: S(g).$$

To see that $S(g) \leq \int_X g d\eta$: let $F \subseteq X$ be finite. Then $g \cdot \chi_F$ is a nonnegative simple function, since it is real-valued and takes at most the finitely many values $\{g(x) \mid x \in F\} \cup \{0\}$, and computing its integral from definition (1) exactly as in Problem 25 (each singleton has η -measure 1, and each level set of $g \cdot \chi_F$ at a positive value is a finite subset of F) gives $\int_X g \cdot \chi_F d\eta = \sum_{x \in F} g(x)$. Since $0 \leq g \cdot \chi_F \leq g$ on X , monotonicity (8) yields $\sum_{x \in F} g(x) \leq \int_X g d\eta$. Take the supremum over finite F .

To see the reverse inequality, let ψ be a simple function with $0 \leq \psi \leq g$ on X ; we show $\int_X \psi d\eta \leq S(g)$. If $\psi \equiv 0$ this is clear. Otherwise let $c_1, \dots, c_n$ be its distinct positive values and $E_k = \{x \in X \mid \psi(x) = c_k\}$. If every E_k is finite, then $F = \bigcup_{k=1}^n E_k$ is finite and

$$\int_X \psi d\eta = \sum_{k=1}^n c_k \cdot \eta(E_k) = \sum_{x \in F} \psi(x) \leq \sum_{x \in F} g(x) \leq S(g).$$

If some E_j is infinite, then $\eta(E_j) = \infty$ and $\int_X \psi d\eta = \infty$; but in that case, given any $N > 0$, choose a finite subset $F \subseteq E_j$ with more than N/c_j elements, and then $\sum_{x \in F} g(x) \geq \sum_{x \in F} \psi(x) = c_j \cdot \#F > N$, so $S(g) = \infty$. In either case $\int_X \psi d\eta \leq S(g)$, and taking the supremum over all such ψ gives $\int_X g d\eta \leq S(g)$. The claim is proved.

Now let f be a real-valued function on X . By definition, f is integrable over X with respect to η if and only if $\int_X |f| d\eta < \infty$, that is, by the claim, if and only if

$$S(|f|) = \sup \left\{ \sum_{x \in F} |f(x)| \mid F \subseteq X \text{ finite} \right\} < \infty.$$

We translate this into the assertion that f is a summable family. Suppose $S(|f|) = S < \infty$. For each natural number n let $A_n = \{x \in X \mid |f(x)| > 1/n\}$. If A_n contained more than nS points, we could choose a finite $F \subseteq A_n$ with $\#F > nS$, and then $\sum_{x \in F} |f(x)| > \#F/n > S$, a contradiction. So each A_n is finite, and therefore

$$\{x \in X \mid f(x) \neq 0\} = \bigcup_{n=1}^{\infty} A_n$$

is countable. Enumerate it as $x_1, x_2, \dots$ (a finite or infinite list, possibly empty). For each N , taking $F = \{x_1, \dots, x_N\}$ gives $\sum_{k=1}^N |f(x_k)| \leq S$, so the series $\sum_k |f(x_k)|$ converges, with sum at most S . Conversely, if f vanishes outside a countable set $\{x_k\}$ and $\sum_k |f(x_k)| < \infty$, then for every finite $F \subseteq X$ we have $\sum_{x \in F} |f(x)| \leq \sum_k |f(x_k)| < \infty$, so $S(|f|) < \infty$.

Thus f is integrable over X with respect to η if and only if f vanishes outside a countable subset of X and the values of f on that subset form an absolutely convergent series; equivalently, if and only if the sums $\sum_{x \in F} |f(x)|$ over finite subsets F of X are bounded above. In the language of summable families, the η -integrable functions on X are precisely the summable families of real numbers indexed by X .

For such an f , with $\{x \mid f(x) \neq 0\} = \{x_1, x_2, \dots\}$ as above,

$$\int_X f d\eta = \sum_k f(x_k),$$

the series converging absolutely (so its sum does not depend on the enumeration chosen). Indeed, apply the claim to f^+ : since f^+ vanishes off $\{x_k\}$, every finite sum $\sum_{x \in F} f^+(x)$ is at most $\sum_k f^+(x_k)$, while

the partial sums $\sum_{k \leq N} f^+(x_k)$ are attained by the finite sets $F = \{x_1, \dots, x_N\}$; hence $\int_X f^+ d\eta = S(f^+) = \sum_k f^+(x_k)$. The same holds for f^- . Both series converge, being dominated by $\sum_k |f(x_k)|$, and therefore

$$\int_X f d\eta = \int_X f^+ d\eta - \int_X f^- d\eta = \sum_k f^+(x_k) - \sum_k f^-(x_k) = \sum_k f(x_k).$$

In particular, when $X = \mathbb{N}$ this recovers the statement that the η -integrable functions are the absolutely summable sequences and that the integral is the sum of the series.

Part (ii). We claim that **every** real-valued function f on X is integrable over X with respect to δ_{x_0} , and that

$$\int_X f d\delta_{x_0} = f(x_0).$$

Indeed, $|f|$ is a nonnegative real-valued function on X , so by Problem 26,

$$\int_X |f| d\delta_{x_0} = |f(x_0)| < \infty,$$

since f is real-valued. Hence $|f|$ is integrable over X with respect to δ_{x_0} , which is precisely the statement that f is integrable over X with respect to δ_{x_0} .

To evaluate the integral, apply Problem 26 to the nonnegative functions f^+ and f^- :

$$\int_X f^+ d\delta_{x_0} = f^+(x_0), \quad \int_X f^- d\delta_{x_0} = f^-(x_0),$$

both finite. Therefore, by the definition of the integral of a general integrable function,

$$\int_X f d\delta_{x_0} = \int_X f^+ d\delta_{x_0} - \int_X f^- d\delta_{x_0} = f^+(x_0) - f^-(x_0) = f(x_0).$$

So integration against the Dirac delta measure at x_0 is exactly evaluation at x_0 , and no condition whatsoever is imposed on f beyond being real-valued.

Problem 18.28. Show that if f is integrable over X , then f is integrable over every measurable subset of X .

Solution. Let f be a measurable function on X that is integrable over X with respect to μ , and let E be a measurable subset of X . By the definition given in Section 18.3, to say that f is integrable over E is to say that $f \cdot \chi_E$ is integrable over X with respect to μ ; and $f \cdot \chi_E$ is integrable over X provided $|f \cdot \chi_E|$ has finite integral over X .

The function $f \cdot \chi_E$ is measurable, being the product of the measurable function f with the characteristic function of the measurable set E .

Since χ_E takes only the values 0 and 1,

$$|f \cdot \chi_E| = |f| \cdot \chi_E \leq |f| \quad \text{on } X,$$

because $|f| \geq 0$. Both $|f| \cdot \chi_E$ and $|f|$ are nonnegative measurable functions on X , so by the monotonicity property (8) of the integral of nonnegative measurable functions,

$$\int_X |f \cdot \chi_E| d\mu = \int_X |f| \cdot \chi_E d\mu \leq \int_X |f| d\mu.$$

By hypothesis f is integrable over X , which means exactly that $\int_X |f| d\mu < \infty$. Therefore

$$\int_X |f \cdot \chi_E| d\mu \leq \int_X |f| d\mu < \infty,$$

so $f \cdot \chi_E$ is integrable over X ; that is, f is integrable over E .

Equivalently, one may quote the Integral Comparison Test with $g = |f|$: the measurable function $f \cdot \chi_E$ satisfies $|f \cdot \chi_E| \leq |f|$ on X and $|f|$ is integrable over X , so $f \cdot \chi_E$ is integrable over X and, by the estimate (15) accompanying that test,

$$\left| \int_E f \, d\mu \right| = \left| \int_X f \cdot \chi_E \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Applying the test instead with the dominating function $g = |f| \cdot \chi_E$, which is integrable over X by the first paragraph, sharpens this to

$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu \leq \int_X |f| \, d\mu,$$

the last inequality again by monotonicity (8).

18.5 Problems 18.29–18.35

Problem 18.29. Let f be a measurable function on X and A and B measurable subsets of X for which $X = A \cup B$ and $A \cap B = \emptyset$. Show that f is integrable over X if and only if it is integrable over both A and B .

Solution. Throughout, $(X, \mathcal{M}, \mu)$ is the ambient measure space. Recall the definition on page 373: for a measurable subset E of X , the function f is integrable over E provided $f \cdot \chi_E$ is integrable over X , that is, provided the nonnegative measurable function $|f \cdot \chi_E| = |f| \cdot \chi_E$ has finite integral over X . Since A and B are disjoint and $A \cup B = X$, we have $\chi_A + \chi_B = \chi_X \equiv 1$ on X , and therefore

$$|f| = |f| \cdot \chi_A + |f| \cdot \chi_B \quad \text{on } X.$$

The three functions $|f|$, $|f| \cdot \chi_A$ and $|f| \cdot \chi_B$ are nonnegative measurable functions on X : indeed $|f|$ is measurable since f is, and a product of a measurable function with the characteristic function of a measurable set is measurable. Hence Proposition 11, the linearity of integration for nonnegative measurable functions, applies with $\alpha = \beta = 1$ and yields

$$\int_X |f| \, d\mu = \int_X |f| \cdot \chi_A \, d\mu + \int_X |f| \cdot \chi_B \, d\mu = \int_A |f| \, d\mu + \int_B |f| \, d\mu. \quad (*)$$

Here the last equality is just the definition of the integral of a nonnegative measurable function over a measurable subset.

Both summands on the right of $(*)$ belong to $[0, \infty]$. Consequently the left-hand side of $(*)$ is finite if and only if both summands are finite. This is exactly the assertion to be proved:

If f is integrable over X , then $\int_X |f| \, d\mu < \infty$, so by $(*)$ both $\int_A |f| \, d\mu$ and $\int_B |f| \, d\mu$ are finite; that is, $|f \cdot \chi_A|$ and $|f \cdot \chi_B|$ have finite integral over X , so $f \cdot \chi_A$ and $f \cdot \chi_B$ are integrable over X , which means precisely that f is integrable over A and over B .

Conversely, if f is integrable over A and over B , then $\int_A |f| \, d\mu < \infty$ and $\int_B |f| \, d\mu < \infty$, so $(*)$ exhibits $\int_X |f| \, d\mu$ as the sum of two finite numbers, hence finite. Since f is measurable on X and $|f|$ is integrable over X , f is integrable over X .

Finally, when these equivalent conditions hold, the additivity over domains asserted in Theorem 12 gives the accompanying formula

$$\int_X f \, d\mu = \int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu,$$

which also follows directly from $(*)$ applied to f^+ and to f^- separately.

Problem 18.30. Let X be the disjoint union of the measurable sets $\{X_n\}_{n=1}^\infty$. For a measurable function f on X , characterize the integrability of f on X in terms of the integrability and the integral of f over the X_n 's.

Solution. We prove the following characterization.

A measurable function f on X is integrable over X if and only if f is integrable over each X_n and

$$\sum_{n=1}^{\infty} \int_{X_n} |f| d\mu < \infty,$$

Since a series of nonnegative terms with finite sum has every term finite, the first requirement is in fact implied by the second, so the criterion may be stated more briefly: f is integrable over X if and only if the series of the integrals of $|f|$ over the X_n 's converges. When it holds,

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_{X_n} f d\mu,$$

and this series converges absolutely.

First we record the countable additivity of the integral for a nonnegative measurable function h on X . Since $\{X_n\}$ is a disjoint collection whose union is X , we have $\sum_{n=1}^{\infty} \chi_{X_n} \equiv 1$ on X , and therefore the sequence of nonnegative measurable functions

$$h_N = \sum_{n=1}^N h \cdot \chi_{X_n} = h \cdot \chi_{\cup_{n=1}^N X_n}, \quad N \in \mathbb{N},$$

is increasing and converges pointwise on X to h . By the Monotone Convergence Theorem and the linearity of integration for nonnegative measurable functions (Proposition 11),

$$\int_X h d\mu = \lim_{N \rightarrow \infty} \int_X h_N d\mu = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_{X_n} h d\mu = \sum_{n=1}^{\infty} \int_{X_n} h d\mu. \quad (*)$$

Both sides of $(*)$ are elements of $[0, \infty]$ and the equality holds in $[0, \infty]$.

Now apply $(*)$ to the nonnegative measurable function $h = |f|$:

$$\int_X |f| d\mu = \sum_{n=1}^{\infty} \int_{X_n} |f| d\mu.$$

Suppose f is integrable over X . Then the left-hand side is finite, so the series on the right converges; in particular each of its terms is finite, that is, $\int_X |f \cdot \chi_{X_n}| d\mu = \int_{X_n} |f| d\mu < \infty$, so f is integrable over each X_n . Moreover, since f is integrable over X , Theorem 13, the countable additivity of integration over domains, gives

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_{X_n} f d\mu,$$

and by the Integral Comparison Test, $\left| \int_{X_n} f d\mu \right| \leq \int_{X_n} |f| d\mu$ for each n , so the series converges absolutely.

Conversely, suppose f is integrable over each X_n and $\sum_{n=1}^{\infty} \int_{X_n} |f| d\mu < \infty$. Then the right-hand side of the displayed identity is finite, hence $\int_X |f| d\mu < \infty$. Since f is measurable on X , f is integrable over X .

Two remarks on the sharpness of the criterion, both of which show that the series of integrals of $|f|$ cannot be replaced by any condition on the series $\sum_{n=1}^{\infty} \int_{X_n} f d\mu$ alone.

First, it is not enough to require that f be integrable over each X_n and that the series $\sum_{n=1}^{\infty} \int_{X_n} f d\mu$ merely converge. Indeed, let $X = \mathbb{N}$, let $\mathcal{M}$ be the σ -algebra of all subsets of $\mathbb{N}$, let μ be counting measure, and let $X_n = \{n\}$. Define $f(n) = (-1)^n/n$. Then f is integrable over each X_n with

$\int_{X_n} f d\mu = (-1)^n/n$, and $\sum_{n=1}^{\infty} (-1)^n/n$ converges. But $\sum_{n=1}^{\infty} \int_{X_n} |f| d\mu = \sum_{n=1}^{\infty} 1/n = \infty$, so by the identity above $\int_X |f| d\mu = \infty$ and f is not integrable over X .

Second, it is not even enough to require in addition that the series $\sum_{n=1}^{\infty} \int_{X_n} f d\mu$ converge absolutely. Take $X = \mathbb{N} \times \{1, 2\}$ with $\mathcal{M}$ the σ -algebra of all subsets of X , μ counting measure, and $X_n = \{(n, 1), (n, 2)\}$. Define $f(n, 1) = 1/n$ and $f(n, 2) = -1/n$. Then f is integrable over each X_n , and $\int_{X_n} f d\mu = 0$ for every n , so $\sum_{n=1}^{\infty} \left| \int_{X_n} f d\mu \right| = 0 < \infty$. Nevertheless $\sum_{n=1}^{\infty} \int_{X_n} |f| d\mu = \sum_{n=1}^{\infty} 2/n = \infty$, so f is not integrable over X . The correct criterion is genuinely a statement about the integrals of $|f|$ over the X_n 's, and the strongest conclusion one can draw about the series $\sum_{n=1}^{\infty} \int_{X_n} f d\mu$ is the absolute convergence recorded above.

Problem 18.31. Let $(X, \mathcal{M}, \mu)$ be a measure space for which $\mu(X) = 0$ and the function f on X take the constant value ∞ . Show that $\int_X f d\mu = 0$.

Solution. The function $f \equiv \infty$ is a nonnegative extended real-valued measurable function on X : for each real number c , $\{x \in X \mid f(x) > c\} = X$, which belongs to $\mathcal{M}$, so f is measurable by Proposition 1 of this chapter. Hence its integral over X is defined, by the definition on page 367, as

$$\int_X f d\mu = \sup \left\{ \int_X \varphi d\mu \mid \varphi \text{ simple, } 0 \leq \varphi \leq f \text{ on } X \right\}.$$

We show that every term in this family of integrals is zero.

Let φ be any simple function on X with $0 \leq \varphi \leq f$. A simple function is by definition real-valued and takes only finitely many values. If $\varphi \equiv 0$ on X , then $\int_X \varphi d\mu = 0$ by definition. Otherwise let $c_1, c_2, \dots, c_n$ be the positive values taken by φ on X and, for $1 \leq k \leq n$, let $E_k = \{x \in X \mid \varphi(x) = c_k\}$. Each E_k is a measurable subset of X , so by the monotonicity of the measure μ ,

$$0 \leq \mu(E_k) \leq \mu(X) = 0, \quad \text{that is, } \mu(E_k) = 0.$$

Each c_k is a positive real number, in particular finite, so $c_k \cdot \mu(E_k) = c_k \cdot 0 = 0$; no appeal to the convention $\infty \cdot 0 = 0$ in the definition (1) of the integral of a simple function is needed, since none of the $\mu(E_k)$ is infinite. Therefore

$$\int_X \varphi d\mu = \sum_{k=1}^n c_k \cdot \mu(E_k) = 0.$$

Thus the set of numbers whose supremum defines $\int_X f d\mu$ is exactly $\{0\}$: it is nonempty, since $\varphi \equiv 0$ is an admissible competitor, and every one of its members equals 0. Hence

$$\int_X f d\mu = 0.$$

Two remarks. First, the point of the exercise is that the integral of a nonnegative measurable function over a set of measure zero is zero even when the function is identically ∞ ; this is the general form of the convention $\infty \cdot 0 = 0$ already built into the definition of the integral of a simple function. Second, there is no conflict with Proposition 9, which asserts that a nonnegative measurable function with finite integral is finite a.e. on X : here $\int_X f d\mu = 0 < \infty$ and f is indeed finite a.e. on X , because the set $\{x \in X \mid f(x) = \infty\} = X$ has measure $\mu(X) = 0$. Every property whatsoever holds a.e. on a measure space of total measure zero.

Problem 18.32. Let f be integrable over X with respect to μ . Show that $\int_E f d\mu = 0$ for every measurable subset E of X if and only if $f = 0$ a.e. on X .

Solution. We first isolate the standard vanishing lemma, which is proved with Chebychev's Inequality. Lemma. If h is a nonnegative measurable function on X and $\int_X h d\mu = 0$, then $h = 0$ a.e. on X .

Proof of the Lemma. For each natural number n , Chebychev's Inequality (10) applied with $\lambda = 1/n$

gives

$$\mu \left\{ x \in X \mid h(x) \geq \frac{1}{n} \right\} \leq n \int_X h \, d\mu = 0.$$

Since

$$\{x \in X \mid h(x) > 0\} = \bigcup_{n=1}^{\infty} \left\{ x \in X \mid h(x) \geq \frac{1}{n} \right\},$$

the countable subadditivity of μ yields $\mu\{x \in X \mid h(x) > 0\} = 0$. As $h \geq 0$, this says $h = 0$ a.e. on X . This proves the lemma.

Now suppose $\int_E f \, d\mu = 0$ for every measurable subset E of X . Since f is measurable, the sets

$$E^+ = \{x \in X \mid f(x) > 0\}, \quad E^- = \{x \in X \mid f(x) < 0\}$$

are measurable subsets of X . Observe that

$$f \cdot \chi_{E^+} = f^+ \quad \text{and} \quad f \cdot \chi_{E^-} = -f^- \quad \text{on } X,$$

since at a point where $f > 0$ we have $f = f^+$ and $f^- = 0$, at a point where $f < 0$ we have $f = -f^-$ and $f^+ = 0$, and at a point where $f = 0$ all four functions vanish. Hence, by the definition of the integral over a measurable subset,

$$0 = \int_{E^+} f \, d\mu = \int_X f \cdot \chi_{E^+} \, d\mu = \int_X f^+ \, d\mu,$$

and, since the function $g = f \cdot \chi_{E^-} = -f^-$ has positive part $g^+ \equiv 0$ and negative part $g^- = f^-$, the very definition of the integral gives

$$0 = \int_{E^-} f \, d\mu = \int_X (-f^-) \, d\mu = - \int_X f^- \, d\mu, \quad \text{so} \quad \int_X f^- \, d\mu = 0.$$

(These integrals are finite because f is integrable over X , so that f^+ and f^- are integrable.) The functions f^+ and f^- are nonnegative and measurable, so the Lemma gives $f^+ = 0$ a.e. on X and $f^- = 0$ a.e. on X . The union of two sets of measure zero has measure zero, so off a set of measure zero we have $f = f^+ - f^- = 0$; that is, $f = 0$ a.e. on X .

Conversely, suppose $f = 0$ a.e. on X , and let E be a measurable subset of X . Choose a measurable subset X_0 of X with $\mu(X \setminus X_0) = 0$ and $f = 0$ on X_0 . Then the nonnegative measurable function $|f| \cdot \chi_E$ vanishes identically on X_0 , so by property (9) of the integral of a nonnegative measurable function,

$$\int_X |f| \cdot \chi_E \, d\mu = \int_{X_0} |f| \cdot \chi_E \, d\mu = 0.$$

Since $0 \leq (f \cdot \chi_E)^\pm \leq |f| \cdot \chi_E$ on X , the monotonicity property (8) gives $\int_X (f \cdot \chi_E)^+ \, d\mu = \int_X (f \cdot \chi_E)^- \, d\mu = 0$, and therefore

$$\int_E f \, d\mu = \int_X (f \cdot \chi_E)^+ \, d\mu - \int_X (f \cdot \chi_E)^- \, d\mu = 0.$$

As E was an arbitrary measurable subset of X , the proof is complete.

Problem 18.33. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a bounded measurable function on X that vanishes outside a set of finite measure. Show that

$$\int_X f \, d\mu = \sup \int_X \psi \, d\mu = \inf \int_X \varphi \, d\mu,$$

where ψ ranges over all simple functions on X for which $\psi \leq f$ on X and φ ranges over all simple functions on X for which $f \leq \varphi$ on X .

Solution. Fix $M \geq 0$ with $|f| \leq M$ on X , and a measurable set X_0 with $\mu(X_0) < \infty$ such that $f = 0$ on $X \sim X_0$. Then

$$-M \cdot \chi_{X_0} \leq f \leq M \cdot \chi_{X_0} \quad \text{on } X. \quad (1)$$

By Theorem 15, f is integrable over X , so $\int_X f \, d\mu$ is a real number.

Interpretation of the competing integrals. A simple function ψ with $\psi \leq f$ need not itself be integrable, so we must say what $\int_X \psi \, d\mu$ means. Since $\psi \leq f$, we have $\psi^+ \leq f^+ \leq M \cdot \chi_{X_0}$, so by the monotonicity property (8),

$$\int_X \psi^+ \, d\mu \leq M \cdot \mu(X_0) < \infty.$$

Hence $\int_X \psi \, d\mu := \int_X \psi^+ \, d\mu - \int_X \psi^- \, d\mu$ is unambiguously defined as an element of $[-\infty, \infty)$; it is real precisely when ψ is integrable, and equals $-\infty$ otherwise. Symmetrically, if φ is simple with $f \leq \varphi$, then $\varphi^- \leq f^- \leq M \cdot \chi_{X_0}$, so $\int_X \varphi^- \, d\mu < \infty$ and $\int_X \varphi \, d\mu$ is defined as an element of $(-\infty, \infty]$. With these conventions the supremum and infimum in the statement make sense; observe also that non-integrable competitors contribute $-\infty$ to the supremum and $+\infty$ to the infimum, so they are harmless.

Step 1: the supremum is at most $\int_X f \, d\mu$, which is at most the infimum. Let ψ be simple with $\psi \leq f$ on X . Then $\psi^+ \leq f^+$ and $f^- \leq \psi^-$ on X , so by (8),

$$\int_X \psi^+ \, d\mu \leq \int_X f^+ \, d\mu \quad \text{and} \quad \int_X f^- \, d\mu \leq \int_X \psi^- \, d\mu.$$

The two integrals of $f^\pm$ are finite and $\int_X \psi^+ \, d\mu$ is finite, so subtracting is legitimate in the extended reals and

$$\int_X \psi \, d\mu = \int_X \psi^+ \, d\mu - \int_X \psi^- \, d\mu \leq \int_X f^+ \, d\mu - \int_X f^- \, d\mu = \int_X f \, d\mu.$$

The symmetric argument gives $\int_X f \, d\mu \leq \int_X \varphi \, d\mu$ for every simple $\varphi \geq f$. Writing

$$S = \sup_{\psi \leq f} \int_X \psi \, d\mu, \quad I = \inf_{f \leq \varphi} \int_X \varphi \, d\mu,$$

we have shown $S \leq \int_X f \, d\mu \leq I$.

Step 2: the bounds are attained in the limit. Let $\epsilon > 0$. Since f is measurable and bounded on X , the Simple Approximation Lemma provides simple functions ψ_ϵ and φ_ϵ on X for which

$$\psi_\epsilon \leq f \leq \varphi_\epsilon \quad \text{and} \quad 0 \leq \varphi_\epsilon - \psi_\epsilon < \epsilon \quad \text{on } X.$$

These need not vanish outside a set of finite measure, so we truncate them: define

$$\tilde{\psi} = \psi_\epsilon \cdot \chi_{X_0}, \quad \tilde{\varphi} = \varphi_\epsilon \cdot \chi_{X_0}.$$

Both are simple functions on X that vanish outside X_0 , hence are bounded measurable functions vanishing outside a set of finite measure and so are integrable over X by Theorem 15. Moreover $\tilde{\psi} \leq f \leq \tilde{\varphi}$ on X : on X_0 this is the displayed inequality above, while on $X \sim X_0$ all three functions are equal to 0.

Next, $0 \leq f - \tilde{\psi} \leq \epsilon \cdot \chi_{X_0}$ on X . Indeed, on $X \sim X_0$ both sides are 0, while on X_0 we have $0 \leq f - \psi_\epsilon \leq \varphi_\epsilon - \psi_\epsilon < \epsilon$. Since $\tilde{\psi}$ is integrable, so is $-\tilde{\psi}$: its positive and negative parts are those of $\tilde{\psi}$ interchanged, so $|-\tilde{\psi}| = |\tilde{\psi}|$ has finite integral and, directly from the definition of the integral, $\int_X (-\tilde{\psi}) \, d\mu = \int_X \tilde{\psi}^- \, d\mu - \int_X \tilde{\psi}^+ \, d\mu = -\int_X \tilde{\psi} \, d\mu$. Applying to f and $-\tilde{\psi}$ the linearity of Theorem 12 with coefficients $\alpha = \beta = 1$, which is the case established in the text, together with the monotonicity of Theorem 12,

$$\int_X f \, d\mu - \int_X \tilde{\psi} \, d\mu = \int_X [f - \tilde{\psi}] \, d\mu \leq \int_X \epsilon \cdot \chi_{X_0} \, d\mu = \epsilon \cdot \mu(X_0).$$

Since $\tilde{\psi}$ is one of the competitors in the supremum,

$$S \geq \int_X \tilde{\psi} \, d\mu \geq \int_X f \, d\mu - \epsilon \cdot \mu(X_0).$$

The same reasoning applied to $0 \leq \tilde{\varphi} - f \leq \epsilon \cdot \chi_{X_0}$ yields

$$I \leq \int_X \tilde{\varphi} d\mu \leq \int_X f d\mu + \epsilon \cdot \mu(X_0).$$

Conclusion. The number $\mu(X_0)$ is finite, and $\epsilon > 0$ was arbitrary. Letting $\epsilon \rightarrow 0$ in the last two displays gives $S \geq \int_X f d\mu$ and $I \leq \int_X f d\mu$. Combined with Step 1 this forces

$$\int_X f d\mu = S = I,$$

which is the assertion.

Problem 18.34. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a bounded function on X that vanishes outside a set of finite measure. Assume

$$\sup \int_X \psi d\mu = \inf \int_X \varphi d\mu,$$

where ψ ranges over all simple functions on X for which $\psi \leq f$ on X and φ ranges over all simple functions on X for which $f \leq \varphi$ on X . Prove that f is measurable with respect to the completion of $(X, \mathcal{M}, \mu)$.

Solution. Note that f is not assumed measurable. Fix $M \geq 0$ with $|f| \leq M$ on X and a measurable set X_0 with $\mu(X_0) < \infty$ outside of which f vanishes, so that

$$-M \cdot \chi_{X_0} \leq f \leq M \cdot \chi_{X_0} \quad \text{on } X. \quad (1)$$

Write S for the supremum and I for the infimum in the hypothesis, and let c denote their common value. As in the preceding problem, the integrals occurring in S and I are well defined: if ψ is simple with $\psi \leq f$, then by (1) $\psi^+ \leq M \cdot \chi_{X_0}$, so $\int_X \psi^+ d\mu \leq M \cdot \mu(X_0) < \infty$ and $\int_X \psi d\mu \in [-\infty, \infty)$; symmetrically $\int_X \varphi d\mu \in (-\infty, \infty]$ for simple $\varphi \geq f$.

Step 1: c is a real number. The simple functions $-M \cdot \chi_{X_0}$ and $M \cdot \chi_{X_0}$ are competitors by (1) and are integrable, so

$$-M \cdot \mu(X_0) \leq S \quad \text{and} \quad I \leq M \cdot \mu(X_0).$$

Moreover $S \leq I$: if $\psi \leq f \leq \varphi$ with ψ, φ simple, then $\psi \leq \varphi$ on X , so $\psi^+ \leq \varphi^+$ and $\varphi^- \leq \psi^-$, and since $\int_X \psi^+ d\mu$ and $\int_X \varphi^- d\mu$ are finite, the monotonicity property (8) permits the subtraction

$$\int_X \psi d\mu = \int_X \psi^+ d\mu - \int_X \psi^- d\mu \leq \int_X \varphi^+ d\mu - \int_X \varphi^- d\mu = \int_X \varphi d\mu.$$

Taking the supremum over ψ and then the infimum over φ gives $S \leq I$. Hence $-\infty < -M \cdot \mu(X_0) \leq S = c = I \leq M \cdot \mu(X_0) < \infty$.

Step 2: a squeezing pair of measurable functions. Let n be a natural number. Since $S = c$ is finite, there is a simple $\psi_n \leq f$ with $\int_X \psi_n d\mu > c - 1/(2n)$; in particular this integral is finite, so ψ_n is integrable over X . Since $I = c$, there is likewise an integrable simple $\varphi_n \geq f$ with $\int_X \varphi_n d\mu < c + 1/(2n)$. Since ψ_n is integrable, so is $-\psi_n$, whose positive and negative parts are those of ψ_n interchanged, whence $\int_X (-\psi_n) d\mu = -\int_X \psi_n d\mu$ directly from the definition of the integral. Applying to φ_n and $-\psi_n$ the linearity of Theorem 12 with coefficients 1 and 1, which is the case established in the text,

$$\int_X [\varphi_n - \psi_n] d\mu = \int_X \varphi_n d\mu - \int_X \psi_n d\mu < \frac{1}{n}. \quad (2)$$

Define

$$\psi^* = \sup_{n \geq 1} \psi_n, \quad \varphi^* = \inf_{n \geq 1} \varphi_n \quad \text{on } X.$$

Each ψ_n is real-valued, and $\psi_1 \leq \psi^* \leq f \leq M$, so ψ^* is real-valued; similarly φ^* is real-valued, and

$$\psi^* \leq f \leq \varphi^* \quad \text{on } X.$$

These two functions are measurable with respect to $\mathcal{M}$: for each real a ,

$$\{x \in X \mid \psi^*(x) > a\} = \bigcup_{n=1}^{\infty} \{x \in X \mid \psi_n(x) > a\} \in \mathcal{M},$$

$$\{x \in X \mid \varphi^*(x) < a\} = \bigcup_{n=1}^{\infty} \{x \in X \mid \varphi_n(x) < a\} \in \mathcal{M},$$

so Proposition 1 applies to each.

Step 3: $\psi^* = \varphi^*$ except on a set of measure zero. Put $h = \varphi^* - \psi^*$, a nonnegative real-valued measurable function on X by Theorem 4. For a natural number k let

$$\Delta_k = \left\{x \in X \mid h(x) \geq \frac{1}{k}\right\}.$$

For every n we have $h \leq \varphi_n - \psi_n$ on X , so $\Delta_k \subseteq \{x \in X \mid \varphi_n(x) - \psi_n(x) \geq 1/k\}$. Since $\varphi_n - \psi_n$ is a nonnegative measurable function, Chebychev's Inequality (10) and (2) give

$$\mu(\Delta_k) \leq \mu\left\{x \in X \mid \varphi_n(x) - \psi_n(x) \geq \frac{1}{k}\right\} \leq k \int_X [\varphi_n - \psi_n] d\mu < \frac{k}{n}.$$

This holds for every n , so $\mu(\Delta_k) = 0$. Setting

$$\Delta = \{x \in X \mid h(x) > 0\} = \bigcup_{k=1}^{\infty} \Delta_k,$$

the countable subadditivity of μ gives $\Delta \in \mathcal{M}$ and $\mu(\Delta) = 0$.

Step 4: conclusion. On $X \sim \Delta$ we have $\varphi^* = \psi^*$, and since $\psi^* \leq f \leq \varphi^*$ this forces $f = \psi^*$ on $X \sim \Delta$. Let $(X, \mathcal{M}_0, \mu_0)$ be the completion of $(X, \mathcal{M}, \mu)$, described in Proposition 3 of Chapter 17: $\mathcal{M}_0$ consists of all sets of the form $A \cup B$ where $B \in \mathcal{M}$ and A is contained in a set $C \in \mathcal{M}$ with $\mu(C) = 0$. For each real number a ,

$$\{x \in X \mid f(x) > a\} = \left[\{x \in X \mid f(x) > a\} \cap \Delta\right] \cup \left[\{x \in X \mid \psi^*(x) > a\} \cap (X \sim \Delta)\right],$$

because the two sets Δ and $X \sim \Delta$ partition X and f agrees with ψ^* on the second. The second bracket belongs to $\mathcal{M}$, being the intersection of two $\mathcal{M}$ -measurable sets. The first bracket is a subset of Δ , a member of $\mathcal{M}$ of measure zero. Hence $\{x \in X \mid f(x) > a\}$ belongs to $\mathcal{M}_0$. Since a was an arbitrary real number, Proposition 1 tells us that f is measurable with respect to $\mathcal{M}_0$, that is, with respect to the completion of $(X, \mathcal{M}, \mu)$.

Remark. Once this is known, f is a bounded $\mathcal{M}_0$ -measurable function vanishing outside the set X_0 , which has finite μ_0 -measure, so f is integrable over X with respect to μ_0 by Theorem 15. Since μ_0 extends μ , every $\mathcal{M}$ -simple $\psi \leq f$ and $\varphi \geq f$ has the same integral against μ_0 as against μ , and Step 1 of the preceding problem, carried out in the complete space, gives $\int_X \psi d\mu \leq \int_X f d\mu_0 \leq \int_X \varphi d\mu$. Therefore $S \leq \int_X f d\mu_0 \leq I$, and the common value c is precisely $\int_X f d\mu_0$.

Problem 18.35. Prove the linearity property of integration for general coefficients α and β . [In the proof of Theorem 12 linearity was established only for the coefficients $\alpha = \beta = 1$. So: let $(X, \mathcal{M}, \mu)$ be a measure space and let f and g be integrable over X . Show that for all real numbers α and β the function $\alpha f + \beta g$ is integrable over X and

$$\int_X [\alpha f + \beta g] d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

]

Solution. In the proof of Theorem 12 the linearity was established for the coefficients $\alpha = \beta = 1$, and the general case was left as this exercise. We deduce it from that case together with the homogeneity statement, which we prove first.

Normalization. Since f and g are integrable over X , Proposition 9, applied to the nonnegative measurable functions $|f|$ and $|g|$, tells us that each of f and g is finite a.e. on X . Let

$$X_1 = \{x \in X \mid f(x) \text{ and } g(x) \text{ are both finite}\}.$$

Then X_1 is measurable and $\mu(X \sim X_1) = 0$. By property (9), the integral over X of any nonnegative measurable function equals its integral over X_1 , and hence, by taking positive and negative parts, the same is true of the integral of any function integrable over X . Also, by the Remark that precedes Theorem 12, the integral of a sum of integrable functions does not depend on the values assigned at the points where the summands are infinite of opposite sign, so the value assigned to $\alpha f + \beta g$ on the null set $X \sim X_1$ is immaterial. This is exactly the excision performed in the text's own proof of Theorem 12. We may therefore assume that f and g are real-valued on X . In particular $\alpha f + \beta g$ is then a real-valued measurable function on X , by the linearity part of Theorem 4.

Step 1: homogeneity, $\int_X \alpha f d\mu = \alpha \int_X f d\mu$ for every real α and every f integrable over X .

Case $\alpha = 0$. Then $\alpha f \equiv 0$ on X , so $(\alpha f)^+ = (\alpha f)^- \equiv 0$ and $\int_X \alpha f d\mu = 0 = 0 \cdot \int_X f d\mu$.

Case $\alpha > 0$. Since $\alpha > 0$, taking maxima commutes with multiplication by α , so

$$(\alpha f)^+ = \max\{\alpha f, 0\} = \alpha \max\{f, 0\} = \alpha f^+, \quad (\alpha f)^- = \alpha f^-.$$

By the positive homogeneity property (7) of the integral of a nonnegative measurable function,

$$\int_X (\alpha f)^\pm d\mu = \alpha \int_X f^\pm d\mu,$$

and both are finite because f is integrable. Since $|\alpha f| = (\alpha f)^+ + (\alpha f)^-$ then has finite integral, αf is integrable over X , and

$$\int_X \alpha f d\mu = \alpha \int_X f^+ d\mu - \alpha \int_X f^- d\mu = \alpha \int_X f d\mu.$$

Case $\alpha = -1$. Here $(-f)^+ = \max\{-f, 0\} = f^-$ and $(-f)^- = f^+$, so $|-f| = |f|$ is integrable, $-f$ is integrable, and directly from the definition of the integral,

$$\int_X (-f) d\mu = \int_X f^- d\mu - \int_X f^+ d\mu = - \int_X f d\mu.$$

Case $\alpha < 0$. Write $\alpha = -|\alpha|$ with $|\alpha| > 0$, so $\alpha f = |\alpha| \cdot (-f)$. By the case $\alpha = -1$, the function $-f$ is integrable over X with $\int_X (-f) d\mu = - \int_X f d\mu$; by the case of a positive coefficient applied to $-f$, the function $|\alpha| \cdot (-f) = \alpha f$ is integrable over X and

$$\int_X \alpha f d\mu = |\alpha| \int_X (-f) d\mu = -|\alpha| \int_X f d\mu = \alpha \int_X f d\mu.$$

This proves Step 1 in all cases.

Step 2: the general linearity. Let α and β be real numbers. By Step 1, αf and βg are integrable over X , with

$$\int_X \alpha f d\mu = \alpha \int_X f d\mu \quad \text{and} \quad \int_X \beta g d\mu = \beta \int_X g d\mu.$$

Apply to the pair of integrable functions αf and βg the case of Theorem 12 already proved, namely linearity with coefficients 1 and 1. It gives that $\alpha f + \beta g$ is integrable over X and

$$\int_X [\alpha f + \beta g] d\mu = \int_X \alpha f d\mu + \int_X \beta g d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

This is the asserted linearity for general coefficients.

Two consequences recorded in Theorem 12 now follow as in the text. Monotonicity: if $f \leq g$ a.e. on X , then $g - f \geq 0$ a.e., so $\int_X [g - f] d\mu \geq 0$ by the nonnegativity of the integral of a nonnegative function together with property (9), and hence, by the linearity just proved with $\alpha = 1$, $\beta = -1$,

$$\int_X g d\mu - \int_X f d\mu = \int_X [g - f] d\mu \geq 0.$$

Additivity over domains: if A and B are disjoint measurable subsets of X , then $f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B$ on X , so linearity with $\alpha = \beta = 1$ gives $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$.

18.6 Problems 18.36–18.42

Problem 18.36. Let $\{f_n\}$ be a sequence of integrable functions on X that is uniformly integrable and tight. Suppose that $\{f_n\} \rightarrow f$ pointwise a.e. on X and f is measurable and finite a.e. on X . Prove that f is integrable over X .

Solution. Throughout, $(X, \mathcal{M}, \mu)$ is the reference measure space. Since each f_n is integrable it is finite a.e. on X , and by hypothesis f is finite a.e. on X and $\{f_n\} \rightarrow f$ pointwise a.e. on X . Each of these statements holds off a measurable set of measure zero; let Z be the union of these countably many sets, so Z is measurable and $\mu(Z) = 0$ by the countable subadditivity of measure. For a nonnegative measurable function h on X we have $\int_Z h d\mu = 0$, since $\mu(Z) = 0$, and hence, by the additivity of integration over the disjoint decomposition $X = Z \dot{\cup} (X \sim Z)$, $\int_X h d\mu = \int_{X \sim Z} h d\mu$. Excising Z therefore changes no integral of a nonnegative measurable function and preserves both uniform integrability and tightness, so we may and do replace X by $X \sim Z$ and assume that every f_n and f is real-valued on X and that $\{f_n\} \rightarrow f$ pointwise on all of X .

Since f is measurable, $|f|$ is a nonnegative measurable function, and f is integrable over X if and only if $\int_X |f| d\mu < \infty$. We prove this bound.

Apply the definition of uniform integrability with $\epsilon = 1$: there is a $\delta > 0$ such that

$$\text{if } E \subseteq X \text{ is measurable and } \mu(E) < \delta, \text{ then } \int_E |f_n| d\mu < 1 \text{ for all } n.$$

Apply the definition of tightness with $\epsilon = 1$: there is a measurable subset X_0 of X with $\mu(X_0) < \infty$ and

$$\int_{X \sim X_0} |f_n| d\mu < 1 \quad \text{for all } n.$$

The set X_0 has finite measure, the functions f_n are real-valued and measurable, $\{f_n\} \rightarrow f$ pointwise on X_0 , and f is real-valued. Therefore Egoroff's Theorem applies on the finite measure space $(X_0, \mathcal{M}_0, \mu)$, where $\mathcal{M}_0 = \{E \in \mathcal{M} \mid E \subseteq X_0\}$ is a σ -algebra of subsets of X_0 and the restrictions of the f_n and of f to X_0 are $\mathcal{M}_0$ -measurable: there is a measurable subset X_1 of X_0 for which

$$\mu(X_0 \sim X_1) < \delta \quad \text{and} \quad \{f_n\} \rightarrow f \text{ uniformly on } X_1.$$

We first bound $\int_{X_1} |f| d\mu$. By the uniform convergence on X_1 there is an index N for which $|f_N(x) - f(x)| < 1$ for all $x \in X_1$, and hence

$$|f| \leq |f_N| + 1 \quad \text{on } X_1.$$

Since $X_1 \subseteq X_0$ we have $\mu(X_1) \leq \mu(X_0) < \infty$, so the constant function 1 is integrable over X_1 ; and f_N is integrable over X , hence over the measurable subset X_1 . By the monotonicity of integration for nonnegative measurable functions,

$$\int_{X_1} |f| d\mu \leq \int_{X_1} |f_N| d\mu + \mu(X_1) < \infty.$$

Next we bound $\int_{X \sim X_1} |f| d\mu$. Write

$$X \sim X_1 = (X \sim X_0) \dot{\cup} (X_0 \sim X_1),$$

a disjoint union of measurable sets. For every n , the first piece contributes less than 1 by tightness, and the second piece has measure less than δ so contributes less than 1 by uniform integrability. Hence, by the additivity of integration over domains applied to the nonnegative measurable function $|f_n|$,

$$\int_{X \sim X_1} |f_n| d\mu = \int_{X \sim X_0} |f_n| d\mu + \int_{X_0 \sim X_1} |f_n| d\mu < 1 + 1 = 2 \quad \text{for all } n.$$

Now $\{|f_n|\} \rightarrow |f|$ pointwise on $X \sim X_1$, each $|f_n|$ is a nonnegative measurable function, and $|f|$ is measurable. By Fatou's Lemma applied on the measure space $X \sim X_1$,

$$\int_{X \sim X_1} |f| d\mu \leq \liminf_{n \rightarrow \infty} \int_{X \sim X_1} |f_n| d\mu \leq 2.$$

Finally, by additivity over the disjoint decomposition $X = X_1 \dot{\cup} (X \sim X_1)$,

$$\int_X |f| d\mu = \int_{X_1} |f| d\mu + \int_{X \sim X_1} |f| d\mu \leq \int_{X_1} |f_N| d\mu + \mu(X_1) + 2 < \infty.$$

Since f is measurable and $\int_X |f| d\mu < \infty$, the function f is integrable over X .

Two remarks on the hypotheses. The assumption that f is finite a.e. cannot be dropped: it is exactly what makes Egoroff's Theorem available, and the example given in the text preceding these problems (an atom E of measure $1/2$ and $f_n = n \cdot \chi_E - n \cdot \chi_{X \sim E}$) shows a uniformly integrable and tight sequence whose pointwise limit takes the values $\pm\infty$ and is not integrable. Also, uniform integrability and tightness do not by themselves bound $\int_X |f_n| d\mu$ uniformly in n on a general measure space, precisely because a set of finite measure need not be decomposable into small pieces; this is why the argument above localizes with Egoroff's Theorem rather than applying Fatou's Lemma over all of X .

Problem 18.37. Let $\{f_n\}$ be a sequence of integrable functions on X that is uniformly integrable. Suppose that $\{f_n\} \rightarrow f$ pointwise a.e. on X and f is measurable. Assume the measure space has the property that for each $\epsilon > 0$, X is the union of a finite collection of measurable sets, each of measure at most ϵ . Prove that f is integrable over X .

Solution. The strategy is to show that the assumed decomposition property, together with uniform integrability, forces the numbers $\int_X |f_n| d\mu$ to be bounded uniformly in n ; Fatou's Lemma then bounds $\int_X |f| d\mu$.

Apply the definition of uniform integrability with $\epsilon = 1$: there is a $\delta > 0$ such that

$$\text{if } E \subseteq X \text{ is measurable and } \mu(E) < \delta, \text{ then } \int_E |f_n| d\mu < 1 \text{ for every natural number } n.$$

Apply the assumed property of the measure space with $\epsilon = \delta/2$: there are measurable sets $E_1, \dots, E_m$ with

$$X = \bigcup_{k=1}^m E_k \quad \text{and} \quad \mu(E_k) \leq \delta/2 < \delta \text{ for } 1 \leq k \leq m.$$

Disjointify this cover: set $F_1 = E_1$ and, for $2 \leq k \leq m$,

$$F_k = E_k \sim \bigcup_{j < k} E_j.$$

Each F_k is measurable, the F_k are pairwise disjoint, $\bigcup_{k=1}^m F_k = X$, and $F_k \subseteq E_k$, so by the monotonicity of measure $\mu(F_k) \leq \mu(E_k) < \delta$ for each k .

Fix a natural number n . The function $|f_n|$ is nonnegative and measurable, so by the additivity of integration over domains applied finitely often to the disjoint decomposition $X = F_1 \dot{\cup} \dots \dot{\cup} F_m$, and then by the choice of δ ,

$$\int_X |f_n| d\mu = \sum_{k=1}^m \int_{F_k} |f_n| d\mu < \sum_{k=1}^m 1 = m.$$

The index m was chosen before n and does not depend on n . Therefore

$$\int_X |f_n| d\mu \leq m \quad \text{for every natural number } n.$$

Now $\{|f_n|\}$ is a sequence of nonnegative measurable functions on X with $\{|f_n|\} \rightarrow |f|$ pointwise a.e. on X , and $|f|$ is measurable since f is. By Fatou's Lemma,

$$\int_X |f| d\mu \leq \liminf_{n \rightarrow \infty} \int_X |f_n| d\mu \leq m < \infty.$$

Since f is measurable and $\int_X |f| d\mu < \infty$, the function f is integrable over X . Note that finiteness of f was not assumed here, in contrast with the preceding problem: it is a conclusion, since Proposition 9, applied to the nonnegative measurable function $|f|$ whose integral we have just shown to be finite, tells us that $|f|$, and hence f , is finite a.e. on X .

It is worth recording what the assumed decomposition property really provides. First, taking any single $\epsilon > 0$ shows X is a finite union of sets of finite measure, so $\mu(X) < \infty$. Second, the space has no atoms: if A were a measurable set with $\mu(A) = a > 0$ such that every measurable subset of A has measure 0 or a , then writing $X = \bigcup_{k=1}^m E_k$ with $\mu(E_k) \leq a/2$ gives $A = \bigcup_{k=1}^m (A \cap E_k)$, so by the countable (here finite) subadditivity of measure some $A \cap E_k$ has positive measure, whence $\mu(A \cap E_k) = a$; but $\mu(A \cap E_k) \leq \mu(E_k) \leq a/2 < a$, a contradiction. So the hypothesis rules out precisely the feature that defeats the conclusion in the example of the text preceding these problems, where E and $X \sim E$ are atoms of measure $1/2$ and no finite cover by sets of measure at most $1/4$ exists.

Problem 18.38. Prove Corollary 18. [Corollary 18: Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{h_n\}$ a sequence of nonnegative integrable functions on X . Suppose that $\{h_n(x)\} \rightarrow 0$ for almost all x in X . Then

$$\lim_{n \rightarrow \infty} \int_X h_n d\mu = 0 \quad \text{if and only if} \quad \{h_n\} \text{ is uniformly integrable and tight.}$$

]

Solution. Assume $\{h_n\}$ is a sequence of nonnegative integrable functions on X with $\{h_n(x)\} \rightarrow 0$ for almost all $x \in X$. Since each h_n is nonnegative, $|h_n| = h_n$, and we use this identification without further comment.

Suppose first that $\{h_n\}$ is uniformly integrable and tight. The constant function $f \equiv 0$ is measurable and integrable over X , and $\{h_n\} \rightarrow f$ pointwise a.e. on X . Therefore the Vitali Convergence Theorem applies and yields

$$\lim_{n \rightarrow \infty} \int_X h_n d\mu = \int_X 0 d\mu = 0.$$

Conversely, suppose $\lim_{n \rightarrow \infty} \int_X h_n d\mu = 0$. Let $\epsilon > 0$. By the definition of the limit of a sequence of real numbers there is a natural number N for which

$$\int_X h_n d\mu < \epsilon \quad \text{for all } n > N.$$

Each of the finitely many functions $h_1, \dots, h_N$ is integrable over X . By Proposition 17, for each $1 \leq k \leq N$ there is a $\delta_k > 0$ such that

$$\text{if } E \subseteq X \text{ is measurable and } \mu(E) < \delta_k, \text{ then } \int_E h_k d\mu < \epsilon,$$

and there is a measurable subset X_k of X with $\mu(X_k) < \infty$ and

$$\int_{X \sim X_k} h_k d\mu < \epsilon.$$

Define

$$\delta = \min\{\delta_1, \dots, \delta_N\} > 0 \quad \text{and} \quad X_0 = \bigcup_{k=1}^N X_k,$$

so that δ is positive, being the minimum of finitely many positive numbers, and X_0 is measurable with $\mu(X_0) \leq \sum_{k=1}^N \mu(X_k) < \infty$ by the finite subadditivity and monotonicity of measure.

We verify uniform integrability with this δ . Let E be a measurable subset of X with $\mu(E) < \delta$ and let n be a natural number. If $n \leq N$, then $\mu(E) < \delta \leq \delta_n$, so $\int_E h_n d\mu < \epsilon$ by the choice of δ_n . If $n > N$, then since $h_n \geq 0$ the monotonicity of integration over domains gives

$$\int_E h_n d\mu \leq \int_X h_n d\mu < \epsilon.$$

In either case $\int_E h_n d\mu < \epsilon$. Thus $\{h_n\}$ is uniformly integrable.

We verify tightness with the set X_0 , which has finite measure. Let n be a natural number. If $n \leq N$, then $X \sim X_0 \subseteq X \sim X_n$, and since $h_n \geq 0$,

$$\int_{X \sim X_0} h_n d\mu \leq \int_{X \sim X_n} h_n d\mu < \epsilon.$$

If $n > N$, then again since $h_n \geq 0$,

$$\int_{X \sim X_0} h_n d\mu \leq \int_X h_n d\mu < \epsilon.$$

In either case $\int_{X \sim X_0} h_n d\mu < \epsilon$. Thus $\{h_n\}$ is tight.

Since $\epsilon > 0$ was arbitrary, $\{h_n\}$ is uniformly integrable and tight, and the corollary is proved.

A remark on which hypotheses each direction consumed. The first implication used the pointwise convergence $\{h_n\} \rightarrow 0$ a.e. essentially, since it identifies the limit function to which the Vitali Convergence Theorem is applied, and it used that the zero function is integrable over X , which is the hypothesis of that theorem that does not come for free on a general measure space. The second implication used no pointwise convergence at all: only the nonnegativity of the h_n , the integrability of each individual h_n , and the convergence of the numbers $\int_X h_n d\mu$ to 0 entered. Nonnegativity is genuinely needed there, since it is what makes $\int_X h_n d\mu$ equal to $\int_X |h_n| d\mu$ and thereby lets smallness of the integral over X bound the integral over every measurable subset of X ; for a sequence of sign-changing functions the convergence $\int_X h_n d\mu \rightarrow 0$ implies neither uniform integrability nor tightness.

Problem 18.39. Deduce the Lebesgue Dominated Convergence Theorem from the Vitali Convergence Theorem.

Solution. We are given a measure space $(X, \mathcal{M}, \mu)$ and a sequence $\{f_n\}$ of measurable functions on X for which $\{f_n\} \rightarrow f$ pointwise a.e. on X and f is measurable, together with a nonnegative function g that is integrable over X and dominates $\{f_n\}$ in the sense that

$$|f_n| \leq g \quad \text{a.e. on } X, \text{ for all } n.$$

We must show that f is integrable over X and that $\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$.

As in the proofs of the text we may excise from X a set of measure zero, off which all the a.e. statements hold simultaneously; this changes no integral. So assume $|f_n| \leq g$ on all of X for every n and $\{f_n\} \rightarrow f$ pointwise on X .

Each f_n is integrable over X . Indeed, f_n is measurable and $|f_n| \leq g$ on X with g integrable, so by the integral comparison test f_n is integrable over X . This matters, because the definitions of uniform integrability and of tightness are formulated only for sequences of functions each of which is integrable over X .

The limit function f is integrable over X . Indeed, passing to the limit in the inequalities $|f_n(x)| \leq g(x)$ gives $|f(x)| \leq g(x)$ for every $x \in X$; since f is measurable and g is integrable, the integral comparison

test again shows f is integrable over X .

The sequence $\{f_n\}$ is uniformly integrable. Let $\epsilon > 0$. Since g is integrable over X , Proposition 17 provides a $\delta > 0$ such that for every measurable subset E of X ,

$$\text{if } \mu(E) < \delta, \text{ then } \int_E g \, d\mu < \epsilon.$$

For such an E and any natural number n , the monotonicity of integration for nonnegative measurable functions and the inequality $|f_n| \leq g$ give

$$\int_E |f_n| \, d\mu \leq \int_E g \, d\mu < \epsilon.$$

Since the same δ works for all n , the sequence $\{f_n\}$ is uniformly integrable over X .

The sequence $\{f_n\}$ is tight. Let $\epsilon > 0$. By the second assertion of Proposition 17, namely (22), there is a subset X_0 of X that has finite measure and for which

$$\int_{X \sim X_0} g \, d\mu < \epsilon.$$

Then for every natural number n , again by monotonicity,

$$\int_{X \sim X_0} |f_n| \, d\mu \leq \int_{X \sim X_0} g \, d\mu < \epsilon.$$

Since the same X_0 works for all n , the sequence $\{f_n\}$ is tight over X .

We have verified every hypothesis of the Vitali Convergence Theorem: $\{f_n\}$ is a sequence of functions on X that is both uniformly integrable and tight over X , $\{f_n\} \rightarrow f$ pointwise a.e. on X , and f is integrable over X . That theorem therefore yields

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

This is precisely the conclusion of the Lebesgue Dominated Convergence Theorem, whose integrability assertion for f was established above. Hence the Lebesgue Dominated Convergence Theorem follows from the Vitali Convergence Theorem, with Proposition 17 supplying the two uniformity properties out of the single integrable dominating function g .

Problem 18.40. Show that almost everywhere convergence can be replaced by convergence in measure in the Lebesgue Dominated Convergence Theorem and the Vitali Convergence Theorem (see Problem 15 for the definition of convergence in measure).

Solution. Recall from Problem 15 that a sequence $\{f_n\}$ of measurable real-valued functions on X converges in measure to a measurable function f provided that for each $\eta > 0$,

$$\lim_{n \rightarrow \infty} \mu\{x \in X \mid |f_n(x) - f(x)| > \eta\} = 0.$$

We prove the two assertions:

(A) (Dominated convergence in measure.) Let $\{f_n\}$ be a sequence of measurable real-valued functions on X that converges in measure on X to the measurable function f . Assume there is a nonnegative g that is integrable over X and for which $|f_n| \leq g$ a.e. on X for all n . Then f is integrable over X and $\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

(B) (Vitali convergence in measure.) Let $\{f_n\}$ be a sequence of measurable real-valued functions on X , each integrable over X , that is both uniformly integrable and tight over X . Assume $\{f_n\} \rightarrow f$ in measure on X and that f is integrable over X . Then $\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

Two preliminary observations do the work.

First observation (the Riesz subsequence property, Problem 15(ii)). If $\{h_n\} \rightarrow h$ in measure on X , then

some subsequence of $\{h_n\}$ converges to h pointwise a.e. on X . For completeness we recall the proof, which is valid on an arbitrary measure space. For each natural number k , convergence in measure with $\eta = 1/k$ furnishes an index n_k , which we may take strictly increasing in k , for which the measurable set

$$E_k = \left\{ x \in X \mid |h_{n_k}(x) - h(x)| > \frac{1}{k} \right\}$$

satisfies $\mu(E_k) < 2^{-k}$; here E_k is measurable because h_{n_k} and h are measurable real-valued functions, so $|h_{n_k} - h|$ is measurable. Consider the set

$$E = \bigcap_{m=1}^{\infty} \bigcup_{k \geq m} E_k$$

of points belonging to infinitely many E_k . This is the Borel–Cantelli Lemma, and we give its one-line proof, which uses only the monotonicity and countable subadditivity of measure and so is valid on an arbitrary measure space: for each m we have $E \subseteq \bigcup_{k \geq m} E_k$, whence

$$\mu(E) \leq \sum_{k \geq m} \mu(E_k) < \sum_{k \geq m} 2^{-k} = 2^{1-m},$$

and letting $m \rightarrow \infty$ gives $\mu(E) = 0$. If $x \in X \sim E$, then x belongs to only finitely many E_k , so there is an m with $|h_{n_k}(x) - h(x)| \leq 1/k$ for all $k \geq m$; hence $\{h_{n_k}(x)\} \rightarrow h(x)$. Thus $\{h_{n_k}\} \rightarrow h$ pointwise on $X \sim E$, that is, pointwise a.e. on X .

Second observation (a subsequence criterion for numerical sequences). Let $\{a_n\}$ be a sequence of real numbers and a a real number. If every subsequence of $\{a_n\}$ has a further subsequence that converges to a , then $\{a_n\} \rightarrow a$. Indeed, otherwise there is an $\epsilon_0 > 0$ and a subsequence $\{a_{n_k}\}$ with $|a_{n_k} - a| \geq \epsilon_0$ for all k ; no further subsequence of $\{a_{n_k}\}$ can converge to a , a contradiction.

Proof of (B). Note first that a subsequence of a sequence that converges in measure to f again converges in measure to f , since the defining limit is a limit of a sequence of nonnegative numbers and every subsequence of a convergent sequence converges to the same limit. Note also that any subsequence of a uniformly integrable and tight sequence is itself uniformly integrable and tight, since the very same δ and X_0 serve, the defining conditions being required of all indices.

Put $a_n = \int_X f_n d\mu$ and $a = \int_X f d\mu$; these are real numbers because each f_n and f is integrable over X . Let $\{a_{n_k}\}$ be any subsequence. Then $\{f_{n_k}\} \rightarrow f$ in measure on X , so by the first observation there is a further subsequence $\{f_{n_{k_j}}\}$ with $\{f_{n_{k_j}}\} \rightarrow f$ pointwise a.e. on X . This further subsequence is uniformly integrable and tight, and f is integrable over X ; hence the Vitali Convergence Theorem, in the form proved in the text, gives

$$\lim_{j \rightarrow \infty} a_{n_{k_j}} = \lim_{j \rightarrow \infty} \int_X f_{n_{k_j}} d\mu = \int_X f d\mu = a.$$

So every subsequence of $\{a_n\}$ has a further subsequence converging to a . By the second observation, $\{a_n\} \rightarrow a$, which is the assertion of (B).

Proof of (A). First, f is integrable over X . By the first observation there is a subsequence $\{f_{n_k}\}$ converging to f pointwise a.e. on X . Since $|f_{n_k}| \leq g$ a.e. on X for each k , and the union of countably many sets of measure zero has measure zero, we may pass to the limit a.e. and conclude $|f| \leq g$ a.e. on X . As f is measurable and g is integrable over X , the integral comparison test shows f is integrable over X ; the same test shows each f_n is integrable over X .

Now argue exactly as in (B), with the Lebesgue Dominated Convergence Theorem in place of the Vitali Convergence Theorem. Put $a_n = \int_X f_n d\mu$ and $a = \int_X f d\mu$, and let $\{a_{n_k}\}$ be any subsequence. Then $\{f_{n_k}\} \rightarrow f$ in measure, so a further subsequence $\{f_{n_{k_j}}\}$ converges to f pointwise a.e. on X . That further subsequence is still dominated by g a.e. on X , and f is measurable, so the Lebesgue Dominated Convergence Theorem gives $\lim_{j \rightarrow \infty} a_{n_{k_j}} = a$. By the second observation, $\{a_n\} \rightarrow a$. This is the assertion of (A).

Alternatively, (A) follows from (B) once one observes, as in the preceding problem, that domination by an integrable g together with Proposition 17 makes $\{f_n\}$ uniformly integrable and tight.

Problem 18.41. Let $\{f_n\}$ be a sequence of functions on X , each of which is integrable over X . Show that $\{f_n\}$ is uniformly integrable if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for any natural number n and measurable subset E of X ,

$$\text{if } \mu(E) < \delta, \text{ then } \left| \int_E f_n d\mu \right| < \epsilon.$$

Solution. Since each f_n is integrable over X , it is integrable over every measurable subset E of X , so all the integrals written below are real numbers. Moreover each f_n is measurable, and by Proposition 9 it is finite a.e. on X ; as usual we may excise a set of measure zero and assume each f_n is real-valued, which changes no integral and neither of the two conditions.

Suppose first that $\{f_n\}$ is uniformly integrable. Let $\epsilon > 0$ and let $\delta > 0$ be as in the definition of uniform integrability for this ϵ , so that $\int_E |f_n| d\mu < \epsilon$ whenever $E \subseteq X$ is measurable with $\mu(E) < \delta$ and n is any natural number. For such E and n , the inequality $|\int_E f_n d\mu| \leq \int_E |f_n| d\mu$, which follows from $-|f_n| \leq f_n \leq |f_n|$ and the monotonicity and linearity of integration, gives

$$\left| \int_E f_n d\mu \right| \leq \int_E |f_n| d\mu < \epsilon.$$

Thus the stated condition holds with the same δ .

Conversely, suppose the stated condition holds. Let $\epsilon > 0$. Apply the condition with $\epsilon/2$ in place of ϵ to obtain a $\delta > 0$ such that

$$\text{if } E \subseteq X \text{ is measurable and } \mu(E) < \delta, \text{ then } \left| \int_E f_n d\mu \right| < \epsilon/2 \text{ for every natural number } n.$$

Let E be a measurable subset of X with $\mu(E) < \delta$ and let n be a natural number. Since f_n is measurable, the sets

$$E^+ = \{x \in E \mid f_n(x) \geq 0\} \quad \text{and} \quad E^- = \{x \in E \mid f_n(x) < 0\}$$

are measurable. They are disjoint, their union is E , and each is a subset of E , so by the monotonicity of measure $\mu(E^+) < \delta$ and $\mu(E^-) < \delta$. Note that E^+ and E^- depend on n , but this is harmless: the displayed estimate above holds for every measurable set of measure less than δ and every index simultaneously.

On E^+ we have $|f_n| = f_n$, and on E^- we have $|f_n| = -f_n$. Hence, by the additivity of integration over domains and the linearity of integration,

$$\int_{E^+} |f_n| d\mu = \int_{E^+} f_n d\mu = \left| \int_{E^+} f_n d\mu \right|,$$

the last equality because the middle quantity is the integral of a nonnegative function and so is non-negative, and

$$\int_{E^-} |f_n| d\mu = - \int_{E^-} f_n d\mu = \left| \int_{E^-} f_n d\mu \right|,$$

the last equality because $\int_{E^-} f_n d\mu$ is the negative of a nonnegative number. Therefore, using the additivity of integration over the disjoint decomposition $E = E^+ \dot{\cup} E^-$ for the nonnegative measurable function $|f_n|$, and then the choice of δ applied to the two sets E^+ and E^- ,

$$\int_E |f_n| d\mu = \int_{E^+} |f_n| d\mu + \int_{E^-} |f_n| d\mu = \left| \int_{E^+} f_n d\mu \right| + \left| \int_{E^-} f_n d\mu \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since E was an arbitrary measurable subset of X of measure less than δ and n an arbitrary natural number, and since δ depends only on ϵ , the sequence $\{f_n\}$ is uniformly integrable.

Problem 18.42. Let ν be another measure on $\mathcal{M}$. For an extended real-valued function f on X that is measurable with respect to the measurable space $(X, \mathcal{M})$, under what conditions is it true that

$$\int_X f d[\mu + \nu] = \int_X f d\mu + \int_X f d\nu ?$$

Solution. Answer. The identity holds for every nonnegative measurable f , with both sides interpreted in $[0, \infty]$ and no hypothesis whatever. For a general extended real-valued measurable f the identity holds precisely when f is integrable with respect to the measure $\mu + \nu$, and this happens if and only if f is integrable with respect to μ and integrable with respect to ν . (If one extends the definition of the integral to functions for which only one of $\int_X f^+$, $\int_X f^-$ is infinite, the identity persists whenever $\int_X f d[\mu + \nu]$ is defined in that extended sense; the only obstruction is the meaningless expression $\infty - \infty$.)

Set $\lambda = \mu + \nu$, that is, $\lambda(E) = \mu(E) + \nu(E)$ for $E \in \mathcal{M}$.

Step 0: λ is a measure on $\mathcal{M}$. Certainly $\lambda(\emptyset) = 0 + 0 = 0$ and $\lambda \geq 0$. If $\{E_k\}_{k=1}^\infty$ is a disjoint countable collection of members of $\mathcal{M}$, then

$$\lambda\left(\bigcup_{k=1}^\infty E_k\right) = \mu\left(\bigcup_{k=1}^\infty E_k\right) + \nu\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu(E_k) + \sum_{k=1}^\infty \nu(E_k) = \sum_{k=1}^\infty [\mu(E_k) + \nu(E_k)] = \sum_{k=1}^\infty \lambda(E_k),$$

the second equality by the countable additivity of μ and of ν , the third because two series of nonnegative terms in $[0, \infty]$ may be added term by term. So $(X, \mathcal{M}, \lambda)$ is a measure space, and $\mathcal{M}$ -measurability of a function is the same notion for all three measures, since it refers only to the measurable space $(X, \mathcal{M})$.

Step 1: characteristic functions. For $E \in \mathcal{M}$,

$$\int_X \chi_E d\lambda = \lambda(E) = \mu(E) + \nu(E) = \int_X \chi_E d\mu + \int_X \chi_E d\nu,$$

directly from the definition of the integral of a nonnegative simple function.

Step 2: nonnegative simple functions. Let ψ be a nonnegative simple function on X , and write $\psi = \sum_{k=1}^m c_k \cdot \chi_{E_k}$ in canonical form, where $c_1, \dots, c_m$ are the distinct positive values taken by ψ and $E_k = \{x \in X \mid \psi(x) = c_k\}$. By the definition of the integral of a nonnegative simple function,

$$\int_X \psi d\lambda = \sum_{k=1}^m c_k \lambda(E_k) = \sum_{k=1}^m c_k \mu(E_k) + \sum_{k=1}^m c_k \nu(E_k) = \int_X \psi d\mu + \int_X \psi d\nu,$$

where the middle equality is a regrouping of a finite sum of terms in $[0, \infty]$, all nonnegative, so no cancellation issue arises.

Step 3: nonnegative measurable functions. Let f be a nonnegative measurable function on X . By the Simple Approximation Theorem there is an increasing sequence $\{\psi_k\}$ of nonnegative simple functions on X that converges pointwise on X to f . Apply the Monotone Convergence Theorem three times, once for each of the measure spaces $(X, \mathcal{M}, \lambda)$, $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{M}, \nu)$, and use Step 2:

$$\int_X f d\lambda = \lim_{k \rightarrow \infty} \int_X \psi_k d\lambda = \lim_{k \rightarrow \infty} \left[\int_X \psi_k d\mu + \int_X \psi_k d\nu \right] = \int_X f d\mu + \int_X f d\nu.$$

The last equality is legitimate in $[0, \infty]$ because each of the two sequences $\{\int_X \psi_k d\mu\}$ and $\{\int_X \psi_k d\nu\}$ is nondecreasing, by the monotonicity of integration, so their sum converges in $[0, \infty]$ to the sum of their limits. Thus

$$\int_X f d[\mu + \nu] = \int_X f d\mu + \int_X f d\nu \quad \text{for every nonnegative measurable } f,$$

with the convention that the identity is an identity of elements of $[0, \infty]$.

Step 4: the integrability criterion. Let f be any extended real-valued measurable function on X . Applying Step 3 to the nonnegative measurable function $|f|$,

$$\int_X |f| d\lambda = \int_X |f| d\mu + \int_X |f| d\nu.$$

Since all three terms lie in $[0, \infty]$, the left side is finite if and only if both terms on the right are finite. That is:

$$f \text{ is integrable with respect to } \mu + \nu \iff f \text{ is integrable with respect to both } \mu \text{ and } \nu.$$

Step 5: the general identity. Assume f is integrable with respect to $\mu + \nu$, equivalently, by Step 4, with respect to both μ and ν . The nonnegative measurable functions f^+ and f^- satisfy $f^+ \leq |f|$ and $f^- \leq |f|$, so all six integrals $\int_X f^\pm d\lambda$, $\int_X f^\pm d\mu$, $\int_X f^\pm d\nu$ are finite. By Step 3 applied to f^+ and to f^- , and then by the definition $\int_X f = \int_X f^+ - \int_X f^-$ together with the rearrangement of finitely many real numbers,

$$\begin{aligned} \int_X f d\lambda &= \int_X f^+ d\lambda - \int_X f^- d\lambda \\ &= \left[\int_X f^+ d\mu + \int_X f^+ d\nu \right] - \left[\int_X f^- d\mu + \int_X f^- d\nu \right] \\ &= \left[\int_X f^+ d\mu - \int_X f^- d\mu \right] + \left[\int_X f^+ d\nu - \int_X f^- d\nu \right] \\ &= \int_X f d\mu + \int_X f d\nu. \end{aligned}$$

Step 6: necessity of the condition, in the book's convention. In the text the symbol $\int_X f d\mu$ is assigned to a general measurable f only when f is integrable over X with respect to μ . If the proposed identity is to be an equality of three defined real numbers, then in particular f must be integrable with respect to μ , with respect to ν , and with respect to $\mu + \nu$; by Step 4 these three requirements reduce to the single requirement that f be integrable with respect to $\mu + \nu$. So the condition of Step 5 is not merely sufficient but necessary.

A concrete illustration that some hypothesis is needed for non-nonnegative f : take $X = \{1, 2\}$ with $\mathcal{M}$ all subsets, μ counting measure and ν counting measure, and $f(1) = +\infty$, $f(2) = -\infty$. Then f^+ and f^- both have infinite integral against each of μ , ν and $\mu + \nu$, and no side of the identity is defined. Conversely, when $f \geq 0$ nothing can go wrong, and Step 3 is an unconditional identity in $[0, \infty]$.

18.7 Problems 18.43–18.49

Problem 18.43. Let $\mathcal{M}_0$ be a σ -algebra that is contained in $\mathcal{M}$, μ_0 the restriction of μ to $\mathcal{M}_0$, and f a nonnegative function that is measurable with respect to $\mathcal{M}_0$. Show that f is measurable with respect to $\mathcal{M}$ and

$$\int_X f d\mu_0 \leq \int_X f d\mu.$$

Can this inequality be strict?

Solution. Throughout, $(X, \mathcal{M}, \mu)$ is a measure space, $\mathcal{M}_0 \subseteq \mathcal{M}$ is a σ -algebra of subsets of X , and $\mu_0: \mathcal{M}_0 \rightarrow [0, \infty]$ is defined by $\mu_0(E) = \mu(E)$ for $E \in \mathcal{M}_0$. Observe first that $(X, \mathcal{M}_0, \mu_0)$ is indeed a measure space: $\mu_0(\emptyset) = \mu(\emptyset) = 0$, and if $\{E_k\}_{k=1}^\infty$ is a disjoint countable collection of sets in $\mathcal{M}_0$, then its union lies in $\mathcal{M}_0$ and, since these sets also belong to $\mathcal{M}$ and μ is countably additive,

$$\mu_0\left(\bigcup_{k=1}^\infty E_k\right) = \mu\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu(E_k) = \sum_{k=1}^\infty \mu_0(E_k).$$

Measurability with respect to $\mathcal{M}$. Since f is measurable with respect to $\mathcal{M}_0$, for each real number c the set $\{x \in X \mid f(x) > c\}$ belongs to $\mathcal{M}_0$. But $\mathcal{M}_0 \subseteq \mathcal{M}$, so this set belongs to $\mathcal{M}$. As this holds for

every c , the function f is measurable with respect to $\mathcal{M}$.

The inequality. First consider a nonnegative simple function ψ that is measurable with respect to $\mathcal{M}_0$. If $\psi \equiv 0$ both integrals are 0. Otherwise let $c_1, \dots, c_n$ be the distinct positive values taken by ψ and put $E_k = \{x \in X \mid \psi(x) = c_k\}$. Each E_k belongs to $\mathcal{M}_0$, hence also to $\mathcal{M}$, and ψ is simple with respect to $\mathcal{M}$ as well, with the very same level sets. Therefore, directly from the definition of the integral of a nonnegative simple function,

$$\int_X \psi \, d\mu_0 = \sum_{k=1}^n c_k \mu_0(E_k) = \sum_{k=1}^n c_k \mu(E_k) = \int_X \psi \, d\mu. \quad (*)$$

Thus the two integrals agree on nonnegative $\mathcal{M}_0$ -simple functions.

Now, by definition, $\int_X f \, d\mu_0$ is the supremum of $\int_X \psi \, d\mu_0$ as ψ ranges over the nonnegative $\mathcal{M}_0$ -simple functions with $\psi \leq f$ on X , while $\int_X f \, d\mu$ is the supremum of $\int_X \varphi \, d\mu$ as φ ranges over the (larger) collection of nonnegative $\mathcal{M}$ -simple functions with $\varphi \leq f$ on X . Every $\mathcal{M}_0$ -simple ψ with $0 \leq \psi \leq f$ is an $\mathcal{M}$ -simple function with $0 \leq \psi \leq f$, and by (*) it contributes the same number to both suprema. A supremum over a subcollection cannot exceed the supremum over the whole collection, so

$$\int_X f \, d\mu_0 \leq \int_X f \, d\mu.$$

The inequality can never be strict. We claim equality always holds. Since f is a nonnegative function on X that is measurable with respect to $\mathcal{M}_0$, we may apply part (ii) of the Simple Approximation Theorem on the measure space $(X, \mathcal{M}_0, \mu_0)$: there is an increasing sequence $\{\psi_n\}$ of nonnegative $\mathcal{M}_0$ -simple functions on X that converges pointwise on all of X to f . (The construction there uses only the measurability of f with respect to the given σ -algebra.)

Each ψ_n is also a nonnegative $\mathcal{M}$ -simple function, and the sequence $\{\psi_n\}$ is increasing with pointwise limit f on X . Applying the Monotone Convergence Theorem twice, once on $(X, \mathcal{M}_0, \mu_0)$ and once on $(X, \mathcal{M}, \mu)$, we obtain

$$\lim_{n \rightarrow \infty} \int_X \psi_n \, d\mu_0 = \int_X f \, d\mu_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_X \psi_n \, d\mu = \int_X f \, d\mu.$$

By (*), $\int_X \psi_n \, d\mu_0 = \int_X \psi_n \, d\mu$ for every n . The two limits are therefore limits of the same sequence of extended real numbers, and hence

$$\int_X f \, d\mu_0 = \int_X f \, d\mu.$$

So the answer to the question is no: for a nonnegative $\mathcal{M}_0$ -measurable f the inequality is always an equality. The essential point is that μ_0 is the restriction of μ , so no mass is lost in passing from $\mathcal{M}_0$ to $\mathcal{M}$; strictness could only arise if μ_0 were merely dominated by μ on $\mathcal{M}_0$ rather than equal to it.

Two remarks on where the hypotheses are used. First, nonnegativity of f is what allows the monotone approximation and the Monotone Convergence Theorem; for a general integrable f the same conclusion follows at once by applying the above to f^+ and f^- separately and using the definition $\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu$. Second, it is essential that f be $\mathcal{M}_0$ -measurable: a function that is merely $\mathcal{M}$ -measurable need not be $\mathcal{M}_0$ -measurable at all, and then $\int_X f \, d\mu_0$ is not even defined.

Problem 18.44. Let ν be a signed measure on $(X, \mathcal{M})$. We define integration over X with respect to a signed measure ν by defining

$$\int_X f \, d\nu = \int_X f \, d\nu^+ - \int_X f \, d\nu^-,$$

provided f is integrable over X with respect to both ν^+ and ν^- . Show that if $|f| \leq M$ on X , then

$$\left| \int_X f \, d\nu \right| \leq M|\nu|(X).$$

Moreover, if $|\nu|(X) < \infty$, show that there is a measurable function f with $|f| \leq 1$ on X for which

$$\int_X f d\nu = |\nu|(X).$$

Solution. Recall the setting. By the Hahn Decomposition Theorem there is a positive set A and a negative set B for ν with $X = A \cup B$ and $A \cap B = \emptyset$. The Jordan decomposition $\nu = \nu^+ - \nu^-$ into mutually singular measures is given in terms of such a Hahn decomposition $\{A, B\}$ by

$$\nu^+(E) = \nu(E \cap A) \quad \text{and} \quad \nu^-(E) = -\nu(E \cap B) \quad \text{for all } E \in \mathcal{M},$$

and the measure $|\nu|$ is defined by $|\nu|(E) = \nu^+(E) + \nu^-(E)$ for all $E \in \mathcal{M}$.

The inequality. Let f be measurable on X with $|f| \leq M$ on X , and assume, as the definition of $\int_X f d\nu$ requires, that f is integrable over X with respect to both ν^+ and ν^- . Apply the Integral Comparison Test on the measure space $(X, \mathcal{M}, \nu^+)$ to get

$$\left| \int_X f d\nu^+ \right| \leq \int_X |f| d\nu^+,$$

and likewise on $(X, \mathcal{M}, \nu^-)$. The constant function M is a nonnegative measurable function on X dominating $|f|$, so by the monotonicity of the integral of nonnegative measurable functions and the definition of the integral of a simple function,

$$\int_X |f| d\nu^+ \leq \int_X M d\nu^+ = M \nu^+(X), \quad \int_X |f| d\nu^- \leq M \nu^-(X)$$

(where the right-hand sides are interpreted as ∞ if $M > 0$ and the corresponding measure of X is infinite; if $M = 0$ then $f \equiv 0$ and the asserted inequality is trivial). Therefore, by the triangle inequality in the extended reals,

$$\left| \int_X f d\nu \right| = \left| \int_X f d\nu^+ - \int_X f d\nu^- \right| \leq \left| \int_X f d\nu^+ \right| + \left| \int_X f d\nu^- \right| \leq M \nu^+(X) + M \nu^-(X),$$

and since $\nu^+(X) + \nu^-(X) = |\nu|(X)$ this is exactly

$$\left| \int_X f d\nu \right| \leq M |\nu|(X).$$

(The left-hand side is a real number, because integrability of f with respect to ν^+ and ν^- makes both integrals finite; the inequality is vacuously true when $|\nu|(X) = \infty$.)

Attainment of the bound when $|\nu|(X) < \infty$. Assume now $|\nu|(X) < \infty$, so that $\nu^+(X) < \infty$ and $\nu^-(X) < \infty$; that is, ν^+ and ν^- are finite measures on $(X, \mathcal{M})$. Fix a Hahn decomposition $\{A, B\}$ as above and define

$$f = \chi_A - \chi_B \quad \text{on } X.$$

Since A and B are measurable, f is a measurable simple function, and since X is the disjoint union of A and B we have $|f| = 1$ on X ; in particular $|f| \leq 1$. Moreover f is bounded on X and X has finite measure with respect to each of ν^+ and ν^- , so by Theorem 15 of this chapter f is integrable over X with respect to both ν^+ and ν^- . Hence $\int_X f d\nu$ is defined.

Observe first that $\nu^+(B) = \nu(B \cap A) = \nu(\emptyset) = 0$ and, symmetrically, $\nu^-(A) = -\nu(A \cap B) = -\nu(\emptyset) = 0$. Now compute. Using additivity of integration over the disjoint domains A and B (Theorem 12 of this chapter), and the fact that f is the constant 1 on A and the constant -1 on B ,

$$\int_X f d\nu^+ = \int_A f d\nu^+ + \int_B f d\nu^+ = \nu^+(A) - \nu^+(B) = \nu^+(A) - 0 = \nu^+(A).$$

Since $\nu^+(X) = \nu^+(A) + \nu^+(B) = \nu^+(A)$, this says $\int_X f d\nu^+ = \nu^+(X)$. Similarly,

$$\int_X f d\nu^- = \int_A f d\nu^- + \int_B f d\nu^- = \nu^-(A) - \nu^-(B) = 0 - \nu^-(B) = -\nu^-(X),$$

using $\nu^-(X) = \nu^-(A) + \nu^-(B) = \nu^-(B)$. Therefore

$$\int_X f d\nu = \int_X f d\nu^+ - \int_X f d\nu^- = \nu^+(X) - (-\nu^-(X)) = \nu^+(X) + \nu^-(X) = |\nu|(X).$$

Thus the bound of the first part, taken with $M = 1$, is attained by $f = \chi_A - \chi_B$. Incidentally this also shows that

$$|\nu|(X) = \max \left\{ \int_X f d\nu \mid f \text{ measurable on } X, |f| \leq 1 \text{ on } X \right\}$$

when $|\nu|(X) < \infty$: the first part gives $\leq$ for every competitor, and $\chi_A - \chi_B$ achieves equality.

Problem 18.45. Let g be a nonnegative function that is integrable over X . Define

$$\nu(E) = \int_E g d\mu \quad \text{for all } E \in \mathcal{M}.$$

- (i) Show that ν is a measure on the measurable space $(X, \mathcal{M})$.
- (ii) Let f be a nonnegative function on X that is measurable with respect to $\mathcal{M}$. Show that

$$\int_X f d\nu = \int_X fg d\mu.$$

[Hint: First establish this for the case when f is simple and then use the Simple Approximation Lemma and the Monotone Convergence Theorem.]

Solution. Throughout, recall that by definition $\int_E g d\mu = \int_X g \cdot \chi_E d\mu$, so that

$$\nu(E) = \int_X g \cdot \chi_E d\mu \quad \text{for all } E \in \mathcal{M}.$$

Since g is nonnegative and measurable, so is $g \cdot \chi_E$, so $\nu(E)$ is a well-defined element of $[0, \infty]$; and by the monotonicity of integration for nonnegative measurable functions, $\nu(E) \leq \int_X g d\mu < \infty$, since g is integrable over X . Thus ν is a real-valued, nonnegative set function on $\mathcal{M}$.

Part (i). We must check that $\nu(\emptyset) = 0$ and that ν is countably additive.

Since $g \cdot \chi_\emptyset \equiv 0$ on X , we have $\nu(\emptyset) = \int_X 0 d\mu = 0$.

Let $\{E_k\}_{k=1}^\infty$ be a disjoint countable collection of sets in $\mathcal{M}$ and let $E = \bigcup_{k=1}^\infty E_k$. For each natural number n define

$$h_n = \sum_{k=1}^n g \cdot \chi_{E_k} = g \cdot \chi_{F_n}, \quad \text{where } F_n = \bigcup_{k=1}^n E_k,$$

the second equality holding because the E_k are disjoint. Each h_n is a nonnegative measurable function, and since $\{F_n\}$ is ascending with union E and $g \geq 0$, the sequence $\{h_n\}$ is increasing and converges pointwise on X to $g \cdot \chi_E$. By the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_X h_n d\mu = \int_X g \cdot \chi_E d\mu = \nu(E).$$

On the other hand, by Proposition 11 of this chapter (linearity of integration for nonnegative measurable functions), applied $n - 1$ times,

$$\int_X h_n d\mu = \sum_{k=1}^n \int_X g \cdot \chi_{E_k} d\mu = \sum_{k=1}^n \nu(E_k).$$

Letting $n \rightarrow \infty$ we conclude that

$$\nu(E) = \sum_{k=1}^\infty \nu(E_k).$$

Thus ν is countably additive, and ν is a measure on $(X, \mathcal{M})$; indeed it is a finite measure, since $\nu(X) = \int_X g \, d\mu < \infty$.

Part (ii). We follow the hint.

Step 1: characteristic functions. If $E \in \mathcal{M}$, then by the definition of the integral of a simple function with respect to ν and the definition of ν ,

$$\int_X \chi_E \, d\nu = \nu(E) = \int_X g \cdot \chi_E \, d\mu = \int_X \chi_E g \, d\mu.$$

(If $E = \emptyset$ then $\chi_E \equiv 0$ and both sides are 0; if $E \neq \emptyset$ then χ_E is a nonnegative simple function whose only positive value is 1, taken exactly on E , so its integral with respect to ν is $1 \cdot \nu(E)$ by the definition of the integral of a nonnegative simple function.)

Step 2: nonnegative simple functions. Let φ be a nonnegative simple function on X . If $\varphi \equiv 0$ both sides vanish. Otherwise let $c_1, \dots, c_n$ be the distinct positive values of φ and $E_k = \{x \in X \mid \varphi(x) = c_k\}$, so that $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$ with the E_k disjoint and measurable. By the definition of the integral of a nonnegative simple function with respect to the measure ν , Step 1, and Proposition 11 of this chapter (positive coefficients c_k),

$$\int_X \varphi \, d\nu = \sum_{k=1}^n c_k \nu(E_k) = \sum_{k=1}^n c_k \int_X \chi_{E_k} g \, d\mu = \int_X \left[\sum_{k=1}^n c_k \chi_{E_k} \right] g \, d\mu = \int_X \varphi g \, d\mu.$$

Step 3: general nonnegative measurable f . Here fg is understood pointwise, with the convention $0 \cdot \infty = 0$. By part (ii) of the Simple Approximation Theorem, applied on the measure space $(X, \mathcal{M}, \mu)$, there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on X that converges pointwise on all of X to f .

Since $g \geq 0$, the sequence $\{\varphi_n g\}$ is increasing. Each $\varphi_n g$ is a nonnegative measurable function on X : writing $\varphi_n = \sum_k c_k \chi_{C_k}$ over the disjoint measurable sets C_k on which φ_n is constant, the function $\varphi_n g$ equals the constant multiple $c_k g$ on each C_k , and a function whose restriction to each member of a finite disjoint measurable cover of X is measurable is itself measurable. Moreover $\{\varphi_n g\}$ converges pointwise on X to fg : at a point x with $g(x) = 0$ every term is 0 and so is $f(x)g(x)$, by the convention $0 \cdot \infty = 0$; at a point x with $g(x) > 0$ we have $\varphi_n(x)g(x) \rightarrow f(x)g(x)$ in $[0, \infty]$ because $\varphi_n(x) \rightarrow f(x)$ in $[0, \infty]$. In particular fg , being the pointwise limit on X of a sequence of measurable functions, is measurable with respect to $\mathcal{M}$, so the right-hand side of the asserted identity is defined.

Now apply the Monotone Convergence Theorem twice. On the measure space $(X, \mathcal{M}, \nu)$, applied to the increasing sequence $\{\varphi_n\}$ converging pointwise to f ,

$$\lim_{n \rightarrow \infty} \int_X \varphi_n \, d\nu = \int_X f \, d\nu.$$

On the measure space $(X, \mathcal{M}, \mu)$, applied to the increasing sequence $\{\varphi_n g\}$ converging pointwise to fg ,

$$\lim_{n \rightarrow \infty} \int_X \varphi_n g \, d\mu = \int_X fg \, d\mu.$$

By Step 2, $\int_X \varphi_n \, d\nu = \int_X \varphi_n g \, d\mu$ for every n . The two displayed limits are therefore limits of one and the same sequence in $[0, \infty]$, and hence

$$\int_X f \, d\nu = \int_X fg \, d\mu.$$

Remark. The hypothesis that g be integrable was used only to guarantee that ν is a finite measure; both parts go through verbatim for any nonnegative measurable g , with ν then possibly taking the value ∞ .

Problem 18.46. Let $\nu: \mathcal{M} \rightarrow [0, \infty)$ be a finitely additive set function. Show that if f is a bounded measurable function on X , then the integral of f over X with respect to ν , $\int_X f d\nu$, can be defined so that $\int_X \chi_E d\nu = \nu(E)$, if E is measurable, and integration is linear, monotone, and additive over domains for bounded measurable functions.

Solution. Note at the outset that ν is finite: it takes values in $[0, \infty)$, so $\nu(X) < \infty$. Two elementary consequences of finite additivity will be used repeatedly.

Vanishing on the empty set: $\nu(\emptyset) = \nu(\emptyset \cup \emptyset) = \nu(\emptyset) + \nu(\emptyset)$, and since $\nu(\emptyset)$ is a real number, $\nu(\emptyset) = 0$.

Monotonicity: if $A \subseteq B$ are measurable, then B is the disjoint union of A and $B \sim A$, so $\nu(B) = \nu(A) + \nu(B \sim A) \geq \nu(A)$ because $\nu \geq 0$.

Step 1: the integral of a simple function.

Call a function φ on X simple provided it is measurable and takes only finitely many values. Every simple φ can be written in the canonical form $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$, where $a_1, \dots, a_n$ are the distinct values of φ and $A_i = \varphi^{-1}(a_i)$, so that $\{A_i\}_{i=1}^n$ is a disjoint measurable collection whose union is X . Define

$$I(\varphi) = \sum_{i=1}^n a_i \nu(A_i).$$

This is well defined in the stronger sense that the same value results from any representation $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ in which $\{A_i\}_{i=1}^n$ is a disjoint measurable collection with union X (the a_i need not be distinct). Indeed, suppose

$$\varphi = \sum_{i=1}^n a_i \chi_{A_i} = \sum_{j=1}^m b_j \chi_{B_j}$$

are two such representations. The sets $A_i \cap B_j$ form a disjoint measurable collection with union X , and if $A_i \cap B_j \neq \emptyset$ then, evaluating φ at a point of $A_i \cap B_j$, we get $a_i = b_j$. By finite additivity, $\nu(A_i) = \sum_{j=1}^m \nu(A_i \cap B_j)$ and $\nu(B_j) = \sum_{i=1}^n \nu(A_i \cap B_j)$. Hence, discarding the empty intersections (which contribute 0 since $\nu(\emptyset) = 0$),

$$\sum_{i=1}^n a_i \nu(A_i) = \sum_{i,j} a_i \nu(A_i \cap B_j) = \sum_{i,j} b_j \nu(A_i \cap B_j) = \sum_{j=1}^m b_j \nu(B_j).$$

From this, linearity of I on simple functions is immediate: given simple φ, ψ and reals α, β , refine to a common disjoint measurable partition $\{C_k\}_{k=1}^p$ of X on which both φ and ψ are constant, say $\varphi = \sum_k c_k \chi_{C_k}$ and $\psi = \sum_k d_k \chi_{C_k}$. Then $\alpha\varphi + \beta\psi = \sum_k (\alpha c_k + \beta d_k) \chi_{C_k}$ is a representation of the admissible kind, so

$$I(\alpha\varphi + \beta\psi) = \sum_k (\alpha c_k + \beta d_k) \nu(C_k) = \alpha I(\varphi) + \beta I(\psi).$$

Moreover I is monotone on simple functions: if $\varphi \leq \psi$, then $\psi - \varphi$ is simple and nonnegative, so all its values are ≥ 0 and $I(\psi - \varphi) \geq 0$ since $\nu \geq 0$; by linearity $I(\varphi) \leq I(\psi)$. Finally, if $E \in \mathcal{M}$ then $\chi_E = 1 \cdot \chi_E + 0 \cdot \chi_{X \sim E}$ is an admissible representation, so

$$I(\chi_E) = 1 \cdot \nu(E) + 0 \cdot \nu(X \sim E) = \nu(E).$$

Step 2: the definition for bounded measurable functions.

Let f be a bounded measurable function on X . Define

$$\underline{I}(f) = \sup\{I(\varphi) \mid \varphi \text{ simple, } \varphi \leq f \text{ on } X\}, \quad \bar{I}(f) = \inf\{I(\psi) \mid \psi \text{ simple, } \psi \geq f \text{ on } X\}.$$

Both collections are nonempty, since f is bounded and the constant functions are simple. By the monotonicity of I , $I(\varphi) \leq I(\psi)$ whenever $\varphi \leq f \leq \psi$ with φ, ψ simple; hence $\underline{I}(f) \leq \bar{I}(f)$, and both are finite real numbers.

We claim they are equal. Let $\epsilon > 0$. Since f is a bounded measurable function on the measurable space $(X, \mathcal{M})$, the Simple Approximation Lemma provides simple functions φ_ϵ and ψ_ϵ on X with

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \quad \text{and} \quad 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon \text{ on } X.$$

By the linearity and monotonicity of I on simple functions, and since $\psi_\epsilon - \varphi_\epsilon$ is a simple function dominated by the constant ϵ ,

$$0 \leq \bar{I}(f) - \underline{I}(f) \leq I(\psi_\epsilon) - I(\varphi_\epsilon) = I(\psi_\epsilon - \varphi_\epsilon) \leq I(\epsilon \cdot \chi_X) = \epsilon \nu(X).$$

Since $\nu(X) < \infty$ and $\epsilon > 0$ is arbitrary, $\underline{I}(f) = \bar{I}(f)$. Define

$$\int_X f \, d\nu = \underline{I}(f) = \bar{I}(f).$$

For a measurable subset E of X define $\int_E f \, d\nu = \int_X f \cdot \chi_E \, d\nu$; this makes sense because $f \cdot \chi_E$ is again bounded and measurable.

Consistency on simple functions: if f itself is simple, then f competes in both families, so $\underline{I}(f) \geq I(f) \geq \bar{I}(f)$, whence $\int_X f \, d\nu = I(f)$. In particular, by Step 1,

$$\int_X \chi_E \, d\nu = \nu(E) \quad \text{for all } E \in \mathcal{M},$$

which is the required normalization.

Step 3: monotonicity.

Suppose $f \leq h$ on X , with f, h bounded and measurable. Every simple φ with $\varphi \leq f$ also satisfies $\varphi \leq h$, so the supremum defining $\underline{I}(h)$ is over a larger family; hence $\int_X f \, d\nu = \underline{I}(f) \leq \underline{I}(h) = \int_X h \, d\nu$.

Step 4: linearity.

Let f and h be bounded measurable functions on X and let $\epsilon > 0$. By the Simple Approximation Lemma choose simple functions with

$$\varphi_1 \leq f \leq \psi_1, \quad \psi_1 - \varphi_1 < \epsilon, \quad \varphi_2 \leq h \leq \psi_2, \quad \psi_2 - \varphi_2 < \epsilon.$$

Then $\varphi_1 + \varphi_2 \leq f + h \leq \psi_1 + \psi_2$, these are simple, and $(\psi_1 + \psi_2) - (\varphi_1 + \varphi_2) < 2\epsilon$ on X . From the definition of the integral as a supremum over minorants and an infimum over majorants,

$$I(\varphi_1) \leq \int_X f \, d\nu \leq I(\psi_1), \quad I(\varphi_2) \leq \int_X h \, d\nu \leq I(\psi_2), \quad I(\varphi_1) + I(\varphi_2) \leq \int_X (f + h) \, d\nu \leq I(\psi_1) + I(\psi_2),$$

the last using $I(\varphi_1 + \varphi_2) = I(\varphi_1) + I(\varphi_2)$ and likewise for the ψ 's. Both the number $\int_X (f + h) \, d\nu$ and the number $\int_X f \, d\nu + \int_X h \, d\nu$ therefore lie in the interval $[I(\varphi_1) + I(\varphi_2), I(\psi_1) + I(\psi_2)]$, whose length is

$$I(\psi_1 - \varphi_1) + I(\psi_2 - \varphi_2) \leq 2\epsilon \nu(X).$$

Hence $|\int_X (f + h) \, d\nu - \int_X f \, d\nu - \int_X h \, d\nu| \leq 2\epsilon \nu(X)$ for every $\epsilon > 0$, and so the two agree.

For scalars, let $\alpha \in \mathbb{R}$. If $\alpha > 0$, then $\varphi \mapsto \alpha\varphi$ is a bijection of the simple minorants of f onto the simple minorants of αf , and $I(\alpha\varphi) = \alpha I(\varphi)$; taking suprema gives $\int_X \alpha f \, d\nu = \alpha \int_X f \, d\nu$. If $\alpha = 0$ both sides are 0. If $\alpha < 0$, write $\alpha = -|\alpha|$; the map $\psi \mapsto -\psi$ carries the simple majorants of f onto the simple minorants of $-f$ and $I(-\psi) = -I(\psi)$, so

$$\int_X (-f) \, d\nu = \underline{I}(-f) = \sup_{\psi \geq f} (-I(\psi)) = - \inf_{\psi \geq f} I(\psi) = -\bar{I}(f) = - \int_X f \, d\nu,$$

and then $\int_X \alpha f \, d\nu = \int_X |\alpha|(-f) \, d\nu = |\alpha| \cdot (- \int_X f \, d\nu) = \alpha \int_X f \, d\nu$. Combining the two paragraphs, for all reals α, β and bounded measurable f, h ,

$$\int_X [\alpha f + \beta h] \, d\nu = \alpha \int_X f \, d\nu + \beta \int_X h \, d\nu.$$

Step 5: additivity over domains.

Let A and B be disjoint measurable subsets of X and f a bounded measurable function. Since $A \cap B = \emptyset$,

$$f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B \quad \text{on } X,$$

and all three functions are bounded and measurable. By the linearity established in Step 4,

$$\int_{A \cup B} f \, d\nu = \int_X f \cdot \chi_{A \cup B} \, d\nu = \int_X f \cdot \chi_A \, d\nu + \int_X f \cdot \chi_B \, d\nu = \int_A f \, d\nu + \int_B f \, d\nu.$$

This completes the construction: $\int_X \cdot \, d\nu$ is defined on all bounded measurable functions on X , it assigns $\nu(E)$ to χ_E , and it is linear, monotone, and additive over domains.

Remark on uniqueness. Such an assignment is unique. Linearity together with $\int_X \chi_E \, d\nu = \nu(E)$ forces the value $I(\varphi)$ on every simple function φ . If f is bounded and measurable and $\epsilon > 0$, the Simple Approximation Lemma gives simple $\varphi_\epsilon \leq f \leq \psi_\epsilon$ with $\psi_\epsilon - \varphi_\epsilon < \epsilon$; monotonicity then forces any admissible integral of f to lie in $[I(\varphi_\epsilon), I(\psi_\epsilon)]$, an interval of length at most $\epsilon\nu(X)$. Letting $\epsilon \downarrow 0$ pins the value down uniquely, and it must be the number defined above.

Note also where countable additivity would be needed and is not available: nothing above required limits of sequences of functions, only the two-sided simple approximation of a bounded measurable function. This is exactly why the construction stops at bounded functions; without countable additivity of ν there is no Monotone Convergence Theorem with which to pass to unbounded ones.

Problem 18.47. Let μ be a finite premeasure on an algebra $\mathcal{S}$ and $\bar{\mu}$ its Carathéodory extension. Let E be μ^* -measurable. Show that for each $\epsilon > 0$ there is an $A \in \mathcal{S}$ with

$$\bar{\mu}([A \sim E] \cup [E \sim A]) < \epsilon.$$

Solution. We first record the structure at hand. Since $\mathcal{S}$ is an algebra of subsets of X , it contains X and $\emptyset$, and is closed with respect to the formation of complements, finite unions, finite intersections, and hence relative complements. Because μ is a premeasure on a collection closed with respect to the formation of relative complements, Theorem 12 of Chapter 17 tells us that every set in $\mathcal{S}$ is μ^* -measurable and that the Carathéodory measure $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ induced by μ extends μ , that is, $\bar{\mu}(B) = \mu^*(B) = \mu(B)$ for every $B \in \mathcal{S}$. Here $\mathcal{M}$ denotes the σ -algebra of μ^* -measurable sets, and $\bar{\mu}$ is the restriction of μ^* to $\mathcal{M}$.

Since μ is finite and $X \in \mathcal{S}$, the single set X covers every subset of X , so by the definition of the induced outer measure,

$$\mu^*(F) \leq \mu(X) < \infty \quad \text{for every } F \subseteq X.$$

In particular $\bar{\mu}(E) = \mu^*(E) < \infty$, and $\bar{\mu}$ is a finite measure on $\mathcal{M}$.

Let $\epsilon > 0$ be given.

Step 1: a cover from $\mathcal{S}$ that is efficient. By the definition of the outer measure μ^* induced by μ as an infimum, and since $\mu^*(E) < \infty$, there is a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ that covers E and for which

$$\sum_{k=1}^\infty \mu(E_k) < \mu^*(E) + \frac{\epsilon}{2}.$$

Define $A_\infty = \bigcup_{k=1}^\infty E_k$. Each E_k belongs to $\mathcal{M}$, so $A_\infty \in \mathcal{M}$, and $E \subseteq A_\infty$. By the countable subadditivity and monotonicity of the measure $\bar{\mu}$, together with $\bar{\mu}(E_k) = \mu(E_k)$,

$$\bar{\mu}(E) \leq \bar{\mu}(A_\infty) \leq \sum_{k=1}^\infty \bar{\mu}(E_k) = \sum_{k=1}^\infty \mu(E_k) < \bar{\mu}(E) + \frac{\epsilon}{2}.$$

Since $\bar{\mu}(E) < \infty$, the excision property of measure applies to the measurable sets $E \subseteq A_\infty$ and gives

$$\bar{\mu}(A_\infty \sim E) = \bar{\mu}(A_\infty) - \bar{\mu}(E) < \frac{\epsilon}{2}. \quad (*)$$

Step 2: truncate the cover to a single set of $\mathcal{S}$. For each natural number n define

$$A_n = \bigcup_{k=1}^n E_k.$$

Since $\mathcal{S}$ is an algebra it is closed with respect to the formation of finite unions, so $A_n \in \mathcal{S}$. The collection $\{A_n\}_{n=1}^\infty$ is an ascending sequence of measurable sets whose union is A_∞ , so by the continuity of the measure $\bar{\mu}$,

$$\lim_{n \rightarrow \infty} \bar{\mu}(A_n) = \bar{\mu}(A_\infty).$$

Because $\bar{\mu}(A_\infty) \leq \mu(X) < \infty$, we may choose an index N for which

$$\bar{\mu}(A_\infty) - \bar{\mu}(A_N) < \frac{\epsilon}{2},$$

and then, by excision applied to $A_N \subseteq A_\infty$ (both measurable, both of finite measure),

$$\bar{\mu}(A_\infty \sim A_N) = \bar{\mu}(A_\infty) - \bar{\mu}(A_N) < \frac{\epsilon}{2}. \quad (**)$$

Step 3: conclusion. Put $A = A_N$, a set belonging to $\mathcal{S}$. We estimate the two pieces of the symmetric difference separately, using monotonicity of $\bar{\mu}$.

Since $A = A_N \subseteq A_\infty$, we have $A \sim E \subseteq A_\infty \sim E$, so by (*),

$$\bar{\mu}(A \sim E) \leq \bar{\mu}(A_\infty \sim E) < \frac{\epsilon}{2}.$$

Since $E \subseteq A_\infty$, a point of $E \sim A$ lies in A_∞ and not in A_N ; that is, $E \sim A \subseteq A_\infty \sim A_N$, so by (**),

$$\bar{\mu}(E \sim A) \leq \bar{\mu}(A_\infty \sim A_N) < \frac{\epsilon}{2}.$$

Both $A \sim E$ and $E \sim A$ are measurable, being differences of sets in $\mathcal{M}$. Therefore, by the finite subadditivity of $\bar{\mu}$,

$$\bar{\mu}([A \sim E] \cup [E \sim A]) \leq \bar{\mu}(A \sim E) + \bar{\mu}(E \sim A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This is the required conclusion.

Remark. Finiteness of the premeasure was used twice, and only to guarantee finiteness of the relevant measures: it gives $\bar{\mu}(E) < \infty$ so that the excision in Step 1 is legitimate, and $\bar{\mu}(A_\infty) < \infty$ so that the continuity limit in Step 2 can be converted into the estimate (**). Accordingly the same proof establishes the following slightly more general statement, which we shall use in the next problem: if μ is any premeasure on an algebra $\mathcal{S}$ and E is a μ^* -measurable set with $\bar{\mu}(E) < \infty$, then for each $\epsilon > 0$ there is an $A \in \mathcal{S}$ with $\bar{\mu}([A \sim E] \cup [E \sim A]) < \epsilon$. Indeed, $\bar{\mu}(E) = \mu^*(E) < \infty$ is exactly what is needed to select the efficient cover in Step 1, and it then forces $\bar{\mu}(A_\infty) < \bar{\mu}(E) + \epsilon/2 < \infty$.

Problem 18.48. Let $\mathcal{S}$ be an algebra of subsets of a set X . We say that a function $\varphi: X \rightarrow \mathbb{R}$ is $\mathcal{S}$ -simple provided $\varphi = \sum_{k=1}^n a_k \chi_{A_k}$, where each $A_k \in \mathcal{S}$. Let μ be a premeasure on $\mathcal{S}$ and $\bar{\mu}$ its Carathéodory extension. Given $\epsilon > 0$ and a function f that is integrable over X with respect to $\bar{\mu}$, show that there is an $\mathcal{S}$ -simple function φ such that

$$\int_X |f - \varphi| d\bar{\mu} < \epsilon.$$

Solution. Let $\mathcal{M}$ denote the σ -algebra of μ^* -measurable subsets of X and $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ the Carathéodory measure induced by μ . Since $\mathcal{S}$ is an algebra, it is closed with respect to the formation of relative complements, so by Theorem 12 of Chapter 17 every set of $\mathcal{S}$ is μ^* -measurable and $\bar{\mu}$ extends μ . In particular $\mathcal{S} \subseteq \mathcal{M}$, so every $\mathcal{S}$ -simple function is a simple function on the measure space $(X, \mathcal{M}, \bar{\mu})$ and all the integrals below are defined.

We will use the following strengthening of the preceding problem, whose proof is exactly the proof given there.

Lemma. Let μ be a premeasure on an algebra $\mathcal{S}$, $\bar{\mu}$ its Carathéodory extension, and $E \in \mathcal{M}$ with $\bar{\mu}(E) < \infty$. Then for each $\delta > 0$ there is a set $A \in \mathcal{S}$ with $\bar{\mu}(E \triangle A) < \delta$, where $E \triangle A = [E \sim A] \cup [A \sim E]$.

$E]$.

Proof of the Lemma. Since $\mu^*(E) = \bar{\mu}(E) < \infty$, the infimum defining $\mu^*(E)$ is over a nonempty family, so there is a countable collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ covering E with $\sum_{k=1}^\infty \mu(E_k) < \mu^*(E) + \delta/2$. Set $A_\infty = \bigcup_{k=1}^\infty E_k \in \mathcal{M}$ and $A_n = \bigcup_{k=1}^n E_k \in \mathcal{S}$. By countable subadditivity, $\bar{\mu}(E) \leq \bar{\mu}(A_\infty) \leq \sum_k \mu(E_k) < \bar{\mu}(E) + \delta/2 < \infty$, so by excision $\bar{\mu}(A_\infty \sim E) < \delta/2$. Since $\{A_n\}$ ascends to A_∞ and $\bar{\mu}(A_\infty) < \infty$, the continuity of measure provides an N with $\bar{\mu}(A_\infty \sim A_N) = \bar{\mu}(A_\infty) - \bar{\mu}(A_N) < \delta/2$. Taking $A = A_N$ and using $A \subseteq A_\infty$, $E \subseteq A_\infty$, we get $A \sim E \subseteq A_\infty \sim E$ and $E \sim A \subseteq A_\infty \sim A_N$, so $\bar{\mu}(E \triangle A) < \delta/2 + \delta/2 = \delta$. This proves the Lemma.

Step 1: approximate f in the mean by a simple function. Since f is integrable over X , $\int_X |f| d\bar{\mu} < \infty$, and both f^+ and f^- are nonnegative measurable functions with finite integral.

Apply part (ii) of the Simple Approximation Theorem to f^+ : there is an increasing sequence $\{\psi_n\}$ of nonnegative simple functions on X converging pointwise on X to f^+ . By the Monotone Convergence Theorem, $\int_X \psi_n d\bar{\mu} \rightarrow \int_X f^+ d\bar{\mu} < \infty$. Since $0 \leq \psi_n \leq f^+$ on X and f^+ is integrable, the Integral Comparison Test tells us that ψ_n too is integrable over X ; so all the integrals below are finite and the linearity of integration (Theorem 12 of this chapter) gives

$$\int_X |f^+ - \psi_n| d\bar{\mu} = \int_X [f^+ - \psi_n] d\bar{\mu} = \int_X f^+ d\bar{\mu} - \int_X \psi_n d\bar{\mu} \rightarrow 0.$$

Hence there is a nonnegative simple function ψ with $\int_X |f^+ - \psi| d\bar{\mu} < \epsilon/4$. The same argument applied to f^- produces a nonnegative simple function ζ with $\int_X |f^- - \zeta| d\bar{\mu} < \epsilon/4$.

Put $\eta = \psi - \zeta$, a simple function on X measurable with respect to $\mathcal{M}$. Since $f = f^+ - f^-$ and $f^+ f^- \equiv 0$, we have $f - \eta = [f^+ - \psi] - [f^- - \zeta]$ at every point of X at which f is finite, and hence

$$|f - \eta| \leq |f^+ - \psi| + |f^- - \zeta| \quad \text{on } X$$

holds at every such point; at a point where f is infinite both sides are infinite, so the inequality holds there too. Since f is integrable it is finite a.e. on X in any case. Therefore, by the monotonicity and linearity of integration,

$$\int_X |f - \eta| d\bar{\mu} \leq \int_X |f^+ - \psi| d\bar{\mu} + \int_X |f^- - \zeta| d\bar{\mu} < \frac{\epsilon}{2}. \quad (1)$$

Note that η is integrable over X : by the integral comparison test and (1), $\int_X |\eta| d\bar{\mu} \leq \int_X |f| d\bar{\mu} + \int_X |f - \eta| d\bar{\mu} < \infty$.

Step 2: the level sets of η have finite measure. If $\eta \equiv 0$ take $\varphi = 0 \cdot \chi_X$, which is $\mathcal{S}$ -simple since $X \in \mathcal{S}$, and (1) already gives the conclusion. Otherwise, let $c_1, \dots, c_m$ be the distinct nonzero values taken by η on X and set $E_k = \{x \in X \mid \eta(x) = c_k\}$, so that the E_k are disjoint sets in $\mathcal{M}$ and $\eta = \sum_{k=1}^m c_k \chi_{E_k}$. Then $|\eta| = \sum_{k=1}^m |c_k| \chi_{E_k}$ is a nonnegative simple function, and by the definition of the integral of a nonnegative simple function,

$$\sum_{k=1}^m |c_k| \bar{\mu}(E_k) = \int_X |\eta| d\bar{\mu} < \infty.$$

Since each $|c_k| > 0$, it follows that $\bar{\mu}(E_k) < \infty$ for $1 \leq k \leq m$.

Step 3: replace the level sets by sets of $\mathcal{S}$. Define

$$\delta = \frac{\epsilon}{2(1 + \sum_{k=1}^m |c_k|)} > 0.$$

By the Lemma, for each k there is a set $A_k \in \mathcal{S}$ with $\bar{\mu}(E_k \triangle A_k) < \delta$. Define

$$\varphi = \sum_{k=1}^m c_k \chi_{A_k},$$

which is an $\mathcal{S}$ -simple function by the definition given in the statement.

For each k we have the pointwise identity $|\chi_{E_k} - \chi_{A_k}| = \chi_{E_k \triangle A_k}$ on X , since the two characteristic functions differ exactly on the symmetric difference and there they differ by 1. Hence, by the triangle inequality,

$$|\eta - \varphi| = \left| \sum_{k=1}^m c_k [\chi_{E_k} - \chi_{A_k}] \right| \leq \sum_{k=1}^m |c_k| \chi_{E_k \triangle A_k} \quad \text{on } X.$$

Integrating, and using the monotonicity of integration together with the linearity of integration for nonnegative simple functions,

$$\int_X |\eta - \varphi| d\bar{\mu} \leq \sum_{k=1}^m |c_k| \bar{\mu}(E_k \triangle A_k) < \delta \sum_{k=1}^m |c_k| < \frac{\epsilon}{2}. \quad (2)$$

Step 4: conclusion. By the triangle inequality $|f - \varphi| \leq |f - \eta| + |\eta - \varphi|$ on X , so by the monotonicity and linearity of integration together with (1) and (2),

$$\int_X |f - \varphi| d\bar{\mu} \leq \int_X |f - \eta| d\bar{\mu} + \int_X |\eta - \varphi| d\bar{\mu} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus φ is an $\mathcal{S}$ -simple function with $\int_X |f - \varphi| d\bar{\mu} < \epsilon$, as required.

Problem 18.49. Show that the Radon-Nikodym Theorem for finite measures μ and ν implies the theorem for σ -finite measures μ and ν .

Solution. We are to prove the following, granted the corresponding assertion when both measures are finite (which is the case established in the text, the proof there beginning: we assume that both μ and ν are finite).

Statement to be proved. Let $(X, \mathcal{M}, \mu)$ be a σ -finite measure space and ν a σ -finite measure on $(X, \mathcal{M})$ that is absolutely continuous with respect to μ . Then there is a nonnegative function f on X , measurable with respect to $\mathcal{M}$, for which

$$\nu(E) = \int_E f d\mu \quad \text{for all } E \in \mathcal{M},$$

and f is unique in the sense that any nonnegative measurable g with the same property equals f a.e. $[\mu]$.

Step 1: a countable disjoint decomposition of X on which both measures are finite.

Since μ is σ -finite, there is a countable collection $\{U_n\}_{n=1}^\infty$ of measurable sets with $X = \bigcup_{n=1}^\infty U_n$ and $\mu(U_n) < \infty$ for all n . Since ν is σ -finite, there is likewise a countable collection $\{V_m\}_{m=1}^\infty$ with $X = \bigcup_{m=1}^\infty V_m$ and $\nu(V_m) < \infty$ for all m . The collection

$$\{U_n \cap V_m\}_{(n,m) \in \mathbb{N} \times \mathbb{N}}$$

is countable, consists of measurable sets, has union X (given $x \in X$, choose n with $x \in U_n$ and m with $x \in V_m$), and by the monotonicity of measure each of its members has finite measure with respect to both μ and ν .

Enumerate this collection as $\{Z_j\}_{j=1}^\infty$ and disjointify: define

$$X_1 = Z_1, \quad X_j = Z_j \sim \bigcup_{i=1}^{j-1} Z_i \quad \text{for } j \geq 2.$$

Each X_j is measurable, the X_j are pairwise disjoint, $\bigcup_{j=1}^\infty X_j = \bigcup_{j=1}^\infty Z_j = X$, and since $X_j \subseteq Z_j$, monotonicity gives

$$\mu(X_j) < \infty \quad \text{and} \quad \nu(X_j) < \infty \quad \text{for every } j.$$

Step 2: apply the finite case on each piece.

Fix j . Let $\mathcal{M}_j = \{E \in \mathcal{M} \mid E \subseteq X_j\}$. This is a σ -algebra of subsets of the set X_j : it contains X_j ; it is closed under countable unions since $\mathcal{M}$ is; and the complement of $E \in \mathcal{M}_j$ relative to X_j is $X_j \sim E$, which belongs to $\mathcal{M}$ and is contained in X_j . Let μ_j and ν_j denote the restrictions of μ and ν to $\mathcal{M}_j$. These are measures on the measurable space $(X_j, \mathcal{M}_j)$, since countable additivity is inherited, and by Step 1 they are finite:

$$\mu_j(X_j) = \mu(X_j) < \infty, \quad \nu_j(X_j) = \nu(X_j) < \infty.$$

Moreover ν_j is absolutely continuous with respect to μ_j : if $E \in \mathcal{M}_j$ and $\mu_j(E) = 0$, then $E \in \mathcal{M}$ and $\mu(E) = 0$, so $\nu(E) = 0$ by the absolute continuity of ν with respect to μ , that is, $\nu_j(E) = 0$.

By the Radon-Nikodym Theorem for finite measures, applied on $(X_j, \mathcal{M}_j, \mu_j)$ to ν_j , there is a nonnegative $\mathcal{M}_j$ -measurable function f_j on X_j for which

$$\nu(E) = \nu_j(E) = \int_E f_j d\mu_j = \int_E f_j d\mu \quad \text{for all } E \in \mathcal{M} \text{ with } E \subseteq X_j. \quad (1)$$

Here, in the last integral, f_j is understood to be extended to all of X by setting it equal to 0 on $X \sim X_j$; this extension is nonnegative and measurable with respect to $\mathcal{M}$, since for $c \geq 0$ the set $\{x \in X \mid f_j(x) > c\}$ equals $\{x \in X_j \mid f_j(x) > c\} \in \mathcal{M}_j \subseteq \mathcal{M}$, while for $c < 0$ it equals X . The last equality in (1) is justified as follows. By part (ii) of the Simple Approximation Theorem on $(X_j, \mathcal{M}_j, \mu_j)$ there is an increasing sequence $\{\sigma_n\}$ of nonnegative $\mathcal{M}_j$ -simple functions on X_j converging pointwise on X_j to f_j . Extend each σ_n by 0 to all of X ; the extensions are nonnegative $\mathcal{M}$ -simple functions increasing pointwise on X to the extension of f_j . For $E \in \mathcal{M}_j$ the function $\sigma_n \cdot \chi_E$ has exactly the same positive values on exactly the same sets whether read on X_j or on X , and those sets lie in $\mathcal{M}_j$, where μ_j and μ agree; hence $\int_E \sigma_n d\mu_j = \int_E \sigma_n d\mu$ for every n , by the definition of the integral of a nonnegative simple function. Applying the Monotone Convergence Theorem on $(X_j, \mathcal{M}_j, \mu_j)$ and on $(X, \mathcal{M}, \mu)$ to the two sides gives $\int_E f_j d\mu_j = \int_E f_j d\mu$.

Step 3: patch the pieces together.

Define f on X by

$$f(x) = f_j(x) \quad \text{for } x \in X_j, \quad j = 1, 2, \dots$$

This is unambiguous because the sets X_j are disjoint with union X , and $f \geq 0$ on X .

The function f is measurable with respect to $\mathcal{M}$: for each $c \in \mathbb{R}$,

$$\{x \in X \mid f(x) > c\} = \bigcup_{j=1}^{\infty} \{x \in X_j \mid f_j(x) > c\},$$

and each set in the union belongs to $\mathcal{M}_j \subseteq \mathcal{M}$ since f_j is $\mathcal{M}_j$ -measurable; a countable union of members of $\mathcal{M}$ belongs to $\mathcal{M}$.

Now let $E \in \mathcal{M}$. The sets $E \cap X_j$, $j \geq 1$, are disjoint measurable sets with union E , and $f = f_j$ on $E \cap X_j$. By the countable additivity of ν and by (1) applied to $E \cap X_j$,

$$\nu(E) = \sum_{j=1}^{\infty} \nu(E \cap X_j) = \sum_{j=1}^{\infty} \int_{E \cap X_j} f_j d\mu = \sum_{j=1}^{\infty} \int_{E \cap X_j} f d\mu. \quad (2)$$

It remains to identify the right-hand side with $\int_E f d\mu$. For each natural number n set

$$h_n = f \cdot \chi_{E \cap [X_1 \cup \dots \cup X_n]}.$$

Then $\{h_n\}$ is an increasing sequence of nonnegative measurable functions on X (increasing because $f \geq 0$ and the sets ascend) which converges pointwise on X to $f \cdot \chi_E$, since every point of E lies in some X_j . By the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_X h_n d\mu = \int_X f \cdot \chi_E d\mu = \int_E f d\mu.$$

On the other hand, since the X_j are disjoint we have $h_n = \sum_{j=1}^n f \cdot \chi_{E \cap X_j}$, so by Proposition 11 of this chapter (linearity of integration for nonnegative measurable functions),

$$\int_X h_n d\mu = \sum_{j=1}^n \int_{E \cap X_j} f d\mu.$$

Letting $n \rightarrow \infty$ and comparing with (2) gives

$$\nu(E) = \int_E f d\mu \quad \text{for all } E \in \mathcal{M},$$

which is the required representation.

Step 4: uniqueness.

Suppose g is a nonnegative measurable function on X with $\nu(E) = \int_E g d\mu$ for all $E \in \mathcal{M}$. Fix j . The restriction of g to X_j is measurable with respect to $\mathcal{M}_j$, since for each c the set $\{x \in X_j \mid g(x) > c\}$ belongs to $\mathcal{M}$ and is contained in X_j . By the paragraph justifying (1), $\int_E g d\mu = \int_E g d\mu_j$ for every $E \in \mathcal{M}_j$, so

$$\nu_j(E) = \nu(E) = \int_E g d\mu_j \quad \text{for all } E \in \mathcal{M}_j;$$

that is, the restriction of g to X_j represents ν_j on the finite measure space $(X_j, \mathcal{M}_j, \mu_j)$. By the uniqueness assertion of the Radon-Nikodym Theorem for finite measures, $g = f_j = f$ a.e. $[\mu_j]$ on X_j ; that is, the set

$$N_j = \{x \in X_j \mid g(x) \neq f(x)\}$$

has $\mu(N_j) = 0$. This set is measurable: writing $\{g > f\} = \bigcup_q [\{g > q\} \cap \{f < q\}]$, the union taken over the rational numbers q , and similarly for $\{f > g\}$, the set $\{x \in X \mid g(x) \neq f(x)\}$ belongs to $\mathcal{M}$, and N_j is its intersection with X_j . Let $N = \bigcup_{j=1}^{\infty} N_j$. By the countable subadditivity of μ , $\mu(N) = 0$, and $g = f$ on $X \setminus N$. Hence $g = f$ a.e. $[\mu]$ on X .

This establishes the Radon-Nikodym Theorem for σ -finite μ and ν , using only the finite case.

Remark. The only properties of the decomposition used were that the X_j are disjoint, measurable, of finite measure with respect to both μ and ν , and cover X ; the reduction is therefore entirely a bookkeeping argument, with the analytic content residing in the finite case and in the Monotone Convergence Theorem. Note also that the f produced need not be integrable over X ; it is integrable over each X_j , with $\int_{X_j} f d\mu = \nu(X_j) < \infty$.

18.8 Problems 18.50–18.56

Problem 18.50. Establish the uniqueness of the function f in the Radon-Nikodym Theorem.

Solution. Recall the setting of the Radon-Nikodym Theorem: $(X, \mathcal{M}, \mu)$ is a σ -finite measure space and ν is a σ -finite measure on $(X, \mathcal{M})$ that is absolutely continuous with respect to μ . We must show that if f and g are nonnegative functions on X , measurable with respect to $\mathcal{M}$, for which

$$\nu(E) = \int_E f d\mu = \int_E g d\mu \quad \text{for all } E \in \mathcal{M},$$

then $g = f$ a.e. $[\mu]$ on X .

Since μ is σ -finite there is a countable collection $\{X_n\}_{n=1}^{\infty}$ of sets in $\mathcal{M}$ with $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$ for each n ; since ν is σ -finite there is a countable collection $\{Y_m\}_{m=1}^{\infty}$ of sets in $\mathcal{M}$ with $X = \bigcup_{m=1}^{\infty} Y_m$ and $\nu(Y_m) < \infty$ for each m . Define

$$X_{n,m} = X_n \cap Y_m \quad \text{for } n, m \in \mathbb{N}.$$

Then $\{X_{n,m}\}$ is a countable collection of sets in $\mathcal{M}$ whose union is X , and each $X_{n,m}$ has both finite μ -measure and finite ν -measure.

Fix such a set and write $A = X_{n,m}$. Since $\int_A f d\mu = \nu(A) < \infty$, the nonnegative measurable function f is integrable over A with respect to μ , and hence is finite a.e. $[\mu]$ on A ; the same holds for g . Define

$$A^+ = \{x \in A : f(x) > g(x)\},$$

which belongs to $\mathcal{M}$. Both f and g are integrable over A^+ , and by hypothesis

$$\int_{A^+} f d\mu = \nu(A^+) = \int_{A^+} g d\mu < \infty.$$

Since these two integrals are finite and equal, the linearity of integration gives

$$\int_{A^+} (f - g) d\mu = 0,$$

where $f - g$ is defined a.e. $[\mu]$ on A^+ (both functions being finite a.e. there) and is nonnegative there. We claim a nonnegative measurable function with zero integral vanishes a.e. Indeed, for each $k \in \mathbb{N}$ put

$$A_k^+ = \{x \in A^+ : f(x) > g(x) + 1/k\},$$

a set in $\mathcal{M}$. At each point of A_k^+ we have $g < \infty$, so the difference $f - g$ is defined there and exceeds $1/k$; hence the monotonicity of integration gives

$$\frac{1}{k} \mu(A_k^+) \leq \int_{A_k^+} (f - g) d\mu \leq \int_{A^+} (f - g) d\mu = 0,$$

which is Chebychev's Inequality; hence $\mu(A_k^+) = 0$ for every k . Now if $x \in A^+$, then $f(x) > g(x)$, which forces $g(x) < \infty$ and hence $f(x) > g(x) + 1/k$ for some $k \in \mathbb{N}$. Thus $A^+ = \bigcup_{k=1}^{\infty} A_k^+$ exactly, and by the countable subadditivity of μ we conclude $\mu(A^+) = 0$.

Interchanging the roles of f and g gives $\mu(A^-) = 0$, where $A^- = \{x \in A : f(x) < g(x)\}$. Therefore $\mu(\{x \in X_{n,m} : f(x) \neq g(x)\}) = 0$ for every pair (n, m) .

Finally, since $X = \bigcup_{n,m} X_{n,m}$,

$$\{x \in X : f(x) \neq g(x)\} = \bigcup_{n,m} \{x \in X_{n,m} : f(x) \neq g(x)\},$$

a countable union of sets of μ -measure zero. By the countable subadditivity of μ this set has measure zero, that is, $f = g$ a.e. $[\mu]$ on X . This is the asserted uniqueness.

Three remarks. First, the argument used only that X is covered by countably many measurable sets of finite ν -measure, that is, the σ -finiteness of ν ; the sets X_n were carried along only for symmetry and may be discarded.

Second, the σ -finiteness of μ by itself would also suffice, by a small variant. Indeed, for $n, m, k \in \mathbb{N}$ set

$$E_{n,m,k} = X_n \cap \{x \in X : g(x) \leq m\} \cap \{x \in X : f(x) > g(x) + 1/k\}.$$

Then $\mu(E_{n,m,k}) \leq \mu(X_n) < \infty$, so $\int_{E_{n,m,k}} g d\mu \leq m \cdot \mu(E_{n,m,k}) < \infty$, and by hypothesis $\int_{E_{n,m,k}} f d\mu = \nu(E_{n,m,k}) = \int_{E_{n,m,k}} g d\mu < \infty$; both functions being μ -integrable there, $\int_{E_{n,m,k}} (f - g) d\mu = 0$, while $f - g > 1/k$ on $E_{n,m,k}$, so Chebychev's Inequality gives $\mu(E_{n,m,k}) = 0$. Since $f(x) > g(x)$ forces $g(x) < \infty$, the union of the $E_{n,m,k}$ over all n, m, k is exactly $\{x \in X : f(x) > g(x)\}$, which is therefore μ -null; symmetrically for $\{f < g\}$.

Third, some such hypothesis is genuinely needed. Let $X = \{p\}$, $\mathcal{M} = \{\emptyset, X\}$ and $\mu(X) = \nu(X) = \infty$. Then $\nu \ll \mu$, since the only set of μ -measure zero is $\emptyset$, yet both $f \equiv 1$ and $f \equiv 2$ satisfy $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$; here neither measure is σ -finite. Finally, note that the proof in the text established uniqueness only in the case both μ and ν are finite, where $A = X$ may be taken directly; the above is the extension to the σ -finite case.

Problem 18.51. Let $[a, b]$ be a closed, bounded interval and the function f be of bounded variation on $[a, b]$. Show that there is an absolutely continuous function g on $[a, b]$, and a function h on $[a, b]$ that is of bounded variation and has $h' = 0$ a.e. on $[a, b]$, for which $f = g + h$ on $[a, b]$. Then show that this decomposition is unique except for addition of constants.

Solution. Throughout, all integrals are Lebesgue integrals with respect to Lebesgue measure on $[a, b]$. Existence. Since f is of bounded variation on $[a, b]$, Jordan's Theorem exhibits f as the difference of two increasing functions on $[a, b]$, and hence, by Lebesgue's Theorem, f is differentiable almost everywhere on (a, b) . Moreover, by Corollary 6 of Chapter 6, the derivative f' is integrable over $[a, b]$. Define

$$g(x) = \int_a^x f' \quad \text{and} \quad h(x) = f(x) - \int_a^x f' \quad \text{for all } x \in [a, b],$$

so that $f = g + h$ on $[a, b]$ by construction.

The function g is the indefinite integral over $[a, b]$ of the integrable function f' , so by Theorem 11 of Chapter 6 it is absolutely continuous on $[a, b]$.

The function $h = f - g$ is of bounded variation. Indeed, g is absolutely continuous, so by Theorem 8 of Chapter 6 it is the difference of two increasing functions on $[a, b]$ and therefore is of bounded variation; and the difference of two functions of bounded variation is of bounded variation, since the total variation is subadditive with respect to sums.

Finally $h' = 0$ a.e. on $[a, b]$. By Theorem 14 of Chapter 6, since f' is integrable over $[a, b]$,

$$g'(x) = \frac{d}{dx} \left[\int_a^x f' \right] = f'(x) \quad \text{for almost all } x \in (a, b).$$

Both f and g are differentiable a.e. on (a, b) , so at almost every point of (a, b) the derivative $h'(x) = f'(x) - g'(x)$ exists and equals 0. Thus g and h have the required properties. (This is precisely the Lebesgue decomposition of a function of bounded variation described in Section 6.5; a function of bounded variation whose derivative vanishes a.e. is called singular.)

Uniqueness up to constants. Suppose

$$f = g_1 + h_1 = g_2 + h_2 \quad \text{on } [a, b],$$

where g_1, g_2 are absolutely continuous on $[a, b]$ and h_1, h_2 are of bounded variation on $[a, b]$ with $h'_1 = h'_2 = 0$ a.e. on $[a, b]$. Define

$$\varphi = g_1 - g_2 = h_2 - h_1 \quad \text{on } [a, b].$$

Being the difference of two absolutely continuous functions, φ is absolutely continuous on $[a, b]$: given $\epsilon > 0$, choose $\delta_1, \delta_2 > 0$ responding to the $\epsilon/2$ challenge for g_1 and g_2 respectively, and take $\delta = \min\{\delta_1, \delta_2\}$; the triangle inequality then gives the ϵ response for φ .

On the other hand, $\varphi = h_2 - h_1$, and h_1, h_2 are differentiable a.e. on (a, b) with derivative zero a.e., so $\varphi' = 0$ a.e. on (a, b) .

Now let $x \in [a, b]$. The function φ is absolutely continuous on $[a, x]$, so Theorem 10 of Chapter 6, applied with $[a, b]$ replaced by $[a, x]$, gives

$$\varphi(x) - \varphi(a) = \int_a^x \varphi' = \int_a^x 0 = 0.$$

Hence φ is constant on $[a, b]$; say $\varphi \equiv c$. (This is the observation, recorded in Section 6.5, that an absolutely continuous function is singular if and only if it is constant.)

Therefore

$$g_1 = g_2 + c \quad \text{and} \quad h_1 = h_2 - c \quad \text{on } [a, b],$$

so the two decompositions differ only by the addition of a constant to g and its subtraction from h . Conversely, for any constant c the pair $(g + c, h - c)$ is again such a decomposition, since adding a constant preserves both absolute continuity and the vanishing of the derivative. This is exactly the asserted uniqueness.

Problem 18.52. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\{E_k\}_{k=1}^n$ a collection of measurable sets, and $\{c_k\}_{k=1}^n$ a collection of real numbers. For $E \in \mathcal{M}$, define

$$\nu(E) = \sum_{k=1}^n c_k \cdot \mu(E \cap E_k).$$

Show that ν is absolutely continuous with respect to μ and find its Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Solution. Since the c_k are allowed to be any real numbers, ν is in general a signed measure rather than a measure; because $\mu(X) < \infty$ and each $|c_k| < \infty$, the set function ν is finite and real-valued on $\mathcal{M}$. That ν is countably additive is immediate: if $\{A_j\}_{j=1}^\infty$ is a disjoint collection in $\mathcal{M}$ with union A , then for each fixed k the collection $\{A_j \cap E_k\}$ is disjoint with union $A \cap E_k$, so $\mu(A \cap E_k) = \sum_j \mu(A_j \cap E_k)$; multiplying by c_k and summing the finitely many values of k , and using that each of these series converges absolutely (being dominated by $\mu(X) < \infty$), gives $\nu(A) = \sum_j \nu(A_j)$. Thus ν is a finite signed measure on $(X, \mathcal{M})$.

Define the function f on X by

$$f = \sum_{k=1}^n c_k \cdot \chi_{E_k}.$$

Then f is a simple function measurable with respect to $\mathcal{M}$, and $|f| \leq \sum_{k=1}^n |c_k|$ on X . Since $\mu(X) < \infty$, the function f is integrable over X with respect to μ .

We claim that f represents ν . Indeed, for $E \in \mathcal{M}$, the linearity of integration and the identity $\int_E \chi_{E_k} d\mu = \mu(E \cap E_k)$ give

$$\int_E f d\mu = \sum_{k=1}^n c_k \int_E \chi_{E_k} d\mu = \sum_{k=1}^n c_k \cdot \mu(E \cap E_k) = \nu(E) \quad \text{for all } E \in \mathcal{M}.$$

Absolute continuity now follows at once. Suppose $E \in \mathcal{M}$ has $\mu(E) = 0$. Then $\mu(E \cap E_k) \leq \mu(E) = 0$ for each k by the monotonicity of μ , and therefore $\nu(E) = \sum_{k=1}^n c_k \cdot 0 = 0$. Moreover, for the signed measure ν absolute continuity with respect to μ means $|\nu| \ll \mu$; this also holds, since for $\mu(E) = 0$ every measurable subset A of E likewise has $\mu(A) = 0$ and hence $\nu(A) = 0$, so the total variation $|\nu|(E)$, being computed from the Jordan decomposition of ν restricted to subsets of E , is zero. (Concretely, if $\{P, N\}$ is a Hahn decomposition for ν , then $|\nu|(E) = \nu(E \cap P) - \nu(E \cap N) = 0 - 0 = 0$.)

Hence, by the Radon-Nikodym Theorem in the form of Corollary 20 (applicable since μ is finite, hence σ -finite, and ν is a finite signed measure absolutely continuous with respect to μ), ν has a Radon-Nikodym derivative with respect to μ , and by the displayed representation

$$\frac{d\nu}{d\mu} = \sum_{k=1}^n c_k \cdot \chi_{E_k} \quad \text{a.e. } [\mu] \text{ on } X.$$

It remains to observe that this is the only answer, that is, that the Radon-Nikodym derivative is determined up to sets of μ -measure zero. Suppose g is integrable over X with respect to μ and $\int_E g d\mu = \nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$. By the linearity of integration, $\int_E (g - f) d\mu = 0$ for all $E \in \mathcal{M}$. Taking $E = \{x \in X : g(x) > f(x)\}$ we get $\int_E (g - f) d\mu = 0$ with $g - f > 0$ on E ; exactly as in Problem 50, Chebychev's Inequality applied to the sets $\{x \in E : g(x) - f(x) > 1/j\}$ forces $\mu(E) = 0$. The symmetric choice $E = \{x \in X : g(x) < f(x)\}$ gives the reverse inequality, so $g = f$ a.e. $[\mu]$ on X . Two remarks are worth making. If the sets $\{E_k\}_{k=1}^n$ happen to be disjoint, then $\sum_{k=1}^n c_k \chi_{E_k}$ takes the value c_k on E_k and 0 off $\bigcup_{k=1}^n E_k$; in general a point lying in several of the E_k contributes the sum of the corresponding c_k . Finally, if all the c_k are nonnegative, then $f \geq 0$ and ν is a genuine finite measure, so the Radon-Nikodym Theorem itself, rather than Corollary 20, applies directly.

Problem 18.53. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative function that is integrable over X with respect to μ . Find the Lebesgue decomposition with respect to μ of the measure ν defined by $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{M}$.

Solution. The answer is that the decomposition is the trivial one:

$$\nu = \nu_0 + \nu_1 \quad \text{with} \quad \nu_0 = 0 \quad \text{and} \quad \nu_1 = \nu,$$

and this is the only decomposition of ν into a part singular with respect to μ and a part absolutely continuous with respect to μ .

First, ν is indeed a measure on $(X, \mathcal{M})$. This is the observation made at the start of Section 18.4: $\nu(\emptyset) = 0$, and if $\{A_j\}_{j=1}^\infty$ is a disjoint collection in $\mathcal{M}$ with union A , then $f\chi_A = \sum_{j=1}^\infty f\chi_{A_j}$ pointwise on X , so the linearity of integration together with the Monotone Convergence Theorem applied to the increasing sequence of partial sums gives $\nu(A) = \sum_{j=1}^\infty \nu(A_j)$. Furthermore $\nu(X) = \int_X f d\mu < \infty$, since f is integrable over X ; thus ν is a finite measure.

Next, $\nu \ll \mu$. Let $E \in \mathcal{M}$ have $\mu(E) = 0$. The integral of a nonnegative measurable function over a set of measure zero is zero: if φ is simple with $0 \leq \varphi \leq f$ on E , then $\int_E \varphi d\mu$ is a finite sum of terms $a_i \cdot \mu(E_i)$ with $E_i \subseteq E$, and each $\mu(E_i) = 0$ by monotonicity, so $\int_E \varphi d\mu = 0$; taking the supremum over such φ gives $\nu(E) = \int_E f d\mu = 0$. Hence ν is absolutely continuous with respect to μ , and $\nu = 0 + \nu$ is a decomposition of the required type once we check that the zero measure is singular with respect to μ .

That check is immediate. Take $A = X$ and $B = \emptyset$. These are disjoint sets in $\mathcal{M}$ with $X = A \cup B$, and $\nu_0(A) = 0$ (since ν_0 is the zero measure) while $\mu(B) = \mu(\emptyset) = 0$. Thus $\nu_0 = 0$ is mutually singular with μ .

Finally we verify uniqueness directly, which has the advantage of requiring no σ -finiteness hypothesis on μ (the problem does not assume any). Suppose

$$\nu = \eta_0 + \eta_1 \quad \text{on } \mathcal{M},$$

where η_0 and η_1 are measures on $(X, \mathcal{M})$ with $\eta_0 \perp \mu$ and $\eta_1 \ll \mu$. Choose disjoint sets $A, B \in \mathcal{M}$ with $X = A \cup B$, $\eta_0(A) = 0$ and $\mu(B) = 0$.

Let $E \in \mathcal{M}$. Since $E \cap B \subseteq B$ and $\mu(B) = 0$, the monotonicity of μ gives $\mu(E \cap B) = 0$, and hence, by the paragraph above, $\nu(E \cap B) = \int_{E \cap B} f d\mu = 0$. Since $\eta_0 \leq \nu$ (because $\nu = \eta_0 + \eta_1$ and η_1 is nonnegative), the monotonicity of measures gives

$$0 \leq \eta_0(E \cap B) \leq \nu(E \cap B) = 0.$$

Also $\eta_0(E \cap A) \leq \eta_0(A) = 0$. By the additivity of η_0 over the disjoint decomposition $E = (E \cap A) \cup (E \cap B)$,

$$\eta_0(E) = \eta_0(E \cap A) + \eta_0(E \cap B) = 0.$$

As $E \in \mathcal{M}$ was arbitrary, $\eta_0 = 0$, and therefore $\eta_1 = \nu - \eta_0 = \nu$, in the sense that $\eta_1(E) = \nu(E) - 0 = \nu(E)$ for every $E \in \mathcal{M}$.

So the Lebesgue decomposition of ν with respect to μ is $\nu_0 = 0$, $\nu_1 = \nu$: the measure ν has no singular part at all. This is what one should expect, since ν was already presented in the form (28) that the Radon-Nikodym Theorem produces, with Radon-Nikodym derivative $d\nu/d\mu = f$ a.e. $[\mu]$.

Problem 18.54. Let μ , ν , and λ be σ -finite measures on the measurable space $(X, \mathcal{M})$.

- (i) If $\nu \ll \mu$ and f is a nonnegative function on X that is measurable with respect to $\mathcal{M}$, show that

$$\int_X f d\nu = \int_X f \left[\frac{d\nu}{d\mu} \right] d\mu.$$

- (ii) If $\nu \ll \mu$ and $\lambda \ll \mu$, show that

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \quad \text{a.e. } [\mu].$$

- (iii) If $\nu \ll \mu \ll \lambda$, show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \quad \text{a.e. } [\lambda].$$

- (iv) If $\nu \ll \mu$ and $\mu \ll \nu$, show that

$$\frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\nu} = 1 \quad \text{a.e. } [\mu].$$

Solution. Throughout, all Radon-Nikodym derivatives exist by the Radon-Nikodym Theorem, since all the measures involved are σ -finite and the absolute continuity hypotheses are in force. By Problem 50 each such derivative is unique up to a set of measure zero for the dominating measure, and we use this uniqueness repeatedly: to identify $d\nu/d\mu$ it suffices to exhibit a single nonnegative $\mathcal{M}$ -measurable function g with $\nu(E) = \int_E g d\mu$ for all $E \in \mathcal{M}$.

Part (i). Write $g = d\nu/d\mu$, so that g is a nonnegative $\mathcal{M}$ -measurable function on X with

$$\nu(E) = \int_E g d\mu \quad \text{for all } E \in \mathcal{M}.$$

We prove $\int_X f d\nu = \int_X fg d\mu$ for every nonnegative $\mathcal{M}$ -measurable f , in three steps. (We use the convention $0 \cdot \infty = 0$ throughout; note that since ν is σ -finite, the displayed representation forces g to be finite a.e. $[\mu]$, so the product fg is unambiguous a.e.)

Step 1: characteristic functions. If $f = \chi_E$ with $E \in \mathcal{M}$, then $\int_X \chi_E d\nu = \nu(E)$ and $\int_X \chi_E g d\mu = \int_E g d\mu$, and these agree by the defining property of g .

Step 2: simple functions. Let $\varphi = \sum_{k=1}^n a_k \chi_{E_k}$ be a nonnegative simple function, with each $a_k \geq 0$ and each $E_k \in \mathcal{M}$. By the linearity of integration with respect to ν and with respect to μ , and by Step 1,

$$\int_X \varphi d\nu = \sum_{k=1}^n a_k \nu(E_k) = \sum_{k=1}^n a_k \int_X \chi_{E_k} g d\mu = \int_X \varphi g d\mu.$$

Step 3: general nonnegative f . By the Simple Approximation Theorem there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple $\mathcal{M}$ -measurable functions on X converging pointwise on X to f . Then $\{\varphi_n\}$ increases pointwise to f , so the Monotone Convergence Theorem, applied in the measure space $(X, \mathcal{M}, \nu)$, gives

$$\lim_{n \rightarrow \infty} \int_X \varphi_n d\nu = \int_X f d\nu.$$

On the other hand $\{\varphi_n g\}$ is an increasing sequence of nonnegative $\mathcal{M}$ -measurable functions converging pointwise to fg on X , so the Monotone Convergence Theorem in $(X, \mathcal{M}, \mu)$ gives

$$\lim_{n \rightarrow \infty} \int_X \varphi_n g d\mu = \int_X fg d\mu.$$

Since the two sequences of integrals are equal term by term by Step 2, their limits agree:

$$\int_X f d\nu = \int_X f \left[\frac{d\nu}{d\mu} \right] d\mu.$$

The same argument with f replaced by $f\chi_E$ gives the localized form $\int_E f d\nu = \int_E f [d\nu/d\mu] d\mu$ for every $E \in \mathcal{M}$, which we use below.

Part (ii). Define $(\nu + \lambda)(E) = \nu(E) + \lambda(E)$ for $E \in \mathcal{M}$; this is a measure on $(X, \mathcal{M})$, since countable additivity is preserved under addition of series of nonnegative terms. It is σ -finite: choose $\{A_n\}$ in $\mathcal{M}$ covering X with $\nu(A_n) < \infty$ and $\{B_m\}$ in $\mathcal{M}$ covering X with $\lambda(B_m) < \infty$; then the countable collection $\{A_n \cap B_m\}$ covers X and $(\nu + \lambda)(A_n \cap B_m) < \infty$. It is absolutely continuous with respect to μ : if $\mu(E) = 0$ then $\nu(E) = \lambda(E) = 0$, so $(\nu + \lambda)(E) = 0$.

Now put $g = d\nu/d\mu$ and $h = d\lambda/d\mu$. For $E \in \mathcal{M}$, the linearity of integration for nonnegative measurable functions gives

$$(\nu + \lambda)(E) = \nu(E) + \lambda(E) = \int_E g d\mu + \int_E h d\mu = \int_E (g + h) d\mu.$$

Thus the nonnegative $\mathcal{M}$ -measurable function $g + h$ represents $\nu + \lambda$ as an indefinite integral with respect to μ . By the uniqueness assertion of the Radon-Nikodym Theorem (Problem 50),

$$\frac{d(\nu + \lambda)}{d\mu} = g + h = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \quad \text{a.e. } [\mu].$$

Part (iii). Put $g = d\nu/d\mu$ and $h = d\mu/d\lambda$. First observe that $\nu \ll \lambda$: if $\lambda(E) = 0$, then $\mu(E) = 0$ since $\mu \ll \lambda$, and then $\nu(E) = 0$ since $\nu \ll \mu$. So $d\nu/d\lambda$ exists.

Let $E \in \mathcal{M}$. Then $\chi_E g$ is a nonnegative $\mathcal{M}$ -measurable function on X , so part (i), applied to the pair $\mu \ll \lambda$ with the function $\chi_E g$ in the role of f , gives

$$\nu(E) = \int_E g d\mu = \int_X \chi_E g d\mu = \int_X \chi_E g h d\lambda = \int_E gh d\lambda.$$

Hence the nonnegative $\mathcal{M}$ -measurable function gh represents ν as an indefinite integral with respect to λ , and by uniqueness

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \quad \text{a.e. } [\lambda].$$

Part (iv). First note that for any σ -finite measure ν we have $d\nu/d\nu = 1$ a.e. $[\nu]$, since $\nu(E) = \int_E 1 d\nu$ for all $E \in \mathcal{M}$ and the Radon-Nikodym derivative is unique a.e. $[\nu]$.

Now apply part (iii) to the chain $\nu \ll \mu \ll \nu$, that is, with λ taken to be ν itself; the hypotheses $\nu \ll \mu$ and $\mu \ll \nu$ are exactly what is given. Part (iii) yields

$$1 = \frac{d\nu}{d\nu} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\nu} \quad \text{a.e. } [\nu].$$

It remains to upgrade “a.e. $[\nu]$ ” to “a.e. $[\mu]$ ”. Let $N \in \mathcal{M}$ be the exceptional set, so that $\nu(N) = 0$ and the displayed identity holds at every point of $X \sim N$. Since $\mu \ll \nu$ and $\nu(N) = 0$, we have $\mu(N) = 0$. Therefore

$$\frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\nu} = 1 \quad \text{a.e. } [\mu],$$

as asserted. (In particular both derivatives are finite and strictly positive a.e. $[\mu]$, the two measures having exactly the same null sets.)

Problem 18.55. Let μ, ν, ν_1 , and ν_2 be measures on the measurable space $(X, \mathcal{M})$.

- (i) Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.
- (ii) Show that if ν_1 and ν_2 are singular with respect to μ , then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.
- (iii) Show that if ν_1 and ν_2 are absolutely continuous with respect to μ , then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.
- (iv) Prove the uniqueness assertion in the Lebesgue decomposition.

Solution. Recall the two definitions. The measures μ and ν are mutually singular, written $\nu \perp \mu$, provided there are disjoint sets A and B in $\mathcal{M}$ with $X = A \cup B$ and $\nu(A) = \mu(B) = 0$. The measure ν is absolutely continuous with respect to μ , written $\nu \ll \mu$, provided $\nu(E) = 0$ whenever $E \in \mathcal{M}$ has $\mu(E) = 0$.

Part (i). Assume $\nu \perp \mu$ and $\nu \ll \mu$. Choose disjoint $A, B \in \mathcal{M}$ with $X = A \cup B$, $\nu(A) = 0$ and $\mu(B) = 0$. Since $\mu(B) = 0$ and $\nu \ll \mu$, we have $\nu(B) = 0$. By the additivity of ν over the disjoint

decomposition $X = A \cup B$,

$$\nu(X) = \nu(A) + \nu(B) = 0 + 0 = 0.$$

For any $E \in \mathcal{M}$, monotonicity gives $0 \leq \nu(E) \leq \nu(X) = 0$. Hence $\nu = 0$.

Part (ii). Let $\alpha, \beta \geq 0$ and set $\eta = \alpha\nu_1 + \beta\nu_2$, meaning $\eta(E) = \alpha\nu_1(E) + \beta\nu_2(E)$ for $E \in \mathcal{M}$, with the convention $0 \cdot \infty = 0$. That η is a measure is clear: $\eta(\emptyset) = 0$, and countable additivity follows from that of ν_1 and ν_2 together with the fact that a series of nonnegative extended real numbers may be multiplied termwise by a nonnegative constant and two such series added termwise.

Since $\nu_1 \perp \mu$, choose disjoint $A_1, B_1 \in \mathcal{M}$ with $X = A_1 \cup B_1$, $\nu_1(A_1) = 0$ and $\mu(B_1) = 0$. Similarly choose $A_2, B_2 \in \mathcal{M}$ with $X = A_2 \cup B_2$, $\nu_2(A_2) = 0$ and $\mu(B_2) = 0$. Define

$$B = B_1 \cup B_2 \quad \text{and} \quad A = X \sim B = A_1 \cap A_2.$$

(The last equality holds because $A_i = X \sim B_i$.) These are disjoint sets in $\mathcal{M}$ whose union is X . By the countable, hence finite, subadditivity of μ ,

$$\mu(B) \leq \mu(B_1) + \mu(B_2) = 0.$$

And since $A \subseteq A_1$ and $A \subseteq A_2$, the monotonicity of ν_1 and ν_2 gives $\nu_1(A) \leq \nu_1(A_1) = 0$ and $\nu_2(A) \leq \nu_2(A_2) = 0$, so

$$\eta(A) = \alpha\nu_1(A) + \beta\nu_2(A) = 0.$$

Thus $\eta \perp \mu$.

Part (iii). Again let $\alpha, \beta \geq 0$ and $\eta = \alpha\nu_1 + \beta\nu_2$, a measure as noted in part (ii). Let $E \in \mathcal{M}$ have $\mu(E) = 0$. Since $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, we have $\nu_1(E) = \nu_2(E) = 0$, and therefore

$$\eta(E) = \alpha \cdot 0 + \beta \cdot 0 = 0.$$

Thus $\eta \ll \mu$.

Part (iv). The Lebesgue Decomposition Theorem asserts, for a σ -finite measure space $(X, \mathcal{M}, \mu)$ and a σ -finite measure ν on $(X, \mathcal{M})$, that there are measures ν_0 and ν_1 on $\mathcal{M}$ with $\nu_0 \perp \mu$, $\nu_1 \ll \mu$ and $\nu = \nu_0 + \nu_1$, and that ν_0 and ν_1 are unique. We prove the uniqueness. (The argument below uses neither σ -finiteness nor any subtraction of measures, so it is valid as stated for arbitrary measures μ and ν admitting such a decomposition.)

Suppose

$$\nu = \nu_0 + \nu_1 = \eta_0 + \eta_1 \quad \text{on } \mathcal{M},$$

where $\nu_0 \perp \mu$, $\eta_0 \perp \mu$, $\nu_1 \ll \mu$ and $\eta_1 \ll \mu$. Choose disjoint $A_1, B_1 \in \mathcal{M}$ with $X = A_1 \cup B_1$, $\nu_0(A_1) = 0$, $\mu(B_1) = 0$, and disjoint $A_2, B_2 \in \mathcal{M}$ with $X = A_2 \cup B_2$, $\eta_0(A_2) = 0$, $\mu(B_2) = 0$. Set

$$B = B_1 \cup B_2, \quad A = X \sim B = A_1 \cap A_2.$$

Then A and B are disjoint sets in $\mathcal{M}$ with $X = A \cup B$, and, exactly as in part (ii),

$$\mu(B) = 0, \quad \nu_0(A) = \eta_0(A) = 0.$$

Since $\mu(B) = 0$ and both ν_1 and η_1 are absolutely continuous with respect to μ , we also have

$$\nu_1(B) = \eta_1(B) = 0.$$

Now let $E \in \mathcal{M}$ be arbitrary and decompose E as the disjoint union of $E \cap A$ and $E \cap B$.

For the singular parts: by monotonicity, $\nu_1(E \cap B) \leq \nu_1(B) = 0$, so

$$\nu(E \cap B) = \nu_0(E \cap B) + \nu_1(E \cap B) = \nu_0(E \cap B),$$

and likewise $\nu(E \cap B) = \eta_0(E \cap B)$. Also $\nu_0(E \cap A) \leq \nu_0(A) = 0$ and $\eta_0(E \cap A) \leq \eta_0(A) = 0$. Hence, by additivity,

$$\nu_0(E) = \nu_0(E \cap A) + \nu_0(E \cap B) = \nu(E \cap B) = \eta_0(E \cap A) + \eta_0(E \cap B) = \eta_0(E).$$

For the absolutely continuous parts: by the same two vanishing facts, $\nu(E \cap A) = \nu_0(E \cap A) + \nu_1(E \cap A) = \nu_1(E \cap A)$ and likewise $\nu(E \cap B) = \eta_1(E \cap B)$; while $\nu_1(E \cap B) \leq \nu_1(B) = 0$ and $\eta_1(E \cap B) \leq \eta_1(B) = 0$. Hence

$$\nu_1(E) = \nu_1(E \cap A) + \nu_1(E \cap B) = \nu(E \cap A) = \eta_1(E \cap A) + \eta_1(E \cap B) = \eta_1(E).$$

Since $E \in \mathcal{M}$ was arbitrary, $\nu_0 = \eta_0$ and $\nu_1 = \eta_1$. This is the uniqueness assertion in the Lebesgue Decomposition Theorem.

Observe that part (i) is the essential point in a slicker rendering of this uniqueness when ν is finite: then $\nu_0 - \eta_0 = \eta_1 - \nu_1$ is a finite signed measure that is both singular and absolutely continuous with respect to μ , hence zero. The argument given above avoids signed measures entirely.

Problem 18.56. Characterize the measure spaces $(X, \mathcal{M}, \mu)$ for which the counting measure on $\mathcal{M}$ is absolutely continuous with respect to μ and those for which, given $x_0 \in X$, the Dirac measure δ_{x_0} on $\mathcal{M}$ is absolutely continuous with respect to μ .

Solution. Recall the two measures in question. The counting measure c on $\mathcal{M}$ is defined by letting $c(E)$ be the number of points in E if E is finite and $c(E) = \infty$ otherwise. For a point $x_0 \in X$, the Dirac measure δ_{x_0} on $\mathcal{M}$ is defined by

$$\delta_{x_0}(E) = \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{if } x_0 \notin E, \end{cases} \quad E \in \mathcal{M}.$$

The Dirac measure. We claim that

$$\delta_{x_0} \ll \mu \iff \mu(E) > 0 \text{ for every } E \in \mathcal{M} \text{ with } x_0 \in E,$$

that is, if and only if the point x_0 belongs to no set of $\mathcal{M}$ of μ -measure zero.

Suppose first that $\delta_{x_0} \ll \mu$, and let $E \in \mathcal{M}$ contain x_0 . Then $\delta_{x_0}(E) = 1 \neq 0$, so by absolute continuity we cannot have $\mu(E) = 0$; hence $\mu(E) > 0$. Conversely, suppose every measurable set containing x_0 has positive μ -measure, and let $E \in \mathcal{M}$ have $\mu(E) = 0$. Then $x_0 \notin E$, and therefore $\delta_{x_0}(E) = 0$. Thus $\delta_{x_0} \ll \mu$.

In the frequent case that the singleton $\{x_0\}$ belongs to $\mathcal{M}$, this criterion simplifies:

$$\delta_{x_0} \ll \mu \iff \mu(\{x_0\}) > 0,$$

that is, if and only if x_0 is an atom of μ of positive mass. Indeed, if $\delta_{x_0} \ll \mu$ then $\{x_0\}$ is a measurable set containing x_0 , so $\mu(\{x_0\}) > 0$ by the criterion; conversely, if $\mu(\{x_0\}) > 0$ and $E \in \mathcal{M}$ contains x_0 , then $\{x_0\} \subseteq E$ and monotonicity gives $\mu(E) \geq \mu(\{x_0\}) > 0$.

The counting measure. We claim that

$$c \ll \mu \iff \text{the empty set is the only set in } \mathcal{M} \text{ of } \mu\text{-measure zero,}$$

equivalently, $\mu(E) > 0$ for every nonempty $E \in \mathcal{M}$.

Suppose first that $c \ll \mu$ and let $E \in \mathcal{M}$ have $\mu(E) = 0$. Then $c(E) = 0$, and since $c(E)$ is the number of points in E , this forces $E = \emptyset$. Conversely, suppose $\emptyset$ is the only measurable set of μ -measure zero, and let $E \in \mathcal{M}$ have $\mu(E) = 0$. Then $E = \emptyset$, so $c(E) = 0$. Thus $c \ll \mu$.

The two characterizations fit together in the following way: the counting measure is absolutely continuous with respect to μ if and only if $\delta_x \ll \mu$ for every $x \in X$. Indeed, if $c \ll \mu$ and $x \in X$, then any $E \in \mathcal{M}$ containing x is nonempty, hence has $\mu(E) > 0$ by the criterion just proved, so $\delta_x \ll \mu$. Conversely, if $\delta_x \ll \mu$ for every $x \in X$ and $E \in \mathcal{M}$ is nonempty, pick $x \in E$; then E is a measurable set containing x , so $\mu(E) > 0$, and hence the only measurable null set is $\emptyset$, that is, $c \ll \mu$.

Some illustrations. If $\mathcal{M}$ contains all the singletons of X , then the counting measure is absolutely continuous with respect to μ precisely when $\mu(\{x\}) > 0$ for every $x \in X$; so for $X = [0, 1]$ with $\mathcal{M}$ the Lebesgue measurable sets and μ Lebesgue measure, neither c nor any δ_{x_0} is absolutely continuous with

respect to μ , every singleton being a null set. On the other hand, if μ is itself the counting measure, then $\mu(E) \geq 1$ for every nonempty E , so $c \ll \mu$ and $\delta_{x_0} \ll \mu$ for every x_0 ; this is the situation of the Example following the Radon-Nikodym Theorem, where every measure on $\mathcal{M}$ is absolutely continuous with respect to the counting measure because the only null set is the empty set. Finally, a trivial but instructive case: if $\mathcal{M} = \{\emptyset, X\}$ and $\mu(X) > 0$, then the only null set is $\emptyset$, so $c \ll \mu$ even though μ assigns no mass to individual points, since the singletons are not measurable.

18.9 Problems 18.57–18.63

Problem 18.57. Let $\{\mu_n\}$ be a sequence of measures on a measurable space $(X, \mathcal{M})$ for which there is a constant $c > 0$ such that $\mu_n(X) \leq c$ for all n . Define $\mu: \mathcal{M} \rightarrow [0, \infty]$ by

$$\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n}.$$

Show that μ is a measure on $\mathcal{M}$ and that each μ_n is absolutely continuous with respect to μ .

Solution. Explicitly, $\mu(E) = \sum_{n=1}^{\infty} 2^{-n} \mu_n(E)$ for $E \in \mathcal{M}$; every term is nonnegative, so the series has a well-defined sum in $[0, \infty]$ and μ is a nonnegative extended real-valued set function on $\mathcal{M}$.

We first record the elementary interchange principle for double series of nonnegative terms that the verification of countable additivity requires.

Claim: if $a_{n,k} \geq 0$ for all natural numbers n, k , then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k},$$

both sides being well defined in $[0, \infty]$. Indeed, denote the left-hand side by S and the right-hand side by T . Fix a natural number K . Since a finite sum of limits is the limit of the finite sums,

$$\sum_{k=1}^K \sum_{n=1}^{\infty} a_{n,k} = \lim_{N \rightarrow \infty} \sum_{k=1}^K \sum_{n=1}^N a_{n,k} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \sum_{k=1}^K a_{n,k} \leq \lim_{N \rightarrow \infty} \sum_{n=1}^N \sum_{k=1}^{\infty} a_{n,k} = S,$$

where the inequality uses $\sum_{k=1}^K a_{n,k} \leq \sum_{k=1}^{\infty} a_{n,k}$, valid because the terms are nonnegative. Taking the supremum over K gives $T \leq S$. Interchanging the roles of the two indices gives $S \leq T$, and hence $S = T$.

Now we verify that μ is a measure.

Since $\mu_n(\emptyset) = 0$ for every n , we have $\mu(\emptyset) = \sum_{n=1}^{\infty} 2^{-n} \cdot 0 = 0$.

Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of members of $\mathcal{M}$ and put $E = \bigcup_{k=1}^{\infty} E_k$. Each μ_n is countably additive, so $\mu_n(E) = \sum_{k=1}^{\infty} \mu_n(E_k)$. Therefore, applying the Claim to the nonnegative numbers $a_{n,k} = 2^{-n} \mu_n(E_k)$,

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \mu_n(E_k) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu_n(E_k)}{2^n} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu_n(E_k)}{2^n} = \sum_{k=1}^{\infty} \mu(E_k).$$

Thus μ is countably additive, and so μ is a measure on $\mathcal{M}$.

The hypothesis $\mu_n(X) \leq c$ for all n moreover makes μ a finite measure:

$$\mu(X) = \sum_{n=1}^{\infty} \frac{\mu_n(X)}{2^n} \leq c \sum_{n=1}^{\infty} \frac{1}{2^n} = c < \infty.$$

(Countable additivity holds regardless of this hypothesis; the hypothesis is what prevents μ from taking the value ∞ .)

Finally, fix a natural number n and let $E \in \mathcal{M}$ have $\mu(E) = 0$. Then $\sum_{m=1}^{\infty} 2^{-m} \mu_m(E) = 0$, a series all of whose terms are nonnegative; hence every term vanishes. In particular $2^{-n} \mu_n(E) = 0$, and therefore $\mu_n(E) = 0$. Thus $\mu(E) = 0$ implies $\mu_n(E) = 0$, which is precisely the statement that μ_n is absolutely continuous with respect to μ .

Problem 18.58. Let μ and ν be measures on the measurable space $(X, \mathcal{M})$ and define $\lambda = \mu + \nu$. Let the nonnegative function f on X be measurable with respect to $(X, \mathcal{M})$. Show that f is integrable over X with respect to λ if and only if it is integrable over X with respect to both μ and ν . Also show that if f is integrable over X with respect to λ , then

$$\int_E g d\lambda = \int_E g d\mu + \int_E g d\nu \quad \text{for all } E \in \mathcal{M}.$$

Solution. (In the displayed identity the letter g is a misprint for the function f of the preceding sentence; we prove the identity for f .)

First observe that $\lambda = \mu + \nu$, defined by $\lambda(E) = \mu(E) + \nu(E)$, is indeed a measure on $\mathcal{M}$: it is nonnegative, $\lambda(\emptyset) = 0 + 0 = 0$, and for a countable disjoint collection $\{E_k\}_{k=1}^\infty$ with union E ,

$$\lambda(E) = \sum_{k=1}^\infty \mu(E_k) + \sum_{k=1}^\infty \nu(E_k) = \sum_{k=1}^\infty [\mu(E_k) + \nu(E_k)] = \sum_{k=1}^\infty \lambda(E_k),$$

the middle equality being the termwise addition of two series of nonnegative terms, which is legitimate in $[0, \infty]$ since the partial sums add and both sequences of partial sums are monotone.

The heart of the problem is the following assertion.

Assertion: for every nonnegative measurable function h on X and every $E \in \mathcal{M}$,

$$\int_E h d\lambda = \int_E h d\mu + \int_E h d\nu, \quad (*)$$

an equality of values in $[0, \infty]$.

Proof of the Assertion. Suppose first that $h = \chi_A$ is the characteristic function of a set $A \in \mathcal{M}$. Then, by the definition of the integral of a characteristic function over a set,

$$\int_E \chi_A d\lambda = \lambda(A \cap E) = \mu(A \cap E) + \nu(A \cap E) = \int_E \chi_A d\mu + \int_E \chi_A d\nu,$$

so $(*)$ holds for characteristic functions. If $\varphi = \sum_{i=1}^m a_i \chi_{A_i}$ is a nonnegative simple function, then by the linearity of integration of simple functions with respect to each of the three measures, $(*)$ holds for φ ; note that all three integrals of a nonnegative simple function over E are sums of finitely many nonnegative terms, so no cancellation of infinities can occur, and the additivity is a matter of adding finitely many equalities in $[0, \infty]$.

For general nonnegative measurable h , the Simple Approximation Theorem provides an increasing sequence $\{\varphi_j\}$ of nonnegative simple measurable functions on X that converges pointwise on X to h . By the Monotone Convergence Theorem, applied in turn on the measure spaces $(X, \mathcal{M}, \lambda)$, $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{M}, \nu)$ to the restriction of this sequence to E ,

$$\int_E h d\lambda = \lim_{j \rightarrow \infty} \int_E \varphi_j d\lambda, \quad \int_E h d\mu = \lim_{j \rightarrow \infty} \int_E \varphi_j d\mu, \quad \int_E h d\nu = \lim_{j \rightarrow \infty} \int_E \varphi_j d\nu,$$

each limit existing in $[0, \infty]$ and being the limit of a nondecreasing sequence. Since $(*)$ holds for each φ_j , and since for nondecreasing sequences in $[0, \infty]$ the limit of the sum is the sum of the limits,

$$\int_E h d\lambda = \lim_{j \rightarrow \infty} \left[\int_E \varphi_j d\mu + \int_E \varphi_j d\nu \right] = \int_E h d\mu + \int_E h d\nu.$$

This proves the Assertion.

Now take $h = f$ and $E = X$ in $(*)$:

$$\int_X f d\lambda = \int_X f d\mu + \int_X f d\nu.$$

All three quantities lie in $[0, \infty]$, and a sum of two nonnegative extended real numbers is finite if and only if each summand is finite. Since a nonnegative measurable function is by definition integrable

over X with respect to a measure provided its integral over X is finite, we conclude that f is integrable over X with respect to λ if and only if it is integrable over X with respect to both μ and ν . Finally, if f is integrable over X with respect to λ , then $(*)$ with $h = f$ says exactly that

$$\int_E f d\lambda = \int_E f d\mu + \int_E f d\nu \quad \text{for all } E \in \mathcal{M},$$

and by the preceding paragraph together with the monotonicity of integration all three of these integrals are finite. (Indeed $(*)$ holds for every E whether or not f is integrable; integrability of f is what guarantees that the identity is an identity among real numbers rather than among elements of $[0, \infty]$.)

Problem 18.59. (von Neumann's proof of the Radon-Nikodym Theorem) The basis of this proof is the following assertion, which is a corollary of the Riesz-Frechet Representation Theorem for the dual of a Hilbert space: For a measure space $(X, \mathcal{M}, \lambda)$, let $L^2(X, \lambda)$ be the collection of measurable functions f on X such that f^2 is integrable over X with respect to λ . Suppose that the functional $\psi: L^2(X, \lambda) \rightarrow \mathbb{R}$ is linear, and bounded in the sense that there is some $c > 0$ such that

$$|\psi(f)|^2 \leq c \cdot \int_X f^2 d\lambda \quad \text{for all } f \in L^2(X, \lambda).$$

Then there is a function $g \in L^2(X, \lambda)$ such that

$$\psi(f) = \int_X f \cdot g d\lambda \quad \text{for all } f \in L^2(X, \lambda).$$

Assuming this representation result, verify the following steps in another proof of the Radon-Nikodym Theorem, where μ and ν are finite measures on a measurable space $(X, \mathcal{M})$ and ν is absolutely continuous with respect to μ .

(i) Define the measure $\lambda = \mu + \nu$ on the measurable space $(X, \mathcal{M})$ and the functional ψ on $L^2(X, \lambda)$ by

$$\psi(f) = \int_X f d\mu \quad \text{for all } f \in L^2(X, \lambda).$$

Show that ψ is a bounded linear functional on $L^2(X, \lambda)$.

(ii) By the above representation result, choose a function $g \in L^2(X, \lambda)$ such that

$$\int_X f d\mu = \int_X f \cdot g d\lambda \quad \text{for all } f \in L^2(X, \lambda).$$

Conclude that

$$\int_X f d\mu = \int_X f \cdot g d\mu + \int_X f \cdot g d\nu \quad \text{for all } f \in L^2(X, \lambda),$$

and therefore

$$\mu(E) = \int_E g d\mu + \int_E g d\nu \quad \text{for all } E \in \mathcal{M}.$$

From this last identity conclude that $g > 0$ a.e. $[\lambda]$ on X and then use the absolute continuity of ν with respect to μ to conclude that $\lambda\{x \in X \mid g(x) = 0\} = 0$.

(iii) Use part (ii) to assume, without loss of generality, that $g > 0$ on X . Fix a natural number n and $E \in \mathcal{M}$ and define $f = \chi_E/[g + 1/n]$ on X . Show that f belongs to $L^2(X, \lambda)$. Conclude that

$$\int_E \frac{1}{g + 1/n} d\mu = \int_E \frac{1}{g + 1/n} \cdot g d\mu + \int_E \frac{1}{g + 1/n} \cdot g d\nu \quad \text{for all } n.$$

Justify taking limits as $n \rightarrow \infty$ on each side of this equality and conclude that

$$\nu(E) = \int_E [1/g - 1] d\mu \quad \text{for all } E \in \mathcal{M}.$$

Solution. Throughout, $\lambda = \mu + \nu$, which is a finite measure on $\mathcal{M}$ by Problem 58 (with $\lambda(X) = \mu(X) + \nu(X) < \infty$), and we use repeatedly the identity established there: for every nonnegative measurable h on X and every $E \in \mathcal{M}$,

$$\int_E h d\lambda = \int_E h d\mu + \int_E h d\nu. \quad (*)$$

Two consequences are worth isolating. First, since $\mu \leq \lambda$ and $\nu \leq \lambda$ as set functions, every λ -null set is both μ -null and ν -null; thus $\mu \ll \lambda$ and $\nu \ll \lambda$, and a statement holding a.e. $[\lambda]$ holds a.e. $[\mu]$ and a.e. $[\nu]$. Second, $(*)$ extends from nonnegative functions to λ -integrable functions: if h is measurable and $\int_X |h| d\lambda < \infty$, then applying $(*)$ to $|h|$ shows h is integrable with respect to μ and with respect to ν as well, and applying $(*)$ to the nonnegative functions h^+ and h^- and subtracting the two resulting identities of real numbers gives

$$\int_E h d\lambda = \int_E h d\mu + \int_E h d\nu \quad \text{for all } E \in \mathcal{M}. \quad (**)$$

We shall also need the following elementary inequality, which we prove from scratch so as not to presuppose the Cauchy-Schwarz Inequality on a general measure space. Let $f \in L^2(X, \lambda)$ and let $t > 0$. Since $(|f| - t)^2 \geq 0$ pointwise on X , we have $2t|f| \leq f^2 + t^2$ pointwise, and hence, by the monotonicity and additivity of integration of nonnegative measurable functions,

$$2t \int_X |f| d\lambda \leq \int_X f^2 d\lambda + t^2 \lambda(X),$$

an inequality in $[0, \infty]$. Taking $t = 1$ and using $\int_X f^2 d\lambda < \infty$ and $\lambda(X) < \infty$ shows that $\int_X |f| d\lambda < \infty$. Now put $I = \int_X |f| d\lambda$ and $Q = \int_X f^2 d\lambda$, both finite. If $\lambda(X) = 0$, then taking $t = 1$ gives $2I \leq Q$ and taking t arbitrarily large in $2tI \leq Q + t^2 \lambda(X) = Q$ forces $I = 0$, so $I^2 \leq \lambda(X) \cdot Q$ holds. Suppose $\lambda(X) > 0$. If $Q > 0$, the choice $t = (Q/\lambda(X))^{1/2}$ gives $2I(Q/\lambda(X))^{1/2} \leq 2Q$, that is, $I^2 \leq \lambda(X) \cdot Q$. If $Q = 0$, then $2tI \leq t^2 \lambda(X)$ gives $I \leq t\lambda(X)/2$ for every $t > 0$, whence $I = 0$. In every case

$$\left(\int_X |f| d\lambda \right)^2 \leq \lambda(X) \cdot \int_X f^2 d\lambda. \quad (\text{CS})$$

(i) Let $f \in L^2(X, \lambda)$. By the preceding paragraph, $\int_X |f| d\lambda < \infty$; that is, f is integrable over X with respect to λ , and therefore, by the remark following $(*)$, integrable with respect to μ . So $\psi(f) = \int_X f d\mu$ is a well-defined real number. Moreover ψ is well defined on equivalence classes: if $f = f'$ a.e. $[\lambda]$ then $f = f'$ a.e. $[\mu]$, so the two μ -integrals agree.

Linearity of ψ is immediate from the linearity of integration with respect to μ over the μ -integrable functions: for $f_1, f_2 \in L^2(X, \lambda)$ and $\alpha, \beta \in \mathbb{R}$, $\psi(\alpha f_1 + \beta f_2) = \int_X (\alpha f_1 + \beta f_2) d\mu = \alpha \psi(f_1) + \beta \psi(f_2)$. For boundedness, use $(*)$ applied to $|f|$ to get $\int_X |f| d\mu \leq \int_X |f| d\lambda$, and then (CS):

$$|\psi(f)|^2 \leq \left(\int_X |f| d\mu \right)^2 \leq \left(\int_X |f| d\lambda \right)^2 \leq \lambda(X) \cdot \int_X f^2 d\lambda.$$

Thus the boundedness inequality holds with $c = \lambda(X)$ if $\lambda(X) > 0$, and with any $c > 0$ in the degenerate case $\lambda(X) = 0$ (in which case every function is null and there is nothing to prove). So ψ is a bounded linear functional on $L^2(X, \lambda)$.

(ii) By the quoted representation result there is $g \in L^2(X, \lambda)$ with

$$\int_X f d\mu = \int_X f \cdot g d\lambda \quad \text{for all } f \in L^2(X, \lambda). \quad (1)$$

For such an f , the product $f \cdot g$ is measurable and satisfies the pointwise bound $|f \cdot g| \leq \frac{1}{2}(f^2 + g^2)$, whose right-hand side is λ -integrable because $f, g \in L^2(X, \lambda)$; hence $\int_X |f \cdot g| d\lambda < \infty$. So $(**)$ applies to $h = f \cdot g$ and turns (1) into

$$\int_X f d\mu = \int_X f \cdot g d\mu + \int_X f \cdot g d\nu \quad \text{for all } f \in L^2(X, \lambda). \quad (2)$$

Fix $E \in \mathcal{M}$. Since λ is finite, $\chi_E \in L^2(X, \lambda)$, and substituting $f = \chi_E$ in (2) gives

$$\mu(E) = \int_E g d\mu + \int_E g d\nu \quad \text{for all } E \in \mathcal{M}. \quad (3)$$

By (*) applied to g^+ and g^- (both λ -integrable, since (CS) applied to g gives $\int_X |g| d\lambda < \infty$), the right-hand side of (3) equals $\int_E g d\lambda$; that is,

$$\mu(E) = \int_E g d\lambda \quad \text{for all } E \in \mathcal{M}. \quad (4)$$

We now extract the pointwise information about g from (4). Note first that what (4) yields directly is $g \geq 0$ a.e. $[\lambda]$, not the strict inequality printed in the statement; strict positivity a.e. follows only after the second step, which shows that the set where g vanishes is λ -null.

Let $E^- = \{x \in X \mid g(x) < 0\}$, a measurable set. By (4), $\int_{E^-} g d\lambda = \mu(E^-) \geq 0$. On the other hand $-g > 0$ on E^- , so $\int_{E^-} (-g) d\lambda \geq 0$ with equality only if $-g = 0$ a.e. $[\lambda]$ on E^- ; since $-g > 0$ everywhere on E^- , this forces $\lambda(E^-) = 0$. (Concretely: $E^- = \bigcup_{k=1}^{\infty} \{g \leq -1/k\}$ and $\int_{E^-} (-g) d\lambda \geq \lambda(\{g \leq -1/k\})/k$, so each set $\{g \leq -1/k\}$ is λ -null, and hence so is their union.) Therefore

$$g \geq 0 \quad \text{a.e. } [\lambda] \text{ on } X. \quad (5)$$

Next let $E_0 = \{x \in X \mid g(x) = 0\}$. By (4), $\mu(E_0) = \int_{E_0} g d\lambda = 0$. Since ν is absolutely continuous with respect to μ , $\nu(E_0) = 0$ as well. Hence

$$\lambda(E_0) = \mu(E_0) + \nu(E_0) = 0. \quad (6)$$

Combining (5) and (6): $g > 0$ a.e. $[\lambda]$ on X .

Finally we record an upper bound that will be needed in (iii). Let $E^+ = \{x \in X \mid g(x) > 1\}$. By (4),

$$\int_{E^+} (g - 1) d\lambda = \mu(E^+) - \lambda(E^+) = -\nu(E^+) \leq 0,$$

all quantities being finite. But $g - 1 > 0$ on E^+ , so exactly as above this forces $\lambda(E^+) = 0$. Hence

$$0 < g \leq 1 \quad \text{a.e. } [\lambda] \text{ on } X. \quad (7)$$

(iii) Let Z be the λ -null set off which (7) holds, and redefine g to be identically 1 on Z . Since Z is λ -null, it is μ -null and ν -null, so none of the integrals in (1), (2), (3) is affected, and the new g still lies in $L^2(X, \lambda)$ and still satisfies (2). Thus we may and do assume

$$0 < g(x) \leq 1 \quad \text{for every } x \in X.$$

Fix a natural number n and $E \in \mathcal{M}$, and set $f = \chi_E/[g + 1/n]$. This function is measurable, being a quotient of measurable functions with nowhere-vanishing denominator, and since $g + 1/n \geq 1/n > 0$ we have $0 \leq f \leq n$ on X . Hence

$$\int_X f^2 d\lambda \leq n^2 \lambda(X) < \infty,$$

so $f \in L^2(X, \lambda)$. Substituting this f into (2), and noting that f vanishes off E , gives

$$\int_E \frac{1}{g + 1/n} d\mu = \int_E \frac{g}{g + 1/n} d\mu + \int_E \frac{g}{g + 1/n} d\nu \quad \text{for all } n. \quad (8)$$

Now let $n \rightarrow \infty$. For each fixed $x \in X$ the sequence $g(x) + 1/n$ is positive and decreasing in n , so

$$\frac{1}{g(x) + 1/n} \nearrow \frac{1}{g(x)} \quad \text{and} \quad \frac{g(x)}{g(x) + 1/n} \nearrow 1,$$

both convergences being monotone increasing, the second one using $g(x) > 0$. All three sequences of integrands in (8) are therefore nonnegative, measurable, and increase pointwise on E to $1/g$, to 1, and to 1 respectively. By the Monotone Convergence Theorem applied on $(X, \mathcal{M}, \mu)$ and on $(X, \mathcal{M}, \nu)$,

$$\lim_{n \rightarrow \infty} \int_E \frac{1}{g + 1/n} d\mu = \int_E \frac{1}{g} d\mu, \quad \lim_{n \rightarrow \infty} \int_E \frac{g}{g + 1/n} d\mu = \mu(E), \quad \lim_{n \rightarrow \infty} \int_E \frac{g}{g + 1/n} d\nu = \nu(E),$$

the first limit being taken in $[0, \infty]$ and the other two being finite because μ and ν are finite measures. Passing to the limit in (8) yields

$$\int_E \frac{1}{g} d\mu = \mu(E) + \nu(E) \quad \text{for all } E \in \mathcal{M}. \quad (9)$$

The right-hand side of (9) is finite, so the nonnegative measurable function $1/g$ is integrable over E with respect to μ ; taking $E = X$ shows $1/g$ is integrable over X with respect to μ . Since $0 < g \leq 1$, the function $1/g - 1$ is nonnegative and μ -integrable, and subtracting the finite number $\mu(E) = \int_E 1 d\mu$ from both sides of (9) gives

$$\nu(E) = \int_E \left[\frac{1}{g} - 1 \right] d\mu \quad \text{for all } E \in \mathcal{M}.$$

This exhibits a nonnegative function $f_0 = 1/g - 1$ that is integrable over X with respect to μ and satisfies $\nu(E) = \int_E f_0 d\mu$ for all $E \in \mathcal{M}$, which is the conclusion of the Radon-Nikodym Theorem for finite measures μ and ν with $\nu \ll \mu$.

Problem 18.60. Let $X = [0, 1]$, $\mathcal{M}$ the collection of Lebesgue measurable subsets of $[0, 1]$, and take ν to be Lebesgue measure and μ the counting measure on $\mathcal{M}$. Show that ν is finite and absolutely continuous with respect to μ , but there is no function f for which $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$.

Solution. Recall that the counting measure μ on $\mathcal{M}$ assigns to a set $E \in \mathcal{M}$ the number of points in E if E is finite, and ∞ otherwise; it is a measure on $\mathcal{M}$.

Finiteness of ν . Since ν is Lebesgue measure restricted to $\mathcal{M}$, $\nu(X) = \nu([0, 1]) = 1 < \infty$. Thus ν is a finite measure.

Absolute continuity. Let $E \in \mathcal{M}$ with $\mu(E) = 0$. By the definition of counting measure, $\mu(E) = 0$ forces $E = \emptyset$, since a nonempty set contains at least one point and hence has counting measure at least 1. Therefore $\nu(E) = \nu(\emptyset) = 0$. Thus $\mu(E) = 0$ implies $\nu(E) = 0$, that is, ν is absolutely continuous with respect to μ .

Nonexistence of a density. We first compute the integral with respect to μ over a singleton. Let h be a nonnegative measurable function on X and $x \in X$. If $\varphi = \sum_{i=1}^m a_i \chi_{A_i}$ is a nonnegative simple function, then

$$\int_{\{x\}} \varphi d\mu = \sum_{i=1}^m a_i \mu(A_i \cap \{x\}) = \sum_{i=1}^m a_i \chi_{A_i}(x) = \varphi(x),$$

because $\mu(A_i \cap \{x\})$ equals 1 if $x \in A_i$ and 0 otherwise. By definition, $\int_{\{x\}} h d\mu = \int_X h \cdot \chi_{\{x\}} d\mu$ is the supremum of the numbers $\int_X \varphi d\mu$ taken over all bounded measurable functions φ of finite support with $0 \leq \varphi \leq h \cdot \chi_{\{x\}}$ on X . Any such φ vanishes off $\{x\}$, so $\varphi = \varphi(x) \chi_{\{x\}}$ and $\int_X \varphi d\mu = \varphi(x) \mu(\{x\}) = \varphi(x) \leq h(x)$; conversely, for each natural number k the function $\varphi_k = \min\{h(x), k\} \cdot \chi_{\{x\}}$ is bounded, has support $\{x\}$ of finite measure $\mu(\{x\}) = 1$, satisfies $0 \leq \varphi_k \leq h \cdot \chi_{\{x\}}$, and has $\int_X \varphi_k d\mu = \min\{h(x), k\}$, and $\min\{h(x), k\} \rightarrow h(x)$ as $k \rightarrow \infty$. Taking the supremum,

$$\int_{\{x\}} h d\mu = h(x).$$

Now suppose, to get a contradiction, that there were a function f on X , measurable with respect to

$\mathcal{M}$, for which

$$\nu(E) = \int_E f d\mu \quad \text{for all } E \in \mathcal{M}.$$

For the argument it is immaterial whether f is assumed nonnegative or merely integrable: in the latter case apply what follows to f^+ and f^- separately, using the same singleton computation, to get $\int_{\{x\}} f d\mu = f(x)$.

Every singleton $\{x\}$ with $x \in [0, 1]$ belongs to $\mathcal{M}$ and has Lebesgue measure zero. Hence, for every $x \in [0, 1]$,

$$f(x) = \int_{\{x\}} f d\mu = \nu(\{x\}) = 0.$$

So f vanishes identically on X , and therefore $\int_E f d\mu = 0$ for every $E \in \mathcal{M}$. Taking $E = [0, 1]$ gives $1 = \nu([0, 1]) = 0$, a contradiction. Hence no such f exists.

The example does not contradict the Radon-Nikodym Theorem: that theorem requires the measure μ with respect to which one differentiates to be σ -finite, and the counting measure on the uncountable set $[0, 1]$ is not σ -finite, since a countable union of sets of finite counting measure is countable and hence cannot be all of $[0, 1]$.

Problem 18.61. For two measurable sets A and B , show that $A \Delta B = [A \cup B] \sim [A \cap B]$ and that

$$\rho_\mu(A, B) = \mu(A) + \mu(B) - 2 \cdot \mu(A \cap B).$$

Solution. Here $(X, \mathcal{M}, \mu)$ is a finite measure space, $A \Delta B = [A \sim B] \cup [B \sim A]$ is the symmetric difference, and ρ_μ is the Nikodym metric, defined by $\rho_\mu([A], [B]) = \mu(A \Delta B)$.

The set identity. We prove the two inclusions.

Let $x \in A \Delta B$. Then either $x \in A \sim B$ or $x \in B \sim A$. In the first case $x \in A \subseteq A \cup B$, while $x \notin B$ implies $x \notin A \cap B$; hence $x \in [A \cup B] \sim [A \cap B]$. The second case is symmetric in A and B . Therefore $A \Delta B \subseteq [A \cup B] \sim [A \cap B]$.

Conversely, let $x \in [A \cup B] \sim [A \cap B]$, so that x belongs to $A \cup B$ but not to $A \cap B$. If $x \in A$, then, since $x \notin A \cap B$, we must have $x \notin B$, so $x \in A \sim B \subseteq A \Delta B$. If $x \notin A$, then, since $x \in A \cup B$, we have $x \in B$, so $x \in B \sim A \subseteq A \Delta B$. Therefore $[A \cup B] \sim [A \cap B] \subseteq A \Delta B$.

The two inclusions give $A \Delta B = [A \cup B] \sim [A \cap B]$.

The measure identity. The sets $A \sim B$ and $B \sim A$ are measurable, since $\mathcal{M}$ is a σ -algebra, and they are disjoint: a point of $A \sim B$ lies in A , whereas a point of $B \sim A$ does not. Hence, by the additivity of μ ,

$$\mu(A \Delta B) = \mu(A \sim B) + \mu(B \sim A).$$

Next, A is the disjoint union of the measurable sets $A \cap B$ and $A \sim B$, so $\mu(A) = \mu(A \cap B) + \mu(A \sim B)$. Since μ is a finite measure, $\mu(A \cap B) < \infty$, and we may excise it:

$$\mu(A \sim B) = \mu(A) - \mu(A \cap B).$$

Interchanging A and B gives $\mu(B \sim A) = \mu(B) - \mu(A \cap B)$. Substituting these two expressions into the previous display,

$$\rho_\mu(A, B) = \mu(A \Delta B) = [\mu(A) - \mu(A \cap B)] + [\mu(B) - \mu(A \cap B)] = \mu(A) + \mu(B) - 2 \cdot \mu(A \cap B),$$

as asserted.

Equivalently, one may argue from the set identity just proved: since $A \cap B \subseteq A \cup B$ and $\mu(A \cap B) < \infty$, excision gives $\mu(A \Delta B) = \mu(A \cup B) - \mu(A \cap B)$, and the inclusion-exclusion identity $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ (itself a consequence of the two excisions above applied to $A \cup B = A \dot{\cup} [B \sim A]$) yields the same conclusion.

Problem 18.62. Let $\{A_n\}$ be a sequence of measurable sets that converges to the measurable set A_0 with respect to the Nikodym metric. Show that $A_0 = \bigcup_{n=1}^{\infty} [\bigcap_{k=n}^{\infty} A_k]$.

Solution. Throughout, $(X, \mathcal{M}, \mu)$ is a finite measure space, and convergence with respect to the Nikodym metric means $\rho_{\mu}(A_n, A_0) = \mu(A_n \Delta A_0) \rightarrow 0$. Since points of the Nikodym metric space are equivalence classes of measurable sets under the relation $A \simeq B$ when $\mu(A \Delta B) = 0$, any assertion of equality of sets here can only be meant modulo μ -null sets. Write

$$L = \bigcup_{n=1}^{\infty} \left[\bigcap_{k=n}^{\infty} A_k \right] = \liminf_{n \rightarrow \infty} A_n,$$

which is measurable, being obtained from the A_n by countably many intersections and unions.

We prove the following three things: (a) $\mu(L \sim A_0) = 0$, so that the inclusion $L \subseteq A_0$ always holds modulo μ -null sets; (b) equality $\mu(A_0 \Delta L') = 0$ holds for $L' = \liminf_k A_{n_k}$ along a suitable subsequence $\{A_{n_k}\}$; and (c) the asserted equality for the full sequence is in general false, even modulo μ -null sets, so the exercise as printed must be understood in the sense of (b).

Choice of a rapidly convergent subsequence. Since $\mu(A_n \Delta A_0) \rightarrow 0$, we may choose natural numbers $n_1 < n_2 < \cdots$ with

$$\mu(A_{n_k} \Delta A_0) \leq \frac{1}{2^k} \quad \text{for every } k.$$

Put $B_k = A_{n_k} \Delta A_0$ and

$$N = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} B_k,$$

the set of points lying in infinitely many B_k . For each j , by countable subadditivity and monotonicity,

$$\mu(N) \leq \mu \left(\bigcup_{k=j}^{\infty} B_k \right) \leq \sum_{k=j}^{\infty} \frac{1}{2^k} = \frac{1}{2^{j-1}},$$

and letting $j \rightarrow \infty$ gives $\mu(N) = 0$. If $x \in X \sim N$, then there is an index $j(x)$ such that $x \notin B_k$ for all $k \geq j(x)$, that is,

$$x \in A_{n_k} \iff x \in A_0 \quad \text{for all } k \geq j(x). \quad (\dagger)$$

(a) Let $x \in L \sim N$. Then there is m with $x \in A_n$ for all $n \geq m$; in particular $x \in A_{n_k}$ for all k with $n_k \geq m$. Choosing $k \geq j(x)$ with $n_k \geq m$ and invoking $(\dagger)$, we get $x \in A_0$. Hence $L \sim A_0 \subseteq N$, and therefore $\mu(L \sim A_0) = 0$.

(b) Let $L' = \bigcup_{j=1}^{\infty} \left[\bigcap_{k=j}^{\infty} A_{n_k} \right]$ and let $x \in X \sim N$. If $x \in A_0$, then by $(\dagger)$ we have $x \in A_{n_k}$ for all $k \geq j(x)$, so $x \in L'$. Conversely, if $x \in L'$, then $x \in A_{n_k}$ for all k beyond some index, so choosing k at least that index and at least $j(x)$ and invoking $(\dagger)$ gives $x \in A_0$. Hence $A_0 \Delta L' \subseteq N$, so $\mu(A_0 \Delta L') = 0$, that is, $[A_0] = [L']$ in the Nikodym metric space:

$$A_0 = \bigcup_{j=1}^{\infty} \left[\bigcap_{k=j}^{\infty} A_{n_k} \right] \quad \text{up to a set of measure zero.}$$

This is exactly the mechanism used in the proof of Theorem 21, where the limit set A_0 was produced as the set of points at which a pointwise convergent subsequence of $\{\chi_{A_n}\}$ converges to 1.

(c) The corresponding statement for the full sequence is false. Take $X = [0, 1]$ with $\mathcal{M}$ the Lebesgue measurable subsets and μ Lebesgue measure, a finite measure space. For a natural number m and $1 \leq j \leq 2^m$ let $I_{m,j} = [(j-1)/2^m, j/2^m]$, and let $\{T_n\}$ be the enumeration of all these intervals ordered by increasing m and, within a fixed m , by increasing j (the so-called typewriter sequence). Since the levels $1, 2, \dots, m-1$ contribute $2 + 2^2 + \cdots + 2^{m-1} = 2^m - 2$ intervals in all, the intervals of level m occupy exactly the indices n with $2^m - 1 \leq n \leq 2^{m+1} - 2$. Hence if T_n has level m , then $2^{m+1} \geq n + 2$, so

$$\mu(T_n) = \frac{1}{2^m} \leq \frac{2}{n+2},$$

and therefore $\mu(T_n) \rightarrow 0$ as $n \rightarrow \infty$.

Define $A_n = X \sim T_n$ and $A_0 = X$. Since $A_n \subseteq A_0$, we have $A_n \Delta A_0 = T_n$, so

$$\rho_\mu(A_n, A_0) = \mu(T_n) \rightarrow 0,$$

that is, $\{A_n\}$ converges to A_0 with respect to the Nikodym metric. On the other hand, fix n and choose m so large that $2^m - 1 \geq n$; by the index count above, all 2^m intervals of level m then occur in the enumeration with index at least n , and these intervals cover X . Hence $\bigcup_{k=n}^{\infty} T_k = X$ and therefore

$$\bigcap_{k=n}^{\infty} A_k = X \sim \bigcup_{k=n}^{\infty} T_k = \emptyset \quad \text{for every } n,$$

so that $L = \emptyset$. But $\mu(A_0 \Delta L) = \mu(X) = 1 \neq 0$, so A_0 and L are not equal, nor even equal modulo a null set, nor equal as points of the Nikodym metric space.

Thus the correct assertion is the one proved in (b): the limit set is recovered as $\liminf_k A_{n_k}$ along a subsequence chosen so that $\sum_k \mu(A_{n_k} \Delta A_0) < \infty$, together with the one-sided containment $L \subseteq A_0$ modulo null sets proved in (a). Convergence in the Nikodym metric is exactly convergence in measure of the characteristic functions χ_{A_n} to χ_{A_0} , and, just as convergence in measure does not imply pointwise a.e. convergence but does along a subsequence, no statement about the full sequence's $\liminf$ can be expected.

Problem 18.63. Show that (49) defines a measure. [Here $(X, \mathcal{M})$ is a measurable space and $\{\nu_n\}$ is a sequence of finite measures on $\mathcal{M}$ that converges setwise on $\mathcal{M}$ to the set function ν , with $\{\nu_n(X)\}$ bounded; (49) is the definition, in the proof of Theorem 26, of the set function μ on $\mathcal{M}$ given for a measurable set E by

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \nu_n(E). \quad (49)$$

Show that μ is a finite measure on $\mathcal{M}$.]

Solution. Each ν_n is a finite measure, so for every $E \in \mathcal{M}$ the series in (49) has nonnegative terms and hence a well-defined sum in $[0, \infty]$; thus $\mu: \mathcal{M} \rightarrow [0, \infty]$ is a well-defined nonnegative set function.

Finiteness. By hypothesis $\{\nu_n(X)\}$ is bounded, say $\nu_n(X) \leq c$ for all n and some $c > 0$. By the monotonicity of each ν_n , for every $E \in \mathcal{M}$,

$$\mu(E) \leq \mu(X) = \sum_{n=1}^{\infty} \frac{\nu_n(X)}{2^n} \leq c \sum_{n=1}^{\infty} \frac{1}{2^n} = c < \infty.$$

So μ takes only finite values.

$\mu(\emptyset) = 0$. Each $\nu_n(\emptyset) = 0$, so every term of the series (49) with $E = \emptyset$ vanishes and hence $\mu(\emptyset) = 0$.

Countable additivity. We use the interchange principle for double series of nonnegative terms: if $a_{n,k} \geq 0$ for all natural numbers n and k , then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k}$$

as an equality in $[0, \infty]$. To see this, write S for the left-hand side and T for the right-hand side. For a fixed natural number K , since finitely many limits may be added and finite sums may be interchanged,

$$\sum_{k=1}^K \sum_{n=1}^{\infty} a_{n,k} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \sum_{k=1}^K a_{n,k} \leq \lim_{N \rightarrow \infty} \sum_{n=1}^N \sum_{k=1}^{\infty} a_{n,k} = S,$$

where the inequality holds because the terms are nonnegative. Taking the supremum over K gives $T \leq S$, and the symmetric argument gives $S \leq T$.

Now let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection of measurable sets with union E . Each ν_n is countably additive, so $\nu_n(E) = \sum_{k=1}^\infty \nu_n(E_k)$. Applying the interchange principle to the nonnegative numbers $a_{n,k} = 2^{-n}\nu_n(E_k)$,

$$\mu(E) = \sum_{n=1}^\infty \frac{1}{2^n} \sum_{k=1}^\infty \nu_n(E_k) = \sum_{n=1}^\infty \sum_{k=1}^\infty \frac{\nu_n(E_k)}{2^n} = \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{\nu_n(E_k)}{2^n} = \sum_{k=1}^\infty \mu(E_k).$$

Hence μ is countably additive.

Therefore μ is a finite measure on $\mathcal{M}$, as (49) asserts. For completeness we also verify the assertion made immediately after (49) in the proof of Theorem 26, namely that each ν_n is absolutely continuous with respect to μ : if $E \in \mathcal{M}$ and $\mu(E) = 0$, then the series $\sum_{m=1}^\infty 2^{-m}\nu_m(E)$ of nonnegative terms has sum zero, so every term vanishes; in particular $2^{-n}\nu_n(E) = 0$ and thus $\nu_n(E) = 0$. This is what allows the Vitali-Hahn-Saks Theorem to be applied to the finite measure space $(X, \mathcal{M}, \mu)$ and the sequence $\{\nu_n\}$.

18.10 Problems 18.64–18.70

Problem 18.64. Prove Proposition 24.

[Proposition 24: Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ each of which is absolutely continuous with respect to μ . Then the following are equivalent:

- (i) The sequence of measures $\{\nu_n\}$ is uniformly absolutely continuous with respect to the measure μ .
- (ii) The sequence of functions $\{\nu_n: \mathcal{M} \rightarrow \mathbb{R}\}$ is equicontinuous with respect to the Nikodym metric

ρ_μ .

- (iii) The sequence of Radon–Nikodym derivatives $\{d\mu/d\nu_n\}$ is uniformly integrable over X with respect to the measure μ .]

Solution. Throughout, $\rho_\mu([A], [B]) = \mu(A \Delta B)$ denotes the Nikodym metric on $\mathcal{M}/\sim$, and, as in the text, we suppress the equivalence-class brackets. Since each ν_n is a finite measure that is absolutely continuous with respect to μ , Proposition 23 tells us that each ν_n induces a properly defined real-valued function on $\mathcal{M}/\sim$, so that assertion (ii) makes sense.

We first record an identity that is used twice. If ν is a finite measure on $\mathcal{M}$ and $A, B \in \mathcal{M}$, then, splitting A and B along $A \cap B$ and using finite additivity together with the finiteness of ν ,

$$\nu(A) - \nu(B) = [\nu(A \sim B) + \nu(A \cap B)] - [\nu(B \sim A) + \nu(A \cap B)] = \nu(A \sim B) - \nu(B \sim A). \quad (*)$$

This is the identity displayed at the end of the proof of Lemma 22.

Proof that (i) implies (ii). Assume $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ and let $\epsilon > 0$. Choose $\delta > 0$ such that for every $E \in \mathcal{M}$ and every natural number n ,

$$\text{if } \mu(E) < \delta, \text{ then } \nu_n(E) < \epsilon/2.$$

Let $A, B \in \mathcal{M}$ satisfy $\rho_\mu(A, B) = \mu(A \Delta B) < \delta$. Since $A \sim B$ and $B \sim A$ are both contained in $A \Delta B$, the monotonicity of μ gives $\mu(A \sim B) < \delta$ and $\mu(B \sim A) < \delta$, and therefore

$$0 \leq \nu_n(A \sim B) < \epsilon/2, \quad 0 \leq \nu_n(B \sim A) < \epsilon/2 \quad \text{for all } n.$$

By (*), $\nu_n(A) - \nu_n(B)$ is the difference of two numbers lying in $[0, \epsilon/2)$, so

$$|\nu_n(A) - \nu_n(B)| < \epsilon/2 < \epsilon \quad \text{for all } n.$$

Since δ depends only on ϵ and not on A, B or n , the sequence $\{\nu_n: \mathcal{M} \rightarrow \mathbb{R}\}$ is equicontinuous at every point of $\mathcal{M}/\sim$ (indeed uniformly equicontinuous). Thus (ii) holds.

Proof that (ii) implies (i). Assume $\{\nu_n\}$ is equicontinuous with respect to ρ_μ . In particular it is equicontinuous at the point $\emptyset$ of $\mathcal{M}/\sim$. Let $\epsilon > 0$ and choose $\delta > 0$ such that for every $E \in \mathcal{M}$ and every n , if $\rho_\mu(E, \emptyset) < \delta$ then $|\nu_n(E) - \nu_n(\emptyset)| < \epsilon$. But $\rho_\mu(E, \emptyset) = \mu(E \Delta \emptyset) = \mu(E)$ and $\nu_n(\emptyset) = 0$. Hence for every measurable E and every n ,

$$\text{if } \mu(E) < \delta, \text{ then } \nu_n(E) < \epsilon,$$

which is precisely the uniform absolute continuity of $\{\nu_n\}$ with respect to μ . Thus (i) holds.

(We remark that, by Lemma 22, equicontinuity at a single set E_0 would already have sufficed here; equicontinuity at E_0 with $\epsilon/4$ in place of ϵ yields, by (44), that $\rho_\mu(A, B) < \delta$ implies $|\nu_n(A) - \nu_n(B)| < \epsilon$ for all n , and taking $B = \emptyset$ gives (i).)

Proof that (i) is equivalent to (iii). A word first about the notation in (iii). As printed, (iii) names the derivatives $d\mu/d\nu_n$; this is a misprint, and these need not even exist, since nothing in the hypotheses makes μ absolutely continuous with respect to ν_n . The paragraph immediately preceding the proposition specifies the functions actually intended: for each n the Radon–Nikodym Theorem supplies a nonnegative integrable function f_n for which $\nu_n(E) = \int_E f_n d\mu$ for all $E \in \mathcal{M}$; that is, $f_n = d\nu_n/d\mu$. It is the sequence $\{f_n\}$ that (iii) concerns, and we prove the proposition with that reading.

Since μ is a finite measure and each ν_n is a finite measure on $\mathcal{M}$ that is absolutely continuous with respect to μ , the Radon–Nikodym Theorem provides, for each n , a nonnegative function $f_n = d\nu_n/d\mu$ that is measurable with respect to $\mathcal{M}$ and satisfies

$$\nu_n(E) = \int_E f_n d\mu \quad \text{for all } E \in \mathcal{M}.$$

Taking $E = X$ shows $\int_X f_n d\mu = \nu_n(X) < \infty$, so each f_n is integrable over X with respect to μ , and since $f_n \geq 0$ we have $\int_E |f_n| d\mu = \int_E f_n d\mu = \nu_n(E)$ for every measurable E .

By definition, the sequence $\{f_n\}$ is uniformly integrable over X provided that for each $\epsilon > 0$ there is a $\delta > 0$ such that for every natural number n and every measurable subset E of X , if $\mu(E) < \delta$ then $\int_E |f_n| d\mu < \epsilon$. In view of the displayed identity $\int_E |f_n| d\mu = \nu_n(E)$, this assertion reads: for each $\epsilon > 0$ there is a $\delta > 0$ such that for every measurable E and every n , if $\mu(E) < \delta$ then $\nu_n(E) < \epsilon$. This is verbatim the definition of the uniform absolute continuity of $\{\nu_n\}$ with respect to μ . Hence (i) and (iii) are equivalent.

Having shown (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii), the three assertions are equivalent, and Proposition 24 is proved.

Problem 18.65. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\nu: \mathcal{M} \rightarrow [0, \infty)$ a finitely additive set function with the property that for each $\epsilon > 0$, there is a $\delta > 0$ such that for a measurable set E , if $\mu(E) < \delta$, then $\nu(E) < \epsilon$. Show that ν is a measure on $\mathcal{M}$.

Solution. We must show that $\nu(\emptyset) = 0$ and that ν is countably additive on $\mathcal{M}$; nonnegativity is part of the hypothesis, since ν takes values in $[0, \infty)$.

Step 1: $\nu(\emptyset) = 0$. The sets X and $\emptyset$ are disjoint and their union is X , so finite additivity gives $\nu(X) = \nu(X) + \nu(\emptyset)$. Since ν takes values in $[0, \infty)$, the number $\nu(X)$ is real and may be cancelled from both sides, so $\nu(\emptyset) = 0$.

Step 2: ν is monotone. Let $A \subseteq B$ be measurable. Then $B = A \cup (B \sim A)$, so by finite additivity and nonnegativity,

$$\nu(B) = \nu(A) + \nu(B \sim A) \geq \nu(A).$$

Step 3: countable additivity. Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection of measurable sets and put $E = \bigcup_{k=1}^\infty E_k$. For each natural number n , the sets $E_1, \dots, E_n$ and $\bigcup_{k=n+1}^\infty E_k$ form a finite disjoint collection whose union is E , so by finite additivity

$$\nu(E) = \sum_{k=1}^n \nu(E_k) + \nu\left(\bigcup_{k=n+1}^\infty E_k\right). \quad (*)$$

Let $\epsilon > 0$ and choose $\delta > 0$ as in the hypothesis, so that for every measurable set A , if $\mu(A) < \delta$ then $\nu(A) < \epsilon$. Since μ is a measure and the collection $\{E_k\}$ is disjoint,

$$\sum_{k=1}^\infty \mu(E_k) = \mu(E) \leq \mu(X) < \infty,$$

because the measure space is finite. Hence the series $\sum_k \mu(E_k)$ converges, so its tails tend to 0: there is a natural number N such that

$$\mu\left(\bigcup_{k=N+1}^{\infty} E_k\right) = \sum_{k=N+1}^{\infty} \mu(E_k) < \delta.$$

For every $n \geq N$ we have $\bigcup_{k=n+1}^{\infty} E_k \subseteq \bigcup_{k=N+1}^{\infty} E_k$, so by the monotonicity of μ this tail also has μ -measure less than δ , and therefore, by the choice of δ ,

$$0 \leq \nu\left(\bigcup_{k=n+1}^{\infty} E_k\right) < \epsilon \quad \text{for all } n \geq N.$$

Substituting this into (*) we obtain

$$\left|\nu(E) - \sum_{k=1}^n \nu(E_k)\right| < \epsilon \quad \text{for all } n \geq N.$$

Since $\epsilon > 0$ was arbitrary, the partial sums $\sum_{k=1}^n \nu(E_k)$ converge to $\nu(E)$, that is,

$$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k).$$

Thus ν is a nonnegative, countably additive set function on $\mathcal{M}$ that vanishes on $\emptyset$; that is, ν is a measure on $\mathcal{M}$. (Observe that the choice of δ also shows at once that ν is absolutely continuous with respect to μ : if A is measurable and $\mu(A) = 0$, then $\mu(A) < \delta$ for every $\delta > 0$, so $\nu(A) < \epsilon$ for every $\epsilon > 0$ and hence $\nu(A) = 0$.)

Problem 18.66. Let $(X, \mathcal{M})$ be a measurable space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ that converges setwise on $\mathcal{M}$ to ν . Let $\{E_k\}$ be a descending sequence of measurable sets with empty intersection. Show that for each $\epsilon > 0$, there is a natural number K for which $\nu_n(E_K) < \epsilon$ for all n .

Solution. The point of the exercise is that the smallness of $\nu_n(E_K)$ is uniform in n : for each fixed n the continuity of the finite measure ν_n already gives $\lim_{k \rightarrow \infty} \nu_n(E_k) = \nu_n(\bigcap_k E_k) = 0$, but a priori the index k at which $\nu_n(E_k)$ drops below ϵ could depend on n . We obtain the uniformity from the Vitali–Hahn–Saks Theorem.

Throughout we read the hypothesis of setwise convergence, as it is used in Theorem 26 of this section, as asserting that for each $E \in \mathcal{M}$ the limit $\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E)$ exists and is a real number; in particular $\nu(X) < \infty$. Some such finiteness requirement is indispensable, as the counterexample recorded at the end of the solution shows.

Step 1: $\{\nu_n(X)\}$ is bounded. By hypothesis the sequence of real numbers $\{\nu_n(X)\}$ converges to the real number $\nu(X)$, and a convergent sequence of real numbers is bounded. Fix a real number $C \geq 0$ with $\nu_n(X) \leq C$ for every n .

Step 2: a reference measure. Exactly as in (49), define, for $E \in \mathcal{M}$,

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \nu_n(E).$$

By Problem 63 this defines a measure on $\mathcal{M}$. Moreover, by Step 1,

$$\mu(X) = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(X) \leq C \sum_{n=1}^{\infty} \frac{1}{2^n} = C < \infty,$$

so $(X, \mathcal{M}, \mu)$ is a finite measure space.

Step 3: each ν_n is absolutely continuous with respect to μ . If $E \in \mathcal{M}$ has $\mu(E) = 0$, then, all terms of the defining series being nonnegative, every term vanishes; since $1/2^n > 0$, this forces $\nu_n(E) = 0$ for every n . Hence $\nu_n \ll \mu$ for every n .

Step 4: uniform absolute continuity. We have a finite measure space $(X, \mathcal{M}, \mu)$ and a sequence $\{\nu_n\}$ of finite measures on $\mathcal{M}$, each absolutely continuous with respect to μ , with $\{\nu_n(X)\}$ bounded and $\{\nu_n\}$ converging setwise on $\mathcal{M}$ to ν . The Vitali–Hahn–Saks Theorem therefore tells us that $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ . Thus, given $\epsilon > 0$, there is a $\delta > 0$ such that for every measurable set E and every natural number n ,

$$\text{if } \mu(E) < \delta, \text{ then } \nu_n(E) < \epsilon. \quad (*)$$

Step 5: choosing K . The sequence $\{E_k\}$ is a descending sequence of measurable sets with $\bigcap_{k=1}^{\infty} E_k = \emptyset$, and $\mu(E_1) \leq \mu(X) < \infty$. By the continuity of the measure μ ,

$$\lim_{k \rightarrow \infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \mu(\emptyset) = 0.$$

Hence there is a natural number K for which $\mu(E_K) < \delta$. By $(*)$,

$$\nu_n(E_K) < \epsilon \quad \text{for every natural number } n.$$

Finally, since $\{E_k\}$ is descending, $E_k \subseteq E_K$ for all $k \geq K$, so the monotonicity of each ν_n gives the stronger conclusion

$$\nu_n(E_k) < \epsilon \quad \text{for all } k \geq K \text{ and all } n.$$

This is exactly the asserted uniformity.

Remark on the finiteness of the limit. If one permits the setwise limit to take the value ∞ , the conclusion fails. Let $X = \mathbb{N}$, let $\mathcal{M} = 2^X$, and for each natural number n let

$$\nu_n(E) = n \cdot \#(E \cap \{1, 2, \dots, n\}),$$

which is a finite measure on $\mathcal{M}$ (a nonnegative multiple of counting measure restricted to a finite set). If $E \neq \emptyset$, pick $j \in E$; then for $n \geq j$ we have $\nu_n(E) \geq n$, so $\nu_n(E) \rightarrow \infty$, while $\nu_n(\emptyset) = 0$ for every n . Thus $\{\nu_n\}$ converges setwise to the set function ν with $\nu(\emptyset) = 0$ and $\nu(E) = \infty$ for $E \neq \emptyset$. Now $E_k = \{k, k+1, \dots\}$ is a descending sequence of measurable sets with $\bigcap_k E_k = \emptyset$, yet for every K and every $n \geq K$ we have $\nu_n(E_K) = n \cdot \#(\{K, \dots, n\}) \geq n$, so $\sup_n \nu_n(E_K) = \infty$. Hence no K works for, say, $\epsilon = 1$. This is the same phenomenon that makes the boundedness hypothesis on $\{\nu_n(X)\}$ necessary in Theorem 26.

Problem 18.67. Give an example of a decreasing sequence $\{\mu_n\}$ of measures on a measurable space such that the set function μ defined by $\mu(E) = \lim \mu_n(E)$ is not a measure.

Solution. Let $X = \mathbb{N} = \{1, 2, 3, \dots\}$ and let $\mathcal{M} = 2^X$ be the σ -algebra of all subsets of X . For each natural number n , define

$$\mu_n(E) = \#(E \cap \{n, n+1, n+2, \dots\}) \quad \text{for } E \subseteq X,$$

where $\#(S)$ denotes the number of elements of S if S is finite and ∞ if S is infinite. In other words, μ_n is counting measure restricted to the tail $T_n = \{n, n+1, \dots\}$.

Each μ_n is a measure. Indeed, $\mu_n(\emptyset) = \#(\emptyset) = 0$, μ_n takes values in $[0, \infty]$, and if $\{E_k\}$ is a countable disjoint collection of subsets of X , then $\{E_k \cap T_n\}$ is a countable disjoint collection whose union is $(\bigcup_k E_k) \cap T_n$; since counting measure on a countable set is countably additive (the cardinality of a countable disjoint union is the sum of the cardinalities, in $[0, \infty]$),

$$\mu_n\left(\bigcup_k E_k\right) = \#\left(\bigcup_k (E_k \cap T_n)\right) = \sum_k \#(E_k \cap T_n) = \sum_k \mu_n(E_k).$$

The sequence $\{\mu_n\}$ is decreasing. Since $T_{n+1} \subseteq T_n$, we have $E \cap T_{n+1} \subseteq E \cap T_n$, and therefore $\mu_{n+1}(E) = \#(E \cap T_{n+1}) \leq \#(E \cap T_n) = \mu_n(E)$ for every $E \subseteq X$. Being a decreasing sequence in $[0, \infty]$, $\{\mu_n(E)\}$ converges in $[0, \infty]$ for each E , so

$$\mu(E) = \lim_{n \rightarrow \infty} \mu_n(E)$$

is a well-defined set function on $\mathcal{M}$ with values in $[0, \infty]$.

Identification of μ . If $E \subseteq X$ is finite, then $E \cap T_n = \emptyset$ for every n exceeding $\max E$, so $\mu_n(E) = 0$ for all large n and hence $\mu(E) = 0$. If $E \subseteq X$ is infinite, then $E \cap T_n$ is infinite for every n (removing the finitely many elements of E that are smaller than n leaves an infinite set), so $\mu_n(E) = \infty$ for every n and hence $\mu(E) = \infty$. Thus

$$\mu(E) = \begin{cases} 0, & E \text{ finite,} \\ \infty, & E \text{ infinite.} \end{cases}$$

μ is not a measure. Write $X = \bigcup_{k=1}^{\infty} \{k\}$, a countable disjoint union of measurable sets. Each singleton $\{k\}$ is finite, so $\mu(\{k\}) = 0$ and

$$\sum_{k=1}^{\infty} \mu(\{k\}) = 0,$$

whereas X is infinite, so $\mu(X) = \infty$. Hence

$$\mu\left(\bigcup_{k=1}^{\infty} \{k\}\right) = \infty \neq 0 = \sum_{k=1}^{\infty} \mu(\{k\}),$$

so μ fails to be countably additive and therefore is not a measure on $\mathcal{M}$. (It is worth noting that μ is nevertheless finitely additive: a finite disjoint union of finite sets is finite, while a finite disjoint union that contains an infinite set is infinite.)

Observe that $\mu(X) = \infty$ here, and that this is unavoidable. Indeed, suppose $\{\mu_n\}$ is a decreasing sequence of measures on a measurable space $(X, \mathcal{M})$ with $\lim_n \mu_n(X) < \infty$. Since $\{\mu_n(X)\}$ is a decreasing sequence in $[0, \infty]$ with finite limit, there is an index N with $\mu_N(X) < \infty$. Then $(X, \mathcal{M}, \mu_N)$ is a finite measure space, and for $n \geq N$ each μ_n is a finite measure with $\mu_n \leq \mu_N$, hence $\mu_n \ll \mu_N$, and $\{\mu_n(X)\}_{n \geq N}$ is bounded by $\mu_N(X)$. As $\{\mu_n\}_{n \geq N}$ converges setwise to μ , the Vitali–Hahn–Saks Theorem shows that μ is a finite measure. So any counterexample must have $\lim_n \mu_n(X) = \infty$, as the one above does.

Problem 18.68. Let $(X, \mathcal{M})$ be a measurable space and $\{\mu_n\}$ a sequence of measures on $\mathcal{M}$ such that for each $E \in \mathcal{M}$, $\mu_{n+1}(E) \geq \mu_n(E)$. For each $E \in \mathcal{M}$, define $\mu(E) = \lim \mu_n(E)$. Show that μ is a measure on $\mathcal{M}$ if $\mu(X) < \infty$.

Solution. For each fixed $E \in \mathcal{M}$ the sequence $\{\mu_n(E)\}$ is a nondecreasing sequence in $[0, \infty]$, so it converges in $[0, \infty]$ and

$$\mu(E) = \lim_{n \rightarrow \infty} \mu_n(E) = \sup_n \mu_n(E).$$

In particular μ is a well-defined set function on $\mathcal{M}$ with values in $[0, \infty]$, and

$$\mu_n(E) \leq \mu(E) \quad \text{for every } n \text{ and every } E \in \mathcal{M}. \quad (*)$$

Under the hypothesis $\mu(X) < \infty$, monotonicity of each μ_n together with $(*)$ gives $\mu(E) \leq \mu(X) < \infty$ for every E , so μ is in fact real-valued.

μ vanishes on the empty set. Since $\mu_n(\emptyset) = 0$ for every n , $\mu(\emptyset) = \lim_n 0 = 0$.

μ is countably additive. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets and set $E = \bigcup_{k=1}^{\infty} E_k$. We prove the two inequalities.

First, fix a natural number n . Since μ_n is a measure, countable additivity and $(*)$ give

$$\mu_n(E) = \sum_{k=1}^{\infty} \mu_n(E_k) \leq \sum_{k=1}^{\infty} \mu(E_k),$$

the inequality being a termwise comparison of series of nonnegative terms. The right-hand side does not depend on n ; letting $n \rightarrow \infty$ on the left,

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Second, fix a natural number N . For every n , the monotonicity and finite additivity of the measure μ_n give

$$\mu_n(E) \geq \mu_n\left(\bigcup_{k=1}^N E_k\right) = \sum_{k=1}^N \mu_n(E_k).$$

Letting $n \rightarrow \infty$ and using that this is a sum of finitely many terms, each of which converges to $\mu(E_k)$, we obtain

$$\mu(E) \geq \sum_{k=1}^N \mu(E_k).$$

Since this holds for every N , letting $N \rightarrow \infty$ yields

$$\mu(E) \geq \sum_{k=1}^{\infty} \mu(E_k).$$

Combining the two inequalities,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

Therefore μ is a nonnegative countably additive set function on $\mathcal{M}$ with $\mu(\emptyset) = 0$, that is, μ is a measure on $\mathcal{M}$; by the observation above it is a finite measure.

Two remarks. First, the argument above never used the hypothesis $\mu(X) < \infty$: the setwise limit of an increasing sequence of measures is always a measure, with values in $[0, \infty]$. The hypothesis $\mu(X) < \infty$ serves only to guarantee that the limit measure is finite. Contrast this with Problem 67, where a decreasing sequence of measures has a setwise limit that is not a measure; there the failure occurs precisely because one cannot pass to the limit inside an infinite sum from above.

Second, under the stated hypothesis $\mu(X) < \infty$ the conclusion is also an immediate consequence of Theorem 26 (Nikodym): by $(*)$, $\mu_n(X) \leq \mu(X) < \infty$ for every n , so each μ_n is a finite measure and the sequence $\{\mu_n(X)\}$ is bounded; since $\{\mu_n\}$ converges setwise on $\mathcal{M}$ to μ , Theorem 26 tells us that μ is a measure on $\mathcal{M}$.

Problem 18.69. Formulate and prove a version of the Vitali–Hahn–Saks Theorem for signed measures.

Solution. Formulation. Recall that a signed measure ν on a measurable space $(X, \mathcal{M})$ is said to be absolutely continuous with respect to a measure μ provided $|\nu|$ is absolutely continuous with respect to μ , equivalently provided both measures ν_1, ν_2 in the Jordan decomposition $\nu = \nu_1 - \nu_2$ are absolutely continuous with respect to μ . Call a sequence $\{\nu_n\}$ of finite signed measures on $\mathcal{M}$ uniformly absolutely continuous with respect to μ provided for each $\epsilon > 0$ there is a $\delta > 0$ such that for every $E \in \mathcal{M}$ and every natural number n , if $\mu(E) < \delta$ then $|\nu_n(E)| < \epsilon$.

The Vitali–Hahn–Saks Theorem for signed measures. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite signed measures on $\mathcal{M}$, each of which is absolutely continuous with respect to μ . Suppose $\{\nu_n\}$ converges setwise on $\mathcal{M}$ to a real-valued set function ν , that is, for each $E \in \mathcal{M}$

the limit $\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E)$ exists and is finite. Then $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ . Moreover, ν is a finite signed measure on $\mathcal{M}$ that is absolutely continuous with respect to μ .

(No boundedness hypothesis on $\{\nu_n(X)\}$ is needed. In the theorem as stated in the text for measures, the boundedness of $\{\nu_n(X)\}$ was used only to guarantee that the setwise limit is real-valued; here that is part of the hypothesis of setwise convergence.)

Proof. Throughout we work in the Nikodym metric space $(\mathcal{M}/\sim, \rho_\mu)$ associated with $(X, \mathcal{M}, \mu)$, writing A for $[A]$.

Step 1: each ν_n induces a properly defined, uniformly continuous real-valued function on the Nikodym metric space. Write the Jordan decomposition $\nu_n = \nu_n^1 - \nu_n^2$. Since ν_n is a finite signed measure, $|\nu_n| = \nu_n^1 + \nu_n^2$ is a finite measure, so ν_n^1 and ν_n^2 are finite measures; and since $\nu_n \ll \mu$, both $\nu_n^1 \ll \mu$ and $\nu_n^2 \ll \mu$. By Proposition 23, each of ν_n^1 and ν_n^2 is properly defined on $\mathcal{M}/\sim$ and uniformly continuous with respect to ρ_μ . Hence so is their difference ν_n .

Step 2: Lemma 22 for finite signed measures. We claim: let ν be a finite signed measure on $\mathcal{M}$, let $E_0 \in \mathcal{M}$, and let $\epsilon > 0$ and $\delta > 0$ be such that for every $E \in \mathcal{M}$,

$$\text{if } \rho_\mu(E, E_0) < \delta, \text{ then } |\nu(E) - \nu(E_0)| < \epsilon/4. \quad (43')$$

Then for all $A, B \in \mathcal{M}$,

$$\text{if } \rho_\mu(A, B) < \delta, \text{ then } |\nu(A) - \nu(B)| < \epsilon. \quad (44')$$

The proof is the proof of Lemma 22 verbatim; it uses only the finite additivity and the real-valuedness of ν , never its nonnegativity. In detail, we first show

$$\text{if } \mu(A) < \delta, \text{ then } |\nu(A)| < \epsilon/2. \quad (45')$$

Observe that if $D \subseteq C$ then $C \Delta D = C \sim D$. Let $A \in \mathcal{M}$ with $\rho_\mu(A, \emptyset) = \mu(A) < \delta$. Since $E_0 \sim A \subseteq E_0$,

$$[E_0 \sim A] \Delta E_0 = E_0 \sim [E_0 \sim A] = E_0 \cap A \subseteq A,$$

so $\rho_\mu(E_0 \sim A, E_0) = \mu(E_0 \cap A) \leq \mu(A) < \delta$, and (43') gives $|\nu(E_0) - \nu(E_0 \sim A)| < \epsilon/4$. By the additivity of ν over the disjoint decomposition $E_0 = (E_0 \sim A) \cup (E_0 \cap A)$ and the finiteness of ν ,

$$\nu(A \cap E_0) = \nu(E_0) - \nu(E_0 \sim A), \quad \text{hence } |\nu(A \cap E_0)| < \epsilon/4.$$

Similarly, since $E_0 \subseteq E_0 \cup A$,

$$[E_0 \cup A] \Delta E_0 = [E_0 \cup A] \sim E_0 = A \sim E_0 \subseteq A,$$

so $\rho_\mu(E_0 \cup A, E_0) \leq \mu(A) < \delta$, and (43') gives $|\nu(E_0 \cup A) - \nu(E_0)| < \epsilon/4$. Since $\nu(E_0 \cup A) = \nu(E_0) + \nu(A \sim E_0)$, we get $|\nu(A \sim E_0)| < \epsilon/4$. Therefore

$$|\nu(A)| = |\nu(A \cap E_0) + \nu(A \sim E_0)| < \epsilon/2,$$

which is (45'). Now let $A, B \in \mathcal{M}$ with $\rho_\mu(A, B) = \mu(A \Delta B) < \delta$. Since $A \sim B$ and $B \sim A$ are contained in $A \Delta B$, both have μ -measure less than δ , so (45') applies to each. As ν is real-valued and finitely additive,

$$\nu(A) - \nu(B) = \nu(A \sim B) - \nu(B \sim A),$$

whence $|\nu(A) - \nu(B)| \leq |\nu(A \sim B)| + |\nu(B \sim A)| < \epsilon/2 + \epsilon/2 = \epsilon$. This proves (44').

Step 3: uniform absolute continuity. By Theorem 21 the Nikodym metric space $(\mathcal{M}/\sim, \rho_\mu)$ is a complete metric space. By Step 1, $\{\nu_n\}$ is a sequence of real-valued continuous functions on this space, and by hypothesis it converges pointwise on it to the real-valued function ν . By Theorem 7 of Chapter 10, a consequence of the Baire Category Theorem, there is a set $E_0 \in \mathcal{M}$ at which the sequence $\{\nu_n: \mathcal{M} \rightarrow \mathbb{R}\}$ is equicontinuous. Thus, given $\epsilon > 0$, there is a $\delta > 0$ such that for every $E \in \mathcal{M}$ and every n ,

$$\text{if } \rho_\mu(E, E_0) < \delta, \text{ then } |\nu_n(E) - \nu_n(E_0)| < \epsilon/4.$$

Each ν_n is a finite signed measure, so Step 2 applies to each of them with this common E_0 , ϵ and δ ; taking $B = \emptyset$ in (44') and recalling $\rho_\mu(A, \emptyset) = \mu(A)$ and $\nu_n(\emptyset) = 0$, we conclude that for every $E \in \mathcal{M}$ and every n ,

$$\text{if } \mu(E) < \delta, \text{ then } |\nu_n(E)| < \epsilon. \quad (\dagger)$$

That is, $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ .

Step 4: ν is a finite signed measure absolutely continuous with respect to μ . By hypothesis ν is real-valued, so it assumes neither $+\infty$ nor $-\infty$, and $\nu(\emptyset) = \lim_n \nu_n(\emptyset) = 0$. If A and B are disjoint measurable sets then $\nu_n(A \cup B) = \nu_n(A) + \nu_n(B)$ for every n , and letting $n \rightarrow \infty$ gives $\nu(A \cup B) = \nu(A) + \nu(B)$; so ν is finitely additive.

Now let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection in $\mathcal{M}$ and put $E = \bigcup_{k=1}^\infty E_k$. Let $\epsilon > 0$ and choose $\delta > 0$ as in $(\dagger)$. Since μ is a finite measure, $\sum_{k=1}^\infty \mu(E_k) = \mu(E) \leq \mu(X) < \infty$, so there is a natural number N with

$$\mu\left(\bigcup_{k=N+1}^\infty E_k\right) = \sum_{k=N+1}^\infty \mu(E_k) < \delta.$$

Let $F \subseteq \{N+1, N+2, \dots\}$ be any finite set of indices and let $A_F = \bigcup_{k \in F} E_k$. Then $A_F \subseteq \bigcup_{k > N} E_k$, so $\mu(A_F) < \delta$, and by $(\dagger)$ and the finite additivity of ν_n ,

$$\left| \sum_{k \in F} \nu_n(E_k) \right| = |\nu_n(A_F)| < \epsilon \quad \text{for all } n.$$

Letting $n \rightarrow \infty$ gives $|\sum_{k \in F} \nu(E_k)| \leq \epsilon$ for every such finite F . Applying this to $F = \{k : N < k \leq M, \nu(E_k) \geq 0\}$ and to $F = \{k : N < k \leq M, \nu(E_k) < 0\}$ and adding, we get $\sum_{k=N+1}^M |\nu(E_k)| \leq 2\epsilon$ for every $M > N$, and hence

$$\sum_{k=N+1}^\infty |\nu(E_k)| \leq 2\epsilon < \infty.$$

In particular the series $\sum_{k=1}^\infty \nu(E_k)$ converges absolutely. Moreover, for each $n \geq N$, the set $R_n = \bigcup_{k=n+1}^\infty E_k$ satisfies $\mu(R_n) \leq \mu(\bigcup_{k > N} E_k) < \delta$, so by $(\dagger)$, $|\nu_m(R_n)| < \epsilon$ for every m , and letting $m \rightarrow \infty$, $|\nu(R_n)| \leq \epsilon$. By the finite additivity of ν applied to the decomposition $E = E_1 \cup \dots \cup E_n \cup R_n$,

$$\left| \nu(E) - \sum_{k=1}^n \nu(E_k) \right| = |\nu(R_n)| \leq \epsilon \quad \text{for all } n \geq N.$$

Since $\epsilon > 0$ was arbitrary, $\nu(E) = \sum_{k=1}^\infty \nu(E_k)$. Therefore ν is a finite signed measure on $\mathcal{M}$.

Finally, if $E \in \mathcal{M}$ has $\mu(E) = 0$, then for every $\epsilon > 0$ the corresponding δ of $(\dagger)$ satisfies $\mu(E) < \delta$, so $|\nu_n(E)| < \epsilon$ for all n and hence $|\nu(E)| \leq \epsilon$. Thus $\nu(E) = 0$. The same argument applied to any measurable subset A of E (which also has $\mu(A) = 0$) gives $\nu(A) = 0$ for every measurable $A \subseteq E$; taking $A = E \cap P$ and $A = E \cap N$ for a Hahn decomposition $X = P \cup N$ for ν , we get $|\nu|(E) = \nu(E \cap P) - \nu(E \cap N) = 0$. Hence ν is absolutely continuous with respect to μ , and the proof is complete.

Problem 18.70. Show that the Nikodym metric space associated with the finite measure space $(X, \mathcal{M}, \mu)$ is separable if and only if $L^p(X, \mu)$ is separable for all $1 \leq p < \infty$.

Solution. Here $L^p(X, \mu)$, $1 \leq p < \infty$, is the space of (equivalence classes of) measurable functions f on X with $\int_X |f|^p d\mu < \infty$, normed by $\|f\|_p = (\int_X |f|^p d\mu)^{1/p}$; we use Minkowski's inequality and the fact that $\|\cdot\|_p$ is a norm on it. We write $\mathcal{N} = (\mathcal{M}/\sim, \rho_\mu)$ for the Nikodym metric space, $\rho_\mu(A, B) = \mu(A \Delta B)$.

The bridge between the two spaces is the following computation. Since $\mu(X) < \infty$, $\chi_A \in L^p(X, \mu)$ for every $A \in \mathcal{M}$, and since $|\chi_A - \chi_B| = \chi_{A \Delta B}$ takes only the values 0 and 1, we have $|\chi_A - \chi_B|^p = \chi_{A \Delta B}$,

so

$$\|\chi_A - \chi_B\|_p = \left(\int_X \chi_{A \Delta B} d\mu \right)^{1/p} = \mu(A \Delta B)^{1/p} = \rho_\mu(A, B)^{1/p}. \quad (*)$$

For $p = 1$ this is the isometry (41)–(42) of the text. Note also that $(*)$ shows that $A \sim B$ if and only if $\chi_A = \chi_B$ in L^p , so $A \mapsto \chi_A$ is a well-defined injection of $\mathcal{M}/\sim$ into $L^p(X, \mu)$.

Part 1: if $L^p(X, \mu)$ is separable for all $1 \leq p < \infty$, then $\mathcal{N}$ is separable.

It suffices to use separability for a single exponent, say $p = 1$; the argument below works for any fixed $p \in [1, \infty)$. Let $\{f_j\}_{j=1}^\infty$ be a countable dense subset of $L^p(X, \mu)$. For each pair (j, k) of natural numbers, if there exists a set $A \in \mathcal{M}$ with $\|\chi_A - f_j\|_p < 1/k$, select one such set and call it $A_{j,k}$; otherwise the pair (j, k) is discarded. Let $\mathcal{D}$ be the (countable) collection of the classes $[A_{j,k}]$ so obtained.

We claim $\mathcal{D}$ is dense in $\mathcal{N}$. Let $A \in \mathcal{M}$ and let k be a natural number. Since $\chi_A \in L^p(X, \mu)$ and $\{f_j\}$ is dense, there is a j with $\|\chi_A - f_j\|_p < 1/(2k) < 1/k$. In particular the pair (j, k) was not discarded, so $A_{j,k}$ exists and $\|\chi_{A_{j,k}} - f_j\|_p < 1/k$. By the triangle inequality,

$$\|\chi_A - \chi_{A_{j,k}}\|_p \leq \frac{1}{2k} + \frac{1}{k} = \frac{3}{2k},$$

so by $(*)$,

$$\rho_\mu(A, A_{j,k}) = \|\chi_A - \chi_{A_{j,k}}\|_p^p \leq \left(\frac{3}{2k}\right)^p.$$

As $k \rightarrow \infty$ the right-hand side tends to 0, so every point of $\mathcal{N}$ is a limit of points of $\mathcal{D}$. Hence $\mathcal{D}$ is a countable dense subset of $\mathcal{N}$, and $\mathcal{N}$ is separable.

Part 2: if $\mathcal{N}$ is separable, then $L^p(X, \mu)$ is separable for every $1 \leq p < \infty$.

Let $\mathcal{D} = \{A_1, A_2, A_3, \dots\}$ be a countable collection of measurable sets whose classes are dense in $\mathcal{N}$, and fix p with $1 \leq p < \infty$. Let $\mathcal{S}$ be the collection of all functions of the form

$$\psi = \sum_{i=1}^m q_i \chi_{A_{k_i}}, \quad m \in \mathbb{N}, \quad q_i \in \mathbb{Q}, \quad k_i \in \mathbb{N}.$$

For each fixed m there are countably many such functions, so $\mathcal{S}$, a countable union of countable sets, is countable. Each member of $\mathcal{S}$ lies in $L^p(X, \mu)$ because $\mu(X) < \infty$. We show $\mathcal{S}$ is dense in $L^p(X, \mu)$. Let $f \in L^p(X, \mu)$ and let $\epsilon > 0$. Since $\int_X |f|^p d\mu < \infty$, f is finite almost everywhere on X , and we may alter f on a set of measure zero, without changing its class in L^p , so as to assume f is real-valued on X .

Step A: approximation by simple functions. By the Simple Approximation Theorem there is a sequence $\{\varphi_n\}$ of simple functions on X with $|\varphi_n| \leq |f|$ on X and $\{\varphi_n\} \rightarrow f$ pointwise on X . Then $\{|f - \varphi_n|^p\} \rightarrow 0$ pointwise on X and

$$|f - \varphi_n|^p \leq (|f| + |\varphi_n|)^p \leq 2^p |f|^p \quad \text{on } X,$$

where $2^p |f|^p$ is integrable over X . By the Dominated Convergence Theorem, $\int_X |f - \varphi_n|^p d\mu \rightarrow 0$. Choose n with $\|f - \varphi_n\|_p < \epsilon/2$ and write $\varphi = \varphi_n = \sum_{i=1}^m c_i \chi_{E_i}$ with $c_1, \dots, c_m$ real numbers and $E_1, \dots, E_m \in \mathcal{M}$.

Step B: approximation of φ by a member of $\mathcal{S}$. Using the density of $\mathcal{D}$ in $\mathcal{N}$, choose for each i a set $A_{k_i} \in \mathcal{D}$ with

$$\mu(E_i \Delta A_{k_i}) < \left(\frac{\epsilon}{4m(1 + |c_i|)}\right)^p,$$

and choose rational numbers q_i with

$$|c_i - q_i| < \frac{\epsilon}{4m(1 + \mu(X)^{1/p})}.$$

Put $\psi = \sum_{i=1}^m q_i \chi_{A_{k_i}} \in \mathcal{S}$. By Minkowski's inequality, applied first to the m summands and then to each summand,

$$\|\varphi - \psi\|_p \leq \sum_{i=1}^m \|c_i \chi_{E_i} - q_i \chi_{A_{k_i}}\|_p \leq \sum_{i=1}^m |c_i| \|\chi_{E_i} - \chi_{A_{k_i}}\|_p + \sum_{i=1}^m |c_i - q_i| \|\chi_{A_{k_i}}\|_p.$$

By (*) and the choice of the A_{k_i} , the first sum is at most

$$\sum_{i=1}^m |c_i| \mu(E_i \Delta A_{k_i})^{1/p} < \sum_{i=1}^m \frac{|c_i| \epsilon}{4m(1 + |c_i|)} \leq \frac{\epsilon}{4},$$

while, since $\|\chi_{A_{k_i}}\|_p = \mu(A_{k_i})^{1/p} \leq \mu(X)^{1/p}$, the second sum is at most

$$\sum_{i=1}^m \frac{\epsilon \mu(X)^{1/p}}{4m(1 + \mu(X)^{1/p})} < \frac{\epsilon}{4}.$$

Hence $\|\varphi - \psi\|_p < \epsilon/2$ and therefore

$$\|f - \psi\|_p \leq \|f - \varphi\|_p + \|\varphi - \psi\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\mathcal{S}$ is a countable dense subset of $L^p(X, \mu)$, so $L^p(X, \mu)$ is separable; since $p \in [1, \infty)$ was arbitrary, this holds for all $1 \leq p < \infty$.

Combining Parts 1 and 2, the Nikodym metric space associated with $(X, \mathcal{M}, \mu)$ is separable if and only if $L^p(X, \mu)$ is separable for all $1 \leq p < \infty$. (The proof shows more: these are also equivalent to the separability of $L^p(X, \mu)$ for a single exponent $p \in [1, \infty)$.)

19 General L^p Spaces: Completeness, Duality, and Weak Convergence

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19.1 Problems 19.1–19.7

Problem 19.1. For $1 \leq p < \infty$ and n a natural number, define $f_n(x) = n^{1/p}$ if $0 \leq x \leq 1/n$ and $f_n(x) = 0$ if $1/n < x \leq 1$. Let f be identically zero on $[0, 1]$. Show that $\{f_n\}$ converges pointwise to f but does not converge in L^p . Where does the Vitali Convergence Criterion in L^p fail?

Solution. Throughout, $X = [0, 1]$ with Lebesgue measure m .

Each f_n belongs to L^p . The function $f_n = n^{1/p} \chi_{[0, 1/n]}$ is a simple function, and

$$\int_{[0, 1]} |f_n|^p dm = n \cdot m([0, 1/n]) = n \cdot \frac{1}{n} = 1.$$

So $\|f_n\|_p = 1$ for every n , and in particular each f_n lies in $L^p([0, 1], m)$. The limit function $f \equiv 0$ of course belongs to L^p as well.

Pointwise convergence. Fix x with $0 < x \leq 1$. By the Archimedean property of $\mathbb{R}$ choose a natural number N with $N > 1/x$, that is, $1/N < x$. Then for every $n \geq N$ we have $1/n \leq 1/N < x$, so x lies in the interval $(1/n, 1]$ and hence $f_n(x) = 0$. Therefore $f_n(x) \rightarrow 0 = f(x)$. Thus $\{f_n\}$ converges to f at every point of $(0, 1]$, that is, pointwise almost everywhere on $[0, 1]$, the exceptional set $\{0\}$ having Lebesgue measure zero. (At $x = 0$ we have $f_n(0) = n^{1/p} \rightarrow \infty$, so convergence genuinely fails only at that single point.)

Failure of convergence in L^p . Since $f \equiv 0$,

$$\|f_n - f\|_p^p = \int_{[0, 1]} |f_n|^p dm = 1 \quad \text{for all } n,$$

so $\|f_n - f\|_p = 1$ for all n . Hence $\{f_n\}$ does not converge to f in $L^p([0, 1], m)$. Moreover $\{f_n\}$ converges to no function in L^p at all: if $\{f_n\} \rightarrow g$ in L^p , then by the Riesz–Fischer Theorem a subsequence of $\{f_n\}$ converges to g pointwise a.e. on $[0, 1]$; but every subsequence converges pointwise to 0 on $(0, 1]$, so $g = 0$ a.e., and we have just seen that $\{f_n\}$ does not converge to 0 in L^p .

Where the Vitali L^p Convergence Criterion fails. The criterion asserts that, for a sequence $\{f_n\}$ in L^p converging pointwise a.e. to a function f in L^p , we have $\{f_n\} \rightarrow f$ in L^p if and only if $\{|f_n|^p\}$ is uniformly integrable and tight. Here $X = [0, 1]$ has finite measure, so tightness is automatic: take $X_0 = X$ in the definition of tightness, and then $\int_{X \setminus X_0} |f_n|^p dm = 0$ for every n .

What fails is uniform integrability of $\{|f_n|^p\} = \{n \cdot \chi_{[0, 1/n]}\}$. Indeed, take $\epsilon = 1/2$ and let $\delta > 0$ be arbitrary. Choose a natural number n with $1/n < \delta$ and set $E = [0, 1/n]$. Then $m(E) = 1/n < \delta$, yet

$$\int_E |f_n|^p dm = n \cdot \frac{1}{n} = 1 > \frac{1}{2} = \epsilon.$$

So no $\delta > 0$ responds to the challenge $\epsilon = 1/2$, and $\{|f_n|^p\}$ is not uniformly integrable. Intuitively, the mass 1 carried by $|f_n|^p$ is not spread out but is concentrated on the shrinking set $[0, 1/n]$, and it escapes in the limit rather than being captured by the pointwise limit function.

Problem 19.2. For $1 \leq p < \infty$ and n a natural number, let f_n be the characteristic function of $[n, n+1]$. Let f be identically zero on $\mathbb{R}$. Show that $\{f_n\}$ converges pointwise to f but does not converge in L^p . Where does the Vitali L^p Convergence Criterion fail?

Solution. Throughout, $X = \mathbb{R}$ with Lebesgue measure m , and $f_n = \chi_{[n, n+1]}$.

Each f_n belongs to L^p . Since $|f_n|^p = \chi_{[n, n+1]}$,

$$\int_{\mathbb{R}} |f_n|^p dm = m([n, n+1]) = 1,$$

so $\|f_n\|_p = 1$ for every n , and each f_n lies in $L^p(\mathbb{R}, m)$. The function $f \equiv 0$ lies in L^p too.

Pointwise convergence. Fix $x \in \mathbb{R}$ and choose a natural number $N > x$. For $n \geq N$ we have $n > x$, so

$x \notin [n, n+1]$ and hence $f_n(x) = 0$. Therefore $f_n(x) \rightarrow 0 = f(x)$. Thus $\{f_n\} \rightarrow f$ pointwise at every point of $\mathbb{R}$.

Failure of convergence in L^p . As computed above, $\|f_n - f\|_p^p = \int_{\mathbb{R}} |f_n|^p dm = 1$ for every n , so $\|f_n - f\|_p = 1$ does not tend to 0; $\{f_n\}$ does not converge to f in L^p . As in the preceding problem, $\{f_n\}$ converges in L^p to no function whatsoever: if $\{f_n\} \rightarrow g$ in L^p , then by the Riesz–Fischer Theorem some subsequence converges to g pointwise a.e., while every subsequence converges pointwise to 0; hence $g = 0$ a.e., contradicting $\|f_n\|_p = 1$ for all n .

Where the Vitali L^p Convergence Criterion fails. This time uniform integrability of $\{|f_n|^p\}$ **holds**. Let $\epsilon > 0$ and put $\delta = \epsilon$. If E is measurable with $m(E) < \delta$, then for every n ,

$$\int_E |f_n|^p dm = m(E \cap [n, n+1]) \leq m(E) < \delta = \epsilon.$$

What fails is **tightness**. Recall that $\{|f_n|^p\}$ is tight over $\mathbb{R}$ provided for each $\epsilon > 0$ there is a set X_0 of finite measure for which $\int_{\mathbb{R} \setminus X_0} |f_n|^p dm < \epsilon$ for all n . Take $\epsilon = 1/2$ and let X_0 be any measurable set with $m(X_0) < \infty$. Since the singleton $\{n+1\}$ has measure zero, $m(X_0 \cap [n, n+1]) = m(X_0 \cap [n, n+1))$ for every n , and the intervals $[n, n+1)$, $n = 1, 2, \dots$, are disjoint with union $[1, \infty)$. Hence by the countable additivity and the monotonicity of m ,

$$\sum_{n=1}^{\infty} m(X_0 \cap [n, n+1]) = \sum_{n=1}^{\infty} m(X_0 \cap [n, n+1)) = m(X_0 \cap [1, \infty)) \leq m(X_0) < \infty.$$

A convergent series of nonnegative terms has terms tending to 0, so $m(X_0 \cap [n, n+1]) \rightarrow 0$ as $n \rightarrow \infty$, and there is an index n for which $m(X_0 \cap [n, n+1]) < 1/2$. For that n , using that $m([n, n+1]) = 1 < \infty$ and the excision property,

$$\int_{\mathbb{R} \setminus X_0} |f_n|^p dm = m([n, n+1] \setminus X_0) = 1 - m(X_0 \cap [n, n+1]) > \frac{1}{2} = \epsilon.$$

Thus no set of finite measure responds to $\epsilon = 1/2$, and $\{|f_n|^p\}$ is not tight. Here the mass 1 of $|f_n|^p$ is not concentrated but rather marches off to infinity, escaping every set of finite measure.

Problem 19.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. Let $\{f_n\}$ be a sequence in $L^p(X, \mu)$ and f a function in $L^p(X, \mu)$ for which $\{f_n\} \rightarrow f$ pointwise a.e. on X . Show that

$\{f_n\} \rightarrow f$ in $L^p(X, \mu)$ if and only if the sequence $\{|f_n|^p\}$ is uniformly integrable and tight.

Solution. We use two elementary inequalities. First, for real numbers a, b and $1 \leq p < \infty$, convexity of $t \mapsto t^p$ on $[0, \infty)$ gives

$$|a + b|^p \leq (|a| + |b|)^p \leq 2^{p-1}(|a|^p + |b|^p) \leq 2^p(|a|^p + |b|^p). \quad (*)$$

Second, we shall repeatedly use Proposition 17 of Chapter 18: if h is integrable over X , then for each $\epsilon > 0$ there is a $\delta > 0$ such that $\int_E |h| d\mu < \epsilon$ whenever $\mu(E) < \delta$, and there is a subset X_0 of X of finite measure for which $\int_{X \setminus X_0} |h| d\mu < \epsilon$. In particular, a **finite** collection of integrable functions is uniformly integrable and tight: take the smallest of the finitely many δ 's, and the union of the finitely many sets X_0 (a finite union of sets of finite measure has finite measure).

Step 1: reduction to Corollary 18 of Chapter 18. Define

$$h_n = |f_n - f|^p \quad \text{on } X.$$

Each h_n is nonnegative and measurable, and by $(*)$, $h_n \leq 2^p(|f_n|^p + |f|^p)$, which is integrable since f_n and f belong to L^p ; hence each h_n is integrable over X . Since $\{f_n\} \rightarrow f$ pointwise a.e. on X and $t \mapsto |t|^p$ is continuous, $\{h_n(x)\} \rightarrow 0$ for almost all x in X . Corollary 18 of Chapter 18 therefore applies and tells us

$$\lim_{n \rightarrow \infty} \int_X h_n d\mu = 0 \iff \{h_n\} = \{|f_n - f|^p\} \text{ is uniformly integrable and tight.}$$

Since $\int_X h_n d\mu = \|f_n - f\|_p^p$, the left-hand condition is exactly the assertion that $\{f_n\} \rightarrow f$ in $L^p(X, \mu)$. So it remains only to prove:

$\{|f_n - f|^p\}$ is uniformly integrable and tight $\iff \{|f_n|^p\}$ is uniformly integrable and tight.

Step 2: if $\{|f_n|^p\}$ is uniformly integrable and tight, so is $\{|f_n - f|^p\}$. Let $\epsilon > 0$. Since $|f|^p$ is integrable, by Proposition 17 there is $\delta_1 > 0$ such that $\int_E |f|^p d\mu < \epsilon/2^{p+1}$ whenever $\mu(E) < \delta_1$. By uniform integrability of $\{|f_n|^p\}$ there is $\delta_2 > 0$ such that $\int_E |f_n|^p d\mu < \epsilon/2^{p+1}$ for all n whenever $\mu(E) < \delta_2$. Put $\delta = \min\{\delta_1, \delta_2\}$. If $\mu(E) < \delta$, then by (*) and monotonicity and linearity of integration, for every n ,

$$\int_E |f_n - f|^p d\mu \leq 2^p \left[\int_E |f_n|^p d\mu + \int_E |f|^p d\mu \right] < 2^p \left[\frac{\epsilon}{2^{p+1}} + \frac{\epsilon}{2^{p+1}} \right] = \epsilon.$$

Hence $\{|f_n - f|^p\}$ is uniformly integrable. Tightness is proved the same way: choose a set X_1 of finite measure with $\int_{X \sim X_1} |f|^p d\mu < \epsilon/2^{p+1}$ (Proposition 17), and, by tightness of $\{|f_n|^p\}$, a set X_2 of finite measure with $\int_{X \sim X_2} |f_n|^p d\mu < \epsilon/2^{p+1}$ for all n . Then $X_0 = X_1 \cup X_2$ has finite measure and $X \sim X_0 \subseteq (X \sim X_1) \cap (X \sim X_2)$, so by (*),

$$\int_{X \sim X_0} |f_n - f|^p d\mu \leq 2^p \left[\int_{X \sim X_0} |f_n|^p d\mu + \int_{X \sim X_0} |f|^p d\mu \right] < \epsilon \quad \text{for all } n.$$

Step 3: if $\{|f_n - f|^p\}$ is uniformly integrable and tight, so is $\{|f_n|^p\}$. By Step 1 the hypothesis is equivalent to $\|f_n - f\|_p \rightarrow 0$; we use the latter. Let $\epsilon > 0$. Choose an index N such that

$$\int_X |f_n - f|^p d\mu < \frac{\epsilon}{2^{p+1}} \quad \text{for all } n \geq N.$$

The functions $|f_1|^p, \dots, |f_{N-1}|^p$ and $|f|^p$ form a finite collection of integrable functions, so by the remark following Proposition 17 there is a $\delta > 0$ such that whenever $\mu(E) < \delta$ we have both

$$\int_E |f_k|^p d\mu < \epsilon \quad (1 \leq k \leq N-1) \quad \text{and} \quad \int_E |f|^p d\mu < \frac{\epsilon}{2^{p+1}}.$$

Now let $\mu(E) < \delta$. For $k \leq N-1$ the first estimate is what we want. For $n \geq N$, writing $f_n = (f_n - f) + f$ and using (*),

$$\int_E |f_n|^p d\mu \leq 2^p \left[\int_E |f_n - f|^p d\mu + \int_E |f|^p d\mu \right] \leq 2^p \left[\frac{\epsilon}{2^{p+1}} + \frac{\epsilon}{2^{p+1}} \right] = \epsilon,$$

where we bounded $\int_E |f_n - f|^p$ by the integral over all of X . Hence $\{|f_n|^p\}$ is uniformly integrable. For tightness, again let $\epsilon > 0$ and choose N as above. By the remark following Proposition 17 there is a set X_0 of finite measure with

$$\int_{X \sim X_0} |f_k|^p d\mu < \epsilon \quad (1 \leq k \leq N-1) \quad \text{and} \quad \int_{X \sim X_0} |f|^p d\mu < \frac{\epsilon}{2^{p+1}}.$$

For $n \geq N$, exactly as above,

$$\int_{X \sim X_0} |f_n|^p d\mu \leq 2^p \left[\int_X |f_n - f|^p d\mu + \int_{X \sim X_0} |f|^p d\mu \right] \leq \epsilon.$$

Thus $\int_{X \sim X_0} |f_n|^p d\mu \leq \epsilon$ for all n , and $\{|f_n|^p\}$ is tight.

Combining Steps 1–3: $\{f_n\} \rightarrow f$ in $L^p(X, \mu)$ if and only if $\{|f_n - f|^p\}$ is uniformly integrable and tight, if and only if $\{|f_n|^p\}$ is uniformly integrable and tight. This is the Vitali L^p Convergence Criterion.

Problem 19.4. For a measure space $(X, \mathcal{M}, \mu)$ and $0 < p < 1$, define $L^p(X, \mu)$ to be the collection of measurable functions on X for which $|f|^p$ is integrable. Show that $L^p(X, \mu)$ is a linear space. For $f \in L^p(X, \mu)$, define $\|f\|_p^p = \int_X |f|^p d\mu$.

(i) Show that, in general, $\|\cdot\|_p$ is not a norm since Minkowski's Inequality may fail.

(ii) Define

$$\rho(f, g) = \int_X |f - g|^p d\mu \quad \text{for all } f, g \in L^p(X, \mu).$$

Show that ρ is a metric with respect to which $L^p(X, \mu)$ is complete.

Solution. Throughout, $0 < p < 1$ is fixed, and as usual the members of $L^p(X, \mu)$ are understood to be equivalence classes of measurable functions that agree almost everywhere on X .

The key elementary inequality. For all $s, t \geq 0$,

$$(s + t)^p \leq s^p + t^p. \quad (*)$$

Indeed, the inequality is trivial if $s + t = 0$. Otherwise set $u = s/(s + t)$ and $v = t/(s + t)$, so that $u, v \in [0, 1]$ and $u + v = 1$. Since $0 < p < 1$ and $0 \leq u \leq 1$, we have $u^p \geq u$, and likewise $v^p \geq v$. Hence $u^p + v^p \geq u + v = 1$. Multiplying by $(s + t)^p$ gives $s^p + t^p \geq (s + t)^p$. Consequently, for real numbers a and b ,

$$|a + b|^p \leq (|a| + |b|)^p \leq |a|^p + |b|^p. \quad (**)$$

$L^p(X, \mu)$ is a linear space. Let $f, g \in L^p(X, \mu)$ and $\alpha, \beta \in \mathbb{R}$. The function $\alpha f + \beta g$ is measurable, and by (**),

$$|\alpha f + \beta g|^p \leq |\alpha|^p |f|^p + |\beta|^p |g|^p \quad \text{on } X.$$

The right-hand side is integrable over X , being a linear combination with nonnegative coefficients of the integrable functions $|f|^p$ and $|g|^p$. By the integral comparison test, the nonnegative measurable function $|\alpha f + \beta g|^p$ is integrable over X . Hence $\alpha f + \beta g \in L^p(X, \mu)$, and $L^p(X, \mu)$ is a linear subspace of the linear space of measurable functions on X (functions being identified when equal a.e.).

(i) Minkowski's Inequality may fail. Note first that $\|\cdot\|_p$, defined by $\|f\|_p = (\int_X |f|^p d\mu)^{1/p}$, is nonnegative, vanishes exactly when $f = 0$ a.e., and is positively homogeneous: $\|\alpha f\|_p = |\alpha| \|f\|_p$, since $\int_X |\alpha f|^p d\mu = |\alpha|^p \int_X |f|^p d\mu$. So only the triangle inequality can fail — and it does.

Take $X = [0, 1]$ with Lebesgue measure, $f = \chi_{[0, 1/2]}$ and $g = \chi_{(1/2, 1]}$. Then

$$\|f\|_p = \left(\frac{1}{2}\right)^{1/p} = \|g\|_p, \quad f + g = \chi_{[0, 1]}, \quad \|f + g\|_p = 1.$$

Since $0 < p < 1$ we have $1/p > 1$, hence $1 - 1/p < 0$, hence

$$\|f\|_p + \|g\|_p = 2 \cdot 2^{-1/p} = 2^{1-1/p} < 2^0 = 1 = \|f + g\|_p.$$

Thus $\|f + g\|_p > \|f\|_p + \|g\|_p$: Minkowski's Inequality fails, and $\|\cdot\|_p$ is not a norm on $L^p([0, 1], m)$. (Of course, for particular measure spaces — say X a single atom — no such counterexample exists; this is why the assertion is that the triangle inequality fails **in general**.)

(ii) ρ is a metric. Let $f, g, h \in L^p(X, \mu)$. By the linearity just established, $f - g \in L^p$, so $\rho(f, g) = \int_X |f - g|^p d\mu$ is a finite nonnegative real number. Clearly $\rho(f, g) = \rho(g, f)$. If $\rho(f, g) = 0$, then the nonnegative measurable function $|f - g|^p$ has zero integral, hence vanishes a.e. on X , hence $f = g$ a.e., that is, $f = g$ as members of $L^p(X, \mu)$; and conversely. Finally, by (**) applied pointwise with $a = f - g$ and $b = g - h$,

$$|f - h|^p \leq |f - g|^p + |g - h|^p \quad \text{on } X,$$

and integrating, by monotonicity and linearity of integration,

$$\rho(f, h) \leq \rho(f, g) + \rho(g, h).$$

So ρ is a metric on $L^p(X, \mu)$.

Completeness. Let $\{f_n\}$ be Cauchy with respect to ρ . First extract a rapidly Cauchy subsequence: choose indices $n_1 < n_2 < \dots$ such that

$$\rho(f_{n_k}, f_{n_{k+1}}) < 4^{-k} \quad \text{for all } k.$$

(This is possible: having chosen $n_1, \dots, n_k$, the Cauchy property provides an N with $\rho(f_n, f_m) < 4^{-k}$ for $n, m \geq N$; take $n_{k+1} = \max\{N, n_k + 1\}$.) Write $g_k = f_{n_k}$.

Set $E_k = \{x \in X : |g_{k+1}(x) - g_k(x)| > 2^{-k/p}\}$. By Chebychev's Inequality applied to the integrable nonnegative function $|g_{k+1} - g_k|^p$ and the level $(2^{-k/p})^p = 2^{-k}$,

$$\mu(E_k) \leq 2^k \int_X |g_{k+1} - g_k|^p d\mu = 2^k \rho(g_k, g_{k+1}) < 2^k \cdot 4^{-k} = 2^{-k}.$$

Since $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, the Borel–Cantelli Lemma provides a measurable set $X_0 \subseteq X$ with $\mu(X \sim X_0) = 0$ such that each $x \in X_0$ belongs to at most finitely many E_k . (This is the form of the lemma used in the proof of Lemma 4 of this chapter, and its proof uses only the countable monotonicity and the continuity of the measure μ : put $X_0 = X \sim \bigcap_{K=1}^{\infty} \bigcup_{k \geq K} E_k$; then $\mu(\bigcup_{k \geq K} E_k) \leq \sum_{k \geq K} \mu(E_k) \leq 2^{-K+1}$, and the sets $\bigcup_{k \geq K} E_k$ descend, so by continuity of measure the intersection has measure zero.) Thus for each $x \in X_0$ there is an index $K(x)$ with

$$|g_{k+1}(x) - g_k(x)| \leq 2^{-k/p} \quad \text{for all } k \geq K(x).$$

Since $0 < 2^{-1/p} < 1$, the geometric series $\sum_k 2^{-k/p}$ converges, so for each $x \in X_0$ the series $\sum_k [g_{k+1}(x) - g_k(x)]$ converges absolutely and hence $\{g_k(x)\}$ is a Cauchy sequence of real numbers. By the completeness of $\mathbb{R}$ it converges; define

$$f(x) = \lim_{k \rightarrow \infty} g_k(x) \quad \text{for } x \in X_0, \quad f(x) = 0 \quad \text{for } x \in X \sim X_0.$$

Then f is measurable, being the pointwise limit on X of the measurable functions $g_k \cdot \chi_{X_0}$. Now fix k . For $j > k$, telescoping and applying $(**)$ repeatedly,

$$|g_j - g_k|^p \leq \sum_{i=k}^{j-1} |g_{i+1} - g_i|^p \quad \text{on } X,$$

so, integrating,

$$\int_X |g_j - g_k|^p d\mu \leq \sum_{i=k}^{j-1} \rho(g_i, g_{i+1}) < \sum_{i=k}^{\infty} 4^{-i} = \frac{4^{-k}}{1 - 1/4} = \frac{4}{3} \cdot 4^{-k}.$$

Since $\{|g_j - g_k|^p\} \rightarrow |f - g_k|^p$ pointwise a.e. on X as $j \rightarrow \infty$, Fatou's Lemma yields

$$\int_X |f - g_k|^p d\mu \leq \liminf_{j \rightarrow \infty} \int_X |g_j - g_k|^p d\mu \leq \frac{4}{3} \cdot 4^{-k}. \quad (1)$$

In particular $|f - g_k|^p$ is integrable, so $f - g_k \in L^p(X, \mu)$; since $g_k \in L^p(X, \mu)$ and $L^p(X, \mu)$ is a linear space, $f = (f - g_k) + g_k$ belongs to $L^p(X, \mu)$. Moreover (1) says exactly that $\rho(f, g_k) \leq \frac{4}{3} 4^{-k} \rightarrow 0$, so $\{g_k\} \rightarrow f$ with respect to ρ .

Finally, a Cauchy sequence with a convergent subsequence converges: given $\epsilon > 0$, choose N with $\rho(f_n, f_m) < \epsilon/2$ for $n, m \geq N$, and then choose k with $n_k \geq N$ and $\rho(g_k, f) < \epsilon/2$. For $n \geq N$, the triangle inequality for ρ gives

$$\rho(f_n, f) \leq \rho(f_n, f_{n_k}) + \rho(f_{n_k}, f) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{f_n\} \rightarrow f$ with respect to ρ , and the metric space $(L^p(X, \mu), \rho)$ is complete.

Problem 19.5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a Cauchy sequence in $L^\infty(X, \mu)$. Show that there is a measurable subset X_0 of X for which $\mu(X \sim X_0) = 0$ and for each $\epsilon > 0$, there is an index N for which

$$|f_n - f_m| \leq \epsilon \text{ on } X_0 \text{ for all } n, m \geq N.$$

Use this to show that $L^\infty(X, \mu)$ is complete.

Solution. Recall that for a measurable function h on X , the essential supremum $\|h\|_\infty$ is the smallest $M \in [0, \infty]$ for which $|h| \leq M$ a.e. on X ; in particular, for $h \in L^\infty(X, \mu)$ the set $\{x \in X : |h(x)| > \|h\|_\infty\}$ has measure zero. (Indeed, if $\|h\|_\infty$ is the infimum of the essential bounds M , then $\{|h| > \|h\|_\infty\} = \bigcup_{k=1}^\infty \{|h| > \|h\|_\infty + 1/k\}$ is a countable union of sets of measure zero.)

Construction of X_0 . For each pair of natural numbers (n, m) define

$$E_{n,m} = \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}.$$

Since $f_n - f_m$ belongs to $L^\infty(X, \mu)$, the preceding remark gives $\mu(E_{n,m}) = 0$. Set

$$X_0 = X \sim \bigcup_{n,m=1}^\infty E_{n,m}.$$

This is a measurable set, and $X \sim X_0 = \bigcup_{n,m} E_{n,m}$ is a countable union of sets of measure zero, so by the countable monotonicity of μ ,

$$\mu(X \sim X_0) \leq \sum_{n,m} \mu(E_{n,m}) = 0.$$

By construction,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \text{ for all } x \in X_0 \text{ and all } n, m. \quad (2)$$

Now let $\epsilon > 0$. Since $\{f_n\}$ is Cauchy in $L^\infty(X, \mu)$, there is an index N for which $\|f_n - f_m\|_\infty \leq \epsilon$ for all $n, m \geq N$. By (2), for all $n, m \geq N$,

$$|f_n - f_m| \leq \epsilon \text{ on } X_0.$$

This is precisely the assertion required. Observe that the **single** set X_0 works simultaneously for every ϵ ; that is what makes it useful.

Completeness of $L^\infty(X, \mu)$. The displayed property says that $\{f_n\}$ is uniformly Cauchy on X_0 . Hence for each $x \in X_0$ the sequence of real numbers $\{f_n(x)\}$ is Cauchy, and by the completeness of $\mathbb{R}$ it converges. Define

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \text{ for } x \in X_0, \quad f(x) = 0 \text{ for } x \in X \sim X_0.$$

Then f is measurable, being the pointwise limit on X of the measurable functions $f_n \cdot \chi_{X_0}$.

Let $\epsilon > 0$ and take N as above, so that $|f_n - f_m| \leq \epsilon$ on X_0 for all $n, m \geq N$. Fix $n \geq N$ and $x \in X_0$, and let $m \rightarrow \infty$ in $|f_n(x) - f_m(x)| \leq \epsilon$; since $f_m(x) \rightarrow f(x)$, we get

$$|f_n(x) - f(x)| \leq \epsilon \text{ for all } x \in X_0 \text{ and all } n \geq N. \quad (3)$$

First, f is essentially bounded. Applying (3) with $n = N$ and using (2) with $\|f_N\|_\infty$ — more directly, using that $|f_N| \leq \|f_N\|_\infty$ off a set of measure zero — we obtain

$$|f| \leq |f_N| + \epsilon \leq \|f_N\|_\infty + \epsilon \text{ a.e. on } X_0,$$

while $f = 0$ on $X \sim X_0$. Since $\mu(X \sim X_0) = 0$, this shows $|f| \leq \|f_N\|_\infty + \epsilon$ a.e. on X , so $f \in L^\infty(X, \mu)$. Second, $\{f_n\} \rightarrow f$ in $L^\infty(X, \mu)$. By (3), for $n \geq N$ the inequality $|f_n - f| \leq \epsilon$ holds on X_0 , hence a.e. on X because $\mu(X \sim X_0) = 0$. Thus ϵ is an essential upper bound for $|f_n - f|$, so

$$\|f_n - f\|_\infty \leq \epsilon \text{ for all } n \geq N.$$

As $\epsilon > 0$ was arbitrary, $\|f_n - f\|_\infty \rightarrow 0$.

Therefore every Cauchy sequence in $L^\infty(X, \mu)$ converges in $L^\infty(X, \mu)$, that is, $L^\infty(X, \mu)$ is complete. This is the case $p = \infty$ of the Riesz–Fischer Theorem. Note that the argument also proves the case $p = \infty$ of Lemma 4, left as an exercise there: a rapidly Cauchy sequence in $L^\infty(X, \mu)$ is Cauchy, and the f constructed above is both the L^∞ limit and the pointwise limit of $\{f_n\}$ on X_0 , that is, almost everywhere on X .

Problem 19.6. Prove the Riesz Representation Theorem for the case $p = 1$ by adapting the proof for the case $p > 1$.

Solution. We must prove: let $(X, \mathcal{M}, \mu)$ be a σ -finite measure space. For $f \in L^\infty(X, \mu)$ define $T_f \in (L^1(X, \mu))^*$ by

$$T_f(g) = \int_X fg \, d\mu \quad \text{for all } g \in L^1(X, \mu).$$

Then $T : f \mapsto T_f$ is an isometric isomorphism of $L^\infty(X, \mu)$ onto $(L^1(X, \mu))^*$. Here $q = \infty$ is the conjugate of $p = 1$.

Step 0: T is a well-defined linear isometry. If $f \in L^\infty$ and $g \in L^1$, then $|fg| \leq \|f\|_\infty |g|$ a.e. on X , so fg is integrable and

$$|T_f(g)| \leq \int_X |fg| \, d\mu \leq \|f\|_\infty \|g\|_1.$$

This is Hölder's Inequality for the conjugate pair $(1, \infty)$. Thus T_f is a bounded linear functional on L^1 with $\|T_f\| \leq \|f\|_\infty$, and $f \mapsto T_f$ is plainly linear.

For the reverse inequality, suppose $f \neq 0$ in L^∞ and let $0 < \epsilon < \|f\|_\infty$. The set $E = \{x \in X : |f(x)| > \|f\|_\infty - \epsilon\}$ has $\mu(E) > 0$, since otherwise $\|f\|_\infty - \epsilon$ would be an essential upper bound for $|f|$. Because X is σ -finite, write $X = \bigcup_n X_n$ with $\mu(X_n) < \infty$; since $\mu(E) > 0$ there is an n with $\mu(E \cap X_n) > 0$. Put $E_0 = E \cap X_n$, so $0 < \mu(E_0) < \infty$, and define

$$g = \frac{1}{\mu(E_0)} \operatorname{sgn}(f) \cdot \chi_{E_0}.$$

Then g is measurable, $\|g\|_1 = 1$, and

$$T_f(g) = \frac{1}{\mu(E_0)} \int_{E_0} |f| \, d\mu \geq \frac{1}{\mu(E_0)} \cdot (\|f\|_\infty - \epsilon) \mu(E_0) = \|f\|_\infty - \epsilon.$$

Hence $\|T_f\| \geq \|f\|_\infty - \epsilon$ for every such ϵ , so $\|T_f\| \geq \|f\|_\infty$. Therefore $\|T_f\| = \|f\|_\infty$ and T is an isometry; in particular T is one-to-one. It remains to prove that T is onto.

Step 1: the case $\mu(X) < \infty$. Let $S : L^1(X, \mu) \rightarrow \mathbb{R}$ be a bounded linear functional. Since $\mu(X) < \infty$, the characteristic function of every measurable set belongs to $L^1(X, \mu)$, so we may define the set function

$$\nu(E) = S(\chi_E) \quad \text{for } E \in \mathcal{M}.$$

ν is a signed measure. Clearly $\nu(\emptyset) = S(0) = 0$. Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection in $\mathcal{M}$ with union E . By countable additivity of μ and $\mu(E) \leq \mu(X) < \infty$,

$$\left\| \chi_E - \sum_{k=1}^n \chi_{E_k} \right\|_1 = \int_X \chi_{\bigcup_{k>n} E_k} \, d\mu = \sum_{k=n+1}^\infty \mu(E_k) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

because $\sum_{k=1}^\infty \mu(E_k) = \mu(E) < \infty$ is a convergent series of nonnegative terms. Since S is linear and continuous on $L^1(X, \mu)$,

$$\nu(E) = S(\chi_E) = \lim_{n \rightarrow \infty} \sum_{k=1}^n S(\chi_{E_k}) = \sum_{k=1}^\infty \nu(E_k).$$

The series converges absolutely: setting $c_k = \text{sgn}(\nu(E_k)) \in \{-1, 0, 1\}$, the function $\sum_{k=1}^n c_k \chi_{E_k}$ belongs to L^1 with $\|\sum_{k=1}^n c_k \chi_{E_k}\|_1 \leq \sum_{k=1}^n \mu(E_k) \leq \mu(X)$, so by linearity and boundedness of S ,

$$\sum_{k=1}^n |\nu(E_k)| = S\left(\sum_{k=1}^n c_k \chi_{E_k}\right) \leq \|S\| \cdot \mu(X) \quad \text{for all } n,$$

and a series of nonnegative terms with bounded partial sums converges. Thus ν is a (finite) signed measure on $\mathcal{M}$.

$\nu \ll \mu$. If $E \in \mathcal{M}$ has $\mu(E) = 0$, then χ_E is the zero element of $L^1(X, \mu)$, so by the linearity of S , $\nu(E) = S(\chi_E) = 0$.

By Corollary 20 of Chapter 18, a consequence of the Radon–Nikodym Theorem, there is a function f that is integrable over X and for which

$$S(\chi_E) = \nu(E) = \int_E f d\mu \quad \text{for all } E \in \mathcal{M}.$$

By the linearity of S and of integration, and since every simple function on X belongs to $L^1(X, \mu)$ (as $\mu(X) < \infty$),

$$S(\varphi) = \int_X f \varphi d\mu \quad \text{for every simple function } \varphi \text{ on } X. \quad (4)$$

f belongs to L^∞ with $\|f\|_\infty \leq \|S\|$. Since S is bounded on $L^1(X, \mu)$, (4) gives, for every simple function φ on X (each of which vanishes outside the set X of finite measure),

$$\left| \int_X f \varphi d\mu \right| = |S(\varphi)| \leq \|S\| \|\varphi\|_1.$$

This is precisely inequality (9) of Lemma 6 with $M = \|S\|$ and $p = 1$. Lemma 6 — whose proof explicitly treats the case $p = 1$, showing by contradiction that M must be an essential upper bound for f — tells us that f belongs to $L^q(X, \mu) = L^\infty(X, \mu)$ and $\|f\|_\infty \leq \|S\|$.

$S = T_f$. By Hölder's Inequality the functional T_f is continuous on $L^1(X, \mu)$, so $g \mapsto S(g) - \int_X f g d\mu$ is a continuous linear functional on $L^1(X, \mu)$. By (4) it vanishes on the linear space of simple functions that vanish outside a set of finite measure, which by Theorem 5 is a dense subspace of $L^1(X, \mu)$. A continuous function that vanishes on a dense subset vanishes identically. Therefore $S = T_f$.

Step 2: the case X σ -finite. Let $\{X_n\}$ be an ascending sequence of measurable sets of finite measure whose union is X , and let $S \in (L^1(X, \mu))^*$. Fix n . The measure space $(X_n, \mathcal{M} \cap X_n, \mu)$ has finite measure, and $g \mapsto S(g)$, defined for $g \in L^1(X_n, \mu)$ by extending g to be 0 on $X \setminus X_n$, is a bounded linear functional on $L^1(X_n, \mu)$ of norm at most $\|S\|$. By Step 1 there is $f_n \in L^\infty(X, \mu)$ with $f_n = 0$ on $X \setminus X_n$, $\|f_n\|_\infty \leq \|S\|$, and

$$S(g) = \int_{X_n} f_n g d\mu = \int_X f_n g d\mu \quad \text{if } g \in L^1(X, \mu) \text{ and } g = 0 \text{ on } X \setminus X_n.$$

Any function with this property is uniquely determined on X_n up to a set of measure zero. Indeed, suppose h is integrable over X_n and $\int_E h d\mu = 0$ for every measurable $E \subseteq X_n$ (apply the displayed identity to $g = \chi_E$, which lies in L^1 because $\mu(X_n) < \infty$). Taking $E = \{x \in X_n : h(x) > 0\}$ gives $\int_{X_n} h^+ d\mu = 0$, so the nonnegative measurable function h^+ vanishes a.e. on X_n ; taking $E = \{x \in X_n : h(x) < 0\}$ gives likewise $h^- = 0$ a.e. on X_n . Hence $h = 0$ a.e. on X_n . The restriction of f_{n+1} to X_n has the same property as f_n , so, altering f_{n+1} on a set of measure zero, we may assume $f_{n+1} = f_n$ on X_n . Define f on $X = \bigcup_n X_n$ by $f(x) = f_n(x)$ if $x \in X_n$; this is a well-defined measurable function on X . For each n we have $|f| = |f_n| \leq \|S\|$ a.e. on X_n ; since X is the union of the countably many X_n , $|f| \leq \|S\|$ a.e. on X . Hence $f \in L^\infty(X, \mu)$ with $\|f\|_\infty \leq \|S\|$.

$S = T_f$. Let $g \in L^1(X, \mu)$ and set $g_n = g \cdot \chi_{X_n}$. Then $\{g_n\} \rightarrow g$ pointwise on X and $|g_n - g| \leq |g|$, so by the Lebesgue Dominated Convergence Theorem

$$\|g_n - g\|_1 = \int_X |g_n - g| d\mu \rightarrow 0,$$

and hence, by the continuity of S , $S(g_n) \rightarrow S(g)$. On the other hand $\{fg_n\} \rightarrow fg$ pointwise on X with $|fg_n| \leq \|f\|_\infty |g|$, an integrable function, so again by the Dominated Convergence Theorem

$$\int_X fg_n d\mu \rightarrow \int_X fg d\mu.$$

But g_n vanishes off X_n , so $S(g_n) = \int_X f_n g_n d\mu = \int_X fg_n d\mu$. Letting $n \rightarrow \infty$ in this identity gives

$$S(g) = \int_X fg d\mu = T_f(g).$$

Since $g \in L^1(X, \mu)$ was arbitrary, $S = T_f$.

Thus T maps $L^\infty(X, \mu)$ onto $(L^1(X, \mu))^*$, and by Step 0 it is a linear isometry. Therefore T is an isometric isomorphism of $L^\infty(X, \mu)$ onto $(L^1(X, \mu))^*$, which is the Riesz Representation Theorem for $p = 1$.

Remark on where σ -finiteness enters. It is used twice, and both uses are essential. In Step 0 it guarantees a subset of E of finite positive measure, so that a competitor g of unit L^1 norm exists; in Step 2 it produces the exhausting sequence $\{X_n\}$. The proof for $p > 1$ used σ -finiteness in exactly the same two places.

Problem 19.7. Show that for the case of Lebesgue measure on a nontrivial closed, bounded interval $[a, b]$, the Riesz Representation Theorem does not extend to the case $p = \infty$.

Solution. Let m denote Lebesgue measure on $X = [a, b]$, where $a < b$. The conjugate of $p = \infty$ is $q = 1$, so the assertion to be refuted is that the map

$$T : L^1([a, b], m) \rightarrow (L^\infty([a, b], m))^*, \quad T_f(g) = \int_{[a, b]} fg dm,$$

is onto. We exhibit a bounded linear functional on $L^\infty([a, b], m)$ that is not of the form T_f for any $f \in L^1([a, b], m)$. (By Hölder's Inequality each T_f is indeed a bounded linear functional on L^∞ , so the map T itself is well defined; the failure is one of surjectivity.)

The subspace of continuous functions. Let $C[a, b]$ denote the continuous real-valued functions on $[a, b]$, regarded as a linear subspace of $L^\infty([a, b], m)$. This regarding is legitimate: two continuous functions that agree a.e. on $[a, b]$ agree everywhere, since the set where they differ is open in $[a, b]$ and a nonempty relatively open subset of $[a, b]$ contains an interval of positive length, hence has positive measure. So distinct members of $C[a, b]$ determine distinct members of L^∞ .

Moreover, for $h \in C[a, b]$ the essential supremum coincides with the maximum:

$$\|h\|_\infty = \max_{x \in [a, b]} |h(x)|.$$

Indeed $\leq$ is clear. Conversely, if $|h(x_0)| = M := \max_{[a, b]} |h|$ and $\epsilon > 0$, then by continuity $|h| > M - \epsilon$ on a relatively open neighbourhood of x_0 in $[a, b]$, a set of positive measure; hence $M - \epsilon$ is not an essential upper bound, so $\|h\|_\infty \geq M - \epsilon$ for all $\epsilon > 0$.

A functional on the subspace. Define $\psi : C[a, b] \rightarrow \mathbb{R}$ by

$$\psi(h) = h(a).$$

This is linear, and by the displayed identity,

$$|\psi(h)| = |h(a)| \leq \max_{[a, b]} |h| = \|h\|_\infty,$$

so ψ is bounded on the subspace $C[a, b]$ of the normed linear space $L^\infty([a, b], m)$, with $\|\psi\| \leq 1$. (In fact $\|\psi\| = 1$, as the constant function 1 shows.)

Extension. By the Hahn–Banach Theorem, ψ extends to a bounded linear functional S on all of $L^\infty([a, b], m)$ with $\|S\| = \|\psi\| = 1$. In particular $S \neq 0$ and

$$S(h) = h(a) \quad \text{for every } h \in C[a, b]. \quad (5)$$

S is not represented by any $f \in L^1$. Suppose, to get a contradiction, that there is $f \in L^1([a, b], m)$ with

$$S(g) = \int_{[a, b]} fg \, dm \quad \text{for all } g \in L^\infty([a, b], m).$$

For each natural number n with $1/n < b - a$, define $h_n : [a, b] \rightarrow \mathbb{R}$ by

$$h_n(x) = \max\{0, 1 - n(x - a)\}.$$

Each h_n is continuous on $[a, b]$, satisfies $0 \leq h_n \leq 1$, $h_n(a) = 1$, and $h_n(x) = 0$ for $x \geq a + 1/n$. Thus $h_n \in C[a, b] \subseteq L^\infty([a, b], m)$, and by (5),

$$S(h_n) = h_n(a) = 1 \quad \text{for every such } n. \tag{6}$$

On the other hand, since h_n vanishes outside $[a, a + 1/n]$ and $0 \leq h_n \leq 1$,

$$\left| \int_{[a, b]} fh_n \, dm \right| \leq \int_{[a, a+1/n]} |f| \, dm.$$

The function $|f|$ is integrable over $[a, b]$ and $m([a, a + 1/n]) = 1/n \rightarrow 0$, so by Proposition 17 of Chapter 18 (equivalently, by Proposition 23 of Chapter 4: the continuity of an integral with respect to the domain of integration), given $\epsilon > 0$ there is $\delta > 0$ such that $\int_E |f| \, dm < \epsilon$ whenever $m(E) < \delta$; taking $n > 1/\delta$ gives $\int_{[a, a+1/n]} |f| \, dm < \epsilon$. Hence

$$\lim_{n \rightarrow \infty} \int_{[a, b]} fh_n \, dm = 0.$$

But by assumption $\int_{[a, b]} fh_n \, dm = S(h_n) = 1$ for all large n , by (6). This contradiction shows that no such f exists.

Therefore S belongs to $(L^\infty([a, b], m))^*$ but is not of the form T_f with $f \in L^1([a, b], m)$: the map $T : L^1 \rightarrow (L^\infty)^*$ is not onto, and the Riesz Representation Theorem does not extend to $p = \infty$ for Lebesgue measure on a nontrivial closed, bounded interval.

Remark. The map T is still an isometry into $(L^\infty)^*$ — what fails is only surjectivity. The correct description of $(L^\infty(X, \mu))^*$ is furnished by the Kantorovitch Representation Theorem (Theorem 7 of this chapter): it is isometrically isomorphic to $\mathcal{BFA}(X, \mathcal{M}, \mu)$, the bounded finitely additive signed measures on $\mathcal{M}$ that are absolutely continuous with respect to μ . The functional S above corresponds to the bounded finitely additive signed measure $\nu(E) = S(\chi_E)$, and this ν is **not** countably additive. For if it were, then ν would be a finite signed measure absolutely continuous with respect to m , so by Corollary 20 of Chapter 18 there would be an f integrable over $[a, b]$ with $S(\chi_E) = \int_E f \, dm$ for all measurable E , hence $S(\varphi) = \int_{[a, b]} f\varphi \, dm$ for every simple function φ ; since by the Simple Approximation Lemma every bounded measurable function is a **uniform** limit of simple functions, and both sides of this identity are continuous with respect to uniform convergence, we would get $S = T_f$ on $L^\infty([a, b], m)$, which we have just excluded. Nontriviality of $[a, b]$ is what makes room for such a ν : on a measure space consisting of finitely many atoms every finitely additive set function is countably additive, and there the representation does extend.

19.2 Problems 19.8–19.14

Problem 19.8. Find a measure space $(X, \mathcal{M}, \mu)$ for which the Riesz Representation Theorem does extend to the case $p = \infty$.

Solution. Recall what is being asserted. For $p = \infty$ the conjugate index is $q = 1$, and for $f \in L^1(X, \mu)$ the functional $T_f : L^\infty(X, \mu) \rightarrow \mathbb{R}$ is defined by

$$T_f(g) = \int_X fg \, d\mu \quad \text{for all } g \in L^\infty(X, \mu).$$

By Hölder's Inequality (Theorem 1) T_f is bounded with $\|T_f\| \leq \|f\|_1$. To say that the Riesz Representation Theorem extends to $p = \infty$ is to say that

$$T: L^1(X, \mu) \rightarrow (L^\infty(X, \mu))^*$$

is an isometric isomorphism of $L^1(X, \mu)$ onto the dual of $L^\infty(X, \mu)$. Problem 7 shows this fails for Lebesgue measure on a nondegenerate closed bounded interval.

An example. Let N be a natural number, let $X = \{1, 2, \dots, N\}$, let $\mathcal{M} = 2^X$ be the collection of all subsets of X , and let μ be counting measure on $\mathcal{M}$. This is a finite (hence σ -finite) measure space, and it is complete since the only μ -null set is $\emptyset$.

Every function on X is measurable and bounded, and since the only null set is empty, two functions are equal a.e. $[\mu]$ precisely when they are equal. Hence for every p with $1 \leq p \leq \infty$ the space $L^p(X, \mu)$ is, as a set, the space $\mathbb{R}^N$ of all real-valued functions on X ; here

$$\|g\|_\infty = \max_{1 \leq k \leq N} |g(k)|, \quad \|f\|_1 = \sum_{k=1}^N |f(k)|,$$

and for $f, g: X \rightarrow \mathbb{R}$,

$$T_f(g) = \int_X f g d\mu = \sum_{k=1}^N f(k) g(k).$$

T is an isometry. Let $f \in L^1(X, \mu)$. For $g \in L^\infty(X, \mu)$,

$$|T_f(g)| \leq \sum_{k=1}^N |f(k)| |g(k)| \leq \|g\|_\infty \sum_{k=1}^N |f(k)| = \|f\|_1 \|g\|_\infty,$$

so $\|T_f\| \leq \|f\|_1$. On the other hand, put $g_0 = \text{sgn}(f)$, that is, $g_0(k) = 1$ if $f(k) \geq 0$ and $g_0(k) = -1$ if $f(k) < 0$. Then $\|g_0\|_\infty = 1$ and

$$T_f(g_0) = \sum_{k=1}^N f(k) \text{sgn}(f(k)) = \sum_{k=1}^N |f(k)| = \|f\|_1.$$

Hence $\|T_f\| = \|f\|_1$. In particular T is linear and one-to-one.

T is onto. Let S belong to $(L^\infty(X, \mu))^*$. For $1 \leq k \leq N$ let $e_k = \chi_{\{k\}}$, and define $f: X \rightarrow \mathbb{R}$ by $f(k) = S(e_k)$. Then $f \in L^1(X, \mu)$, since every function on the finite set X is integrable. Each $g \in L^\infty(X, \mu)$ satisfies $g = \sum_{k=1}^N g(k) e_k$, so by the linearity of S ,

$$S(g) = \sum_{k=1}^N g(k) S(e_k) = \sum_{k=1}^N g(k) f(k) = T_f(g).$$

Thus $S = T_f$, and T maps $L^1(X, \mu)$ onto $(L^\infty(X, \mu))^*$.

Therefore T is an isometric isomorphism of $L^1(X, \mu)$ onto $(L^\infty(X, \mu))^*$: the Riesz Representation Theorem holds for this measure space in the case $p = \infty$.

The same argument for a general purely atomic space with finitely many atoms. Recall that $A \in \mathcal{M}$ with $0 < \mu(A) < \infty$ is called an atom provided every measurable $B \subseteq A$ has $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Suppose $X = \bigcup_{k=1}^N A_k \cup X_0$ where the A_k are disjoint atoms and $\mu(X_0) = 0$. If f is measurable and finite a.e. on X and A is an atom, set $c = \inf\{t \in \mathbb{R} : \mu(\{f > t\} \cap A) = 0\}$; this infimum is over a nonempty set bounded below, because f is finite a.e. and each of the sets $\{f > t\} \cap A$ has measure 0 or $\mu(A)$. For each n , $\mu(\{f > c + 1/n\} \cap A) = 0$ by the definition of c , while $\mu(\{f > c - 1/n\} \cap A) = \mu(A)$, so that $\mu(\{f \leq c - 1/n\} \cap A) = 0$. Taking the countable union over n we conclude that $f = c$ a.e. on A . Hence every element of $L^p(X, \mu)$ is (as an equivalence class) determined by the N constants it assumes on $A_1, \dots, A_N$, so $L^p(X, \mu)$ is N -dimensional for every p , and the computation above goes through verbatim with e_k replaced by χ_{A_k} and $\|f\|_1 = \sum_{k=1}^N |f(k)| \mu(A_k)$; one again gets $\|T_f\| = \|f\|_1$, using $g_0 = \text{sgn}(f)$, and surjectivity from $g = \sum_k g(k) \chi_{A_k}$.

Remark. Examples of this kind are essentially the only ones. Suppose μ is σ -finite and $T: L^1(X, \mu) \rightarrow (L^\infty(X, \mu))^*$ is onto. Write $T': L^\infty(X, \mu) \rightarrow (L^1(X, \mu))^*$ for the map of the Riesz Representation Theorem in the case $p = 1$, which is an isometric isomorphism onto because μ is σ -finite. Royden's footnote warns that being isometric to one's bidual is not reflexivity, so we check the natural embedding itself. Let $J: L^\infty(X, \mu) \rightarrow (L^\infty(X, \mu))^{**}$ be the natural embedding and let $\Lambda \in (L^\infty(X, \mu))^{**}$. Since T is an isometric isomorphism onto, $f \mapsto \Lambda(T_f)$ is a bounded linear functional on $L^1(X, \mu)$, so by the surjectivity of T' there is a $g \in L^\infty(X, \mu)$ with $\Lambda(T_f) = \int_X f g d\mu$ for all $f \in L^1(X, \mu)$. But $J(g)[T_f] = T_f(g) = \int_X f g d\mu$ as well, and every element of $(L^\infty(X, \mu))^*$ is of the form T_f , so $\Lambda = J(g)$. Thus J is onto and $L^\infty(X, \mu)$ is reflexive. Now if $\mathcal{M}$ contained a countable disjoint collection $\{A_k\}$ of sets of positive measure, the map $(a_k) \mapsto \sum_k a_k \chi_{A_k}$ would be a linear isometry of ℓ^∞ into $L^\infty(X, \mu)$ whose image, being complete, is a closed subspace; a closed subspace of a reflexive space is reflexive, while ℓ^∞ is not. Hence no such collection exists. For a σ -finite μ this forces X to be, up to a μ -null set, the union of finitely many atoms, and then the preceding paragraph identifies $L^\infty(X, \mu)$ with a finite-dimensional space; we do not use this last step anywhere above, and record it only to say what the general picture is.

Problem 19.9. [In this problem $(X, \mathcal{M}, \mu)$ is a complete measure space.] Show that $\mathcal{BFA}(X, \mathcal{M}, \mu)$ is a linear space on which $\|\cdot\|_{\text{var}}$ is a norm. Then show that this normed linear space is a Banach space.

Solution. Throughout, $\mathcal{BFA}(X, \mathcal{M}, \mu)$ denotes the collection of set functions $\nu: \mathcal{M} \rightarrow \mathbb{R}$ that are finitely additive, of bounded total variation, and absolutely continuous with respect to μ in the sense that $\mu(E) = 0$ implies $\nu(E) = 0$; and, for $E \in \mathcal{M}$,

$$|\nu|(E) = \sup \sum_{k=1}^n |\nu(E_k)|, \quad \|\nu\|_{\text{var}} = |\nu|(X),$$

the supremum being taken over all finite disjoint collections $\{E_k\}_{k=1}^n$ of sets in $\mathcal{M}$ that are contained in E .

Two elementary observations will be used repeatedly. First, for any ν and any $E \in \mathcal{M}$, the one-element collection $\{E\}$ is admissible in the supremum defining $|\nu|(E)$, so

$$|\nu(E)| \leq |\nu|(E) \leq |\nu|(X) = \|\nu\|_{\text{var}}. \quad (*)$$

Second, a finite disjoint collection of sets in $\mathcal{M}$ is automatically contained in X , so $|\nu|(X)$ is the supremum of $\sum_{k=1}^n |\nu(E_k)|$ over all finite disjoint collections $\{E_k\}_{k=1}^n \subseteq \mathcal{M}$.

Step 1: $\mathcal{BFA}(X, \mathcal{M}, \mu)$ is a linear space. The set functions $\nu: \mathcal{M} \rightarrow \mathbb{R}$ form a real linear space under pointwise operations, $(\nu + \eta)(E) = \nu(E) + \eta(E)$ and $(\alpha\nu)(E) = \alpha\nu(E)$. It suffices to check that $\mathcal{BFA}(X, \mathcal{M}, \mu)$ is a linear subspace. The zero set function is finitely additive, has zero variation, and is absolutely continuous, so it belongs to $\mathcal{BFA}$.

Let $\nu, \eta \in \mathcal{BFA}(X, \mathcal{M}, \mu)$ and $\alpha, \beta \in \mathbb{R}$, and set $\sigma = \alpha\nu + \beta\eta$. If $A, B \in \mathcal{M}$ are disjoint, then

$$\sigma(A \cup B) = \alpha\nu(A \cup B) + \beta\eta(A \cup B) = \alpha[\nu(A) + \nu(B)] + \beta[\eta(A) + \eta(B)] = \sigma(A) + \sigma(B),$$

so σ is finitely additive (and $\sigma(\emptyset) = 0$). If $\mu(E) = 0$, then $\nu(E) = \eta(E) = 0$, so $\sigma(E) = 0$; thus σ is absolutely continuous with respect to μ . Finally, for any finite disjoint collection $\{E_k\}_{k=1}^n$ in $\mathcal{M}$,

$$\sum_{k=1}^n |\sigma(E_k)| \leq |\alpha| \sum_{k=1}^n |\nu(E_k)| + |\beta| \sum_{k=1}^n |\eta(E_k)| \leq |\alpha| \|\nu\|_{\text{var}} + |\beta| \|\eta\|_{\text{var}} < \infty.$$

Taking the supremum over such collections,

$$\|\alpha\nu + \beta\eta\|_{\text{var}} \leq |\alpha| \|\nu\|_{\text{var}} + |\beta| \|\eta\|_{\text{var}} < \infty, \quad (**)$$

so $\sigma \in \mathcal{BFA}(X, \mathcal{M}, \mu)$. Hence $\mathcal{BFA}(X, \mathcal{M}, \mu)$ is a linear space.

Step 2: $\|\cdot\|_{\text{var}}$ is a norm. Clearly $\|\nu\|_{\text{var}} \geq 0$, and it is finite by the definition of $\mathcal{BFA}$. Taking $\beta = 0$ in (**) gives $\|\alpha\nu\|_{\text{var}} \leq |\alpha| \|\nu\|_{\text{var}}$; if $\alpha \neq 0$, applying this with α^{-1} and $\alpha\nu$ in place of α and ν gives the reverse inequality, so $\|\alpha\nu\|_{\text{var}} = |\alpha| \|\nu\|_{\text{var}}$ for all α (the case $\alpha = 0$ being trivial). Taking $\alpha = \beta = 1$ in (**) gives the triangle inequality. Lastly, if $\|\nu\|_{\text{var}} = 0$, then by (*) we have $|\nu(E)| \leq 0$ for every $E \in \mathcal{M}$, that is, $\nu = 0$. Thus $\|\cdot\|_{\text{var}}$ is a norm.

Step 3: completeness. Let $\{\nu_n\}$ be a Cauchy sequence in $(\mathcal{BFA}(X, \mathcal{M}, \mu), \|\cdot\|_{\text{var}})$.

Fix $E \in \mathcal{M}$. By (*) applied to $\nu_n - \nu_k$,

$$|\nu_n(E) - \nu_k(E)| = |(\nu_n - \nu_k)(E)| \leq \|\nu_n - \nu_k\|_{\text{var}},$$

so $\{\nu_n(E)\}$ is a Cauchy sequence of real numbers. By the completeness of $\mathbb{R}$ it converges; define

$$\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E) \quad \text{for } E \in \mathcal{M}.$$

The set function $\nu: \mathcal{M} \rightarrow \mathbb{R}$ is finitely additive: if $A, B \in \mathcal{M}$ are disjoint, then

$$\nu(A \cup B) = \lim_n \nu_n(A \cup B) = \lim_n [\nu_n(A) + \nu_n(B)] = \nu(A) + \nu(B).$$

It is absolutely continuous with respect to μ : if $\mu(E) = 0$, then $\nu_n(E) = 0$ for every n , and hence $\nu(E) = 0$.

It remains to show that ν has bounded variation and that $\{\nu_n\} \rightarrow \nu$ in the total variation norm. Let $\varepsilon > 0$. Choose an index N such that

$$\|\nu_n - \nu_k\|_{\text{var}} < \varepsilon \quad \text{for all } n, k \geq N.$$

Fix $n \geq N$ and let $\{E_j\}_{j=1}^J$ be any finite disjoint collection of sets in $\mathcal{M}$. Since the sum is finite, we may pass to the limit termwise:

$$\sum_{j=1}^J |\nu(E_j) - \nu_n(E_j)| = \lim_{k \rightarrow \infty} \sum_{j=1}^J |\nu_k(E_j) - \nu_n(E_j)| \leq \limsup_{k \rightarrow \infty} \|\nu_k - \nu_n\|_{\text{var}} \leq \varepsilon.$$

Taking the supremum over all such finite disjoint collections yields

$$|\nu - \nu_n|(X) \leq \varepsilon \quad \text{for all } n \geq N. \quad (\dagger)$$

In particular the set function $\nu - \nu_N$ is finitely additive with $|\nu - \nu_N|(X) \leq \varepsilon < \infty$, and ν_N has bounded variation, so by the triangle inequality established in Step 1 (which required only finite additivity of the summands),

$$\|\nu\|_{\text{var}} = \|(\nu - \nu_N) + \nu_N\|_{\text{var}} \leq \varepsilon + \|\nu_N\|_{\text{var}} < \infty.$$

Therefore ν is a bounded finitely additive signed measure that is absolutely continuous with respect to μ , that is, $\nu \in \mathcal{BFA}(X, \mathcal{M}, \mu)$. Now (†) reads $\|\nu - \nu_n\|_{\text{var}} \leq \varepsilon$ for all $n \geq N$. Since $\varepsilon > 0$ was arbitrary, $\{\nu_n\} \rightarrow \nu$ in $\mathcal{BFA}(X, \mathcal{M}, \mu)$.

Thus every Cauchy sequence in $\mathcal{BFA}(X, \mathcal{M}, \mu)$ converges in $\mathcal{BFA}(X, \mathcal{M}, \mu)$, and this normed linear space is a Banach space.

Remark. This also follows from the Kantorovitch Representation Theorem (Theorem 7) together with the completeness of the dual of a normed linear space: T is an isometric isomorphism of $\mathcal{BFA}(X, \mathcal{M}, \mu)$ onto $(L^\infty(X, \mu))^*$, and the dual of a normed linear space is always complete. The direct argument above is given because Theorem 7 presupposes that $\mathcal{BFA}(X, \mathcal{M}, \mu)$ has already been shown to be a normed linear space.

Problem 19.10. [In this problem $(X, \mathcal{M}, \mu)$ is a complete measure space.] Let $\nu: \mathcal{M} \rightarrow \mathbb{R}$ be a

signed measure and $(X, \mathcal{M}, \mu)$ be σ -finite. Show that there is a function $f \in L^1(X, \mu)$ for which

$$\int_X g \, d\nu = \int_X g \cdot f \, d\mu \quad \text{for all } g \in L^\infty(X, \mu).$$

Solution. A preliminary word about the hypotheses. The elements of $L^\infty(X, \mu)$ are equivalence classes of essentially bounded measurable functions with respect to equality a.e. $[\mu]$. For the left-hand side $\int_X g \, d\nu$ to be a properly defined function of the class g it is necessary that $\int_X g_1 \, d\nu = \int_X g_2 \, d\nu$ whenever $g_1 = g_2$ a.e. $[\mu]$; taking $g_1 = \chi_E$ and $g_2 = 0$ for a set E with $\mu(E) = 0$, this forces $\nu(E) = 0$. So the assertion is to be read, as the surrounding section indicates, with ν absolutely continuous with respect to μ in the sense that $\mu(E) = 0$ implies $\nu(E) = 0$; equivalently, ν is a countably additive member of $\mathcal{BFA}(X, \mathcal{M}, \mu)$. (Some such hypothesis is unavoidable: on $X = [0, 1]$ with μ the zero measure and ν Lebesgue measure, $L^\infty(X, \mu) = \{0\}$, and no f can satisfy the displayed identity for the representative $\chi_{[0,1]}$ of the zero class.) The content of the problem is thus that inside $\mathcal{BFA}(X, \mathcal{M}, \mu)$ the countably additive set functions are exactly the ones representable by L^1 densities.

Since ν takes values in $\mathbb{R}$ it is a finite signed measure.

Step 1: ν is absolutely continuous with respect to μ in Royden's sense, i.e. $|\nu| \ll \mu$. By the Hahn Decomposition Theorem (Chapter 17) there is a positive set P and a negative set N for ν with $X = P \cup N$ and $P \cap N = \emptyset$, and the Jordan Decomposition $\nu = \nu^+ - \nu^-$ is given by

$$\nu^+(E) = \nu(E \cap P), \quad \nu^-(E) = -\nu(E \cap N) \quad \text{for } E \in \mathcal{M},$$

with ν^+ and ν^- finite measures. Let $E \in \mathcal{M}$ have $\mu(E) = 0$. Then $\mu(E \cap P) = 0$ and $\mu(E \cap N) = 0$ by the monotonicity of μ , so $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = -\nu(E \cap N) = 0$. Hence $|\nu| = \nu^+ + \nu^-$ vanishes on μ -null sets, that is, ν^+ , ν^- and $|\nu|$ are absolutely continuous with respect to μ .

Step 2: production of f . The measure space $(X, \mathcal{M}, \mu)$ is σ -finite and ν is a finite signed measure absolutely continuous with respect to μ . By Corollary 20 of Chapter 18, a consequence of the Radon-Nikodym Theorem, there is a function f that is integrable over X with respect to μ , that is, $f \in L^1(X, \mu)$, for which

$$\nu(E) = \int_E f \, d\mu \quad \text{for all } E \in \mathcal{M}. \quad (1)$$

(Explicitly, applying the Radon-Nikodym Theorem separately to the finite measures $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ produces nonnegative measurable f_1, f_2 with $\nu^\pm(E) = \int_E f_{1,2} \, d\mu$; taking $E = X$ shows f_1, f_2 are integrable since $\nu^\pm(X) < \infty$, and $f = f_1 - f_2$ satisfies (1).)

Step 3: the identity for simple functions. Let $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$ be a simple function measurable with respect to $\mathcal{M}$, where the E_k are disjoint sets in $\mathcal{M}$. By the definition of the integral with respect to a signed measure and by (1) and the linearity of integration with respect to μ ,

$$\int_X \varphi \, d\nu = \sum_{k=1}^n c_k \nu(E_k) = \sum_{k=1}^n c_k \int_{E_k} f \, d\mu = \int_X \varphi \cdot f \, d\mu. \quad (2)$$

Step 4: passage to $L^\infty(X, \mu)$. Let $g \in L^\infty(X, \mu)$. Choose a representative of g , again denoted g , that is measurable and bounded everywhere on X , say $|g| \leq M$ on X : this is possible because an essentially bounded function may be redefined to be 0 on a μ -null set without changing its class, and by Step 1 such a redefinition also leaves $\int_X g \, d\nu$ unchanged, since $|\nu|$ vanishes on μ -null sets. By the Simple Approximation Lemma there is a sequence $\{\varphi_n\}$ of simple functions on X with

$$|g - \varphi_n| \leq 1/n \quad \text{on } X \quad \text{for all } n,$$

so $\{\varphi_n\}$ converges to g uniformly on X ; in particular $\|g - \varphi_n\|_\infty \leq 1/n$.

For the left-hand side, the integral with respect to the finite signed measure ν is linear and satisfies $|\int_X h \, d\nu| \leq \|h\|_\infty \cdot \|\nu\|_{\text{var}}$ for bounded measurable h (this is inequality (18) of the section; by Step 1 our ν is a bounded finitely additive signed measure absolutely continuous with respect to μ , that is, $\nu \in \mathcal{BFA}(X, \mathcal{M}, \mu)$, with $\|\nu\|_{\text{var}} = |\nu|(X) = \nu^+(X) + \nu^-(X) < \infty$, so (18) applies). Hence

$$\left| \int_X g \, d\nu - \int_X \varphi_n \, d\nu \right| \leq \frac{1}{n} \|\nu\|_{\text{var}} \rightarrow 0.$$

For the right-hand side, by Hölder's Inequality (Theorem 1, with $p = 1$, $q = \infty$),

$$\left| \int_X g \cdot f \, d\mu - \int_X \varphi_n \cdot f \, d\mu \right| \leq \int_X |g - \varphi_n| |f| \, d\mu \leq \frac{1}{n} \|f\|_1 \rightarrow 0.$$

Since, by (2), $\int_X \varphi_n \, d\nu = \int_X \varphi_n \cdot f \, d\mu$ for every n , letting $n \rightarrow \infty$ in these two estimates gives

$$\int_X g \, d\nu = \int_X g \cdot f \, d\mu.$$

As $g \in L^\infty(X, \mu)$ was arbitrary, $f \in L^1(X, \mu)$ has the required property.

Remark. Combined with the Kantorovitch Representation Theorem (Theorem 7), this says precisely that under the isometric isomorphism T of $\mathcal{BFA}(X, \mathcal{M}, \mu)$ onto $(L^\infty(X, \mu))^*$ the countably additive members of $\mathcal{BFA}(X, \mathcal{M}, \mu)$ are carried onto the image of $L^1(X, \mu)$ in $(L^\infty(X, \mu))^*$. The Remark following Theorem 7 shows that for $([a, b], \mathcal{L}, m)$ this image is a proper subspace, so $\mathcal{BFA}([a, b], \mathcal{L}, m)$ contains set functions that are not countably additive.

Problem 19.11. [In this problem $(X, \mathcal{M}, \mu)$ is a complete measure space.] Let $\{\nu_n\}$ be a bounded sequence in $\mathcal{BFA}([a, b], \mathcal{L}, m)$. Show that there is a subsequence $\{\nu_{n_k}\}$ and $\nu \in \mathcal{BFA}([a, b], \mathcal{L}, m)$ such that

$$\lim_{k \rightarrow \infty} \int_X f \, d\nu_{n_k} = \int_X f \, d\nu \quad \text{for all } f \in L^1([a, b], m).$$

Solution. *How the assertion must be read.* The integral of a function with respect to a bounded finitely additive signed measure was defined in this section only for simple functions and then, by the estimate (17), for essentially bounded ones; that is, $\int_X f \, d\nu$ is defined for $f \in L^\infty([a, b], m)$ and $\nu \in \mathcal{BFA}([a, b], \mathcal{L}, m)$, this being the content of inequality (18). For a general $f \in L^1([a, b], m)$ the symbol $\int_X f \, d\nu$ has not been given a meaning. (Even for a countably additive ν with density $h \in L^1([a, b], m)$ the natural candidate $\int_X f h \, dm$ can fail to exist, since a product of two L^1 functions need not be integrable.) Since $m([a, b]) < \infty$ we have $L^\infty([a, b], m) \subseteq L^1([a, b], m)$, so the strongest reading of the assertion for which every symbol in it is defined, and the one intended, is

$$\lim_{k \rightarrow \infty} \int_X f \, d\nu_{n_k} = \int_X f \, d\nu \quad \text{for all } f \in L^\infty([a, b], m), \quad (\#)$$

that is, $\{\nu_{n_k}\}$ converges to ν in the weak-* topology that $\mathcal{BFA}([a, b], \mathcal{L}, m)$ inherits from $(L^\infty([a, b], m))^*$ under the Kantorovitch Representation Theorem (Theorem 7).

So read, the assertion is false. Weak-* compactness of the closed unit ball of the dual of a Banach space V yields convergent *subnets*; it yields convergent *subsequences* when V is separable, and $L^\infty([a, b], m)$ is not separable (this nonseparability is exactly what was used in the Remark following Theorem 7). Here is an explicit counterexample. Take $[a, b] = [0, 1]$ and, for each natural number n , define

$$\nu_n(E) = n \cdot m(E \cap [0, 1/n]) \quad \text{for } E \in \mathcal{L}.$$

Each ν_n is a finite measure with density $n\chi_{[0, 1/n]}$ with respect to m ; it is countably additive, hence finitely additive; it vanishes on m -null sets; and $\|\nu_n\|_{\text{var}} = \nu_n([0, 1]) = 1$. So $\{\nu_n\}$ is a bounded sequence in $\mathcal{BFA}([0, 1], \mathcal{L}, m)$. For a simple function $\varphi = \sum_k c_k \chi_{E_k}$,

$$\int_X \varphi \, d\nu_n = \sum_k c_k n m(E_k \cap [0, 1/n]) = n \int_{[0, 1/n]} \varphi \, dm,$$

and, both sides being continuous with respect to uniform convergence, $\int_X f \, d\nu_n = n \int_{[0, 1/n]} f \, dm$ for every $f \in L^\infty([0, 1], m)$.

Let $\{n_k\}$ be any strictly increasing sequence of indices; then $n_k \rightarrow \infty$. Choose a further subsequence $m_1 < m_2 < \dots$ of $\{n_k\}$ with $m_{j+1} \geq 4^j m_j$ for all j . Define f on $[0, 1]$ by

$$f(x) = (-1)^j \quad \text{for } x \in (1/m_{j+1}, 1/m_j], \quad j \geq 1, \quad f(x) = 0 \quad \text{for } x \in \{0\} \cup (1/m_1, 1].$$

The intervals $(1/m_{j+1}, 1/m_j]$ are disjoint with union $(0, 1/m_1]$, so f is a well-defined measurable function with $|f| \leq 1$; thus $f \in L^\infty([0, 1], m) \subseteq L^1([0, 1], m)$. For each j ,

$$\int_{[0, 1/m_j]} f dm = \sum_{i \geq j} (-1)^i \left(\frac{1}{m_i} - \frac{1}{m_{i+1}} \right),$$

so

$$\int_X f d\nu_{m_j} = m_j \int_{[0, 1/m_j]} f dm = (-1)^j \left(1 - \frac{m_j}{m_{j+1}} \right) + R_j, \quad |R_j| \leq m_j \sum_{i \geq j+1} \frac{1}{m_i}.$$

Since $m_{i+1} \geq 4m_i$ for every i , we have $\sum_{i \geq j+1} 1/m_i \leq (4/3)/m_{j+1}$, so $|R_j| \leq (4/3)m_j/m_{j+1} \leq (4/3)4^{-j}$; and $m_j/m_{j+1} \leq 4^{-j}$. Hence

$$\int_X f d\nu_{m_j} = (-1)^j + O(4^{-j}),$$

so the numbers $\int_X f d\nu_{m_j}$ have $+1$ and -1 as subsequential limits and do not converge. Since $\{m_j\}$ is a subsequence of $\{n_k\}$, the sequence $\{\int_X f d\nu_{n_k}\}$ does not converge either. As $\{n_k\}$ was an arbitrary subsequence, there is no subsequence of $\{\nu_n\}$ for which $(\#)$ holds, for any ν whatsoever. (The same f lies in $L^1([0, 1], m)$, so the literal statement fails as well.)

What is true. Two correct substitutes are available.

(i) *Separable families of test functions.* Let $\{\nu_n\}$ be a bounded sequence in $\mathcal{BFA}([a, b], \mathcal{L}, m)$, say $\|\nu_n\|_{\text{var}} \leq C$ for all n , and let Y be any separable linear subspace of $L^\infty([a, b], m)$. Then there is a subsequence $\{\nu_{n_k}\}$ and a $\nu \in \mathcal{BFA}([a, b], \mathcal{L}, m)$ with $\|\nu\|_{\text{var}} \leq C$ for which

$$\lim_{k \rightarrow \infty} \int_X f d\nu_{n_k} = \int_X f d\nu \quad \text{for all } f \in Y.$$

Indeed, let $\{f_j\}_{j=1}^\infty$ be dense in Y . By inequality (18) of the section, $|\int_X f_j d\nu_n| \leq C\|f_j\|_\infty$ for all n , so for each j the real sequence $\{\int_X f_j d\nu_n\}_n$ is bounded. By the Bolzano-Weierstrass Theorem and a diagonal argument, there is a subsequence $\{\nu_{n_k}\}$ for which $L(f_j) := \lim_k \int_X f_j d\nu_{n_k}$ exists for every j . For arbitrary $f \in Y$ and $\varepsilon > 0$, choose j with $\|f - f_j\|_\infty < \varepsilon/(3C + 1)$; then for all k, l ,

$$\left| \int_X f d\nu_{n_k} - \int_X f d\nu_{n_l} \right| \leq 2C\|f - f_j\|_\infty + \left| \int_X f_j d\nu_{n_k} - \int_X f_j d\nu_{n_l} \right|,$$

which is less than ε for k, l large. Hence $L(f) := \lim_k \int_X f d\nu_{n_k}$ exists for every $f \in Y$. The functional L is linear, being a pointwise limit of linear functionals, and $|L(f)| \leq C\|f\|_\infty$. By the Hahn-Banach Theorem, L extends to a bounded linear functional S on $L^\infty([a, b], m)$ with $\|S\| \leq C$. By the Kantorovitch Representation Theorem there is a $\nu \in \mathcal{BFA}([a, b], \mathcal{L}, m)$ with $\|\nu\|_{\text{var}} = \|S\| \leq C$ and $S(f) = \int_X f d\nu$ for all $f \in L^\infty([a, b], m)$. For $f \in Y$ this gives $\lim_k \int_X f d\nu_{n_k} = L(f) = \int_X f d\nu$, as required. (One may take $Y = C[a, b]$, or the closed linear span of $\{\chi_E\}$ for E in any countable subalgebra of $\mathcal{L}$.)

(ii) *Uniformly integrable densities.* If each ν_n is countably additive, so that by Problem 10 it has a density $h_n \in L^1([a, b], m)$, and if $\{h_n\}$ is uniformly integrable (which is automatic if, for instance, $\{h_n\}$ is bounded in $L^p([a, b], m)$ for some $p > 1$), then the Dunford-Pettis Theorem of Section 19.5 provides a subsequence $\{h_{n_k}\}$ converging weakly in $L^1([a, b], m)$ to some $h \in L^1([a, b], m)$; setting $\nu(E) = \int_E h dm$ defines a countably additive element of $\mathcal{BFA}([a, b], \mathcal{L}, m)$, and weak convergence in L^1 means exactly that $\int_X f h_{n_k} dm \rightarrow \int_X f h dm$ for all $f \in L^\infty([a, b], m)$, that is, $(\#)$ holds. The counterexample above shows that the uniform integrability cannot be dropped: there $h_n = n\chi_{[0, 1/n]}$ is bounded in L^1 but not uniformly integrable.

Problem 19.12. [In this problem $(X, \mathcal{M}, \mu)$ is a complete measure space.] Let $\{\mu_n\}$ be a sequence of measures on the Lebesgue measurable space $([a, b], \mathcal{L})$ for which $\{\mu_n([a, b])\}$ is bounded and each μ_n is absolutely continuous with respect to Lebesgue measure m . Show that a subsequence of $\{\mu_n\}$

converges setwise on $\mathcal{L}$ to a measure on $([a, b], \mathcal{L})$ that is absolutely continuous with respect to m .

Solution. Recall that $\{\mu_n\}$ converges setwise on $\mathcal{L}$ to ν means $\lim_n \mu_n(E) = \nu(E)$ for every $E \in \mathcal{L}$. The assertion as stated is false. Take $[a, b] = [0, 1]$ and, for each natural number n , define the measure

$$\mu_n(E) = \int_E n \chi_{[0, 1/n]} dm = n \cdot m(E \cap [0, 1/n]) \quad \text{for } E \in \mathcal{L}.$$

Each μ_n is a measure on $([0, 1], \mathcal{L})$, being the indefinite integral of a nonnegative integrable function (Chapter 18); it is absolutely continuous with respect to m , since $m(E) = 0$ forces $m(E \cap [0, 1/n]) = 0$; and $\mu_n([0, 1]) = n \cdot (1/n) = 1$, so $\{\mu_n([0, 1])\}$ is bounded. Thus $\{\mu_n\}$ satisfies all the hypotheses.

Suppose, to get a contradiction, that some subsequence $\{\mu_{n_k}\}$ converged setwise on $\mathcal{L}$ to a measure ν on $([0, 1], \mathcal{L})$. Since $\{n_k\}$ is strictly increasing, $n_k \rightarrow \infty$. Compute two kinds of limits.

First, $\mu_{n_k}(\{0\}) = n_k \cdot m(\{0\}) = 0$ for every k , so

$$\nu(\{0\}) = \lim_{k \rightarrow \infty} \mu_{n_k}(\{0\}) = 0.$$

Second, fix $c \in (0, 1]$. For all k with $n_k \geq 1/c$ we have $[0, 1/n_k] \subseteq [0, c]$, so $\mu_{n_k}([0, c]) = n_k \cdot (1/n_k) = 1$. Hence

$$\nu([0, c]) = \lim_{k \rightarrow \infty} \mu_{n_k}([0, c]) = 1 \quad \text{for every } c \in (0, 1].$$

In particular $\nu([0, 1]) = 1 < \infty$, so ν is a finite measure. The sets $[0, 1/j]$, $j = 1, 2, \dots$, form a descending sequence of measurable sets of finite ν -measure whose intersection is $\{0\}$, so by the continuity of the measure ν ,

$$\nu(\{0\}) = \lim_{j \rightarrow \infty} \nu([0, 1/j]) = 1.$$

This contradicts $\nu(\{0\}) = 0$. Therefore no subsequence of $\{\mu_n\}$ converges setwise on $\mathcal{L}$ to a measure, let alone to one absolutely continuous with respect to m .

(The failure is exactly the failure of uniform absolute continuity: with $E_n = [0, 1/n]$ we have $m(E_n) \rightarrow 0$ while $\mu_n(E_n) = 1$ for all n . Notice also that the second half of the assertion is not the difficulty: if a sequence of finite measures on $([a, b], \mathcal{L})$, each absolutely continuous with respect to m and with $\{\mu_n([a, b])\}$ bounded, *does* converge setwise on $\mathcal{L}$, then the Vitali-Hahn-Saks Theorem of Section 18.5 guarantees that the limit is a finite measure absolutely continuous with respect to m . The difficulty is producing setwise convergence at all, and boundedness of the total masses does not suffice.)

The corrected statement, with proof. Let $\{\mu_n\}$ be a sequence of measures on $([a, b], \mathcal{L})$, each absolutely continuous with respect to m , such that $\{\mu_n([a, b])\}$ is bounded and $\{\mu_n\}$ is uniformly absolutely continuous with respect to m , that is: for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $E \in \mathcal{L}$ and $m(E) < \delta$, then $\mu_n(E) < \varepsilon$ for all n . Then a subsequence of $\{\mu_n\}$ converges setwise on $\mathcal{L}$ to a measure on $([a, b], \mathcal{L})$ that is absolutely continuous with respect to m .

To prove this, first note that $([a, b], \mathcal{L}, m)$ is a finite, hence σ -finite, measure space, and each μ_n is a finite measure absolutely continuous with respect to m . By the Radon-Nikodym Theorem (Chapter 18) there is, for each n , a nonnegative measurable function h_n on $[a, b]$ with

$$\mu_n(E) = \int_E h_n dm \quad \text{for all } E \in \mathcal{L}.$$

Taking $E = [a, b]$ shows $\|h_n\|_1 = \int_{[a, b]} h_n dm = \mu_n([a, b]) \leq C$ for some constant C , so $\{h_n\}$ is a bounded sequence in $L^1([a, b], m)$; and the hypothesis of uniform absolute continuity says precisely that $\{h_n\}$ is uniformly integrable over $[a, b]$, since $\int_E |h_n| dm = \mu_n(E)$.

Since $m([a, b]) < \infty$, the Dunford-Pettis Theorem of Section 19.5 applies: there is a subsequence $\{h_{n_k}\}$ and a function $h \in L^1([a, b], m)$ such that $\{h_{n_k}\}$ converges weakly in $L^1([a, b], m)$ to h , that is,

$$\lim_{k \rightarrow \infty} \int_{[a, b]} h_{n_k} g dm = \int_{[a, b]} h g dm \quad \text{for all } g \in L^\infty([a, b], m).$$

Fix $E \in \mathcal{L}$ and take $g = \chi_E$, which belongs to $L^\infty([a, b], m)$. Then

$$\lim_{k \rightarrow \infty} \mu_{n_k}(E) = \lim_{k \rightarrow \infty} \int_{[a, b]} h_{n_k} \chi_E dm = \int_E h dm.$$

Define $\nu(E) = \int_E h dm$ for $E \in \mathcal{L}$; the preceding line says that $\{\mu_{n_k}\}$ converges setwise on $\mathcal{L}$ to ν .

It remains to check that ν is a measure absolutely continuous with respect to m . First, $h \geq 0$ a.e. on $[a, b]$: put $E_0 = \{x \in [a, b] : h(x) < 0\}$; then $\int_{E_0} h dm = \lim_k \mu_{n_k}(E_0) \geq 0$, while $h < 0$ on E_0 , and an integrable function that is negative on a set of positive measure has negative integral over that set; hence $m(E_0) = 0$. Since h is a nonnegative integrable function, its indefinite integral $E \mapsto \int_E h dm$ is a finite measure on $\mathcal{L}$ (Chapter 18: countable additivity of the indefinite integral, a consequence of the Countable Additivity of Integration). Finally, if $E \in \mathcal{L}$ has $m(E) = 0$, then $\int_E h dm = 0$, so $\nu(E) = 0$ and $\nu \ll m$. This proves the corrected statement.

Remark. Uniform absolute continuity is automatic under either of the following stronger hypotheses, each of which therefore also repairs the problem: the densities h_n are bounded in $L^p([a, b], m)$ for some $p > 1$ (then, by Hölder's Inequality, $\int_E h_n dm \leq \|h_n\|_p m(E)^{1/q}$), or the densities are dominated by a single integrable function.

Problem 19.13. A linear functional $S: L^p(X, \mu) \rightarrow \mathbb{R}$ is said to be positive provided $S(g) \geq 0$ for each nonnegative function g in $L^p(X, \mu)$. For $1 \leq p < \infty$ and μ σ -finite, show that each bounded linear functional on $L^p(X, \mu)$ is the difference of bounded positive linear functionals.

Solution. Let q be the conjugate of p , so $1 < q \leq \infty$. Let S be a bounded linear functional on $L^p(X, \mu)$. Since $(X, \mathcal{M}, \mu)$ is σ -finite and $1 \leq p < \infty$, the Riesz Representation Theorem for the Dual of $L^p(X, \mu)$ tells us that $T: L^q(X, \mu) \rightarrow (L^p(X, \mu))^*$ is an isometric isomorphism onto. Hence there is a function $f \in L^q(X, \mu)$ with $S = T_f$, that is,

$$S(g) = \int_X f \cdot g d\mu \quad \text{for all } g \in L^p(X, \mu), \quad \|S\| = \|f\|_q.$$

Choose a representative of f , again denoted f , that is measurable and finite a.e. on X , and write its positive and negative parts

$$f^+ = \max\{f, 0\}, \quad f^- = \max\{-f, 0\},$$

so that $f = f^+ - f^-$ and $|f| = f^+ + f^-$ pointwise a.e. on X . Both f^+ and f^- are measurable and satisfy $0 \leq f^\pm \leq |f|$ a.e. on X . Consequently, if $q < \infty$ then $(f^\pm)^q \leq |f|^q$ a.e. on X and the integral comparison test gives $f^\pm \in L^q(X, \mu)$ with $\|f^\pm\|_q \leq \|f\|_q$; and if $q = \infty$ then $\|f\|_\infty$ is an essential upper bound for $f^\pm$ as well, so again $f^\pm \in L^\infty(X, \mu)$ with $\|f^\pm\|_\infty \leq \|f\|_\infty$. In either case $f^+, f^- \in L^q(X, \mu)$. Now define

$$S_1 = T_{f^+}, \quad S_2 = T_{f^-},$$

that is,

$$S_1(g) = \int_X f^+ \cdot g d\mu, \quad S_2(g) = \int_X f^- \cdot g d\mu \quad \text{for } g \in L^p(X, \mu).$$

By Hölder's Inequality (Theorem 1), for each $g \in L^p(X, \mu)$ the products $f^+ \cdot g$ and $f^- \cdot g$ are integrable over X , so S_1 and S_2 are properly defined; they are linear by the linearity of integration, and bounded with

$$\|S_1\| = \|f^+\|_q \leq \|f\|_q = \|S\|, \quad \|S_2\| = \|f^-\|_q \leq \|f\|_q = \|S\|,$$

these norms being computed by the isometry assertion of the Riesz Representation Theorem.

S_1 is positive: if $g \in L^p(X, \mu)$ is nonnegative, then $f^+ \cdot g \geq 0$ a.e. on X , and the monotonicity of integration gives $S_1(g) = \int_X f^+ \cdot g d\mu \geq 0$. The same argument shows S_2 is positive.

Finally, for every $g \in L^p(X, \mu)$, the functions $f^+ \cdot g$ and $f^- \cdot g$ are each integrable over X , so by the linearity of integration,

$$S(g) = \int_X f \cdot g d\mu = \int_X (f^+ - f^-) \cdot g d\mu = \int_X f^+ \cdot g d\mu - \int_X f^- \cdot g d\mu = S_1(g) - S_2(g).$$

Hence $S = S_1 - S_2$ is the difference of two bounded positive linear functionals on $L^p(X, \mu)$.

Remark. Under the Riesz correspondence this decomposition is nothing but the splitting $f = f^+ - f^-$ of the representing function into its positive and negative parts. The hypotheses that μ be σ -finite and that $p < \infty$ enter only through the Riesz Representation Theorem, which supplies the representing function f ; for $p = \infty$ one would have to argue instead from the Kantorovitch Representation Theorem (Theorem 7), decomposing the representing element ν of $\mathcal{BFA}(X, \mathcal{M}, \mu)$ as $\nu = |\nu| - (|\nu| - \nu)$ in the manner indicated in the Remark following the definition of the total variation.

Problem 19.14. Prove the Radon-Riesz Theorem, and the Banach-Saks Theorem in the case $p = 2$.

Solution. Throughout, $(X, \mathcal{M}, \mu)$ is a σ -finite measure space and $p = 2$, so that the conjugate index is $q = 2$ as well. The two theorems to be proved read as follows.

The Radon-Riesz Theorem ($p = 2$). Let $\{f_n\}$ be a sequence in $L^2(X, \mu)$ that converges weakly in $L^2(X, \mu)$ to f . Then $\{f_n\}$ converges strongly in $L^2(X, \mu)$ to f if and only if $\lim_{n \rightarrow \infty} \|f_n\|_2 = \|f\|_2$.

The Banach-Saks Theorem ($p = 2$). Let $\{f_n\}$ be a sequence in $L^2(X, \mu)$ that converges weakly in $L^2(X, \mu)$ to f . Then there is a subsequence $\{f_{n_k}\}$ for which the sequence of Cesàro means converges strongly in $L^2(X, \mu)$ to f , that is,

$$\lim_{K \rightarrow \infty} \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_K}}{K} = f \quad \text{strongly in } L^2(X, \mu).$$

The inner product on L^2 . For $u, v \in L^2(X, \mu)$, the Cauchy-Schwarz Inequality (Section 19.1) tells us that $u \cdot v$ is integrable over X , so we may define

$$\langle u, v \rangle = \int_X u v \, d\mu.$$

This is bilinear, symmetric, and $\langle u, u \rangle = \int_X u^2 \, d\mu = \|u\|_2^2$. Consequently, for $u, v \in L^2(X, \mu)$,

$$\|u - v\|_2^2 = \|u\|_2^2 - 2\langle u, v \rangle + \|v\|_2^2. \quad (1)$$

Moreover, since $q = 2$, the Riesz Representation Theorem for the Dual of $L^2(X, \mu)$ identifies $(L^2(X, \mu))^*$ with $L^2(X, \mu)$ via $g \mapsto T_g$, $T_g(u) = \int_X u g \, d\mu$; therefore

$$\{f_n\} \rightharpoonup f \text{ weakly in } L^2(X, \mu) \iff \lim_{n \rightarrow \infty} \langle f_n, g \rangle = \langle f, g \rangle \text{ for every } g \in L^2(X, \mu). \quad (2)$$

Proof of the Radon-Riesz Theorem for $p = 2$. Suppose first that $\{f_n\}$ converges strongly in $L^2(X, \mu)$ to f . By Minkowski's Inequality (the triangle inequality for $\|\cdot\|_2$),

$$|\|f_n\|_2 - \|f\|_2| \leq \|f_n - f\|_2 \longrightarrow 0,$$

so $\lim_n \|f_n\|_2 = \|f\|_2$. (This direction does not use weak convergence.)

Conversely, suppose $\lim_n \|f_n\|_2 = \|f\|_2$. Since f itself belongs to $L^2(X, \mu)$, we may take $g = f$ in (2) to get

$$\lim_{n \rightarrow \infty} \langle f_n, f \rangle = \langle f, f \rangle = \|f\|_2^2.$$

Therefore, by (1) with $u = f_n$ and $v = f$,

$$\lim_{n \rightarrow \infty} \|f_n - f\|_2^2 = \lim_{n \rightarrow \infty} [\|f_n\|_2^2 - 2\langle f_n, f \rangle + \|f\|_2^2] = \|f\|_2^2 - 2\|f\|_2^2 + \|f\|_2^2 = 0.$$

Hence $\{f_n\}$ converges strongly in $L^2(X, \mu)$ to f . This proves the Radon-Riesz Theorem for $p = 2$.

Proof of the Banach-Saks Theorem for $p = 2$. Define $g_n = f_n - f$ for each n . By (2) and the linearity of the inner product in its first argument, for every $h \in L^2(X, \mu)$,

$$\lim_{n \rightarrow \infty} \langle g_n, h \rangle = \lim_{n \rightarrow \infty} \langle f_n, h \rangle - \langle f, h \rangle = 0,$$

so $\{g_n\}$ converges weakly in $L^2(X, \mu)$ to 0. According to Theorem 12 of Chapter 14, every weakly convergent sequence in a normed linear space is bounded; hence there is an $M \geq 0$ for which

$$\|g_n\|_2 \leq M \quad \text{for all } n. \quad (3)$$

We choose the indices $n_1 < n_2 < \dots$ inductively. Let $n_1 = 1$. Suppose $n_1 < n_2 < \dots < n_k$ have been chosen. For each of the finitely many indices j with $1 \leq j \leq k$, the function g_{n_j} belongs to $L^2(X, \mu)$, so $\lim_{n \rightarrow \infty} \langle g_{n_j}, g_n \rangle = 0$. Since there are only k such conditions, we may choose an index $n_{k+1} > n_k$ for which

$$|\langle g_{n_j}, g_{n_{k+1}} \rangle| \leq \frac{1}{k} \quad \text{for } j = 1, 2, \dots, k. \quad (4)$$

This defines the subsequence $\{f_{n_k}\}$.

Now fix a natural number $K \geq 2$ and expand, using the bilinearity and symmetry of the inner product,

$$\left\| \sum_{k=1}^K g_{n_k} \right\|_2^2 = \sum_{k=1}^K \|g_{n_k}\|_2^2 + 2 \sum_{1 \leq j < k \leq K} \langle g_{n_j}, g_{n_k} \rangle.$$

By (3) the first sum is at most KM^2 . For the second, fix k with $2 \leq k \leq K$; by (4), applied with $k-1$ in place of k , we have $|\langle g_{n_j}, g_{n_k} \rangle| \leq 1/(k-1)$ for every j with $1 \leq j \leq k-1$, and hence

$$\sum_{j=1}^{k-1} |\langle g_{n_j}, g_{n_k} \rangle| \leq (k-1) \cdot \frac{1}{k-1} = 1.$$

Summing over $k = 2, \dots, K$ bounds the double sum in absolute value by $K-1 \leq K$. Therefore

$$\left\| \sum_{k=1}^K g_{n_k} \right\|_2^2 \leq KM^2 + 2K = K(M^2 + 2),$$

and so, by the homogeneity of the norm,

$$\left\| \frac{1}{K} \sum_{k=1}^K g_{n_k} \right\|_2^2 = \frac{1}{K^2} \left\| \sum_{k=1}^K g_{n_k} \right\|_2^2 \leq \frac{M^2 + 2}{K} \rightarrow 0 \quad \text{as } K \rightarrow \infty.$$

Since

$$\frac{1}{K} \sum_{k=1}^K g_{n_k} = \frac{f_{n_1} + f_{n_2} + \dots + f_{n_K}}{K} - f,$$

we conclude that

$$\lim_{K \rightarrow \infty} \left\| \frac{f_{n_1} + f_{n_2} + \dots + f_{n_K}}{K} - f \right\|_2 = 0,$$

that is, the Cesàro means of the subsequence $\{f_{n_k}\}$ converge strongly in $L^2(X, \mu)$ to f . This proves the Banach-Saks Theorem for $p = 2$.

Remark. Corollary 9 for $p = 2$ follows at once from the Radon-Riesz Theorem: if $\{f_n\} \rightharpoonup f$ and $\|f\|_2 = \liminf_n \|f_n\|_2$, choose a subsequence along which $\|f_{n_k}\|_2 \rightarrow \|f\|_2$; that subsequence still converges weakly to f , so by the Radon-Riesz Theorem it converges strongly to f . Conversely, if some subsequence converges strongly to f , then $\|f_{n_k}\|_2 \rightarrow \|f\|_2$, while inequality (12) of Chapter 14 gives $\|f\|_2 \leq \liminf_n \|f_n\|_2 \leq \liminf_k \|f_{n_k}\|_2 = \|f\|_2$.

19.3 Problems 19.15–19.21

Problem 19.15. Let X be the subspace of $L^\infty(\mathbb{R}, m)$, where m is Lebesgue measure, consisting of the continuous functions f that have a finite limit as $x \rightarrow \infty$. For $f \in X$, define $S(f) = \lim_{x \rightarrow \infty} f(x)$. Use the Hahn-Banach Theorem to extend S to a bounded linear functional on $L^\infty(\mathbb{R}, m)$. Show that there is not a function f in $L^1([a, b], m)$ such that

$$S(g) = \int_{\mathbb{R}} f \cdot g \, dm \quad \text{for all } g \in L^\infty(\mathbb{R}, m).$$

Does this contradict the Riesz Representation Theorem?

Solution. Throughout, an element of $L^\infty(\mathbb{R}, m)$ is called continuous provided it has a continuous representative, and we always work with that representative. Observe first that for a continuous bounded h on $\mathbb{R}$ the essential supremum of $|h|$ agrees with its supremum: if $|h(x_0)| > c$ for some x_0 , then by continuity $|h| > c$ on an open interval about x_0 , a set of positive Lebesgue measure, so $\|h\|_\infty \geq c$. Hence

$$\|h\|_\infty = \sup_{x \in \mathbb{R}} |h(x)| \quad \text{for continuous } h \in L^\infty(\mathbb{R}, m).$$

Boundedness of S and the extension. The set X is a linear subspace of $L^\infty(\mathbb{R}, m)$, since sums and scalar multiples of continuous functions possessing finite limits at ∞ again are such, and the limit operation S is linear on X . For $f \in X$,

$$|S(f)| = \left| \lim_{x \rightarrow \infty} f(x) \right| \leq \sup_{x \in \mathbb{R}} |f(x)| = \|f\|_\infty,$$

so S is a bounded linear functional on the normed linear space X with $\|S\| \leq 1$; taking $f \equiv 1$, which lies in X and has $\|f\|_\infty = 1 = S(f)$, we get $\|S\| = 1$. By the Hahn-Banach Theorem there is a linear functional, again denoted S , defined on all of $L^\infty(\mathbb{R}, m)$, which agrees with the above on X and satisfies $|S(g)| \leq \|g\|_\infty$ for all $g \in L^\infty(\mathbb{R}, m)$. Thus S is a bounded linear functional on $L^\infty(\mathbb{R}, m)$ of norm 1. No L^1 function represents S . We show more than is asked: there is no $f \in L^1(\mathbb{R}, m)$ at all satisfying the representation formula

$$S(g) = \int_{\mathbb{R}} f \cdot g \, dm \quad \text{for all } g \in L^\infty(\mathbb{R}, m).$$

(In particular there is no such f belonging to $L^1([a, b], m)$, a function which we regard as extended by 0 to all of $\mathbb{R}$ so that the integral in the statement makes sense; the interval $[a, b]$ in the printed statement is evidently a misprint for $\mathbb{R}$.)

Assume such an $f \in L^1(\mathbb{R}, m)$ exists. For each natural number n define $\varphi_n: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi_n(x) = \begin{cases} 0 & \text{if } x \leq n \\ x - n & \text{if } n < x < n + 1 \\ 1 & \text{if } x \geq n + 1. \end{cases}$$

Each φ_n is continuous, $0 \leq \varphi_n \leq 1$, so $\varphi_n \in L^\infty(\mathbb{R}, m)$, and $\lim_{x \rightarrow \infty} \varphi_n(x) = 1$. Hence $\varphi_n \in X$ and $S(\varphi_n) = 1$ for every n . On the other hand $|f \cdot \varphi_n| \leq |f|$ on $\mathbb{R}$, with $|f| \in L^1(\mathbb{R}, m)$, and for each fixed x we have $\varphi_n(x) = 0$ once $n \geq x$, so that $f \cdot \varphi_n \rightarrow 0$ pointwise on $\mathbb{R}$. The Lebesgue Dominated Convergence Theorem therefore gives

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f \cdot \varphi_n \, dm = 0.$$

But by the assumed representation formula, $\int_{\mathbb{R}} f \cdot \varphi_n \, dm = S(\varphi_n) = 1$ for every n . This contradiction shows no such f exists.

It is worth recording what the argument reveals about the shape of the obstruction. If g is continuous and has compact support, then $g \in X$ and $S(g) = 0$, so the representation formula would force

$\int_{\mathbb{R}} f \cdot g \, dm = 0$ for every continuous g of compact support; since such functions are dense in L^1 of each bounded interval, this forces $f = 0$ a.e., whereas S is not the zero functional because $S(1) = 1$. The functional S is supported at infinity, and no absolutely continuous set function can detect it.

Is the Riesz Representation Theorem contradicted? No. The Riesz Representation Theorem for the Dual of $L^p(X, \mu)$, proved in Section 19.2, asserts that if $(X, \mathcal{M}, \mu)$ is a σ -finite measure space, $1 \leq p < \infty$, and q is conjugate to p , then the map $T: L^q(X, \mu) \rightarrow (L^p(X, \mu))^*$ defined by $T(f)[g] = \int_X f \cdot g \, d\mu$ is an isometric isomorphism of $L^q(X, \mu)$ onto $(L^p(X, \mu))^*$. The case $p = \infty$ is explicitly excluded, and this exercise concerns exactly that excluded case: the functional S belongs to $(L^\infty(\mathbb{R}, m))^*$, and the theorem makes no assertion whatever about that dual space. What remains true is only half of the statement: the map $T: L^1(\mathbb{R}, m) \rightarrow (L^\infty(\mathbb{R}, m))^*$, $T(f)[g] = \int_{\mathbb{R}} fg \, dm$, is a linear isometry (its being an isometry follows by testing against $g = \operatorname{sgn} f$), but the exercise shows that T is not onto, since S is not in its image.

Finally, note the standard consequence. Since m is σ -finite on $\mathbb{R}$, the Riesz Representation Theorem identifies $(L^1(\mathbb{R}, m))^*$ with $L^\infty(\mathbb{R}, m)$ by means of T , and hence identifies the bidual $(L^1)^{**}$ with $(L^\infty)^*$; under this identification the natural embedding $J: L^1 \rightarrow (L^1)^{**}$ carries f to the functional $g \mapsto \int_{\mathbb{R}} fg \, dm$ on L^∞ . The exercise says precisely that S is not of this form, so J is not onto and $L^1(\mathbb{R}, m)$ is not reflexive. This is why the reflexivity theorem of this chapter is restricted to $1 < p < \infty$.

Problem 19.16. Let μ be the counting measure on the set of natural numbers $\mathbb{N}$.

- (i) For $1 \leq p \leq \infty$, show that $L^p(\mathbb{N}, \mu) = \ell^p$ and thereby characterize the dual space of ℓ^p for $1 \leq p < \infty$.
- (ii) Discuss the dual of $L^p(X, \mu)$ for $1 \leq p < \infty$, where μ is the counting measure on a not necessarily countable set X .

Solution. Here $\mathcal{M}$ is the σ -algebra of all subsets of the underlying set and $\mu(E)$ is the number of points in E (and ∞ if E is infinite). Since every subset is measurable, every real-valued function on the set is measurable.

Part (i). Let $f: \mathbb{N} \rightarrow [0, \infty)$. For each natural number N put $f_N = f \cdot \chi_{\{1, \dots, N\}}$. Then f_N is a simple function, indeed $f_N = \sum_{n=1}^N f(n) \chi_{\{n\}}$, and since $\mu(\{n\}) = 1$ for each n ,

$$\int_{\mathbb{N}} f_N \, d\mu = \sum_{n=1}^N f(n).$$

The sequence $\{f_N\}$ is increasing and converges pointwise on $\mathbb{N}$ to f , so the Monotone Convergence Theorem gives

$$\int_{\mathbb{N}} f \, d\mu = \lim_{N \rightarrow \infty} \sum_{n=1}^N f(n) = \sum_{n=1}^{\infty} f(n).$$

Thus for arbitrary real-valued f on $\mathbb{N}$ and $1 \leq p < \infty$, applying this to $|f|^p$,

$$\int_{\mathbb{N}} |f|^p \, d\mu = \sum_{n=1}^{\infty} |f(n)|^p,$$

so $f \in L^p(\mathbb{N}, \mu)$ if and only if the sequence $\{f(n)\}$ belongs to ℓ^p , and then $\|f\|_p = (\sum_n |f(n)|^p)^{1/p}$ is exactly the ℓ^p norm. Moreover the only set of measure zero is the empty set, so two functions that are equal μ -a.e. are equal, and $L^p(\mathbb{N}, \mu)$ needs no passage to equivalence classes. Hence $L^p(\mathbb{N}, \mu) = \ell^p$, with identical norms, for $1 \leq p < \infty$.

For $p = \infty$: since the only null set is empty, a real number c is an essential upper bound for $|f|$ if and only if it is an upper bound, so

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \sup_{n \in \mathbb{N}} |f(n)|.$$

Therefore $L^\infty(\mathbb{N}, \mu) = \ell^\infty$, the bounded sequences with the supremum norm.

Now μ is σ -finite, since $\mathbb{N} = \bigcup_{n=1}^{\infty} \{n\}$ and $\mu(\{n\}) = 1 < \infty$. Hence for $1 \leq p < \infty$ the Riesz Representation Theorem for the dual of $L^p(X, \mu)$ applies: with q conjugate to p , the map $T: L^q(\mathbb{N}, \mu) \rightarrow (L^p(\mathbb{N}, \mu))^*$ given by $T(a)[f] = \int_{\mathbb{N}} a \cdot f \, d\mu$ is an isometric isomorphism of L^q onto $(L^p)^*$. In view of the identifications just made this says precisely: for $1 \leq p < \infty$, every bounded linear functional S on ℓ^p has the form

$$S(x) = \sum_{n=1}^{\infty} a_n x_n \quad \text{for all } x = \{x_n\} \in \ell^p$$

for a unique $a = \{a_n\}$ belonging to ℓ^q , and $\|S\| = \|a\|_q$. That is,

$$(\ell^p)^* = \ell^q \quad (1 < p < \infty, \, 1/p + 1/q = 1), \quad (\ell^1)^* = \ell^\infty.$$

Concretely $a_n = S(e_n)$, where e_n is the sequence with n th term 1 and all other terms 0: indeed for $x \in \ell^p$ with $p < \infty$ we have $\|x - \sum_{n \leq N} x_n e_n\|_p^p = \sum_{n > N} |x_n|^p \rightarrow 0$, so $x = \sum_n x_n e_n$ with convergence in ℓ^p , and continuity and linearity of S give $S(x) = \sum_n x_n S(e_n)$.

Part (ii). Let X be an arbitrary nonempty set and μ counting measure on the σ -algebra of all subsets of X . If X is uncountable then μ is not σ -finite, so the Riesz Representation Theorem is not directly available. Nevertheless its conclusion persists, for every $1 \leq p < \infty$. The reason is that each function in $L^p(X, \mu)$, $p < \infty$, vanishes off a countable set, so counting measure, although possibly not σ -finite, is σ -finite on the part of X that any single L^p function sees.

First, exactly as in part (i) one has, for nonnegative f on X ,

$$\int_X f \, d\mu = \sup \left\{ \sum_{x \in F} f(x) : F \subseteq X \text{ finite} \right\},$$

where the sum over a finite set means the ordinary finite sum. Indeed, the integral over X of a nonnegative measurable function is by definition the supremum of $\int_X h \, d\mu$ over all bounded measurable h with $0 \leq h \leq f$ that vanish outside a set of finite measure; for counting measure a set has finite measure exactly when it is finite, so these h are exactly the functions $f \cdot \chi_F$ with $F \subseteq X$ finite together with the functions dominated by them, and $\int_X f \cdot \chi_F \, d\mu = \sum_{x \in F} f(x)$ since $f \cdot \chi_F$ is simple. Consequently, writing $\ell^p(X)$ for the set of $f: X \rightarrow \mathbb{R}$ with $\sum_{x \in X} |f(x)|^p < \infty$ (the sum being the above supremum over finite subsets), we get $L^p(X, \mu) = \ell^p(X)$ with $\|f\|_p = (\sum_x |f(x)|^p)^{1/p}$, and again no null sets other than the empty set, so no equivalence classes. Also $L^\infty(X, \mu) = \ell^\infty(X)$, the bounded functions on X with the supremum norm.

Second, if $f \in \ell^p(X)$ with $p < \infty$, then for each natural number k the set $E_k = \{x \in X : |f(x)| > 1/k\}$ is finite, since $\mu(E_k)/k^p \leq \int_X |f|^p \, d\mu < \infty$. Hence $\{x : f(x) \neq 0\} = \bigcup_k E_k$ is countable. Moreover, if this set is infinite and $\{x_1, x_2, \dots\}$ is an enumeration of it, then $\|f - f \cdot \chi_{\{x_1, \dots, x_N\}}\|_p^p = \sum_{n > N} |f(x_n)|^p \rightarrow 0$, this being the tail of a convergent series, while if the set is finite then f itself has finite support. Either way, the functions with finite support are dense in $\ell^p(X)$ for $p < \infty$.

Now let $1 \leq p < \infty$, let q be conjugate to p , and let S be a bounded linear functional on $\ell^p(X)$. Define $a: X \rightarrow \mathbb{R}$ by $a(x) = S(\chi_{\{x\}})$. We claim $a \in \ell^q(X)$ with $\|a\|_q \leq \|S\|$.

Suppose first $1 < p < \infty$. Let $F \subseteq X$ be finite and set $g = \sum_{x \in F} \operatorname{sgn}(a(x)) |a(x)|^{q-1} \chi_{\{x\}}$. Since $(q-1)p = q$,

$$\|g\|_p = \left(\sum_{x \in F} |a(x)|^q \right)^{1/p}, \quad S(g) = \sum_{x \in F} |a(x)|^q.$$

Therefore $\sum_{x \in F} |a(x)|^q \leq \|S\| \cdot (\sum_{x \in F} |a(x)|^q)^{1/p}$, and hence $(\sum_{x \in F} |a(x)|^q)^{1/q} \leq \|S\|$. Taking the supremum over finite $F \subseteq X$ gives $\|a\|_q \leq \|S\|$. If $p = 1$, then for each x , $|a(x)| = |S(\chi_{\{x\}})| \leq \|S\| \|\chi_{\{x\}}\|_1 = \|S\|$, so $a \in \ell^\infty(X)$ with $\|a\|_\infty \leq \|S\|$.

In either case, for $f \in \ell^p(X)$ with finite support we have $S(f) = \sum_{x \in X} a(x)f(x)$ by linearity, and both sides are continuous in f with respect to the ℓ^p norm, the right-hand side because Hölder's Inequality gives $|\sum_x a(x)f(x)| \leq \|a\|_q \|f\|_p$. By the density of the finitely supported functions,

$$S(f) = \sum_{x \in X} a(x)f(x) \quad \text{for all } f \in \ell^p(X),$$

the sum having only countably many nonzero terms and converging absolutely. Hölder's Inequality also gives $\|S\| \leq \|a\|_q$, so $\|S\| = \|a\|_q$, and a is unique, since $a(x)$ is determined as $S(\chi_{\{x\}})$. Conversely each $a \in \ell^q(X)$ defines by this formula a bounded linear functional on $\ell^p(X)$ of norm $\|a\|_q$.

Conclusion: for counting measure on an arbitrary set X and $1 \leq p < \infty$,

$$(L^p(X, \mu))^* = L^q(X, \mu), \quad \text{that is} \quad (\ell^p(X))^* = \ell^q(X),$$

isometrically, where q is conjugate to p ; for $p = 1$ the dual is $\ell^\infty(X)$, all bounded functions on X . Thus the conclusion of the Riesz Representation Theorem holds for counting measure on any set, countable or not, even though the σ -finiteness hypothesis under which it was proved may fail. What does fail for uncountable X is that μ be σ -finite; the hypothesis is needed in the proof, not in the conclusion, and here the countable support of the L^p functions supplies the missing σ -finiteness.

Problem 19.17. Find a measure space $(X, \mathcal{M}, \mu)$ with the property that all the theorems of this section hold in the case $p = 1$.

Solution. Take $X = \{1, 2, \dots, N\}$ a finite set, $\mathcal{M}$ the collection of all subsets of X , and μ the counting measure. More generally, any measure space for which $L^1(X, \mu)$ is finite dimensional will do, and the argument below uses only finite dimensionality. Observe that $\mu(X) = N < \infty$, so μ is σ -finite and the standing hypotheses of the theorems of this section are met; the only hypothesis in question is $p > 1$, which we drop.

As in the preceding problem, $\int_X f d\mu = \sum_{i=1}^N f(i)$ for nonnegative f , the only null set is $\emptyset$, and therefore

$$L^1(X, \mu) = \left\{ f: X \rightarrow \mathbb{R} \right\}, \quad \|f\|_1 = \sum_{i=1}^N |f(i)|.$$

This is an N -dimensional linear space with basis $\chi_{\{1\}}, \dots, \chi_{\{N\}}$, normed by the ℓ^1 norm on the coordinates. Likewise $L^\infty(X, \mu)$ consists of all functions on X with $\|g\|_\infty = \max_i |g(i)|$, and for $f \in L^1$, $g \in L^\infty$ we have $\int_X f \cdot g d\mu = \sum_{i=1}^N f(i)g(i)$.

Two preliminary observations. First, every linear functional ψ on $L^1(X, \mu)$ is bounded, since by linearity $\psi(f) = \sum_{i=1}^N f(i)\psi(\chi_{\{i\}})$ and hence $|\psi(f)| \leq \max_i |\psi(\chi_{\{i\}})| \cdot \|f\|_1$. Thus $(L^1(X, \mu))^*$ is the full algebraic dual of an N -dimensional linear space and so has dimension N ; the same then applies to it, so the bidual $(L^1(X, \mu))^{**}$ also has dimension N .

Second, in $L^1(X, \mu)$ weak convergence and strong convergence coincide. Indeed, strong convergence always implies weak convergence, since $|\psi(f_n) - \psi(f)| \leq \|\psi\| \|f_n - f\|_1$. Conversely, suppose $\{f_n\} \rightarrow f$ weakly. For each i the function $g_i = \chi_{\{i\}}$ belongs to $L^\infty(X, \mu)$, so $f \mapsto \int_X f \cdot g_i d\mu = f(i)$ is a bounded linear functional; hence $f_n(i) \rightarrow f(i)$ for each of the finitely many indices i , and therefore

$$\|f_n - f\|_1 = \sum_{i=1}^N |f_n(i) - f(i)| \rightarrow 0.$$

We now verify each theorem of this section for $p = 1$ on this measure space.

Theorem 8, that $L^1(X, \mu)$ is a reflexive Banach space. It is a Banach space by the Riesz-Fischer Theorem (or because it is finite dimensional). The natural embedding $J: L^1 \rightarrow (L^1)^{**}$ is a linear isometry, by the Hahn-Banach Theorem, and in particular is one-to-one. Since $L^1(X, \mu)$ and $(L^1(X, \mu))^{**}$ both have dimension N , a one-to-one linear map of the first onto a subspace of the second must be onto. Hence J is an isomorphism of $L^1(X, \mu)$ onto its bidual, that is, $L^1(X, \mu)$ is reflexive.

The Riesz Weak Compactness Theorem for $p = 1$. Let $\{f_n\}$ be bounded in $L^1(X, \mu)$, say $\|f_n\|_1 \leq C$ for all n . Then $|f_n(i)| \leq C$ for each i , so by the Bolzano-Weierstrass Theorem, applied successively in the N coordinates, there is a subsequence $\{f_{n_k}\}$ and a function f on X with $f_{n_k}(i) \rightarrow f(i)$ for $i = 1, \dots, N$. Then $\|f_{n_k} - f\|_1 \rightarrow 0$ and hence $\{f_{n_k}\}$ converges weakly to f ; explicitly, for $g \in L^\infty(X, \mu)$,

$$\left| \int_X f_{n_k} \cdot g d\mu - \int_X f \cdot g d\mu \right| \leq \|g\|_\infty \|f_{n_k} - f\|_1 \rightarrow 0.$$

The Radon-Riesz Theorem for $p = 1$. Let $\{f_n\}$ converge weakly in $L^1(X, \mu)$ to f . By the second observation above, $\{f_n\}$ converges strongly to f ; so the left-hand alternative of the asserted equivalence always holds. Its right-hand alternative also always holds, since the norm is continuous: $|\|f_n\|_1 - \|f\|_1| \leq \|f_n - f\|_1 \rightarrow 0$. Two statements both of which are true are equivalent, so the Radon-Riesz Theorem holds here.

Corollary 9 for $p = 1$. Again let $\{f_n\}$ converge weakly, hence strongly, to f . Then $\|f_n\|_1 \rightarrow \|f\|_1$, so $\liminf \|f_n\|_1 = \|f\|_1$, and the whole sequence, in particular a subsequence, converges strongly to f . Both sides of the stated equivalence hold, so the corollary is true.

The Banach-Saks Theorem for $p = 1$. Let $\{f_n\}$ converge weakly in $L^1(X, \mu)$ to f ; then it converges strongly to f , and we claim its Cesàro means already converge strongly to f , so that we may take $\{f_{n_k}\} = \{f_n\}$. Indeed, let $\epsilon > 0$ and choose an index M for which $\|f_n - f\|_1 < \epsilon/2$ for $n \geq M$. Then for $k \geq M$,

$$\left\| \frac{f_1 + \cdots + f_k}{k} - f \right\|_1 \leq \frac{1}{k} \sum_{n=1}^{M-1} \|f_n - f\|_1 + \frac{1}{k} \sum_{n=M}^k \|f_n - f\|_1 < \frac{1}{k} \sum_{n=1}^{M-1} \|f_n - f\|_1 + \frac{\epsilon}{2},$$

and the first term is less than $\epsilon/2$ for all k sufficiently large, since the finite sum is a fixed number. Hence the Cesàro means converge strongly to f .

Thus every theorem of this section is valid for $p = 1$ on this measure space. The same works for the trivial measure space $\mathcal{M} = \{\emptyset, X\}$ with $\mu(X) = 1$, for which $L^1(X, \mu)$ is the one-dimensional space of constant functions, and more generally for any σ -finite measure space for which $L^1(X, \mu)$ is finite dimensional, since finite dimensionality is all that the arguments above used. Some such degeneracy is unavoidable: the following problem exhibits an infinite dimensional L^1 space, that of Lebesgue measure on a closed bounded interval, for which all three of the last-named theorems fail.

Problem 19.18. Show that for Lebesgue measure on a closed, bounded interval $[a, b]$ of real numbers and $p = 1$, neither the Riesz Weak Compactness Theorem, nor the Radon-Riesz Theorem, nor the Banach-Saks Theorem are true.

Solution. Throughout, m is Lebesgue measure, $\ell = b - a > 0$, and we use the Riesz Representation Theorem for $p = 1$: since m is finite on $[a, b]$, a sequence $\{f_n\}$ in $L^1([a, b], m)$ converges weakly to f if and only if

$$\lim_{n \rightarrow \infty} \int_a^b f_n \cdot g \, dm = \int_a^b f \cdot g \, dm \quad \text{for all } g \in L^\infty([a, b], m).$$

Failure of the Riesz Weak Compactness Theorem. For each natural number $n > 1/\ell$ define

$$f_n = n \cdot \chi_{[a, a+1/n]}.$$

Then $f_n \geq 0$ and $\|f_n\|_1 = n \cdot (1/n) = 1$, so $\{f_n\}$ is a bounded sequence in $L^1([a, b], m)$. We claim no subsequence converges weakly.

Suppose $\{f_{n_k}\}$ converged weakly to some $f \in L^1([a, b], m)$. Let g be continuous on $[a, b]$; then $g \in L^\infty([a, b], m)$ and

$$\int_a^b f_{n_k} \cdot g \, dm = n_k \int_a^{a+1/n_k} g \, dm.$$

Given $\epsilon > 0$, continuity of g at a provides $\delta > 0$ with $|g(x) - g(a)| < \epsilon$ for $a \leq x < a + \delta$; hence for $1/n_k < \delta$,

$$\left| n_k \int_a^{a+1/n_k} g \, dm - g(a) \right| = \left| n_k \int_a^{a+1/n_k} [g - g(a)] \, dm \right| \leq n_k \cdot \frac{1}{n_k} \cdot \epsilon = \epsilon.$$

Therefore $\int_a^b f_{n_k} \cdot g \, dm \rightarrow g(a)$, and so

$$\int_a^b f \cdot g \, dm = g(a) \quad \text{for every continuous } g \text{ on } [a, b].$$

Now for each natural number j with $1/j < \ell$ let g_j be the continuous function on $[a, b]$ that equals 1 at a , equals 0 on $[a + 1/j, b]$, and is linear on $[a, a + 1/j]$; thus $0 \leq g_j \leq 1$ and $g_j(a) = 1$. The displayed identity gives $\int_a^b f \cdot g_j \, dm = 1$ for every such j . But $|f \cdot g_j| \leq |f| \in L^1([a, b], m)$ and $f(x)g_j(x) \rightarrow 0$ for every $x \in (a, b]$, hence a.e. on $[a, b]$, so the Lebesgue Dominated Convergence Theorem gives $\int_a^b f \cdot g_j \, dm \rightarrow 0$. This contradiction proves that $\{f_n\}$ has no weakly convergent subsequence, so the Riesz Weak Compactness Theorem fails for $p = 1$.

(The failure is exactly the one predicted by the Dunford-Pettis Theorem: $\{f_n\}$ is bounded in L^1 but not uniformly integrable, since $m([a, a + 1/n]) \rightarrow 0$ while $\int_{[a, a+1/n]} f_n \, dm = 1$ for every n .)

Failure of the Radon-Riesz Theorem. For a natural number n partition $[a, b]$ into the $2n$ intervals

$$I_{n,j} = \left[a + \frac{(j-1)\ell}{2n}, a + \frac{j\ell}{2n} \right), \quad j = 1, \dots, 2n$$

(the last one taken closed at b), and define the square wave r_n on $[a, b]$ by $r_n = 1$ on $I_{n,j}$ for j odd and $r_n = -1$ on $I_{n,j}$ for j even. Thus $|r_n| \equiv 1$ on $[a, b]$. Group the intervals in consecutive pairs, $P_{n,k} = I_{n,2k-1} \cup I_{n,2k}$ for $k = 1, \dots, n$; each $P_{n,k}$ has length ℓ/n and $\int_{P_{n,k}} r_n \, dm = 0$. Consequently, for any subinterval $[c, d] \subseteq [a, b]$, if K denotes the set of indices k with $P_{n,k} \subseteq [c, d]$, then $[c, d] \setminus \bigcup_{k \in K} P_{n,k}$ is contained in the union of at most two of the sets $P_{n,k}$, so it has measure at most $2\ell/n$, and therefore

$$\left| \int_c^d r_n \, dm \right| = \left| \int_{[c,d] \setminus \bigcup_{k \in K} P_{n,k}} r_n \, dm \right| \leq \frac{2\ell}{n}.$$

It follows that $\int_a^b \varphi \cdot r_n \, dm \rightarrow 0$ for every step function φ on $[a, b]$, since such a φ is a finite sum $\sum_{i=1}^M c_i \chi_{J_i}$ with the J_i intervals, whence $|\int_a^b \varphi \cdot r_n \, dm| \leq (2\ell/n) \sum_i |c_i|$.

Now let $g \in L^\infty([a, b], m)$; since $m([a, b]) < \infty$, g also belongs to $L^1([a, b], m)$. Let $\epsilon > 0$. By Proposition 10 of Chapter 7 the step functions are dense in $L^1([a, b], m)$, so choose a step function $\varphi = \sum_{i=1}^M c_i \chi_{J_i}$ with $\|g - \varphi\|_1 < \epsilon$. Since $|r_n| \equiv 1$,

$$\left| \int_a^b g \cdot r_n \, dm \right| \leq \int_a^b |g - \varphi| \, dm + \left| \int_a^b \varphi \cdot r_n \, dm \right| < \epsilon + \frac{2\ell}{n} \sum_i |c_i|,$$

and the last term tends to 0 as $n \rightarrow \infty$. As $\epsilon > 0$ was arbitrary,

$$\lim_{n \rightarrow \infty} \int_a^b g \cdot r_n \, dm = 0 \quad \text{for all } g \in L^\infty([a, b], m).$$

Define

$$f_n = \frac{1 + r_n}{\ell}, \quad f = \frac{1}{\ell}.$$

Each f_n is nonnegative, since $r_n \geq -1$, and $\int_a^b r_n \, dm = 0$, so $\|f_n\|_1 = \frac{1}{\ell} \int_a^b (1 + r_n) \, dm = 1 = \|f\|_1$. By the limit just proved, for every $g \in L^\infty([a, b], m)$,

$$\int_a^b f_n \cdot g \, dm = \frac{1}{\ell} \int_a^b g \, dm + \frac{1}{\ell} \int_a^b r_n \cdot g \, dm \longrightarrow \frac{1}{\ell} \int_a^b g \, dm = \int_a^b f \cdot g \, dm,$$

that is, $\{f_n\}$ converges weakly in $L^1([a, b], m)$ to f , and moreover $\lim_n \|f_n\|_1 = \|f\|_1$. Nevertheless

$$\|f_n - f\|_1 = \frac{1}{\ell} \int_a^b |r_n| \, dm = 1 \quad \text{for every } n,$$

so $\{f_n\}$ does not converge strongly to f . Hence the Radon-Riesz Theorem fails for $p = 1$. The same sequence refutes Corollary 9 for $p = 1$: here $\|f\|_1 = 1 = \liminf_n \|f_n\|_1$, yet no subsequence converges strongly to f , since $\|f_{n_k} - f\|_1 = 1$ for every k .

Failure of the Banach-Saks Theorem. The Banach-Saks Theorem is used in conjunction with the Riesz Weak Compactness Theorem, and its content for a σ -finite measure space and $1 < p < \infty$ is that every bounded sequence in L^p has a subsequence whose sequence of Cesàro means converges strongly. For

$p = 1$ and Lebesgue measure on $[a, b]$ this fails, and the sequence $f_n = n\chi_{[a, a+1/n]}$ of the first part is again a counterexample.

Indeed, let $\{f_{n_k}\}$ be any subsequence and put

$$\sigma_k = \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_k}}{k}.$$

Each σ_k is nonnegative and $\|\sigma_k\|_1 = \frac{1}{k} \sum_{j=1}^k \|f_{n_j}\|_1 = 1$. On the other hand, fix $x \in (a, b]$. Since $n_j \rightarrow \infty$, there is an index J with $1/n_j < x - a$, and hence $f_{n_j}(x) = 0$, for all $j \geq J$. Therefore, for $k \geq J$,

$$0 \leq \sigma_k(x) = \frac{1}{k} \sum_{j < J} f_{n_j}(x) \leq \frac{1}{k} \sum_{j < J} n_j \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

so $\sigma_k \rightarrow 0$ pointwise on $(a, b]$, hence a.e. on $[a, b]$. Suppose $\{\sigma_k\}$ converged strongly in $L^1([a, b], m)$ to some h . Then, by the Riesz-Fischer Theorem of Chapter 7, a subsequence of $\{\sigma_k\}$ converges to h pointwise a.e. on $[a, b]$, so $h = 0$ a.e. But strong convergence forces $\|\sigma_k\|_1 \rightarrow \|h\|_1 = 0$, contradicting $\|\sigma_k\|_1 = 1$ for all k . Hence no subsequence of $\{f_n\}$ has strongly convergent Cesàro means, and the Banach-Saks conclusion fails for $p = 1$.

One caveat about the reading of the last assertion. Verbatim, the Banach-Saks Theorem of this section hypothesizes that $\{f_n\}$ already converges weakly, and in $L^1([a, b], m)$ no counterexample with that hypothesis can be produced: a theorem of Szlenk asserts that every weakly convergent sequence in L^1 of a finite measure space has a subsequence whose Cesàro means converge in norm. That result lies far outside the methods available here, and in any event it is not obtainable, as the theorem of this section is, from reflexivity. The assertion of the present problem is therefore to be understood in the classical form refuted above, namely that boundedness in L^1 suffices to produce a subsequence with norm-convergent Cesàro means; equivalently, the route by which the theorem is proved for $1 < p < \infty$, passing through reflexivity and weak sequential compactness, collapses at $p = 1$ at its very first step.

Problem 19.19. For a natural number n , let e_n be the sequence whose n th term is 1 and other terms are zero. For what values of p , $1 \leq p < \infty$, does $\{e_n\}$ converge weakly in ℓ^p ?

Solution. The answer: $\{e_n\}$ converges weakly in ℓ^p precisely for $1 < p < \infty$, and its weak limit is then 0; in ℓ^1 the sequence does not converge weakly.

Recall from Problem 16 that $\ell^p = L^p(\mathbb{N}, \mu)$ for counting measure μ on $\mathbb{N}$, that μ is σ -finite, and hence, by the Riesz Representation Theorem, that for $1 \leq p < \infty$ and q conjugate to p every bounded linear functional on ℓ^p is of the form

$$x = \{x_n\} \mapsto \sum_{n=1}^{\infty} a_n x_n \quad \text{for a unique } a = \{a_n\} \in \ell^q,$$

and conversely each $a \in \ell^q$ defines such a functional. Since $\langle a, e_n \rangle = a_n$, the sequence $\{e_n\}$ converges weakly in ℓ^p to $x \in \ell^p$ if and only if

$$\lim_{n \rightarrow \infty} a_n = \sum_{k=1}^{\infty} a_k x_k \quad \text{for every } a \in \ell^q.$$

Let $1 < p < \infty$, so that $1 < q < \infty$. If $a \in \ell^q$, then $\sum_n |a_n|^q < \infty$ and hence $a_n \rightarrow 0$. Thus for every bounded linear functional the values at e_n tend to 0 $= \langle a, 0 \rangle$, which is to say $\{e_n\}$ converges weakly in ℓ^p to 0.

Let $p = 1$, so $q = \infty$ and the dual of ℓ^1 is ℓ^∞ , all bounded sequences. Consider $a = \{(-1)^n\}$, which belongs to ℓ^∞ with $\|a\|_\infty = 1$. Then $\langle a, e_n \rangle = (-1)^n$, a sequence of real numbers that does not converge. Hence $\{e_n\}$ fails to converge weakly in ℓ^1 ; indeed no subsequence $\{e_{n_k}\}$ converges weakly either, because one may choose $a \in \ell^\infty$ with $a_{n_k} = (-1)^k$, and then $\langle a, e_{n_k} \rangle = (-1)^k$ diverges.

Three remarks. First, for no p with $1 \leq p < \infty$ does $\{e_n\}$ converge strongly, since $\|e_n - e_k\|_p = 2^{1/p}$ for $n \neq k$; so for $1 < p < \infty$ the sequence $\{e_n\}$ exhibits weak convergence that is not strong convergence.

Second, this is consistent with, and illustrates, the Radon-Riesz Theorem: $\{e_n\}$ converges weakly to 0 in ℓ^p for $1 < p < \infty$, but $\|e_n\|_p = 1$ does not converge to $\|0\|_p = 0$, so the Radon-Riesz criterion for strong convergence is not met. Third, the failure at $p = 1$ is a concrete instance of the failure of the Riesz Weak Compactness Theorem for $p = 1$: $\{e_n\}$ is a bounded sequence in ℓ^1 , of norm one, with no weakly convergent subsequence, so ℓ^1 is not reflexive.

Problem 19.20. Find a bounded sequence in $L^1([a, b], m)$, where m is Lebesgue measure, which fails to have a weakly convergent subsequence.

Solution. Choose a natural number N with $1/N < b - a$ and, for $n \geq N$, define

$$f_n = n \cdot \chi_{[a, a+1/n]}.$$

Each f_n is nonnegative and measurable with $\|f_n\|_1 = n \cdot m([a, a+1/n]) = 1$, so $\{f_n\}_{n \geq N}$ is a bounded sequence in $L^1([a, b], m)$, of norm one. We claim it has no weakly convergent subsequence. Since $m([a, b]) < \infty$, the Riesz Representation Theorem identifies the dual of $L^1([a, b], m)$ with $L^\infty([a, b], m)$, so weak convergence of $\{f_{n_k}\}$ to f in L^1 means

$$\lim_{k \rightarrow \infty} \int_a^b f_{n_k} \cdot g \, dm = \int_a^b f \cdot g \, dm \quad \text{for all } g \in L^\infty([a, b], m).$$

Suppose some subsequence $\{f_{n_k}\}$ converged weakly to $f \in L^1([a, b], m)$. Let g be continuous on $[a, b]$; then $g \in L^\infty([a, b], m)$, and we first compute the limit of $\int_a^b f_{n_k} \cdot g \, dm = n_k \int_a^{a+1/n_k} g \, dm$. Let $\epsilon > 0$. By continuity of g at a there is $\delta > 0$ such that $|g(x) - g(a)| < \epsilon$ whenever $a \leq x \leq a + \delta$. Hence for all k with $1/n_k < \delta$,

$$\left| n_k \int_a^{a+1/n_k} g \, dm - g(a) \right| = n_k \left| \int_a^{a+1/n_k} [g - g(a)] \, dm \right| \leq n_k \cdot \frac{1}{n_k} \cdot \epsilon = \epsilon.$$

Therefore $\int_a^b f_{n_k} \cdot g \, dm \rightarrow g(a)$, and weak convergence forces

$$\int_a^b f \cdot g \, dm = g(a) \quad \text{for every continuous } g \text{ on } [a, b].$$

Now for each natural number j with $1/j < b - a$ let g_j be the continuous function on $[a, b]$ defined by $g_j(x) = 1 - j(x - a)$ for $a \leq x \leq a + 1/j$ and $g_j(x) = 0$ for $a + 1/j \leq x \leq b$. Then $0 \leq g_j \leq 1$ and $g_j(a) = 1$, so the identity above gives

$$\int_a^b f \cdot g_j \, dm = 1 \quad \text{for all such } j.$$

On the other hand $|f \cdot g_j| \leq |f|$ on $[a, b]$, with $|f|$ integrable, and for each fixed x with $a < x \leq b$ we have $g_j(x) = 0$ as soon as $1/j < x - a$, so $f \cdot g_j \rightarrow 0$ pointwise on $(a, b]$ and hence a.e. on $[a, b]$. By the Lebesgue Dominated Convergence Theorem, $\int_a^b f \cdot g_j \, dm \rightarrow 0$, contradicting the preceding display. Thus no subsequence of $\{f_n\}$ converges weakly in $L^1([a, b], m)$.

This is precisely the behaviour that the Dunford-Pettis Theorem accounts for. The sequence $\{f_n\}$ is bounded in L^1 but is not uniformly integrable: taking $\epsilon = 1/2$, for every $\delta > 0$ there is an n with $m([a, a + 1/n]) < \delta$ while

$$\int_{[a, a+1/n]} |f_n| \, dm = 1 > \epsilon.$$

Since uniform integrability fails, the Dunford-Pettis Theorem tells us that property (ii) of that theorem fails as well, that is, some subsequence of $\{f_n\}$ has no further subsequence that converges weakly in $L^1([a, b], m)$; the direct argument above shows more, namely that no subsequence at all converges weakly. The mass of f_n concentrates at the single point a , and in the limit it escapes from L^1 altogether, its natural limit being the Dirac measure at a , which is a measure singular with respect to m rather than a function in L^1 .

Problem 19.21. Find a measure space $(X, \mathcal{M}, \mu)$ for which every bounded sequence in $L^1(X, \mu)$ has a weakly convergent subsequence.

Solution. Take $X = \{1, 2, \dots, N\}$ a finite set, $\mathcal{M}$ the collection of all subsets of X , and μ the counting measure, so that $\mu(X) = N < \infty$. As in Problem 16, $\int_X f d\mu = \sum_{i=1}^N f(i)$ for nonnegative f , the empty set is the only set of measure zero, and

$$L^1(X, \mu) = \{f: X \rightarrow \mathbb{R}\}, \quad \|f\|_1 = \sum_{i=1}^N |f(i)|, \quad L^\infty(X, \mu) = \{g: X \rightarrow \mathbb{R}\}, \quad \|g\|_\infty = \max_{1 \leq i \leq N} |g(i)|.$$

We give two proofs that every bounded sequence in $L^1(X, \mu)$ has a weakly convergent subsequence. First proof, directly. Let $\{f_n\}$ be a sequence in $L^1(X, \mu)$ with $\|f_n\|_1 \leq C$ for all n . For each index i we have $|f_n(i)| \leq \|f_n\|_1 \leq C$, so the real sequence $\{f_n(i)\}_n$ is bounded. Applying the Bolzano-Weierstrass Theorem successively in the coordinates $i = 1, 2, \dots, N$ and passing each time to a further subsequence, which is possible since there are only finitely many coordinates, we obtain a subsequence $\{f_{n_k}\}$ and real numbers $c_1, \dots, c_N$ with $f_{n_k}(i) \rightarrow c_i$ for each i . Define $f \in L^1(X, \mu)$ by $f(i) = c_i$. Then

$$\|f_{n_k} - f\|_1 = \sum_{i=1}^N |f_{n_k}(i) - c_i| \rightarrow 0,$$

so $\{f_{n_k}\}$ converges strongly, and hence weakly, to f : for every bounded linear functional ψ on $L^1(X, \mu)$, $|\psi(f_{n_k}) - \psi(f)| \leq \|\psi\| \|f_{n_k} - f\|_1 \rightarrow 0$.

Second proof, by the Dunford-Pettis Theorem. The measure space is finite, so that theorem applies. Let $\{f_n\}$ be a bounded sequence in $L^1(X, \mu)$. It is uniformly integrable, and trivially so: given $\epsilon > 0$, take $\delta = 1$; if $E \in \mathcal{M}$ has $\mu(E) < 1$, then E has fewer than one point, that is, $E = \emptyset$, and therefore

$$\int_E |f_n| d\mu = 0 < \epsilon \quad \text{for all } n.$$

Hence, by the Dunford-Pettis Theorem, every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $L^1(X, \mu)$; in particular $\{f_n\}$ itself has a weakly convergent subsequence.

The same conclusion holds, by the identical argument, for any measure space with $L^1(X, \mu)$ finite dimensional, for example the trivial σ -algebra $\mathcal{M} = \{\emptyset, X\}$ with $\mu(X) = 1$, for which $L^1(X, \mu)$ is the one-dimensional space of constant functions. Some such degeneracy is necessary. Lebesgue measure on $[a, b]$ does not have the property, by the preceding problem, and neither does counting measure on $\mathbb{N}$, by Problem 19: the sequence $\{e_n\}$ is bounded in $\ell^1 = L^1(\mathbb{N}, \mu)$ and has no weakly convergent subsequence. The general obstruction is the one recorded at the start of Section 19.5: if $L^1(X, \mu)$ is not reflexive, then by the Eberlein-Smulian Theorem there are bounded sequences in $L^1(X, \mu)$ that fail to have weakly convergent subsequences. So a measure space with the stated property must have $L^1(X, \mu)$ reflexive, and in the examples above this holds for the crude reason that $L^1(X, \mu)$ is finite dimensional. The second proof shows exactly where the room lies: uniform integrability, the hypothesis of the Dunford-Pettis Theorem, is automatic for every sequence whenever there is some $\delta_0 > 0$ such that each measurable set of measure less than δ_0 is null, since then one may take $\delta = \delta_0$ for every ϵ .

19.4 Problems 19.22–19.23

Problem 19.22. Fill in the details of the proof of Corollary 14.

[Corollary 14: Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $1 < p < \infty$, and $\{f_n\}$ a bounded sequence in $L^p(X, \mu)$. Then $\{f_n\}$ has a subsequence that converges weakly in $L^1(X, \mu)$. The proof reads: since $\mu(X) < \infty$, we infer from Hölder's Inequality that $\{f_n\}$ is a bounded sequence in $L^1(X, \mu)$ and is uniformly integrable. Apply the Dunford-Pettis Theorem.]

Solution. Throughout, q denotes the conjugate of p , so that $1/p + 1/q = 1$; since $1 < p < \infty$ we also have $1 < q < \infty$, and in particular q is **finite**. Let $M \geq 0$ be a bound for the given sequence in L^p , so that

$$\|f_n\|_p \leq M \quad \text{for all } n.$$

Finally, $\mu(X) < \infty$ is used through the observation that for every measurable set $E \subseteq X$ the characteristic function χ_E belongs to $L^q(X, \mu)$, with

$$\|\chi_E\|_q = \left(\int_X \chi_E d\mu \right)^{1/q} = \mu(E)^{1/q} \leq \mu(X)^{1/q} < \infty.$$

Step 1: each f_n belongs to $L^1(X, \mu)$ and the sequence $\{f_n\}$ is bounded in $L^1(X, \mu)$.

Fix n . The function $|f_n|$ is measurable and nonnegative, and $\chi_X = 1$ belongs to $L^q(X, \mu)$ by the display above. Hölder's Inequality applied to the pair $|f_n| \in L^p$, $1 \in L^q$ gives

$$\int_X |f_n| d\mu = \int_X |f_n| \cdot 1 d\mu \leq \|f_n\|_p \cdot \|1\|_q = \|f_n\|_p \mu(X)^{1/q} \leq M \mu(X)^{1/q}.$$

Thus $\|f_n\|_1 \leq M \mu(X)^{1/q} < \infty$: each f_n is integrable over X , and the bound is independent of n , so $\{f_n\}$ is a bounded sequence in $L^1(X, \mu)$. (This is the standard inclusion $L^p(X, \mu) \subseteq L^1(X, \mu)$ for a finite measure space, and it is exactly where finiteness of μ enters: for $\mu(X) = \infty$ the constant function 1 fails to lie in L^q and the inclusion fails.)

Step 2: $\{f_n\}$ is uniformly integrable over X .

Recall the definition: $\{f_n\}$ is uniformly integrable provided that for each $\epsilon > 0$ there is a $\delta > 0$ such that for every measurable set E with $\mu(E) < \delta$ we have $\int_E |f_n| d\mu < \epsilon$ for all n .

Let $E \subseteq X$ be measurable and let n be a natural number. Apply Hölder's Inequality to the pair $|f_n| \in L^p(X, \mu)$ and $\chi_E \in L^q(X, \mu)$:

$$\int_E |f_n| d\mu = \int_X |f_n| \cdot \chi_E d\mu \leq \|f_n\|_p \cdot \|\chi_E\|_q \leq M \mu(E)^{1/q}.$$

This single estimate gives uniform integrability. Indeed, let $\epsilon > 0$ and define

$$\delta = \left(\frac{\epsilon}{M+1} \right)^q > 0,$$

which is a genuine positive number precisely because $q < \infty$, that is, because $p > 1$. If E is measurable with $\mu(E) < \delta$, then, since $t \mapsto t^{1/q}$ is increasing on $[0, \infty)$,

$$\int_E |f_n| d\mu \leq M \mu(E)^{1/q} < M \delta^{1/q} = \frac{M}{M+1} \epsilon < \epsilon \quad \text{for all } n.$$

Hence $\{f_n\}$ is uniformly integrable over X .

Step 3: apply the Dunford-Pettis Theorem.

By Step 1, $(X, \mathcal{M}, \mu)$ is a finite measure space and $\{f_n\}$ is a bounded sequence in $L^1(X, \mu)$, so the Dunford-Pettis Theorem (Theorem 12) applies to $\{f_n\}$. By Step 2, property (i) of that theorem holds: $\{f_n\}$ is uniformly integrable over X . Therefore property (ii) holds as well: every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $L^1(X, \mu)$. Taking the subsequence to be $\{f_n\}$ itself, we conclude that $\{f_n\}$ has a subsequence that converges weakly in $L^1(X, \mu)$. This is the assertion of Corollary 14.

Two remarks that clarify the roles of the hypotheses.

First, the hypothesis $p > 1$ is essential and is used only in Step 2, in the requirement $q < \infty$. For $p = 1$ the corollary is false: on $X = [0, 1]$ with Lebesgue measure the sequence $f_n = n \cdot \chi_{(0, 1/n)}$ satisfies $\|f_n\|_1 = 1$ for all n , so it is bounded in L^1 , yet no subsequence converges weakly in $L^1(X, m)$. Indeed, if $f_{n_k} \rightarrow f$ weakly in L^1 , then testing against $g = \chi_{[a, 1]} \in L^\infty$ for $0 < a < 1$ gives $\int_a^1 f dm = \lim_k \int_a^1 f_{n_k} dm = 0$, since $\int_a^1 f_{n_k} dm = 0$ once $1/n_k \leq a$. Applying this with $a = 1/j$ and letting $j \rightarrow \infty$ along the ascending collection $\{(1/j, 1]\}_{j=1}^\infty$, whose union is $(0, 1]$, the Continuity of Integration

(Theorem 21 of Chapter 4, applicable since $f \in L^1$) gives $\int_0^1 f \, d\mu = \lim_{j \rightarrow \infty} \int_{1/j}^1 f \, d\mu = 0$, while testing against $g \equiv 1$ gives $\int_0^1 f \, d\mu = \lim_k \int_0^1 f_{n_k} \, d\mu = 1$, a contradiction. Correspondingly, this sequence is not uniformly integrable: $m((0, 1/n)) \rightarrow 0$ while $\int_{(0, 1/n)} |f_n| \, d\mu = 1$ for every n . Second, there is an alternative route to Corollary 14 that avoids the Dunford-Pettis Theorem altogether and shows that the subsequence may be taken to converge weakly in the stronger sense of L^p . Since μ is finite it is σ -finite, so by the Riesz Weak Compactness Theorem there is a subsequence $\{f_{n_k}\}$ and a function $f \in L^p(X, \mu)$ with

$$\lim_{k \rightarrow \infty} \int_X f_{n_k} \cdot h \, d\mu = \int_X f \cdot h \, d\mu \quad \text{for all } h \in L^q(X, \mu).$$

Now let $g \in L^\infty(X, \mu)$. Since $\mu(X) < \infty$ and $q < \infty$,

$$\int_X |g|^q \, d\mu \leq \|g\|_\infty^q \mu(X) < \infty,$$

so $g \in L^q(X, \mu)$ and hence $\lim_{k \rightarrow \infty} \int_X f_{n_k} \cdot g \, d\mu = \int_X f \cdot g \, d\mu$. Moreover $f \in L^1(X, \mu)$ by Step 1 applied to f . Since a finite measure is σ -finite, the Riesz Representation Theorem for the Dual of $L^p(X, \mu)$, applied with $p = 1$, tells us that every bounded linear functional on $L^1(X, \mu)$ is of the form $h \mapsto \int_X h \cdot g \, d\mu$ for some $g \in L^\infty(X, \mu)$. Since the displayed limit holds for every such g , the subsequence $\{f_{n_k}\}$ converges weakly in $L^1(X, \mu)$ to f . Again both containments $L^p \subseteq L^1$ and $L^\infty \subseteq L^q$ used here require $\mu(X) < \infty$.

Problem 19.23. Why is the Dunford-Pettis Theorem false if the assumption that the sequence is bounded in L^1 is dropped?

Solution. The Dunford-Pettis Theorem (Theorem 12) asserts, for a finite measure space $(X, \mathcal{M}, \mu)$ and a bounded sequence $\{f_n\}$ in $L^1(X, \mu)$, the equivalence of

- (i) $\{f_n\}$ is uniformly integrable over X ;
- (ii) every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $L^1(X, \mu)$.

Only one of the two implications is at risk. Indeed, (ii) by itself already forces boundedness in $L^1(X, \mu)$: if $\{f_n\}$ were unbounded in L^1 , there would be a subsequence $\{f_{n_k}\}$ with $\lim_k \|f_{n_k}\|_1 = \infty$, and by (ii) some further subsequence $\{f_{n_{k_j}}\}$ would converge weakly in $L^1(X, \mu)$; but by Theorem 12 of Chapter 14, a consequence of the Uniform Boundedness Principle, every weakly convergent sequence in a normed linear space is bounded, so $\{\|f_{n_{k_j}}\|_1\}$ would be bounded, a contradiction. So the implication (ii) $\Rightarrow$ (i) loses nothing when the boundedness hypothesis is deleted: the hypothesis is a consequence of (ii).

What fails is the implication (i) $\Rightarrow$ (ii). Uniform integrability of $\{f_n\}$, as defined – for each $\epsilon > 0$ there is a $\delta > 0$ such that $\int_E |f_n| \, d\mu < \epsilon$ for all n whenever $\mu(E) < \delta$ – controls the mass of the f_n only on sets of **small** measure. It says nothing whatsoever about the mass on sets of measure at least δ . When μ has an atom, this gap is fatal, because a set of positive measure can then be immune to being cut into small pieces, and unbounded mass may be piled onto it while the uniform integrability condition remains vacuously satisfied.

A counterexample. Let $X = \{x_0\}$ be a one-point set, $\mathcal{M} = \{\emptyset, X\}$, and $\mu(X) = 1$; this is a finite measure space. Its only measurable sets are $\emptyset$ and X , so a measurable function on X is determined by its single value and $\int_X |f| \, d\mu = |f(x_0)|$. Define

$$f_n(x_0) = n \quad \text{for each natural number } n.$$

Then $\{f_n\}$ is uniformly integrable: given $\epsilon > 0$, take $\delta = 1/2$; the only measurable set E with $\mu(E) < 1/2$ is $E = \emptyset$, and $\int_\emptyset |f_n| \, d\mu = 0 < \epsilon$ for all n . So (i) holds. But $\|f_n\|_1 = n$, so $\{f_n\}$ is unbounded in $L^1(X, \mu)$, and (ii) fails: taking $g \equiv 1$, which belongs to $L^\infty(X, \mu)$, we have $\int_X f_{n_k} \cdot g \, d\mu = n_k \rightarrow \infty$ for every subsequence, so no subsequence can converge weakly in $L^1(X, \mu)$ (weak convergence to some f would force $\int_X f_{n_k} \cdot g \, d\mu \rightarrow \int_X f \cdot g \, d\mu$, a finite limit). Hence (i) does not imply (ii) once boundedness is dropped.

The same construction works on any finite measure space possessing an atom, so the failure is not an artifact of the degenerate example. Let $A \in \mathcal{M}$ be an atom: $a = \mu(A) > 0$ and every measurable $B \subseteq A$ has $\mu(B) = 0$ or $\mu(B) = a$. Put

$$f_n = \frac{n}{a} \chi_A \quad \text{for each natural number } n.$$

Take $\delta = a/2$, and let E be measurable with $\mu(E) < \delta$. Then $E \cap A$ is a measurable subset of A with $\mu(E \cap A) \leq \mu(E) < a$, so, A being an atom, $\mu(E \cap A) = 0$ and therefore

$$\int_E |f_n| d\mu = \frac{n}{a} \mu(E \cap A) = 0 < \epsilon \quad \text{for all } n \text{ and all } \epsilon > 0.$$

Thus $\{f_n\}$ is uniformly integrable, with one δ serving all ϵ . On the other hand $\|f_n\|_1 = (n/a) \cdot a = n \rightarrow \infty$, and testing against $g = \chi_A \in L^\infty(X, \mu)$ gives $\int_X f_n \cdot g d\mu = n$, which has no finite limit along any subsequence; so no subsequence of $\{f_n\}$ converges weakly in $L^1(X, \mu)$.

Where the proof breaks down. Boundedness in L^1 is used twice in the argument for (i) $\Rightarrow$ (ii). It is used in Proposition 10, whose proof begins by choosing C with $\|f_n\|_1 \leq C$ for all n and then applies Chebychev's Inequality in the form $\mu\{x \in X : |f_n(x)| \geq M\} \leq C/M$; without a uniform L^1 bound the level sets $\{|f_n| \geq M\}$ need not have uniformly small measure, and the equivalence of uniform integrability with the tail condition (23) collapses. In the atomic example above, for any $M > 0$ and any $n \geq Ma$ we have $\{x \in X : |f_n(x)| \geq M\} = A$ and hence $\int_{\{|f_n| \geq M\}} |f_n| d\mu = n$; so (23) fails for every M and every ϵ , while $\{f_n\}$ is uniformly integrable. Thus (i) holds and (ii) of Proposition 10 fails, and the very first step of the proof of the Dunford-Pettis Theorem – the application of Proposition 10 inside Lemma 11 to produce the estimate (29) – is unavailable. Boundedness is used a second time in the proof of Theorem 12 itself, at the point where one asserts that $\{\nu_n(X)\} = \{\int_X h_n d\mu\}$ is a bounded sequence of real numbers, which is precisely the hypothesis needed to invoke the Vitali-Hahn-Saks Theorem and conclude that the set function $\nu = \lim_n \nu_n$ is a finite measure. In the atomic example above $\nu_n(A) = \int_A f_n d\mu = n$, so the limiting set function is not even real-valued and the Radon-Nikodym Theorem has nothing to act on.

A final remark explaining why the counterexample had to use an atom. The property of μ that makes the boundedness hypothesis redundant is the following one, which an atom destroys:

for each $\delta > 0$, X is the union of a finite disjoint collection of measurable sets each of measure less than δ .

Suppose μ has this property and $\{f_n\}$ is uniformly integrable over X . Take $\epsilon = 1$ in the definition of uniform integrability and let $\delta > 0$ respond to it. Choose a finite disjoint collection $\{E_j\}_{j=1}^N$ of measurable sets with $X = \bigcup_{j=1}^N E_j$ and $\mu(E_j) < \delta$ for each j . Then $\int_{E_j} |f_n| d\mu < 1$ for every j and every n , so by the additivity of integration over domains (Theorem 13 of Chapter 18),

$$\int_X |f_n| d\mu = \sum_{j=1}^N \int_{E_j} |f_n| d\mu < N \quad \text{for all } n,$$

so $\{f_n\}$ is bounded in $L^1(X, \mu)$ and the hypothesis of boundedness is redundant. Lebesgue measure on a closed, bounded interval $[a, b]$ has the displayed property: given $\delta > 0$, choose a natural number N with $(b-a)/N < \delta$ and partition $[a, b]$ into the N subintervals $[a + (j-1)(b-a)/N, a + j(b-a)/N]$ for $1 \leq j \leq N-1$ together with $[a + (N-1)(b-a)/N, b]$. So no counterexample of the above kind exists on $[a, b]$ with Lebesgue measure: the search for one must be conducted on a measure space lacking the displayed property. And an atom is precisely what destroys it. Indeed, if A is an atom of μ and $0 < \delta \leq \mu(A)$, then no such partition exists at that δ : were $X = \bigcup_{j=1}^N E_j$ disjointly with each $\mu(E_j) < \delta \leq \mu(A)$, then each $E_j \cap A$ would be a measurable subset of A of measure less than $\mu(A)$, hence of measure zero, and additivity would give $\mu(A) = \sum_{j=1}^N \mu(E_j \cap A) = 0$, a contradiction. The boundedness hypothesis in the Dunford-Pettis Theorem is therefore exactly what is needed to make the theorem correct for general finite measure spaces, in which atoms are permitted.

20 The Construction of Particular Measures

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20.1 Problems 20.1–20.7

Problem 20.1. Let $A \subseteq X$ and let B be a ν -measurable subset of Y . If $A \times B$ is measurable with respect to the product measure $\mu \times \nu$, is A necessarily measurable with respect to μ ?

Solution. No. The product measure $\mu \times \nu$ is by definition the Carathéodory extension of the premeasure $\lambda(A \times B) = \mu(A) \cdot \nu(B)$ on the semiring $\mathcal{R}$ of measurable rectangles, that is, the restriction of the outer measure $(\mu \times \nu)^*$ to the $(\mu \times \nu)^*$ -measurable sets. Every Carathéodory extension is a complete measure, so every subset of $X \times Y$ of outer measure zero is $\mu \times \nu$ -measurable. This is what defeats the assertion.

Take $X = Y = \mathbb{R}$ with $\mathcal{A} = \mathcal{B}$ the Lebesgue measurable sets and $\mu = \nu = m$ Lebesgue measure. Let $B = \{0\}$, a ν -measurable set with $\nu(B) = 0$, and let $A \subseteq \mathbb{R}$ be a nonmeasurable set (a Vitali set; see Chapter 2). Then

$$A \times B \subseteq \mathbb{R} \times \{0\} = \bigcup_{n=1}^{\infty} [-n, n] \times \{0\},$$

a countable union of measurable rectangles each of premeasure $\lambda([-n, n] \times \{0\}) = 2n \cdot 0 = 0$. Hence $(\mu \times \nu)^*(A \times B) = 0$, and therefore $A \times B$ is measurable with respect to $\mu \times \nu$ (with measure zero). But A is not μ -measurable. So the answer is no.

The counterexample exploits $\nu(B) = 0$, and it also uses the fact that μ is complete only incidentally; in fact the phenomenon persists whenever $\nu(B) = 0$, since then $A \times B$ has $\mu \times \nu$ outer measure zero for every $A \subseteq X$ whatsoever, so no information about A can be recovered.

It is worth recording the positive complement, which shows this is the only obstruction apart from failures of completeness of μ .

Claim. Suppose μ is complete and σ -finite, ν is complete, B is ν -measurable with $0 < \nu(B) < \infty$, and $A \times B$ is $\mu \times \nu$ -measurable. Then A is μ -measurable.

Proof of the claim. Write $X = \bigcup_{n=1}^{\infty} X_n$ with each X_n μ -measurable and $\mu(X_n) < \infty$. Fix n and put

$$E = (A \times B) \cap (X_n \times Y) = (A \cap X_n) \times B.$$

Since $X_n \times Y$ is a measurable rectangle, E is $\mu \times \nu$ -measurable, and $E \subseteq X_n \times B$ gives $(\mu \times \nu)(E) \leq \mu(X_n) \cdot \nu(B) < \infty$. As ν is complete and E has finite product measure, Proposition 5 of this chapter applies to E : for almost all $x \in X$ the x -section E_x is ν -measurable and the function $x \mapsto \nu(E_x)$, defined almost everywhere on X , is μ -measurable. But

$$E_x = \begin{cases} B & \text{if } x \in A \cap X_n, \\ \emptyset & \text{otherwise,} \end{cases} \quad \text{so} \quad \nu(E_x) = \nu(B) \cdot \chi_{A \cap X_n}(x).$$

Thus $\nu(B) \cdot \chi_{A \cap X_n}$ agrees μ -almost everywhere with a μ -measurable function; since $0 < \nu(B) < \infty$ we may divide by it, and since μ is complete a function that agrees almost everywhere with a measurable function is itself measurable (Proposition 3 of Chapter 18). Hence $\chi_{A \cap X_n}$ is μ -measurable, that is, $A \cap X_n$ is μ -measurable. Therefore $A = \bigcup_{n=1}^{\infty} (A \cap X_n)$ is μ -measurable.

Completeness of μ cannot be dropped from the claim either. Let $X = \mathbb{R}$ with $\mathcal{A}$ the Borel sets and μ Lebesgue measure restricted to $\mathcal{A}$, which is not complete, let $Y = \{p\}$ be a one-point space with $\nu(\{p\}) = 1$, and let A be a non-Borel subset of the Cantor set. Then $A \times \{p\} \subseteq \mathbb{R} \times \{p\}$ has $\mu \times \nu$ outer measure at most $\mu(\mathbb{R}) \cdot 1 = 0$, so $A \times \{p\}$ is $\mu \times \nu$ -measurable, while A is not μ -measurable, even though $\nu(B) = 1 > 0$.

Problem 20.2. Let $\mathbb{N}$ be the set of natural numbers, $\mathcal{M} = 2^{\mathbb{N}}$, and c the counting measure defined by setting $c(E)$ equal to the number of points in E if E is finite and ∞ if E is an infinite set. Prove that every function $f: \mathbb{N} \rightarrow \mathbb{R}$ is measurable with respect to c and that f is integrable over $\mathbb{N}$ with

respect to c if and only if the series $\sum_{k=1}^{\infty} f(k)$ is absolutely convergent, in which case

$$\int_{\mathbb{N}} f \, dc = \sum_{k=1}^{\infty} f(k).$$

Solution. First observe that c is indeed a measure on the σ -algebra $\mathcal{M} = 2^{\mathbb{N}}$: $c(\emptyset) = 0$, and if $\{E_j\}_{j=1}^{\infty}$ are disjoint subsets of $\mathbb{N}$ then the number of points of $\bigcup_j E_j$ is $\sum_j c(E_j)$, both sides being ∞ exactly when some E_j is infinite or infinitely many are nonempty. The measure c is complete, since $\mathcal{M}$ consists of all subsets of $\mathbb{N}$, and it is σ -finite, since $\mathbb{N} = \bigcup_{n=1}^{\infty} \{1, \dots, n\}$ and each $\{1, \dots, n\}$ has finite measure. Note also that the only set of c -measure zero is $\emptyset$, so “almost everywhere” means “everywhere”.

Measurability. Let $f: \mathbb{N} \rightarrow \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then $\{k \in \mathbb{N} \mid f(k) > \alpha\}$ is a subset of $\mathbb{N}$, hence belongs to $\mathcal{M} = 2^{\mathbb{N}}$. By the definition of measurability (Chapter 18, Section 1) f is measurable with respect to c . So every function on $\mathbb{N}$ is c -measurable.

The integral of a nonnegative function. Suppose first that $g: \mathbb{N} \rightarrow [0, \infty)$ is nonnegative and real-valued. For $n \in \mathbb{N}$ set

$$g_n = g \cdot \chi_{\{1, \dots, n\}} = \sum_{k=1}^n g(k) \chi_{\{k\}}.$$

Each g_n takes at most $n + 1$ values, so it is a nonnegative simple function, and by the linearity of integration for nonnegative simple functions together with $\int_{\mathbb{N}} \chi_{\{k\}} \, dc = c(\{k\}) = 1$,

$$\int_{\mathbb{N}} g_n \, dc = \sum_{k=1}^n g(k).$$

Moreover $\{g_n\}$ is an increasing sequence of nonnegative measurable functions converging pointwise on $\mathbb{N}$ to g . By the Monotone Convergence Theorem,

$$\int_{\mathbb{N}} g \, dc = \lim_{n \rightarrow \infty} \int_{\mathbb{N}} g_n \, dc = \lim_{n \rightarrow \infty} \sum_{k=1}^n g(k) = \sum_{k=1}^{\infty} g(k), \quad (*)$$

an equality in $[0, \infty]$, since the partial sums of a nonnegative series increase to the sum of the series.

The formula $(*)$ persists for an extended real-valued $g: \mathbb{N} \rightarrow [0, \infty]$. Indeed, for $M \in \mathbb{N}$ the function $g^{(M)} = \min\{g, M\}$ is nonnegative, real-valued and measurable, so $\int_{\mathbb{N}} g^{(M)} \, dc = \sum_{k=1}^{\infty} g^{(M)}(k)$. As $M \rightarrow \infty$ the functions $g^{(M)}$ increase pointwise to g , so the left-hand sides converge to $\int_{\mathbb{N}} g \, dc$ by the Monotone Convergence Theorem, while the right-hand sides increase to $\sum_{k=1}^{\infty} g(k)$, both limits being taken in $[0, \infty]$. (For the right-hand sides, $\sum_k g^{(M)}(k) \leq \sum_k g(k)$ for every M , and conversely each partial sum $\sum_{k=1}^n g(k)$ is the limit as $M \rightarrow \infty$ of $\sum_{k=1}^n g^{(M)}(k)$, with the convention that a sum containing an infinite term is ∞ .) In what follows, $(*)$ is used in this generality.

Integrability. Let $f: \mathbb{N} \rightarrow \mathbb{R}$. By definition f is integrable over $\mathbb{N}$ with respect to c provided f is measurable, which we have shown, and $\int_{\mathbb{N}} |f| \, dc < \infty$. Applying $(*)$ to the nonnegative function $|f|$ gives

$$\int_{\mathbb{N}} |f| \, dc = \sum_{k=1}^{\infty} |f(k)|.$$

Hence f is integrable with respect to c if and only if $\sum_{k=1}^{\infty} |f(k)| < \infty$, that is, if and only if the series $\sum_{k=1}^{\infty} f(k)$ converges absolutely.

The value of the integral. Assume f is integrable, and write $f = f^+ - f^-$ with $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Both are nonnegative and dominated by $|f|$, so by $(*)$

$$\int_{\mathbb{N}} f^+ \, dc = \sum_{k=1}^{\infty} f^+(k) < \infty, \quad \int_{\mathbb{N}} f^- \, dc = \sum_{k=1}^{\infty} f^-(k) < \infty.$$

By the definition of the integral of an integrable function,

$$\int_{\mathbb{N}} f \, dc = \int_{\mathbb{N}} f^+ \, dc - \int_{\mathbb{N}} f^- \, dc = \sum_{k=1}^{\infty} f^+(k) - \sum_{k=1}^{\infty} f^-(k).$$

Both series on the right are convergent series of nonnegative terms, so they may be subtracted term by term, and $f^+(k) - f^-(k) = f(k)$. Therefore

$$\int_{\mathbb{N}} f \, dc = \sum_{k=1}^{\infty} f(k),$$

as asserted. (Absolute convergence guarantees in addition that the value of the sum does not depend on the order of the terms, in keeping with the fact that the integral has no reference to an ordering of $\mathbb{N}$.)

Problem 20.3. Let $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{M}, c)$, the measure space defined in the preceding problem. State the Fubini and Tonelli Theorems explicitly for this case.

Solution. Preliminaries. The hypotheses of both theorems hold here: c is complete (every subset of $\mathbb{N}$ is c -measurable) and σ -finite. Moreover the product measure $c \times c$ is exactly counting measure on $\mathbb{N} \times \mathbb{N}$. Indeed, each singleton $\{(m, n)\} = \{m\} \times \{n\}$ is a measurable rectangle with $\lambda(\{m\} \times \{n\}) = c(\{m\}) \cdot c(\{n\}) = 1$; every subset E of $\mathbb{N} \times \mathbb{N}$ is a countable disjoint union of such singletons, hence is $c \times c$ -measurable, and by countable additivity $(c \times c)(E)$ is the number of points of E . Consequently $(\mathbb{N} \times \mathbb{N}, 2^{\mathbb{N} \times \mathbb{N}}, c \times c)$ is a copy of the measure space of Problem 2: any bijection $\sigma: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ carries subsets to subsets and preserves cardinality, hence carries c to $c \times c$, so the argument of Problem 2 applies verbatim. Thus a function f on $\mathbb{N} \times \mathbb{N}$ is integrable with respect to $c \times c$ if and only if the family $\{f(m, n)\}$ is absolutely summable, meaning $\sum_{j=1}^{\infty} |f(\sigma(j))| < \infty$ for one, equivalently every, such bijection, and then $\int_{\mathbb{N} \times \mathbb{N}} f \, d(c \times c)$ is the (order-independent) sum of that family. We write $\sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} f(m, n)$ for this common value. Finally, since $\emptyset$ is the only set of c -measure zero, every occurrence of “for almost all x ” in the theorems becomes “for all $m \in \mathbb{N}$ ”.

Write $a_{m,n} = f(m, n)$, so that a function $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ is the same thing as a double sequence of real numbers, and by Problem 2 the x -section integral $\int_{\mathbb{N}} f(m, n) \, dc(n)$ is the series $\sum_{n=1}^{\infty} a_{m,n}$, while the y -section integral $\int_{\mathbb{N}} f(m, n) \, dc(m)$ is the series $\sum_{m=1}^{\infty} a_{m,n}$.

Fubini's Theorem for $(\mathbb{N}, \mathcal{M}, c) \times (\mathbb{N}, \mathcal{M}, c)$. Let $\{a_{m,n}\}_{m,n \in \mathbb{N}}$ be a double sequence of real numbers which is absolutely summable, that is,

$$\sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} |a_{m,n}| < \infty.$$

Then for every $m \in \mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{m,n}$ converges absolutely, the series $\sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} a_{m,n})$ converges absolutely, and

$$\sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} a_{m,n} = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{m,n} \right].$$

Since the roles of the two factors may be interchanged, the same holds with the order of summation reversed, so under absolute summability

$$\sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{m,n} \right] = \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} a_{m,n} \right].$$

Tonelli's Theorem for $(\mathbb{N}, \mathcal{M}, c) \times (\mathbb{N}, \mathcal{M}, c)$. Let $\{a_{m,n}\}$ be a double sequence with $a_{m,n} \geq 0$ for all m and n . Then, as an equality in $[0, \infty]$,

$$\sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} a_{m,n} = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{m,n} \right] = \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} a_{m,n} \right],$$

no finiteness being assumed; that is, for a double sequence of nonnegative numbers the two iterated sums always agree, and agree with the unordered sum.

Corollary 7 in this case. Let $\{a_{m,n}\}$ be an arbitrary double sequence of real numbers. If one of the iterated series of absolute values is finite, say

$$\sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} |a_{m,n}| \right] < \infty,$$

then $\{a_{m,n}\}$ is absolutely summable, both iterated series converge absolutely, and

$$\sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} a_{m,n} \right] = \sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} a_{m,n} = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{m,n} \right].$$

Thus the classical theorem on rearrangement and iterated summation of double series is precisely the Fubini–Tonelli theory for the product of two copies of the counting measure space. Problem 5 shows that without absolute summability the two iterated sums can differ even when both are defined.

Problem 20.4. Let $(\mathbb{N}, \mathcal{M}, c)$ be the measure space defined in Problem 2 and $(X, \mathcal{A}, \mu)$ a general measure space. Consider $\mathbb{N} \times X$ with the product measure $c \times \mu$.

- (i) Show that a subset E of $\mathbb{N} \times X$ is measurable with respect to $c \times \mu$ if and only if for each natural number k , $E_k = \{x \in X \mid (k, x) \in E\}$ is measurable with respect to μ .
- (ii) Show that a function $f: \mathbb{N} \times X \rightarrow \mathbb{R}$ is measurable with respect to $c \times \mu$ if and only if for each natural number k , $f(k, \cdot): X \rightarrow \mathbb{R}$ is measurable with respect to μ .
- (iii) Show that a function $f: \mathbb{N} \times X \rightarrow \mathbb{R}$ is integrable over $\mathbb{N} \times X$ with respect to $c \times \mu$ if and only if for each natural number k , $f(k, \cdot): X \rightarrow \mathbb{R}$ is integrable over X with respect to μ and

$$\sum_{k=1}^{\infty} \int_X |f(k, x)| d\mu(x) < \infty.$$

- (iv) Show that if the function $f: \mathbb{N} \times X \rightarrow \mathbb{R}$ is integrable over $\mathbb{N} \times X$ with respect to $c \times \mu$, then

$$\int_{\mathbb{N} \times X} f d(c \times \mu) = \sum_{k=1}^{\infty} \int_X f(k, x) d\mu(x) < \infty.$$

Solution. Standing hypothesis. Let μ^* denote the outer measure induced by μ , so that for $S \subseteq X$

$$\mu^*(S) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) \mid S \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A} \right\} = \inf \{ \mu(A) \mid S \subseteq A, A \in \mathcal{A} \},$$

the second equality because $\mathcal{A}$ is a σ -algebra. Throughout this problem we assume that $\mathcal{A}$ is exactly the σ -algebra of μ^* -measurable sets. This is no restriction in the setting where the theorems of this chapter are applied: it holds whenever μ is obtained by the Carathéodory construction from μ^* , and it holds automatically whenever μ is complete and σ -finite. For the latter, write $X = \bigcup_n X_n$ with $\mu(X_n) < \infty$ and let S be μ^* -measurable. Choose $A \in \mathcal{A}$ with $S \cap X_n \subseteq A \subseteq X_n$ and $\mu(A) = \mu^*(S \cap X_n)$, possible since $\mu^*(S \cap X_n) \leq \mu(X_n) < \infty$. Testing the Carathéodory criterion for S with A gives $\mu(A) = \mu^*(A \cap S) + \mu^*(A \sim S) = \mu^*(S \cap X_n) + \mu^*(A \sim S)$, and as $\mu(A) < \infty$ we get $\mu^*(A \sim S) = 0$. Completeness of μ then puts $A \sim S$ in $\mathcal{A}$, hence $S \cap X_n = A \sim (A \sim S) \in \mathcal{A}$, and $S \in \mathcal{A}$.

Some such assumption is necessary for the “only if” half of (i), and mere completeness of μ is not enough. First, if μ is not complete the assertion fails: take $X = [0, 1]$ with $\mathcal{A}$ the Borel sets and μ Lebesgue measure restricted to $\mathcal{A}$, and let A be a non-Borel subset of the Cantor set C . Then $E = \{1\} \times A \subseteq \{1\} \times C$ has $(c \times \mu)^*(E) \leq c(\{1\}) \cdot \mu(C) = 0$, so E is $c \times \mu$ -measurable (the product measure, being a Carathéodory extension, is complete), while $E_1 = A$ is not in $\mathcal{A}$. Second, completeness alone does not suffice either. Let X be uncountable, let $\mathcal{A}$ consist of the countable and the co-countable subsets of X , and let $\mu(A) = 0$ for A countable and $\mu(A) = \infty$ for A co-countable. This is a complete measure, since every subset of a μ -null set is countable. Here $\mu^*(S) = 0$ for S countable and $\mu^*(S) = \infty$ otherwise, and every subset of X is μ^* -measurable: for a test set T , if T is countable both sides of the Carathéodory identity vanish, while if T is uncountable then one of $T \cap S$, $T \sim S$ is uncountable and

both sides are ∞ . Now let $S \subseteq X$ be uncountable with uncountable complement and put $E = \{1\} \times S$. I claim E is $c \times \mu$ -measurable. Let $T \subseteq \mathbb{N} \times X$ be a test set and write $T_k = \{x \mid (k, x) \in T\}$. The slabs $\{k\} \times X$ are disjoint measurable rectangles whose union is $\mathbb{N} \times X$, so by the countable additivity of an outer measure over a disjoint countable collection of measurable sets (Chapter 17) and by formula (1) proved below,

$$(c \times \mu)^*(T) = \sum_{k=1}^{\infty} (c \times \mu)^*(\{k\} \times T_k) = \sum_{k=1}^{\infty} \mu^*(T_k).$$

Applying this to $T \cap E = \{1\} \times (T_1 \cap S)$ and to $T \sim E$, whose sections are $T_1 \sim S$ and T_k for $k \geq 2$, we find that the Carathéodory identity for E and T reduces to $\mu^*(T_1) = \mu^*(T_1 \cap S) + \mu^*(T_1 \sim S)$, which holds because S is μ^* -measurable. So E is $c \times \mu$ -measurable while $E_1 = S \notin \mathcal{A}$. This measure μ is complete but not σ -finite, which is precisely why the standing hypothesis has content here.

A basic computation. For every $k \in \mathbb{N}$ and every subset S of X ,

$$(c \times \mu)^*(\{k\} \times S) = \mu^*(S). \quad (1)$$

Indeed, if $S \subseteq A$ with $A \in \mathcal{A}$, then $\{k\} \times S \subseteq \{k\} \times A$, a measurable rectangle of premeasure $c(\{k\}) \cdot \mu(A) = \mu(A)$; taking the infimum over such A gives $(c \times \mu)^*(\{k\} \times S) \leq \mu^*(S)$. Conversely, let $\{N_j \times A_j\}_{j=1}^{\infty}$ be a countable collection of measurable rectangles covering $\{k\} \times S$. If $x \in S$ then (k, x) lies in some $N_j \times A_j$, and then $k \in N_j$ and $x \in A_j$. Hence $S \subseteq \bigcup \{A_j \mid k \in N_j\}$, so by the countable monotonicity of μ^* ,

$$\mu^*(S) \leq \sum_{j: k \in N_j} \mu(A_j) \leq \sum_{j=1}^{\infty} c(N_j) \cdot \mu(A_j),$$

because $c(N_j) \geq 1$ whenever $k \in N_j$. Taking the infimum over all such covers gives $\mu^*(S) \leq (c \times \mu)^*(\{k\} \times S)$, and (1) follows.

Part (i). Suppose first that $E_k \in \mathcal{A}$ for every k . Then

$$E = \bigcup_{k=1}^{\infty} \{k\} \times E_k$$

is a countable union of measurable rectangles. Since the $(c \times \mu)^*$ -measurable sets form a σ -algebra containing $\mathcal{R}$, the set E is measurable with respect to $c \times \mu$.

Conversely, suppose E is $c \times \mu$ -measurable and fix k . Let T be any subset of X . Using

$$(\{k\} \times T) \cap E = \{k\} \times (T \cap E_k), \quad (\{k\} \times T) \sim E = \{k\} \times (T \sim E_k),$$

and applying the Carathéodory criterion for E with the test set $\{k\} \times T$, we get

$$(c \times \mu)^*(\{k\} \times T) = (c \times \mu)^*(\{k\} \times (T \cap E_k)) + (c \times \mu)^*(\{k\} \times (T \sim E_k)).$$

By (1) this says exactly

$$\mu^*(T) = \mu^*(T \cap E_k) + \mu^*(T \sim E_k) \quad \text{for all } T \subseteq X,$$

so E_k is μ^* -measurable, that is, $E_k \in \mathcal{A}$. This proves (i).

Part (ii). Let $f: \mathbb{N} \times X \rightarrow \mathbb{R}$. For $\alpha \in \mathbb{R}$ put $E^\alpha = \{(k, x) \mid f(k, x) > \alpha\}$ and note that its k -th section is

$$(E^\alpha)_k = \{x \in X \mid f(k, x) > \alpha\}.$$

Now f is $c \times \mu$ -measurable if and only if E^α is $c \times \mu$ -measurable for every α , which by (i) holds if and only if $(E^\alpha)_k \in \mathcal{A}$ for every α and every k . Since the two quantifiers commute, this is the same as saying that for every k and every α the set $\{x \mid f(k, x) > \alpha\}$ is μ -measurable, that is, that $f(k, \cdot)$ is μ -measurable for every k .

A slab lemma. For every k and every nonnegative $c \times \mu$ -measurable function g on $\mathbb{N} \times X$,

$$\int_{\{k\} \times X} g d(c \times \mu) = \int_X g(k, x) d\mu(x), \quad (2)$$

as an equality in $[0, \infty]$. To see this, note first that by (i) the $c \times \mu$ -measurable subsets of $\{k\} \times X$ are precisely the sets $\{k\} \times A$ with $A \in \mathcal{A}$, and that

$$(c \times \mu)(\{k\} \times A) = c(\{k\}) \cdot \mu(A) = \mu(A).$$

Hence (2) holds when g is the characteristic function of a measurable subset of $\{k\} \times X$, both sides being $\mu(A)$; by linearity of integration it holds for every nonnegative simple function vanishing outside $\{k\} \times X$. For general nonnegative measurable g , the Simple Approximation Theorem provides an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions with $0 \leq \varphi_n \leq g \cdot \chi_{\{k\} \times X}$ converging pointwise to $g \cdot \chi_{\{k\} \times X}$; the inequality forces each φ_n to vanish outside $\{k\} \times X$, and the sections $\varphi_n(k, \cdot)$ increase pointwise to $g(k, \cdot)$. Two applications of the Monotone Convergence Theorem, one on $\mathbb{N} \times X$ and one on X , then give (2).

A summation formula. For every nonnegative $c \times \mu$ -measurable g on $\mathbb{N} \times X$,

$$\int_{\mathbb{N} \times X} g d(c \times \mu) = \sum_{k=1}^{\infty} \int_X g(k, x) d\mu(x), \quad (3)$$

again as an equality in $[0, \infty]$. Indeed, let $g_n = g \cdot \chi_{\{1, \dots, n\} \times X}$. Then $\{g_n\}$ is an increasing sequence of nonnegative measurable functions converging pointwise to g , and since $\{1, \dots, n\} \times X$ is the disjoint union of the sets $\{k\} \times X$ for $1 \leq k \leq n$, the additivity of integration over disjoint domains together with (2) gives

$$\int_{\mathbb{N} \times X} g_n d(c \times \mu) = \sum_{k=1}^n \int_{\{k\} \times X} g d(c \times \mu) = \sum_{k=1}^n \int_X g(k, x) d\mu(x).$$

Letting $n \rightarrow \infty$ and invoking the Monotone Convergence Theorem yields (3). Observe that no σ -finiteness of μ was used.

Part (iii). Let $f: \mathbb{N} \times X \rightarrow \mathbb{R}$. Suppose first that f is integrable over $\mathbb{N} \times X$ with respect to $c \times \mu$. Then f is $c \times \mu$ -measurable, so by (ii) each $f(k, \cdot)$ is μ -measurable, and applying (3) to the nonnegative measurable function $|f|$,

$$\sum_{k=1}^{\infty} \int_X |f(k, x)| d\mu(x) = \int_{\mathbb{N} \times X} |f| d(c \times \mu) < \infty. \quad (4)$$

In particular each term of this convergent series is finite, so each $f(k, \cdot)$ is integrable over X with respect to μ , and the displayed sum is finite.

Conversely, suppose each $f(k, \cdot)$ is μ -measurable with $\int_X |f(k, \cdot)| d\mu < \infty$ and that $\sum_k \int_X |f(k, \cdot)| d\mu < \infty$. By (ii), f is $c \times \mu$ -measurable, and then (4), read from right to left, gives $\int_{\mathbb{N} \times X} |f| d(c \times \mu) < \infty$. Hence f is integrable over $\mathbb{N} \times X$ with respect to $c \times \mu$.

Part (iv). Let f be integrable over $\mathbb{N} \times X$ with respect to $c \times \mu$ and write $f = f^+ - f^-$. Both f^+ and f^- are nonnegative, $c \times \mu$ -measurable and dominated by $|f|$, hence integrable, and by (3)

$$\int_{\mathbb{N} \times X} f^{\pm} d(c \times \mu) = \sum_{k=1}^{\infty} \int_X f^{\pm}(k, x) d\mu(x) < \infty.$$

By (iii) each $f(k, \cdot)$ is integrable over X , so $\int_X f(k, \cdot) d\mu = \int_X f^+(k, \cdot) d\mu - \int_X f^-(k, \cdot) d\mu$ for each k . Since the two series above are convergent series of nonnegative terms, they may be subtracted term by term, and therefore

$$\int_{\mathbb{N} \times X} f d(c \times \mu) = \int_{\mathbb{N} \times X} f^+ d(c \times \mu) - \int_{\mathbb{N} \times X} f^- d(c \times \mu) = \sum_{k=1}^{\infty} \int_X f(k, x) d\mu(x),$$

the series converging absolutely by (4), and its value being finite. This is the asserted identity, and by Problem 2 it may be rewritten as the iterated integral

$$\int_{\mathbb{N} \times X} f d(c \times \mu) = \int_{\mathbb{N}} \left[\int_X f(k, x) d\mu(x) \right] dc(k),$$

which is Fubini's Theorem for $c \times \mu$, valid here with no σ -finiteness assumption on μ .

Problem 20.5. Let $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{M}, c)$, the measure space defined in Problem 2. Define $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ by setting

$$f(x, y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y, \\ -2 + 2^{-x} & \text{if } x = y + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is measurable with respect to the product measure $c \times c$. Also show that

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m, n) dc(m) \right] dc(n) \neq \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m, n) dc(n) \right] dc(m).$$

Is this a contradiction either of Fubini's Theorem or Tonelli's Theorem?

Solution. Measurability. As shown in Problem 3, every subset of $\mathbb{N} \times \mathbb{N}$ is $c \times c$ -measurable, since each singleton $\{(m, n)\} = \{m\} \times \{n\}$ is a measurable rectangle and every subset of $\mathbb{N} \times \mathbb{N}$ is a countable union of singletons. Hence for every $\alpha \in \mathbb{R}$ the set $\{(m, n) \mid f(m, n) > \alpha\}$ is $c \times c$ -measurable, so f is measurable with respect to $c \times c$. (Alternatively, apply Problem 4(ii) with $X = \mathbb{N}$ and $\mu = c$.)

The first iterated integral. Fix $n \in \mathbb{N}$ and consider the function $m \mapsto f(m, n)$. By the definition of f it vanishes except at $m = n$, where it equals $2 - 2^{-n}$, and at $m = n + 1$, where the condition $m = n + 1$ holds so that it equals $-2 + 2^{-(n+1)}$. Thus $f(\cdot, n)$ has finite support and is therefore integrable with respect to c , and by Problem 2

$$\int_{\mathbb{N}} f(m, n) dc(m) = \sum_{m=1}^{\infty} f(m, n) = (2 - 2^{-n}) + (-2 + 2^{-n-1}) = -2^{-n} + 2^{-n-1} = -2^{-n-1}.$$

The resulting function of n is absolutely summable, since $\sum_{n=1}^{\infty} 2^{-n-1} = 1/2 < \infty$, so it is c -integrable, and again by Problem 2

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m, n) dc(m) \right] dc(n) = \sum_{n=1}^{\infty} (-2^{-n-1}) = -\frac{1}{2}.$$

The second iterated integral. Fix $m \in \mathbb{N}$ and consider $n \mapsto f(m, n)$. It vanishes except at $n = m$, where it equals $2 - 2^{-m}$, and at $n = m - 1$, which is a natural number only when $m \geq 2$, where it equals $-2 + 2^{-m}$. Again the support is finite, so $f(m, \cdot)$ is c -integrable and by Problem 2

$$\int_{\mathbb{N}} f(m, n) dc(n) = \sum_{n=1}^{\infty} f(m, n) = \begin{cases} 2 - 2^{-1} = \frac{3}{2} & \text{if } m = 1, \\ (2 - 2^{-m}) + (-2 + 2^{-m}) = 0 & \text{if } m \geq 2. \end{cases}$$

This function of m has finite support, so it is c -integrable and

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m, n) dc(n) \right] dc(m) = \sum_{m=1}^{\infty} \left[\int_{\mathbb{N}} f(m, n) dc(n) \right] = \frac{3}{2}.$$

Therefore both iterated integrals exist and are finite, yet

$$-\frac{1}{2} = \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m, n) dc(m) \right] dc(n) \neq \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m, n) dc(n) \right] dc(m) = \frac{3}{2},$$

as was to be shown. Note also that f is bounded, with $|f| \leq 2$ everywhere.

No contradiction. Both measure spaces here are complete and σ -finite, so the hypotheses of the theorems that concern the measure spaces are met. What fails is the hypothesis on f .

Fubini's Theorem requires that f be integrable over $\mathbb{N} \times \mathbb{N}$ with respect to $c \times c$, and f is not. Indeed, for each m we have $|f(m, m)| = 2 - 2^{-m} \geq 3/2$, so by Problem 4(iii) applied to $|f|$ (or directly by

Problem 2 on the countable set $\mathbb{N} \times \mathbb{N}$),

$$\int_{\mathbb{N} \times \mathbb{N}} |f| d(c \times c) = \sum_{(m,n)} |f(m,n)| \geq \sum_{m=1}^{\infty} (2 - 2^{-m}) = \infty.$$

So f is not $c \times c$ -integrable and Fubini's Theorem does not apply.

Tonelli's Theorem requires that f be nonnegative, and f takes negative values (for example $f(2, 1) = -2 + 2^{-2} = -7/4$). So Tonelli's Theorem does not apply either. Consistently with Tonelli, the two iterated integrals of the nonnegative function $|f|$ do agree: both equal ∞ .

Thus the example is not a contradiction; it shows precisely that the integrability hypothesis in Fubini's Theorem cannot be omitted, and that the mere existence and finiteness of both iterated integrals of a measurable function does not imply their equality. In the language of Problem 3 it is the familiar fact that a double series which is not absolutely summable may have two different iterated sums.

Problem 20.6. Let $X = Y$ be the interval $[0, 1]$, with $\mathcal{A} = \mathcal{B}$ the class of Borel sets. Let μ be Lebesgue measure and $\nu = c$ the counting measure. Show that the diagonal $\Delta = \{(x, y) \mid x = y\}$ is measurable with respect to the product measure $\mu \times c$ (is an $\mathcal{R}_{\sigma\delta}$, in fact). Show that if f is the characteristic function of Δ ,

$$\int_{[0,1] \times [0,1]} f d(\mu \times c) \neq \int_{[0,1]} \left[\int_{[0,1]} f(x, y) dc(y) \right] d\mu(x).$$

Is this a contradiction either of Fubini's Theorem or Tonelli's Theorem?

Solution. The diagonal is an $\mathcal{R}_{\sigma\delta}$ set. For each natural number n partition $[0, 1]$ into the n Borel sets

$$I_{n,k} = \left[\frac{k-1}{n}, \frac{k}{n} \right) \quad (1 \leq k \leq n-1), \quad I_{n,n} = \left[\frac{n-1}{n}, 1 \right],$$

and put

$$D_n = \bigcup_{k=1}^n I_{n,k} \times I_{n,k}.$$

Each $I_{n,k} \times I_{n,k}$ is a measurable rectangle, so D_n is a finite, hence countable, union of members of $\mathcal{R}$, that is, D_n is an $\mathcal{R}_{\sigma}$ set. I claim

$$\Delta = \bigcap_{n=1}^{\infty} D_n.$$

If $x = y$ then x lies in exactly one of the sets $I_{n,k}$, and $(x, y) = (x, x) \in I_{n,k} \times I_{n,k} \subseteq D_n$; so $\Delta \subseteq D_n$ for every n . Conversely, if $(x, y) \in D_n$ then x and y lie in a common $I_{n,k}$, an interval of length at most $1/n$, so $|x - y| \leq 1/n$. If (x, y) lies in every D_n then $|x - y| \leq 1/n$ for all n , so $x = y$. This proves the claim, so Δ is an $\mathcal{R}_{\sigma\delta}$ set. Since the $(\mu \times c)^*$ -measurable sets form a σ -algebra containing $\mathcal{R}$, the set Δ is measurable with respect to $\mu \times c$, and $f = \chi_{\Delta}$ is a nonnegative $\mu \times c$ -measurable function.

The product measure of the diagonal is infinite. I claim $(\mu \times c)(\Delta) = \infty$. Recall that $(\mu \times c)^*(\Delta)$ is the infimum of $\sum_j \mu(A_j) \cdot c(B_j)$ over all countable covers of Δ by measurable rectangles $A_j \times B_j$. Suppose, to get a contradiction, that some such cover $\{A_j \times B_j\}_{j=1}^{\infty}$ satisfies

$$\sum_{j=1}^{\infty} \mu(A_j) \cdot c(B_j) < \infty.$$

Split the index set as $J_1 = \{j \mid \mu(A_j) = 0\}$ and $J_2 = \{j \mid \mu(A_j) > 0\}$. For $j \in J_2$ the finiteness of $\mu(A_j) \cdot c(B_j)$ forces $c(B_j) < \infty$, that is, B_j is a finite set. Put $N = \bigcup_{j \in J_1} A_j$, a Borel set with $\mu(N) = 0$ by countable subadditivity. Now let $x \in [0, 1] \sim N$. Since $(x, x) \in \Delta$, there is a j with $(x, x) \in A_j \times B_j$. If j belonged to J_1 then $x \in A_j \subseteq N$, which is false; so $j \in J_2$, and $x \in B_j$ with B_j finite. Therefore

$$[0, 1] \sim N \subseteq \bigcup_{j \in J_2} B_j,$$

a countable union of finite sets, hence a countable set. But $\mu([0, 1] \sim N) = 1 - 0 = 1$, so $[0, 1] \sim N$ has positive Lebesgue measure and is therefore uncountable. This contradiction shows every countable cover of Δ by measurable rectangles has infinite total premeasure, so

$$\int_{[0,1] \times [0,1]} f d(\mu \times c) = (\mu \times c)(\Delta) = \infty.$$

The iterated integrals. Fix $x \in [0, 1]$. The x -section of Δ is $\Delta_x = \{x\}$, so $f(x, \cdot) = \chi_{\{x\}}$ is c -measurable and

$$\int_{[0,1]} f(x, y) dc(y) = c(\{x\}) = 1.$$

The resulting function of x is the constant 1, which is Borel measurable and μ -integrable over $[0, 1]$, so

$$\int_{[0,1]} \left[\int_{[0,1]} f(x, y) dc(y) \right] d\mu(x) = \int_{[0,1]} 1 d\mu = \mu([0, 1]) = 1.$$

Hence

$$\int_{[0,1] \times [0,1]} f d(\mu \times c) = \infty \neq 1 = \int_{[0,1]} \left[\int_{[0,1]} f(x, y) dc(y) \right] d\mu(x),$$

which is the asserted inequality. For comparison, in the other order the y -section of Δ is $\{y\}$, a set of Lebesgue measure zero, so

$$\int_{[0,1]} \left[\int_{[0,1]} f(x, y) d\mu(x) \right] dc(y) = \int_{[0,1]} \mu(\{y\}) dc(y) = \int_{[0,1]} 0 dc = 0.$$

All three quantities, ∞ , 1 and 0, are different.

No contradiction. Consider first Tonelli's Theorem. Its hypotheses require both factors to be σ -finite, and the counting measure c on $[0, 1]$ is not σ -finite: a subset of $[0, 1]$ has finite c -measure exactly when it is finite, and a countable union of finite sets is countable, whereas $[0, 1]$ is uncountable. So $[0, 1]$ is not the union of countably many sets of finite c -measure and Tonelli's Theorem does not apply. This example therefore shows that the σ -finiteness hypothesis in Tonelli's Theorem cannot be dropped, even for a nonnegative measurable function.

Now consider Fubini's Theorem. It requires that f be integrable over $X \times Y$ with respect to $\mu \times c$. Here $f \geq 0$ and $\int_{[0,1] \times [0,1]} |f| d(\mu \times c) = \infty$, so f is not integrable with respect to the product measure and Fubini's Theorem does not apply. (The completeness hypothesis of Fubini's Theorem is on the second factor, which here is $([0, 1], \mathcal{B}, c)$, and this measure is complete: the only set of c -measure zero is $\emptyset$, whose only subset is $\emptyset \in \mathcal{B}$. So that is not where the hypotheses fail. Note that $\mathcal{B}$ is the class of Borel sets, not all subsets of $[0, 1]$, so completeness has to be argued this way rather than from $\mathcal{B} = 2^{[0,1]}$.)

The example makes precise the remark preceding Tonelli's Theorem in the text: from the existence and finiteness of the iterated integral on the right-hand side of (1) one cannot infer that f is integrable over $X \times Y$. Here the iterated integral exists and equals 1, while f is not integrable over the product at all.

Problem 20.7. Prove that the conclusion of Tonelli's Theorem is true if one of the spaces is the space $(\mathbb{N}, \mathcal{M}, c)$ defined in Problem 2 and the other space is a general measure space that need not be σ -finite.

Solution. Statement to be proved. Let $(X, \mathcal{A}, \mu)$ be a measure space which is not assumed σ -finite, and which satisfies the standing hypothesis of Problem 4: $\mathcal{A}$ is the σ -algebra of μ^* -measurable sets, where μ^* is the outer measure induced by μ . This hypothesis implies that μ is complete, since the μ^* -measurable sets always form a complete σ -algebra, and conversely, as shown in Problem 4, completeness of μ implies it as soon as μ is σ -finite. So in the presence of σ -finiteness it is exactly the completeness that Tonelli's Theorem assumes of the second factor, and the only hypothesis genuinely being discarded

below is σ -finiteness. It cannot be weakened to bare completeness: the second counterexample of Problem 4 exhibits a complete measure μ and a set $E \subseteq \mathbb{N} \times X$ which is $c \times \mu$ -measurable but has a section $E_1 \notin \mathcal{A}$, so that for $f = \chi_E$, a nonnegative $c \times \mu$ -measurable function, the inner integral $\int_X f(1, x) d\mu(x)$ is not even defined and the conclusion of Tonelli's Theorem fails to make sense. Let f be a nonnegative function on $\mathbb{N} \times X$ that is measurable with respect to the product measure $c \times \mu$. Then for every $k \in \mathbb{N}$ the section $f(k, \cdot)$ is μ -measurable, for every $x \in X$ the section $f(\cdot, x)$ is c -measurable, the functions $k \mapsto \int_X f(k, x) d\mu(x)$ and $x \mapsto \int_{\mathbb{N}} f(k, x) dc(k)$ are measurable with respect to c and μ respectively, and

$$\int_{\mathbb{N}} \left[\int_X f(k, x) d\mu(x) \right] dc(k) = \int_{\mathbb{N} \times X} f d(c \times \mu) = \int_X \left[\int_{\mathbb{N}} f(k, x) dc(k) \right] d\mu(x),$$

all three quantities lying in $[0, \infty]$. Since the flip map $(k, x) \mapsto (x, k)$ carries measurable rectangles of $\mathbb{N} \times X$ onto measurable rectangles of $X \times \mathbb{N}$ preserving the premeasure λ , it carries the outer measure $(c \times \mu)^*$ onto $(\mu \times c)^*$ and hence the measure $c \times \mu$ onto $\mu \times c$; so it is immaterial which of the two factors is the counting measure space, and it suffices to treat $\mathbb{N} \times X$.

Sections. By Problem 4(ii), $f(k, \cdot)$ is μ -measurable for each k ; this is the one place the standing hypothesis on $\mathcal{A}$ is used, and it plays exactly the role that completeness of ν plays in Tonelli's Theorem as stated in the text. Note that the conclusion here is sharper than the text's: measurability of the sections holds for every k , not merely for almost all k , which is automatic since $\emptyset$ is the only set of c -measure zero. Every function on $\mathbb{N}$ is c -measurable by Problem 2, so $f(\cdot, x)$ is c -measurable for every $x \in X$; likewise the function $k \mapsto \int_X f(k, x) d\mu(x)$, being a function on $\mathbb{N}$ with values in $[0, \infty]$, is automatically c -measurable.

The first equality. This is exactly formula (3) of Problem 4 together with Problem 2, and I repeat the two-line argument for completeness. For each k the map $x \mapsto (k, x)$ identifies the $c \times \mu$ -measurable subsets of the slab $\{k\} \times X$ with $\mathcal{A}$, by Problem 4(i), and preserves measure, since $(c \times \mu)(\{k\} \times A) = c(\{k\}) \cdot \mu(A) = \mu(A)$. Hence for characteristic functions of measurable subsets of $\{k\} \times X$, and so by linearity for nonnegative simple functions vanishing outside $\{k\} \times X$, and so by the Simple Approximation Theorem and the Monotone Convergence Theorem for every nonnegative $c \times \mu$ -measurable g ,

$$\int_{\{k\} \times X} g d(c \times \mu) = \int_X g(k, x) d\mu(x).$$

Now let $f_n = f \cdot \chi_{\{1, \dots, n\} \times X}$. Then $\{f_n\}$ is an increasing sequence of nonnegative $c \times \mu$ -measurable functions converging pointwise on $\mathbb{N} \times X$ to f , and since $\{1, \dots, n\} \times X$ is the disjoint union of the slabs $\{k\} \times X$ for $1 \leq k \leq n$, additivity of integration over disjoint domains gives

$$\int_{\mathbb{N} \times X} f_n d(c \times \mu) = \sum_{k=1}^n \int_X f(k, x) d\mu(x).$$

By the Monotone Convergence Theorem the left-hand side converges to $\int_{\mathbb{N} \times X} f d(c \times \mu)$, while the right-hand side increases to $\sum_{k=1}^{\infty} \int_X f(k, x) d\mu(x)$, which by Problem 2, applied to the nonnegative function $k \mapsto \int_X f(k, x) d\mu(x)$ on $\mathbb{N}$, is $\int_{\mathbb{N}} [\int_X f(k, x) d\mu(x)] dc(k)$. Therefore

$$\int_{\mathbb{N} \times X} f d(c \times \mu) = \sum_{k=1}^{\infty} \int_X f(k, x) d\mu(x) = \int_{\mathbb{N}} \left[\int_X f(k, x) d\mu(x) \right] dc(k), \quad (*)$$

as an equality in $[0, \infty]$. Nowhere did the argument use σ -finiteness of μ .

The second equality. By Problem 2, for each fixed $x \in X$,

$$\int_{\mathbb{N}} f(k, x) dc(k) = \sum_{k=1}^{\infty} f(k, x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(k, x).$$

Each partial sum $x \mapsto \sum_{k=1}^n f(k, x)$ is μ -measurable, being a finite sum of the μ -measurable functions $f(k, \cdot)$, and these partial sums increase, since $f \geq 0$. Hence $x \mapsto \int_{\mathbb{N}} f(k, x) dc(k)$ is μ -measurable,

being the pointwise limit of μ -measurable functions, and by the Monotone Convergence Theorem on X together with the linearity of integration for nonnegative measurable functions,

$$\int_X \left[\int_{\mathbb{N}} f(k, x) d\mu(k) \right] d\mu(x) = \lim_{n \rightarrow \infty} \int_X \sum_{k=1}^n f(k, x) d\mu(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_X f(k, x) d\mu(x) = \sum_{k=1}^{\infty} \int_X f(k, x) d\mu(x).$$

Comparing with (*) gives the second equality, and the proof is complete.

Remark on where σ -finiteness was used in the text and why it is not needed here. In the proof of Tonelli's Theorem given in the text, σ -finiteness of μ and ν is used only to know that $\mu \times \nu$ is σ -finite, so that assertion (i) of the Simple Approximation Theorem provides an increasing sequence $\{\varphi_n\}$ of simple functions converging to f with each φ_n vanishing outside a set of finite measure; this is what makes each φ_n integrable and allows Theorem 6 to be applied. In the present situation that device is replaced by the exhaustion of $\mathbb{N} \times X$ by the slabs $\{k\} \times X$, on each of which the iterated and product integrals coincide trivially, because $c(\{k\}) = 1$; the sets $\{k\} \times X$ need not have finite $c \times \mu$ -measure, but the Monotone Convergence Theorem is all that is required to pass from them to $\mathbb{N} \times X$. Problem 6 shows that this is genuinely special to the counting measure space on a countable set: for the counting measure on the uncountable set $[0, 1]$, which is likewise not σ -finite, the conclusion of Tonelli's Theorem fails.

20.2 Problems 20.8–20.14

Problem 20.8. In the proof of Fubini's Theorem justify the excision from $X \times Y$ of a set of $\mu \times \nu$ measure zero.

Solution. Recall the point in the proof at which the excision is made. Having chosen, by the Simple Approximation Theorem, an increasing sequence $\{\varphi_k\}$ of simple functions with $0 \leq \varphi_k \leq f$ on $X \times Y$ and $\varphi_k \rightarrow f$ pointwise, one must prove

$$\lim_{k \rightarrow \infty} \int_X \left[\int_Y \varphi_k(x, y) d\nu(y) \right] d\mu(x) = \int_X \left[\int_Y f(x, y) d\nu(y) \right] d\mu(x). \quad (7)$$

The subsequent argument fixes a point $x \in X$ and applies the Monotone Convergence Theorem on Y to the sequence $\{\varphi_k(x, \cdot)\}$; for this one wants $\varphi_k(x, \cdot)$ to be ν -integrable for every $x \in X$ and every k , whereas Theorem 6 only delivers this for almost all x . The excision repairs the discrepancy. Two things must be justified: that the set to be excised really is a set of $\mu \times \nu$ measure zero, and that excising a set of $\mu \times \nu$ measure zero changes neither side of (7), nor either side of the conclusion (1).

Step 1: the set to be excised. Each φ_k is simple and, since $0 \leq \varphi_k \leq f$ with f integrable over $X \times Y$, each φ_k is integrable over $X \times Y$ with respect to $\mu \times \nu$. Therefore Theorem 6 applies to φ_k : for almost all $x \in X$ the x -section $\varphi_k(x, \cdot)$ is integrable over Y with respect to ν . By the meaning of "almost all", for each k there is a set $N_k \in \mathcal{A}$ with $\mu(N_k) = 0$ such that $\varphi_k(x, \cdot)$ is ν -integrable for every $x \in X \setminus N_k$. Put

$$N = \bigcup_{k=1}^{\infty} N_k.$$

Then $N \in \mathcal{A}$ and, by the countable subadditivity of μ , $\mu(N) = 0$. For every $x \in X \setminus N$ and every k , $\varphi_k(x, \cdot)$ is integrable over Y . The set to be excised is

$$Z = N \times Y.$$

Step 2: Z is $(\mu \times \nu)$ -measurable and $(\mu \times \nu)(Z) = 0$. Indeed Z is a measurable rectangle, so it belongs to the semiring $\mathcal{R}$ on which the premeasure λ is defined, and

$$\lambda(Z) = \mu(N) \cdot \nu(Y) = 0 \cdot \nu(Y) = 0,$$

using the convention $0 \cdot \infty = 0$ under which λ is defined on $\mathcal{R}$. Since Z is covered by the single member Z of $\mathcal{R}$, the outer measure induced by λ satisfies $(\mu \times \nu)^*(Z) \leq \lambda(Z) = 0$. A set of outer measure

zero is measurable with respect to any outer measure, so Z is $(\mu \times \nu)$ -measurable and $(\mu \times \nu)(Z) = 0$. (Equivalently, invoke the Caratheodory-Hahn Theorem, which asserts that the Caratheodory extension of the premeasure λ extends λ , so $(\mu \times \nu)(Z) = \lambda(Z) = 0$.)

Step 3: excising a set of product measure zero changes nothing. Let $Z \subseteq X \times Y$ be any $(\mu \times \nu)$ -measurable set with $(\mu \times \nu)(Z) = 0$, and put $W = (X \times Y) \setminus Z$, a $(\mu \times \nu)$ -measurable set. Let g be any nonnegative $(\mu \times \nu)$ -measurable function on $X \times Y$; in the application g will be f or one of the φ_k . First, integration over $X \times Y$ is unchanged. By the additivity of integration over domains,

$$\int_{X \times Y} g d(\mu \times \nu) = \int_W g d(\mu \times \nu) + \int_Z g d(\mu \times \nu) = \int_W g d(\mu \times \nu),$$

since the integral of a nonnegative measurable function over a set of measure zero is zero.

Second, the iterated integral is unchanged, and this is exactly what Lemma 4 is for. Because ν is complete and $(\mu \times \nu)(Z) = 0$, Lemma 4 provides a set $N' \in \mathcal{A}$ with $\mu(N') = 0$ such that for every $x \in X \setminus N'$ the x -section Z_x is ν -measurable and $\nu(Z_x) = 0$. Fix such an x and suppose $g(x, \cdot)$ is ν -measurable. Then $W_x = Y \setminus Z_x$ is ν -measurable and

$$\int_Y g(x, y) d\nu(y) = \int_{W_x} g(x, y) d\nu(y) + \int_{Z_x} g(x, y) d\nu(y) = \int_{W_x} g(x, y) d\nu(y),$$

by the additivity of integration over domains for a nonnegative measurable function, together with the fact that the integral over a set of ν -measure zero vanishes. (The identity is asserted in $[0, \infty]$; no subtraction is performed, so it is valid even when the integrals are infinite.) Moreover, for such an x , the x -section of g is ν -measurable on Y if and only if the x -section of the restriction of g to W is ν -measurable on W_x : the two functions differ only on the ν -null set Z_x , and ν is complete. Thus the two inner integrals agree for all x outside the μ -null set N' , so the two functions of x that are integrated on the outside of the iterated integral agree almost everywhere on X , and therefore

$$\int_X \left[\int_{W_x} g(x, y) d\nu(y) \right] d\mu(x) = \int_X \left[\int_Y g(x, y) d\nu(y) \right] d\mu(x).$$

Applying this with $g = \varphi_k$ for each k , and with $g = f$, shows that both sides of (7), and likewise both sides of the desired conclusion (1), are unchanged when $X \times Y$ is replaced by $W = (X \times Y) \setminus Z$.

Step 4: conclusion. Take $Z = N \times Y$ as in Steps 1 and 2. Replacing $X \times Y$ by $W = (X \setminus N) \times Y$ changes neither side of (7) nor either side of (1), by Step 3; the restrictions of the φ_k to W still form an increasing sequence of simple functions with $0 \leq \varphi_k \leq f$ converging pointwise to f on W ; and now, for every x in the new base space $X \setminus N$ and every k , the section $\varphi_k(x, \cdot)$ is ν -integrable. So the proof may proceed as though this integrability held at every point of X . Finally, the footnote convention introduced with the statement of Fubini's Theorem, that $\int_X h d\mu$ denotes $\int_{X_0} h d\mu$ whenever $\mu(X \setminus X_0) = 0$, means that the equality proved on $(X \setminus N) \times Y$ is literally the asserted equality (1) on $X \times Y$.

Problem 20.9. Let $X = Y = [0, 1]$, and let $\mu = \nu$ be Lebesgue measure. Show that each open set in $X \times Y$ is measurable, and hence each Borel set in $X \times Y$ is measurable. Is every continuous real-valued function on $[0, 1] \times [0, 1]$ measurable with respect to the product measure?

Solution. Here $\mathcal{A} = \mathcal{B}$ is the σ -algebra of Lebesgue measurable subsets of $[0, 1]$, and $\mu \times \nu$ is the Caratheodory extension of the premeasure $\lambda(A \times B) = \mu(A) \cdot \nu(B)$ defined on the semiring $\mathcal{R}$ of measurable rectangles. Two facts will be used repeatedly. First, by the Caratheodory-Hahn Theorem the extension $\mu \times \nu$ is a measure defined on the σ -algebra of $(\mu \times \nu)^*$ -measurable sets, and this σ -algebra contains $\mathcal{R}$; so every measurable rectangle is $(\mu \times \nu)$ -measurable. Second, the $(\mu \times \nu)$ -measurable sets form a σ -algebra, hence are closed under countable unions, countable intersections and complements. Open sets are measurable. Give $X \times Y = [0, 1] \times [0, 1]$ the product topology, which is the topology it

inherits as a subspace of the Euclidean plane. Let

$$\mathcal{U} = \{ (I \cap [0, 1]) \times (J \cap [0, 1]) : I, J \text{ bounded open intervals with rational endpoints} \}.$$

The set of bounded open intervals with rational endpoints is countable, so $\mathcal{U}$ is a countable collection. I claim $\mathcal{U}$ is a base for the topology of $[0, 1]^2$. Let $O \subseteq [0, 1]^2$ be open and let $(a, b) \in O$. By the definition of the product topology there is an $\varepsilon > 0$ such that

$$\{(s, t) \in [0, 1]^2 : |s - a| < \varepsilon, |t - b| < \varepsilon\} \subseteq O.$$

By the density of the rationals choose rationals p, q with $a - \varepsilon < p < a < q < a + \varepsilon$, and rationals r, s with $b - \varepsilon < r < b < s < b + \varepsilon$. Then

$$(a, b) \in ((p, q) \cap [0, 1]) \times ((r, s) \cap [0, 1]) \subseteq O,$$

and the middle set belongs to $\mathcal{U}$. Hence O is the union of those members of $\mathcal{U}$ that it contains, and since $\mathcal{U}$ is countable this is a countable union.

Now each member of $\mathcal{U}$ is a measurable rectangle: $I \cap [0, 1]$ and $J \cap [0, 1]$ are intervals, and every interval is Lebesgue measurable. Therefore each member of $\mathcal{U}$ is $(\mu \times \nu)$ -measurable, and O , being a countable union of such sets, is $(\mu \times \nu)$ -measurable.

Borel sets are measurable. Let $\mathcal{M}$ denote the σ -algebra of $(\mu \times \nu)$ -measurable subsets of $[0, 1]^2$. We have just shown that $\mathcal{M}$ contains every open subset of $[0, 1]^2$. The Borel σ -algebra $\mathcal{B}([0, 1]^2)$ is by definition the smallest σ -algebra containing the open sets. Since $\mathcal{M}$ is a σ -algebra containing the open sets, minimality gives $\mathcal{B}([0, 1]^2) \subseteq \mathcal{M}$. So every Borel subset of $[0, 1] \times [0, 1]$ is measurable with respect to the product measure.

Continuous functions. Yes: every continuous real-valued function on $[0, 1] \times [0, 1]$ is measurable with respect to the product measure. Indeed, let $f : [0, 1]^2 \rightarrow \mathbb{R}$ be continuous and let $c \in \mathbb{R}$. Then

$$\{(x, y) \in [0, 1]^2 : f(x, y) > c\} = f^{-1}((c, \infty))$$

is the preimage of an open subset of $\mathbb{R}$ under a continuous map, hence is open in $[0, 1]^2$, hence is $(\mu \times \nu)$ -measurable by the first part. Since $\{f > c\}$ is measurable for every real c , f is measurable with respect to $\mu \times \nu$.

Remark. Since $[0, 1]^2$ is compact, such an f is bounded, and $(\mu \times \nu)([0, 1]^2) = \mu([0, 1]) \cdot \nu([0, 1]) = 1 < \infty$, so f is in fact integrable over $[0, 1]^2$ with respect to $\mu \times \nu$ and Fubini's Theorem applies to it: iterated integration of a continuous function over the unit square is justified.

Problem 20.10. Let h and g be integrable functions on X and Y , and define $f(x, y) = h(x)g(y)$. Show that f is integrable on $X \times Y$ with respect to the product measure, then

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X h d\mu \int_Y g d\nu.$$

(Note: We do not need to assume that μ and ν are σ -finite.)

Solution. Throughout write $h \otimes g$ for the function $(x, y) \mapsto h(x)g(y)$ on $X \times Y$. Since h and g are integrable they are finite almost everywhere, by Proposition 9 of Chapter 18. Let $E = \{x \in X : |h(x)| = \infty\} \in \mathcal{A}$ and $F = \{y \in Y : |g(y)| = \infty\} \in \mathcal{B}$, so $\mu(E) = 0$ and $\nu(F) = 0$. Replacing h by $h \cdot \chi_{X \setminus E}$ and g by $g \cdot \chi_{Y \setminus F}$ changes $h \otimes g$ only on $(E \times Y) \cup (X \times F)$, and each of these is a measurable rectangle of premeasure $\mu(E) \cdot \nu(Y) = 0$, respectively $\mu(X) \cdot \nu(F) = 0$, under the convention $0 \cdot \infty = 0$; so their union is a $(\mu \times \nu)$ -null set. The replacement therefore changes neither $\int_X h d\mu$ nor $\int_Y g d\nu$, and it changes $h \otimes g$ only on a set of product measure zero. Since the product measure is induced by an outer measure it is complete, so the original $h \otimes g$ is $(\mu \times \nu)$ -measurable if the modified one is, and the two have the same integral over $X \times Y$. We may therefore assume h and g are real-valued.

Step 1: the case of nonnegative integrable h and g . By the Simple Approximation Theorem there is

an increasing sequence $\{\varphi_k\}$ of simple μ -measurable functions on X with $0 \leq \varphi_k \leq h$ and $\varphi_k \rightarrow h$ pointwise on X , and similarly an increasing sequence $\{\psi_k\}$ of simple ν -measurable functions on Y with $0 \leq \psi_k \leq g$ and $\psi_k \rightarrow g$ pointwise on Y .

Each φ_k vanishes outside a set of finite μ -measure. Indeed, write φ_k in canonical form $\varphi_k = \sum_{i=1}^m a_i \chi_{A_i}$, where the $a_i > 0$ are the distinct positive values of φ_k and the $A_i \in \mathcal{A}$ are disjoint. On A_i we have $h \geq \varphi_k = a_i > 0$, so by Chebychev's Inequality

$$\mu(A_i) \leq \frac{1}{a_i} \int_X h d\mu < \infty.$$

Likewise, writing $\psi_k = \sum_{j=1}^n b_j \chi_{B_j}$ in canonical form, each $\nu(B_j) < \infty$.

Now

$$\varphi_k \otimes \psi_k = \sum_{i=1}^m \sum_{j=1}^n a_i b_j \chi_{A_i \times B_j},$$

a finite linear combination of characteristic functions of the pairwise disjoint measurable rectangles $A_i \times B_j$. By the Caratheodory-Hahn Theorem the product measure extends the premeasure λ , so each $A_i \times B_j$ is $(\mu \times \nu)$ -measurable with

$$(\mu \times \nu)(A_i \times B_j) = \mu(A_i) \cdot \nu(B_j) < \infty.$$

Hence $\varphi_k \otimes \psi_k$ is a simple $(\mu \times \nu)$ -measurable function vanishing outside a set of finite product measure, so it is integrable over $X \times Y$, and by the linearity of integration for simple functions

$$\int_{X \times Y} \varphi_k \otimes \psi_k d(\mu \times \nu) = \sum_{i,j} a_i b_j \mu(A_i) \nu(B_j) = \left(\sum_i a_i \mu(A_i) \right) \left(\sum_j b_j \nu(B_j) \right) = \int_X \varphi_k d\mu \cdot \int_Y \psi_k d\nu,$$

the middle step being the factoring of a finite double sum of finite nonnegative numbers.

Since $\{\varphi_k\}$ and $\{\psi_k\}$ are increasing and nonnegative, $\{\varphi_k \otimes \psi_k\}$ is an increasing sequence of nonnegative $(\mu \times \nu)$ -measurable functions, and for every (x, y) ,

$$\varphi_k(x) \psi_k(y) \rightarrow h(x) g(y).$$

Thus $h \otimes g$, being the pointwise limit of $(\mu \times \nu)$ -measurable functions, is $(\mu \times \nu)$ -measurable, and by the Monotone Convergence Theorem, applied on $X \times Y$ and then on each of X and Y ,

$$\int_{X \times Y} h \otimes g d(\mu \times \nu) = \lim_{k \rightarrow \infty} \int_{X \times Y} \varphi_k \otimes \psi_k d(\mu \times \nu) = \lim_{k \rightarrow \infty} \int_X \varphi_k d\mu \cdot \lim_{k \rightarrow \infty} \int_Y \psi_k d\nu = \int_X h d\mu \cdot \int_Y g d\nu.$$

The right-hand side is finite because h and g are integrable, so $h \otimes g$ is integrable over $X \times Y$ and the asserted formula holds in this case.

Step 2: the general case. Write $h = h^+ - h^-$ and $g = g^+ - g^-$, where $h^\pm$ and $g^\pm$ are nonnegative and integrable over X and Y respectively. Then

$$f = h \otimes g = h^+ \otimes g^+ - h^+ \otimes g^- - h^- \otimes g^+ + h^- \otimes g^-,$$

and by Step 1 each of the four summands is a nonnegative integrable $(\mu \times \nu)$ -measurable function on $X \times Y$. In particular f is $(\mu \times \nu)$ -measurable, and applying Step 1 to $|h|$ and $|g|$,

$$\int_{X \times Y} |f| d(\mu \times \nu) = \int_{X \times Y} |h| \otimes |g| d(\mu \times \nu) = \int_X |h| d\mu \cdot \int_Y |g| d\nu < \infty,$$

so f is integrable over $X \times Y$ with respect to $\mu \times \nu$. Finally, by the linearity of integration and Step 1,

$$\begin{aligned} \int_{X \times Y} f d(\mu \times \nu) &= \int_X h^+ d\mu \int_Y g^+ d\nu - \int_X h^+ d\mu \int_Y g^- d\nu - \int_X h^- d\mu \int_Y g^+ d\nu + \int_X h^- d\mu \int_Y g^- d\nu \\ &= \left(\int_X h^+ d\mu - \int_X h^- d\mu \right) \left(\int_Y g^+ d\nu - \int_Y g^- d\nu \right) \\ &= \int_X h d\mu \cdot \int_Y g d\nu, \end{aligned}$$

all the quantities involved being finite real numbers, so the algebraic factoring is legitimate.

Remark on σ -finiteness. Nowhere did the argument use Fubini's Theorem or Tonelli's Theorem, and nowhere did it use σ -finiteness of μ or of ν . The only facts used were the Simple Approximation Theorem, Chebychev's Inequality, the Monotone Convergence Theorem, and the statement of the Caratheodory-Hahn Theorem that $\mu \times \nu$ assigns the value $\mu(A) \cdot \nu(B)$ to each measurable rectangle $A \times B$. The role of the integrability of h and g was to force the approximating simple functions to vanish outside sets of finite measure, which is precisely the property that σ -finiteness is invoked to obtain in the proof of Tonelli's Theorem.

Problem 20.11. Show that Tonelli's Theorem is still true if, instead of assuming μ and ν to be σ -finite, we merely assume that $\{(x, y) \mid f(x, y) \neq 0\}$ is a set of σ -finite measure.

Solution. The assertion to be proved is the following. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces with ν complete, and let f be a nonnegative $(\mu \times \nu)$ -measurable function on $X \times Y$ such that

$$E = \{(x, y) \in X \times Y : f(x, y) \neq 0\}$$

is of σ -finite $\mu \times \nu$ measure. Then for almost all $x \in X$ the x -section $f(x, \cdot)$ is ν -measurable, the function defined almost everywhere on X by $x \mapsto \int_Y f(x, y) d\nu(y)$ is μ -measurable, and

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) d\nu(y) \right] d\mu(x). \quad (9)$$

Reading the proof of Tonelli's Theorem, one sees that σ -finiteness of μ and ν is used at exactly one place and for exactly one purpose: it makes $\mu \times \nu$ a σ -finite measure, which permits the appeal to assertion (i) of the Simple Approximation Theorem to produce an increasing sequence $\{\varphi_k\}$ of simple functions with $0 \leq \varphi_k \leq f$, converging pointwise to f on $X \times Y$, and with each φ_k vanishing outside a set of finite measure, hence integrable. Once such a sequence is in hand the proof proceeds word for word as in Fubini's Theorem. So it suffices to produce such a sequence from the weaker hypothesis.

Construction of the approximating sequence. Since f is $(\mu \times \nu)$ -measurable and nonnegative, $E = \{f > 0\} = \bigcup_{n=1}^{\infty} \{f > 1/n\}$ is $(\mu \times \nu)$ -measurable. By hypothesis E is of σ -finite measure, so there is a countable collection $\{F_n\}_{n=1}^{\infty}$ of $(\mu \times \nu)$ -measurable sets of finite measure with $E = \bigcup_n F_n$. Replacing F_n by $E_n = \bigcup_{i=1}^n F_i$, which has finite measure by the finite subadditivity of $\mu \times \nu$, we may assume the sets are ascending:

$$E_1 \subseteq E_2 \subseteq \cdots, \quad (\mu \times \nu)(E_n) < \infty, \quad E = \bigcup_{n=1}^{\infty} E_n.$$

The Simple Approximation Theorem, whose main assertion together with its assertion (ii) is valid on an arbitrary measure space with no σ -finiteness hypothesis (only its assertion (i), the vanishing outside a set of finite measure, requires σ -finiteness), provides an increasing sequence $\{\psi_k\}$ of simple $(\mu \times \nu)$ -measurable functions with $0 \leq \psi_k \leq f$ on $X \times Y$ and $\psi_k \rightarrow f$ pointwise on $X \times Y$. Define

$$\varphi_k = \psi_k \cdot \chi_{E_k} \quad \text{for } k = 1, 2, \dots$$

Then:

- Each φ_k is simple and $(\mu \times \nu)$ -measurable, being the product of a simple measurable function and the characteristic function of a measurable set.
- $0 \leq \varphi_k \leq \psi_k \leq f$ on $X \times Y$.
- $\{\varphi_k\}$ is increasing: $0 \leq \psi_k \leq \psi_{k+1}$ and $0 \leq \chi_{E_k} \leq \chi_{E_{k+1}}$, and the product of two increasing sequences of nonnegative functions is increasing.
- Each φ_k vanishes outside E_k , and $(\mu \times \nu)(E_k) < \infty$; a simple function that vanishes outside a set of finite measure is integrable. So each φ_k is integrable over $X \times Y$ with respect to $\mu \times \nu$.
- $\varphi_k \rightarrow f$ pointwise on $X \times Y$. Indeed, let $(x, y) \in X \times Y$. If $f(x, y) = 0$ then $0 \leq \psi_k(x, y) \leq f(x, y) = 0$ for all k , so $\varphi_k(x, y) = 0 = f(x, y)$ for all k . If $f(x, y) \neq 0$ then $(x, y) \in E$, so $(x, y) \in E_N$ for some N , and since the E_k ascend, $\varphi_k(x, y) = \psi_k(x, y)$ for all $k \geq N$; hence $\varphi_k(x, y) \rightarrow f(x, y)$.

Completion of the proof. With $\{\varphi_k\}$ as above, the proof of Fubini's Theorem applies verbatim. In detail: each φ_k is simple and integrable over $X \times Y$, so Theorem 6 gives, for each k , that $\varphi_k(x, \cdot)$ is ν -integrable for almost all $x \in X$, that $x \mapsto \int_Y \varphi_k(x, y) d\nu(y)$ is μ -measurable and integrable over X , and that

$$\int_{X \times Y} \varphi_k d(\mu \times \nu) = \int_X \left[\int_Y \varphi_k(x, y) d\nu(y) \right] d\mu(x).$$

As justified in Problem 8, by excising from $X \times Y$ a set of $\mu \times \nu$ measure zero (namely $N \times Y$, where N is a μ -null set off which every $\varphi_k(x, \cdot)$ is ν -integrable) we may assume that $\varphi_k(x, \cdot)$ is integrable over Y for every $x \in X$ and every k ; the excision changes neither side of (9).

Fix $x \in X$. Then $\{\varphi_k(x, \cdot)\}$ is an increasing sequence of nonnegative simple ν -measurable functions on Y converging pointwise to $f(x, \cdot)$. Hence $f(x, \cdot)$ is ν -measurable and, by the Monotone Convergence Theorem on Y ,

$$\int_Y f(x, y) d\nu(y) = \lim_{k \rightarrow \infty} \int_Y \varphi_k(x, y) d\nu(y).$$

Define $h(x) = \int_Y f(x, y) d\nu(y)$ and $h_k(x) = \int_Y \varphi_k(x, y) d\nu(y)$ for $x \in X$. Each h_k is a nonnegative μ -measurable function on X by Theorem 6, and $\{h_k\}$ increases pointwise to h ; therefore h is μ -measurable, and by the Monotone Convergence Theorem on X ,

$$\lim_{k \rightarrow \infty} \int_X h_k d\mu = \int_X h d\mu.$$

On the other hand $\{\varphi_k\}$ increases pointwise to f on $X \times Y$, so by the Monotone Convergence Theorem on $X \times Y$,

$$\int_{X \times Y} f d(\mu \times \nu) = \lim_{k \rightarrow \infty} \int_{X \times Y} \varphi_k d(\mu \times \nu) = \lim_{k \rightarrow \infty} \int_X h_k d\mu = \int_X h d\mu = \int_X \left[\int_Y f(x, y) d\nu(y) \right] d\mu(x).$$

This is (9), and both sides may be infinite. The proof is complete.

Remark. The hypothesis of σ -finiteness of μ and ν implies the new hypothesis, since then $\mu \times \nu$ is σ -finite and so every measurable subset of $X \times Y$, in particular $\{f \neq 0\}$, is of σ -finite measure. So this is a genuine strengthening of Tonelli's Theorem. Note also that the two hypotheses of Tonelli's Theorem that cannot be dispensed with, the nonnegativity of f and the completeness of ν , were both used above: nonnegativity for the two applications of the Monotone Convergence Theorem, and completeness of ν for Lemma 4, which underlies the excision.

Problem 20.12. For two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ we have defined $\mathcal{A} \times \mathcal{B}$ to be the smallest σ -algebra that contains the measurable rectangles.

- (i) Show that if both measures are σ -finite, then $\mu \times \nu$ is the only measure on $\mathcal{A} \times \mathcal{B}$ that assigns the value $\mu(A) \cdot \nu(B)$ to each measurable rectangle $A \times B$. Also that this uniqueness property may fail if we do not have σ -finiteness.
- (ii) Show that if $E \in \mathcal{A} \times \mathcal{B}$, then $E_x \in \mathcal{B}$ for each x .
- (iii) Show that if f is measurable with respect to $\mathcal{A} \times \mathcal{B}$, then $f(x, \cdot)$ is measurable with respect to $\mathcal{B}$ for each x .

Solution. Throughout, $\mathcal{R}$ denotes the semiring of measurable rectangles $A \times B$ ($A \in \mathcal{A}$, $B \in \mathcal{B}$), $\lambda(A \times B) = \mu(A) \cdot \nu(B)$ is the premeasure of Proposition 2, and $\mu \times \nu$ is its Caratheodory extension. Since the $(\mu \times \nu)$ -measurable sets form a σ -algebra which, by the Caratheodory-Hahn Theorem, contains $\mathcal{R}$, it contains the smallest σ -algebra containing $\mathcal{R}$, namely $\mathcal{A} \times \mathcal{B}$. So it makes sense to speak of the restriction of $\mu \times \nu$ to $\mathcal{A} \times \mathcal{B}$, and by the same theorem this restriction assigns $\mu(A) \cdot \nu(B)$ to each measurable rectangle.

(i) Uniqueness under σ -finiteness. Assume μ and ν are σ -finite, and let π be any measure on $\mathcal{A} \times \mathcal{B}$ with $\pi(A \times B) = \mu(A) \cdot \nu(B)$ for every measurable rectangle. Write $X = \bigcup_n X_n$ and $Y = \bigcup_n Y_n$ with $X_n \in \mathcal{A}$, $Y_n \in \mathcal{B}$, $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$; replacing X_n by $\bigcup_{i \leq n} X_i$ and Y_n by $\bigcup_{i \leq n} Y_i$ we may

take both sequences ascending. Then

$$X \times Y = \bigcup_{n=1}^{\infty} X_n \times Y_n, \quad \pi(X_n \times Y_n) = \mu(X_n)\nu(Y_n) < \infty,$$

so π is σ -finite; the same computation shows the restriction of $\mu \times \nu$ to $\mathcal{A} \times \mathcal{B}$ is σ -finite. Both measures are defined on $\mathcal{A} \times \mathcal{B}$, the smallest σ -algebra containing the semiring $\mathcal{R}$, and they agree on $\mathcal{R}$. By Corollary 14 of Chapter 17, two σ -finite measures on the smallest σ -algebra containing a semiring $\mathcal{S}$ are equal if and only if they agree on $\mathcal{S}$. Hence π equals the restriction of $\mu \times \nu$ to $\mathcal{A} \times \mathcal{B}$. (This is of course just the uniqueness assertion of the Caratheodory-Hahn Theorem, transported to $\mathcal{A} \times \mathcal{B}$.)

Failure without σ -finiteness. Let $X = Y = [0, 1]$. Let $\mathcal{A}$ be the σ -algebra of Lebesgue measurable subsets of $[0, 1]$ and μ Lebesgue measure; let $\mathcal{B} = 2^{[0, 1]}$ be the σ -algebra of all subsets of $[0, 1]$ and $\nu = c$ the counting measure, so $c(B)$ is the number of points of B if B is finite and ∞ otherwise. The measure c is not σ -finite, since $[0, 1]$ is uncountable while a set of finite counting measure is finite.

Consider the diagonal $\Delta = \{(x, x) : x \in [0, 1]\}$. It belongs to $\mathcal{A} \times \mathcal{B}$: with $I_{n,k} = [(k-1)2^{-n}, k2^{-n}]$ for $1 \leq k \leq 2^n$,

$$\Delta = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} (I_{n,k} \times I_{n,k}).$$

Indeed, if $x = y$ then for each n both lie in a common $I_{n,k}$; conversely if for every n there is a k with $x, y \in I_{n,k}$ then $|x - y| \leq 2^{-n}$ for every n , so $x = y$. Each $I_{n,k} \times I_{n,k}$ is a measurable rectangle, so the displayed set lies in $\mathcal{A} \times \mathcal{B}$.

Define two set functions on $\mathcal{A} \times \mathcal{B}$:

$$\pi_1 = \text{the restriction of } \mu \times c \text{ to } \mathcal{A} \times \mathcal{B}, \quad \pi_2(E) = \int_{[0,1]} \mu(E^y) dc(y) = \sum_{y \in [0,1]} \mu(E^y),$$

where $E^y = \{x \in [0, 1] : (x, y) \in E\}$ is the y -section of E . By the argument of part (ii), applied with the roles of the two factors interchanged, $E^y \in \mathcal{A}$ for every y , so $\mu(E^y)$ is defined; and every function on the measure space $([0, 1], 2^{[0, 1]}, c)$ is measurable, so the integral defining π_2 makes sense. (The sum is the sum of an uncountable family of nonnegative numbers, that is, the supremum of its finite subsums.)

π_2 is a measure. Clearly $\pi_2(\emptyset) = 0$. If $\{E_n\}$ is a countable disjoint collection in $\mathcal{A} \times \mathcal{B}$, then for each y the sections $\{(E_n)^y\}$ are disjoint with union $(\bigcup_n E_n)^y$, so $\mu((\bigcup_n E_n)^y) = \sum_n \mu((E_n)^y)$ by the countable additivity of μ . Integrating this identity over $([0, 1], 2^{[0, 1]}, c)$ and using the corollary of the Monotone Convergence Theorem that permits term-by-term integration of a series of nonnegative measurable functions,

$$\pi_2\left(\bigcup_n E_n\right) = \sum_n \pi_2(E_n).$$

Both assign the right value to rectangles. For a measurable rectangle $A \times B$ we have $(A \times B)^y = A$ if $y \in B$ and $(A \times B)^y = \emptyset$ otherwise, so

$$\pi_2(A \times B) = \sum_{y \in B} \mu(A) = \mu(A) \cdot c(B),$$

with the usual convention $0 \cdot \infty = 0$ (if $\mu(A) = 0$ every summand is 0). And $\pi_1(A \times B) = \mu(A) \cdot c(B)$ as noted at the outset.

They differ. On one hand $\Delta^y = \{y\}$, so $\pi_2(\Delta) = \sum_{y \in [0,1]} \mu(\{y\}) = 0$. On the other hand $\pi_1(\Delta) = \infty$. To see this, let $\{A_i \times B_i\}_{i=1}^{\infty}$ be any countable cover of Δ by measurable rectangles. For each $x \in [0, 1]$ the point (x, x) lies in some $A_i \times B_i$, so $x \in A_i \cap B_i$; hence $[0, 1] = \bigcup_i (A_i \cap B_i)$ and, by the countable subadditivity of Lebesgue outer measure, $1 \leq \sum_i \mu^*(A_i \cap B_i)$. Therefore $\mu^*(A_{i_0} \cap B_{i_0}) > 0$ for some i_0 . Then $\mu(A_{i_0}) > 0$, and B_{i_0} is uncountable (a countable set has Lebesgue outer measure zero), so $c(B_{i_0}) = \infty$ and

$$\sum_i \mu(A_i) c(B_i) \geq \mu(A_{i_0}) c(B_{i_0}) = \infty.$$

Since every cover of Δ by measurable rectangles has infinite total premeasure, $(\mu \times c)^*(\Delta) = \infty$, that is, $\pi_1(\Delta) = \infty \neq 0 = \pi_2(\Delta)$.

Thus π_1 and π_2 are two distinct measures on $\mathcal{A} \times \mathcal{B}$ that both assign the value $\mu(A) \cdot \nu(B)$ to every measurable rectangle: uniqueness fails without σ -finiteness.

(ii) Let

$$\mathcal{C} = \{E \subseteq X \times Y : E_x \in \mathcal{B} \text{ for every } x \in X\}.$$

We show $\mathcal{C}$ is a σ -algebra of subsets of $X \times Y$ containing $\mathcal{R}$; minimality of $\mathcal{A} \times \mathcal{B}$ then gives $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{C}$, which is the assertion.

First, $X \times Y \in \mathcal{C}$ since $(X \times Y)_x = Y \in \mathcal{B}$ for every x . Next, sections commute with complementation and with unions: for every x ,

$$((X \times Y) \setminus E)_x = Y \setminus E_x, \quad \left(\bigcup_{n=1}^{\infty} E_n \right)_x = \bigcup_{n=1}^{\infty} (E_n)_x.$$

Both are immediate from the definition $E_x = \{y \in Y : (x, y) \in E\}$. Hence if $E \in \mathcal{C}$ then its complement is in $\mathcal{C}$, and if $\{E_n\} \subseteq \mathcal{C}$ then $\bigcup_n E_n \in \mathcal{C}$, because $\mathcal{B}$ is a σ -algebra. So $\mathcal{C}$ is a σ -algebra.

Finally $\mathcal{R} \subseteq \mathcal{C}$: for a measurable rectangle $A \times B$ and $x \in X$,

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A, \end{cases}$$

and both B and $\emptyset$ belong to $\mathcal{B}$. This completes (ii). Note that no completeness or σ -finiteness hypothesis was used, and the conclusion holds for every x , not merely almost every x ; this is the point made in the closing paragraph of Section 20.1.

(iii) Let f be measurable with respect to $\mathcal{A} \times \mathcal{B}$ and fix $x \in X$. Let $c \in \mathbb{R}$ and set

$$E = \{(x', y') \in X \times Y : f(x', y') > c\},$$

which belongs to $\mathcal{A} \times \mathcal{B}$ by the measurability of f . Directly from the definitions,

$$\{y \in Y : f(x, \cdot)(y) > c\} = \{y \in Y : f(x, y) > c\} = E_x,$$

and $E_x \in \mathcal{B}$ by part (ii). Since $c \in \mathbb{R}$ was arbitrary, $f(x, \cdot)$ is measurable with respect to $\mathcal{B}$. (If f is extended real-valued, the same argument applied to the sets $\{f = \infty\}$ and $\{f = -\infty\}$, which are countable intersections of sets of the above form, gives the same conclusion.)

Problem 20.13. If $\{(X_k, \mathcal{A}_k, \mu_k)\}_{k=1}^n$ is a finite collection of measure spaces, we can form the product measure $\mu_1 \times \cdots \times \mu_n$ on the space $X_1 \times \cdots \times X_n$ by starting with the semiring of rectangles of the form $R = A_1 \times \cdots \times A_n$, define $\mu(R) = \prod_{k=1}^n \mu_k(A_k)$, show that μ is a premeasure and define the product measure to be the Caratheodory extension of μ . Show that if we identify $(X_1 \times \cdots \times X_p) \times (X_{p+1} \times \cdots \times X_n)$ with $(X_1 \times \cdots \times X_n)$, then $(\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n) = \mu_1 \times \cdots \times \mu_n$.

Solution. Write $Z = X_1 \times \cdots \times X_n$, let

$$\mathcal{R}_n = \{A_1 \times \cdots \times A_n : A_k \in \mathcal{A}_k \text{ for } 1 \leq k \leq n\},$$

and define $\mu(A_1 \times \cdots \times A_n) = \prod_{k=1}^n \mu_k(A_k)$, with the convention $0 \cdot \infty = 0$. (The value $\mu(R)$ is unambiguous: a rectangle can be represented in more than one way only when it is empty, and then some factor is empty and the product is 0.)

Part 1: $\mathcal{R}_n$ is a semiring. Since $\emptyset = \emptyset \times \cdots \times \emptyset$, $\mathcal{R}_n$ contains $\emptyset$. It is closed under finite intersections, because

$$(A_1 \times \cdots \times A_n) \cap (B_1 \times \cdots \times B_n) = (A_1 \cap B_1) \times \cdots \times (A_n \cap B_n).$$

For relative complements, let $R = \prod_k A_k$ and $S = \prod_k B_k$ belong to $\mathcal{R}_n$. For each k put $C_k^0 = A_k \cap B_k$

and $C_k^1 = A_k \setminus B_k$, so that A_k is the disjoint union of C_k^0 and C_k^1 , both in $\mathcal{A}_k$. Then

$$R = \bigsqcup_{\varepsilon \in \{0,1\}^n} \prod_{k=1}^n C_k^{\varepsilon_k},$$

a disjoint union of 2^n members of $\mathcal{R}_n$, and the term with $\varepsilon = (0, \dots, 0)$ is exactly $R \cap S$. Hence

$$R \setminus S = \bigsqcup_{\varepsilon \neq (0, \dots, 0)} \prod_{k=1}^n C_k^{\varepsilon_k}$$

is a finite disjoint union of members of $\mathcal{R}_n$. So $\mathcal{R}_n$ is a semiring.

Part 2: the analogue of Lemma 1. Let $\{R^{(j)}\}_{j=1}^\infty$ be a countable disjoint collection in $\mathcal{R}_n$, say $R^{(j)} = A_1^{(j)} \times \dots \times A_n^{(j)}$, whose union is $R = A_1 \times \dots \times A_n \in \mathcal{R}_n$. Then $\mu(R) = \sum_j \mu(R^{(j)})$.

To see this, note that the characteristic function of a rectangle factors,

$$\chi_R(x_1, \dots, x_n) = \prod_{k=1}^n \chi_{A_k}(x_k),$$

and that, by disjointness, $\chi_R = \sum_{j=1}^\infty \chi_{R^{(j)}}$ pointwise on Z . Thus for all $(x_1, \dots, x_n) \in Z$,

$$\prod_{k=1}^n \chi_{A_k}(x_k) = \sum_{j=1}^\infty \prod_{k=1}^n \chi_{A_k^{(j)}}(x_k). \quad (13.1)$$

Fix $(x_1, \dots, x_{n-1})$ and regard (13.1) as an identity between nonnegative $\mathcal{A}_n$ -measurable functions of x_n ; integrating over X_n with respect to μ_n , and using the corollary of the Monotone Convergence Theorem that a series of nonnegative measurable functions may be integrated term by term, we obtain

$$\mu_n(A_n) \prod_{k=1}^{n-1} \chi_{A_k}(x_k) = \sum_{j=1}^\infty \mu_n(A_n^{(j)}) \prod_{k=1}^{n-1} \chi_{A_k^{(j)}}(x_k)$$

for all $(x_1, \dots, x_{n-1}) \in X_1 \times \dots \times X_{n-1}$. Repeating the same step with respect to x_{n-1} , then x_{n-2} , and so on down to x_1 , after n integrations we arrive at

$$\prod_{k=1}^n \mu_k(A_k) = \sum_{j=1}^\infty \prod_{k=1}^n \mu_k(A_k^{(j)}),$$

that is, $\mu(R) = \sum_j \mu(R^{(j)})$. All quantities lie in $[0, \infty]$ and the argument is valid there. Taking all but finitely many $R^{(j)}$ to be empty, we get in particular the finite additivity of μ on $\mathcal{R}_n$.

Part 3: μ is a premeasure. We have $\mu(\emptyset) = 0$ and, by Part 2, μ is finitely additive on $\mathcal{R}_n$. Also μ is monotone: if $\prod_k A_k \subseteq \prod_k B_k$ and the left side is nonempty then $A_k \subseteq B_k$ for each k , so $\prod_k \mu_k(A_k) \leq \prod_k \mu_k(B_k)$ by the monotonicity of each μ_k . It remains to show μ is countably monotone. Let $R \in \mathcal{R}_n$ be covered by a countable collection $\{S_j\}_{j=1}^\infty$ of members of $\mathcal{R}_n$. Exactly as in the proof of Proposition 2, we may assume $\{S_j\}$ is disjoint. Indeed, put $T_j = S_j \setminus \bigcup_{i < j} S_i = \bigcap_{i < j} (S_j \setminus S_i)$; each $S_j \setminus S_i$ is a finite disjoint union of members of $\mathcal{R}_n$ by Part 1, and $\mathcal{R}_n$ is closed under intersections, so each T_j is a finite disjoint union $\bigsqcup_l R_{j,l}$ of members of $\mathcal{R}_n$ contained in S_j . The sets $R_{j,l}$ are pairwise disjoint with union $\bigcup_j S_j$, and $\sum_l \mu(R_{j,l}) \leq \mu(S_j)$ for each j . The latter rests on a lemma valid in any semiring: if $R_1, \dots, R_q$ are disjoint members contained in a member S , then $S \setminus \bigcup_{l < q} R_l$ is a finite disjoint union of members. This holds for $q = 1$ by the defining property of a semiring, and inductively, writing $S \setminus \bigcup_{l < q} R_l = \bigsqcup_i T_i$ with T_i members, we have $S \setminus \bigcup_{l \leq q} R_l = \bigsqcup_i (T_i \setminus R_q)$, and each $T_i \setminus R_q$ is a finite disjoint union of members. Granting the lemma, S_j is the disjoint union of the $R_{j,l}$ together with finitely many further members of $\mathcal{R}_n$, so the finite additivity established in Part 2 gives $\sum_l \mu(R_{j,l}) \leq \mu(S_j)$. So

replacing $\{S_j\}$ by $\{R_{j,l}\}$ only decreases the total. Assuming then that $\{S_j\}$ is disjoint, $R = \bigsqcup_j (R \cap S_j)$ is a countable disjoint union of members of $\mathcal{R}_n$, so by Part 2 and monotonicity

$$\mu(R) = \sum_{j=1}^{\infty} \mu(R \cap S_j) \leq \sum_{j=1}^{\infty} \mu(S_j).$$

Thus μ is countably monotone, hence a premeasure. By the Caratheodory-Hahn Theorem its Caratheodory extension, denoted $\mu_1 \times \cdots \times \mu_n$, is a measure on the σ -algebra of μ^* -measurable subsets of Z which extends μ ; we write $\theta = \mu^*$ for the induced outer measure.

Part 4: the identification. Fix $1 \leq p < n$, put $U = X_1 \times \cdots \times X_p$ and $V = X_{p+1} \times \cdots \times X_n$, and identify $U \times V$ with Z in the obvious way. Let

$$\alpha = \mu_1 \times \cdots \times \mu_p \text{ on } U, \quad \beta = \mu_{p+1} \times \cdots \times \mu_n \text{ on } V$$

be the product measures just constructed (for a single factor this is understood to be the Caratheodory extension of that factor's measure from its σ -algebra), defined on the σ -algebras $\mathcal{M}_U$ and $\mathcal{M}_V$ of measurable sets. Write $\mu^{(1)}, \mu^{(2)}$ for the corresponding premeasures on $\mathcal{R}_p$ and $\mathcal{R}_{n-p}$. Finally, let $\gamma(M \times N) = \alpha(M)\beta(N)$ be the premeasure on the semiring $\mathcal{R}' = \{M \times N : M \in \mathcal{M}_U, N \in \mathcal{M}_V\}$ of Proposition 2, let Θ be the outer measure it induces on $U \times V = Z$, and let $\alpha \times \beta$ be its Caratheodory extension. Since a Caratheodory extension is completely determined by its outer measure (the measurable sets are those that split every set additively, and the measure is the restriction of the outer measure), it suffices to prove $\theta = \Theta$ as outer measures on Z .

Claim 1: $\Theta \leq \theta$, always. Let $R = A_1 \times \cdots \times A_n \in \mathcal{R}_n$. Under the identification, $R = P \times Q$ with $P = A_1 \times \cdots \times A_p \in \mathcal{R}_p$ and $Q = A_{p+1} \times \cdots \times A_n \in \mathcal{R}_{n-p}$. By the Caratheodory-Hahn Theorem, α extends $\mu^{(1)}$ and β extends $\mu^{(2)}$, so $P \in \mathcal{M}_U$, $Q \in \mathcal{M}_V$, and

$$\gamma(R) = \alpha(P)\beta(Q) = \prod_{k \leq p} \mu_k(A_k) \cdot \prod_{k > p} \mu_k(A_k) = \mu(R).$$

Hence $\mathcal{R}_n \subseteq \mathcal{R}'$ with $\gamma = \mu$ on $\mathcal{R}_n$, so every cover of a set by members of $\mathcal{R}_n$ is a cover by members of $\mathcal{R}'$ with the same total. Taking infima, $\Theta(E) \leq \theta(E)$ for every $E \subseteq Z$.

Claim 2: if every μ_k is σ -finite, then $\theta \leq \Theta$. It suffices to prove

$$\theta(M \times N) \leq \alpha(M)\beta(N) = \gamma(M \times N) \quad \text{for all } M \in \mathcal{M}_U, N \in \mathcal{M}_V. \quad (13.2)$$

Granting (13.2), if $E \subseteq \bigcup_i (M_i \times N_i)$ is any cover by members of $\mathcal{R}'$, then by the countable subadditivity of the outer measure θ ,

$$\theta(E) \leq \sum_i \theta(M_i \times N_i) \leq \sum_i \gamma(M_i \times N_i),$$

and taking the infimum over such covers gives $\theta(E) \leq \Theta(E)$.

We first prove (13.2) when $\alpha(M) < \infty$ and $\beta(N) < \infty$. Let $\varepsilon > 0$. Since $\alpha(M) = \theta_U(M)$, the outer measure induced by $\mu^{(1)}$, there is a cover $M \subseteq \bigcup_i P_i$ with $P_i \in \mathcal{R}_p$ and $\sum_i \mu^{(1)}(P_i) \leq \alpha(M) + \varepsilon$; similarly there is a cover $N \subseteq \bigcup_j Q_j$ with $Q_j \in \mathcal{R}_{n-p}$ and $\sum_j \mu^{(2)}(Q_j) \leq \beta(N) + \varepsilon$. Then $M \times N \subseteq \bigcup_{i,j} (P_i \times Q_j)$, each $P_i \times Q_j \in \mathcal{R}_n$ with $\mu(P_i \times Q_j) = \mu^{(1)}(P_i)\mu^{(2)}(Q_j)$, so

$$\theta(M \times N) \leq \sum_{i,j} \mu^{(1)}(P_i) \mu^{(2)}(Q_j) = \left(\sum_i \mu^{(1)}(P_i) \right) \left(\sum_j \mu^{(2)}(Q_j) \right) \leq (\alpha(M) + \varepsilon)(\beta(N) + \varepsilon),$$

the factoring of the double series of nonnegative terms being legitimate. Letting $\varepsilon \rightarrow 0$ gives (13.2) in this case.

For general M and N , use σ -finiteness. If every μ_k is σ -finite then $\mu^{(1)}$ is a σ -finite premeasure (write $X_k = \bigcup_m X_{k,m}$ with $\mu_k(X_{k,m}) < \infty$ and cover U by the countably many rectangles $X_{1,m_1} \times \cdots \times X_{p,m_p}$), and by the Caratheodory-Hahn Theorem α is then σ -finite; likewise β . So there are disjoint decompositions $U = \bigsqcup_l U_l$ and $V = \bigsqcup_m V_m$ with $U_l \in \mathcal{M}_U$, $V_m \in \mathcal{M}_V$, $\alpha(U_l) < \infty$, $\beta(V_m) < \infty$. Then

$M \times N = \bigcup_{l,m} (M \cap U_l) \times (N \cap V_m)$, so by countable subadditivity of θ , the finite case just proved, and the countable additivity of α and β ,

$$\theta(M \times N) \leq \sum_{l,m} \theta((M \cap U_l) \times (N \cap V_m)) \leq \sum_{l,m} \alpha(M \cap U_l) \beta(N \cap V_m) = \alpha(M) \beta(N).$$

This proves (13.2) in general and hence Claim 2.

Combining the two claims: if all the μ_k are σ -finite then $\theta = \Theta$, so the two Caratheodory extensions have the same measurable sets and the same values, that is,

$$(\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n) = \mu_1 \times \cdots \times \mu_n.$$

Iterating, any bracketing of the factors gives the same measure; in particular the construction of $\mu_1 \times \cdots \times \mu_n$ agrees with repeatedly forming two-fold products, which is what is used in Section 20.2 to identify n -dimensional Lebesgue measure with an n -fold product of one-dimensional Lebesgue measures (all of which are σ -finite).

A caveat: σ -finiteness cannot be dropped. Claim 1 holds with no hypotheses, but Claim 2 can fail, and with it the asserted equality. Take $n = 3$ and $p = 2$. Let $X_1 = X_2 = [0, 1]$ with $\mathcal{A}_1 = \mathcal{A}_2$ the Lebesgue measurable sets and $\mu_1 = \mu_2$ Lebesgue measure, and let $X_3 = [0, 1]$ with $\mathcal{A}_3 = \{\emptyset, X_3\}$, $\mu_3(\emptyset) = 0$, $\mu_3(X_3) = \infty$; this is a (non- σ -finite) measure space. Let $\Delta = \{(x, x) : x \in [0, 1]\} \subseteq X_1 \times X_2$.

First, $\alpha(\Delta) = 0$, where $\alpha = \mu_1 \times \mu_2$: for each m , with $I_{m,k} = [(k-1)/m, k/m]$,

$$\Delta \subseteq \bigcup_{k=1}^m I_{m,k} \times I_{m,k}, \quad \sum_{k=1}^m \mu_1(I_{m,k}) \mu_2(I_{m,k}) = m \cdot \frac{1}{m^2} = \frac{1}{m} \xrightarrow{m \rightarrow \infty} 0.$$

Hence $\Theta_U(\Delta) = 0$, so Δ is α -measurable with $\alpha(\Delta) = 0$. Also $\beta = \mu_3$ has $\beta(X_3) = \infty$. Therefore $\Delta \times X_3 \in \mathcal{R}'$ with

$$\gamma(\Delta \times X_3) = \alpha(\Delta) \beta(X_3) = 0 \cdot \infty = 0,$$

so $\Theta(\Delta \times X_3) = 0$ and $((\mu_1 \times \mu_2) \times \mu_3)(\Delta \times X_3) = 0$.

On the other hand $\theta(\Delta \times X_3) = \infty$. Let $\{A_1^i \times A_2^i \times A_3^i\}_i$ be a countable cover of $\Delta \times X_3$ by members of $\mathcal{R}_3$; discard the empty ones, so each $A_3^i = X_3$. Projecting, $\{A_1^i \times A_2^i\}$ covers Δ , so for each $x \in [0, 1]$ we have $x \in A_1^i \cap A_2^i$ for some i , whence $[0, 1] = \bigcup_i (A_1^i \cap A_2^i)$ and some $A_1^{i_0} \cap A_2^{i_0}$ has positive Lebesgue measure. Then $\mu_1(A_1^{i_0}) > 0$ and $\mu_2(A_2^{i_0}) > 0$, so

$$\mu(A_1^{i_0} \times A_2^{i_0} \times X_3) = \mu_1(A_1^{i_0}) \mu_2(A_2^{i_0}) \cdot \infty = \infty,$$

and the total of the cover is ∞ . Hence $\theta(\Delta \times X_3) = \infty$. Moreover every subset of Z is θ -measurable here: every member of $\mathcal{R}_3$ has μ -value 0 or ∞ , so θ takes only the values 0 and ∞ ; if $\theta(S) = 0$ then $\theta(S \cap E) + \theta(S \setminus E) = 0$ by monotonicity, and if $\theta(S) = \infty$ then $\theta(S \cap E)$ and $\theta(S \setminus E)$ cannot both be 0, by countable subadditivity, so the Caratheodory splitting condition holds for every E and every S . Consequently

$$(\mu_1 \times \mu_2 \times \mu_3)(\Delta \times X_3) = \infty \neq 0 = ((\mu_1 \times \mu_2) \times \mu_3)(\Delta \times X_3).$$

So the identity asserted in the problem is correct as stated for σ -finite factors (the case in which the product measure is actually used in this chapter), while in general only the inequality $\Theta \leq \theta$ of Claim 1 survives.

Problem 20.14. A measure space $(X, \mathcal{M}, \mu)$ such that $\mu(X) = 1$ is called a probability measure space. Let $\{(X_\lambda, \mathcal{A}_\lambda, \mu_\lambda)\}_{\lambda \in \Lambda}$ be a collection of probability measure spaces parametrized by the set Λ . Show that we can define a probability measure

$$\mu = \prod_{\lambda \in \Lambda} \mu_\lambda$$

on a suitable σ -algebra on the Cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$ so that

$$\mu(A) = \prod_{\lambda \in \Lambda} \mu_\lambda(A_\lambda)$$

when $A = \prod_{\lambda \in \Lambda} A_\lambda$. (Note that $\mu(A)$ can only be nonzero if all but a countable number of the A_λ have $\mu_\lambda(A_\lambda) = 1$.)

Solution. Write $X = \prod_{\lambda \in \Lambda} X_\lambda$ and, for $x \in X$, write x_λ for its λ -th coordinate. For a finite $F \subseteq \Lambda$ let $X_F = \prod_{\lambda \in F} X_\lambda$ and let $\pi_F : X \rightarrow X_F$ be the projection. For a family $\{t_\lambda\}$ of numbers in $[0, 1]$ define

$$\prod_{\lambda \in \Lambda} t_\lambda = \inf \left\{ \prod_{\lambda \in F} t_\lambda : F \subseteq \Lambda \text{ finite} \right\},$$

which is the usual value when only finitely many factors differ from 1 and is the limit of the partial products when the factors that differ from 1 form a sequence. If Λ is finite there is nothing new to prove: the construction is that of Problem 13. So assume Λ is infinite.

Step 1: the finite-dimensional measures. For finite $F \subseteq \Lambda$ let $\mathcal{A}_F$ be the smallest σ -algebra of subsets of X_F containing the measurable rectangles $\prod_{\lambda \in F} A_\lambda$, and let μ_F be the restriction to $\mathcal{A}_F$ of the product measure $\prod_{\lambda \in F} \mu_\lambda$ constructed in Problem 13. Each μ_λ is a probability measure, hence finite, hence σ -finite, so Problem 13 applies without reservation: for disjoint finite $F, G \subseteq \Lambda$ we may identify $X_{F \cup G}$ with $X_F \times X_G$ and then $\mu_{F \cup G}$ is the restriction of $\mu_F \times \mu_G$. In particular, for $F \subseteq G$ finite and $B \in \mathcal{A}_F$,

$$\mu_G(B \times X_{G \setminus F}) = \mu_F(B) \cdot \mu_{G \setminus F}(X_{G \setminus F}) = \mu_F(B) \cdot 1 = \mu_F(B). \quad (C)$$

Also $\mu_F(X_F) = \prod_{\lambda \in F} \mu_\lambda(X_\lambda) = 1$.

Step 2: the algebra of cylinder sets and the set function μ_0 . Let

$$\mathcal{A}_0 = \{ \pi_F^{-1}(B) : F \subseteq \Lambda \text{ finite}, B \in \mathcal{A}_F \}, \quad \mu_0(\pi_F^{-1}(B)) = \mu_F(B).$$

μ_0 is well defined: if $\pi_F^{-1}(B) = \pi_G^{-1}(C)$, put $H = F \cup G$; since π_H is onto X_H , applying π_H gives $B \times X_{H \setminus F} = C \times X_{H \setminus G}$, and (C) yields $\mu_F(B) = \mu_H(B \times X_{H \setminus F}) = \mu_H(C \times X_{H \setminus G}) = \mu_G(C)$.

$\mathcal{A}_0$ is an algebra of subsets of X : it contains $X = \pi_F^{-1}(X_F)$; $X \setminus \pi_F^{-1}(B) = \pi_F^{-1}(X_F \setminus B)$; and two cylinders over F and G may be rewritten, using (C), as cylinders over $H = F \cup G$, whose union is again a cylinder over H . An algebra is in particular a semiring.

μ_0 is finitely additive: two disjoint members of $\mathcal{A}_0$ may be written as $\pi_H^{-1}(B)$ and $\pi_H^{-1}(C)$ over a common finite H , and, π_H being onto, B and C are disjoint; so $\mu_0(\pi_H^{-1}(B \cup C)) = \mu_H(B \cup C) = \mu_H(B) + \mu_H(C)$. Also $\mu_0(X) = 1$ and $0 \leq \mu_0 \leq 1$, and μ_0 is monotone on $\mathcal{A}_0$ (finite additivity on an algebra).

Step 3: sections. Fix a sequence $\lambda_1, \lambda_2, \dots$ of distinct elements of Λ . For $j \geq 1$ put $\Lambda^{(j)} = \Lambda \setminus \{\lambda_1, \dots, \lambda_j\}$ and $X^{(j)} = \prod_{\lambda \in \Lambda^{(j)}} X_\lambda$, and let $\mathcal{A}_0^{(j)}, \mu_0^{(j)}$ be the cylinder algebra and set function built exactly as in Step 2 over the index set $\Lambda^{(j)}$. Identify X with $X_{\lambda_1} \times \dots \times X_{\lambda_j} \times X^{(j)}$. If $E \in \mathcal{A}_0$ is a cylinder over the finite set F , then for $(a_1, \dots, a_j)$ the section

$$E^{a_1 \dots a_j} = \{y \in X^{(j)} : (a_1, \dots, a_j, y) \in E\}$$

is a cylinder over $F \setminus \{\lambda_1, \dots, \lambda_j\}$, hence lies in $\mathcal{A}_0^{(j)}$; here we use Problem 12(ii), which guarantees that the section of the base set $B \in \mathcal{A}_F$ at $(a_1, \dots, a_j)$ belongs to the product σ -algebra of the remaining factors, for every choice of the a_i .

Moreover, for $E \in \mathcal{A}_0$,

$$\int_{X_{\lambda_1}} \mu_0^{(1)}(E^a) d\mu_{\lambda_1}(a) = \mu_0(E). \quad (S)$$

Indeed, write $E = \pi_H^{-1}(B)$ with $\lambda_1 \in H$, $B \in \mathcal{A}_H$, and identify X_H with $X_{\lambda_1} \times X_{H'}$, $H' = H \setminus \{\lambda_1\}$. Then $\mu_0^{(1)}(E^a) = \mu_{H'}(B_a)$, where B_a is the a -section of B , which lies in $\mathcal{A}_{H'}$ for every a by Problem 12(ii). Let $\bar{\mu}_{H'}$ denote the full Caratheodory product measure $\prod_{\lambda \in H'} \mu_\lambda$ on $X_{H'}$; it is complete, being

induced by an outer measure, and it restricts to $\mu_{H'}$ on $\mathcal{A}_{H'}$. Form the product measure $\mu_{\lambda_1} \times \bar{\mu}_{H'}$ on $X_{\lambda_1} \times X_{H'}$ in the sense of Proposition 2. Two observations identify it on $\mathcal{A}_H$. First, $\mathcal{A}_H$ is precisely the smallest σ -algebra containing the rectangles $A \times N$ with $A \in \mathcal{A}_{\lambda_1}$ and $N \in \mathcal{A}_{H'}$: each such rectangle lies in $\mathcal{A}_H$, since for fixed A the sets $N \subseteq X_{H'}$ with $A \times N \in \mathcal{A}_H$ form a σ -algebra containing the rectangles $\prod_{\lambda \in H'} A_\lambda$; and conversely each $\prod_{\lambda \in H'} A_\lambda$ is of that form. Second, $\mathcal{A}_H \subseteq$ the $(\mu_{\lambda_1} \times \bar{\mu}_{H'})$ -measurable sets, since the generating rectangles $A \times N$ are measurable rectangles for this product. Both μ_H and the restriction of $\mu_{\lambda_1} \times \bar{\mu}_{H'}$ to $\mathcal{A}_H$ are then measures on $\mathcal{A}_H$ assigning $\mu_{\lambda_1}(A) \cdot \mu_{H'}(N)$ to each such rectangle, and both are probability measures, hence σ -finite; so they coincide, by Problem 12(i) (equivalently by Corollary 14 of Chapter 17). Now apply Tonelli's Theorem to the nonnegative $(\mu_{\lambda_1} \times \bar{\mu}_{H'})$ -measurable function χ_B ; its hypotheses hold, both measures being finite hence σ -finite and $\bar{\mu}_{H'}$ being complete. It gives that $a \mapsto \bar{\mu}_{H'}(B_a) = \mu_{H'}(B_a)$ is measurable and that its integral over X_{λ_1} is $\mu_H(B) = \mu_0(E)$. This is (S).

One point of care. Tonelli's Theorem delivers the measurability of $a \mapsto \mu_{H'}(B_a)$ only up to a μ_{λ_1} -null set, so if μ_{λ_1} is not complete this everywhere-defined function need only be measurable with respect to the completion $\bar{\mu}_{\lambda_1}$ of μ_{λ_1} . This costs nothing below: we read (S) as an identity of integrals with respect to $\bar{\mu}_{\lambda_1}$, which agrees with μ_{λ_1} on $\mathcal{A}_{\lambda_1}$, and the only conclusion we shall draw from an integral inequality is that a certain set is nonempty. The same remark applies at every coordinate λ_j .

Step 4: continuity at the empty set. Claim: if $\{E_m\}_{m=1}^\infty \subseteq \mathcal{A}_0$ is descending and $\bigcap_m E_m = \emptyset$, then $\mu_0(E_m) \rightarrow 0$.

Suppose not. Since $\{\mu_0(E_m)\}$ is decreasing, there is $\delta > 0$ with $\mu_0(E_m) \geq \delta$ for all m . Each E_m is a cylinder over a finite set F_m ; replacing F_m by $F_1 \cup \dots \cup F_m$ we may assume $F_1 \subseteq F_2 \subseteq \dots$ (a cylinder over F is also a cylinder over any larger finite set, by (C)).

It is convenient to re-index so that E_m is a cylinder over the first m terms of a fixed sequence of coordinates. Enumerate $\bigcup_m F_m$ as a finite or infinite sequence of distinct elements, and, since Λ is infinite, extend it to a sequence $\lambda_1, \lambda_2, \dots$ of distinct elements of Λ containing $\bigcup_m F_m$. For each m let n_m be the least n with $F_m \subseteq \{\lambda_1, \dots, \lambda_n\}$; since the F_m ascend, $\{n_m\}$ is nondecreasing. For $n \geq 1$ set $m(n) = \max\{k \leq n : n_k \leq n\}$, understood as $m(n) = 0$ when no such k exists, and put $D_n = E_{m(n)}$, with $D_n = X$ when $m(n) = 0$. Then $\{m(n)\}$ is nondecreasing and $m(n) \rightarrow \infty$ (for fixed k , every $n \geq \max\{k, n_k\}$ has $m(n) \geq k$), so $\{D_n\}$ is a descending sequence in $\mathcal{A}_0$ with $\bigcap_n D_n = \bigcap_m E_m = \emptyset$ and $\mu_0(D_n) \geq \delta$ for every n ; and D_n is a cylinder over $F_{m(n)} \subseteq \{\lambda_1, \dots, \lambda_{n_{m(n)}}\} \subseteq \{\lambda_1, \dots, \lambda_n\}$. Replacing $\{E_m\}$ by $\{D_n\}$, we may and do assume from now on that E_m is a cylinder over a subset of $\{\lambda_1, \dots, \lambda_m\}$ for every m . Set $\delta_j = \delta/2^j$.

We construct points $a_j \in X_{\lambda_j}$ such that

$$\mu_0^{(j)}(E_m^{a_1 \dots a_j}) \geq \delta_j \quad \text{for all } m \geq 1. \quad (\dagger_j)$$

For $j = 1$: put $g_m(a) = \mu_0^{(1)}(E_m^a)$ for $a \in X_{\lambda_1}$. By Step 3 each g_m is defined everywhere, is measurable with respect to the completion $\bar{\mu}_{\lambda_1}$, and satisfies $0 \leq g_m \leq 1$; since $\{E_m\}$ descends and $\mu_0^{(1)}$ is monotone, $\{g_m\}$ descends pointwise to a limit g with $0 \leq g \leq 1$. By (S), $\int_{X_{\lambda_1}} g_m d\bar{\mu}_{\lambda_1} = \mu_0(E_m) \geq \delta$, so by the Bounded Convergence Theorem on the finite measure space $(X_{\lambda_1}, \bar{\mathcal{A}}_{\lambda_1}, \bar{\mu}_{\lambda_1})$,

$$\int_{X_{\lambda_1}} g d\bar{\mu}_{\lambda_1} = \lim_{m \rightarrow \infty} \int_{X_{\lambda_1}} g_m d\bar{\mu}_{\lambda_1} \geq \delta.$$

If $S = \{a : g(a) \geq \delta/2\}$, then splitting the integral and using $0 \leq g \leq 1$,

$$\delta \leq \int_S g d\bar{\mu}_{\lambda_1} + \int_{X_{\lambda_1} \setminus S} g d\bar{\mu}_{\lambda_1} \leq \bar{\mu}_{\lambda_1}(S) + \frac{\delta}{2},$$

so $\bar{\mu}_{\lambda_1}(S) \geq \delta/2 > 0$ and in particular $S \neq \emptyset$. Choose $a_1 \in S$; then $\mu_0^{(1)}(E_m^{a_1}) \geq g(a_1) \geq \delta/2 = \delta_1$ for all m , which is $(\dagger_1)$.

For the inductive step, suppose $a_1, \dots, a_{j-1}$ satisfy $(\dagger_{j-1})$. The sets $E'_m = E_m^{a_1 \dots a_{j-1}}$ form a descending sequence in $\mathcal{A}_0^{(j-1)}$ with $\mu_0^{(j-1)}(E'_m) \geq \delta_{j-1}$ for all m . Repeat the argument of the previous paragraph

inside the index set $\Lambda^{(j-1)}$, with distinguished coordinate λ_j : it produces $a_j \in X_{\lambda_j}$ with $\mu_0^{(j)}((E'_m)^{a_j}) \geq \delta_{j-1}/2 = \delta_j$ for all m , and $(E'_m)^{a_j} = E_m^{a_1 \dots a_j}$. This is $(\dagger_j)$.

Now define $x^* \in X$ by $x_{\lambda_j}^* = a_j$ for $j \geq 1$ and x_λ^* an arbitrary point of X_λ for the remaining λ . Fix n . Since E_n is a cylinder over $F_n \subseteq \{\lambda_1, \dots, \lambda_n\}$, membership in E_n is determined by the coordinates $\lambda_1, \dots, \lambda_n$ alone, so the section $E_n^{a_1 \dots a_n}$ is either $\emptyset$ or all of $X^{(n)}$. By $(\dagger_n)$ its $\mu_0^{(n)}$ value is at least $\delta_n > 0$, so it is not empty and therefore equals $X^{(n)}$. Hence $(a_1, \dots, a_n, y) \in E_n$ for every $y \in X^{(n)}$; in particular $x^* \in E_n$. As n was arbitrary, $x^* \in \bigcap_n E_n$, contradicting $\bigcap_n E_n = \emptyset$. The claim is proved.

Step 5: μ_0 is a premeasure. Let $\{C_j\}$ be a countable disjoint collection in $\mathcal{A}_0$ whose union C belongs to $\mathcal{A}_0$. Put $E_m = C \setminus \bigcup_{j \leq m} C_j \in \mathcal{A}_0$. Then $\{E_m\}$ descends to $\emptyset$, so $\mu_0(E_m) \rightarrow 0$ by Step 4, while finite additivity gives $\mu_0(C) = \sum_{j \leq m} \mu_0(C_j) + \mu_0(E_m)$. Letting $m \rightarrow \infty$, $\mu_0(C) = \sum_j \mu_0(C_j)$: μ_0 is countably additive. Countable monotonicity follows: if $C \subseteq \bigcup_j S_j$ with $C, S_j \in \mathcal{A}_0$, set $T_j = (C \cap S_j) \setminus \bigcup_{i < j} S_i \in \mathcal{A}_0$; these are disjoint with union C , and $T_j \subseteq S_j$, so $\mu_0(C) = \sum_j \mu_0(T_j) \leq \sum_j \mu_0(S_j)$. Thus μ_0 is a premeasure on the semiring $\mathcal{A}_0$.

Step 6: the measure. By the Caratheodory-Hahn Theorem the Caratheodory extension μ of μ_0 is a measure on the σ -algebra $\mathcal{M}$ of μ_0^* -measurable subsets of X , it extends μ_0 , and, μ_0 being finite hence σ -finite, it is the unique measure on $\mathcal{M}$ extending μ_0 . Since $\mu(X) = \mu_0(X) = 1$, the space $(X, \mathcal{M}, \mu)$ is a probability measure space. Moreover $\mathcal{M} \supseteq \mathcal{A}_0$ and hence $\mathcal{M}$ contains

$$\bigotimes_{\lambda \in \Lambda} \mathcal{A}_\lambda := \sigma(\mathcal{A}_0),$$

the smallest σ -algebra for which every projection $x \mapsto x_\lambda$ is measurable; by Corollary 14 of Chapter 17, μ restricted to this σ -algebra is the only measure on it with the prescribed values on cylinders. This is the “suitable σ -algebra” of the statement. We write $\mu = \prod_{\lambda \in \Lambda} \mu_\lambda$.

Step 7: the product formula. Let $A = \prod_{\lambda \in \Lambda} A_\lambda$ with $A_\lambda \in \mathcal{A}_\lambda$ for every λ .

Case (a): $A_\lambda = X_\lambda$ for all λ outside a countable set, say outside $\{\lambda_1, \lambda_2, \dots\}$. Put $C_m = \{x \in X : x_{\lambda_i} \in A_{\lambda_i} \text{ for } 1 \leq i \leq m\}$. Each $C_m \in \mathcal{A}_0$ with $\mu(C_m) = \prod_{i=1}^m \mu_{\lambda_i}(A_{\lambda_i})$, the sequence $\{C_m\}$ descends, and $A = \bigcap_m C_m$, so $A \in \bigotimes_{\lambda} \mathcal{A}_\lambda \subseteq \mathcal{M}$. Since $\mu(C_1) \leq 1 < \infty$, the continuity of measure gives

$$\mu(A) = \lim_{m \rightarrow \infty} \mu(C_m) = \lim_{m \rightarrow \infty} \prod_{i=1}^m \mu_{\lambda_i}(A_{\lambda_i}) = \prod_{\lambda \in \Lambda} \mu_\lambda(A_\lambda),$$

all the omitted factors being 1. This is the asserted formula.

Case (b): uncountably many λ have $\mu_\lambda(A_\lambda) < 1$. Writing $\{\lambda : \mu_\lambda(A_\lambda) < 1\} = \bigcup_{k=1}^\infty \{\lambda : \mu_\lambda(A_\lambda) \leq 1 - 1/k\}$, some k makes the latter set uncountable, in particular infinite; choose distinct $\lambda_1, \lambda_2, \dots$ in it. With C_m as above, $A \subseteq C_m$ and $\mu_0(C_m) \leq (1 - 1/k)^m \rightarrow 0$, so $\mu^*(A) = 0$. A set of outer measure zero is measurable, so $A \in \mathcal{M}$ and $\mu(A) = 0$; and $\prod_{\lambda} \mu_\lambda(A_\lambda) \leq (1 - 1/k)^m$ for every m , so the product is 0 as well. The formula holds, and this is exactly the parenthetical remark in the statement: $\mu(A)$ can be nonzero only when all but countably many A_λ have $\mu_\lambda(A_\lambda) = 1$.

A caveat on the remaining case. If uncountably many A_λ differ from X_λ but only countably many have $\mu_\lambda(A_\lambda) < 1$, then A need not be μ -measurable at all, so the identity must be read as an assertion about those A that are measurable (equivalently, cases (a) and (b) above). Here is an example. Let Λ be uncountable, let every $(X_\lambda, \mathcal{A}_\lambda, \mu_\lambda)$ be $[0, 1]$ with Lebesgue measure, and let $A_\lambda = (0, 1]$ for every λ , so $\mu_\lambda(A_\lambda) = 1$ and $\prod_{\lambda} \mu_\lambda(A_\lambda) = 1$. Let $\{C_i\}$ be any countable cover of $X \setminus A = \{x : x_\lambda = 0 \text{ for some } \lambda\}$ by members of $\mathcal{A}_0$, and let Λ_1 be the countable union of the finite index sets of the C_i . Choose $\lambda_0 \in \Lambda \setminus \Lambda_1$. Given any $y \in X$, let x agree with y off λ_0 and have $x_{\lambda_0} = 0$; then $x \in X \setminus A$, so $x \in C_i$ for some i , and since C_i depends only on coordinates in Λ_1 and x, y agree there, $y \in C_i$. Hence $\bigcup_i C_i = X$ and $\sum_i \mu_0(C_i) \geq 1$ by the countable monotonicity of μ_0 ; therefore $\mu^*(X \setminus A) = 1$. Similarly, if $\{C_i\}$ covers A and Λ_1 is as above, put $D = \{y \in X : y_\lambda \in (0, 1] \text{ for all } \lambda \in \Lambda_1\}$, which by case (a) is measurable with $\mu(D) = 1$; for $y \in D$ let x agree with y on Λ_1 and equal 1 elsewhere, so $x \in A$, whence $x \in C_i$ for some i and therefore $y \in C_i$. Thus $D \subseteq \bigcup_i C_i$ and $\sum_i \mu_0(C_i) \geq \mu(D) = 1$, giving $\mu^*(A) = 1$. Since $\mu^*(A) + \mu^*(X \setminus A) = 2 > 1 = \mu(X)$, the set A is not μ -measurable.

20.3 Problems 20.15–20.21

Problem 20.15. Consider the triangle $\Delta = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, 0 \leq y \leq [b/a]x\}$. By covering Δ with finite collections of rectangles and using the continuity of measure, determine the Lebesgue measure of Δ .

Solution. Here $a > 0$ and $b > 0$. We show that

$$\mu_2(\Delta) = \frac{ab}{2}.$$

The set Δ is closed, hence Borel, hence μ_2 -measurable by Theorem 11; it is contained in the bounded interval $[0, a] \times [0, b]$, so $\mu_2(\Delta) \leq ab < \infty$.

The outer estimate. For a natural number n partition $[0, a]$ by the points $x_k = ka/n$, $0 \leq k \leq n$, and define the finite collection of bounded intervals (rectangles) in $\mathbb{R}^2$

$$R_k^n = [x_{k-1}, x_k] \times \left[0, \frac{b}{a}x_k\right], \quad 1 \leq k \leq n, \quad \mathcal{O}_n = \bigcup_{k=1}^n R_k^n.$$

Since $x \mapsto (b/a)x$ is increasing, every point (x, y) of Δ lies in R_k^n for any k with $x_{k-1} \leq x \leq x_k$: indeed $y \leq (b/a)x \leq (b/a)x_k$. Hence $\Delta \subseteq \mathcal{O}_n$ for every n . By the finite subadditivity and monotonicity of μ_2 , and since $\mu_2(I) = \text{vol}(I)$ for a bounded interval I (Theorem 11),

$$\mu_2(\mathcal{O}_n) \leq \sum_{k=1}^n \text{vol}(R_k^n) = \sum_{k=1}^n \frac{a}{n} \cdot \frac{b}{a} \cdot \frac{ka}{n} = \frac{ab}{n^2} \sum_{k=1}^n k = \frac{ab}{2} \cdot \frac{n+1}{n}.$$

Now restrict to the dyadic values $n = 2^m$, $m \geq 1$, and put $G_m = \mathcal{O}_{2^m}$. The partition belonging to 2^{m+1} refines the one belonging to 2^m : each interval $[x'_{j-1}, x'_j]$ of the finer partition is contained in some interval $[x_{k-1}, x_k]$ of the coarser one, and then $x'_j \leq x_k$, so

$$[x'_{j-1}, x'_j] \times \left[0, \frac{b}{a}x'_j\right] \subseteq [x_{k-1}, x_k] \times \left[0, \frac{b}{a}x_k\right].$$

Therefore $G_{m+1} \subseteq G_m$: the sequence $\{G_m\}$ is descending. Each G_m is measurable, being a finite union of bounded intervals, and $\mu_2(G_1) < \infty$, so by the continuity of measure

$$\mu_2\left(\bigcap_{m=1}^{\infty} G_m\right) = \lim_{m \rightarrow \infty} \mu_2(G_m) \leq \lim_{m \rightarrow \infty} \frac{ab}{2} \cdot \frac{2^m + 1}{2^m} = \frac{ab}{2}.$$

Since $\Delta \subseteq \bigcap_m G_m$, monotonicity gives $\mu_2(\Delta) \leq ab/2$.

In fact $\bigcap_m G_m = \Delta$. If $(x, y) \in G_m$ then $0 \leq x \leq a$, $y \geq 0$, and $y \leq (b/a)\sigma_m(x)$, where $\sigma_m(x)$ is the right endpoint of some interval of the 2^m -partition that contains x ; hence $\sigma_m(x) \leq x + a/2^m$. Letting $m \rightarrow \infty$ yields $y \leq (b/a)x$, that is, $(x, y) \in \Delta$.

The inner estimate. For the same partition define the pairwise disjoint bounded intervals

$$S_k^n = [x_{k-1}, x_k] \times \left[0, \frac{b}{a}x_{k-1}\right], \quad 1 \leq k \leq n, \quad U_n = \bigcup_{k=1}^n S_k^n.$$

They are disjoint because their first factors are disjoint. If $(x, y) \in S_k^n$ then $x \geq x_{k-1}$ and $0 \leq y \leq (b/a)x_{k-1} \leq (b/a)x$, so $U_n \subseteq \Delta$. By the finite additivity and monotonicity of μ_2 ,

$$\mu_2(\Delta) \geq \mu_2(U_n) = \sum_{k=1}^n \text{vol}(S_k^n) = \sum_{k=1}^n \frac{a}{n} \cdot \frac{b}{a} \cdot \frac{(k-1)a}{n} = \frac{ab}{n^2} \sum_{k=0}^{n-1} k = \frac{ab}{2} \cdot \frac{n-1}{n}.$$

Letting $n \rightarrow \infty$ gives $\mu_2(\Delta) \geq ab/2$.

Combining the two estimates, $\mu_2(\Delta) = ab/2$, the familiar formula for the area of a right triangle with legs of length a and b .

Problem 20.16. Let $[a, b]$ be a closed, bounded interval of real numbers. Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is bounded and Lebesgue measurable. Show that the graph of f has measure zero with respect to Lebesgue measure on the plane. Generalize this to bounded real-valued functions of several real variables.

Solution. Write $G = \{(x, f(x)) \mid x \in [a, b]\} \subseteq \mathbb{R}^2$ for the graph. We prove directly from the definition of Lebesgue outer measure that $\mu_2^*(G) = 0$; since sets of outer measure zero are measurable and Lebesgue measure is complete (Theorem 11), this gives $\mu_2(G) = 0$.

Choose $M > 0$ with $|f(x)| \leq M$ for all $x \in [a, b]$. Fix a natural number N and partition the interval $[-M, M]$ into N consecutive pieces of length $2M/N$:

$$J_i = \left[-M + \frac{2M(i-1)}{N}, -M + \frac{2Mi}{N} \right) \quad (1 \leq i \leq N-1), \quad J_N = \left[M - \frac{2M}{N}, M \right].$$

These are bounded intervals in $\mathbb{R}$, pairwise disjoint, of length $2M/N$ each, and their union is $[-M, M]$. Put $E_i = f^{-1}(J_i) \subseteq [a, b]$. Since f is Lebesgue measurable and J_i is an interval, each E_i is a Lebesgue measurable subset of $[a, b]$. The sets $E_1, \dots, E_N$ are pairwise disjoint and their union is $[a, b]$, because the J_i are disjoint with union $[-M, M] \supseteq f([a, b])$. Hence, by the finite additivity of μ_1 ,

$$\sum_{i=1}^N \mu_1(E_i) = \mu_1([a, b]) = b - a.$$

Let $\epsilon > 0$. For each i , by the definition of Lebesgue outer measure choose a countable collection $\{I_{i,j}\}_{j=1}^\infty$ of bounded intervals in $\mathbb{R}$ that covers E_i and satisfies

$$\sum_{j=1}^\infty \ell(I_{i,j}) < \mu_1(E_i) + \frac{\epsilon}{N}.$$

Now the countable collection of bounded intervals $\{I_{i,j} \times J_i \mid 1 \leq i \leq N, j \geq 1\}$ in $\mathbb{R}^2$ covers G . Indeed, if $x \in [a, b]$ then $f(x) \in J_i$ for exactly one i , so $x \in E_i$, so $x \in I_{i,j}$ for some j , and therefore $(x, f(x)) \in I_{i,j} \times J_i$. Consequently, by the definition of μ_2^* as the outer measure induced by the premeasure volume,

$$\mu_2^*(G) \leq \sum_{i=1}^N \sum_{j=1}^\infty \text{vol}(I_{i,j} \times J_i) = \sum_{i=1}^N \frac{2M}{N} \sum_{j=1}^\infty \ell(I_{i,j}) < \frac{2M}{N} \sum_{i=1}^N \left(\mu_1(E_i) + \frac{\epsilon}{N} \right) = \frac{2M}{N} (b - a + \epsilon).$$

Since $\epsilon > 0$ is arbitrary, $\mu_2^*(G) \leq 2M(b-a)/N$, and since N is an arbitrary natural number, $\mu_2^*(G) = 0$. Thus G is Lebesgue measurable in the plane and $\mu_2(G) = 0$.

Generalization. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ with $\mu_n(E) < \infty$ and let $f: E \rightarrow \mathbb{R}$ be bounded and Lebesgue measurable. Then

$$G = \{(x, f(x)) \mid x \in E\} \subseteq \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$$

has $\mu_{n+1}(G) = 0$. The proof is verbatim the one above, with the following two changes: $E_i = f^{-1}(J_i)$ is now a measurable subset of $\mathbb{R}^n$ with $\sum_{i=1}^N \mu_n(E_i) = \mu_n(E)$; and $\{I_{i,j}\}_j$ is a countable cover of E_i by bounded intervals in $\mathbb{R}^n$ with $\sum_j \text{vol}(I_{i,j}) < \mu_n(E_i) + \epsilon/N$. Since $\text{vol}(I_{i,j} \times J_i) = \text{vol}(I_{i,j}) \cdot \ell(J_i)$ for the bounded interval $I_{i,j} \times J_i$ of $\mathbb{R}^{n+1}$, the same computation yields

$$\mu_{n+1}^*(G) \leq \frac{2M}{N} (\mu_n(E) + \epsilon) \quad \text{for all } N \text{ and } \epsilon,$$

whence $\mu_{n+1}(G) = 0$.

Finally, the restriction $\mu_n(E) < \infty$ may be dropped: an arbitrary measurable $E \subseteq \mathbb{R}^n$ is the union of the countable ascending collection $\{E \cap B_k\}_{k=1}^\infty$, where B_k is the open ball of radius k about the origin, each of finite measure. The graph of f over E is the countable union of the graphs of the restrictions $f|_{E \cap B_k}$, each of measure zero by the above, so it has measure zero by the countable subadditivity of μ_{n+1} . (Boundedness of f can likewise be relaxed to finiteness by decomposing E into the sets on which $|f| \leq k$.)

Problem 20.17. Every open set of real numbers is the union of a countable disjoint collection of open intervals. Is the open subset of the plane $\{(x, y) \in \mathbb{R}^2 \mid 0 < x, y < 1\}$ the union of a countable disjoint collection of open balls?

Solution. No. The two-dimensional analogue of the one-dimensional decomposition theorem fails: the open unit square

$$Q = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < 1\}$$

is not the union of any disjoint collection of open balls, countable or otherwise.

Recall that an open ball in $\mathbb{R}^2$ is a set $B(c, r) = \{p \in \mathbb{R}^2 \mid \|p - c\| < r\}$ with $r > 0$; its boundary $\partial B(c, r)$ is the circle $\{p \mid \|p - c\| = r\}$, a set with infinitely many points.

Suppose, to get a contradiction, that

$$Q = \bigcup_{k \in \Lambda} B_k,$$

where Λ is a nonempty index set and $\{B_k\}_{k \in \Lambda}$ is a disjoint collection of open balls, $B_k = B(c_k, r_k)$. Since $Q \neq \emptyset$, $\Lambda \neq \emptyset$; fix $k \in \Lambda$.

Step 1: $\partial B_k \cap Q = \emptyset$. Suppose $p \in \partial B_k$ and $p \in Q$. Then $p \in B_j$ for some $j \in \Lambda$. A point of ∂B_k does not belong to the open set B_k , so $j \neq k$. Since B_j is open there is $\delta > 0$ with $B(p, \delta) \subseteq B_j$. Since p belongs to the boundary of B_k , every neighbourhood of p meets B_k , so $B(p, \delta) \cap B_k \neq \emptyset$ and therefore $B_j \cap B_k \neq \emptyset$. This contradicts the disjointness of the collection. Hence no point of ∂B_k lies in Q .

Step 2: $\partial B_k \subseteq \partial Q$. We have $B_k \subseteq Q$, so $\partial B_k \subseteq \overline{B_k} \subseteq \overline{Q}$. By Step 1, $\partial B_k \cap Q = \emptyset$. Since Q is open, $\overline{Q} \setminus Q = \partial Q$. Therefore $\partial B_k \subseteq \partial Q$.

Step 3: The contradiction. The boundary of the open unit square is the union of the four closed segments

$$[0, 1] \times \{0\}, \quad [0, 1] \times \{1\}, \quad \{0\} \times [0, 1], \quad \{1\} \times [0, 1],$$

each of which lies on a straight line. A circle of positive radius meets a straight line in at most two points, since substituting the parametrization of the line into the equation $\|p - c_k\|^2 = r_k^2$ gives a quadratic in one real variable with a nonzero leading coefficient. Hence $\partial B_k \cap \partial Q$ contains at most $4 \cdot 2 = 8$ points. But by Step 2, $\partial B_k \cap \partial Q = \partial B_k$, a circle of radius $r_k > 0$, which is an infinite set. This is a contradiction.

Therefore no such decomposition exists, and the answer to the question is no.

Two remarks. First, the argument used nothing about the square except that it is nonempty and that its boundary contains no circle of positive radius; so the same proof shows that no nonempty open subset of $\mathbb{R}^2$ whose boundary contains no such circle is the union of a disjoint collection of open balls. Second, the obstruction is topological rather than measure-theoretic. What the proof rules out is an exact exhaustion of Q : for every disjoint collection of open balls contained in Q , the leftover set is nonempty. It does not rule out an exhaustion up to a null set, and in fact no such obstruction exists; but that refinement uses covering machinery beyond the present section and is not needed for the question asked, which the above answers in the negative.

Problem 20.18. Verify inequality (13): for a bounded interval I in $\mathbb{R}$ with end-points a and b , the integral count of I satisfies

$$(b - a) - 1 \leq \mu^{\text{integral}}(I) \leq (b - a) + 1.$$

Solution. Recall that a point of $\mathbb{R}^n$ is called an integral point provided each of its coordinates is an integer, and that $\mu^{\text{integral}}(I)$ denotes the number of integral points in I . For $n = 1$ the integral points are exactly the integers, so

$$\mu^{\text{integral}}(I) = \#(\mathbb{Z} \cap I).$$

Let I be a bounded interval in $\mathbb{R}$ with end-points $a \leq b$; thus I is one of $[a, b]$, $[a, b)$, $(a, b]$, (a, b) , and $\ell(I) = b - a$. Since I is bounded, $\mathbb{Z} \cap I$ is finite, so $\mu^{\text{integral}}(I)$ is a well-defined nonnegative integer.

Throughout we use the inclusions

$$(a, b) \subseteq I \subseteq [a, b].$$

The upper estimate. By monotonicity of the count it suffices to bound $\#(\mathbb{Z} \cap [a, b])$. If $\mathbb{Z} \cap [a, b] = \emptyset$ the count is $0 \leq (b - a) + 1$, since $b \geq a$. Otherwise let p be the least and q the greatest integer in $[a, b]$. Every integer strictly between p and q lies in $[a, b]$, and no integer outside $[p, q]$ lies in $[a, b]$, so

$$\mathbb{Z} \cap [a, b] = \{p, p + 1, \dots, q\}, \quad \#(\mathbb{Z} \cap [a, b]) = q - p + 1.$$

Since $a \leq p$ and $q \leq b$ we get $q - p \leq b - a$, hence

$$\mu^{\text{integral}}(I) \leq \#(\mathbb{Z} \cap [a, b]) = q - p + 1 \leq (b - a) + 1.$$

The lower estimate. By monotonicity it suffices to bound $\#(\mathbb{Z} \cap (a, b))$ from below. If $b - a \leq 1$, then $(b - a) - 1 \leq 0 \leq \mu^{\text{integral}}(I)$ and there is nothing to prove. So assume $b - a > 1$.

We first note that (a, b) then contains an integer. Indeed, let m be the least integer with $m > a$ (such an m exists by the Archimedean property, and the set of integers exceeding a has a least element since it is a nonempty set of integers bounded below). Then $m - 1 \leq a$, so $m \leq a + 1 < b$, and therefore $m \in (a, b)$.

Since $\mathbb{Z} \cap (a, b)$ is a nonempty finite set, it has a least element p and a greatest element q , and as above

$$\mathbb{Z} \cap (a, b) = \{p, p + 1, \dots, q\}, \quad \#(\mathbb{Z} \cap (a, b)) = q - p + 1.$$

Now $p - 1$ is an integer with $p - 1 < p < b$; were $p - 1 > a$ we would have $p - 1 \in \mathbb{Z} \cap (a, b)$, contradicting the minimality of p . Hence $p - 1 \leq a$, that is, $p \leq a + 1$. Symmetrically, $q + 1$ is an integer with $q + 1 > q > a$; were $q + 1 < b$ we would have $q + 1 \in \mathbb{Z} \cap (a, b)$, contradicting the maximality of q . Hence $q + 1 \geq b$, that is, $q \geq b - 1$. Therefore

$$\mu^{\text{integral}}(I) \geq \#(\mathbb{Z} \cap (a, b)) = q - p + 1 \geq (b - 1) - (a + 1) + 1 = (b - a) - 1.$$

The two estimates together give (13). Note also that the degenerate cases are covered: if $I = \emptyset$ or I is a single point, then $a = b$, the count is 0 or 1, and indeed $-1 \leq 0 \leq 1$ and $-1 \leq 1 \leq 1$.

For completeness we record how (13) is used in Lemma 8. If $I = I_1 \times \dots \times I_n$ is a bounded interval in $\mathbb{R}^n$ with I_k having end-points a_k and b_k , then a point of $\mathbb{R}^n$ is an integral point of I precisely when its k -th coordinate is an integral point of I_k for each k , so

$$\mu^{\text{integral}}(I) = \mu^{\text{integral}}(I_1) \dots \mu^{\text{integral}}(I_n).$$

Multiplying the n upper bounds furnished by (13) gives the upper estimate in (14) at once, all the quantities involved being nonnegative.

The lower estimate in (14) needs a word of care, since a product of lower bounds need not bound the product of the quantities bounded when some of the lower bounds are negative. For instance, with $n = 2$ and $I_1 = I_2 = (0, 1/2)$ the left-hand side of (14) equals $(-1/2)(-1/2) = 1/4$, while $\mu^{\text{integral}}(I) = 0$. Multiplication is legitimate precisely when every factor $(b_k - a_k) - 1$ is nonnegative, and that is the only regime in which Lemma 8 uses the estimate. Indeed, (14) is applied there not to I but to the dilated interval $T_\epsilon(I)$, whose k -th edge has length $\epsilon(b_k - a_k)$, and the conclusion (12) is a statement about the limit as $\epsilon \rightarrow \infty$. If $b_k > a_k$ for every k , then $\epsilon(b_k - a_k) - 1 \geq 0$ for all ϵ sufficiently large, so for such ϵ the n inequalities may be multiplied and (15) holds; dividing by ϵ^n and letting $\epsilon \rightarrow \infty$ squeezes $\mu^{\text{integral}}(T_\epsilon(I))/\epsilon^n$ between $\prod_k [(b_k - a_k) - 1/\epsilon]$ and $\prod_k [(b_k - a_k) + 1/\epsilon]$, both of which converge to $\prod_k (b_k - a_k) = \text{vol}(I)$.

If instead $b_k = a_k$ for some k , then $\text{vol}(I) = 0$ and the upper estimate alone suffices: the k -th factor of $\prod_j [\epsilon(b_j - a_j) + 1]$ equals 1, so this product is a polynomial in ϵ of degree at most $n - 1$, and therefore $\mu^{\text{integral}}(T_\epsilon(I))/\epsilon^n \rightarrow 0 = \text{vol}(I)$. In both cases (12) follows.

Problem 20.19. Verify the set equality (16). That is, let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, let $\{z_k\}_{k=1}^\infty$ be an enumeration of the collection of points of $\mathcal{O}$ that have rational coordinates, and for each k and each natural number m let $I_{k,m}$ be the open cube centered at z_k of edge-length $1/m$. Show that

$$\mathcal{O} = \bigcup_{I_{k,m} \subseteq \mathcal{O}} I_{k,m},$$

the union being taken over all pairs (k, m) for which $I_{k,m} \subseteq \mathcal{O}$. [In the text this index m is written n ; we rename it to avoid a clash with the dimension.]

Solution. If $\mathcal{O} = \emptyset$ both sides are empty and there is nothing to prove, so assume $\mathcal{O} \neq \emptyset$. Then, since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ and $\mathcal{O}$ is open and nonempty, the set of points of $\mathcal{O}$ with rational coordinates is a countably infinite dense subset of $\mathcal{O}$, so an enumeration $\{z_k\}_{k=1}^\infty$ exists.

We use two norms on $\mathbb{R}^n$: the Euclidean norm $\|x\| = (\sum_{j=1}^n x_j^2)^{1/2}$, which induces the topology, and the maximum norm $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$. They are related by

$$\|x\|_\infty \leq \|x\| \leq \sqrt{n} \|x\|_\infty \quad \text{for all } x \in \mathbb{R}^n.$$

With this notation the open cube centered at z of edge-length $1/m$ is

$$C(z, m) = \prod_{j=1}^n \left(z_j - \frac{1}{2m}, z_j + \frac{1}{2m} \right) = \left\{ y \in \mathbb{R}^n \mid \|y - z\|_\infty < \frac{1}{2m} \right\},$$

so that $I_{k,m} = C(z_k, m)$.

The inclusion $\supseteq$. Each set occurring in the union on the right is, by the very condition defining the union, a subset of $\mathcal{O}$. Hence their union is contained in $\mathcal{O}$.

The inclusion $\subseteq$. Let $x \in \mathcal{O}$. Since $\mathcal{O}$ is open, there is an $r > 0$ such that the open ball

$$B(x, r) = \{ y \in \mathbb{R}^n \mid \|y - x\| < r \}$$

is contained in $\mathcal{O}$. By the Archimedean property choose a natural number m with

$$\frac{\sqrt{n}}{m} < r.$$

Since the rational numbers are dense in $\mathbb{R}$, for each coordinate j we may choose a rational q_j with $|x_j - q_j| < 1/(2m)$. Put $z = (q_1, \dots, q_n)$, a point of $\mathbb{R}^n$ with rational coordinates satisfying

$$\|x - z\|_\infty < \frac{1}{2m}.$$

First, $z \in \mathcal{O}$: indeed

$$\|x - z\| \leq \sqrt{n} \|x - z\|_\infty < \frac{\sqrt{n}}{2m} < \frac{r}{2} < r,$$

so $z \in B(x, r) \subseteq \mathcal{O}$. Since z is a point of $\mathcal{O}$ with rational coordinates, $z = z_k$ for some index k .

Second, $x \in I_{k,m} = C(z, m)$, since $\|x - z\|_\infty < 1/(2m)$.

Third, $I_{k,m} \subseteq \mathcal{O}$. Let $y \in C(z, m)$. Then $\|y - z\|_\infty < 1/(2m)$, so $\|y - z\| \leq \sqrt{n} \|y - z\|_\infty < \sqrt{n}/(2m)$, and therefore, by the triangle inequality,

$$\|y - x\| \leq \|y - z\| + \|z - x\| < \frac{\sqrt{n}}{2m} + \frac{\sqrt{n}}{2m} = \frac{\sqrt{n}}{m} < r.$$

Hence $y \in B(x, r) \subseteq \mathcal{O}$. As $y \in C(z, m)$ was arbitrary, $I_{k,m} = C(z, m) \subseteq \mathcal{O}$.

Thus $I_{k,m}$ is one of the cubes occurring in the union on the right-hand side of (16), and $x \in I_{k,m}$. Since $x \in \mathcal{O}$ was arbitrary, $\mathcal{O}$ is contained in that union, and the two inclusions give the asserted equality.

Finally, the union in (16) is indexed by a subset of the countable set $\mathbb{N} \times \mathbb{N}$ of pairs (k, m) , hence is a countable union. Each open cube $I_{k,m}$ is a bounded interval in $\mathbb{R}^n$ and so is μ_n -measurable; therefore $\mathcal{O}$, being a countable union of measurable sets, is μ_n -measurable. This is exactly what the proof of Theorem 11 requires.

Problem 20.20. Let $E \subseteq \mathbb{R}^n$ and $z \in \mathbb{R}^n$.

- (i) Show that $E + z$ is open if E is open.
- (ii) Show that $E + z$ is G_δ if E is G_δ .
- (iii) Show that $\mu_n^*(E + z) = \mu_n^*(E)$.
- (iv) Show that E is μ_n -measurable if and only if $E + z$ is μ_n -measurable.

Solution. Throughout, $E + z = \{x + z \mid x \in E\}$ and $\tau_z: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the translation $\tau_z(x) = x + z$, so that $E + z = \tau_z(E)$. We record two elementary facts used repeatedly.

First, τ_z is a bijection with inverse τ_{-z} , and consequently, for any collection $\{A_\lambda\}$ of subsets of $\mathbb{R}^n$ and any $A, B \subseteq \mathbb{R}^n$,

$$\left(\bigcap_\lambda A_\lambda\right) + z = \bigcap_\lambda (A_\lambda + z), \quad \left(\bigcup_\lambda A_\lambda\right) + z = \bigcup_\lambda (A_\lambda + z), \quad (A \setminus B) + z = (A + z) \setminus (B + z).$$

Each is immediate from the equivalence $p \in A + z \iff p - z \in A$. For instance, $p \in (\bigcap_\lambda A_\lambda) + z$ iff $p - z \in A_\lambda$ for every λ iff $p \in A_\lambda + z$ for every λ . In particular $(\mathbb{R}^n \setminus A) + z = \mathbb{R}^n \setminus (A + z)$.

Second, τ_z is an isometry: $\|\tau_z(x) - \tau_z(y)\| = \|(x + z) - (y + z)\| = \|x - y\|$. Hence for every x and $r > 0$,

$$B(x, r) + z = B(x + z, r),$$

where $B(x, r) = \{y \mid \|y - x\| < r\}$.

(i) Let E be open and let $p \in E + z$. Then $p = x + z$ for some $x \in E$, and since E is open there is $r > 0$ with $B(x, r) \subseteq E$. Translating, $B(p, r) = B(x, r) + z \subseteq E + z$. Since p was an arbitrary point of $E + z$, the set $E + z$ is open.

(ii) Let E be G_δ , say $E = \bigcap_{k=1}^\infty \mathcal{O}_k$ with each $\mathcal{O}_k$ open. By the first fact above,

$$E + z = \bigcap_{k=1}^\infty (\mathcal{O}_k + z),$$

and by part (i) each $\mathcal{O}_k + z$ is open. Hence $E + z$ is the intersection of a countable collection of open sets, that is, $E + z$ is G_δ .

(iii) Lebesgue outer measure μ_n^* is the outer measure induced by the premeasure volume on the semiring $\mathcal{I}$ of bounded intervals in $\mathbb{R}^n$, so that for $A \subseteq \mathbb{R}^n$,

$$\mu_n^*(A) = \inf \left\{ \sum_{k=1}^\infty \text{vol}(I^k) \mid \{I^k\}_{k=1}^\infty \subseteq \mathcal{I}, A \subseteq \bigcup_{k=1}^\infty I^k \right\}.$$

Observe that $\mathcal{I}$ is carried onto itself by τ_z with volume unchanged. Indeed, if $I = I_1 \times \cdots \times I_n$ with each I_j a bounded interval in $\mathbb{R}$ having end-points $a_j \leq b_j$, then

$$I + z = (I_1 + z_1) \times \cdots \times (I_n + z_n),$$

and $I_j + z_j$ is a bounded interval in $\mathbb{R}$ of the same type with end-points $a_j + z_j$ and $b_j + z_j$, so $\ell(I_j + z_j) = (b_j + z_j) - (a_j + z_j) = \ell(I_j)$. Multiplying,

$$\text{vol}(I + z) = \prod_{j=1}^n \ell(I_j + z_j) = \prod_{j=1}^n \ell(I_j) = \text{vol}(I).$$

Now let $\{I^k\}_{k=1}^\infty$ be any countable collection of bounded intervals covering E . Then $\{I^k + z\}_{k=1}^\infty$ is a countable collection of bounded intervals and, by the union identity above,

$$E + z \subseteq \left(\bigcup_k I^k\right) + z = \bigcup_k (I^k + z),$$

so it covers $E + z$. Since $\sum_k \text{vol}(I^k + z) = \sum_k \text{vol}(I^k)$, we conclude

$$\mu_n^*(E + z) \leq \sum_{k=1}^\infty \text{vol}(I^k).$$

Taking the infimum over all such covers of E gives $\mu_n^*(E + z) \leq \mu_n^*(E)$.

This holds for every set and every translation vector; applying it to the set $E + z$ and the vector $-z$, and using $(E + z) + (-z) = E$, gives $\mu_n^*(E) \leq \mu_n^*(E + z)$. Therefore $\mu_n^*(E + z) = \mu_n^*(E)$.

(iv) Suppose E is μ_n -measurable. By Corollary 14 there is a G_δ subset G of $\mathbb{R}^n$ with

$$E \subseteq G \quad \text{and} \quad \mu_n^*(G \setminus E) = 0.$$

By part (ii), $G + z$ is G_δ . Translating the inclusion, $E + z \subseteq G + z$. Moreover, by the set-difference identity and part (iii),

$$\mu_n^*((G + z) \setminus (E + z)) = \mu_n^*((G \setminus E) + z) = \mu_n^*(G \setminus E) = 0.$$

Thus $G + z$ is a G_δ set containing $E + z$ whose difference with $E + z$ has outer measure zero, so by Corollary 14 (the implication (ii) implies (i)) the set $E + z$ is μ_n -measurable.

Conversely, if $E + z$ is μ_n -measurable, then applying what was just proved to the set $E + z$ and the vector $-z$ shows that $(E + z) + (-z) = E$ is μ_n -measurable. Hence E is measurable if and only if $E + z$ is.

Combining (iii) and (iv): if E is μ_n -measurable then so is $E + z$, and

$$\mu_n(E + z) = \mu_n^*(E + z) = \mu_n^*(E) = \mu_n(E),$$

which is the translation invariance of Lebesgue measure asserted in the text.

Remark. Part (iv) can also be proved directly from the Caratheodory criterion, without appeal to Corollary 14. Suppose E is μ_n -measurable and let $A \subseteq \mathbb{R}^n$ be arbitrary. Applying the Caratheodory criterion for E to the test set $A - z$ and then translating each of the three sets involved by z , using (iii) and the identities $((A - z) \cap E) + z = A \cap (E + z)$ and $((A - z) \cap E^c) + z = A \cap (E + z)^c$, we obtain

$$\mu_n^*(A) = \mu_n^*(A - z) = \mu_n^*((A - z) \cap E) + \mu_n^*((A - z) \cap E^c) = \mu_n^*(A \cap (E + z)) + \mu_n^*(A \cap (E + z)^c),$$

so $E + z$ is μ_n -measurable.

Problem 20.21. For each natural number n , show that every subset of $\mathbb{R}^n$ of positive outer Lebesgue measure contains a subset that is not Lebesgue measurable.

Solution. This is the n -dimensional version of Vitali's theorem (Theorem 17 of Chapter 2). We use the translation invariance established in Problem 20: for $A \subseteq \mathbb{R}^n$ and $z \in \mathbb{R}^n$, the set $A + z$ is μ_n -measurable if and only if A is, and $\mu_n^*(A + z) = \mu_n^*(A)$.

Let $E \subseteq \mathbb{R}^n$ with $\mu_n^*(E) > 0$.

Step 1: reduction to a bounded set. For each natural number k let $B_k = \{x \in \mathbb{R}^n \mid \|x\| < k\}$. Then $E = \bigcup_{k=1}^{\infty} (E \cap B_k)$, so by the countable subadditivity of the outer measure μ_n^* ,

$$0 < \mu_n^*(E) \leq \sum_{k=1}^{\infty} \mu_n^*(E \cap B_k).$$

Hence $\mu_n^*(E \cap B_k) > 0$ for some k . Put $E_0 = E \cap B_k$ and $R = k$, so that E_0 is bounded, $E_0 \subseteq B(0, R)$, and $\mu_n^*(E_0) > 0$. A subset of E_0 that fails to be measurable is a subset of E that fails to be measurable, so it suffices to produce one inside E_0 .

Step 2: the rational equivalence relation and a choice set. Since $\mathbb{Q}^n$ is a subgroup of the additive group $\mathbb{R}^n$, the relation

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q}^n$$

is an equivalence relation on $\mathbb{R}^n$: it is reflexive because $0 \in \mathbb{Q}^n$, symmetric because $\mathbb{Q}^n$ is closed under negation, and transitive because $\mathbb{Q}^n$ is closed under addition. It therefore partitions $\mathbb{R}^n$ into disjoint equivalence classes. By the Axiom of Choice, select a set $C \subseteq E_0$ containing exactly one point of E_0 from each equivalence class that has nonempty intersection with E_0 . Thus:

- (a) $C \subseteq E_0 \subseteq B(0, R)$;

- (b) distinct points of C are inequivalent, that is, if $c_1, c_2 \in C$ and $c_1 - c_2 \in \mathbb{Q}^n$ then $c_1 = c_2$;
- (c) every $x \in E_0$ satisfies $x - c \in \mathbb{Q}^n$ for exactly one $c \in C$.

Step 3: a bounded countably infinite set of translation vectors. Let

$$\Lambda = \mathbb{Q}^n \cap B(0, 2R).$$

As a subset of the countable set $\mathbb{Q}^n$, Λ is countable; and since $B(0, 2R)$ is a nonempty open set and $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, Λ is infinite. So Λ is countably infinite, and it is bounded.

Step 4: the translates of C by Λ are disjoint and cover E_0 . Suppose $\lambda_1, \lambda_2 \in \Lambda$ and $(C + \lambda_1) \cap (C + \lambda_2) \neq \emptyset$, say $c_1 + \lambda_1 = c_2 + \lambda_2$ with $c_1, c_2 \in C$. Then $c_1 - c_2 = \lambda_2 - \lambda_1 \in \mathbb{Q}^n$, so by (b) $c_1 = c_2$, and hence $\lambda_1 = \lambda_2$. Therefore $\{C + \lambda\}_{\lambda \in \Lambda}$ is a disjoint collection.

Next, let $x \in E_0$. By (c) there is $c \in C$ with $\lambda := x - c \in \mathbb{Q}^n$. Since $x \in B(0, R)$ and $c \in B(0, R)$, the triangle inequality gives $\|\lambda\| \leq \|x\| + \|c\| < 2R$, so $\lambda \in \Lambda$ and $x = c + \lambda \in C + \lambda$. Therefore

$$E_0 \subseteq \bigcup_{\lambda \in \Lambda} (C + \lambda).$$

Finally, since $C \subseteq B(0, R)$ and $\|\lambda\| < 2R$ for $\lambda \in \Lambda$,

$$\bigcup_{\lambda \in \Lambda} (C + \lambda) \subseteq B(0, 3R) \subseteq [-3R, 3R]^n.$$

Step 5: C is not μ_n -measurable. Suppose, to the contrary, that C is μ_n -measurable. By Problem 20, each translate $C + \lambda$ is μ_n -measurable and $\mu_n(C + \lambda) = \mu_n(C)$. The collection $\{C + \lambda\}_{\lambda \in \Lambda}$ is a countable disjoint collection of measurable sets, so by the countable additivity and monotonicity of the measure μ_n , together with the containment in the cube $[-3R, 3R]^n$ and Theorem 11,

$$\sum_{\lambda \in \Lambda} \mu_n(C) = \sum_{\lambda \in \Lambda} \mu_n(C + \lambda) = \mu_n\left(\bigcup_{\lambda \in \Lambda} (C + \lambda)\right) \leq \mu_n([-3R, 3R]^n) = (6R)^n < \infty.$$

The index set Λ is infinite, so a sum of infinitely many copies of the nonnegative number $\mu_n(C)$ can be finite only if $\mu_n(C) = 0$.

But then, by the countable subadditivity and monotonicity of the outer measure μ_n^* and the covering established in Step 4,

$$\mu_n^*(E_0) \leq \sum_{\lambda \in \Lambda} \mu_n^*(C + \lambda) = \sum_{\lambda \in \Lambda} \mu_n(C) = 0,$$

contradicting $\mu_n^*(E_0) > 0$.

Therefore C is not μ_n -measurable. Since $C \subseteq E_0 \subseteq E$, the set E contains a subset that is not Lebesgue measurable, as was to be shown.

Remark. The argument shows a little more. From $E_0 \subseteq \bigcup_{\lambda \in \Lambda} (C + \lambda)$ and subadditivity we get $0 < \mu_n^*(E_0) \leq \sum_{\lambda \in \Lambda} \mu_n^*(C)$, so $\mu_n^*(C) > 0$. On the other hand, if $D \subseteq C$ were measurable, then the translates $\{D + \lambda\}_{\lambda \in \Lambda}$ would be disjoint, measurable, and contained in $[-3R, 3R]^n$, so exactly the computation of Step 5 would force $\mu_n(D) = 0$. Thus C has positive outer measure yet contains no measurable subset of positive measure. Note also that positivity of the outer measure is essential and not merely convenient: if $\mu_n^*(E) = 0$, then by the monotonicity of μ_n^* every subset of E has outer measure zero, and every set of outer measure zero is μ_n^* -measurable, so such an E has no nonmeasurable subset at all. (This is the completeness of Lebesgue measure recorded in Theorem 11.)

20.4 Problems 20.22–20.28

Problem 20.22. For each natural number n , show that there is a subset of $\mathbb{R}^n$ that is not a Borel set but is μ_n -measurable.

Solution. We first recall the one-dimensional case and then transplant it into $\mathbb{R}^n$.

Step 1: a Lebesgue measurable subset of $\mathbb{R}$ that is not Borel. This is Proposition 22 of Chapter 2,

and we recall its proof. Let C be the Cantor set, let φ be the Cantor-Lebesgue function, and define ψ on $[0, 1]$ by $\psi(x) = \varphi(x) + x$. By Proposition 21 of Chapter 2, ψ is a strictly increasing continuous function mapping $[0, 1]$ onto $[0, 2]$, and $\psi(C)$ is a measurable set with $\mu_1(\psi(C)) = 1$. Since $\psi(C)$ has positive outer measure, it contains a subset W that is not Lebesgue measurable (Vitali's theorem on the existence of nonmeasurable sets; see also Problem 21 of this chapter). Put

$$A = \psi^{-1}(W) \subseteq C.$$

Since $A \subseteq C$ and $\mu_1(C) = 0$, we have $\mu_1^*(A) = 0$, so A is Lebesgue measurable, every set of outer measure zero satisfying the Caratheodory criterion. On the other hand A is not a Borel set: a strictly increasing continuous function defined on an interval maps Borel sets onto Borel sets (Problem 47 of Chapter 2), so if A were Borel then $W = \psi(A)$ would be a Borel subset of $\mathbb{R}$ and hence Lebesgue measurable, contradicting the choice of W .

Since $\mu_1 = m$ on $\mathbb{R}$, the set A is μ_1 -measurable and not Borel, which settles the case $n = 1$. From now on fix this set $A \subseteq C \subseteq [0, 1]$ and assume $n \geq 2$.

Step 2: the candidate in $\mathbb{R}^n$. Define

$$A_n = A \times \{0\}^{n-1} = \{(t, 0, \dots, 0) \in \mathbb{R}^n : t \in A\}.$$

Step 3: A_n is μ_n -measurable. Let $\epsilon > 0$. Then A_n is contained in the bounded interval

$$I_\epsilon = [0, 1] \times [-\epsilon, \epsilon] \times \cdots \times [-\epsilon, \epsilon] \subseteq \mathbb{R}^n,$$

whose volume is $\text{vol}(I_\epsilon) = 1 \cdot (2\epsilon)^{n-1}$. Since $\{I_\epsilon\}$ is a one-term cover of A_n by bounded intervals, the definition of Lebesgue outer measure gives

$$\mu_n^*(A_n) \leq \text{vol}(I_\epsilon) = (2\epsilon)^{n-1}.$$

As $n - 1 \geq 1$, letting $\epsilon \rightarrow 0$ yields $\mu_n^*(A_n) = 0$. A set of outer measure zero is μ_n^* -measurable: for any $E \subseteq \mathbb{R}^n$, monotonicity of outer measure gives

$$\mu_n^*(E \cap A_n) + \mu_n^*(E \sim A_n) \leq \mu_n^*(A_n) + \mu_n^*(E) = \mu_n^*(E),$$

which is the Caratheodory criterion. Hence A_n belongs to $\mathcal{L}^n$. Equivalently, this is the completeness of the measure space $(\mathbb{R}^n, \mathcal{L}^n, \mu_n)$ asserted in Theorem 11.

Step 4: A_n is not a Borel subset of $\mathbb{R}^n$. Consider the map

$$j : \mathbb{R} \rightarrow \mathbb{R}^n, \quad j(t) = (t, 0, \dots, 0),$$

which is continuous, and observe that

$$j^{-1}(A_n) = \{t \in \mathbb{R} : (t, 0, \dots, 0) \in A_n\} = A.$$

Now let

$$\mathcal{C} = \{E \subseteq \mathbb{R}^n : j^{-1}(E) \text{ is a Borel subset of } \mathbb{R}\}.$$

Since the formation of preimages commutes with complements and with countable unions, $\mathcal{C}$ is a σ -algebra of subsets of $\mathbb{R}^n$. Since j is continuous, $j^{-1}(\mathcal{O})$ is open, hence Borel, for each open $\mathcal{O} \subseteq \mathbb{R}^n$; thus $\mathcal{C}$ contains every open subset of $\mathbb{R}^n$. Because the Borel σ -algebra $\mathcal{B}(\mathbb{R}^n)$ is by definition the smallest σ -algebra of subsets of $\mathbb{R}^n$ containing the open sets, we conclude $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{C}$. This is the n -dimensional form of Problem 46 of Chapter 2.

Consequently, if A_n were a Borel subset of $\mathbb{R}^n$, then $A = j^{-1}(A_n)$ would be a Borel subset of $\mathbb{R}$, contradicting Step 1. Therefore A_n is not Borel.

Thus for $n \geq 2$ the set A_n is μ_n -measurable but is not a Borel subset of $\mathbb{R}^n$, and for $n = 1$ the set A itself has these two properties. Hence for every natural number n there is a subset of $\mathbb{R}^n$ that is μ_n -measurable but not Borel.

Problem 20.23. If (27) holds for each interval in $\mathbb{R}^n$, use the uniqueness assertion of the Caratheodory-Hahn Theorem to show directly that it also holds for every measurable subset of $\mathbb{R}^n$.

[Here (27) is the assertion that for an invertible linear operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and each Lebesgue measurable subset E of $\mathbb{R}^n$, the set $T(E)$ is Lebesgue measurable and $\mu_n(T(E)) = |\det(T)| \cdot \mu_n(E)$.]

Solution. Fix an invertible linear operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and assume that

$$\mu_n(T(I)) = |\det(T)| \cdot \mu_n(I) \quad \text{for every bounded interval } I \text{ in } \mathbb{R}^n. \quad (*)$$

We must show the same formula holds for every $E \in \mathcal{L}^n$.

Step 1: T carries $\mathcal{L}^n$ into $\mathcal{L}^n$. By Proposition 17 a linear operator on $\mathbb{R}^n$ is Lipschitz, and by Proposition 18 a Lipschitz mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ maps Lebesgue measurable sets to Lebesgue measurable sets. Hence $T(E) \in \mathcal{L}^n$ whenever $E \in \mathcal{L}^n$, and the following definition makes sense.

Step 2: definition of a competing measure. Since $\det(T) \neq 0$, define

$$\nu(E) = \frac{1}{|\det(T)|} \cdot \mu_n(T(E)) \quad \text{for all } E \in \mathcal{L}^n.$$

We claim ν is a measure on the σ -algebra $\mathcal{L}^n$.

Clearly $\nu(E) \in [0, \infty]$ and $\nu(\emptyset) = \mu_n(T(\emptyset))/|\det T| = 0$. For countable additivity, let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection of sets in $\mathcal{L}^n$. Since T is one-to-one and onto, it preserves unions and disjointness: $T(\bigcup_{k=1}^\infty E_k) = \bigcup_{k=1}^\infty T(E_k)$, and if $j \neq k$ then $T(E_j) \cap T(E_k) = T(E_j \cap E_k) = \emptyset$ by injectivity. Each $T(E_k)$ is measurable by Step 1. Therefore, by the countable additivity of μ_n ,

$$\nu\left(\bigcup_{k=1}^\infty E_k\right) = \frac{1}{|\det T|} \mu_n\left(\bigcup_{k=1}^\infty T(E_k)\right) = \frac{1}{|\det T|} \sum_{k=1}^\infty \mu_n(T(E_k)) = \sum_{k=1}^\infty \nu(E_k).$$

Thus ν is a measure on $\mathcal{L}^n$.

Step 3: ν extends the premeasure volume. Let $\mathcal{I}_n$ denote the semiring of bounded intervals in $\mathbb{R}^n$ (Proposition 9) and $\text{vol} : \mathcal{I}_n \rightarrow [0, \infty)$ the set function volume, which by Proposition 10 is a premeasure on $\mathcal{I}_n$. Each bounded interval is Lebesgue measurable with $\mu_n(I) = \text{vol}(I)$ (Theorem 11). Hence, using hypothesis (*), for every $I \in \mathcal{I}_n$,

$$\nu(I) = \frac{1}{|\det T|} \mu_n(T(I)) = \frac{1}{|\det T|} \cdot |\det T| \cdot \mu_n(I) = \mu_n(I) = \text{vol}(I).$$

So ν is a measure on $\mathcal{L}^n$ that extends vol .

Step 4: invoke uniqueness. The premeasure vol is σ -finite: indeed

$$\mathbb{R}^n = \bigcup_{k=1}^\infty [-k, k) \times \cdots \times [-k, k),$$

a countable union of bounded intervals each of finite volume. By definition, μ_n^* is the outer measure induced by the premeasure vol on the semiring $\mathcal{I}_n$, and $\mathcal{L}^n$ is precisely the σ -algebra of μ_n^* -measurable sets, with μ_n the induced Caratheodory measure. The uniqueness assertion of the Caratheodory-Hahn Theorem states that, for a σ -finite premeasure on a semiring, the induced Caratheodory measure is the unique measure on the σ -algebra of μ^* -measurable sets that extends the premeasure.

Both μ_n and ν are measures on $\mathcal{L}^n$ extending vol : the former by Theorem 11, the latter by Steps 2 and 3. Therefore $\nu = \mu_n$ on $\mathcal{L}^n$, that is,

$$\frac{1}{|\det T|} \mu_n(T(E)) = \mu_n(E) \quad \text{for all } E \in \mathcal{L}^n,$$

which upon multiplying by $|\det T|$ is exactly

$$\mu_n(T(E)) = |\det(T)| \cdot \mu_n(E) \quad \text{for all Lebesgue measurable } E \subseteq \mathbb{R}^n.$$

This is (27).

Problem 20.24. Let $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be Lipschitz with Lipschitz constant c . Show that there is a constant c' that depends only on the dimension n and c for which the estimate (23) holds. [Here (23) is the estimate $\mu_n^*(\Psi(I)) \leq c' \cdot \text{vol}(I)$ for every bounded interval I in $\mathbb{R}^n$.]

Solution. We show that

$$c' = (c\sqrt{n})^n = c^n n^{n/2}$$

works. Throughout, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ and $\text{diam}(S) = \sup\{\|u - v\| : u, v \in S\}$.

Step 1: a set of small diameter has small outer measure. Let S be a nonempty bounded subset of $\mathbb{R}^n$ with $\text{diam}(S) \leq d$. For $1 \leq i \leq n$ let $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be the i -th coordinate projection, and set $a_i = \inf \pi_i(S)$. For $u, v \in S$ we have $|\pi_i(u) - \pi_i(v)| \leq \|u - v\| \leq d$, so $\sup \pi_i(S) - a_i \leq d$ and hence $\pi_i(S) \subseteq [a_i, a_i + d]$. Therefore

$$S \subseteq J := [a_1, a_1 + d] \times \cdots \times [a_n, a_n + d],$$

a bounded interval in $\mathbb{R}^n$ with $\text{vol}(J) = d^n$. Since $\{J\}$ is a cover of S by bounded intervals, the definition of Lebesgue outer measure gives

$$\mu_n^*(S) \leq \text{vol}(J) = d^n. \quad (\text{i})$$

Step 2: the estimate for cubes. Let Q be a bounded interval in $\mathbb{R}^n$ all of whose edges have the same length s , i.e. a cube of edge length s . For $u, v \in Q$ we have $|u_i - v_i| \leq s$ for each i , so

$$\|u - v\| = \left(\sum_{i=1}^n (u_i - v_i)^2 \right)^{1/2} \leq s\sqrt{n},$$

that is, $\text{diam}(Q) \leq s\sqrt{n}$. Since Ψ is Lipschitz with constant c , for $u, v \in Q$,

$$\|\Psi(u) - \Psi(v)\| \leq c\|u - v\| \leq cs\sqrt{n},$$

so $\text{diam}(\Psi(Q)) \leq c\sqrt{n}s$. If Q is empty the estimate below is trivial; otherwise by (i),

$$\mu_n^*(\Psi(Q)) \leq (c\sqrt{n}s)^n = (c\sqrt{n})^n s^n = c' \cdot \text{vol}(Q). \quad (\text{ii})$$

Step 3: passage to a general bounded interval. Let $I = I_1 \times \cdots \times I_n$ be a bounded interval in $\mathbb{R}^n$, where each I_i is a bounded interval of real numbers with endpoints $a_i \leq b_i$ and length $\ell_i = b_i - a_i$, so that $\text{vol}(I) = \ell_1 \cdots \ell_n$. If $I = \emptyset$ then $\Psi(I) = \emptyset$ and (23) is trivial, so assume $I \neq \emptyset$. Fix a natural number m and consider the grid of half-open cubes of edge length $1/m$,

$$Q_{\mathbf{k}} = \left[\frac{k_1}{m}, \frac{k_1+1}{m} \right) \times \cdots \times \left[\frac{k_n}{m}, \frac{k_n+1}{m} \right), \quad \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n.$$

These cubes are pairwise disjoint and their union is $\mathbb{R}^n$. Let $\mathcal{Q}_m$ be the collection of those $Q_{\mathbf{k}}$ that meet I . Since I is bounded, $\mathcal{Q}_m$ is a finite collection, and

$$I \subseteq \bigcup_{Q \in \mathcal{Q}_m} Q, \quad \text{hence} \quad \Psi(I) \subseteq \bigcup_{Q \in \mathcal{Q}_m} \Psi(Q).$$

By the finite subadditivity and monotonicity of the outer measure μ_n^* , together with (ii),

$$\mu_n^*(\Psi(I)) \leq \sum_{Q \in \mathcal{Q}_m} \mu_n^*(\Psi(Q)) \leq c' \sum_{Q \in \mathcal{Q}_m} \text{vol}(Q). \quad (\text{iii})$$

Step 4: control of the total volume of the grid cubes. Let $Q \in \mathcal{Q}_m$ and choose $p \in Q \cap I$. Since $\text{diam}(Q) \leq \sqrt{n}/m$, each $q \in Q$ satisfies $|q_i - p_i| \leq \|q - p\| \leq \sqrt{n}/m$ for every i , while $a_i \leq p_i \leq b_i$ because $p \in I$. Hence every $Q \in \mathcal{Q}_m$ is contained in

$$\tilde{I}_m = \left[a_1 - \frac{\sqrt{n}}{m}, b_1 + \frac{\sqrt{n}}{m} \right] \times \cdots \times \left[a_n - \frac{\sqrt{n}}{m}, b_n + \frac{\sqrt{n}}{m} \right],$$

a bounded interval of volume $\prod_{i=1}^n (\ell_i + 2\sqrt{n}/m)$. The cubes in $\mathcal{Q}_m$ are pairwise disjoint and Lebesgue measurable with $\mu_n(Q) = \text{vol}(Q)$ (Theorem 11), so by the additivity and monotonicity of μ_n ,

$$\sum_{Q \in \mathcal{Q}_m} \text{vol}(Q) = \mu_n\left(\bigcup_{Q \in \mathcal{Q}_m} Q\right) \leq \mu_n(\tilde{I}_m) = \prod_{i=1}^n \left(\ell_i + \frac{2\sqrt{n}}{m}\right).$$

Step 5: conclusion. Substituting into (iii),

$$\mu_n^*(\Psi(I)) \leq c' \prod_{i=1}^n \left(\ell_i + \frac{2\sqrt{n}}{m}\right) \quad \text{for every natural number } m.$$

The right-hand side is a polynomial in $1/m$ whose value at $1/m = 0$ is $c' \prod_{i=1}^n \ell_i$; letting $m \rightarrow \infty$ we obtain

$$\mu_n^*(\Psi(I)) \leq c' \prod_{i=1}^n \ell_i = c' \cdot \text{vol}(I).$$

This is exactly (23), with $c' = (c\sqrt{n})^n = c^n n^{n/2}$, a constant depending only on c and on the dimension n .

Problem 20.25. Prove that the Cartesian product of two semirings is a semiring. Based on this use an induction argument to prove that the collection of intervals in $\mathbb{R}^n$ is a semiring.

Solution. Recall the definition: a nonempty collection $\mathcal{S}$ of subsets of a set X is a semiring provided that whenever A and B belong to $\mathcal{S}$, then $A \cap B$ also belongs to $\mathcal{S}$ and there is a finite disjoint collection $\{C_k\}_{k=1}^p$ of sets in $\mathcal{S}$ for which $A \sim B = \bigcup_{k=1}^p C_k$.

Part 1: the product of two semirings is a semiring.

Let $\mathcal{S}_1$ be a semiring of subsets of a set X and $\mathcal{S}_2$ a semiring of subsets of a set Y . Define

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 = \{A \times B : A \in \mathcal{S}_1, B \in \mathcal{S}_2\},$$

a collection of subsets of $X \times Y$. It is nonempty since $\mathcal{S}_1$ and $\mathcal{S}_2$ are.

Closure under intersection. Let $A \times B$ and $A' \times B'$ belong to $\mathcal{S}$. A point (x, y) lies in both if and only if $x \in A \cap A'$ and $y \in B \cap B'$, so

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

Since $A \cap A' \in \mathcal{S}_1$ and $B \cap B' \in \mathcal{S}_2$, this set belongs to $\mathcal{S}$.

Relative complements. We claim that

$$(A \times B) \sim (A' \times B') = [(A \sim A') \times B] \cup [(A \cap A') \times (B \sim B')], \quad (\dagger)$$

and that the two sets on the right are disjoint.

Indeed, suppose $(x, y) \in (A \times B) \sim (A' \times B')$. Then $x \in A$, $y \in B$, and either $x \notin A'$ or $y \notin B'$. If $x \notin A'$ then $(x, y) \in (A \sim A') \times B$. If $x \in A'$, then necessarily $y \notin B'$, and $(x, y) \in (A \cap A') \times (B \sim B')$. Conversely, if $(x, y) \in (A \sim A') \times B$ then $x \in A$, $y \in B$ and $x \notin A'$, so $(x, y) \notin A' \times B'$; and if $(x, y) \in (A \cap A') \times (B \sim B')$ then $x \in A$, $y \in B$ and $y \notin B'$, so again $(x, y) \notin A' \times B'$. This proves $(\dagger)$. The two sets on the right are disjoint because the first consists of points whose first coordinate lies outside A' while the second consists of points whose first coordinate lies in A' .

Now, since $\mathcal{S}_1$ is a semiring, there is a finite disjoint collection $\{C_k\}_{k=1}^p$ in $\mathcal{S}_1$ with $A \sim A' = \bigcup_{k=1}^p C_k$; since $\mathcal{S}_2$ is a semiring, there is a finite disjoint collection $\{D_j\}_{j=1}^q$ in $\mathcal{S}_2$ with $B \sim B' = \bigcup_{j=1}^q D_j$. Moreover $A \cap A' \in \mathcal{S}_1$. Substituting into $(\dagger)$ and distributing the product over unions,

$$(A \times B) \sim (A' \times B') = \bigcup_{k=1}^p (C_k \times B) \cup \bigcup_{j=1}^q ((A \cap A') \times D_j).$$

Every set appearing here belongs to $\mathcal{S}$, since $C_k \in \mathcal{S}_1$, $B \in \mathcal{S}_2$, $A \cap A' \in \mathcal{S}_1$, $D_j \in \mathcal{S}_2$. The collection is finite, with $p + q$ members, and it is disjoint: the sets $C_k \times B$ are pairwise disjoint because the C_k are; the sets $(A \cap A') \times D_j$ are pairwise disjoint because the D_j are; and a set of the first kind is disjoint from a set of the second kind because $C_k \subseteq A \sim A'$ is disjoint from $A \cap A'$, so already the first factors are disjoint.

(If $A \sim A'$ is empty we may take $p = 0$, and likewise $q = 0$ if $B \sim B'$ is empty; the empty union is then the empty set, as required.)

Hence $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is a semiring of subsets of $X \times Y$.

Part 2: the base case $n = 1$.

Let $\mathcal{I}_1$ be the collection of bounded intervals of real numbers, i.e. sets of the form $[a, b]$, $[a, b)$, $(a, b]$ or (a, b) with $a \leq b$; this collection contains the empty set and all singletons. It is nonempty.

If $A, B \in \mathcal{I}_1$, then $A \cap B$ is a bounded convex subset of $\mathbb{R}$, hence again a bounded interval, so $A \cap B \in \mathcal{I}_1$.

For relative complements, let $A, B \in \mathcal{I}_1$ and put $J = A \cap B \in \mathcal{I}_1$; then $A \sim B = A \sim J$. If $J = \emptyset$ then $A \sim B = A$, a single member of $\mathcal{I}_1$. If $J \neq \emptyset$, set

$$L = \{x \in \mathbb{R} : x < t \text{ for all } t \in J\}, \quad R = \{x \in \mathbb{R} : x > t \text{ for all } t \in J\}.$$

Both L and R are intervals (possibly empty), and they are disjoint since $J \neq \emptyset$. We claim $\mathbb{R} \sim J = L \cup R$. Certainly L and R are disjoint from J . Conversely, if $x \notin J$ but $x \notin L \cup R$, then there are $t_1, t_2 \in J$ with $t_1 \leq x$ and $x \leq t_2$; since J is an interval, hence convex, this forces $x \in J$, a contradiction. Therefore

$$A \sim B = A \cap (\mathbb{R} \sim J) = (A \cap L) \cup (A \cap R),$$

a disjoint union of two sets, each an intersection of the bounded interval A with an interval, hence a bounded interval and so a member of $\mathcal{I}_1$. Thus $\mathcal{I}_1$ is a semiring.

Part 3: the induction.

By definition, a bounded interval in $\mathbb{R}^n$ is a set of the form $I_1 \times \cdots \times I_n$ where each I_i is a bounded interval of real numbers. Let $\mathcal{I}_n$ denote the collection of all such sets.

We argue by induction on n that $\mathcal{I}_n$ is a semiring of subsets of $\mathbb{R}^n$. The case $n = 1$ is Part 2. Assume $n \geq 2$ and that $\mathcal{I}_{n-1}$ is a semiring of subsets of $\mathbb{R}^{n-1}$. Identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ by the natural correspondence $(x_1, \dots, x_n) \leftrightarrow ((x_1, \dots, x_{n-1}), x_n)$. Under this identification,

$$I_1 \times \cdots \times I_n = (I_1 \times \cdots \times I_{n-1}) \times I_n,$$

so that

$$\mathcal{I}_n = \mathcal{I}_{n-1} \times \mathcal{I}_1,$$

the Cartesian product of the collection $\mathcal{I}_{n-1}$ of subsets of $\mathbb{R}^{n-1}$ with the collection $\mathcal{I}_1$ of subsets of $\mathbb{R}$, in the sense of Part 1. Since $\mathcal{I}_{n-1}$ is a semiring by the induction hypothesis and $\mathcal{I}_1$ is a semiring by Part 2, Part 1 tells us that $\mathcal{I}_n$ is a semiring of subsets of $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$.

By induction, the collection of bounded intervals in $\mathbb{R}^n$ is a semiring for every natural number n . This is Proposition 9.

Problem 20.26. Show that if the function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous with respect to each variable separately, then it is measurable with respect to Lebesgue measure μ_2 .

Solution. The hypothesis is that for each fixed $y \in [0, 1]$ the function $x \mapsto f(x, y)$ is continuous on $[0, 1]$, and for each fixed $x \in [0, 1]$ the function $y \mapsto f(x, y)$ is continuous on $[0, 1]$. We exhibit f as the pointwise limit on $[0, 1] \times [0, 1]$ of a sequence of continuous functions; measurability then follows.

Step 1: construction of the approximants. Fix a natural number m and for $0 \leq k \leq m$ put $x_k = k/m$. For $0 \leq k \leq m$ define the hat function $h_k : [0, 1] \rightarrow \mathbb{R}$ by

$$h_k(x) = \max\{0, 1 - m|x - x_k|\}.$$

Each h_k is continuous on $[0, 1]$, being the maximum of two continuous functions. Moreover $h_k(x_j) = 1$

if $j = k$ and $h_k(x_j) = 0$ if $j \neq k$, and for every $x \in [0, 1]$,

$$\sum_{k=0}^m h_k(x) = 1,$$

since if $x_{j-1} \leq x \leq x_j$ then $h_k(x) = 0$ for $k \notin \{j-1, j\}$ while $h_{j-1}(x) + h_j(x) = (1 - m(x - x_{j-1})) + (1 - m(x_j - x)) = 2 - m(x_j - x_{j-1}) = 1$. Thus for such x the numbers $h_{j-1}(x)$ and $h_j(x)$ are nonnegative and sum to 1.

Now define $f_m : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by

$$f_m(x, y) = \sum_{k=0}^m h_k(x) f(x_k, y).$$

Explicitly, if $x_{j-1} \leq x \leq x_j$ then

$$f_m(x, y) = (1 - m(x - x_{j-1})) f(x_{j-1}, y) + m(x - x_{j-1}) f(x_j, y),$$

so f_m is the linear interpolant of f in the first variable on the grid $\{x_k\}$; in particular $f_m(x_k, y) = f(x_k, y)$ for all k and all y . Since f_m is given by a single formula valid on the whole square, no matching of definitions on overlapping pieces is needed.

Step 2: each f_m is continuous on $[0, 1] \times [0, 1]$. For $0 \leq k \leq m$ set $g_k(y) = f(x_k, y)$; by the hypothesis of continuity in the second variable, each g_k is continuous on $[0, 1]$. The map $(x, y) \mapsto h_k(x)$ is continuous on the square, being the composition of the continuous projection $(x, y) \mapsto x$ with the continuous h_k , and likewise $(x, y) \mapsto g_k(y)$ is continuous on the square. Hence each product $(x, y) \mapsto h_k(x)g_k(y)$ is continuous, and f_m , being the finite sum

$$f_m(x, y) = \sum_{k=0}^m h_k(x) g_k(y),$$

of these continuous functions, is continuous on $[0, 1] \times [0, 1]$.

The square $[0, 1] \times [0, 1]$ is a bounded interval in $\mathbb{R}^2$, hence Lebesgue measurable by Theorem 11. Therefore, by Corollary 12, each f_m is measurable with respect to two-dimensional Lebesgue measure μ_2 .

Step 3: $f_m \rightarrow f$ pointwise on $[0, 1] \times [0, 1]$. Fix $(x, y) \in [0, 1] \times [0, 1]$ and let $\epsilon > 0$. The function $t \mapsto f(t, y)$ is continuous on the compact interval $[0, 1]$, hence uniformly continuous there; choose $\delta > 0$ such that

$$|f(s, y) - f(t, y)| < \epsilon \quad \text{whenever } s, t \in [0, 1], \quad |s - t| < \delta.$$

Let $m > 1/\delta$ and choose k with $x_{k-1} \leq x \leq x_k$. Then $|x - x_{k-1}| \leq 1/m < \delta$ and $|x - x_k| \leq 1/m < \delta$, so

$$|f(x_{k-1}, y) - f(x, y)| < \epsilon \quad \text{and} \quad |f(x_k, y) - f(x, y)| < \epsilon.$$

Writing $\lambda = m(x - x_{k-1}) \in [0, 1]$, we have $f_m(x, y) = (1 - \lambda)f(x_{k-1}, y) + \lambda f(x_k, y)$ and also $f(x, y) = (1 - \lambda)f(x, y) + \lambda f(x, y)$, whence

$$|f_m(x, y) - f(x, y)| \leq (1 - \lambda)|f(x_{k-1}, y) - f(x, y)| + \lambda|f(x_k, y) - f(x, y)| < (1 - \lambda)\epsilon + \lambda\epsilon = \epsilon.$$

Since this holds for all $m > 1/\delta$, we conclude $f_m(x, y) \rightarrow f(x, y)$.

(Note that only the continuity of f in the first variable was used in this step; the continuity in the second variable was used in Step 2.)

Step 4: conclusion. The sequence $\{f_m\}$ consists of μ_2 -measurable functions on $[0, 1] \times [0, 1]$ and converges pointwise on all of $[0, 1] \times [0, 1]$ to f . By Theorem 6 of Chapter 18, the pointwise limit of a sequence of measurable functions on a measure space is measurable. Hence f is measurable with respect to Lebesgue measure μ_2 .

Problem 20.27. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping of $\mathbb{R}$ onto $\mathbb{R}$ for which there is a constant $c > 0$ for which

$$|g(u) - g(v)| \geq c \cdot |u - v| \quad \text{for all } u, v \in \mathbb{R}.$$

Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue measurable, then so is the composition $f \circ g : \mathbb{R} \rightarrow \mathbb{R}$.

Solution. Step 1: g is a bijection and its inverse is Lipschitz. If $g(u) = g(v)$, then $0 = |g(u) - g(v)| \geq c|u - v|$, and since $c > 0$ this forces $u = v$. Thus g is one-to-one, and by hypothesis it is onto $\mathbb{R}$; so g is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ and possesses an inverse

$$h := g^{-1} : \mathbb{R} \rightarrow \mathbb{R}.$$

Let $s, t \in \mathbb{R}$ and set $u = h(s)$, $v = h(t)$, so that $g(u) = s$ and $g(v) = t$. The hypothesis gives

$$|s - t| = |g(u) - g(v)| \geq c|u - v| = c|h(s) - h(t)|,$$

and therefore

$$|h(s) - h(t)| \leq \frac{1}{c} |s - t| \quad \text{for all } s, t \in \mathbb{R}.$$

So h is a Lipschitz mapping of $\mathbb{R}$ into $\mathbb{R}$, with Lipschitz constant $1/c$.

Step 2: preimages under g are images under h . Since g is a bijection with inverse h , for any set $E \subseteq \mathbb{R}$ the preimage of E under g coincides with the image of E under h :

$$g^{-1}(E) = \{t \in \mathbb{R} : g(t) \in E\} = \{h(s) : s \in E\} = h(E).$$

Indeed, if $g(t) \in E$ then $t = h(g(t)) \in h(E)$; conversely if $t = h(s)$ with $s \in E$ then $g(t) = g(h(s)) = s \in E$.

Step 3: measurability of the composition. Let $\mathcal{O}$ be an open subset of $\mathbb{R}$. Then

$$(f \circ g)^{-1}(\mathcal{O}) = g^{-1}(f^{-1}(\mathcal{O})).$$

Since f is Lebesgue measurable, $E := f^{-1}(\mathcal{O})$ is a Lebesgue measurable subset of $\mathbb{R}$. By Step 2, $(f \circ g)^{-1}(\mathcal{O}) = h(E)$. By Step 1, $h : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz, so Proposition 18, applied with $n = 1$ and $\Psi = h$, tells us that $h(E)$ is Lebesgue measurable.

Therefore $(f \circ g)^{-1}(\mathcal{O})$ is Lebesgue measurable for every open $\mathcal{O} \subseteq \mathbb{R}$. Taking in particular $\mathcal{O} = (a, \infty)$ for an arbitrary real number a , the set

$$\{x \in \mathbb{R} : (f \circ g)(x) > a\}$$

is Lebesgue measurable, and this is precisely the criterion for $f \circ g$ to be a Lebesgue measurable function. Hence $f \circ g$ is Lebesgue measurable.

Remark. Both hypotheses are used: that g is onto is what makes $h = g^{-1}$ defined on all of $\mathbb{R}$, and the expansiveness estimate is what makes h Lipschitz rather than merely continuous. The distinction matters, because a continuous mapping of $\mathbb{R}$ onto $\mathbb{R}$ need not carry Lebesgue measurable sets to Lebesgue measurable sets, and correspondingly the composition of Lebesgue measurable functions of a real variable need not be Lebesgue measurable; see the example following Proposition 22 of Chapter 2 and the discussion preceding Proposition 5 of Chapter 18.

Problem 20.28. By using the bilinearity of the inner product, prove (28).

[Here (28) is the polarization identity

$$\langle u, v \rangle = \frac{1}{4} \left[\|u + v\|^2 - \|u - v\|^2 \right] \quad \text{for all } u, v \in \mathbb{R}^n.$$

]

Solution. Recall that the inner product on $\mathbb{R}^n$ is the bilinear form

$$\langle x, y \rangle = \sum_{k=1}^n x_k y_k \quad \text{for all } x, y \in \mathbb{R}^n,$$

that it is symmetric, $\langle x, y \rangle = \langle y, x \rangle$, as is immediate from the commutativity of multiplication of real numbers, and that the Euclidean norm is defined by $\|x\| = \sqrt{\langle x, x \rangle}$, so that

$$\|x\|^2 = \langle x, x \rangle \quad \text{for all } x \in \mathbb{R}^n.$$

Let $u, v \in \mathbb{R}^n$. Using $\|u + v\|^2 = \langle u + v, u + v \rangle$ and expanding by linearity in the first argument and then in the second argument,

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u + v \rangle + \langle v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + 2\langle u, v \rangle + \|v\|^2, \end{aligned}$$

where the last equality uses the symmetry $\langle v, u \rangle = \langle u, v \rangle$.

Similarly, replacing v by $-v$ and using that $\langle u, -v \rangle = -\langle u, v \rangle$ and $\langle -v, -v \rangle = \langle v, v \rangle$, both consequences of bilinearity,

$$\begin{aligned} \|u - v\|^2 &= \langle u - v, u - v \rangle \\ &= \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 - 2\langle u, v \rangle + \|v\|^2. \end{aligned}$$

Subtracting the second identity from the first, the terms $\|u\|^2$ and $\|v\|^2$ cancel and we obtain

$$\|u + v\|^2 - \|u - v\|^2 = (\|u\|^2 + 2\langle u, v \rangle + \|v\|^2) - (\|u\|^2 - 2\langle u, v \rangle + \|v\|^2) = 4\langle u, v \rangle.$$

Dividing by 4 gives

$$\langle u, v \rangle = \frac{1}{4} [\|u + v\|^2 - \|u - v\|^2] \quad \text{for all } u, v \in \mathbb{R}^n,$$

which is (28).

As noted in the text, this identity has the consequence used in the proof of Corollary 24: if a linear operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves the Euclidean norm, that is $\|T(x)\| = \|x\|$ for all x , then for all $u, v \in \mathbb{R}^n$, using the linearity of T ,

$$\langle T(u), T(v) \rangle = \frac{1}{4} [\|T(u) + T(v)\|^2 - \|T(u) - T(v)\|^2] = \frac{1}{4} [\|T(u+v)\|^2 - \|T(u-v)\|^2] = \frac{1}{4} [\|u+v\|^2 - \|u-v\|^2] = \langle u, v \rangle,$$

so that T preserves the inner product.

20.5 Problems 20.29–20.35

Problem 20.29. Let the mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear. Define $c = \sup\{\|T(x)\| \mid \|x\| \leq 1\}$. Show that c is the smallest Lipschitz constant for T .

Solution. First observe that c is a well-defined real number. Indeed, by Proposition 17 the linear operator T is Lipschitz, so there is a constant $M \geq 0$ with

$$\|T(u) - T(v)\| \leq M \|u - v\| \quad \text{for all } u, v \in \mathbb{R}^n.$$

Since T is linear, $T(0) = 0$; hence for $\|x\| \leq 1$ we get $\|T(x)\| = \|T(x) - T(0)\| \leq M\|x\| \leq M$. Thus $\{\|T(x)\| : \|x\| \leq 1\}$ is a nonempty set of reals (it contains $\|T(0)\| = 0$) bounded above by M , and therefore $0 \leq c \leq M < \infty$.

Step 1: c is a Lipschitz constant for T .

Let $u, v \in \mathbb{R}^n$. If $u = v$, the inequality $\|T(u) - T(v)\| \leq c\|u - v\|$ reads $0 \leq 0$ and holds. Suppose $u \neq v$ and set

$$w = \frac{u - v}{\|u - v\|},$$

so that $\|w\| = 1 \leq 1$ and hence $\|T(w)\| \leq c$ by the definition of c as a supremum. By the linearity of T ,

$$T(u) - T(v) = T(u - v) = T(\|u - v\| w) = \|u - v\| T(w),$$

and therefore, by the positive homogeneity of the norm,

$$\|T(u) - T(v)\| = \|u - v\| \|T(w)\| \leq c \|u - v\|.$$

Since u, v were arbitrary, c is a Lipschitz constant for T .

Step 2: c is the smallest such constant.

Let c' be any Lipschitz constant for T , that is,

$$\|T(u) - T(v)\| \leq c' \|u - v\| \quad \text{for all } u, v \in \mathbb{R}^n.$$

Take $v = 0$ and let x be any vector with $\|x\| \leq 1$. Since $T(0) = 0$,

$$\|T(x)\| = \|T(x) - T(0)\| \leq c' \|x - 0\| = c' \|x\| \leq c'.$$

Thus c' is an upper bound for the set $\{\|T(x)\| : \|x\| \leq 1\}$, and since c is the least upper bound of that set, $c \leq c'$.

By Step 1, c belongs to the set of Lipschitz constants for T ; by Step 2, it is a lower bound for that set. Hence c is the smallest Lipschitz constant for T .

Problem 20.30. Show that a subspace W of $\mathbb{R}^n$ of dimension less than n has n -dimensional Lebesgue measure zero by first showing this is so for the subspace $\{x \in \mathbb{R}^n \mid x_n = 0\}$.

Solution. Throughout, μ_n^* denotes n -dimensional Lebesgue outer measure, defined for $E \subseteq \mathbb{R}^n$ by

$$\mu_n^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(I_k) \mid E \subseteq \bigcup_{k=1}^{\infty} I_k, \text{ each } I_k \text{ an interval in } \mathbb{R}^n \right\},$$

where an interval $I = J_1 \times \cdots \times J_n$ is a product of bounded intervals of real numbers and $\text{vol}(I) = \prod_{j=1}^n \ell(J_j)$.

Step 1: the hyperplane $V = \{x \in \mathbb{R}^n \mid x_n = 0\}$ is null.

For each natural number k put

$$V_k = \{x \in \mathbb{R}^n \mid x_n = 0 \text{ and } |x_j| \leq k \text{ for } 1 \leq j \leq n-1\},$$

so that $V = \bigcup_{k=1}^{\infty} V_k$. Fix k and let $\varepsilon > 0$. Choose

$$\delta = \frac{\varepsilon}{2(2k)^{n-1}} > 0$$

and consider the single interval

$$I = [-k, k] \times \cdots \times [-k, k] \times [-\delta, \delta] \subseteq \mathbb{R}^n.$$

Every point of V_k has its first $n-1$ coordinates in $[-k, k]$ and its last coordinate equal to $0 \in [-\delta, \delta]$, so $V_k \subseteq I$. Padding the cover with degenerate intervals of volume 0, we obtain a countable cover of V_k by intervals of total volume

$$\text{vol}(I) = (2k)^{n-1} \cdot 2\delta = \varepsilon.$$

Hence $\mu_n^*(V_k) \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\mu_n^*(V_k) = 0$.

By the countable subadditivity of outer measure,

$$\mu_n^*(V) \leq \sum_{k=1}^{\infty} \mu_n^*(V_k) = 0,$$

so $\mu_n^*(V) = 0$. A set of outer measure zero is Carathéodory measurable: for any $A \subseteq \mathbb{R}^n$, monotonicity gives

$$\mu_n^*(A \cap V) + \mu_n^*(A \cap V^c) \leq \mu_n^*(V) + \mu_n^*(A) = \mu_n^*(A).$$

Therefore V is Lebesgue measurable and $\mu_n(V) = 0$. The same computation shows every subset of V is measurable with measure zero, that is, Lebesgue measure is complete.

Step 2: an arbitrary proper subspace is null.

Let W be a subspace of $\mathbb{R}^n$ with $m = \dim W < n$. Choose a basis $w_1, \dots, w_m$ of W and extend it to a basis $w_1, \dots, w_n$ of $\mathbb{R}^n$. (If $m = 0$, then $W = \{0\} \subseteq V$ and Step 1 already gives $\mu_n(W) = 0$; so assume $m \geq 1$.) Define the linear operator $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$T(e_j) = w_j \quad \text{for } 1 \leq j \leq n,$$

where $e_1, \dots, e_n$ is the standard basis. Since T carries a basis onto a basis, T is invertible, that is, $T \in GL(n, \mathbb{R})$.

Let

$$V_0 = \{x \in \mathbb{R}^n \mid x_{m+1} = \dots = x_n = 0\} = \text{span}\{e_1, \dots, e_m\}.$$

Because $m < n$, every $x \in V_0$ has $x_n = 0$, so $V_0 \subseteq V$ and therefore $\mu_n(V_0) = 0$ by Step 1 and the completeness of Lebesgue measure. On the other hand, by linearity,

$$T(V_0) = \text{span}\{w_1, \dots, w_m\} = W.$$

By Corollary 23, since T is invertible and V_0 is Lebesgue measurable, $T(V_0)$ is Lebesgue measurable and

$$\mu_n(W) = \mu_n(T(V_0)) = |\det T| \cdot \mu_n(V_0) = |\det T| \cdot 0 = 0.$$

(One may avoid Corollary 23 altogether: by Proposition 17 the operator T is Lipschitz, and the proof of Proposition 18 shows that a Lipschitz mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ carries a set of Lebesgue outer measure zero onto a set of Lebesgue outer measure zero. Applying this to V_0 gives $\mu_n^*(W) = \mu_n^*(T(V_0)) = 0$ directly, and hence W is measurable with $\mu_n(W) = 0$.)

Problem 20.31. Prove the two change of variables formulas (24) first for characteristic functions of sets of finite measure, then for simple functions that vanish outside a set of finite measure and finally for nonnegative integrable functions of a single real variable.

[Formula (24), from Proposition 20: if $f: \mathbb{R} \rightarrow \mathbb{R}$ is integrable over $\mathbb{R}$ with respect to one-dimensional Lebesgue measure and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, then

$$\int_{\mathbb{R}} f d\mu_1 = \frac{1}{|\alpha|} \cdot \int_{\mathbb{R}} f(\alpha x) d\mu_1(x) \quad \text{and} \quad \int_{\mathbb{R}} f d\mu_1 = \int_{\mathbb{R}} f(x + \beta) d\mu_1(x). \quad (24)$$

]

Solution. A remark on the first formula as printed. The factor $1/|\alpha|$ stands on the wrong side of the first identity: taking $f = \chi_{[0,1]}$ and $\alpha = 2$ gives $\int_{\mathbb{R}} f d\mu_1 = 1$, while $f(2x) = \chi_{[0,1/2]}(x)$, so $\frac{1}{|\alpha|} \int_{\mathbb{R}} f(\alpha x) d\mu_1(x) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. The identity that is true, and that is used in the proofs of Proposition 21 and Theorem 22, is

$$\int_{\mathbb{R}} f(\alpha x) d\mu_1(x) = \frac{1}{|\alpha|} \int_{\mathbb{R}} f d\mu_1, \quad (24')$$

equivalently $\int_{\mathbb{R}} f d\mu_1 = |\alpha| \int_{\mathbb{R}} f(\alpha x) d\mu_1(x)$. We prove (24') and the (correctly printed) translation formula.

Preliminaries on dilations and translations.

For $\lambda \neq 0$ and $A \subseteq \mathbb{R}$ write $\lambda A = \{\lambda a \mid a \in A\}$, and for $z \in \mathbb{R}$ write $A + z = \{a + z \mid a \in A\}$.

Lemma A. For each $\lambda \neq 0$ and each $A \subseteq \mathbb{R}$, $\mu_1^*(\lambda A) = |\lambda| \mu_1^*(A)$.

Indeed, if $\{I_k\}_{k=1}^\infty$ is a countable collection of bounded intervals covering A , then $\{\lambda I_k\}_{k=1}^\infty$ is a countable collection of bounded intervals covering λA , and $\ell(\lambda I_k) = |\lambda| \ell(I_k)$ for each k . Hence

$$\mu_1^*(\lambda A) \leq \sum_{k=1}^\infty |\lambda| \ell(I_k) = |\lambda| \sum_{k=1}^\infty \ell(I_k).$$

Taking the infimum over all such covers gives $\mu_1^*(\lambda A) \leq |\lambda| \mu_1^*(A)$. Applying this inequality with λ^{-1} in place of λ and λA in place of A yields $\mu_1^*(A) \leq |\lambda|^{-1} \mu_1^*(\lambda A)$, that is, $|\lambda| \mu_1^*(A) \leq \mu_1^*(\lambda A)$. Equality follows.

Lemma B. For each $\lambda \neq 0$, if $E \subseteq \mathbb{R}$ is Lebesgue measurable then so is λE .

Let $A \subseteq \mathbb{R}$ be arbitrary and set $B = \lambda^{-1}A$, so $A = \lambda B$. Since $x \mapsto \lambda x$ is a bijection of $\mathbb{R}$,

$$A \cap \lambda E = \lambda(B \cap E) \quad \text{and} \quad A \cap (\lambda E)^c = \lambda(B \cap E^c).$$

Hence, using Lemma A three times and the measurability of E ,

$$\mu_1^*(A \cap \lambda E) + \mu_1^*(A \cap (\lambda E)^c) = |\lambda| [\mu_1^*(B \cap E) + \mu_1^*(B \cap E^c)] = |\lambda| \mu_1^*(B) = \mu_1^*(\lambda B) = \mu_1^*(A).$$

So λE satisfies the Carathéodory criterion.

For translations, Problem 20 of this section provides the corresponding facts: $\mu_1^*(E + z) = \mu_1^*(E)$ for all $E \subseteq \mathbb{R}$ and $z \in \mathbb{R}$, and E is measurable if and only if $E + z$ is. (The proof is the one just given, with $\ell(I + z) = \ell(I)$ in place of $\ell(\lambda I) = |\lambda| \ell(I)$.)

Lemma C. If f is Lebesgue measurable on $\mathbb{R}$, $\alpha \neq 0$ and $\beta \in \mathbb{R}$, then $x \mapsto f(\alpha x)$ and $x \mapsto f(x + \beta)$ are Lebesgue measurable.

For each $c \in \mathbb{R}$,

$$\{x \mid f(\alpha x) > c\} = \alpha^{-1}\{y \mid f(y) > c\}, \quad \{x \mid f(x + \beta) > c\} = \{y \mid f(y) > c\} - \beta,$$

and both right-hand sets are measurable by Lemma B and Problem 20.

Step 1: characteristic functions of sets of finite measure.

Let E be Lebesgue measurable with $\mu_1(E) < \infty$ and put $f = \chi_E$. For $x \in \mathbb{R}$ we have $f(\alpha x) = 1$ exactly when $\alpha x \in E$, that is, exactly when $x \in \alpha^{-1}E$. Hence

$$f(\alpha \cdot) = \chi_{\alpha^{-1}E}.$$

By Lemmas A and B the set $\alpha^{-1}E$ is measurable with $\mu_1(\alpha^{-1}E) = |\alpha|^{-1} \mu_1(E) < \infty$. Therefore

$$\int_{\mathbb{R}} f(\alpha x) d\mu_1(x) = \mu_1(\alpha^{-1}E) = \frac{1}{|\alpha|} \mu_1(E) = \frac{1}{|\alpha|} \int_{\mathbb{R}} f d\mu_1,$$

which is (24'). Similarly $f(x + \beta) = 1$ exactly when $x \in E - \beta$, so $f(\cdot + \beta) = \chi_{E - \beta}$ and, by the translation invariance of Lebesgue measure,

$$\int_{\mathbb{R}} f(x + \beta) d\mu_1(x) = \mu_1(E - \beta) = \mu_1(E) = \int_{\mathbb{R}} f d\mu_1.$$

Step 2: simple functions vanishing outside a set of finite measure.

Let ψ be simple, measurable, and vanishing outside a set of finite measure. Write ψ in canonical form

$$\psi = \sum_{k=1}^m c_k \chi_{E_k},$$

where $c_1, \dots, c_m$ are the distinct nonzero values of ψ and $E_k = \{x \mid \psi(x) = c_k\}$; the E_k are disjoint and measurable, and each is contained in the set where ψ does not vanish, so $\mu_1(E_k) < \infty$. By the computation in Step 1,

$$\psi(\alpha \cdot) = \sum_{k=1}^m c_k \chi_{\alpha^{-1}E_k}, \quad \psi(\cdot + \beta) = \sum_{k=1}^m c_k \chi_{E_k - \beta},$$

and these are again simple functions vanishing outside sets of finite measure. By the linearity of integration over the linear space of integrable functions and Step 1,

$$\int_{\mathbb{R}} \psi(\alpha x) d\mu_1(x) = \sum_{k=1}^m c_k \mu_1(\alpha^{-1} E_k) = \frac{1}{|\alpha|} \sum_{k=1}^m c_k \mu_1(E_k) = \frac{1}{|\alpha|} \int_{\mathbb{R}} \psi d\mu_1,$$

and

$$\int_{\mathbb{R}} \psi(x + \beta) d\mu_1(x) = \sum_{k=1}^m c_k \mu_1(E_k - \beta) = \sum_{k=1}^m c_k \mu_1(E_k) = \int_{\mathbb{R}} \psi d\mu_1.$$

Step 3: nonnegative integrable functions.

Let $f \geq 0$ be integrable over $\mathbb{R}$ with respect to μ_1 . By the Simple Approximation Theorem there is an increasing sequence $\{\psi_j\}$ of nonnegative simple measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f . For each j we have $0 \leq \psi_j \leq f$, so ψ_j is integrable. Moreover an integrable nonnegative simple function vanishes outside a set of finite measure. Indeed, if ψ_j is identically zero this is trivial; otherwise ψ_j takes finitely many values, so it has a smallest positive value $c > 0$, and then $c \chi_{\{\psi_j \neq 0\}} \leq \psi_j$, whence by the monotonicity of integration

$$\mu_1(\{x \mid \psi_j(x) \neq 0\}) \leq \frac{1}{c} \int_{\mathbb{R}} \psi_j d\mu_1 < \infty.$$

Thus Step 2 applies to each ψ_j .

Fix $\alpha \neq 0$. The sequence $\{\psi_j(\alpha \cdot)\}$ consists of nonnegative measurable functions (Lemma C), is increasing, and converges pointwise on $\mathbb{R}$ to $f(\alpha \cdot)$, because $\psi_j(\alpha x) \rightarrow f(\alpha x)$ for each fixed x . By the Monotone Convergence Theorem applied to $\{\psi_j(\alpha \cdot)\}$ and then to $\{\psi_j\}$, together with Step 2,

$$\int_{\mathbb{R}} f(\alpha x) d\mu_1(x) = \lim_{j \rightarrow \infty} \int_{\mathbb{R}} \psi_j(\alpha x) d\mu_1(x) = \lim_{j \rightarrow \infty} \frac{1}{|\alpha|} \int_{\mathbb{R}} \psi_j d\mu_1 = \frac{1}{|\alpha|} \int_{\mathbb{R}} f d\mu_1.$$

The identical argument with $\{\psi_j(\cdot + \beta)\}$, which increases pointwise to $f(\cdot + \beta)$, gives

$$\int_{\mathbb{R}} f(x + \beta) d\mu_1(x) = \lim_{j \rightarrow \infty} \int_{\mathbb{R}} \psi_j(x + \beta) d\mu_1(x) = \lim_{j \rightarrow \infty} \int_{\mathbb{R}} \psi_j d\mu_1 = \int_{\mathbb{R}} f d\mu_1.$$

Remark: arbitrary nonnegative measurable functions. Both formulas persist, as equalities in $[0, \infty]$, for every nonnegative measurable u on $\mathbb{R}$, integrable or not. Indeed, put

$$u_j = \min\{u, j\} \cdot \chi_{[-j, j]} \quad \text{for } j \in \mathbb{N}.$$

Each u_j is nonnegative, measurable, bounded by j , and vanishes outside $[-j, j]$, hence is integrable; and $\{u_j\}$ increases pointwise to u . Since $u_j(\alpha \cdot)$ increases pointwise to $u(\alpha \cdot)$ and $u_j(\cdot + \beta)$ increases pointwise to $u(\cdot + \beta)$, two applications of the Monotone Convergence Theorem together with Step 3 give

$$\int_{\mathbb{R}} u(\alpha x) d\mu_1(x) = \frac{1}{|\alpha|} \int_{\mathbb{R}} u d\mu_1 \quad \text{and} \quad \int_{\mathbb{R}} u(x + \beta) d\mu_1(x) = \int_{\mathbb{R}} u d\mu_1.$$

Step 4: general integrable functions.

If f is integrable over $\mathbb{R}$, write $f = f^+ - f^-$ with $f^{\pm} \geq 0$ integrable. Then $f(\alpha \cdot)^{\pm} = f^{\pm}(\alpha \cdot)$ and $f(\cdot + \beta)^{\pm} = f^{\pm}(\cdot + \beta)$, so by Step 3 each of $f(\alpha \cdot)$, $f(\cdot + \beta)$ is integrable, and by the linearity of integration

$$\int_{\mathbb{R}} f(\alpha x) d\mu_1(x) = \frac{1}{|\alpha|} \int_{\mathbb{R}} f^+ d\mu_1 - \frac{1}{|\alpha|} \int_{\mathbb{R}} f^- d\mu_1 = \frac{1}{|\alpha|} \int_{\mathbb{R}} f d\mu_1,$$

and likewise $\int_{\mathbb{R}} f(x + \beta) d\mu_1(x) = \int_{\mathbb{R}} f d\mu_1$. This proves (24') and the translation formula in full generality.

Problem 20.32. For a subset E of $\mathbb{R}$, define

$$\sigma(E) = \left\{ (x, y) \in \mathbb{R}^2 \mid x - y \in E \right\}.$$

- (i) If E is a Lebesgue measurable subset of $\mathbb{R}$, show that $\sigma(E)$ is a measurable subset of $\mathbb{R}^2$. (Hint: Consider first the cases when E open, E a G_δ , E of measure zero, and E measurable.)
- (ii) If f is a Lebesgue measurable function on $\mathbb{R}$, show that the function F defined by $F(x, y) = f(x - y)$ is a Lebesgue measurable function on $\mathbb{R}^2$.
- (iii) If f and g belong to $L^1(\mathbb{R}, \mu_1)$, show that for almost all x in $\mathbb{R}$, the function φ given by $\varphi(y) = f(x - y)g(y)$ belongs to $L^1(\mathbb{R}, \mu_1)$. If we denote its integral by $h(x)$, show that h is integrable and

$$\int_{\mathbb{R}} |h| d\mu_1 \leq \int_{\mathbb{R}} |f| d\mu_1 \cdot \int_{\mathbb{R}} |g| d\mu_1.$$

Solution. Throughout, let $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the mapping $T(x, y) = x - y$, so that $\sigma(E) = T^{-1}(E)$ for every $E \subseteq \mathbb{R}$. Since σ is a preimage operation it commutes with all set-theoretic operations:

$$\sigma\left(\bigcap_k E_k\right) = \bigcap_k \sigma(E_k), \quad \sigma\left(\bigcup_k E_k\right) = \bigcup_k \sigma(E_k), \quad \sigma(A \sim B) = \sigma(A) \sim \sigma(B).$$

It will also be convenient to introduce the linear operator

$$S: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad S(x, y) = (x - y, y),$$

which is invertible with $S^{-1}(u, v) = (u + v, v)$ and $\det S = \det S^{-1} = 1$. For any $E \subseteq \mathbb{R}$,

$$S^{-1}(E \times \mathbb{R}) = \{(u + v, v) \mid u \in E, v \in \mathbb{R}\} = \{(x, y) \mid x - y \in E\} = \sigma(E), \quad (*)$$

since $(x, y) = (u + v, v)$ with $u \in E$ precisely when $y = v$ and $x - y = u \in E$.

Part (i).

Case 1: E open. The mapping T is continuous, so $\sigma(E) = T^{-1}(E)$ is an open subset of $\mathbb{R}^2$. Every open set is Borel and every Borel set is Lebesgue measurable, so $\sigma(E)$ is μ_2 -measurable.

Case 2: E a G_δ . Write $E = \bigcap_{k=1}^{\infty} O_k$ with each O_k open. Then $\sigma(E) = \bigcap_{k=1}^{\infty} \sigma(O_k)$ is a countable intersection of open subsets of $\mathbb{R}^2$, hence a G_δ subset of $\mathbb{R}^2$, hence Borel and so μ_2 -measurable.

Case 3: $\mu_1^*(E) = 0$. We first show $\mu_2^*(E \times \mathbb{R}) = 0$. Fix a natural number k and let $\varepsilon > 0$. Choose a countable collection $\{I_j\}$ of bounded intervals of real numbers covering E with $\sum_j \ell(I_j) < \varepsilon$. Then $\{I_j \times [-k, k]\}$ is a countable collection of intervals in $\mathbb{R}^2$ covering $E \times [-k, k]$, of total volume

$$\sum_j \ell(I_j) \cdot 2k < 2k\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, $\mu_2^*(E \times [-k, k]) = 0$; and since $E \times \mathbb{R} = \bigcup_{k=1}^{\infty} E \times [-k, k]$, countable subadditivity gives $\mu_2^*(E \times \mathbb{R}) = 0$. In particular $E \times \mathbb{R}$ is μ_2 -measurable with $\mu_2(E \times \mathbb{R}) = 0$.

Now S^{-1} is linear, hence Lipschitz by Proposition 17. The proof of Proposition 18 shows that a Lipschitz mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ carries a set of Lebesgue outer measure zero onto a set of Lebesgue outer measure zero. Therefore, by (*),

$$\mu_2^*(\sigma(E)) = \mu_2^*(S^{-1}(E \times \mathbb{R})) = 0,$$

so $\sigma(E)$ is μ_2 -measurable with $\mu_2(\sigma(E)) = 0$. (Alternatively, apply Corollary 23 to the invertible operator S^{-1} : $\mu_2(S^{-1}(E \times \mathbb{R})) = |\det S^{-1}| \mu_2(E \times \mathbb{R}) = 0$.)

Case 4: E Lebesgue measurable. By Corollary 14 there is a G_δ subset G of $\mathbb{R}$ with

$$E \subseteq G \quad \text{and} \quad \mu_1^*(G \sim E) = 0.$$

Since $E \subseteq G$ we have $E = G \sim (G \sim E)$, and therefore

$$\sigma(E) = \sigma(G) \sim \sigma(G \sim E).$$

By Case 2, $\sigma(G)$ is μ_2 -measurable; by Case 3, $\sigma(G \sim E)$ is μ_2 -measurable (indeed of measure zero). The μ_2 -measurable sets form a σ -algebra, so the difference $\sigma(E)$ is μ_2 -measurable. This proves (i). Part (ii).

Let f be Lebesgue measurable on $\mathbb{R}$ and $F(x, y) = f(x - y)$, that is, $F = f \circ T$. For each $c \in \mathbb{R}$,

$$\{(x, y) \in \mathbb{R}^2 \mid F(x, y) > c\} = T^{-1}(\{u \in \mathbb{R} \mid f(u) > c\}) = \sigma(\{f > c\}).$$

The set $\{f > c\}$ is Lebesgue measurable, so by part (i) the set $\sigma(\{f > c\})$ is μ_2 -measurable. Since this holds for every c , F is Lebesgue measurable on $\mathbb{R}^2$.

Part (iii).

Let $f, g \in L^1(\mathbb{R}, \mu_1)$ and define

$$\Phi(x, y) = f(x - y) g(y), \quad (x, y) \in \mathbb{R}^2.$$

The factor $(x, y) \mapsto f(x - y)$ is μ_2 -measurable by part (ii). The factor $(x, y) \mapsto g(y)$ is also μ_2 -measurable: for each c , $\{(x, y) \mid g(y) > c\} = \mathbb{R} \times \{g > c\}$, and for any Lebesgue measurable $A \subseteq \mathbb{R}$ the set $\mathbb{R} \times A$ is μ_2 -measurable; indeed, by Corollary 14 write $A = F_0 \cup D$ with F_0 an F_σ and $\mu_1^*(D) = 0$; then $\mathbb{R} \times F_0$ is an F_σ subset of $\mathbb{R}^2$, hence Borel, while $\mu_2^*(\mathbb{R} \times D) = 0$ by the covering argument of Case 3 above (with the roles of the coordinates interchanged). A product of μ_2 -measurable functions is μ_2 -measurable, so Φ , and hence $|\Phi|$, is μ_2 -measurable.

Since $|\Phi| \geq 0$ is μ_2 -measurable, Tonelli's Theorem, as formulated for Lebesgue measure in Theorem 16, gives

$$\int_{\mathbb{R}^2} |\Phi| d\mu_2 = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x - y)| |g(y)| d\mu_1(x) \right] d\mu_1(y).$$

For each fixed y , the second formula of (24) (translation invariance, with $\beta = -y$) gives

$$\int_{\mathbb{R}} |f(x - y)| d\mu_1(x) = \int_{\mathbb{R}} |f| d\mu_1.$$

Therefore

$$\int_{\mathbb{R}^2} |\Phi| d\mu_2 = \int_{\mathbb{R}} |g(y)| \left[\int_{\mathbb{R}} |f| d\mu_1 \right] d\mu_1(y) = \int_{\mathbb{R}} |f| d\mu_1 \cdot \int_{\mathbb{R}} |g| d\mu_1 < \infty. \quad (\dagger)$$

Thus Φ is integrable over $\mathbb{R}^2$ with respect to μ_2 .

Now apply Fubini's Theorem, as formulated in Theorem 16, to the integrable function Φ : for almost all $x \in \mathbb{R}$ the function

$$\varphi(y) = \Phi(x, y) = f(x - y)g(y)$$

is integrable over $\mathbb{R}$ with respect to μ_1 , that is, $\varphi \in L^1(\mathbb{R}, \mu_1)$; moreover the function

$$h(x) = \int_{\mathbb{R}} f(x - y)g(y) d\mu_1(y),$$

which is defined for almost all x (set $h = 0$ on the exceptional set of measure zero), is integrable over $\mathbb{R}$ and

$$\int_{\mathbb{R}} h d\mu_1 = \int_{\mathbb{R}^2} \Phi d\mu_2.$$

Finally, for almost every x the monotonicity of the integral gives

$$|h(x)| = \left| \int_{\mathbb{R}} \Phi(x, y) d\mu_1(y) \right| \leq \int_{\mathbb{R}} |\Phi(x, y)| d\mu_1(y),$$

so, integrating over $\mathbb{R}$ and using Tonelli's Theorem once more together with $(\dagger)$,

$$\int_{\mathbb{R}} |h| d\mu_1 \leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\Phi(x, y)| d\mu_1(y) \right] d\mu_1(x) = \int_{\mathbb{R}^2} |\Phi| d\mu_2 = \int_{\mathbb{R}} |f| d\mu_1 \cdot \int_{\mathbb{R}} |g| d\mu_1.$$

This is the required inequality.

Problem 20.33. Let f and g be functions in $L^1(\mathbb{R}, \mu_1)$, and define $f * g$ on $\mathbb{R}$ by

$$(f * g)(y) = \int_{\mathbb{R}} f(y - x)g(x) d\mu_1(x).$$

- (i) Show that $f * g = g * f$.
- (ii) Show that $(f * g) * h = f * (g * h)$ for each $h \in L^1(\mathbb{R}, \mu_1)$.

Solution. By the preceding problem, for $f, g \in L^1(\mathbb{R}, \mu_1)$ the integral defining $(f * g)(y)$ exists for almost all $y \in \mathbb{R}$, and $f * g$, defined to be 0 on the exceptional set of measure zero, belongs to $L^1(\mathbb{R}, \mu_1)$ with

$$\int_{\mathbb{R}} |f * g| d\mu_1 \leq \int_{\mathbb{R}} |f| d\mu_1 \cdot \int_{\mathbb{R}} |g| d\mu_1.$$

All the identities below are therefore identities between elements of $L^1(\mathbb{R}, \mu_1)$, that is, they hold almost everywhere on $\mathbb{R}$.

A reflection lemma. For each $\beta \in \mathbb{R}$ and each nonnegative Lebesgue measurable function u on $\mathbb{R}$,

$$\int_{\mathbb{R}} u d\mu_1 = \int_{\mathbb{R}} u(\beta - x) d\mu_1(x) \quad (*)$$

as an equality in $[0, \infty]$; consequently a measurable u is integrable if and only if $x \mapsto u(\beta - x)$ is, and then $(*)$ holds for u as well.

To see this, recall from Problem 31 that for nonnegative measurable u and for $\alpha \neq 0$, $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}} u(\alpha x) d\mu_1(x) = \frac{1}{|\alpha|} \int_{\mathbb{R}} u d\mu_1 \quad \text{and} \quad \int_{\mathbb{R}} u(x + \gamma) d\mu_1(x) = \int_{\mathbb{R}} u d\mu_1.$$

Take $\alpha = -1$ in the first to get $\int_{\mathbb{R}} u(-x) d\mu_1(x) = \int_{\mathbb{R}} u d\mu_1$. Now put $v(x) = u(-x)$ and apply the second to v with $\gamma = -\beta$:

$$\int_{\mathbb{R}} u d\mu_1 = \int_{\mathbb{R}} v d\mu_1 = \int_{\mathbb{R}} v(x - \beta) d\mu_1(x) = \int_{\mathbb{R}} u(\beta - x) d\mu_1(x).$$

Applying $(*)$ to $|u|$ gives the assertion about integrability, and then $(*)$ for integrable u follows by splitting $u = u^+ - u^-$.

Part (i).

Fix $y \in \mathbb{R}$ and set $u(x) = f(y - x)g(x)$, a Lebesgue measurable function of x . By $(*)$ with $\beta = y$,

$$u(y - x) = f(y - (y - x))g(y - x) = f(x)g(y - x),$$

so u is integrable over $\mathbb{R}$ if and only if $x \mapsto f(x)g(y - x)$ is, and in that case

$$(f * g)(y) = \int_{\mathbb{R}} u d\mu_1 = \int_{\mathbb{R}} u(y - x) d\mu_1(x) = \int_{\mathbb{R}} g(y - x)f(x) d\mu_1(x) = (g * f)(y).$$

Thus $(f * g)(y)$ is defined exactly when $(g * f)(y)$ is, and the two agree there. Hence $f * g = g * f$ almost everywhere on $\mathbb{R}$, that is, as elements of $L^1(\mathbb{R}, \mu_1)$.

Part (ii).

Let $f, g, h \in L^1(\mathbb{R}, \mu_1)$. Fix $z \in \mathbb{R}$ and define

$$\Psi_z(x, y) = f(z - x - y)g(y)h(x), \quad (x, y) \in \mathbb{R}^2.$$

Measurability of Ψ_z . As in Problem 32(iii), $(x, y) \mapsto g(y)$ and $(x, y) \mapsto h(x)$ are μ_2 -measurable. For the remaining factor, let $f_z(w) = f(z - w)$; this is Lebesgue measurable on $\mathbb{R}$, being the composition of f with the invertible affine map $w \mapsto z - w$ (measurability of such compositions was established in Lemma C of Problem 31). Then $f(z - x - y) = f_z(x + y)$. Let $F(x, y) = f_z(x - y)$, which is μ_2 -measurable by Problem 32(ii), and let $A(x, y) = (x, -y)$, an invertible linear operator on $\mathbb{R}^2$. Since

$F(A(x, y)) = f_z(x + y)$, Corollary 19 shows $(x, y) \mapsto f(z - x - y)$ is μ_2 -measurable. Hence Ψ_z is μ_2 -measurable.

Ψ_z is integrable for almost all z . Since Ψ_z is μ_2 -measurable, Tonelli's Theorem, as formulated in Theorem 16, gives

$$\int_{\mathbb{R}^2} |\Psi_z| d\mu_2 = \int_{\mathbb{R}} |h(x)| \left[\int_{\mathbb{R}} |f(z - x - y)| |g(y)| d\mu_1(y) \right] d\mu_1(x) = \int_{\mathbb{R}} (|f| * |g|)(z - x) |h(x)| d\mu_1(x),$$

that is, $\int_{\mathbb{R}^2} |\Psi_z| d\mu_2 = ((|f| * |g|) * |h|)(z)$, an identity in $[0, \infty]$. By Problem 32 applied to the nonnegative functions $|f|, |g| \in L^1$, the function $|f| * |g|$ belongs to $L^1(\mathbb{R}, \mu_1)$; applying Problem 32 once more to $|f| * |g|$ and $|h|$, the function $(|f| * |g|) * |h|$ belongs to $L^1(\mathbb{R}, \mu_1)$ and is therefore finite for almost all z . Fix such a z ; then Ψ_z is integrable over $\mathbb{R}^2$ with respect to μ_2 .

Fubini's Theorem. For such a z , Fubini's Theorem (Theorem 16) asserts that both iterated integrals of Ψ_z exist and are equal:

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \Psi_z(x, y) d\mu_1(y) \right] d\mu_1(x) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \Psi_z(x, y) d\mu_1(x) \right] d\mu_1(y).$$

Consider the left-hand side. For almost all x , the inner integral exists and equals

$$h(x) \int_{\mathbb{R}} f((z - x) - y) g(y) d\mu_1(y) = h(x) (f * g)(z - x),$$

this last equality holding for almost all x because the set N of points at which $f * g$ is undefined has measure zero, and hence so does $\{x \mid z - x \in N\}$, by the reflection lemma (*) applied to χ_N . Therefore the left-hand side equals

$$\int_{\mathbb{R}} (f * g)(z - x) h(x) d\mu_1(x) = ((f * g) * h)(z).$$

Symmetrically, for almost all y the inner integral on the right equals

$$g(y) \int_{\mathbb{R}} f((z - y) - x) h(x) d\mu_1(x) = g(y) (f * h)(z - y),$$

so the right-hand side equals $((f * h) * g)(z)$. We conclude that for almost all $z \in \mathbb{R}$,

$$(f * g) * h = (f * h) * g. \quad (\dagger)$$

Conclusion. Apply $(\dagger)$ with f, g, h replaced by g, h, f respectively:

$$(g * h) * f = (g * f) * h.$$

By part (i), $g * f = f * g$ and $(g * h) * f = f * (g * h)$. Substituting these into the last identity gives

$$f * (g * h) = (f * g) * h$$

almost everywhere on $\mathbb{R}$, that is, as elements of $L^1(\mathbb{R}, \mu_1)$. This is the asserted associativity.

Problem 20.34. Let f be a nonnegative function that is integrable over $\mathbb{R}$ with respect to μ_1 . Show that

$$\mu_2 \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq f(x) \right\} = \mu_2 \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < y < f(x) \right\} = \int_{\mathbb{R}} f(x) dx.$$

For each $t \geq 0$, define $\varphi(t) = \mu_1 \{x \in \mathbb{R} \mid f(x) \geq t\}$. Show that φ is a decreasing function and

$$\int_0^\infty \varphi(t) d\mu_1(t) = \int_{\mathbb{R}} f(x) d\mu_1(x).$$

Solution. Since f is nonnegative and integrable over $\mathbb{R}$, it is finite almost everywhere; let $N = \{x \mid f(x) = \infty\}$, so $\mu_1(N) = 0$. Redefining f to be 0 on N changes no integral of f , changes each set $\{x \mid f(x) \geq t\}$ only within the null set N and hence changes none of the numbers $\varphi(t)$, and alters each of the plane sets considered below only within $N \times \mathbb{R}$, which has μ_2 -measure zero by the covering argument used in Problem 32. We may therefore assume throughout that f is real-valued.

A measurability observation. The function $\tilde{f}(x, y) = f(x)$ is μ_2 -measurable on $\mathbb{R}^2$: for each $c \in \mathbb{R}$,

$$\{(x, y) \mid \tilde{f}(x, y) > c\} = \{f > c\} \times \mathbb{R},$$

and a set of the form $A \times \mathbb{R}$ with $A \subseteq \mathbb{R}$ Lebesgue measurable is μ_2 -measurable. (This is the argument given in Problem 32(iii), with the roles of the two coordinates interchanged: by Corollary 14 write $A = F_0 \cup D$ with F_0 an F_σ subset of $\mathbb{R}$ and $\mu_1^*(D) = 0$; then $F_0 \times \mathbb{R}$ is an F_σ subset of $\mathbb{R}^2$, hence Borel, while covering D by intervals of total length less than ε and multiplying by $[-k, k]$ shows $\mu_2^*(D \times \mathbb{R}) = 0$.) The function $\pi(x, y) = y$ is continuous, hence Borel measurable, hence μ_2 -measurable. Since sums and differences of measurable functions are measurable, $(x, y) \mapsto f(x) - y$ is μ_2 -measurable. Part 1: the two regions under the graph.

Put

$$E = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq f(x)\}, \quad E_0 = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < f(x)\}.$$

Then

$$E = \{(x, y) \mid y \geq 0\} \cap \{(x, y) \mid f(x) - y \geq 0\}, \quad E_0 = \{(x, y) \mid y > 0\} \cap \{(x, y) \mid f(x) - y > 0\},$$

and each of the four sets on the right is μ_2 -measurable by the observation above. Hence E and E_0 are μ_2 -measurable.

The function χ_E is nonnegative and μ_2 -measurable, so Tonelli's Theorem, as formulated for Lebesgue measure in Theorem 16, applies:

$$\mu_2(E) = \int_{\mathbb{R}^2} \chi_E d\mu_2 = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \chi_E(x, y) d\mu_1(y) \right] d\mu_1(x).$$

Fix $x \in \mathbb{R}$. Since $f(x) \geq 0$,

$$\{y \in \mathbb{R} \mid \chi_E(x, y) = 1\} = \{y \mid 0 \leq y \leq f(x)\} = [0, f(x)],$$

an interval of length $f(x)$, so the inner integral equals $\mu_1([0, f(x)]) = f(x)$. Therefore

$$\mu_2(E) = \int_{\mathbb{R}} f(x) d\mu_1(x).$$

Exactly the same computation with E_0 gives, for fixed x , the slice

$$\{y \mid 0 < y < f(x)\} = (0, f(x)),$$

which is empty when $f(x) = 0$ and otherwise an interval of length $f(x)$; in either case its measure is $f(x)$. Hence

$$\mu_2(E_0) = \int_{\mathbb{R}} f(x) d\mu_1(x)$$

as well, and the two stated equalities follow. (In particular, since $E_0 \subseteq E$ and $\mu_2(E) = \int_{\mathbb{R}} f d\mu_1 < \infty$, the excision property of measure gives $\mu_2(E \setminus E_0) = \mu_2(E) - \mu_2(E_0) = 0$; that is, the union of the line $\{y = 0\}$ with the graph of f meets the region under the graph in a set of μ_2 -measure zero.)

Part 2: φ is decreasing.

Let $0 \leq t_1 \leq t_2$. If $f(x) \geq t_2$ then $f(x) \geq t_1$, so

$$\{x \mid f(x) \geq t_2\} \subseteq \{x \mid f(x) \geq t_1\}.$$

Both sets are Lebesgue measurable since f is measurable, so by the monotonicity of measure $\varphi(t_2) \leq \varphi(t_1)$. Thus φ is decreasing on $[0, \infty)$.

Note that $\varphi(t) < \infty$ for $t > 0$: by Chebychev's Inequality,

$$\varphi(t) = \mu_1\{x \mid f(x) \geq t\} \leq \frac{1}{t} \int_{\mathbb{R}} f \, d\mu_1 < \infty.$$

Also, being monotone, φ is Lebesgue measurable on $(0, \infty)$: for each $c \in \mathbb{R}$ the set $\{t > 0 \mid \varphi(t) > c\}$ is an interval with left endpoint 0, hence measurable. (Its measurability will in any case be delivered by Tonelli's Theorem below.)

Part 3: the layer-cake formula.

Consider

$$A = \{(x, t) \in \mathbb{R}^2 \mid t > 0 \text{ and } f(x) \geq t\} = \{(x, t) \mid t > 0\} \cap \{(x, t) \mid f(x) - t \geq 0\},$$

which is μ_2 -measurable by the measurability observation at the start (with the second coordinate now named t). Apply Tonelli's Theorem to the nonnegative μ_2 -measurable function χ_A , in each of the two orders.

Integrating first with respect to x : for fixed t ,

$$\int_{\mathbb{R}} \chi_A(x, t) \, d\mu_1(x) = \begin{cases} \mu_1\{x \mid f(x) \geq t\} = \varphi(t), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Hence

$$\mu_2(A) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \chi_A(x, t) \, d\mu_1(x) \right] d\mu_1(t) = \int_{(0, \infty)} \varphi(t) \, d\mu_1(t) = \int_0^\infty \varphi(t) \, d\mu_1(t).$$

Integrating first with respect to t : for fixed x ,

$$\{t \in \mathbb{R} \mid \chi_A(x, t) = 1\} = \{t \mid 0 < t \leq f(x)\} = (0, f(x)],$$

which is empty if $f(x) = 0$ and otherwise an interval of length $f(x)$; in either case its measure is $f(x)$.

Hence

$$\mu_2(A) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \chi_A(x, t) \, d\mu_1(t) \right] d\mu_1(x) = \int_{\mathbb{R}} f(x) \, d\mu_1(x).$$

Equating the two evaluations of $\mu_2(A)$ gives

$$\int_0^\infty \varphi(t) \, d\mu_1(t) = \int_{\mathbb{R}} f(x) \, d\mu_1(x),$$

as required. In particular φ is integrable over $(0, \infty)$, since the right-hand side is finite.

Problem 20.35. Prove Proposition 26.

[Proposition 26. Let μ be a Borel measure on $\mathcal{B}(I)$ and g_μ its cumulative distribution function. Then the measure μ is absolutely continuous with respect to Lebesgue measure if and only if the function g_μ is absolutely continuous.]

Solution. Here $I = [a, b]$ is a closed, bounded interval, $\mathcal{B}(I)$ is the σ -algebra of Borel subsets of I , μ is a finite measure on $\mathcal{B}(I)$, and

$$g_\mu(x) = \mu[a, x] \quad \text{for all } x \in I.$$

We write m for Lebesgue measure, restricted to $\mathcal{B}(I)$, and abbreviate $g = g_\mu$. Recall the two notions involved.

The measure μ is absolutely continuous with respect to m provided $m(E) = 0$ implies $\mu(E) = 0$ for $E \in \mathcal{B}(I)$. Since μ is finite, Proposition 19 of Chapter 18 tells us this is equivalent to the following ε - δ criterion: for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $E \in \mathcal{B}(I)$ and $m(E) < \delta$, then $\mu(E) < \varepsilon$.

The function g is absolutely continuous on $[a, b]$ provided for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every finite disjoint collection $\{(c_k, d_k)\}_{k=1}^n$ of open intervals in (a, b) with $\sum_{k=1}^n (d_k - c_k) < \delta$ we have

$$\sum_{k=1}^n |g(d_k) - g(c_k)| < \varepsilon.$$

Two elementary identities will be used repeatedly. Since μ is finite and $[a, c] \subseteq [a, d]$ for $c \leq d$ in I , additivity gives

$$g(d) - g(c) = \mu[a, d] - \mu[a, c] = \mu((c, d]) \quad \text{for } a \leq c \leq d \leq b, \quad (*)$$

and in particular g is increasing. Also $g(a) = \mu\{a\}$.

Part 1: if μ is absolutely continuous with respect to m , then g is absolutely continuous.

Let $\varepsilon > 0$. By Proposition 19 of Chapter 18 there is a $\delta > 0$ such that

$$E \in \mathcal{B}(I), \quad m(E) < \delta \implies \mu(E) < \varepsilon.$$

Let $\{(c_k, d_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a, b) with $\sum_{k=1}^n (d_k - c_k) < \delta$. The half-open intervals $(c_k, d_k]$ are also disjoint: if $k \neq j$ and, say, $c_k < c_j$, then disjointness of (c_k, d_k) and (c_j, d_j) forces $d_k \leq c_j$, whence $(c_k, d_k] \cap (c_j, d_j] = \emptyset$. Put

$$E = \bigcup_{k=1}^n (c_k, d_k] \in \mathcal{B}(I).$$

Then $m(E) = \sum_{k=1}^n (d_k - c_k) < \delta$, so $\mu(E) < \varepsilon$. By $(*)$ and the additivity of μ ,

$$\sum_{k=1}^n |g(d_k) - g(c_k)| = \sum_{k=1}^n [g(d_k) - g(c_k)] = \sum_{k=1}^n \mu((c_k, d_k]) = \mu(E) < \varepsilon,$$

the first equality because g is increasing. Hence g is absolutely continuous on $[a, b]$.

Part 2: a necessary correction to the converse.

The converse as printed is not correct without one further hypothesis, because the value $g(a) = \mu\{a\}$ is invisible to the increments that define absolute continuity of g . For an explicit counterexample take $I = [0, 1]$ and

$$\mu(E) = \delta_0(E) + m(E) \quad \text{for } E \in \mathcal{B}([0, 1]),$$

where δ_0 is the Dirac measure concentrated at 0. This μ is a finite Borel measure, and

$$g_\mu(x) = \mu[0, x] = 1 + x \quad \text{for } x \in [0, 1],$$

which is Lipschitz and hence absolutely continuous. Yet $m(\{0\}) = 0$ while $\mu(\{0\}) = 1$, so μ is not absolutely continuous with respect to m .

The correct converse, which we now prove, is: if g is absolutely continuous and $g(a) = 0$ (equivalently, $\mu\{a\} = 0$), then μ is absolutely continuous with respect to m . Note that $\mu\{a\} = 0$ is certainly necessary for $\mu \ll m$, since $m\{a\} = 0$; so with this hypothesis added the two assertions of Proposition 26 are genuinely equivalent.

Part 3: if g is absolutely continuous and $g(a) = 0$, then μ is absolutely continuous with respect to m . First observe that μ has no atoms. Indeed, g is absolutely continuous and therefore continuous on $[a, b]$. For $c \in (a, b)$ and $x \in [a, c)$ we have $\{c\} \subseteq (x, c]$, so by $(*)$ and monotonicity

$$0 \leq \mu\{c\} \leq \mu((x, c]) = g(c) - g(x) \xrightarrow{x \rightarrow c^-} 0,$$

so $\mu\{c\} = 0$; and $\mu\{a\} = g(a) = 0$ by hypothesis.

Now let $E \in \mathcal{B}(I)$ with $m(E) = 0$, and let $\varepsilon > 0$. Choose $\delta > 0$ as in the definition of absolute continuity of g for this ε . Since $m(E) = 0 < \delta$, by the outer regularity of Lebesgue measure there is an open subset O of $\mathbb{R}$ with $E \subseteq O$ and $m(O) < \delta$. Put

$$U = O \cap (a, b),$$

an open subset of (a, b) with $m(U) < \delta$, and note that $E \subseteq U \cup \{a, b\}$.

Every open set of real numbers is the union of a countable disjoint collection of open intervals; write

$$U = \bigcup_{k=1}^{\infty} (c_k, d_k), \quad \text{the intervals disjoint and contained in } (a, b),$$

so that $\sum_{k=1}^{\infty} (d_k - c_k) = m(U) < \delta$. For each natural number n , the finite subcollection $\{(c_k, d_k)\}_{k=1}^n$ is disjoint with total length less than δ , so the absolute continuity of g gives

$$\sum_{k=1}^n [g(d_k) - g(c_k)] < \varepsilon.$$

Letting $n \rightarrow \infty$, since all terms are nonnegative,

$$\sum_{k=1}^{\infty} [g(d_k) - g(c_k)] \leq \varepsilon.$$

By (*), $g(d_k) - g(c_k) = \mu((c_k, d_k]) \geq \mu((c_k, d_k))$, so by the countable additivity of μ over the disjoint collection $\{(c_k, d_k)\}$,

$$\mu(U) = \sum_{k=1}^{\infty} \mu((c_k, d_k)) \leq \sum_{k=1}^{\infty} [g(d_k) - g(c_k)] \leq \varepsilon.$$

Since $\mu\{a\} = \mu\{b\} = 0$ and $E \subseteq U \cup \{a, b\}$, monotonicity and subadditivity give

$$\mu(E) \leq \mu(U) + \mu\{a\} + \mu\{b\} \leq \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, $\mu(E) = 0$. Therefore μ is absolutely continuous with respect to Lebesgue measure.

(The argument in Part 3 in fact proves the ε - δ form directly: if $E \in \mathcal{B}(I)$ has $m(E) < \delta/2$, choose the open $O \supseteq E$ with $m(O) < \delta$ and run the same estimate to get $\mu(E) \leq \varepsilon$.)

20.6 Problems 20.36–20.42

Problem 20.36. Suppose f is a bounded Borel measurable function on $[a, b]$ and g is increasing and absolutely continuous on $[a, b]$. Prove that if m denotes Lebesgue measure, then

$$\int_{[a, b]} f dg = \int_{[a, b]} fg' dm.$$

Solution. A word first about normalization, because the identity as printed is not literally true for every absolutely continuous increasing g . In Proposition 25 the Borel measure attached to an increasing, right-continuous g on $I = [a, b]$ is built from the set function on the semiring $\mathcal{S}$ that assigns $\mu\{a\} = g(a)$ and $\mu(c, d] = g(d) - g(c)$, so that its cumulative distribution function is $\mu_g[a, x] = g(x)$. Taking $f \equiv 1$ in the asserted identity, that convention makes the left side $\mu_g([a, b]) = g(b)$ and the right side $\int_{[a, b]} g' dm = g(b) - g(a)$, by Theorem 10 of Chapter 6. So with the atom at a retained the two sides differ by $f(a)g(a)$. What we prove is the sharp statement

$$\int_{[a, b]} f d\mu_g = f(a)g(a) + \int_{[a, b]} fg' dm,$$

of which the identity in the problem, and the text's formula (31), are the case $g(a) = 0$; equivalently, they are the statements about the Lebesgue-Stieltjes measure of g normalized to carry no mass at the left endpoint, namely the Proposition 25 measure $\mu_{g-g(a)}$ of the increasing right-continuous function $g - g(a)$, which satisfies

$$\mu_{g-g(a)}(c, d] = g(d) - g(c) \text{ for } (c, d] \subseteq [a, b], \quad \mu_{g-g(a)}\{a\} = 0.$$

That normalization is the one the section really intends: Proposition 26 asserts that a Borel measure is absolutely continuous with respect to m exactly when its cumulative distribution function is, and a measure carrying an atom at a is never absolutely continuous with respect to m , however smooth g may be. Moreover

$$\mu_g = g(a)\delta_a + \mu_{g-g(a)},$$

where δ_a is the Dirac measure at a : the two sides are Borel measures agreeing on the semiring $\mathcal{S}$ of Proposition 25, hence, both being finite and so σ -finite, they agree on $\mathcal{B}([a, b]) = \sigma(\mathcal{S})$ by the uniqueness assertion of the Caratheodory-Hahn Theorem. So the sharp statement above will follow once we prove

$$\int_{[a, b]} f d\mu_{g-g(a)} = \int_{[a, b]} f g' dm,$$

and it is this that we now prove, by exhibiting $\mu_{g-g(a)}$ as the measure with density g' .

Step 1: the derivative g' is an integrable density. Since g is absolutely continuous on $[a, b]$, Theorem 10 of Chapter 6 tells us that g is differentiable almost everywhere on (a, b) , that g' is Lebesgue integrable over $[a, b]$, and that

$$\int_{[a, x]} g' dm = g(x) - g(a) \quad \text{for all } x \in [a, b].$$

Moreover $g' \geq 0$ almost everywhere, since g is increasing and hence all of its difference quotients are nonnegative.

Step 2: the measure with density g' . Define, for $E \in \mathcal{B}([a, b])$,

$$\nu(E) = \int_E g' dm.$$

Then $\nu(\emptyset) = 0$ and $\nu(E) \geq 0$ for every E , because $g' \geq 0$ a.e. If $\{E_k\}_{k=1}^\infty$ is a disjoint collection of Borel subsets of $[a, b]$ with union E , then the partial sums of $\sum_k g' \chi_{E_k}$ are nonnegative and increase pointwise a.e. to $g' \chi_E$, so the Monotone Convergence Theorem gives

$$\nu(E) = \int_{[a, b]} g' \chi_E dm = \sum_{k=1}^\infty \int_{[a, b]} g' \chi_{E_k} dm = \sum_{k=1}^\infty \nu(E_k).$$

Thus ν is a measure on $\mathcal{B}([a, b])$, and it is finite because $\nu([a, b]) = \int_{[a, b]} g' dm = g(b) - g(a) < \infty$. So ν is a Borel measure in the sense of this section.

Step 3: ν is the normalized Lebesgue-Stieltjes measure of g . By Step 1 the cumulative distribution function of ν is

$$g_\nu(x) = \nu([a, x]) = \int_{[a, x]} g' dm = g(x) - g(a) \quad \text{for all } x \in [a, b].$$

The function $g - g(a)$ is increasing and, being absolutely continuous, is continuous and in particular continuous on the right; so Proposition 25 applies to it and asserts that there is exactly one Borel measure on $\mathcal{B}([a, b])$ whose cumulative distribution function is $g - g(a)$. Consequently

$$\nu = \mu_{g-g(a)},$$

and unwinding the construction in Proposition 25, $\nu(c, d] = g(d) - g(c)$ for $(c, d] \subseteq [a, b]$ while $\nu\{a\} = g(a) - g(a) = 0$. Thus ν is precisely the normalized measure singled out above. Equivalently, g' is the Radon-Nikodym derivative of ν with respect to m , which is the assertion of Problem 44.

Step 4: from sets to bounded Borel functions. It remains to prove

$$\int_{[a, b]} f d\nu = \int_{[a, b]} f g' dm$$

for every bounded Borel measurable f on $[a, b]$; call this identity $(*)$.

If $f = \chi_E$ with $E \in \mathcal{B}([a, b])$, then $(*)$ reads $\nu(E) = \int_E g' dm$, which is the definition of ν . By the linearity of integration with respect to ν and with respect to m , $(*)$ therefore holds for every Borel simple function on $[a, b]$.

Now let f be bounded and Borel measurable, say $|f| \leq M$ on $[a, b]$. By the Simple Approximation Lemma there is, for each natural number n , a Borel simple function φ_n on $[a, b]$ with $|\varphi_n - f| < 1/n$ on all of $[a, b]$; truncating φ_n at the levels $\pm M$ leaves it simple and only decreases the error, so we may also assume $|\varphi_n| \leq M$. Then $\varphi_n \rightarrow f$ pointwise on $[a, b]$.

On the left-hand side: ν is a finite measure, so the constant function M is ν -integrable, and $|\varphi_n| \leq M$ for all n ; the Lebesgue Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{[a,b]} \varphi_n d\nu = \int_{[a,b]} f d\nu.$$

On the right-hand side: $|\varphi_n g'| \leq M g'$ almost everywhere, and $M g'$ is integrable over $[a, b]$ with respect to m by Step 1, while $\varphi_n g' \rightarrow f g'$ pointwise almost everywhere; the Lebesgue Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{[a,b]} \varphi_n g' dm = \int_{[a,b]} f g' dm.$$

Since $\int_{[a,b]} \varphi_n d\nu = \int_{[a,b]} \varphi_n g' dm$ for every n , the two limits coincide, which is (*). Combining Steps 3 and 4,

$$\int_{[a,b]} f dg = \int_{[a,b]} f d\nu = \int_{[a,b]} f g' dm,$$

which is the asserted identity for the Lebesgue-Stieltjes integral taken with respect to the normalized measure ν . Both sides are finite: $f g'$ is measurable and $|f g'| \leq M g'$ with g' integrable. If instead one keeps the atom prescribed in Proposition 25, then $\mu_g = g(a)\delta_a + \nu$ gives

$$\int_{[a,b]} f d\mu_g = f(a)g(a) + \int_{[a,b]} f g' dm,$$

which reduces to the displayed identity exactly when $g(a) = 0$.

Problem 20.37. Suppose f is a continuous function on $[a, b]$ and g is increasing and absolutely continuous on $[a, b]$. Prove that if m denotes Lebesgue measure, then

$$\int_a^b f(x) dg(x) = \int_{[a,b]} f g' dm.$$

Solution. Here the left-hand side is the Riemann-Stieltjes integral described in this section: it is the number A , if one exists, such that for each $\epsilon > 0$ there is a $\delta > 0$ for which every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with mesh $\|P\| < \delta$ and every choice $C = \{c_1, \dots, c_n\}$ of tags $c_i \in [x_{i-1}, x_i]$ satisfy $|S(f, g, P, C) - A| < \epsilon$, where

$$S(f, g, P, C) = \sum_{i=1}^n f(c_i) [g(x_i) - g(x_{i-1})].$$

We show directly that the Riemann-Stieltjes sums converge to $\int_{[a,b]} f g' dm$; this proves at once that the Riemann-Stieltjes integral exists and that it has the asserted value. No normalization convention enters here, since the Riemann-Stieltjes integral involves only differences of values of g .

Preliminaries. Since g is absolutely continuous on $[a, b]$, by Theorem 10 of Chapter 6 the derivative g' exists almost everywhere, is Lebesgue integrable over $[a, b]$, and

$$\int_{[c,d]} g' dm = g(d) - g(c) \quad \text{whenever } a \leq c \leq d \leq b.$$

Since g is increasing, $g' \geq 0$ almost everywhere; consequently

$$\int_{[a,b]} |g'| dm = \int_{[a,b]} g' dm = g(b) - g(a).$$

Also f is continuous on the compact interval $[a, b]$, hence bounded and Borel measurable, so $f g'$ is integrable over $[a, b]$ and the right-hand side is a well-defined real number. If $g(b) = g(a)$ then g is

constant, every Riemann-Stieltjes sum is 0, and $g' = 0$ a.e., so both sides vanish; assume from now on that $g(b) - g(a) > 0$.

The estimate. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition and $C = \{c_i\}$ a choice of tags, $c_i \in [x_{i-1}, x_i]$. Introduce the step function

$$\varphi_{P,C} = \sum_{i=1}^n f(c_i) \chi_{(x_{i-1}, x_i]},$$

a Borel simple function on $(a, b]$. Using the preliminary identity on each subinterval,

$$S(f, g, P, C) = \sum_{i=1}^n f(c_i) [g(x_i) - g(x_{i-1})] = \sum_{i=1}^n f(c_i) \int_{(x_{i-1}, x_i]} g' dm = \int_{(a, b]} \varphi_{P,C} g' dm,$$

where we used that $\int_{(c, d]} g' dm = \int_{[c, d]} g' dm$ because the single point c has Lebesgue measure zero, and then additivity of the integral over the disjoint sets $(x_{i-1}, x_i]$. For the same reason $\int_{(a, b]} f g' dm = \int_{[a, b]} f g' dm$. Hence

$$\left| S(f, g, P, C) - \int_{[a, b]} f g' dm \right| = \left| \int_{(a, b]} (\varphi_{P,C} - f) g' dm \right| \leq \left(\sup_{x \in (a, b]} |\varphi_{P,C}(x) - f(x)| \right) \int_{[a, b]} g' dm.$$

Uniform continuity. Since f is continuous on the compact interval $[a, b]$, it is uniformly continuous there. Let $\epsilon > 0$ and choose $\delta > 0$ so that

$$|f(u) - f(v)| < \frac{\epsilon}{g(b) - g(a)} \quad \text{whenever } u, v \in [a, b] \text{ and } |u - v| < \delta.$$

Suppose $\|P\| < \delta$. If $x \in (x_{i-1}, x_i]$, then $\varphi_{P,C}(x) = f(c_i)$ and both x and c_i belong to $[x_{i-1}, x_i]$, an interval of length less than δ ; hence $|\varphi_{P,C}(x) - f(x)| = |f(c_i) - f(x)| < \epsilon / (g(b) - g(a))$. Since the intervals $(x_{i-1}, x_i]$, $1 \leq i \leq n$, partition $(a, b]$, we conclude

$$\sup_{x \in (a, b]} |\varphi_{P,C}(x) - f(x)| \leq \frac{\epsilon}{g(b) - g(a)}.$$

Conclusion. For every partition P with $\|P\| < \delta$ and every choice of tags C ,

$$\left| S(f, g, P, C) - \int_{[a, b]} f g' dm \right| \leq \frac{\epsilon}{g(b) - g(a)} \cdot (g(b) - g(a)) = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, f is Riemann-Stieltjes integrable over $[a, b]$ with respect to g and

$$\int_a^b f(x) dg(x) = \int_{[a, b]} f g' dm,$$

which is what was to be shown. Combining this with Problem 36 (applied to the bounded Borel measurable function f , a continuous function on $[a, b]$ being Borel measurable) yields the identity asserted in the text,

$$\int_a^b f(x) dg(x) = \int_{[a, b]} f dg,$$

for f continuous and g increasing and absolutely continuous. As in Problem 36, the right-hand side must be read with the Lebesgue-Stieltjes measure of g normalized to carry no mass at a ; if one keeps the atom $\mu_g\{a\} = g(a)$ of Proposition 25, the two sides differ by $f(a)g(a)$, since the Riemann-Stieltjes sums involve only differences of values of g and so cannot see an additive constant.

Problem 20.38. Let f and g be functions on $[-1, 1]$ such that $f = 0$ on $[-1, 0]$, $f = 1$ on $(0, 1]$, and $g = 0$ on $[-1, 0)$, $g = 1$ on $[0, 1]$. Show that f is not Riemann-Stieltjes integrable with respect to g over $[-1, 1]$ but is Riemann-Stieltjes integrable with respect to g on $[-1, 0]$ and on $[0, 1]$.

Solution. Recall the definitions: for a partition $P = \{x_0 < x_1 < \cdots < x_n\}$ of the interval in question and tags $C = \{c_i\}$ with $c_i \in [x_{i-1}, x_i]$,

$$S(f, g, P, C) = \sum_{i=1}^n f(c_i) [g(x_i) - g(x_{i-1})],$$

and f is Riemann-Stieltjes integrable with respect to g with integral A provided that for each $\epsilon > 0$ there is a $\delta > 0$ such that $|S(f, g, P, C) - A| < \epsilon$ whenever $\|P\| < \delta$, for every admissible choice of tags C .

Part 1: the two subintervals.

On $[-1, 0]$ we have $f \equiv 0$, since $f = 0$ on $[-1, 0]$. Hence for every partition P of $[-1, 0]$ and every choice of tags C , each tag c_i lies in $[-1, 0]$ and so $f(c_i) = 0$; therefore $S(f, g, P, C) = 0$. Taking $A = 0$, the definition is satisfied with any $\delta > 0$. Thus f is Riemann-Stieltjes integrable with respect to g over $[-1, 0]$ and

$$\int_{-1}^0 f(x) dg(x) = 0.$$

On $[0, 1]$ we have $g \equiv 1$, since $g = 1$ on $[0, 1]$. Hence for every partition P of $[0, 1]$, each increment $g(x_i) - g(x_{i-1}) = 1 - 1 = 0$, so again $S(f, g, P, C) = 0$ for every P and C . Thus f is Riemann-Stieltjes integrable with respect to g over $[0, 1]$ and

$$\int_0^1 f(x) dg(x) = 0.$$

Part 2: failure on $[-1, 1]$.

First we compute the increments of g across a partition of $[-1, 1]$. Let $P = \{-1 = x_0 < x_1 < \cdots < x_n = 1\}$. Since $-1 < 0 \leq 1$, there is exactly one index i_0 with

$$x_{i_0-1} < 0 \leq x_{i_0},$$

namely the least index i with $x_i \geq 0$. For $i < i_0$ we have $x_{i-1} < x_i < 0$, so $g(x_i) - g(x_{i-1}) = 0 - 0 = 0$. For $i > i_0$ we have $0 \leq x_{i_0} \leq x_{i-1} < x_i$, so $g(x_i) - g(x_{i-1}) = 1 - 1 = 0$. For $i = i_0$ we have $x_{i_0-1} < 0$ and $x_{i_0} \geq 0$, so $g(x_{i_0}) - g(x_{i_0-1}) = 1 - 0 = 1$. Therefore

$$S(f, g, P, C) = f(c_{i_0}), \quad c_{i_0} \in [x_{i_0-1}, x_{i_0}].$$

Now let $\delta > 0$ be arbitrary; we exhibit two partitions of mesh less than δ , with tags, whose sums are 0 and 1 respectively.

Choose a natural number n with $2/n < \delta$ and $1/n < 1$, and let P_1 be the uniform partition of $[-1, 1]$ into $2n$ subintervals of length $1/n$, so $\|P_1\| = 1/n < \delta$. The points of P_1 are $-1 + k/n$ for $k = 0, 1, \dots, 2n$, and 0 occurs among them ($k = n$). Thus for P_1 the distinguished index is the one with $x_{i_0} = 0$, $x_{i_0-1} = -1/n$. Tag this subinterval with $c_{i_0} = -1/n < 0$; then $f(c_{i_0}) = 0$ and, by the computation above, $S(f, g, P_1, C_1) = 0$. (Any other choice of tag in $[-1/n, 0]$ also gives 0, since f vanishes on $[-1, 0]$.)

Next let P_2 be the partition of $[-1, 1]$ obtained by shifting so that 0 is interior to a subinterval: take

$$P_2 = \left\{ -1 < -1 + \frac{1}{2n} < -1 + \frac{3}{2n} < \cdots \right\},$$

more precisely the points $-1, -1 + \frac{1}{2n} + \frac{k}{n}$ for $k = 0, 1, \dots, 2n-1$, and 1. Every gap has length at most $1/n < \delta$, so $\|P_2\| < \delta$. Here $0 = -1 + \frac{n}{n}$ is not a partition point, because the points other than ± 1 are of the form $-1 + \frac{2k+1}{2n}$ with odd numerator over $2n$, hence never equal to $-1 + \frac{2n}{2n} = 0$. Consequently

the distinguished index i_0 satisfies $x_{i_0-1} < 0 < x_{i_0}$, and we may tag that subinterval with $c_{i_0} = x_{i_0} > 0$, whence $f(c_{i_0}) = 1$ and $S(f, g, P_2, C_2) = 1$.

Suppose, for contradiction, that f were Riemann-Stieltjes integrable with respect to g over $[-1, 1]$, with integral A . Apply the definition with $\epsilon = 1/2$ to obtain $\delta > 0$. Building P_1, C_1 and P_2, C_2 as above for this δ , we would get

$$|0 - A| < \frac{1}{2} \quad \text{and} \quad |1 - A| < \frac{1}{2},$$

so that $1 = |1 - 0| \leq |1 - A| + |A - 0| < 1$, a contradiction. Hence f is not Riemann-Stieltjes integrable with respect to g over $[-1, 1]$.

Remark. The obstruction is that f and g share the point 0 as a discontinuity, and there f is continuous from the left while g is continuous from the right; on each of $[-1, 0]$ and $[0, 1]$ separately one of the two functions is constant, so no such clash can occur. In particular, Riemann-Stieltjes integrability over $[-1, 0]$ and over $[0, 1]$ does not imply integrability over the union.

Problem 20.39. Prove the inequality (29). [Hint: Choose $\epsilon > 0$. By the continuity on the right of g , choose $\eta_i > 0$ so that $g(b_i + \eta_i) < g(b_i) + \epsilon 2^{-i}$, and choose $\delta > 0$ so that $g(a + \delta) < g(a) + \epsilon$. Then the open intervals $(a_i, b_i + \eta_i)$ cover the closed interval $[a + \delta, b]$.]

Solution. The assertion (29) is the following. Let $I = [a, b]$ and let $g : I \rightarrow \mathbb{R}$ be increasing and continuous on the right. If $(c, d] \subseteq I$ is covered by a countable collection $\{(c_k, d_k]\}_{k=1}^\infty$ of intervals of the same form contained in I , then

$$g(d) - g(c) \leq \sum_{k=1}^{\infty} [g(d_k) - g(c_k)].$$

Extension of g . It is convenient to have g defined slightly to the right of b . Extend g to all of $\mathbb{R}$ by setting $g(x) = g(a)$ for $x < a$ and $g(x) = g(b)$ for $x > b$. The extended function is again increasing, and it is continuous on the right at every point of $\mathbb{R}$: at points of $[a, b)$ this is the hypothesis, at b and at points $x > b$ the extension is locally constant to the right of the point, and at points $x < a$ likewise. We continue to write g for the extension.

Two reductions. If $c = d$ the left-hand side is 0, while every summand on the right is nonnegative because g is increasing; so the inequality holds. Assume then $c < d$. Also, every summand $g(d_k) - g(c_k)$ is nonnegative, so if the right-hand series diverges there is nothing to prove; and dropping terms from a nonnegative series only decreases it, a fact we use below.

A finite-cover lemma. We first prove the following elementary statement.

Lemma. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be increasing, let $\alpha \leq \beta$, and suppose $[\alpha, \beta] \subseteq \bigcup_{j=1}^N (u_j, v_j)$ for finitely many open intervals. Then

$$h(\beta) - h(\alpha) \leq \sum_{j=1}^N [h(v_j) - h(u_j)].$$

Proof of the Lemma, by induction on N . If $N = 1$, then $u_1 < \alpha \leq \beta < v_1$, so by monotonicity $h(\beta) - h(\alpha) \leq h(v_1) - h(u_1)$. Assume the statement for coverings by fewer than N intervals, and let $[\alpha, \beta] \subseteq \bigcup_{j=1}^N (u_j, v_j)$. Since $\beta \in [\alpha, \beta]$, there is an index j_0 with $\beta \in (u_{j_0}, v_{j_0})$. Note first that all the summands $h(v_j) - h(u_j)$ are nonnegative, since $u_j < v_j$ and h is increasing. If $u_{j_0} < \alpha$, then $u_{j_0} < \alpha \leq \beta < v_{j_0}$ gives $h(\beta) - h(\alpha) \leq h(v_{j_0}) - h(u_{j_0}) \leq \sum_{j=1}^N [h(v_j) - h(u_j)]$ and we are done. Otherwise $\alpha \leq u_{j_0} < \beta$. Every point of $[\alpha, u_{j_0}]$ lies in $[\alpha, \beta]$ but is $\leq u_{j_0}$, hence does not lie in (u_{j_0}, v_{j_0}) ; therefore

$$[\alpha, u_{j_0}] \subseteq \bigcup_{j \neq j_0} (u_j, v_j),$$

a covering by $N - 1$ intervals, and the induction hypothesis yields $h(u_{j_0}) - h(\alpha) \leq \sum_{j \neq j_0} [h(v_j) - h(u_j)]$. Since $u_{j_0} < \beta < v_{j_0}$, monotonicity gives $h(\beta) - h(u_{j_0}) \leq h(v_{j_0}) - h(u_{j_0})$. Adding the last two inequalities,

$$h(\beta) - h(\alpha) = [h(\beta) - h(u_{j_0})] + [h(u_{j_0}) - h(\alpha)] \leq \sum_{j=1}^N [h(v_j) - h(u_j)].$$

This completes the induction and proves the Lemma.

Proof of (29). Let $\epsilon > 0$. Since g is continuous on the right at each d_k , choose $\eta_k > 0$ so that

$$g(d_k + \eta_k) < g(d_k) + \epsilon 2^{-k} \quad (k = 1, 2, \dots).$$

Since g is continuous on the right at c , choose $\delta > 0$ with $c + \delta < d$ and

$$g(c + \delta) < g(c) + \epsilon.$$

The open intervals $(c_k, d_k + \eta_k)$, $k = 1, 2, \dots$, cover the closed bounded interval $[c + \delta, d]$: indeed, if $x \in [c + \delta, d]$ then $c < x \leq d$, so $x \in (c, d] \subseteq \bigcup_k (c_k, d_k]$, and $(c_k, d_k] \subseteq (c_k, d_k + \eta_k)$.

By the Heine-Borel Theorem, $[c + \delta, d]$ is compact, so finitely many of these open intervals already cover it: there is a finite set F of indices with

$$[c + \delta, d] \subseteq \bigcup_{k \in F} (c_k, d_k + \eta_k).$$

Applying the Lemma to the increasing function g (the extension constructed above, so that g is defined at each $d_k + \eta_k$) with $\alpha = c + \delta$ and $\beta = d$,

$$g(d) - g(c + \delta) \leq \sum_{k \in F} [g(d_k + \eta_k) - g(c_k)].$$

By the choice of η_k , for each $k \in F$

$$g(d_k + \eta_k) - g(c_k) < [g(d_k) - g(c_k)] + \epsilon 2^{-k},$$

and since each $g(d_k) - g(c_k) \geq 0$ and $\sum_{k \in F} \epsilon 2^{-k} \leq \epsilon$, we obtain

$$g(d) - g(c + \delta) \leq \sum_{k=1}^{\infty} [g(d_k) - g(c_k)] + \epsilon.$$

Finally, $g(c) > g(c + \delta) - \epsilon$, so

$$g(d) - g(c) < g(d) - g(c + \delta) + \epsilon \leq \sum_{k=1}^{\infty} [g(d_k) - g(c_k)] + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, letting $\epsilon \rightarrow 0$ gives

$$g(d) - g(c) \leq \sum_{k=1}^{\infty} [g(d_k) - g(c_k)],$$

which is (29).

Remark. Right continuity of g is essential. If g is the increasing function on $[0, 2]$ with $g \equiv 0$ on $[0, 1]$ and $g \equiv 1$ on $(1, 2]$, then $(1, 3/2]$ is the disjoint union of the intervals $(1 + \frac{1}{n+1}, 1 + \frac{1}{n}]$ for $n \geq 2$, and the corresponding sum of increments of g is 0, while $g(3/2) - g(1) = 1$.

Problem 20.40. For an increasing function $g: [a, b] \rightarrow \mathbb{R}$, define

$$g^*(x) = \lim_{y \rightarrow x^+} g(y).$$

Show that g^* is an increasing function that is continuous on the right and agrees with g wherever g is continuous on the right. Conclude that $g = g^*$, except possibly at a countable number of points. Show that $(g^*)^* = g^*$, and if g and G are increasing functions that agree wherever they are both

continuous, then $g^* = G^*$. If f is a bounded Borel measurable function on $[a, b]$, show that

$$\int_{[a, b]} f dg = \int_{[a, b]} f dg^*.$$

Solution. Since g is increasing on $[a, b]$, for $x \in [a, b]$ the function g is bounded below on $(x, b]$ by $g(x)$, and the limit from the right exists and equals an infimum:

$$g^*(x) = \lim_{y \rightarrow x^+} g(y) = \inf_{x < y \leq b} g(y) \geq g(x), \quad x \in [a, b].$$

At the right endpoint there is nothing to the right of b , and we use the standard convention $g^*(b) = g(b)$; equivalently, extend g to $[a, \infty)$ by $g(x) = g(b)$ for $x > b$ and then $g^*(b) = \lim_{y \rightarrow b^+} g(y) = g(b)$ automatically. With this convention $g^* \geq g$ on all of $[a, b]$.

(1) g^* is increasing. Let $a \leq x < x' \leq b$. If $x' < b$, then $\{y : x' < y \leq b\} \subseteq \{y : x < y \leq b\}$, and an infimum over a smaller set is at least as large, so $g^*(x) \leq g^*(x')$. If $x' = b$, pick any y with $x < y \leq b$; then $g^*(x) \leq g(y) \leq g(b) = g^*(b)$. In both cases $g^*(x) \leq g^*(x')$.

(2) g^* is continuous on the right. Let $x \in [a, b)$ and $\epsilon > 0$. Since $g^*(x) = \inf_{y > x} g(y)$, there is $y_0 \in (x, b]$ with $g(y_0) < g^*(x) + \epsilon$. Let $z \in (x, y_0)$. On the one hand $g^*(z) \geq g^*(x)$ by (1). On the other hand $y_0 > z$, so $g^*(z) = \inf_{y > z} g(y) \leq g(y_0) < g^*(x) + \epsilon$. Hence $|g^*(z) - g^*(x)| < \epsilon$ for all $z \in (x, y_0)$, which is precisely right continuity of g^* at x . At $x = b$ the requirement is vacuous. So g^* is continuous on the right on $[a, b]$.

(3) $g^* = g$ wherever g is continuous on the right. If g is continuous on the right at $x \in [a, b)$, then by definition $\lim_{y \rightarrow x^+} g(y) = g(x)$, i.e. $g^*(x) = g(x)$. At $x = b$ the convention gives $g^*(b) = g(b)$.

(4) $g = g^*$ off a countable set. We first recall why an increasing function on $[a, b]$ has at most countably many discontinuities. For $x \in (a, b)$ both one-sided limits $g(x^-) = \sup_{y < x} g(y)$ and $g(x^+) = \inf_{y > x} g(y)$ exist and $g(x^-) \leq g(x) \leq g(x^+)$; put $j(x) = g(x^+) - g(x^-) \geq 0$, and g is continuous at x exactly when $j(x) = 0$. If $a < x_1 < x_2 < \cdots < x_N < b$, choose points

$$a \leq s_1 < x_1 < t_1 \leq s_2 < x_2 < t_2 \leq \cdots \leq s_N < x_N < t_N \leq b,$$

which is possible since the x_i are distinct interior points. For each i , monotonicity gives $j(x_i) = g(x_i^+) - g(x_i^-) \leq g(t_i) - g(s_i)$, since $g(x_i^-) = \sup_{y < x_i} g(y) \geq g(s_i)$ and $g(x_i^+) = \inf_{y > x_i} g(y) \leq g(t_i)$. Moreover

$$g(b) - g(a) = \sum_{i=1}^N [g(t_i) - g(s_i)] + [g(s_1) - g(a)] + \sum_{i=1}^{N-1} [g(s_{i+1}) - g(t_i)] + [g(b) - g(t_N)],$$

and every bracket in the last three groups is nonnegative because g is increasing and $a \leq s_1$, $t_i \leq s_{i+1}$, $t_N \leq b$; hence $\sum_{i=1}^N [g(t_i) - g(s_i)] \leq g(b) - g(a)$. Combining,

$$\sum_{i=1}^N j(x_i) \leq g(b) - g(a).$$

Therefore, for each natural number n , the set $\{x \in (a, b) : j(x) > 1/n\}$ has at most $n(g(b) - g(a))$ elements, and the set D of interior discontinuities of g , being $\bigcup_n \{x : j(x) > 1/n\}$, is countable. Adjoining the two endpoints, the set of all points of $[a, b]$ at which g fails to be continuous is countable. At every point of continuity g is in particular continuous on the right, so $g = g^*$ there by (3). Hence $\{x \in [a, b] : g(x) \neq g^*(x)\}$ is contained in a countable set, so $g = g^*$ except at countably many points.

(5) $(g^*)^* = g^*$. By (1) and (2), g^* is increasing and continuous on the right at every point of $[a, b]$, so applying (3) to the function g^* in place of g gives $(g^*)^*(x) = g^*(x)$ for all x .

(6) If g and G are increasing on $[a, b]$ and agree wherever they are both continuous, then $g^* = G^*$ on $[a, b]$. Let D_g and D_G be the sets of discontinuities of g and of G ; by the argument in (4) each is countable, hence so is $D = D_g \cup D_G$. Fix $x \in [a, b)$. For every η with $0 < \eta < b - x$ the interval $(x, x + \eta)$ is uncountable, so it is not contained in D ; choose $y \in (x, x + \eta) \setminus D$. Doing this for $\eta = (b - x)/n$,

$n = 1, 2, \dots$, produces a sequence $\{y_n\}$ in $(x, b]$ with $y_n \rightarrow x$, $y_n > x$, and $y_n \notin D$, so that g and G are both continuous at y_n and therefore $g(y_n) = G(y_n)$. Since the right-hand limits exist, they may be computed along any such sequence:

$$g^*(x) = \lim_{n \rightarrow \infty} g(y_n) = \lim_{n \rightarrow \infty} G(y_n) = G^*(x).$$

Thus $g^* = G^*$ on $[a, b)$. At the endpoint b the conclusion requires the additional (and harmless) information $g(b) = G(b)$; without it the assertion can fail, as the example $g = \chi_{\{1\}}$ and $G \equiv 0$ on $[0, 1]$ shows, these being increasing functions that are both continuous only on $[0, 1)$, where they agree, while $g^*(1) = 1 \neq 0 = G^*(1)$. We therefore read the statement, as it is used below, as the assertion on $[a, b)$ together with the normalization $g(b) = G(b)$, which holds automatically when b is a point of continuity of both.

(7) The integral identity. By (1) and (2) the function g^* is increasing and continuous on the right, so Proposition 25 provides the unique Borel measure μ_{g^*} on $\mathcal{B}([a, b])$ with cumulative distribution function g^* , and by formula (30) the right-hand integral $\int_{[a, b]} f dg^* = \int_{[a, b]} f d\mu_{g^*}$ is defined for every bounded Borel measurable f .

The left-hand integral needs interpretation, because for a general increasing g the definition (30) does not apply verbatim: Proposition 25 requires right continuity, and no Borel measure μ can have $\mu[a, x] = g(x)$ for all x unless g is continuous on the right, since by the first half of Proposition 25 a cumulative distribution function is always continuous on the right. The construction inside Proposition 25 breaks down for the same reason: the set function on the semiring $\mathcal{S}$ given by $\mu\{a\} = g(a)$ and $\mu(c, d] = g(d) - g(c)$ is not countably additive on $\mathcal{S}$ when g has a jump from the right, as the example $g = \chi_{(1, 2]}$ on $[0, 2]$ shows: there $(1, 2]$ is the disjoint union of the intervals $(1 + \frac{1}{n+1}, 1 + \frac{1}{n}]$, $n \geq 2$, across which the prescribed increments all vanish, while $g(2) - g(1) = 1$. So no measure reproduces the prescription exactly, and the most one can require of a measure attached to g is that its cumulative distribution function agree with g where g is well behaved, and that it carry the right total mass.

Accordingly, $\int_{[a, b]} f dg$ is understood as $\int_{[a, b]} f d\mu$, where μ is a Borel measure on $\mathcal{B}([a, b])$ whose cumulative distribution function g_μ agrees with g at every point at which both are continuous, and which has the same total mass, $\mu([a, b]) = g(b)$. (When g is itself continuous on the right this returns $\mu = \mu_g$ and the original definition (30), by (3) and the uniqueness in Proposition 25, so nothing is lost.) The two requirements determine μ completely, and the content of the exercise is that they determine it to be μ_{g^*} . Indeed, g_μ is increasing and continuous on the right by Proposition 25, so $(g_\mu)^* = g_\mu$ by (5); and g_μ and g are increasing functions agreeing wherever both are continuous with $g_\mu(b) = \mu([a, b]) = g(b)$, so by (6)

$$g_\mu = (g_\mu)^* = g^* \quad \text{on } [a, b].$$

Thus μ has cumulative distribution function g^* , and the uniqueness assertion of Proposition 25 gives $\mu = \mu_{g^*}$. Such a μ does exist, namely μ_{g^*} itself: its cumulative distribution function is g^* , which by (3) agrees with g at every point of continuity of g , and its total mass is $g^*(b) = g(b)$. Consequently

$$\int_{[a, b]} f dg = \int_{[a, b]} f d\mu = \int_{[a, b]} f d\mu_{g^*} = \int_{[a, b]} f dg^*$$

for every bounded Borel measurable f on $[a, b]$. When g is itself continuous on the right we have $g^* = g$ by (3), so this is consistent with the original definition (30).

In short: only the right-continuous regularization g^* of an increasing function g is seen by Lebesgue-Stieltjes integration, and g and g^* differ at only countably many points.

Problem 20.41. (i) Show that each bounded function g of bounded variation gives rise to a finite signed Borel measure ν such that

$$\nu(c, d] = g(d^+) - g(c^+) \quad \text{for all } (c, d] \subseteq [a, b].$$

(ii) Extend the definition of the Lebesgue-Stieltjes integral $\int_{[a, b]} f dg$ to functions g of bounded variation and bounded Borel measurable functions f .

(iii) Show that if $|f| \leq M$ on $[a, b]$ and if the total variation of g is T , then $\left| \int_{[a,b]} f dg \right| \leq MT$.

Solution. Throughout, g is of bounded variation on $[a, b]$ (hence automatically bounded), $T = \text{TV}(g)$ denotes its total variation over $[a, b]$, and we write

$$g_1(x) = \text{TV}(g_{[a,x]}), \quad g_2(x) = g_1(x) - g(x) \quad (a \leq x \leq b)$$

for the Jordan decomposition of g : by the lemma preceding Jordan's Theorem in Chapter 6, which supplies the explicit representation (23) of a function of bounded variation as a difference of increasing functions, g_1 and g_2 are increasing on $[a, b]$, $g = g_1 - g_2$, $g_1(a) = 0$, and $g_1(b) = T$. For $c < d$ in $[a, b]$ the additivity of total variation gives

$$g_1(d) - g_1(c) = \text{TV}(g_{[c,d]}) \geq |g(d) - g(c)|. \quad (\dagger)$$

Since g_1 and g_2 are increasing and bounded, the one-sided limits $g_i(x^+)$ exist at every $x \in [a, b]$, and therefore so does $g(x^+) = g_1(x^+) - g_2(x^+)$. As in Problem 40 we write $h^*(x) = h(x^+)$ for $x \in [a, b]$ and $h^*(b) = h(b)$, so that the symbol $g(b^+)$ occurring in the statement is to be read as $g(b)$; thus $g^* = g_1^* - g_2^*$, and by Problem 40 each g_i^* is increasing and continuous on the right.

Part (i). By Proposition 25 there are unique finite Borel measures $\mu_1 = \mu_{g_1^*}$ and $\mu_2 = \mu_{g_2^*}$ on $\mathcal{B}([a, b])$ with cumulative distribution functions g_1^* and g_2^* . Let ρ and σ denote their restrictions to $(a, b]$, that is,

$$\rho(E) = \mu_1(E \cap (a, b]), \quad \sigma(E) = \mu_2(E \cap (a, b]) \quad (E \in \mathcal{B}([a, b])),$$

which are again finite Borel measures, now carrying no mass at the point a . Define

$$\nu = \rho - \sigma.$$

Since ρ and σ are finite measures, ν is a finite signed measure on $\mathcal{B}([a, b])$, and for $(c, d] \subseteq [a, b]$,

$$\nu(c, d] = [g_1^*(d) - g_1^*(c)] - [g_2^*(d) - g_2^*(c)] = g^*(d) - g^*(c) = g(d^+) - g(c^+),$$

which is the required identity. (We also record the normalization $\nu(\{a\}) = 0$; this is the choice that makes part (iii) correct, and it is not prescribed by the displayed formula, which involves only intervals of the form $(c, d]$.)

The signed measure ν is uniquely determined by these two requirements, and in particular does not depend on the Jordan decomposition used to construct it. Indeed, suppose ν' is another finite signed Borel measure on $[a, b]$ with $\nu'(c, d] = g(d^+) - g(c^+)$ for all $(c, d] \subseteq [a, b]$ and $\nu'(\{a\}) = 0$. Write $\nu = \nu^+ - \nu^-$ and $\nu' = (\nu')^+ - (\nu')^-$ for the Jordan decompositions of Chapter 17. Then $\nu^+ + (\nu')^-$ and $(\nu')^+ + \nu^-$ are finite measures on $\mathcal{B}([a, b])$ that agree on the semiring $\mathcal{S}$ consisting of $\emptyset$, $\{a\}$, and the intervals $(c, d] \subseteq [a, b]$, since ν and ν' agree there. A finite measure is σ -finite, so the uniqueness assertion of the Caratheodory-Hahn Theorem, applied to the premeasure that is their common restriction to $\mathcal{S}$, forces these two measures to agree on the σ -algebra generated by $\mathcal{S}$, which contains $\mathcal{B}([a, b])$. Rearranging, $\nu = \nu'$.

Part (ii). For a bounded Borel measurable f on $[a, b]$ define

$$\int_{[a,b]} f dg = \int_{[a,b]} f d\nu = \int_{[a,b]} f d\nu^+ - \int_{[a,b]} f d\nu^-,$$

where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of the finite signed measure ν of part (i). Both integrals on the right are finite: f is Borel measurable and $|f| \leq M$ for some M , and $\nu^\pm$ are finite measures, so $|\int f d\nu^\pm| \leq M\nu^\pm([a, b]) < \infty$. The definition is unambiguous because ν is uniquely determined by g , by part (i), and the Jordan decomposition of a signed measure is unique.

This extends the earlier definition: if g is increasing and continuous on the right with $g(a) = 0$, then g has bounded variation with $g_1 = g$, $g_2 = 0$, so $\nu = \mu_g$ restricted to $(a, b]$, and since $\mu_g(\{a\}) = g(a) = 0$ we get $\nu = \mu_g$ and the new definition reduces to (30).

Part (iii). We first show that $|\nu| \leq \rho$ as measures, where $|\nu| = \nu^+ + \nu^-$ is the total variation measure of ν .

Claim: $\rho - \nu \geq 0$ and $\rho + \nu \geq 0$ as set functions on $\mathcal{B}([a, b])$.

The first is immediate, since $\rho - \nu = \sigma$ is a measure. For the second, set $h = g_1 + g$. By (†), for $a \leq c < d \leq b$,

$$h(d) - h(c) = [g_1(d) - g_1(c)] + [g(d) - g(c)] \geq [g_1(d) - g_1(c)] - |g(d) - g(c)| \geq 0,$$

so h is increasing on $[a, b]$. By Problem 40, h^* is then increasing and continuous on the right, and $h^* = g_1^* + g^*$ because the two right-hand limits exist separately. Let π be the restriction to $(a, b]$ of the Borel measure μ_{h^*} supplied by Proposition 25; thus π is a finite measure with $\pi(\{a\}) = 0$ and, for every $(c, d] \subseteq [a, b]$,

$$\pi(c, d] = h^*(d) - h^*(c) = [g_1^*(d) - g_1^*(c)] + [g^*(d) - g^*(c)] = \rho(c, d] + \nu(c, d].$$

So the finite measure π and the finite signed measure $\rho + \nu$ agree on the semiring $\mathcal{S}$. Writing $\rho + \nu = (\rho + \nu)^+ - (\rho + \nu)^-$ for its Jordan decomposition, the two finite measures $\pi + (\rho + \nu)^-$ and $(\rho + \nu)^+$ agree on $\mathcal{S}$, hence agree on $\mathcal{B}([a, b]) = \sigma(\mathcal{S})$ by the uniqueness argument already used in part (i). Rearranging, $\rho + \nu = \pi \geq 0$, which proves the claim.

Consequently $-\rho(E) \leq \nu(E) \leq \rho(E)$ for every $E \in \mathcal{B}([a, b])$. Now let $\{P, N\}$ be a Hahn decomposition of $[a, b]$ with respect to ν , furnished by the Hahn Decomposition Theorem of Chapter 17, so that $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Then, using the two displayed bounds on the disjoint Borel sets $E \cap P$ and $E \cap N$ and the additivity of the measure ρ ,

$$|\nu|(E) = \nu(E \cap P) - \nu(E \cap N) \leq \rho(E \cap P) + \rho(E \cap N) = \rho(E),$$

so we conclude

$$|\nu|(E) \leq \rho(E) \quad \text{for all } E \in \mathcal{B}([a, b]).$$

In particular, because g_1 is increasing with $g_1(a) = 0$ and $g_1(b) = T$,

$$|\nu|([a, b]) \leq \rho([a, b]) = \rho((a, b]) = g_1^*(b) - g_1^*(a) = g_1(b) - g_1(a^+) \leq g_1(b) - g_1(a) = T.$$

Finally, let f be bounded Borel measurable with $|f| \leq M$ on $[a, b]$. Then

$$\left| \int_{[a, b]} f dg \right| = \left| \int_{[a, b]} f d\nu^+ - \int_{[a, b]} f d\nu^- \right| \leq \int_{[a, b]} |f| d\nu^+ + \int_{[a, b]} |f| d\nu^- = \int_{[a, b]} |f| d|\nu|,$$

and since $|f| \leq M$ and $|\nu|$ is a finite measure,

$$\left| \int_{[a, b]} f dg \right| \leq M |\nu|([a, b]) \leq MT,$$

which is the desired estimate.

Problem 20.42. Let g be a continuous increasing function on $[a, b]$ with $g(a) = c$, $g(b) = d$, and let f be a nonnegative Borel measurable function on $[c, d]$. Show that

$$\int_{[a, b]} f \circ g dg = \int_{[c, d]} f dm.$$

Solution. As in Problem 36 we use the normalization in which the Lebesgue-Stieltjes measure of g carries no atom at the left endpoint: μ is the Borel measure on $\mathcal{B}([a, b])$ determined by

$$\mu(u, v] = g(v) - g(u) \quad \text{for } (u, v] \subseteq [a, b], \quad \mu(\{a\}) = 0,$$

so that μ is the measure $\mu_{g-g(a)}$ of Proposition 25 (legitimate, since $g - g(a)$ is increasing and, being

continuous, continuous on the right), and $\int_{[a,b]} h dg = \int_{[a,b]} h d\mu$. This is the normalization the asserted identity requires: the total masses must match, and $\mu([a,b]) = g(b) - g(a) = d - c = m([c,d])$. Note μ is a finite measure. For nonnegative Borel measurable h (not necessarily bounded) the integral $\int_{[a,b]} h d\mu$ is defined in $[0, \infty]$ in the usual way, and likewise for m ; the asserted identity is an identity of extended real numbers.

Step 0: measurability of the integrand. The function g is continuous on $[a,b]$, hence Borel measurable, and $g([a,b]) \subseteq [c,d]$ since g is increasing with $g(a) = c$, $g(b) = d$. If f is Borel measurable on $[c,d]$ and $B \subseteq \mathbb{R}$ is a Borel set, then $f^{-1}(B)$ is a Borel subset of $[c,d]$, and $(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$ is a Borel subset of $[a,b]$ because the preimage under a Borel measurable function of a Borel set is Borel. Hence $f \circ g$ is a nonnegative Borel measurable function on $[a,b]$ and the left-hand side is defined.

Step 1: the push-forward measure. Define, for $E \in \mathcal{B}([c,d])$,

$$\lambda(E) = \mu(g^{-1}(E)).$$

By Step 0, $g^{-1}(E) \in \mathcal{B}([a,b])$, so λ is well defined. Clearly $\lambda(\emptyset) = 0$ and $\lambda \geq 0$; and if $\{E_k\}$ is a disjoint countable collection of Borel subsets of $[c,d]$, the preimages $g^{-1}(E_k)$ are disjoint with union $g^{-1}(\bigcup_k E_k)$, so the countable additivity of μ gives that of λ . Thus λ is a Borel measure on $\mathcal{B}([c,d])$, finite since $\lambda([c,d]) \leq \mu([a,b]) < \infty$.

Step 2: λ is Lebesgue measure on $[c,d]$. Because g is continuous and increasing with $g(a) = c$ and $g(b) = d$, the Intermediate Value Theorem shows $g([a,b]) = [c,d]$. For $t \in [c,d]$ set

$$u(t) = \max\{x \in [a,b] : g(x) = t\},$$

which exists: the set $g^{-1}(\{t\})$ is nonempty by surjectivity and is closed and bounded by continuity of g , so it has a largest element, and $g(u(t)) = t$. Moreover

$$\{x \in [a,b] : g(x) \leq t\} = [a, u(t)],$$

for if $x \leq u(t)$ then $g(x) \leq g(u(t)) = t$ since g is increasing, while if $x > u(t)$ then $g(x) \geq t$ and $g(x) \neq t$ by the maximality of $u(t)$, so $g(x) > t$.

Now compute the cumulative distribution function of λ . For $t \in [c,d]$,

$$\lambda([c,t]) = \mu(g^{-1}([c,t])) = \mu(\{x : g(x) \leq t\}) = \mu([a, u(t)]) = \mu(\{a\}) + \mu((a, u(t))) = 0 + [g(u(t)) - g(a)] = t - c,$$

where we used $g^{-1}([c,t]) = \{x : g(x) \leq t\}$, valid because $g \geq c$ everywhere on $[a,b]$. (If $u(t) = a$, the interval $(a, u(t))$ is empty and the computation reads $\lambda([c,t]) = 0 = t - c$, consistent since then $t = g(a) = c$.)

On the other hand, Lebesgue measure m restricted to $\mathcal{B}([c,d])$ is a finite Borel measure whose cumulative distribution function is $m([c,t]) = t - c$. The function $t \mapsto t - c$ is increasing and continuous, in particular continuous on the right, so by the uniqueness assertion of Proposition 25 there is exactly one Borel measure on $\mathcal{B}([c,d])$ with this cumulative distribution function. Therefore

$$\lambda = m \quad \text{on } \mathcal{B}([c,d]).$$

Step 3: change of variables. We claim that for every nonnegative Borel measurable f on $[c,d]$,

$$\int_{[c,d]} f d\lambda = \int_{[a,b]} f \circ g d\mu. \quad (*)$$

If $f = \chi_E$ with $E \in \mathcal{B}([c,d])$, then $\chi_E \circ g = \chi_{g^{-1}(E)}$, and $(*)$ reads $\lambda(E) = \mu(g^{-1}(E))$, which is the definition of λ . By the linearity of integration of nonnegative functions, $(*)$ holds for every nonnegative Borel simple function on $[c,d]$.

For general nonnegative Borel measurable f on $[c,d]$, the Simple Approximation Theorem furnishes an increasing sequence $\{\varphi_n\}$ of nonnegative Borel simple functions on $[c,d]$ with $\varphi_n \rightarrow f$ pointwise on $[c,d]$. Then $\{\varphi_n \circ g\}$ is an increasing sequence of nonnegative Borel simple functions on $[a,b]$ converging pointwise to $f \circ g$. Applying the Monotone Convergence Theorem on each side,

$$\int_{[c,d]} f d\lambda = \lim_{n \rightarrow \infty} \int_{[c,d]} \varphi_n d\lambda = \lim_{n \rightarrow \infty} \int_{[a,b]} \varphi_n \circ g d\mu = \int_{[a,b]} f \circ g d\mu,$$

which is $(*)$, both sides being possibly $+\infty$.

Conclusion. Combining Steps 2 and 3,

$$\int_{[a,b]} f \circ g \, dg = \int_{[a,b]} f \circ g \, d\mu = \int_{[c,d]} f \, d\lambda = \int_{[c,d]} f \, dm.$$

In words: a continuous increasing g carries its own Lebesgue-Stieltjes measure forward onto Lebesgue measure on the image interval $[c, d]$, and the displayed formula is precisely the change of variables attached to that push-forward.

20.7 Problems 20.43–20.49

Problem 20.43. Let g be increasing on $[a, b]$. Find a Borel measure μ on $\mathcal{B}([a, b])$ such that

$$\int_a^b f(x) \, dg(x) = \int_{[a,b]} f \, d\mu \quad \text{for all } f \in C[a, b].$$

Solution. Here $\int_a^b f(x) \, dg(x)$ is the Riemann-Stieltjes integral defined in this section, and a Borel measure means a finite measure on $\mathcal{B}([a, b])$.

Construction of the measure. Since g is increasing on $[a, b]$, for each $x \in [a, b)$ the one-sided limit $g(x^+) = \lim_{y \rightarrow x^+} g(y)$ exists and equals $\inf_{y > x} g(y)$. Define $h: [a, b] \rightarrow \mathbf{R}$ by

$$h(x) = g(x^+) - g(a) \quad \text{for } x \in [a, b), \quad h(b) = g(b) - g(a).$$

We claim h is increasing, nonnegative and continuous on the right. Monotonicity is clear from the monotonicity of g : for $x < y < b$ we have $g(x^+) \leq g(y) \leq g(y^+)$, and $g(x^+) \leq g(b)$ for $x < b$. Also $h(a) = g(a^+) - g(a) \geq 0$, so $h \geq 0$. Right continuity is the content of Problem 40 (the function denoted g^* there is $x \mapsto g(x^+)$, and $h = g^* - g(a)$ except at the right endpoint); we recall the argument. Let $x \in [a, b)$ and $\epsilon > 0$, and choose $y \in (x, b]$ with $g(y) < g(x^+) + \epsilon$. For every t with $x < t < y$ we have $g(x^+) \leq g(t^+) \leq g(y) < g(x^+) + \epsilon$, so $h(x) \leq h(t) < h(x) + \epsilon$. Thus $h(t) \rightarrow h(x)$ as $t \rightarrow x^+$. Right continuity at b is vacuous.

By Proposition 25, since h is increasing, nonnegative and continuous on the right, there is a unique Borel measure μ on $\mathcal{B}([a, b])$ whose cumulative distribution function is h , that is,

$$\mu[a, x] = h(x) \quad \text{for all } x \in [a, b].$$

Consequently

$$\mu(\{a\}) = g(a^+) - g(a), \quad \mu((c, d]) = h(d) - h(c) \quad \text{for } a \leq c < d \leq b,$$

and $\mu([a, b]) = h(b) = g(b) - g(a) < \infty$. This μ is the required measure. Observe that it charges each interior point x_0 with the full jump $g(x_0^+) - g(x_0^-)$ of g there, charges a with $g(a^+) - g(a)$, and charges b with $g(b) - g(b^-)$; this is precisely what the Riemann-Stieltjes sums see.

Proof of the identity. Fix $f \in C[a, b]$ and let $\epsilon > 0$. Since f is continuous on the closed, bounded interval $[a, b]$ it is uniformly continuous there, so we may choose $\delta_1 > 0$ such that

$$|f(u) - f(v)| < \epsilon \quad \text{whenever } u, v \in [a, b] \text{ and } |u - v| \leq \delta_1.$$

Since f is continuous and g is monotone, f is Riemann-Stieltjes integrable over $[a, b]$ with respect to g , as is noted in the text of this section; write $A = \int_a^b f(x) \, dg(x)$. By the definition of that integral there is a $\delta_2 > 0$ for which

$$|S(f, g, P, C) - A| < \epsilon \quad \text{whenever } \|P\| < \delta_2.$$

Put $\delta = \min\{\delta_1, \delta_2\}$.

An increasing function on $[a, b]$ has at most countably many points of discontinuity: to distinct discontinuity points x there belong the nonempty open intervals $(g(x^-), g(x^+))$, these are pairwise disjoint,

and a disjoint collection of nonempty open intervals of real numbers is countable. Hence the points of continuity of g are dense in $[a, b]$, and we may choose a partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}, \quad \|P\| < \delta,$$

all of whose interior points $x_1, \dots, x_{n-1}$ are points of continuity of g . Define the disjoint Borel sets

$$E_1 = [a, x_1], \quad E_i = (x_{i-1}, x_i] \quad (2 \leq i \leq n),$$

whose union is $[a, b]$. We compute their measures. First $\mu(E_1) = h(x_1)$. If $n \geq 2$, then x_1 is a point of continuity of g , so $g(x_1^+) = g(x_1)$ and $h(x_1) = g(x_1) - g(a) = g(x_1) - g(x_0)$; if $n = 1$, then $x_1 = b$ and $\mu(E_1) = h(b) = g(b) - g(a) = g(x_1) - g(x_0)$ as well. For $2 \leq i \leq n-1$ both x_{i-1} and x_i are continuity points of g , so

$$\mu(E_i) = h(x_i) - h(x_{i-1}) = g(x_i^+) - g(x_{i-1}^+) = g(x_i) - g(x_{i-1}).$$

Finally, if $n \geq 2$, then x_{n-1} is a continuity point and

$$\mu(E_n) = h(b) - h(x_{n-1}) = (g(b) - g(a)) - (g(x_{n-1}^+) - g(a)) = g(x_n) - g(x_{n-1}).$$

Thus in every case $\mu(E_i) = g(x_i) - g(x_{i-1})$ for $1 \leq i \leq n$.

Let $C = \{c_1, \dots, c_n\}$ be any choice of points with $c_i \in [x_{i-1}, x_i]$. Since $[a, b]$ is the disjoint union of the sets E_i , and f is bounded and Borel measurable while μ is finite, the additivity over domains of integration gives

$$\int_{[a,b]} f d\mu - S(f, g, P, C) = \sum_{i=1}^n \left[\int_{E_i} f d\mu - f(c_i) \mu(E_i) \right] = \sum_{i=1}^n \int_{E_i} [f - f(c_i)] d\mu.$$

For $x \in E_i \subseteq [x_{i-1}, x_i]$ we have $|x - c_i| \leq x_i - x_{i-1} < \delta \leq \delta_1$, so that $|f(x) - f(c_i)| < \epsilon$. Therefore

$$\left| \int_{[a,b]} f d\mu - S(f, g, P, C) \right| \leq \sum_{i=1}^n \epsilon \mu(E_i) = \epsilon (g(b) - g(a)).$$

On the other hand $\|P\| < \delta \leq \delta_2$, so $|S(f, g, P, C) - A| < \epsilon$, and the triangle inequality yields

$$\left| \int_{[a,b]} f d\mu - A \right| \leq \epsilon (g(b) - g(a)) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, $\int_{[a,b]} f d\mu = A = \int_a^b f(x) dg(x)$, and this holds for every $f \in C[a, b]$.

Uniqueness. The measure μ is the only Borel measure with this property. Indeed, let μ and ν be finite Borel measures on $\mathcal{B}([a, b])$ with $\int f d\mu = \int f d\nu$ for all $f \in C[a, b]$. Fix $x \in [a, b)$ and, for large natural numbers k , let f_k be the continuous function that equals 1 on $[a, x]$, equals 0 on $[x + 1/k, b]$ and is linear in between. Then $\{f_k\}$ is uniformly bounded by 1 and converges pointwise on $[a, b]$ to $\chi_{[a,x]}$. Since μ and ν are finite, the Bounded Convergence Theorem gives

$$\mu[a, x] = \lim_{k \rightarrow \infty} \int f_k d\mu = \lim_{k \rightarrow \infty} \int f_k d\nu = \nu[a, x],$$

while $\mu[a, b] = \nu[a, b]$ follows by taking $f \equiv 1$. So μ and ν have the same cumulative distribution function and hence, by the uniqueness assertion of Proposition 25, $\mu = \nu$.

Problem 20.44. If the Borel measure μ is absolutely continuous with respect to Lebesgue measure, show that its Radon-Nikodym derivative is the derivative of its cumulative distribution function.

Solution. Let $I = [a, b]$, let μ be a Borel measure on $\mathcal{B}(I)$, that is, a finite measure on the Borel subsets of I , and let m denote Lebesgue measure on $\mathcal{B}(I)$. Assume $\mu \ll m$. Let

$$g_\mu(x) = \mu[a, x] \quad \text{for } x \in I$$

be the cumulative distribution function of μ . We must show that g_μ is differentiable at almost every point of (a, b) and that g'_μ is a Radon-Nikodym derivative of μ with respect to m ; since the Radon-Nikodym derivative is determined only up to a set of Lebesgue measure zero (Problem 50 of Chapter 18), this is the precise meaning of the assertion.

Both μ and m are finite measures on $\mathcal{B}(I)$, hence σ -finite, and $\mu \ll m$. By the Radon-Nikodym Theorem there is a nonnegative function f that is integrable over I with respect to m and for which

$$\mu(E) = \int_E f \, dm \quad \text{for all } E \in \mathcal{B}(I),$$

and f is the Radon-Nikodym derivative $d\mu/dm$, unique up to equality almost everywhere with respect to m .

Take $E = [a, x]$ for $x \in I$. Since the singleton $\{a\}$ has Lebesgue measure zero, and integration over a set of measure zero contributes nothing,

$$g_\mu(x) = \mu[a, x] = \int_{[a, x]} f \, dm = \int_a^x f \quad \text{for all } x \in [a, b].$$

Thus g_μ is the indefinite integral of the integrable function f , translated by the constant $g_\mu(a) = \mu(\{a\}) = 0$; note in passing that $g_\mu(a) = \int_{\{a\}} f \, dm = 0$, consistent with $\mu \ll m$ and $m(\{a\}) = 0$.

Now apply Theorem 14 of Chapter 6: if f is integrable over the closed, bounded interval $[a, b]$, then

$$\frac{d}{dx} \left[\int_a^x f \right] = f(x) \quad \text{for almost all } x \in (a, b).$$

Therefore g_μ is differentiable at almost every point of (a, b) and

$$g'_\mu(x) = f(x) = \frac{d\mu}{dm}(x) \quad \text{for almost all } x \in (a, b).$$

Hence the Radon-Nikodym derivative of μ with respect to Lebesgue measure is, up to a set of measure zero, the derivative of the cumulative distribution function of μ . Equivalently, g'_μ is nonnegative and integrable over $[a, b]$ and

$$\mu(E) = \int_E g'_\mu \, dm \quad \text{for all } E \in \mathcal{B}(I).$$

Two remarks. First, the differentiability almost everywhere of g_μ can also be seen from Proposition 26, by which $\mu \ll m$ forces g_μ to be absolutely continuous on $[a, b]$, and an absolutely continuous function is differentiable almost everywhere; Theorem 10 of Chapter 6 then gives $g_\mu(x) - g_\mu(a) = \int_a^x g'_\mu$, which is the displayed formula for E an interval, and the general case follows from the uniqueness assertion of the Caratheodory-Hahn Theorem, since the intervals form a semiring generating $\mathcal{B}(I)$ and both set functions are finite measures agreeing on it. Second, this is exactly the fact used in the text to justify formula (31): when g is increasing and absolutely continuous, g' is the Radon-Nikodym derivative of μ_g with respect to m , so that $\int_{[a, b]} f \, dg = \int_{[a, b]} f g' \, dm$.

Problem 20.45. For a finite measure μ on the collection $\mathcal{B}(\mathbf{R})$ of all Borel subsets of $\mathbf{R}$, define $g: \mathbf{R} \rightarrow \mathbf{R}$ by setting $g(x) = \mu(-\infty, x]$. Show that each bounded, increasing function $g: \mathbf{R} \rightarrow \mathbf{R}$ that is continuous on the right and has $\lim_{x \rightarrow -\infty} g(x) = 0$ is the cumulative distribution function of a unique finite Borel measure on $\mathcal{B}(\mathbf{R})$.

Solution. Call $g(x) = \mu(-\infty, x]$ the cumulative distribution function of the finite Borel measure μ on $\mathcal{B}(\mathbf{R})$.

First observe that a cumulative distribution function does have the four listed properties, so that the class described in the problem is exactly the right one. Indeed, $0 \leq g \leq \mu(\mathbf{R}) < \infty$ and g is increasing by the monotonicity of μ . If $\{x_k\}$ is a decreasing sequence converging to x , then $\bigcap_{k=1}^{\infty} (-\infty, x_k] = (-\infty, x]$, so by the continuity of measure (permissible since μ is finite) $g(x_k) \rightarrow g(x)$; since g is increasing this gives continuity on the right. Similarly $\bigcap_{k=1}^{\infty} (-\infty, -k] = \emptyset$, so $g(-k) \rightarrow 0$, and monotonicity upgrades this to $\lim_{x \rightarrow -\infty} g(x) = 0$.

Now let $g: \mathbf{R} \rightarrow \mathbf{R}$ be bounded, increasing, continuous on the right, with $\lim_{x \rightarrow -\infty} g(x) = 0$. We construct the measure.

The semiring and the premeasure. Let

$$\mathcal{S} = \{\emptyset\} \cup \{(c, d] : -\infty < c < d < \infty\}.$$

Then $\mathcal{S}$ is a semiring of subsets of $\mathbf{R}$: it is closed under intersection, and the relative complement of two of its members is the union of at most two disjoint members of $\mathcal{S}$. Define $\nu_0: \mathcal{S} \rightarrow [0, \infty)$ by

$$\nu_0(\emptyset) = 0, \quad \nu_0((c, d]) = g(d) - g(c),$$

which is nonnegative because g is increasing.

We claim ν_0 is a premeasure, that is, finitely additive and countably monotone. For finite additivity, suppose $(c, d]$ is the disjoint union of $(c_1, d_1], \dots, (c_n, d_n]$. Relabelling, necessarily $c = c_1$, $d_k = c_{k+1}$ for $1 \leq k \leq n-1$, and $d_n = d$, so that the sum $\sum_{k=1}^n [g(d_k) - g(c_k)]$ telescopes to $g(d) - g(c)$. For countable monotonicity, suppose $(c, d] \subseteq \bigcup_{k=1}^{\infty} (c_k, d_k]$; we must show

$$g(d) - g(c) \leq \sum_{k=1}^{\infty} [g(d_k) - g(c_k)].$$

This is inequality (29) of the proof of Proposition 25, established in Problem 39; we recall the argument, which uses only right continuity and monotonicity of g and the compactness of closed, bounded intervals. Let $\epsilon > 0$. By continuity on the right of g at c , choose $\delta > 0$ with $c + \delta < d$ and $g(c + \delta) < g(c) + \epsilon$. By continuity on the right at each d_k , choose $\eta_k > 0$ with $g(d_k + \eta_k) < g(d_k) + \epsilon \cdot 2^{-k}$. The open intervals $(c_k, d_k + \eta_k)$ contain the sets $(c_k, d_k]$ and hence cover the compact interval $[c + \delta, d]$; let F be a finite set of indices for which $\{(c_k, d_k + \eta_k)\}_{k \in F}$ still covers $[c + \delta, d]$.

We now chain these finitely many intervals. In general, let g be increasing and let the closed, bounded interval $[\alpha, \beta]$ be covered by finitely many open intervals (a_i, b_i) , $i \in F$; we claim $g(\beta) - g(\alpha) \leq \sum_{i \in F} [g(b_i) - g(a_i)]$. Choose $i_1 \in F$ with $\alpha \in (a_{i_1}, b_{i_1})$. Having chosen $i_1, \dots, i_j$, stop if $b_{i_j} > \beta$; otherwise $b_{i_j} \in (\alpha, \beta]$ lies in $[\alpha, \beta]$, so we may choose $i_{j+1} \in F$ with $b_{i_j} \in (a_{i_{j+1}}, b_{i_{j+1}})$, and then $b_{i_{j+1}} > b_{i_j}$. Since the numbers $b_{i_1} < b_{i_2} < \dots$ are strictly increasing they are distinct, so the indices $i_1, i_2, \dots$ are distinct and the process stops after some $m \leq \#F$ steps, with $b_{i_m} > \beta$. Because $a_{i_{j+1}} < b_{i_j}$ for $1 \leq j \leq m-1$ and g is increasing, each bracket in

$$\sum_{j=1}^m [g(b_{i_j}) - g(a_{i_j})] = [g(b_{i_m}) - g(a_{i_1})] + \sum_{j=1}^{m-1} [g(b_{i_j}) - g(a_{i_{j+1}})]$$

in the last sum is nonnegative, whence $\sum_{j=1}^m [g(b_{i_j}) - g(a_{i_j})] \geq g(b_{i_m}) - g(a_{i_1}) \geq g(\beta) - g(\alpha)$, using $b_{i_m} > \beta$ and $a_{i_1} < \alpha$. Discarding the remaining indices of F , whose terms $g(b_i) - g(a_i)$ are nonnegative, proves the claim. Applying it to $[\alpha, \beta] = [c + \delta, d]$ and the intervals $(c_k, d_k + \eta_k)$, $k \in F$, we get

$$g(d) - g(c) < \epsilon + [g(d) - g(c + \delta)] \leq \epsilon + \sum_{k \in F} [g(d_k + \eta_k) - g(c_k)] \leq 2\epsilon + \sum_{k=1}^{\infty} [g(d_k) - g(c_k)].$$

Since $\epsilon > 0$ is arbitrary, the asserted inequality holds. Thus ν_0 is a premeasure on the semiring $\mathcal{S}$.

Existence. By the Caratheodory-Hahn Theorem, the Caratheodory measure induced by ν_0 is an extension of ν_0 to the σ -algebra $\mathcal{M}$ of ν_0^* -measurable sets, and $\mathcal{M}$ contains $\mathcal{S}$. Since a σ -algebra containing

$\mathcal{S}$ contains every open interval, $(\alpha, \beta) = \bigcup_k (\alpha, \beta - 1/k]$, and every open subset of $\mathbf{R}$ is a countable union of open intervals, $\mathcal{M}$ contains the topology of $\mathbf{R}$ and hence contains $\mathcal{B}(\mathbf{R})$, the smallest σ -algebra containing the open sets. Let ν be the restriction of the Caratheodory measure to $\mathcal{B}(\mathbf{R})$. Then ν is a measure on $\mathcal{B}(\mathbf{R})$ with

$$\nu((c, d]) = g(d) - g(c) \quad \text{for all } c < d.$$

The measure ν is finite: $\mathbf{R} = \bigcup_{n=1}^{\infty} (-n, n]$ is an ascending union, so by the continuity of measure

$$\nu(\mathbf{R}) = \lim_{n \rightarrow \infty} [g(n) - g(-n)] = \lim_{n \rightarrow \infty} g(n) \leq \sup_{\mathbf{R}} g < \infty,$$

using $\lim_{x \rightarrow -\infty} g(x) = 0$ and the boundedness of g . Finally, for each $x \in \mathbf{R}$ the sets $(-n, x]$, $n > |x|$, ascend to $(-\infty, x]$, so again by the continuity of measure

$$\nu(-\infty, x] = \lim_{n \rightarrow \infty} \nu((-n, x]) = \lim_{n \rightarrow \infty} [g(x) - g(-n)] = g(x).$$

Thus g is the cumulative distribution function of the finite Borel measure ν .

Uniqueness. Suppose ν_1 and ν_2 are finite Borel measures on $\mathcal{B}(\mathbf{R})$ both having cumulative distribution function g . For $c < d$, since $(c, d] = (-\infty, d] \sim (-\infty, c]$ and the measures are finite, excision gives

$$\nu_i((c, d]) = g(d) - g(c) \quad (i = 1, 2),$$

so ν_1 and ν_2 agree on $\mathcal{S}$. Both are finite, hence σ -finite, and the smallest σ -algebra containing $\mathcal{S}$ is $\mathcal{B}(\mathbf{R})$, as observed above (each $(c, d] = \bigcap_k (c, d + 1/k)$ is a Borel set, so that σ -algebra is contained in $\mathcal{B}(\mathbf{R})$ as well). By Corollary 14 of Chapter 17, two σ -finite measures on that σ -algebra that agree on $\mathcal{S}$ are equal. Hence $\nu_1 = \nu_2$, and the measure with cumulative distribution function g is unique.

Problem 20.46. Show that in the definition of Hausdorff measure one can take the coverings to be by open sets or by closed sets.

Solution. Let (X, ρ) be a metric space, let $\alpha > 0$, let $\epsilon > 0$ and let $E \subseteq X$. Recall

$$H_{\alpha}^{(\epsilon)}(E) = \inf \sum_{k=1}^{\infty} [\text{diam}(A_k)]^{\alpha},$$

the infimum taken over all countable collections $\{A_k\}_{k=1}^{\infty}$ of subsets of X that cover E and have $\text{diam}(A_k) < \epsilon$ for every k . Denote by $F_{\alpha}^{(\epsilon)}(E)$ and $G_{\alpha}^{(\epsilon)}(E)$ the corresponding infima taken over such covers by closed sets, respectively by open sets. Since $H_{\alpha}^{*}(E) = \lim_{\epsilon \rightarrow 0} H_{\alpha}^{(\epsilon)}(E)$, it suffices to prove that for each $\epsilon > 0$,

$$F_{\alpha}^{(\epsilon)}(E) = H_{\alpha}^{(\epsilon)}(E) = G_{\alpha}^{(\epsilon)}(E).$$

In each case the inequality $H_{\alpha}^{(\epsilon)}(E) \leq F_{\alpha}^{(\epsilon)}(E)$ and $H_{\alpha}^{(\epsilon)}(E) \leq G_{\alpha}^{(\epsilon)}(E)$ is immediate, since the covers by closed sets and the covers by open sets are subfamilies of the covers over which the infimum defining $H_{\alpha}^{(\epsilon)}(E)$ is taken, and an infimum over a smaller family is at least as large. So only the reverse inequalities require proof.

Closed coverings. We first note that for every subset A of X ,

$$\text{diam}(\overline{A}) = \text{diam}(A).$$

Since $A \subseteq \overline{A}$, we have $\text{diam}(A) \leq \text{diam}(\overline{A})$. Conversely, let $u, v \in \overline{A}$ and $\eta > 0$; choose $u', v' \in A$ with $\rho(u, u') < \eta$ and $\rho(v, v') < \eta$. Then

$$\rho(u, v) \leq \rho(u, u') + \rho(u', v') + \rho(v', v) < \text{diam}(A) + 2\eta.$$

As $\eta > 0$ is arbitrary, $\rho(u, v) \leq \text{diam}(A)$, and taking the supremum over $u, v \in \overline{A}$ gives $\text{diam}(\overline{A}) \leq \text{diam}(A)$.

Now let $\{A_k\}$ be any countable cover of E with $\text{diam}(A_k) < \epsilon$ for all k . Then $\{\overline{A_k}\}$ is a countable cover of E by closed sets with $\text{diam}(\overline{A_k}) = \text{diam}(A_k) < \epsilon$, and

$$\sum_{k=1}^{\infty} [\text{diam}(\overline{A_k})]^\alpha = \sum_{k=1}^{\infty} [\text{diam}(A_k)]^\alpha.$$

Hence $F_\alpha^{(\epsilon)}(E) \leq \sum_k [\text{diam}(A_k)]^\alpha$, and taking the infimum over all such covers, $F_\alpha^{(\epsilon)}(E) \leq H_\alpha^{(\epsilon)}(E)$. Therefore $F_\alpha^{(\epsilon)} = H_\alpha^{(\epsilon)}$.

Open coverings. Let $\eta > 0$. We may assume $H_\alpha^{(\epsilon)}(E) < \infty$, for otherwise the desired inequality $G_\alpha^{(\epsilon)}(E) \leq H_\alpha^{(\epsilon)}(E)$ is trivial. Choose a countable cover $\{A_k\}$ of E with $d_k = \text{diam}(A_k) < \epsilon$ for all k and

$$\sum_{k=1}^{\infty} d_k^\alpha < H_\alpha^{(\epsilon)}(E) + \eta.$$

If $A_k = \emptyset$ we simply put $O_k = \emptyset$, which is open and has diameter 0; so we may assume in what follows that each A_k is nonempty. Since $\alpha > 0$, the function $t \mapsto t^\alpha$ is continuous on $[0, \infty)$; therefore, for each k we may choose $\delta_k > 0$ so small that

$$d_k + 2\delta_k < \epsilon \quad \text{and} \quad (d_k + 2\delta_k)^\alpha < d_k^\alpha + \eta \cdot 2^{-k}.$$

Define

$$O_k = \{x \in X : \rho(x, A_k) < \delta_k\},$$

where $\rho(x, A_k) = \inf_{u \in A_k} \rho(x, u)$. The function $x \mapsto \rho(x, A_k)$ is continuous on X , indeed it satisfies $|\rho(x, A_k) - \rho(y, A_k)| \leq \rho(x, y)$, so O_k is open, and clearly $A_k \subseteq O_k$. Moreover $\text{diam}(O_k) \leq d_k + 2\delta_k$: if $u, v \in O_k$, choose $u', v' \in A_k$ with $\rho(u, u') < \delta_k$ and $\rho(v, v') < \delta_k$, so that

$$\rho(u, v) \leq \rho(u, u') + \rho(u', v') + \rho(v', v) < \delta_k + d_k + \delta_k.$$

In particular $\text{diam}(O_k) < \epsilon$. Thus $\{O_k\}$ is a countable cover of E by open sets, each of diameter less than ϵ , and

$$\sum_{k=1}^{\infty} [\text{diam}(O_k)]^\alpha \leq \sum_{k=1}^{\infty} (d_k + 2\delta_k)^\alpha \leq \sum_{k=1}^{\infty} d_k^\alpha + \eta < H_\alpha^{(\epsilon)}(E) + 2\eta.$$

Hence $G_\alpha^{(\epsilon)}(E) < H_\alpha^{(\epsilon)}(E) + 2\eta$, and since $\eta > 0$ is arbitrary, $G_\alpha^{(\epsilon)}(E) \leq H_\alpha^{(\epsilon)}(E)$. Therefore $G_\alpha^{(\epsilon)} = H_\alpha^{(\epsilon)}$. Since the three set functions agree for every $\epsilon > 0$, their limits as $\epsilon \rightarrow 0$ agree, so the outer measure H_α^* , and with it the Hausdorff measure H_α obtained by restricting H_α^* to $\mathcal{B}(X)$, is unchanged if in the definition we admit only coverings by open sets, or only coverings by closed sets.

Problem 20.47. Show that the set function outer Hausdorff measure H_α^* is countably monotone.

Solution. Let (X, ρ) be a metric space and $\alpha > 0$. We must show that if $E \subseteq X$ and $\{E_n\}_{n=1}^\infty$ is a countable collection of subsets of X for which

$$E \subseteq \bigcup_{n=1}^{\infty} E_n,$$

then

$$H_\alpha^*(E) \leq \sum_{n=1}^{\infty} H_\alpha^*(E_n).$$

Fix $\epsilon > 0$. We first show that the set function $H_\alpha^{(\epsilon)}$ is countably monotone. We may assume the right-hand sum $\sum_n H_\alpha^*(E_n)$ is finite, since otherwise there is nothing to prove; in particular each $H_\alpha^*(E_n)$ is finite, and since $H_\alpha^{(\epsilon)}(E_n) \leq H_\alpha^*(E_n)$, each $H_\alpha^{(\epsilon)}(E_n)$ is finite as well.

Let $\eta > 0$. For each natural number n , by the definition of $H_\alpha^{(\epsilon)}(E_n)$ as an infimum, there is a countable collection $\{A_{n,k}\}_{k=1}^\infty$ of subsets of X that covers E_n , with $\text{diam}(A_{n,k}) < \epsilon$ for all k , and

$$\sum_{k=1}^{\infty} [\text{diam}(A_{n,k})]^\alpha < H_\alpha^{(\epsilon)}(E_n) + \eta \cdot 2^{-n}.$$

The doubly indexed collection $\{A_{n,k}\}_{n,k \geq 1}$ is countable, since a countable union of countable collections is countable, each of its members has diameter less than ϵ , and it covers $\bigcup_n E_n$ and therefore covers E . Hence it is admissible in the infimum defining $H_\alpha^{(\epsilon)}(E)$, and since all the terms are nonnegative the double series may be summed in any order:

$$H_\alpha^{(\epsilon)}(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [\text{diam}(A_{n,k})]^\alpha \leq \sum_{n=1}^{\infty} [H_\alpha^{(\epsilon)}(E_n) + \eta \cdot 2^{-n}] = \sum_{n=1}^{\infty} H_\alpha^{(\epsilon)}(E_n) + \eta.$$

Since $\eta > 0$ is arbitrary,

$$H_\alpha^{(\epsilon)}(E) \leq \sum_{n=1}^{\infty} H_\alpha^{(\epsilon)}(E_n).$$

Now, for each n and each $\epsilon > 0$ we have $H_\alpha^{(\epsilon)}(E_n) \leq \sup_{\epsilon' > 0} H_\alpha^{(\epsilon')}(E_n) = H_\alpha^*(E_n)$. Therefore

$$H_\alpha^{(\epsilon)}(E) \leq \sum_{n=1}^{\infty} H_\alpha^*(E_n) \quad \text{for every } \epsilon > 0.$$

The right-hand side does not depend on ϵ , so taking the supremum over $\epsilon > 0$ on the left, and recalling that $H_\alpha^*(E) = \sup_{\epsilon > 0} H_\alpha^{(\epsilon)}(E) = \lim_{\epsilon \rightarrow 0} H_\alpha^{(\epsilon)}(E)$, we conclude

$$H_\alpha^*(E) \leq \sum_{n=1}^{\infty} H_\alpha^*(E_n).$$

Thus H_α^* is countably monotone.

Two remarks complete the verification, made in the text, that H_α^* is an outer measure. First, H_α^* is monotone: if $E \subseteq F$, then every cover of F by sets of diameter less than ϵ is such a cover of E , so $H_\alpha^{(\epsilon)}(E) \leq H_\alpha^{(\epsilon)}(F)$ for each ϵ , and hence $H_\alpha^*(E) \leq H_\alpha^*(F)$. In fact monotonicity is already a consequence of countable monotonicity, taking $E_1 = F$ and $E_n = \emptyset$ for $n \geq 2$. Second, $H_\alpha^*(\emptyset) = 0$, since for each $\epsilon > 0$ the collection consisting of the empty set alone covers $\emptyset$ and has $\sum [\text{diam}(A_k)]^\alpha = 0$, so $H_\alpha^{(\epsilon)}(\emptyset) = 0$.

Problem 20.48. In the plane $\mathbf{R}^2$ show that a bounded set may be enclosed in a ball of the same diameter. Use this to show that for a bounded subset A of $\mathbf{R}^2$, $H_2(A) \geq (4/\pi) \cdot \mu_2(A)$, where μ_2 is Lebesgue measure on $\mathbf{R}^2$.

Solution. The first assertion, read literally, is false, and we begin by saying why; we then prove the statement that it was invoked for, namely the isodiametric inequality, and from it deduce the inequality for H_2 exactly as the problem intends.

A counterexample to the enclosing claim. Let T be the set of the three vertices v_1, v_2, v_3 of an equilateral triangle of side length 1, so that $\text{diam}(T) = 1$. Suppose T were contained in a closed ball $\overline{B}(c, 1/2)$. Let $\bar{v} = (v_1 + v_2 + v_3)/3$ be the centroid. Expanding, for any $c \in \mathbf{R}^2$,

$$\sum_{i=1}^3 \|v_i - c\|^2 = \sum_{i=1}^3 \|v_i - \bar{v}\|^2 + 3\|\bar{v} - c\|^2 \geq \sum_{i=1}^3 \|v_i - \bar{v}\|^2,$$

because the cross term $2\langle \sum_i (v_i - \bar{v}), \bar{v} - c \rangle$ vanishes. For the equilateral triangle of side 1, $\|v_i - \bar{v}\| = 1/\sqrt{3}$, so the right-hand side equals 1. Hence $\max_i \|v_i - c\| \geq 1/\sqrt{3} > 1/2$ for every c , and no ball of

diameter 1 contains T . The same example shows the claim fails for the full triangle, and by Jung's theorem $1/\sqrt{3}$ is the correct radius here.

What the argument actually needs is only the following consequence that a disk of diameter d would have supplied, and this is true.

The isodiametric inequality in the plane. For every bounded set $S \subseteq \mathbf{R}^2$ with $d = \text{diam}(S)$,

$$\mu_2^*(S) \leq \frac{\pi}{4} d^2,$$

that is, S has outer measure no larger than that of a disk of the same diameter, even though S need not fit inside such a disk.

Reduction to a compact convex set. Let $K = \overline{\text{conv}}(S)$, the closure of the convex hull of S . Then K is compact, convex and $\mu_2^*(S) \leq \mu_2(K)$. Moreover $\text{diam}(K) = d$. Indeed, if $u = \sum_i \lambda_i x_i$ and $v = \sum_j \nu_j y_j$ are convex combinations of points of S , then

$$\|u - v\| = \left\| \sum_i \lambda_i (x_i - v) \right\| \leq \max_i \|x_i - v\| = \max_i \left\| \sum_j \nu_j (x_i - y_j) \right\| \leq \max_{i,j} \|x_i - y_j\| \leq d,$$

so $\text{diam}(\text{conv } S) \leq d$, and passing to the closure does not increase the diameter, as shown in Problem 46. If K has empty interior it is contained in a line and $\mu_2(K) = 0$, in which case the inequality is trivial; the argument below does not, however, require any such case distinction.

A symmetrization lemma. Let $C \subseteq \mathbf{R}^2$ be a bounded convex set. For x in the projection $P = \{x : C_x \neq \emptyset\}$ of C on the first axis, where $C_x = \{y : (x, y) \in C\}$, put $a(x) = \inf C_x$, $b(x) = \sup C_x$ and $\ell(x) = b(x) - a(x)$; each C_x is an interval, since C is convex, so $\mu_1(C_x) = \ell(x)$. Define the Steiner symmetral

$$C^* = \{(x, y) : x \in P, |y| \leq \ell(x)/2\}.$$

We claim: (i) C^* is convex and symmetric about the first axis; (ii) C^* is measurable, and if C is measurable then $\mu_2(C^*) = \mu_2(C)$; (iii) $\text{diam}(C^*) \leq \text{diam}(C)$.

For (i): P is an interval, and convexity of C gives $C_{tx_1 + (1-t)x_2} \supseteq tC_{x_1} + (1-t)C_{x_2}$ for $t \in [0, 1]$, whence

$$\ell(tx_1 + (1-t)x_2) \geq t\ell(x_1) + (1-t)\ell(x_2),$$

so ℓ is concave on P . If $(x_1, y_1), (x_2, y_2) \in C^*$, then

$$|ty_1 + (1-t)y_2| \leq t|y_1| + (1-t)|y_2| \leq \frac{1}{2}[t\ell(x_1) + (1-t)\ell(x_2)] \leq \frac{1}{2}\ell(tx_1 + (1-t)x_2),$$

so C^* is convex; symmetry in y is built into the definition.

For (ii): a concave function on an interval is continuous on its interior, hence ℓ is measurable, so C^* , the region between the graphs of the measurable functions $\pm\ell/2$ over P , is measurable. If C is measurable, then Tonelli's theorem, applied to the characteristic functions of C and of C^* , gives

$$\mu_2(C) = \int_P \mu_1(C_x) dx = \int_P \ell(x) dx = \int_P \mu_1((C^*)_x) dx = \mu_2(C^*),$$

since $(C^*)_x = [-\ell(x)/2, \ell(x)/2]$ has length $\ell(x)$.

For (iii): write $d' = \text{diam}(C)$ and let $(x_1, y_1), (x_2, y_2) \in C^*$. Then

$$|y_1 - y_2| \leq |y_1| + |y_2| \leq \frac{\ell(x_1) + \ell(x_2)}{2} = \frac{[b(x_1) - a(x_2)] + [b(x_2) - a(x_1)]}{2} \leq M,$$

where $M = \max\{b(x_1) - a(x_2), b(x_2) - a(x_1)\}$, since a mean of two numbers is at most their maximum. Suppose the maximum is $M = b(x_2) - a(x_1)$; the other case is identical with the indices interchanged. Note $M \geq |y_1 - y_2| \geq 0$. Let $\eta > 0$ and choose $s \in C_{x_1}$ with $s < a(x_1) + \eta$ and $t \in C_{x_2}$ with $t > b(x_2) - \eta$. Then (x_1, s) and (x_2, t) belong to C and $t - s > M - 2\eta$, so, if η is small enough that $M - 2\eta \geq 0$,

$$(d')^2 \geq (x_1 - x_2)^2 + (t - s)^2 \geq (x_1 - x_2)^2 + (M - 2\eta)^2.$$

Letting $\eta \rightarrow 0$ gives $(d')^2 \geq (x_1 - x_2)^2 + M^2 \geq (x_1 - x_2)^2 + (y_1 - y_2)^2$. Taking the supremum over pairs of points of C^* yields $\text{diam}(C^*) \leq d'$.

Proof of the isodiametric inequality. Apply the lemma to the compact convex set K to obtain $K_1 = K^*$: convex, measurable, symmetric about the first axis, with $\mu_2(K_1) = \mu_2(K)$ and $\text{diam}(K_1) \leq d$. Now apply the lemma to K_1 with the roles of the two coordinates interchanged, obtaining K_2 : convex, measurable, symmetric about the second axis, with $\mu_2(K_2) = \mu_2(K_1)$ and $\text{diam}(K_2) \leq \text{diam}(K_1) \leq d$. Furthermore K_2 is still symmetric about the first axis: since K_1 is symmetric in y , its horizontal slices at heights y and $-y$ coincide and so have equal length, and the symmetral K_2 replaces each horizontal slice by the interval of that length centred at $x = 0$; hence the slices of K_2 at y and at $-y$ coincide. Being symmetric about both coordinate axes, K_2 is symmetric about the origin: if $(x, y) \in K_2$ then $(-x, y) \in K_2$ and then $(-x, -y) \in K_2$. Consequently, for $(x, y) \in K_2$,

$$2\sqrt{x^2 + y^2} = \|(x, y) - (-x, -y)\| \leq \text{diam}(K_2) \leq d,$$

that is, $K_2 \subseteq \overline{B}(0, d/2)$. Therefore

$$\mu_2^*(S) \leq \mu_2(K) = \mu_2(K_1) = \mu_2(K_2) \leq \mu_2(\overline{B}(0, d/2)) = \pi(d/2)^2 = \frac{\pi}{4}d^2,$$

which is the isodiametric inequality. (This is exactly the estimate that would have followed at once from the false enclosing claim; note that the disk itself shows the constant $\pi/4$ is sharp.)

The inequality for Hausdorff measure. Let A be a bounded subset of $\mathbf{R}^2$; suppose A is a Borel set, so that $H_2(A)$ is defined, the argument otherwise applying verbatim to the outer measures H_2^* and μ_2^* . Fix $\epsilon > 0$ and let $\{A_k\}_{k=1}^\infty$ be any countable cover of A with $\text{diam}(A_k) < \epsilon$ for each k . Each A_k is bounded, so by the countable monotonicity of Lebesgue outer measure and the isodiametric inequality,

$$\mu_2(A) \leq \sum_{k=1}^\infty \mu_2^*(A_k) \leq \frac{\pi}{4} \sum_{k=1}^\infty [\text{diam}(A_k)]^2.$$

Hence

$$\sum_{k=1}^\infty [\text{diam}(A_k)]^2 \geq \frac{4}{\pi} \mu_2(A)$$

for every such cover, and taking the infimum over all of them,

$$H_2^{(\epsilon)}(A) \geq \frac{4}{\pi} \mu_2(A) \quad \text{for every } \epsilon > 0.$$

Letting $\epsilon \rightarrow 0$,

$$H_2(A) = H_2^*(A) = \lim_{\epsilon \rightarrow 0} H_2^{(\epsilon)}(A) \geq \frac{4}{\pi} \mu_2(A),$$

as required. In fact equality holds, $H_2 = (4/\pi)\mu_2$ on $\mathcal{B}(\mathbf{R}^2)$, which is the case $n = 2$ of the assertion of Problem 55 that H_n is a constant multiple of μ_n ; the reverse inequality is proved by covering efficiently and is not needed here.

Problem 20.49. Let (X, ρ) be a metric space and $\alpha > 0$. For $E \subseteq X$, define

$$H'_\alpha(E) = \inf \sum_{k=1}^\infty [\text{diam}(A_k)]^\alpha,$$

where $\{A_k\}_{k=1}^\infty$ is a countable collection of subsets of X that covers E : there is no restriction regarding the size of the diameters of the sets in the cover. Compare the set functions H'_α and H_α .

Solution. Write $H_\alpha^{(\epsilon)}$ and H_α^* for the set functions of the text, so that H_α is the restriction of H_α^* to $\mathcal{B}(X)$. The set function H'_α is obtained by dropping the constraint on the diameters altogether; it is

what is usually called the Hausdorff content. The comparison is as follows.

H'_α is an outer measure. That $H'_\alpha(\emptyset) = 0$ and that H'_α is monotone and countably monotone is proved exactly as in Problem 47, word for word, with the requirement that the covering sets have diameter less than ϵ simply deleted: given $E \subseteq \bigcup_n E_n$ and $\eta > 0$, choose for each n a countable cover $\{A_{n,k}\}_k$ of E_n with $\sum_k [\text{diam } A_{n,k}]^\alpha < H'_\alpha(E_n) + \eta 2^{-n}$; the union of these collections is a countable cover of E , whence $H'_\alpha(E) \leq \sum_n H'_\alpha(E_n) + \eta$.

$H'_\alpha \leq H_\alpha^*$. For each $\epsilon > 0$, every cover admissible in the definition of $H_\alpha^{(\epsilon)}(E)$ is admissible in the definition of $H'_\alpha(E)$, and an infimum over a larger family is no larger; hence

$$H'_\alpha(E) \leq H_\alpha^{(\epsilon)}(E) \quad \text{for all } \epsilon > 0, \quad \text{so} \quad H'_\alpha(E) \leq H_\alpha^*(E) \quad \text{for all } E \subseteq X,$$

and in particular $H'_\alpha \leq H_\alpha$ on $\mathcal{B}(X)$. Furthermore, covering E by the single set E itself,

$$H'_\alpha(E) \leq [\text{diam}(E)]^\alpha,$$

so H'_α is finite on every bounded set, while H_α need not be.

The inequality is generally strict. Take $X = \mathbf{R}$ with the usual metric, $\alpha = 1/2$ and $E = [0, 1]$. On the one hand $H'_{1/2}(E) \leq [\text{diam } E]^{1/2} = 1 < \infty$. On the other hand $H_1(E) = \mu_1(E) = 1 > 0$, since, as noted in the text, H_1 is Lebesgue measure on $\mathbf{R}$. Were $H'_{1/2}(E)$ finite, Proposition 30, applied with $\alpha = 1/2 < 1 = \beta$, would force $H_1(E) = 0$, a contradiction. Hence

$$H'_{1/2}([0, 1]) \leq 1 < \infty = H_1([0, 1]).$$

So H'_α and H_α are genuinely different set functions.

They have the same null sets, and hence define the same Hausdorff dimension. Suppose $H_\alpha^*(E) = 0$; then $H'_\alpha(E) = 0$ by the inequality above. Conversely, suppose $H'_\alpha(E) = 0$, and let $\epsilon > 0$. Let δ satisfy $0 < \delta < \epsilon^\alpha$. Choose a countable cover $\{A_k\}$ of E with $\sum_k [\text{diam } A_k]^\alpha < \delta$. Then for each k we have $[\text{diam } A_k]^\alpha < \delta < \epsilon^\alpha$, so $\text{diam}(A_k) < \epsilon$; the cover is therefore admissible in the definition of $H_\alpha^{(\epsilon)}(E)$, and $H_\alpha^{(\epsilon)}(E) \leq \delta$. Since δ may be taken arbitrarily small, $H_\alpha^{(\epsilon)}(E) = 0$ for every $\epsilon > 0$, and hence $H_\alpha^*(E) = 0$. Thus

$$H'_\alpha(E) = 0 \iff H_\alpha^*(E) = 0,$$

and consequently, for every $E \subseteq \mathbf{R}^n$,

$$\dim_H(E) = \inf\{\beta \geq 0 : H_\beta(E) = 0\} = \inf\{\beta \geq 0 : H'_\beta(E) = 0\}.$$

In this sense H'_α carries the same information about dimension as H_α , and the restriction on diameters in the definition of Hausdorff measure is what converts a set function with the right null sets into one with the right size.

H'_α is not a Caratheodory outer measure, so Theorem 28 fails for it. Take again $X = \mathbf{R}$ and $\alpha = 1/2$, and set $E = [0, 1]$, $F = [2, 3]$, so that $\rho(E, F) = 1 > 0$. We first compute $H'_{1/2}(E) = 1$. The upper bound is the single-set cover. For the lower bound, let $\{A_k\}$ cover E and put $d_k = \text{diam}(A_k)$. Each $A_k \subseteq \mathbf{R}$ is contained in an interval of length d_k , namely $[\inf A_k, \sup A_k]$, so $\mu_1^*(A_k) \leq d_k$, and the countable monotonicity of Lebesgue outer measure gives $\sum_k d_k \geq \mu_1^*(E) = 1$. Write $S = \sum_k d_k$, so $S \geq 1$. If $S < \infty$, then $d_k/S \leq 1$ for each k , whence $(d_k/S)^{1/2} \geq d_k/S$; summing over k gives $S^{-1/2} \sum_k d_k^{1/2} \geq 1$, that is, $\sum_k d_k^{1/2} \geq S^{1/2} \geq 1$. If instead $S = \infty$, then either $d_k \geq 1$ for some k , in which case $\sum_k d_k^{1/2} \geq 1$ at once, or $d_k < 1$ for every k , in which case $d_k^{1/2} \geq d_k$ for every k and $\sum_k d_k^{1/2} \geq S = \infty$. In every case $\sum_k d_k^{1/2} \geq 1$, so $H'_{1/2}(E) \geq 1$. Hence $H'_{1/2}(E) = 1$, and likewise $H'_{1/2}(F) = 1$. But covering $E \cup F$ by the single set $[0, 3]$ gives

$$H'_{1/2}(E \cup F) \leq 3^{1/2} < 2 = H'_{1/2}(E) + H'_{1/2}(F),$$

even though E and F are a positive distance apart. So $H'_{1/2}$ is not a Caratheodory outer measure. The same computation shows that the Borel set E is not $H'_{1/2}$ -measurable: testing the Caratheodory condition with $A = E \cup F$,

$$H'_{1/2}(A) \leq \sqrt{3} < 2 = H'_{1/2}(A \cap E) + H'_{1/2}(A \cap E^c).$$

Thus the measure induced by H'_α through the Caratheodory construction need not be defined on the Borel sets, whereas H_α^* is a Caratheodory outer measure by Proposition 29 and so, by Theorem 28, induces a measure on a σ -algebra containing $\mathcal{B}(X)$.

To summarize: H'_α is an outer measure with $H'_\alpha \leq H_\alpha^*$, it is finite on bounded sets, it agrees with H_α^* as to which sets are null and therefore yields the same Hausdorff dimension, but it is in general strictly smaller than H_α^* , it is not additive on positively separated sets, and Borel sets need not be measurable with respect to it.

20.8 Problems 20.50–20.55

Problem 20.50. Show that each Hausdorff measure H_α on Euclidean space $\mathbb{R}^n$ is invariant with respect to rigid motions.

Solution. A rigid motion of $\mathbb{R}^n$ is a mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ that preserves Euclidean distance,

$$\|T(u) - T(v)\| = \|u - v\| \quad \text{for all } u, v \in \mathbb{R}^n.$$

(Each such T has the form $T(x) = Ax + b$ with A orthogonal, but we use nothing beyond the displayed identity and the surjectivity of T .) Since T preserves distance it is one-to-one, and its inverse T^{-1} is again a distance preserving map of $\mathbb{R}^n$ onto $\mathbb{R}^n$, that is, again a rigid motion. In particular T and T^{-1} are continuous, so T is a homeomorphism of $\mathbb{R}^n$ onto itself.

The point of the proof is that the definition of H_α refers to the metric only through diameters of sets, and diameters are preserved by T . Indeed, for every subset A of $\mathbb{R}^n$,

$$\text{diam}(T(A)) = \sup_{u, v \in A} \|T(u) - T(v)\| = \sup_{u, v \in A} \|u - v\| = \text{diam}(A),$$

since every pair of points of $T(A)$ is of the form $T(u), T(v)$ with $u, v \in A$.

Fix $\alpha > 0$, fix $\epsilon > 0$, and let E be a subset of $\mathbb{R}^n$. Let $\{A_k\}_{k=1}^\infty$ be a countable collection of subsets of $\mathbb{R}^n$ that covers E , each set having diameter less than ϵ . Then $\{T(A_k)\}_{k=1}^\infty$ is a countable collection that covers $T(E)$: if $y \in T(E)$, then $y = T(x)$ with $x \in E$, and $x \in A_k$ for some index k , so that $y \in T(A_k)$. Moreover each $T(A_k)$ has diameter $\text{diam}(A_k) < \epsilon$. Therefore, by the definition of $H_\alpha^{(\epsilon)}$ as an infimum,

$$H_\alpha^{(\epsilon)}(T(E)) \leq \sum_{k=1}^\infty [\text{diam}(T(A_k))]^\alpha = \sum_{k=1}^\infty [\text{diam}(A_k)]^\alpha.$$

Taking the infimum over all such covers $\{A_k\}$ of E gives

$$H_\alpha^{(\epsilon)}(T(E)) \leq H_\alpha^{(\epsilon)}(E).$$

Since T^{-1} is also a rigid motion and $T^{-1}(T(E)) = E$, the same inequality applied to T^{-1} and to the set $T(E)$ yields $H_\alpha^{(\epsilon)}(E) \leq H_\alpha^{(\epsilon)}(T(E))$. Hence

$$H_\alpha^{(\epsilon)}(T(E)) = H_\alpha^{(\epsilon)}(E) \quad \text{for every } \epsilon > 0.$$

Now let $\epsilon \rightarrow 0$. Since $H_\alpha^*(F) = \lim_{\epsilon \rightarrow 0} H_\alpha^{(\epsilon)}(F)$ for every set F , we conclude that

$$H_\alpha^*(T(E)) = H_\alpha^*(E) \quad \text{for every subset } E \text{ of } \mathbb{R}^n.$$

Thus the outer measure H_α^* itself is invariant under rigid motions.

Finally, H_α is by definition the restriction of H_α^* to the Borel σ -algebra $\mathcal{B}(\mathbb{R}^n)$; this restriction is a measure because H_α^* is a Caratheodory outer measure by Proposition 29 and hence every Borel set is H_α^* -measurable by Theorem 28. Since T is a homeomorphism of $\mathbb{R}^n$ onto itself, the collection $\{E \subseteq \mathbb{R}^n : T(E) \in \mathcal{B}(\mathbb{R}^n)\}$ is a σ -algebra (because T is a bijection, image commutes with complements and unions) and it contains every open set (because T is an open mapping, its inverse being continuous). Therefore it contains $\mathcal{B}(\mathbb{R}^n)$, so E Borel implies $T(E)$ Borel. Consequently, for every Borel set E ,

$$H_\alpha(T(E)) = H_\alpha^*(T(E)) = H_\alpha^*(E) = H_\alpha(E).$$

That is, H_α is invariant with respect to rigid motions.

Problem 20.51. Give a direct proof to show that if I is a nontrivial interval in $\mathbb{R}^n$, then $H_n(I) > 0$.

Solution. In the notation of Section 20.2, an interval in $\mathbb{R}^n$ is a product $I = I_1 \times \cdots \times I_n$ of bounded intervals of real numbers, and $\text{vol}(I) = \ell(I_1) \cdots \ell(I_n)$. By a nontrivial such interval we mean one for which every side I_j has positive length; equivalently, I has nonempty interior, and $\text{vol}(I) > 0$. Write μ_n for Lebesgue measure on $\mathbb{R}^n$ and μ_n^* for Lebesgue outer measure, so that $\mu_n(I) = \text{vol}(I)$ by Theorem 11 of Chapter 20. We show the stronger statement

$$H_n^*(E) \geq \mu_n^*(E) \quad \text{for every subset } E \text{ of } \mathbb{R}^n,$$

which applied to $E = I$ gives $H_n(I) \geq \mu_n(I) > 0$.

The only geometric fact required is the following.

Claim. If A is a subset of $\mathbb{R}^n$ with $d = \text{diam}(A) < \infty$, then $\mu_n^*(A) \leq d^n$.

To see this we may assume $A \neq \emptyset$. For $1 \leq j \leq n$ let $\pi_j: \mathbb{R}^n \rightarrow \mathbb{R}$ be the j -th coordinate projection, $\pi_j(x) = x_j$. For $u, v \in A$,

$$|\pi_j(u) - \pi_j(v)| = |u_j - v_j| \leq \|u - v\| \leq d,$$

so the set of real numbers $\pi_j(A)$ has diameter at most d . Setting $a_j = \inf \pi_j(A)$ (a real number, since A is nonempty and $\pi_j(A)$ is bounded), we conclude $\pi_j(A) \subseteq [a_j, a_j + d]$. Hence

$$A \subseteq \prod_{j=1}^n [a_j, a_j + d],$$

and by the monotonicity of Lebesgue outer measure together with the value of Lebesgue measure on an interval (Theorem 11 of Chapter 20),

$$\mu_n^*(A) \leq \prod_{j=1}^n ((a_j + d) - a_j) = d^n.$$

This proves the claim. (If $\text{diam}(A) = \infty$ the inequality $\mu_n^*(A) \leq [\text{diam}(A)]^n$ holds trivially.)

Now fix $\epsilon > 0$ and let $E \subseteq \mathbb{R}^n$. Let $\{A_k\}_{k=1}^\infty$ be any countable collection of subsets of $\mathbb{R}^n$ covering E , each of diameter less than ϵ . By the countable monotonicity of Lebesgue outer measure and the claim,

$$\mu_n^*(E) \leq \sum_{k=1}^\infty \mu_n^*(A_k) \leq \sum_{k=1}^\infty [\text{diam}(A_k)]^n.$$

Taking the infimum over all such covers gives $\mu_n^*(E) \leq H_n^{(\epsilon)}(E)$, and since this holds for every $\epsilon > 0$,

$$\mu_n^*(E) \leq \sup_{\epsilon > 0} H_n^{(\epsilon)}(E) = H_n^*(E).$$

In particular, if I is a nontrivial interval in $\mathbb{R}^n$ with sides $I_1, \dots, I_n$ of positive lengths $\ell_1, \dots, \ell_n$, then I is a Borel set and

$$H_n(I) = H_n^*(I) \geq \mu_n^*(I) = \text{vol}(I) = \prod_{j=1}^n \ell_j > 0.$$

Thus $H_n(I) > 0$. Note that this argument is direct: it uses only the definition of $H_n^{(\epsilon)}$ as an infimum over covers, the elementary containment of a set of diameter d in a cube of side d , and the countable monotonicity of Lebesgue outer measure. No comparison theorem between H_n and μ_n is invoked.

Problem 20.52. Show that in any metric space, H_0 is counting measure.

Solution. Let (X, ρ) be a metric space. For $\alpha = 0$ the defining formula

$$H_0^{(\epsilon)}(E) = \inf \sum_{k=1}^{\infty} [\text{diam}(A_k)]^0$$

is to be read with the usual conventions $t^0 = 1$ for $t \geq 0$ and $[\text{diam}(\emptyset)]^0 = 0$; that is, each nonempty set in the cover contributes 1 to the sum and the empty set contributes 0. Thus, discarding the empty members of a cover,

$$H_0^{(\epsilon)}(E) = \inf \left\{ N \in \{0, 1, 2, \dots\} \cup \{\infty\} : E \text{ can be covered by } N \text{ nonempty sets each of diameter } < \epsilon \right\},$$

the coverings being countable. We show $H_0^*(E)$ equals the number of points of E when E is finite and equals ∞ when E is infinite; that is, H_0^* is counting measure on 2^X , and hence its restriction H_0 to $\mathcal{B}(X)$ is counting measure on the Borel sets.

First, $H_0^{(\epsilon)}(\emptyset) = 0$ for every ϵ , since the empty collection covers $\emptyset$ and its associated sum is 0; hence $H_0^*(\emptyset) = 0$.

Upper bound. Suppose E has exactly m points, $E = \{x_1, \dots, x_m\}$ with the x_i distinct. The singletons $\{x_1\}, \dots, \{x_m\}$ cover E and each has diameter $0 < \epsilon$. Therefore

$$H_0^{(\epsilon)}(E) \leq m \quad \text{for every } \epsilon > 0, \quad \text{so} \quad H_0^*(E) \leq m.$$

Lower bound. Let E be any subset of X and let $x_1, \dots, x_m$ be m distinct points of E . Put

$$\epsilon_0 = \min_{1 \leq i < j \leq m} \rho(x_i, x_j) > 0$$

(when $m = 1$ take ϵ_0 to be any positive number). Let $0 < \epsilon \leq \epsilon_0$ and let $\{A_k\}$ be a countable collection of nonempty sets, each of diameter less than ϵ , that covers E . No single A_k can contain two of the points $x_1, \dots, x_m$: if $x_i, x_j \in A_k$ with $i \neq j$, then $\text{diam}(A_k) \geq \rho(x_i, x_j) \geq \epsilon_0 \geq \epsilon$, contradicting $\text{diam}(A_k) < \epsilon$. On the other hand each x_i lies in some A_k . Hence the map sending i to an index k with $x_i \in A_k$ is one-to-one, so the collection $\{A_k\}$ has at least m members. Therefore

$$\sum_k [\text{diam}(A_k)]^0 \geq m,$$

and taking the infimum over all such covers, $H_0^{(\epsilon)}(E) \geq m$ for all $0 < \epsilon \leq \epsilon_0$. Since $H_0^{(\epsilon)}$ increases as ϵ decreases,

$$H_0^*(E) = \sup_{\epsilon > 0} H_0^{(\epsilon)}(E) \geq m.$$

Combining the two bounds: if E is finite with m points, then $m \leq H_0^*(E) \leq m$, so $H_0^*(E) = m$. If E is infinite, then for every positive integer m we may select m distinct points of E , and the lower bound gives $H_0^*(E) \geq m$; hence $H_0^*(E) = \infty$.

Thus $H_0^*(E)$ is exactly the number of points in E , with the value ∞ for infinite E . Restricting to the Borel σ -algebra, H_0 is counting measure on $\mathcal{B}(X)$.

Problem 20.53. Let $[a, b]$ be a closed, bounded interval of real numbers and $R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y = 0\}$. Show that $H_2(R) = 0$. Then show that $H_1(R) = b - a$. Conclude that the Hausdorff dimension of R is 1.

Solution. Throughout, $\mathbb{R}^2$ carries the Euclidean metric and R is the horizontal segment from $(a, 0)$ to $(b, 0)$. We assume $a < b$, since otherwise R is a single point, $H_\alpha(R) = 0$ for every $\alpha > 0$, and the assertions about H_1 and the dimension are vacuous or trivial. Write $L = b - a > 0$.

A convenient family of covers. For each positive integer m let $h = L/m$ and, for $1 \leq k \leq m$,

$$R_k = \{(x, 0) : a + (k-1)h \leq x \leq a + kh\}.$$

Then $R = \bigcup_{k=1}^m R_k$, and each R_k is a horizontal segment of length h , so $\text{diam}(R_k) = h$.

Part 1: $H_2(R) = 0$. Fix $\epsilon > 0$ and choose m so large that $h = L/m < \epsilon$. The finite collection $\{R_k\}_{k=1}^m$ covers R and each member has diameter $h < \epsilon$; extending it to a countable collection by repeating the one point set $\{(a, 0)\}$, whose diameter is 0, adds nothing to the sums below. Hence

$$H_2^{(\epsilon)}(R) \leq \sum_{k=1}^m [\text{diam}(R_k)]^2 = m \cdot h^2 = m \cdot \frac{L^2}{m^2} = \frac{L^2}{m}.$$

This holds for every m with $L/m < \epsilon$; letting $m \rightarrow \infty$ we get $H_2^{(\epsilon)}(R) = 0$. As this is true for every $\epsilon > 0$,

$$H_2(R) = H_2^*(R) = \sup_{\epsilon > 0} H_2^{(\epsilon)}(R) = 0.$$

Part 2: $H_1(R) = b - a$. For the upper bound, use the same covers: for any $\epsilon > 0$ and any m with $L/m < \epsilon$,

$$H_1^{(\epsilon)}(R) \leq \sum_{k=1}^m \text{diam}(R_k) = m \cdot \frac{L}{m} = L.$$

Hence $H_1^{(\epsilon)}(R) \leq L$ for every $\epsilon > 0$, and so $H_1(R) \leq L = b - a$.

For the lower bound, let $\pi: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection $\pi(x, y) = x$. For any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of the plane,

$$|\pi(u) - \pi(v)| = |u_1 - v_1| \leq \|u - v\|,$$

so π is Lipschitz with constant 1 and therefore $\text{diam}(\pi(A)) \leq \text{diam}(A)$ for every $A \subseteq \mathbb{R}^2$. Also, a nonempty set S of real numbers with $\text{diam}(S) = d < \infty$ is contained in an interval of length d , namely $[\inf S, \inf S + d]$, so its Lebesgue outer measure satisfies $m^*(S) \leq d = \text{diam}(S)$.

Now fix $\epsilon > 0$ and let $\{A_k\}_{k=1}^\infty$ be a countable cover of R by sets of diameter less than ϵ . Then $\{\pi(A_k)\}$ covers $\pi(R) = [a, b]$, and by countable monotonicity of Lebesgue outer measure on $\mathbb{R}$,

$$b - a = m^*([a, b]) \leq \sum_{k=1}^\infty m^*(\pi(A_k)) \leq \sum_{k=1}^\infty \text{diam}(\pi(A_k)) \leq \sum_{k=1}^\infty \text{diam}(A_k).$$

Taking the infimum over all such covers gives $H_1^{(\epsilon)}(R) \geq b - a$ for every $\epsilon > 0$, hence $H_1(R) \geq b - a$. Combining with the upper bound,

$$H_1(R) = b - a.$$

Part 3: $\dim_H(R) = 1$. Recall $\dim_H(R) = \inf\{\beta \geq 0 : H_\beta(R) = 0\}$.

If $\beta > 1$, then since $H_1(R) = b - a < \infty$, Proposition 30 (applied with $\alpha = 1 < \beta$) gives $H_\beta(R) = 0$. Hence every $\beta > 1$ belongs to the set whose infimum defines $\dim_H(R)$, so $\dim_H(R) \leq 1$.

If $0 < \beta < 1$, then $H_\beta(R) \neq 0$; in fact $H_\beta(R) = \infty$. For if $H_\beta(R) < \infty$, then Proposition 30 (applied with $\alpha = \beta < 1$) would give $H_1(R) = 0$, contradicting $H_1(R) = b - a > 0$. And $H_0(R) = \infty \neq 0$ since R is an infinite set and H_0 is counting measure (Problem 52). Finally $H_1(R) = b - a \neq 0$. Therefore no $\beta \in [0, 1]$ lies in the set $\{\beta \geq 0 : H_\beta(R) = 0\}$, so $\dim_H(R) \geq 1$.

Consequently $\dim_H(R) = 1$.

Problem 20.54. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous bounded function on the closed, bounded interval $[a, b]$ that has a continuous bounded derivative on the open interval (a, b) . Consider the graph G of f as a subset of the plane. Show that

$$H_1(G) = \int_a^b \sqrt{1 + |f'(x)|^2} dx.$$

Solution. Assume $a < b$ (otherwise both sides vanish). Write

$$G = \{(x, f(x)) : a \leq x \leq b\} \subseteq \mathbb{R}^2.$$

Since $x \mapsto (x, f(x))$ is continuous on the compact interval $[a, b]$, the set G is compact, hence Borel, and $H_1(G)$ is defined.

Set $M = \sup_{x \in (a, b)} |f'(x)| < \infty$ and define $g: [a, b] \rightarrow \mathbb{R}$ by

$$g(x) = \sqrt{1 + |f'(x)|^2} \quad \text{for } x \in (a, b), \quad g(a) = g(b) = \sqrt{1 + M^2}.$$

Then $1 \leq g \leq \sqrt{1 + M^2}$, so g is bounded, and g is continuous on (a, b) because f' is. Thus the set of points of $[a, b]$ at which g fails to be continuous is contained in the two point set $\{a, b\}$, which has measure zero. By Theorem 8 of Chapter 5 (Lebesgue's criterion), g is Riemann integrable over $[a, b]$, and its Riemann integral coincides with its Lebesgue integral; changing g at the two endpoints changes neither. Denote

$$\Lambda = \int_a^b g = \int_a^b \sqrt{1 + |f'(x)|^2} dx, \quad b - a \leq \Lambda \leq (b - a)\sqrt{1 + M^2} < \infty.$$

For a partition $P: a = x_0 < x_1 < \cdots < x_m = b$ we write $U(g, P)$ and $L(g, P)$ for the upper and lower Darboux sums of g associated with P ; since g is Riemann integrable,

$$\inf_P U(g, P) = \sup_P L(g, P) = \Lambda.$$

For $1 \leq k \leq m$ set $P_k = (x_k, f(x_k))$ and let G_k be the piece of the graph over $[x_{k-1}, x_k]$, that is, $G_k = \{(x, f(x)) : x_{k-1} \leq x \leq x_k\}$.

A diameter estimate. Let $s < t$ be points of $[x_{k-1}, x_k]$. Then f is continuous on $[s, t]$ and differentiable on $(s, t) \subseteq (a, b)$, so by the Mean Value Theorem there is $\xi \in (s, t) \subseteq (x_{k-1}, x_k)$ with $f(t) - f(s) = f'(\xi)(t - s)$. Hence

$$\|(t, f(t)) - (s, f(s))\| = \sqrt{(t - s)^2 + |f'(\xi)|^2(t - s)^2} = (t - s) \sqrt{1 + |f'(\xi)|^2}.$$

Consequently, with $S_k = \sup\{g(\xi) : \xi \in (x_{k-1}, x_k)\}$,

$$\text{diam}(G_k) \leq (x_k - x_{k-1}) S_k \leq (x_k - x_{k-1}) \sqrt{1 + M^2}. \quad (*)$$

Observe also that $S_k \leq \sup_{[x_{k-1}, x_k]} g$, so $\sum_{k=1}^m (x_k - x_{k-1}) S_k \leq U(g, P)$.

The upper bound $H_1(G) \leq \Lambda$. Fix $\epsilon > 0$ and $\eta > 0$. Choose a partition P of $[a, b]$ with $U(g, P) < \Lambda + \eta$. Refining a partition does not increase its upper Darboux sum, so we may refine P and assume in addition that its mesh is smaller than $\epsilon/\sqrt{1 + M^2}$. By (*), every G_k then has diameter less than ϵ . Since $\{G_k\}_{k=1}^m$ covers G ,

$$H_1^{(\epsilon)}(G) \leq \sum_{k=1}^m \text{diam}(G_k) \leq \sum_{k=1}^m (x_k - x_{k-1}) S_k \leq U(g, P) < \Lambda + \eta.$$

As $\eta > 0$ was arbitrary, $H_1^{(\epsilon)}(G) \leq \Lambda$ for every $\epsilon > 0$, and therefore

$$H_1(G) = \sup_{\epsilon > 0} H_1^{(\epsilon)}(G) \leq \Lambda.$$

Two lemmas for the lower bound. First, if $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (or φ mapping $\mathbb{R}^2$ into a line) satisfies $\|\varphi(u) - \varphi(v)\| \leq \|u - v\|$, then $\text{diam}(\varphi(A)) \leq \text{diam}(A)$ for every A ; hence, exactly as in Problem 50, the image of a cover of E by sets of diameter less than ϵ is a cover of $\varphi(E)$ by sets of diameter less than ϵ , so

$$H_1^{(\epsilon)}(\varphi(E)) \leq H_1^{(\epsilon)}(E), \quad \text{and therefore} \quad H_1^*(\varphi(E)) \leq H_1^*(E).$$

Second, if S is the closed segment in the plane joining two points p and q , then $H_1(S) = \|p - q\|$: there is a rigid motion of $\mathbb{R}^2$ carrying S onto the horizontal segment $\{(x, 0) : 0 \leq x \leq \|p - q\|\}$, whose H_1 measure is $\|p - q\|$ by Problem 53, and H_1 is invariant under rigid motions by Problem 50.

The lower bound $H_1(G) \geq \Lambda$. Let $P: a = x_0 < \cdots < x_m = b$ be any partition. Each G_k is compact, hence Borel. For $j < k$ the sets G_j and G_k are disjoint unless $k = j + 1$, in which case $G_j \cap G_{j+1} = \{P_j\}$. Put $\tilde{G}_1 = G_1$ and $\tilde{G}_k = G_k \setminus \{P_{k-1}\}$ for $2 \leq k \leq m$. These are pairwise disjoint Borel sets whose union is G . A one point set has H_1 measure 0 (cover it by itself; its diameter is 0), so by the additivity and monotonicity of the measure H_1 we have $H_1(\tilde{G}_k) = H_1(G_k)$, and therefore

$$H_1(G) = \sum_{k=1}^m H_1(\tilde{G}_k) = \sum_{k=1}^m H_1(G_k).$$

Fix k and let ℓ be the line through P_{k-1} and P_k (these points are distinct, their first coordinates differing), and let π_ℓ be the orthogonal projection of the plane onto ℓ . Orthogonal projection onto a line does not increase distances, so by the first lemma $H_1^*(\pi_\ell(G_k)) \leq H_1^*(G_k)$. Now G_k is the image of the interval $[x_{k-1}, x_k]$ under the continuous map $x \mapsto (x, f(x))$, hence is connected; so $\pi_\ell(G_k)$ is a connected subset of the line ℓ containing the two points P_{k-1} and P_k , and therefore contains the whole segment S_k joining them. By monotonicity and the second lemma,

$$H_1(G_k) = H_1^*(G_k) \geq H_1^*(\pi_\ell(G_k)) \geq H_1^*(S_k) = \|P_k - P_{k-1}\|.$$

By the Mean Value Theorem there is $\xi_k \in (x_{k-1}, x_k)$ with $f(x_k) - f(x_{k-1}) = f'(\xi_k)(x_k - x_{k-1})$, so

$$\|P_k - P_{k-1}\| = (x_k - x_{k-1})\sqrt{1 + |f'(\xi_k)|^2} = (x_k - x_{k-1})g(\xi_k) \geq (x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} g.$$

Summing over k ,

$$H_1(G) = \sum_{k=1}^m H_1(G_k) \geq \sum_{k=1}^m \|P_k - P_{k-1}\| \geq L(g, P).$$

Taking the supremum over all partitions P of $[a, b]$ and using $\sup_P L(g, P) = \Lambda$, we obtain $H_1(G) \geq \Lambda$. Combining the two bounds,

$$H_1(G) = \Lambda = \int_a^b \sqrt{1 + |f'(x)|^2} dx.$$

Problem 20.55. Let J be an interval in $\mathbb{R}^n$, each of whose sides has length 1. Define $\gamma_n = H_n(J)$. Show that if I is any bounded interval in $\mathbb{R}^n$, then $H_n(I) = \gamma_n \cdot \mu_n(I)$. From this infer, using the uniqueness assertion of the Caratheodory-Hahn Theorem, that $H_n = \gamma_n \cdot \mu_n$ on the Borel subsets of $\mathbb{R}^n$.

Solution. Throughout, μ_n denotes Lebesgue measure on $\mathbb{R}^n$ and μ_n^* Lebesgue outer measure. By an interval in $\mathbb{R}^n$ we mean a product $\prod_{j=1}^n I_j$ of intervals of real numbers; a cube of side s is such a product with all I_j of length s . All the sets appearing below are Borel, so H_n is a measure on them.

Step 1: H_n is finite on bounded sets, and $1 \leq \gamma_n \leq n^{n/2}$. Let Q be a cube of side $s > 0$. For a positive integer m , partition Q (or its closure) into m^n closed subcubes of side s/m ; each has diameter $s\sqrt{n}/m$. Given $\epsilon > 0$, choose m with $s\sqrt{n}/m < \epsilon$. Then

$$H_n^{(\epsilon)}(Q) \leq m^n \left(\frac{s\sqrt{n}}{m} \right)^n = n^{n/2} s^n.$$

Hence $H_n^*(Q) \leq n^{n/2} s^n < \infty$. Since every bounded set lies in such a cube, H_n^* is finite on bounded sets. In the other direction, Problem 51 gives $H_n^*(E) \geq \mu_n^*(E)$ for every E , so $H_n(Q) \geq \mu_n(Q) = s^n$. Applying this to a unit cube J gives

$$1 \leq \gamma_n \leq n^{n/2}, \quad \text{in particular } 0 < \gamma_n < \infty.$$

Step 2: bounded Borel subsets of hyperplanes are H_n -null. Let F be a bounded Borel subset of a hyperplane $\Pi = \{x \in \mathbb{R}^n : x_j = c\}$ (the Borel hypothesis enters only through Proposition 30; every set to which Step 2 is applied below is either a bounded interval or a face of a closed bounded interval, hence Borel). If $n = 1$ then Π is a single point and $H_1(F) = 0$ because a one point set is covered by itself, of diameter 0. Let $n \geq 2$. With the metric inherited from $\mathbb{R}^n$, Π is isometric to $\mathbb{R}^{n-1}$, so F is contained in an $(n-1)$ -dimensional cube $K \subseteq \Pi$ of some side s , and K is the union of m^{n-1} subcubes of side s/m , each of diameter $s\sqrt{n-1}/m$. Exactly as in Step 1,

$$H_{n-1}^*(F) \leq H_{n-1}^*(K) \leq m^{n-1} \left(\frac{s\sqrt{n-1}}{m} \right)^{n-1} = (n-1)^{(n-1)/2} s^{n-1} < \infty.$$

Thus $H_{n-1}(F) = H_{n-1}^*(F) < \infty$, and since $0 < n-1 < n$, Proposition 30 gives $H_n(F) = 0$.

Step 3: the type of the interval is irrelevant. Let I be a bounded interval with $\bar{I} = \prod_{j=1}^n [a_j, b_j]$ and $\text{int}(I) = \prod_{j=1}^n (a_j, b_j)$. If some $b_j = a_j$, then I lies in a hyperplane and is bounded, so $H_n(I) = 0 = \gamma_n \mu_n(I)$ by Step 2, and such degenerate intervals need not be considered further. Otherwise $\bar{I} \setminus \text{int}(I)$ is the union of the $2n$ faces of $\bar{I}$, each a compact, hence Borel, bounded subset of a hyperplane, so $H_n(\bar{I} \setminus \text{int}(I)) = 0$ by Step 2 and finite subadditivity. Since $\text{int}(I) \subseteq I \subseteq \bar{I}$, monotonicity and additivity of the measure H_n give

$$H_n(\text{int } I) = H_n(I) = H_n(\bar{I}),$$

and the same three quantities agree for μ_n . Thus in proving $H_n(I) = \gamma_n \mu_n(I)$ we may replace I by any interval with the same closure; we shall use half-open intervals $\prod_j [a_j, b_j)$. In particular every unit cube, of whatever type and wherever located, has H_n measure γ_n : its type is immaterial by the above, and its location is immaterial because H_n is invariant under translations, these being rigid motions (Problem 50).

Step 4: half-open cubes of side $1/m$. Let $Q_0 = [0, 1)^n$ and let m be a positive integer. Then Q_0 is the union of the m^n pairwise disjoint half-open cubes

$$\prod_{j=1}^n \left[\frac{k_j}{m}, \frac{k_j+1}{m} \right), \quad 0 \leq k_j \leq m-1,$$

each of which is a translate of $[0, 1/m)^n$. By the finite additivity of the measure H_n on Borel sets, by translation invariance, and by Step 3 (which gives $H_n(Q_0) = \gamma_n$),

$$\gamma_n = H_n(Q_0) = m^n H_n([0, 1/m)^n), \quad \text{so} \quad H_n([0, 1/m)^n) = \frac{\gamma_n}{m^n} = \gamma_n \mu_n([0, 1/m)^n).$$

By translation invariance again, every half-open cube of side $1/m$ has H_n measure γ_n/m^n .

Step 5: $H_n(I) = \gamma_n \mu_n(I)$ for every bounded interval I . By Step 3 we may assume $I = \prod_{j=1}^n [a_j, b_j)$ with $a_j < b_j$ for all j . For a positive integer m let $\mathcal{G}_m$ be the collection of all cubes

$$\prod_{j=1}^n \left[\frac{k_j}{m}, \frac{k_j+1}{m} \right), \quad k = (k_1, \dots, k_n) \in \mathbb{Z}^n;$$

these are pairwise disjoint and their union is $\mathbb{R}^n$. Let $\mathcal{U}_m$ consist of those members of $\mathcal{G}_m$ that are contained in I and $\mathcal{V}_m$ of those that meet I ; since I is bounded, both collections are finite. Put $U_m = \bigcup \mathcal{U}_m$ and $V_m = \bigcup \mathcal{V}_m$, so that $U_m \subseteq I \subseteq V_m$. Because the cubes are disjoint, finite additivity of H_n and of μ_n , together with Step 4, give

$$H_n(U_m) = (\#\mathcal{U}_m) \frac{\gamma_n}{m^n} = \gamma_n \mu_n(U_m), \quad H_n(V_m) = (\#\mathcal{V}_m) \frac{\gamma_n}{m^n} = \gamma_n \mu_n(V_m).$$

By monotonicity of H_n ,

$$\gamma_n \mu_n(U_m) \leq H_n(I) \leq \gamma_n \mu_n(V_m). \quad (\dagger)$$

We estimate the two extreme terms. A cube $\prod_j [k_j/m, (k_j+1)/m)$ lies in I precisely when $a_j \leq k_j/m$ and $(k_j+1)/m \leq b_j$ for every j ; the union of all such cubes is again a half-open interval whose j -th

side has length at least $b_j - a_j - 2/m$. Hence

$$\mu_n(U_m) \geq \prod_{j=1}^n \max\{b_j - a_j - 2/m, 0\}.$$

Likewise, if $\prod_j [k_j/m, (k_j + 1)/m)$ meets I , then for each j we have $k_j/m < b_j$ and $(k_j + 1)/m > a_j$, so that cube is contained in $[a_j - 1/m, b_j + 1/m)$ in the j -th coordinate. Hence $V_m \subseteq \prod_j [a_j - 1/m, b_j + 1/m)$ and

$$\mu_n(V_m) \leq \prod_{j=1}^n \left(b_j - a_j + \frac{2}{m}\right).$$

Both bounds converge to $\prod_{j=1}^n (b_j - a_j) = \mu_n(I)$ as $m \rightarrow \infty$. Letting $m \rightarrow \infty$ in (†) yields

$$H_n(I) = \gamma_n \mu_n(I).$$

Together with the degenerate case handled in Step 3, this holds for every bounded interval I in $\mathbb{R}^n$.

Step 6: passage to all Borel sets. Let $\mathcal{T}$ denote the collection of bounded intervals in $\mathbb{R}^n$. By Proposition 9 of Chapter 20, $\mathcal{T}$ is a semiring, and by Proposition 10 of Chapter 20 the set function $\text{vol}: \mathcal{T} \rightarrow [0, \infty)$ is a premeasure; it is σ -finite, since $\mathbb{R}^n$ is the union of the countably many cubes $\prod_{j=1}^n [k_j, k_j + 1)$, $k \in \mathbb{Z}^n$, each of finite volume. Since $0 < \gamma_n < \infty$ by Step 1, $\mu = \gamma_n \cdot \text{vol}$ is likewise a σ -finite premeasure on $\mathcal{T}$, and the outer measure it induces is $\gamma_n \mu_n^*$, since taking the infimum of $\sum_m \gamma_n \text{vol}(I_m)$ over covers by members of $\mathcal{T}$ is γ_n times the infimum of $\sum_m \text{vol}(I_m)$. As γ_n is a positive real number, a set satisfies the Caratheodory measurability criterion for $\gamma_n \mu_n^*$ if and only if it satisfies it for μ_n^* ; hence the Caratheodory measure induced by μ is $\gamma_n \mu_n$ on the σ -algebra $\mathcal{L}^n$ of Lebesgue measurable sets, which contains $\mathcal{B}(\mathbb{R}^n)$ by Theorem 11 of Chapter 20. On the other hand, Step 5 together with Theorem 11 of Chapter 20 says precisely that the measure H_n on $\mathcal{B}(\mathbb{R}^n)$ also extends μ , since $\mu_n(I) = \text{vol}(I)$ for $I \in \mathcal{T}$.

The uniqueness assertion of the Caratheodory-Hahn Theorem now identifies the two: a σ -finite premeasure admits only one extension to a measure, so a Borel measure on $\mathbb{R}^n$ is determined by its values on $\mathcal{T}$. That assertion is stated for measures defined on the whole σ -algebra of μ^* -measurable sets, whereas H_n is given to us only on $\mathcal{B}(\mathbb{R}^n)$; we therefore record the argument in the present setting, where it uses nothing beyond the countable additivity and monotonicity of the two measures on Borel sets.

First, $H_n \leq \gamma_n \mu_n$ on $\mathcal{B}(\mathbb{R}^n)$. Let E be Borel, and assume $\mu_n(E) < \infty$, the inequality being trivial otherwise. Let $\eta > 0$. Since Lebesgue outer measure is by definition the outer measure induced by the premeasure vol , there is a countable collection $\{I_m\}_{m=1}^\infty$ of bounded intervals covering E with

$$\sum_{m=1}^\infty \text{vol}(I_m) < \mu_n^*(E) + \eta = \mu_n(E) + \eta.$$

Each I_m is Borel, so by the countable monotonicity of the measure H_n and by Step 5,

$$H_n(E) \leq \sum_{m=1}^\infty H_n(I_m) = \gamma_n \sum_{m=1}^\infty \text{vol}(I_m) < \gamma_n (\mu_n(E) + \eta).$$

Since $\eta > 0$ is arbitrary, $H_n(E) \leq \gamma_n \mu_n(E)$.

Second, this inequality is an equality. Let C be one of the half-open unit cubes $\prod_{j=1}^n [k_j, k_j + 1)$ and let A be a Borel subset of C . By Steps 3 and 4, $H_n(C) = \gamma_n = \gamma_n \mu_n(C)$, a finite number. By the additivity of both measures and the inequality just proved,

$$\gamma_n = H_n(C) = H_n(A) + H_n(C \setminus A) \leq \gamma_n \mu_n(A) + \gamma_n \mu_n(C \setminus A) = \gamma_n \mu_n(C) = \gamma_n.$$

All four terms here are finite, so the two inequalities $H_n(A) \leq \gamma_n \mu_n(A)$ and $H_n(C \setminus A) \leq \gamma_n \mu_n(C \setminus A)$ that were added must each be an equality. Hence $H_n = \gamma_n \mu_n$ on the Borel subsets of C .

Finally, $\mathbb{R}^n$ is the union of the countable, pairwise disjoint family of half-open unit cubes $C_k = \prod_{j=1}^n [k_j, k_j + 1)$, $k \in \mathbb{Z}^n$. For E Borel, countable additivity of the two measures gives

$$H_n(E) = \sum_k H_n(E \cap C_k) = \gamma_n \sum_k \mu_n(E \cap C_k) = \gamma_n \mu_n(E).$$

That is,

$$H_n = \gamma_n \cdot \mu_n \quad \text{on } \mathcal{B}(\mathbb{R}^n).$$

Since $1 \leq \gamma_n \leq n^{n/2}$ by Step 1, H_n is a nonzero finite constant multiple of Lebesgue measure on the Borel sets, as asserted in the text preceding these problems. For $n = 1$ one has $\gamma_1 = 1$, in agreement with the fact, recorded in that text, that H_1 is Lebesgue measure on $\mathbb{R}$. For $n = 2$ the constant is not 1: applying the inequality $H_2(A) \geq (4/\pi) \cdot \mu_2(A)$ of Problem 48 to a unit square gives $\gamma_2 \geq 4/\pi > 1$, which is the sense in which H_n fails to equal Lebesgue measure for $n > 1$.

21 Measure and Topology

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21.1 Problems 21.1–21.7

Problem 21.1. Let X be a locally compact Hausdorff space, and F a set that has closed intersection with each compact subset of X . Show that F is closed.

Solution. We show that F contains each of its points of closure, so that $\overline{F} = F$ and hence F is closed. Let x belong to $\overline{F}$. Since X is locally compact, x has a neighborhood U whose closure $\overline{U}$ is compact. Put

$$K = \overline{U}, \quad E = F \cap K.$$

By hypothesis $E = F \cap K$ is closed, since K is a compact subset of X .

We claim x is a point of closure of E . Indeed, let $\mathcal{O}$ be any neighborhood of x . Then $\mathcal{O} \cap U$ is also a neighborhood of x , being an intersection of two open sets each of which contains x . Since x belongs to $\overline{F}$, every neighborhood of x meets F , and therefore

$$(\mathcal{O} \cap U) \cap F \neq \emptyset.$$

But $U \subseteq \overline{U} = K$, so that $(\mathcal{O} \cap U) \cap F \subseteq \mathcal{O} \cap (F \cap K) = \mathcal{O} \cap E$. Hence $\mathcal{O} \cap E \neq \emptyset$. As $\mathcal{O}$ was an arbitrary neighborhood of x , the point x belongs to $\overline{E}$.

Since E is closed, $\overline{E} = E$, and therefore

$$x \in E = F \cap K \subseteq F.$$

Thus $\overline{F} \subseteq F$, so $\overline{F} = F$ and F is closed.

Two remarks on the hypothesis. First, we read the phrase *closed intersection* to mean that $F \cap K$ is closed as a subset of X . The other possible reading, that $F \cap K$ is closed in the subspace K , is equivalent to it here, because K is a compact subset of the Hausdorff space X and hence closed in X , so that a subset of K is closed in the subspace K if and only if it is closed in X . This is the only place the Hausdorff assumption is used; under the first reading the argument above needs only local compactness, in the form of the existence, for each point of X , of a neighborhood with compact closure. Second, the converse implication holds as well: if F is closed in X and K is a compact subset of X , then K is closed in X (again since X is Hausdorff), so $F \cap K$, being the intersection of two closed sets, is closed. Thus for a subset F of a locally compact Hausdorff space, F is closed if and only if $F \cap K$ is closed for every compact K .

Problem 21.2. Regarding the proof of Proposition 3:

- (i) Show that F and $\overline{V} \sim V$ are closed subsets of $\overline{V}$.
- (ii) Show that the function f is continuous.

Solution. Recall the setting of Proposition 3. K is a compact subset of the locally compact Hausdorff space X and $\mathcal{O}$ is a neighborhood of K ; by Proposition 2 there is a neighborhood V of K with

$$K \subseteq V \subseteq \overline{V} \subseteq \mathcal{O} \quad \text{and} \quad \overline{V} \text{ compact.}$$

The set called F in part (i) is the compact set K of the proposition: it is to the pair $K, \overline{V} \sim V$ that Urysohn's Lemma is applied in the printed proof, so the F of the problem is a misprint for K , and we write K below. Throughout, $\overline{V}$ carries its subspace topology inherited from X . Since $\overline{V}$ is compact and Hausdorff (a subspace of a Hausdorff space is Hausdorff), it is normal, and Urysohn's Lemma is applied to it to produce a continuous $f: \overline{V} \rightarrow \mathbb{R}$ with

$$f = 1 \text{ on } K, \quad f = 0 \text{ on } \overline{V} \sim V, \quad 0 \leq f \leq 1 \text{ on } \overline{V},$$

and f is then extended to all of X by setting $f = 0$ on $X \sim \overline{V}$.

(i) First, K is a compact subset of the Hausdorff space X , and a compact subset of a Hausdorff space is closed. Hence K is closed in X , and therefore

$$K \cap \overline{V} = K$$

(the equality holds because $K \subseteq V \subseteq \overline{V}$) is closed in the subspace $\overline{V}$, by the definition of the subspace topology: the closed subsets of $\overline{V}$ are exactly the sets of the form $C \cap \overline{V}$ with C closed in X . Second, V is open in X , so $V \cap \overline{V} = V$ is open in the subspace $\overline{V}$; consequently its complement in that subspace,

$$\overline{V} \sim V,$$

is closed in $\overline{V}$.

Finally these two closed sets are disjoint, since $K \subseteq V$, so $K \cap (\overline{V} \sim V) = \emptyset$. Thus Urysohn's Lemma may indeed be applied to the pair $K, \overline{V} \sim V$ of disjoint closed subsets of the normal space $\overline{V}$.

(ii) Denote by f the extended function on X : it equals the Urysohn function on $\overline{V}$ and equals 0 on $X \sim \overline{V}$. Observe first that

$$X = \overline{V} \cup (X \sim V),$$

and that both sets in this union are closed in X : $\overline{V}$ is a closure, and $X \sim V$ is the complement of the open set V .

The restriction of f to $\overline{V}$ is continuous, being the function furnished by Urysohn's Lemma.

The restriction of f to $X \sim V$ is identically zero. Indeed, a point of $X \sim V$ either lies outside $\overline{V}$, where f was defined to be 0, or lies in $\overline{V} \sim V$, where the Urysohn function vanishes. A constant function is continuous.

Now let C be a closed subset of $\mathbb{R}$. Then

$$f^{-1}(C) = [(f|_{\overline{V}})^{-1}(C)] \cup [(f|_{X \sim V})^{-1}(C)],$$

since the two sets $\overline{V}$ and $X \sim V$ cover X . By the continuity of the two restrictions, $(f|_{\overline{V}})^{-1}(C)$ is closed in the subspace $\overline{V}$ and $(f|_{X \sim V})^{-1}(C)$ is closed in the subspace $X \sim V$. A subset that is closed in a closed subspace is closed in the whole space: if A is closed in the subspace B and B is closed in X , then $A = C' \cap B$ for some closed $C' \subseteq X$, and an intersection of two closed subsets of X is closed in X . Hence both sets on the right are closed in X , and therefore so is their union $f^{-1}(C)$, being the union of two closed sets.

Since the inverse image under f of every closed subset of $\mathbb{R}$ is closed, f is continuous on X .

For completeness we record why f then has the properties asserted in the proposition. The set $\{x \in X \mid f(x) \neq 0\}$ is contained in V , so

$$\text{supp } f = \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq \overline{V},$$

and a closed subset of the compact set $\overline{V}$ is compact; hence f has compact support and belongs to $C_c(X)$. Moreover $f = 1$ on K , $0 \leq f \leq 1$ on X , and $f = 0$ on $X \sim \mathcal{O}$ because $\overline{V} \subseteq \mathcal{O}$ forces $X \sim \mathcal{O} \subseteq X \sim \overline{V}$, where f vanishes.

Problem 21.3. Let X be a locally compact Hausdorff space and X^* the Alexandroff one-point compactification of X :

- (i) Prove that the subsets of X^* that are either open subsets of X or the complements of compact subsets of X are a topology for X^* .
- (ii) Show that the identity mapping from X to the subspace $X^* \sim \{\omega\}$ is a homeomorphism.
- (iii) Show that X^* is compact and Hausdorff.

Solution. Here $X^* = X \cup \{\omega\}$, where ω is a single point not belonging to X . Write

$$\mathcal{T}^* = \mathcal{T}_1 \cup \mathcal{T}_2, \quad \mathcal{T}_1 = \{U \subseteq X \mid U \text{ open in } X\}, \quad \mathcal{T}_2 = \{X^* \sim K \mid K \subseteq X \text{ compact}\}.$$

Throughout we use the following two facts about the Hausdorff space X : a compact subset of a Hausdorff space is closed, and a closed subset of a compact space is compact.

(i) We verify the three axioms for a topology.

The empty set belongs to $\mathcal{T}_1$, being open in X . The whole set X^* belongs to $\mathcal{T}_2$, since $\emptyset$ is a compact subset of X and $X^* = X^* \sim \emptyset$.

Finite intersections. It suffices to treat the intersection of two members of $\mathcal{T}^*$, and there are three

cases.

If U_1, U_2 belong to $\mathcal{T}_1$, then $U_1 \cap U_2$ is open in X , so belongs to $\mathcal{T}_1$.

If K_1, K_2 are compact subsets of X , then

$$(X^* \sim K_1) \cap (X^* \sim K_2) = X^* \sim (K_1 \cup K_2),$$

and the union of two compact sets is compact; so this belongs to $\mathcal{T}_2$.

If U is open in X and $K \subseteq X$ is compact, then, since $\omega \notin U$,

$$U \cap (X^* \sim K) = U \sim K = U \cap (X \sim K).$$

As X is Hausdorff, K is closed in X , so $X \sim K$ is open in X ; hence this set is open in X and belongs to $\mathcal{T}_1$.

Arbitrary unions. Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be a collection of members of $\mathcal{T}^*$, and split the index set as $\Lambda = A \cup B$, where $W_\alpha = U_\alpha$ is open in X for $\alpha \in A$ and $W_\beta = X^* \sim K_\beta$ with $K_\beta \subseteq X$ compact for $\beta \in B$. Put $U = \bigcup_{\alpha \in A} U_\alpha$, an open subset of X (interpreted as $\emptyset$ if $A = \emptyset$).

If $B = \emptyset$, the union is U , a member of $\mathcal{T}_1$.

If $B \neq \emptyset$, then

$$\bigcup_{\beta \in B} (X^* \sim K_\beta) = X^* \sim \bigcap_{\beta \in B} K_\beta,$$

so, writing $K = \bigcap_{\beta \in B} K_\beta$,

$$\bigcup_{\lambda \in \Lambda} W_\lambda = U \cup (X^* \sim K) = X^* \sim (K \sim U).$$

Fix $\beta_0 \in B$. Each K_β is compact, hence closed in the Hausdorff space X , so K is a closed subset of the compact set K_{β_0} and therefore K is compact. Moreover

$$K \sim U = K \cap (X \sim U)$$

is a closed subset of the compact set K , hence compact. Thus the union is the complement in X^* of a compact subset of X , that is, a member of $\mathcal{T}_2$.

Therefore $\mathcal{T}^*$ is a topology for X^* .

(ii) The set $X^* \sim \{\omega\}$ is exactly X as a set, and the map in question is the identity. So we must only show that the subspace topology that X inherits from X^* is the original topology of X .

A subset of the subspace has the form $W \cap X$ with $W \in \mathcal{T}^*$. If $W = U$ is open in X , then $W \cap X = U$ is open in X . If $W = X^* \sim K$ with K compact, then $W \cap X = X \sim K$, which is open in X because K , being compact in a Hausdorff space, is closed. So every relatively open set is open in X .

Conversely, if U is open in X , then $U \in \mathcal{T}^*$ and $U = U \cap X$ is relatively open.

Hence the two topologies on X coincide, and the identity map is a homeomorphism of X onto the subspace $X^* \sim \{\omega\}$. In particular X is an open subset of X^* .

(iii) We prove compactness first. Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be a cover of X^* by members of $\mathcal{T}^*$. Some W_{λ_0} contains ω . Since ω belongs to no member of $\mathcal{T}_1$ (those are subsets of X), W_{λ_0} must belong to $\mathcal{T}_2$; say

$$W_{\lambda_0} = X^* \sim K, \quad K \subseteq X \text{ compact.}$$

By part (ii), each set $W_\lambda \cap X$ is open in X , and these sets cover X , in particular they cover K . By the compactness of K there are indices $\lambda_1, \dots, \lambda_n$ with

$$K \subseteq \bigcup_{i=1}^n (W_{\lambda_i} \cap X) \subseteq \bigcup_{i=1}^n W_{\lambda_i}.$$

Then $\{W_{\lambda_0}, W_{\lambda_1}, \dots, W_{\lambda_n}\}$ covers X^* , since $W_{\lambda_0} = X^* \sim K$ takes care of everything outside K . Thus every open cover of X^* has a finite subcover, and X^* is compact.

Now the Hausdorff property. Let $u \neq v$ be points of X^* .

If both belong to X , then, since X is Hausdorff, there are disjoint sets U_1, U_2 open in X with $u \in U_1$, $v \in U_2$; these are members of $\mathcal{T}^*$, so they separate u and v in X^* .

Otherwise one of the points is ω ; say $v = \omega$ and $u = x \in X$. Since X is locally compact, x has a neighborhood V in X with $\overline{V}$ compact. (Alternatively invoke Lemma 1 with $\mathcal{O} = X$.) Then V belongs to $\mathcal{T}_1$ and $X^* \sim \overline{V}$ belongs to $\mathcal{T}_2$; they are disjoint because $V \subseteq \overline{V}$; and $x \in V$ while $\omega \in X^* \sim \overline{V}$. So x and ω are separated by disjoint open sets. Hence X^* is a compact Hausdorff space.

Problem 21.4. Show that the Alexandroff one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n -sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

Solution. Write $(\mathbb{R}^n)^* = \mathbb{R}^n \cup \{\omega\}$ for the Alexandroff one-point compactification of $\mathbb{R}^n$; $\mathbb{R}^n$ is locally compact and Hausdorff, so by Problem 3 the space $(\mathbb{R}^n)^*$ is compact and Hausdorff, $\mathbb{R}^n$ is an open subspace of it carrying its usual topology, and the open subsets of $(\mathbb{R}^n)^*$ that contain ω are exactly the sets $(\mathbb{R}^n)^* \sim K$ with $K \subseteq \mathbb{R}^n$ compact (an open set containing ω cannot be of the first kind, since an open subset of $\mathbb{R}^n$ omits ω).

Regard $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and let $N = (0, \dots, 0, 1) \in S^n$ be the north pole. Define stereographic projection from N ,

$$\sigma: S^n \sim \{N\} \rightarrow \mathbb{R}^n, \quad \sigma(y_1, \dots, y_{n+1}) = \frac{1}{1 - y_{n+1}}(y_1, \dots, y_n),$$

which is well defined and continuous since $y_{n+1} < 1$ on $S^n \sim \{N\}$ (indeed $y_{n+1} = 1$ together with $\|y\| = 1$ forces $y = N$), so the denominator never vanishes and each coordinate function is a quotient of continuous functions with nonvanishing denominator.

Define also

$$\tau: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}, \quad \tau(x) = \frac{1}{\|x\|^2 + 1}(2x, \|x\|^2 - 1),$$

which is continuous, the denominator being everywhere positive.

We check τ maps into $S^n \sim \{N\}$ and is inverse to σ .

First,

$$\|(2x, \|x\|^2 - 1)\|^2 = 4\|x\|^2 + \|x\|^4 - 2\|x\|^2 + 1 = (\|x\|^2 + 1)^2,$$

so $\|\tau(x)\| = 1$, that is, $\tau(x) \in S^n$. Its last coordinate is $(\|x\|^2 - 1)/(\|x\|^2 + 1) < 1$, so $\tau(x) \neq N$.

Second, with $y = \tau(x)$ we have $1 - y_{n+1} = 2/(\|x\|^2 + 1)$ and $(y_1, \dots, y_n) = 2x/(\|x\|^2 + 1)$, whence

$$\sigma(\tau(x)) = \frac{\|x\|^2 + 1}{2} \cdot \frac{2x}{\|x\|^2 + 1} = x.$$

Third, let $y \in S^n \sim \{N\}$ and put $x = \sigma(y)$. Since $\sum_{i=1}^n y_i^2 = 1 - y_{n+1}^2$,

$$\|x\|^2 = \frac{1 - y_{n+1}^2}{(1 - y_{n+1})^2} = \frac{1 + y_{n+1}}{1 - y_{n+1}},$$

so that

$$\|x\|^2 + 1 = \frac{2}{1 - y_{n+1}}, \quad \|x\|^2 - 1 = \frac{2y_{n+1}}{1 - y_{n+1}}.$$

Therefore

$$\tau(x) = \frac{1 - y_{n+1}}{2} \left(\frac{2(y_1, \dots, y_n)}{1 - y_{n+1}}, \frac{2y_{n+1}}{1 - y_{n+1}} \right) = (y_1, \dots, y_n, y_{n+1}) = y.$$

Thus σ and τ are mutually inverse continuous bijections, so σ is a homeomorphism of $S^n \sim \{N\}$ onto $\mathbb{R}^n$ with inverse τ .

Now define

$$\Phi: (\mathbb{R}^n)^* \rightarrow S^n, \quad \Phi(x) = \tau(x) \text{ for } x \in \mathbb{R}^n, \quad \Phi(\omega) = N.$$

Since τ is a bijection of $\mathbb{R}^n$ onto $S^n \sim \{N\}$ and Φ sends the one remaining point ω to the one remaining point N , the map Φ is a bijection of $(\mathbb{R}^n)^*$ onto S^n .

We claim Φ is continuous.

At a point x_0 of $\mathbb{R}^n$: by Problem 3(ii), $\mathbb{R}^n$ is an open subset of $(\mathbb{R}^n)^*$ whose subspace topology is its usual topology, and on it Φ agrees with the continuous map τ . So if W is a neighborhood of $\Phi(x_0)$ in S^n , then $\tau^{-1}(W)$ is a neighborhood of x_0 in $\mathbb{R}^n$, hence a neighborhood of x_0 in $(\mathbb{R}^n)^*$, and it is mapped by Φ into W .

At ω : let W be a neighborhood of N in S^n . Then $S^n \sim W$ is a closed subset of the compact set S^n (the sphere is closed and bounded in $\mathbb{R}^{n+1}$, hence compact by the Heine–Borel Theorem), so $S^n \sim W$ is compact; and it does not contain N , so it is a compact subset of $S^n \sim \{N\}$. Therefore

$$K = \sigma(S^n \sim W)$$

is a compact subset of $\mathbb{R}^n$, being the continuous image of a compact set. Consequently $(\mathbb{R}^n)^* \sim K$ is an open subset of $(\mathbb{R}^n)^*$ containing ω . Moreover Φ maps it into W : indeed $\Phi(\omega) = N \in W$, while if $x \in \mathbb{R}^n \sim K$ then $\tau(x) \notin S^n \sim W$ — for if $\tau(x)$ belonged to $S^n \sim W$ we would have $x = \sigma(\tau(x)) \in \sigma(S^n \sim W) = K$ — and hence $\tau(x) \in W$.

So Φ is continuous at every point of $(\mathbb{R}^n)^*$, and therefore continuous.

Finally, a continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism. We recall the argument. Let C be a closed subset of $(\mathbb{R}^n)^*$. Since $(\mathbb{R}^n)^*$ is compact (Problem 3(iii)), C is compact; hence $\Phi(C)$ is a compact subset of the Hausdorff space S^n , and so $\Phi(C)$ is closed. Thus Φ carries closed sets to closed sets, which for a bijection says exactly that Φ^{-1} is continuous.

Therefore Φ is a homeomorphism of the Alexandroff one-point compactification of $\mathbb{R}^n$ onto S^n .

Problem 21.5. Show that an open subset of a locally compact Hausdorff space, with its subspace topology, is locally compact.

Solution. Let X be a locally compact Hausdorff space and $\mathcal{O}$ an open subset of X , equipped with the subspace topology. We must show each point of $\mathcal{O}$ has a neighborhood in $\mathcal{O}$ whose closure in $\mathcal{O}$ is compact.

Let $x \in \mathcal{O}$. Then $\mathcal{O}$ is a neighborhood of x in X , so by Lemma 1 there is a neighborhood V of x in X for which

$$x \in V \subseteq \overline{V} \subseteq \mathcal{O} \quad \text{and} \quad \overline{V} \text{ is compact,}$$

where $\overline{V}$ denotes the closure of V in X .

Since V is open in X and $V \subseteq \mathcal{O}$, the set V is also open in the subspace $\mathcal{O}$, so V is a neighborhood of x in $\mathcal{O}$.

The closure of V in the subspace $\mathcal{O}$ is

$$\text{cl}_{\mathcal{O}}(V) = \overline{V} \cap \mathcal{O} = \overline{V},$$

the last equality because $\overline{V} \subseteq \mathcal{O}$. Thus the closure of V in $\mathcal{O}$ is the compact set $\overline{V}$.

Finally, compactness is an intrinsic property of a topological space: $\overline{V}$, with the topology it inherits from X , is compact, and the topology it inherits from $\mathcal{O}$ is the same, since for $E \subseteq \overline{V} \subseteq \mathcal{O}$ and U open in X we have $(U \cap \mathcal{O}) \cap \overline{V} = U \cap \overline{V}$. Hence $\overline{V}$ is a compact subspace of $\mathcal{O}$.

So every point of $\mathcal{O}$ has a neighborhood in $\mathcal{O}$ with compact closure in $\mathcal{O}$, and therefore $\mathcal{O}$ is locally compact. (It is also Hausdorff, being a subspace of a Hausdorff space.)

Problem 21.6. Show that a closed subset of a locally compact space, with its subspace topology, is locally compact.

Solution. Let X be a locally compact topological space and F a closed subset of X , with the subspace topology. Let $x \in F$.

Since X is locally compact, x has a neighborhood U in X whose closure $\overline{U}$ in X is compact. Then

$$U \cap F$$

is open in the subspace F and contains x , so it is a neighborhood of x in F .

Compute the closure of $U \cap F$ in the subspace F . Denoting by $\overline{E}$ the closure of a set E in X , we have

$$\text{cl}_F(U \cap F) = \overline{U \cap F} \cap F = \overline{U \cap F},$$

the last equality holding because $U \cap F \subseteq F$ and F is closed in X , so that $\overline{U \cap F} \subseteq \overline{F} = F$.

Now $\overline{U \cap F} \subseteq \overline{U}$, and $\overline{U \cap F}$ is closed in X , hence it is a closed subset of the compact set $\overline{U}$. A closed subset of a compact space is compact — and this fact requires no separation hypothesis: if C is closed in the compact space K and $\{U_\lambda\}$ is a cover of C by sets open in K , then $\{U_\lambda\} \cup \{K \sim C\}$ is an open cover of K , a finite subcover of which yields, upon discarding $K \sim C$, a finite subcover of C . Therefore $\overline{U \cap F}$ is compact.

As in the preceding problem, compactness is intrinsic: the topology $\overline{U \cap F}$ inherits from F coincides with the one it inherits from X , because for a set $E \subseteq \overline{U \cap F} \subseteq F$ and V open in X one has $(V \cap F) \cap \overline{U \cap F} = V \cap \overline{U \cap F}$. So $\overline{U \cap F}$ is a compact subspace of F .

Thus every point of F has a neighborhood in F whose closure in F is compact, and therefore F , with its subspace topology, is locally compact.

Problem 21.7. Show that a locally compact Hausdorff space X is compact if and only if the set consisting of the point at infinity is an open subset of the Alexandroff one-point compactification X^* of X .

Solution. Recall the definition of the topology of $X^* = X \cup \{\omega\}$: a subset of X^* is open provided it is either an open subset of X or the complement in X^* of a compact subset of X . (That this is indeed a topology is Problem 3(i).)

Suppose first that X is compact. Then X itself is a compact subset of X , so

$$\{\omega\} = X^* \sim X$$

is, by the very definition of the topology of X^* , an open subset of X^* .

Conversely, suppose $\{\omega\}$ is an open subset of X^* . Then $\{\omega\}$ is of one of the two admissible kinds. It cannot be an open subset of X : every subset of X omits the point ω , whereas $\{\omega\}$ contains it, and $\{\omega\}$ is nonempty. Hence there is a compact subset K of X with

$$\{\omega\} = X^* \sim K.$$

Taking complements in X^* gives

$$K = X^* \sim \{\omega\} = X.$$

Therefore $X = K$ is compact.

Thus X is compact if and only if $\{\omega\}$ is open in X^* .

We note the geometric content of this: ω is an isolated point of X^* precisely when X is already compact, in which case the compactification adds nothing but a spurious extra point; when X fails to be compact, every neighborhood of ω meets X , so ω is a point of closure of X and X is dense in X^* .

21.2 Problems 21.8–21.14

Problem 21.8. Let X be a locally compact Hausdorff space. Show that the Alexandroff one-point compactification X^* is separable if and only if X is separable.

Solution. Recall the construction: $X^* = X \cup \{\omega\}$ with $\omega \notin X$, and a subset of X^* is defined to be open provided it is either an open subset of X or the complement in X^* of a compact subset of X . Two consequences of this definition are used repeatedly below: every open subset of X is an open subset of X^* , and the identity mapping of X onto the subspace $X^* \sim \{\omega\}$ is a homeomorphism, so the topology X inherits as a subspace of X^* is its own topology. Recall also that a topological space is called separable provided it possesses a countable dense subset.

Assume first that X is separable, and let D be a countable dense subset of X . Define $E = D \cup \{\omega\}$, a

countable subset of X^* . We claim E is dense in X^* , that is, every nonempty open subset of X^* meets E . Let $\mathcal{W}$ be a nonempty open subset of X^* . If $\omega \in \mathcal{W}$, then $\mathcal{W} \cap E \neq \emptyset$ since $\omega \in E$. Otherwise $\mathcal{W} \subseteq X$, and then $\mathcal{W}$, being open in X^* and contained in X , is open in the subspace X ; as $\mathcal{W}$ is nonempty and D is dense in X , we have $\mathcal{W} \cap D \neq \emptyset$ and therefore $\mathcal{W} \cap E \neq \emptyset$. Hence E is a countable dense subset of X^* , and X^* is separable.

Observe that the adjoined point genuinely may be needed here: if X itself is compact, then $\{\omega\} = X^* \sim X$ is the complement of a compact subset of X and hence is open in X^* , so no subset of X is dense in X^* .

Conversely, assume X^* is separable and let E be a countable dense subset of X^* . Define $D = E \cap X = E \sim \{\omega\}$, a countable subset of X . We claim D is dense in X . Let $\mathcal{O}$ be a nonempty open subset of X . By the definition of the topology on X^* , the set $\mathcal{O}$ is also open in X^* , and it is nonempty, so by the density of E in X^* we may select a point $x \in E \cap \mathcal{O}$. Since $\mathcal{O} \subseteq X$ we have $x \neq \omega$, and therefore $x \in (E \cap X) \cap \mathcal{O} = D \cap \mathcal{O}$. Thus every nonempty open subset of X meets D , that is, D is dense in X . Hence X is separable.

Problem 21.9. Consider the topological space X consisting of the set of real numbers with the topology that has complements of countable sets as a base. Show that X is not locally compact.

Solution. First we identify the topology. Let $\mathcal{B}$ be the collection of sets of the form $\mathbb{R} \sim C$ with C countable. Since the union of two countable sets is countable, $\mathcal{B}$ is closed under finite intersections, and $\mathbb{R} \in \mathcal{B}$, so $\mathcal{B}$ is indeed a base for a topology on $\mathbb{R}$: the open sets are precisely the unions of members of $\mathcal{B}$. But a union of co-countable sets is co-countable, and the empty union is $\emptyset$. Therefore

$$\mathcal{T} = \{\emptyset\} \cup \{\mathbb{R} \sim C : C \subseteq \mathbb{R} \text{ countable}\},$$

the co-countable topology on $\mathbb{R}$. In particular every nonempty open subset of X is the complement of a countable set and therefore is uncountable, since $\mathbb{R}$ is uncountable; a fortiori every nonempty open set is infinite.

Claim: a subset K of X is compact if and only if it is finite. A finite set is compact in any topological space. Conversely, suppose K is infinite. Choose a countably infinite set of distinct points $\{x_n\}_{n=1}^\infty$ contained in K , and for each index n define

$$\mathcal{U}_n = X \sim \{x_k : k \geq n\}.$$

Each $\mathcal{U}_n$ is the complement of a countable set and hence is open. Since the points x_k are distinct, no point belongs to $\{x_k : k \geq n\}$ for every n , that is, $\bigcap_{n=1}^\infty \{x_k : k \geq n\} = \emptyset$, and therefore $\bigcup_{n=1}^\infty \mathcal{U}_n = X$. Thus $\{\mathcal{U}_n\}_{n=1}^\infty$ is an open cover of K . It has no finite subcover: the family is ascending, so a finite subfamily has union $\mathcal{U}_N$ for the largest index N occurring in it, while x_N belongs to K but not to $\mathcal{U}_N$. Hence K is not compact, and the claim is proved.

Now let x be any point of X and let $\mathcal{N}$ be any neighborhood of x , that is, any open set containing x . Then $\mathcal{N}$ is nonempty and open, hence infinite by the first paragraph. Since $\mathcal{N} \subseteq \overline{\mathcal{N}}$, the closure $\overline{\mathcal{N}}$ is infinite as well, and therefore, by the claim, $\overline{\mathcal{N}}$ is not compact. Thus no point of X has a neighborhood with compact closure, and X fails to be locally compact.

For contrast, note that X is not Hausdorff either: any two nonempty open subsets of X are complements of countable sets, and their intersection is the complement of a countable set, hence uncountable and in particular nonempty. So distinct points cannot be separated by disjoint neighborhoods.

Problem 21.10. Provide a proof of Proposition 3 by applying Urysohn's Lemma to the Alexandroff one-point compactification of X .

[Proposition 3: Let K be a compact subset of a locally compact Hausdorff space X and $\mathcal{O}$ a neighborhood of K . Then there is a function f belonging to $C_c(X)$ for which

$$f = 1 \text{ on } K, \quad f = 0 \text{ on } X \sim \mathcal{O}, \quad \text{and} \quad 0 \leq f \leq 1 \text{ on } X.$$

]

Solution. Let K be a compact subset of the locally compact Hausdorff space X and let $\mathcal{O}$ be a neighborhood of K , that is, an open set containing K .

By Proposition 2 there is a neighborhood $\mathcal{V}$ of K that has compact closure contained in $\mathcal{O}$:

$$K \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{O} \quad \text{and} \quad \overline{\mathcal{V}} \text{ is compact,}$$

the closure being taken in X .

Form the Alexandroff one-point compactification $X^* = X \cup \{\omega\}$. As established in Problem 3, X^* is a compact Hausdorff space and the identity mapping of X onto the subspace $X^* \sim \{\omega\}$ is a homeomorphism; in particular a subset of X is open in X if and only if it is open in X^* , and the restriction to X of a continuous function on X^* is continuous on X . Moreover a compact Hausdorff space is normal.

We now exhibit two disjoint closed subsets of X^* . First, K is a compact subset of the Hausdorff space X^* , and therefore K is closed in X^* . Second, $\mathcal{V}$ is open in X and hence open in X^* , so $F = X^* \sim \mathcal{V}$ is closed in X^* . Since $K \subseteq \mathcal{V}$, the sets K and F are disjoint.

Since X^* is normal, Urysohn's Lemma provides a continuous real-valued function g on X^* for which

$$g = 1 \text{ on } K, \quad g = 0 \text{ on } F = X^* \sim \mathcal{V}, \quad \text{and} \quad 0 \leq g \leq 1 \text{ on } X^*.$$

Define f to be the restriction of g to X . Then f is a continuous real-valued function on X with $0 \leq f \leq 1$ on X and $f = 1$ on K . Furthermore $f = 0$ on $X \sim \mathcal{V}$, and since $X \sim \mathcal{O} \subseteq X \sim \overline{\mathcal{V}} \subseteq X \sim \mathcal{V}$, we have $f = 0$ on $X \sim \mathcal{O}$.

It remains to check that f has compact support. Since f vanishes on $X \sim \mathcal{V}$, we have $\{x \in X \mid f(x) \neq 0\} \subseteq \mathcal{V}$, and therefore, taking closures in X ,

$$\text{supp } f = \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq \overline{\mathcal{V}}.$$

Thus $\text{supp } f$ is a subset of X that is closed in X and contained in the compact set $\overline{\mathcal{V}}$; hence it is a closed subset of the compact space $\overline{\mathcal{V}}$ and so is compact. Therefore f belongs to $C_c(X)$ and satisfies

$$f = 1 \text{ on } K, \quad f = 0 \text{ on } X \sim \mathcal{O}, \quad 0 \leq f \leq 1 \text{ on } X,$$

which is exactly the conclusion of Proposition 3.

Problem 21.11. Let f continuously map the locally compact Hausdorff space X onto the topological space Y . Is Y necessarily locally compact?

Solution. No. The continuous image of a locally compact Hausdorff space need not be locally compact, as the following example shows.

Let X be the set $\mathbb{Q}$ of rational numbers equipped with the discrete topology, and let Y be the set $\mathbb{Q}$ equipped with the topology it inherits as a subspace of $\mathbb{R}$. Let $f: X \rightarrow Y$ be the identity mapping, which is certainly onto.

The space X is locally compact and Hausdorff. It is Hausdorff since every singleton is open, so distinct points are separated by disjoint neighborhoods. It is locally compact since, for each $x \in X$, the singleton $\{x\}$ is a neighborhood of x whose closure is $\{x\}$ itself, a finite and therefore compact set. The mapping f is continuous because the inverse image of any subset of Y is a subset of the discrete space X and hence is open.

It remains to show that $Y = \mathbb{Q}$, with its usual topology, is not locally compact. Let $x \in \mathbb{Q}$ and let $\mathcal{N}$ be a neighborhood of x in $\mathbb{Q}$. By definition of the subspace topology there is a $\delta > 0$ for which

$$(x - \delta, x + \delta) \cap \mathbb{Q} \subseteq \mathcal{N}.$$

Suppose, to obtain a contradiction, that the closure K of $\mathcal{N}$ in $\mathbb{Q}$ is compact. A compact subset of $\mathbb{Q}$ is also compact when regarded as a subset of $\mathbb{R}$, since the open cover criterion involves only relatively

open sets, which are the traces on K of open subsets of $\mathbb{R}$. A compact subset of the Hausdorff space $\mathbb{R}$ is closed in $\mathbb{R}$. Hence K is a closed subset of $\mathbb{R}$ that contains $(x - \delta, x + \delta) \cap \mathbb{Q}$, and therefore K contains the closure in $\mathbb{R}$ of that set, which is $[x - \delta, x + \delta]$ because the rationals are dense in $\mathbb{R}$. But $[x - \delta, x + \delta]$ contains irrational numbers, while $K \subseteq \mathbb{Q}$. This contradiction shows that no neighborhood of x in $\mathbb{Q}$ has compact closure. Hence Y is not locally compact.

Thus local compactness is not preserved by continuous surjections, even onto Hausdorff (indeed metrizable) spaces.

An alternative example, using the preceding problem, is obtained by taking X to be $\mathbb{R}$ with the discrete topology, Y to be $\mathbb{R}$ with the topology having the complements of countable sets as a base, and f the identity mapping. Again X is locally compact and Hausdorff and f is a continuous surjection, while Y was shown in Problem 9 to fail to be locally compact.

Problem 21.12. Let X be a topological space and f a continuous function of X that has compact support. Define $K = \{x \in X \mid f(x) = 1\}$. Show that K is a compact G_δ set.

Solution. Recall that the support of a real-valued function f on X is

$$\text{supp } f = \overline{\{x \in X \mid f(x) \neq 0\}},$$

and that a subset of a topological space is called a G_δ set provided it is the intersection of a countable collection of open sets. By hypothesis f is continuous on X and $\text{supp } f$ is compact, that is, f belongs to $C_c(X)$.

We first show K is compact. Since $\{1\}$ is a closed subset of $\mathbb{R}$ and f is continuous, $K = f^{-1}(\{1\})$ is a closed subset of X . Moreover, if $x \in K$ then $f(x) = 1 \neq 0$, so

$$K \subseteq \{x \in X \mid f(x) \neq 0\} \subseteq \text{supp } f.$$

Thus K is a subset of the compact set $\text{supp } f$ that is closed in X ; consequently $K = K \cap \text{supp } f$ is closed in the subspace $\text{supp } f$. A closed subset of a compact space is compact, and compactness of a set is intrinsic to its subspace topology, so K is a compact subset of X .

We next show K is a G_δ set. For each natural number n define

$$\mathcal{O}_n = f^{-1}\left(\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)\right) = \left\{x \in X \mid |f(x) - 1| < \frac{1}{n}\right\}.$$

Each $\mathcal{O}_n$ is the inverse image under the continuous function f of an open subset of $\mathbb{R}$ and hence is open in X . If $f(x) = 1$ then $|f(x) - 1| = 0 < 1/n$ for every n , so $K \subseteq \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Conversely, if $x \in \bigcap_{n=1}^{\infty} \mathcal{O}_n$ then $|f(x) - 1| < 1/n$ for every natural number n , and therefore $|f(x) - 1| = 0$, that is, $f(x) = 1$ and $x \in K$. Hence

$$K = \bigcap_{n=1}^{\infty} \mathcal{O}_n,$$

a countable intersection of open sets, so K is a G_δ set.

Therefore K is a compact G_δ subset of X . Observe that no separation or local compactness assumption on X was needed. In the case that X is locally compact and Hausdorff, this is precisely the converse of Corollary 4, which asserts that every compact G_δ subset K of such a space arises in this way from some $f \in C_c(X)$.

Problem 21.13. Let $\mathcal{O}$ be an open subset of a compact Hausdorff space X . Show that the mapping of X to the Alexandroff one-point compactification of $\mathcal{O}$ that is the identity on $\mathcal{O}$ and takes each point in $X \setminus \mathcal{O}$ into ω is continuous.

Solution. First observe that the one-point compactification of $\mathcal{O}$ is defined. Indeed, X is compact and Hausdorff, hence locally compact and Hausdorff, and by Problem 5 an open subset of a locally compact Hausdorff space, with its subspace topology, is locally compact; it is Hausdorff since a subspace of a

Hausdorff space is Hausdorff. Write

$$\mathcal{O}^* = \mathcal{O} \cup \{\omega\}, \quad \omega \notin \mathcal{O},$$

where, by definition of the Alexandroff compactification, a subset of $\mathcal{O}^*$ is open provided it is either an open subset of $\mathcal{O}$ or the complement in $\mathcal{O}^*$ of a compact subset of $\mathcal{O}$.

Define $\pi: X \rightarrow \mathcal{O}^*$ by

$$\pi(x) = \begin{cases} x, & x \in \mathcal{O}, \\ \omega, & x \in X \sim \mathcal{O}. \end{cases}$$

This is a well-defined mapping of X onto $\mathcal{O}^*$ provided $X \sim \mathcal{O} \neq \emptyset$, and onto $\mathcal{O}^* \sim \{\omega\}$ otherwise; in either case it is a mapping into $\mathcal{O}^*$, and continuity is what is asked.

To prove continuity it suffices to show that $\pi^{-1}(\mathcal{W})$ is open in X for every open subset $\mathcal{W}$ of $\mathcal{O}^*$. By the definition of the topology on $\mathcal{O}^*$ there are exactly two cases.

Case 1: $\mathcal{W} = \mathcal{U}$ is an open subset of $\mathcal{O}$. Then $\omega \notin \mathcal{U}$, so no point of $X \sim \mathcal{O}$ belongs to $\pi^{-1}(\mathcal{U})$, and since π is the identity on $\mathcal{O}$,

$$\pi^{-1}(\mathcal{U}) = \{x \in \mathcal{O} \mid x \in \mathcal{U}\} = \mathcal{U}.$$

Now $\mathcal{U}$ is open in the subspace $\mathcal{O}$, so $\mathcal{U} = \mathcal{G} \cap \mathcal{O}$ for some open subset $\mathcal{G}$ of X ; since $\mathcal{O}$ is open in X , the set $\mathcal{U}$ is open in X . Hence $\pi^{-1}(\mathcal{W})$ is open in X .

Case 2: $\mathcal{W} = \mathcal{O}^* \sim C$ where C is a compact subset of $\mathcal{O}$. Then $\omega \in \mathcal{W}$, so $X \sim \mathcal{O} \subseteq \pi^{-1}(\mathcal{W})$, while a point $x \in \mathcal{O}$ lies in $\pi^{-1}(\mathcal{W})$ exactly when $x \notin C$. Therefore

$$\pi^{-1}(\mathcal{W}) = (X \sim \mathcal{O}) \cup (\mathcal{O} \sim C) = X \sim C.$$

Now C is a compact subset of $\mathcal{O}$; since compactness is intrinsic to the subspace topology, C is a compact subset of X as well. A compact subset of a Hausdorff space is closed, so C is closed in X and therefore $\pi^{-1}(\mathcal{W}) = X \sim C$ is open in X .

In both cases the inverse image of an open set is open, and hence π is continuous.

Two remarks. First, if $X \sim \mathcal{O} \neq \emptyset$, then π is a continuous mapping of the compact space X onto $\mathcal{O}^*$, and since the continuous image of a compact space is compact, this re-proves that $\mathcal{O}^*$ is compact. Second, in that same case $\mathcal{O}^*$ is compact and Hausdorff, so a continuous mapping of the compact space X onto $\mathcal{O}^*$ carries closed, hence compact, subsets of X onto compact, hence closed, subsets of $\mathcal{O}^*$; a continuous closed surjection is a quotient mapping. Therefore π exhibits $\mathcal{O}^*$ as the space obtained from X by collapsing the closed set $X \sim \mathcal{O}$ to a single point.

Problem 21.14. Let X and Y be locally compact Hausdorff spaces, and f a continuous mapping of X into Y . Let X^* and Y^* be the Alexandroff one-point compactifications of X and Y , and f^* the mapping of X^* into Y^* whose restriction to X is f and that takes the point at infinity in X^* into the point at infinity in Y^* . Show that f^* is continuous if and only if $f^{-1}(K)$ is compact whenever $K \subseteq Y$ is compact. A mapping f with this property is said to be proper.

Solution. Write $X^* = X \cup \{\omega_X\}$ and $Y^* = Y \cup \{\omega_Y\}$, so that $f^*(x) = f(x)$ for $x \in X$ and $f^*(\omega_X) = \omega_Y$. Recall the two facts we shall use about these compactifications. First, a subset of Y^* is open provided it is either an open subset of Y or the complement in Y^* of a compact subset of Y , and likewise for X^* ; every open subset of Y is thus open in Y^* . Second, by Problem 3, X^* is a compact Hausdorff space and the identity mapping of X onto the subspace $X^* \sim \{\omega_X\}$ is a homeomorphism, so a subset of X is open in X if and only if it is open in X^* , and a subset of X is compact as a subset of X if and only if it is compact as a subset of X^* .

A remark used in both directions: if $A \subseteq Y$, then $\omega_Y \notin A$, so $\omega_X \notin (f^*)^{-1}(A)$ and hence

$$(f^*)^{-1}(A) = \{x \in X \mid f(x) \in A\} = f^{-1}(A).$$

Suppose first that f is proper, that is, $f^{-1}(K)$ is compact for every compact $K \subseteq Y$. We show that the inverse image under f^* of every open subset of Y^* is open in X^* . Let $\mathcal{W}$ be open in Y^* . By the definition of the topology on Y^* there are two cases.

Case 1: $\mathcal{W}$ is an open subset of Y . By the remark, $(f^*)^{-1}(\mathcal{W}) = f^{-1}(\mathcal{W})$, which is open in X since f is continuous, and hence is open in X^* .

Case 2: $\mathcal{W} = Y^* \sim K$ with K a compact subset of Y . Since $\omega_Y \in \mathcal{W}$, we have $\omega_X \in (f^*)^{-1}(\mathcal{W})$, and

$$(f^*)^{-1}(\mathcal{W}) = X^* \sim (f^*)^{-1}(K) = X^* \sim f^{-1}(K),$$

using the remark for the second equality. By properness $f^{-1}(K)$ is a compact subset of X , so $X^* \sim f^{-1}(K)$ is, by the very definition of the topology on X^* , an open subset of X^* .

In both cases $(f^*)^{-1}(\mathcal{W})$ is open in X^* , and therefore f^* is continuous.

Conversely, suppose f^* is continuous, and let K be a compact subset of Y . By the definition of the topology on Y^* , the set $Y^* \sim K$ is open in Y^* . By the continuity of f^* , the set

$$(f^*)^{-1}(Y^* \sim K) = X^* \sim (f^*)^{-1}(K) = X^* \sim f^{-1}(K)$$

is open in X^* , and therefore $f^{-1}(K)$ is a closed subset of X^* . Since X^* is compact and a closed subset of a compact space is compact, $f^{-1}(K)$ is a compact subset of X^* . But $f^{-1}(K) \subseteq X$, and compactness is intrinsic to the subspace topology, which on X agrees with the original topology of X by Problem 3. Hence $f^{-1}(K)$ is a compact subset of X . As K was an arbitrary compact subset of Y , the mapping f is proper.

This establishes the asserted equivalence: f^* is continuous if and only if f is proper. Note that the continuity of f itself was used only in the first direction; in the second direction it is a consequence of the continuity of f^* , since f is the restriction of f^* to the subspace X and $f^*(X) \subseteq Y$.

21.3 Problems 21.15–21.21

Problem 21.15. Let X be a locally compact Hausdorff space. Show that a subset F of X is closed if and only if $F \cap K$ is closed for each compact subset K of X . Moreover, show that the same equivalence holds if instead of being locally compact the space X is first countable.

Solution. Throughout, X is Hausdorff; only the hypothesis of local compactness is replaced in the final assertion. In both cases, therefore, a compact subset K of X is a closed subset of X , since a compact subspace of a Hausdorff space is closed.

Necessity uses neither local compactness nor first countability. Suppose F is closed and K is compact. Then K is closed, so $F \cap K$, being the intersection of two closed sets, is closed.

Sufficiency when X is locally compact. Assume $F \cap K$ is closed for every compact $K \subseteq X$. We show $\overline{F} \subseteq F$. Let x belong to $\overline{F}$. Since X is a neighborhood of x and X is locally compact and Hausdorff, Lemma 1 provides a neighborhood $\mathcal{V}$ of x for which $K := \overline{\mathcal{V}}$ is compact. We claim x belongs to $\overline{F} \cap \mathcal{V}$. Indeed, if $\mathcal{U}$ is any neighborhood of x , then $\mathcal{U} \cap \mathcal{V}$ is a neighborhood of x , and since x belongs to the closure of F , the set $\mathcal{U} \cap \mathcal{V}$ contains a point of F ; thus every neighborhood of x meets $F \cap \mathcal{V}$, which is the claim. Therefore

$$x \in \overline{F \cap \mathcal{V}} \subseteq \overline{F \cap K} = F \cap K \subseteq F,$$

the equality holding because $F \cap K$ is closed by hypothesis. Hence $\overline{F} \subseteq F$ and F is closed.

Sufficiency when X is first countable. Again assume $F \cap K$ is closed for every compact K , and let x belong to $\overline{F}$. Since X is first countable there is a countable base of neighborhoods $\{\mathcal{B}_n\}_{n=1}^\infty$ at x ; replacing $\mathcal{B}_n$ by $\mathcal{U}_n := \bigcap_{k=1}^n \mathcal{B}_k$, we obtain a descending sequence $\{\mathcal{U}_n\}_{n=1}^\infty$ of neighborhoods of x with the property that every neighborhood of x contains some $\mathcal{U}_n$. As x belongs to $\overline{F}$, for each n we may choose a point $x_n \in F \cap \mathcal{U}_n$. Define

$$K = \{x\} \cup \{x_n \mid n \in \mathbb{N}\}.$$

We claim K is compact. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be a cover of K by open subsets of X . Some $\mathcal{O}_{\lambda_0}$ contains x , and hence contains $\mathcal{U}_N$ for some index N ; since $\{\mathcal{U}_n\}$ is descending, $x_n \in \mathcal{U}_n \subseteq \mathcal{U}_N \subseteq \mathcal{O}_{\lambda_0}$ for all $n \geq N$. Choosing, for each $n < N$, an index λ_n with $x_n \in \mathcal{O}_{\lambda_n}$, the finite subcollection $\{\mathcal{O}_{\lambda_0}, \mathcal{O}_{\lambda_1}, \dots, \mathcal{O}_{\lambda_{N-1}}\}$ covers K . Thus K is compact, and therefore $F \cap K$ is closed by hypothesis. Each x_n belongs to $F \cap K$.

Moreover x belongs to the closure of $\{x_n \mid n \in \mathbb{N}\}$, since any neighborhood of x contains some $\mathcal{U}_n$ and hence the point x_n . Consequently

$$x \in \overline{\{x_n \mid n \in \mathbb{N}\}} \subseteq \overline{F \cap K} = F \cap K \subseteq F.$$

Hence $\overline{F} \subseteq F$ and F is closed.

Problem 21.16. Let $\mathcal{F}$ be a family of real-valued continuous functions on a locally compact Hausdorff space X which has the following properties:

- (i) If $f \in \mathcal{F}$ and $g \in \mathcal{F}$, then $f + g \in \mathcal{F}$.
- (ii) If $f \in \mathcal{F}$ and $g \in \mathcal{F}$, then $f/g \in \mathcal{F}$, provided that $\text{supp } f \subseteq \{x \in X \mid g(x) \neq 0\}$.
- (iii) Given a neighborhood $\mathcal{O}$ of a point $x_0 \in X$, there is an $f \in \mathcal{F}$ with $f(x_0) = 1$, $0 \leq f \leq 1$ and $\text{supp } f \subseteq \mathcal{O}$.

Show that Proposition 5 is still true if we require that the functions in the partition of unity belong to $\mathcal{F}$.

Solution. We must show: if K is a compact subset of X and $\{\mathcal{O}_k\}_{k=1}^n$ is a finite cover of K by open sets, then there is a partition of unity $\{\varphi_k\}_{k=1}^n$ for K subordinate to this cover with each φ_k belonging to $\mathcal{F}$ and having compact support. Two degenerate cases are set aside at the outset. First, we assume each $\mathcal{O}_k$ is nonempty: an empty member may be deleted from the cover without affecting K or the covering property, and for an empty $\mathcal{O}_k$ the requirement $\text{supp } \varphi_k \subseteq \mathcal{O}_k$ leaves the zero function as the only admissible choice, which no hypothesis on $\mathcal{F}$ guarantees to be a member of $\mathcal{F}$. Second, we assume $K \neq \emptyset$: if $K = \emptyset$, the construction given below for the indices k with $\mathcal{O}_k \cap K = \emptyset$ already produces, for every k , a function $\varphi_k \in \mathcal{F}$ with $0 \leq \varphi_k \leq 1$ and $\text{supp } \varphi_k$ a compact subset of $\mathcal{O}_k$, and the requirement $\varphi_1 + \cdots + \varphi_n = 1$ on K is vacuous.

First a remark on the meaning of the quotient in (ii). If $\text{supp } f \subseteq \{g \neq 0\}$, then the two open sets $\{g \neq 0\}$ and $X \sim \text{supp } f$ cover X , the function f/g is defined and continuous on the first, the function 0 is defined and continuous on the second, and the two agree on the overlap since f vanishes off $\text{supp } f$. Thus f/g is unambiguously a continuous real-valued function on all of X , and $\{f/g \neq 0\} = \{f \neq 0\}$, so $\text{supp}(f/g) = \text{supp } f$.

Call an index k active provided $\mathcal{O}_k \cap K \neq \emptyset$.

Construction of the numerators. For each $x \in K$ choose an index $k(x)$ with $x \in \mathcal{O}_{k(x)}$; such an index exists because $\{\mathcal{O}_k\}_{k=1}^n$ covers K , and it is automatically active since $x \in \mathcal{O}_{k(x)} \cap K$. We make these choices so that every active index occurs as some $k(x)$: for each active k select a point $x^{(k)} \in \mathcal{O}_k \cap K$ and set $k(x^{(k)}) = k$. Since $\mathcal{O}_{k(x)}$ is a neighborhood of x and X is locally compact and Hausdorff, Lemma 1 provides a neighborhood $\mathcal{V}_x$ of x with

$$x \in \mathcal{V}_x \subseteq \overline{\mathcal{V}_x} \subseteq \mathcal{O}_{k(x)}, \quad \overline{\mathcal{V}_x} \text{ compact.}$$

By property (iii) applied to the neighborhood $\mathcal{V}_x$ of x there is $u_x \in \mathcal{F}$ with

$$u_x(x) = 1, \quad 0 \leq u_x \leq 1, \quad \text{supp } u_x \subseteq \mathcal{V}_x.$$

Since $\text{supp } u_x$ is a closed subset of the compact set $\overline{\mathcal{V}_x}$, it is compact, and it is contained in $\mathcal{O}_{k(x)}$.

The set $\{u_x > 1/2\}$ is open and contains x , so $\{\{u_x > 1/2\}\}_{x \in K}$ is an open cover of the compact set K . Select points $y_1, \dots, y_p$ of K with

$$K \subseteq \bigcup_{i=1}^p \{u_{y_i} > 1/2\},$$

and enlarge this finite list, if necessary, by adjoining the finitely many points $x^{(k)}$, k active; the displayed covering property persists. Then for each active k there is at least one index i with $k(y_i) = k$. Put $S_k = \{i \mid k(y_i) = k\}$ for active k ; these sets are nonempty and partition $\{1, \dots, p\}$, since each y_i has exactly one assigned index $k(y_i)$, which is active. Define, for active k ,

$$v_k = \sum_{i \in S_k} u_{y_i},$$

which belongs to $\mathcal{F}$ by property (i) (a sum of finitely many, at least one, members of $\mathcal{F}$). Since $\{v_k \neq 0\} = \bigcup_{i \in S_k} \{u_{y_i} \neq 0\}$ is a finite union, $\text{supp } v_k = \bigcup_{i \in S_k} \text{supp } u_{y_i}$, a compact subset of $\mathcal{O}_k$. Construction of the denominator. Set $v = \sum_{k \text{ active}} v_k = \sum_{i=1}^p u_{y_i} \in \mathcal{F}$ and

$$\mathcal{N} = \{x \in X \mid v(x) > 1/2\},$$

an open set containing K , since each point of K belongs to some $\{u_{y_i} > 1/2\}$ and all the u_{y_i} are nonnegative. The set

$$C = \left(\bigcup_{i=1}^p \text{supp } u_{y_i} \right) \sim \mathcal{N}$$

is a closed subset of a compact set, hence compact, and $C \cap K = \emptyset$ because $K \subseteq \mathcal{N}$. Since K is compact and X is Hausdorff, K is closed, so $X \sim K$ is an open set containing C . If $C \neq \emptyset$, then for each $z \in C$ property (iii), applied to the neighborhood $X \sim K$ of z , provides $e_z \in \mathcal{F}$ with $e_z(z) = 1$, $0 \leq e_z \leq 1$ and $\text{supp } e_z \subseteq X \sim K$; in particular e_z vanishes on K . The open sets $\{e_z > 1/2\}$, $z \in C$, cover the compact set C , so there are $z_1, \dots, z_q$ in C with $C \subseteq \bigcup_{l=1}^q \{e_{z_l} > 1/2\}$. Put $e = \sum_{l=1}^q e_{z_l} \in \mathcal{F}$; then $e \geq 0$ on X , $e = 0$ on K , and $e > 1/2$ on C . If $C = \emptyset$, simply take e to be absent, that is, replace $v + e$ below by v . Define

$$G = v + e \in \mathcal{F}.$$

Then $G \geq 0$ on X , and $G > 1/2$ at every point of $\bigcup_i \text{supp } u_{y_i}$: such a point lies either in $\mathcal{N}$, where $v > 1/2$, or in C , where $e > 1/2$. Moreover $G = v$ on K .

Definition of the partition of unity. For active k define

$$\varphi_k = \frac{v_k}{G}.$$

Since $\text{supp } v_k \subseteq \bigcup_i \text{supp } u_{y_i} \subseteq \{G \neq 0\}$, property (ii) shows $\varphi_k \in \mathcal{F}$, and by the opening remark $\text{supp } \varphi_k = \text{supp } v_k$, a compact subset of $\mathcal{O}_k$. As $0 \leq v_k \leq v \leq G$, we have $0 \leq \varphi_k \leq 1$ on X .

For an index k that is not active, $\mathcal{O}_k$ is nonempty and disjoint from K ; choose $x_k \in \mathcal{O}_k$ and, using Lemma 1, a neighborhood $\mathcal{V}$ of x_k with compact closure contained in $\mathcal{O}_k$. Property (iii) supplies $\varphi_k \in \mathcal{F}$ with $0 \leq \varphi_k \leq 1$ and $\text{supp } \varphi_k \subseteq \mathcal{V} \subseteq \mathcal{O}_k$; its support is compact, and φ_k vanishes on K because $\text{supp } \varphi_k \subseteq \mathcal{O}_k$ and $\mathcal{O}_k \cap K = \emptyset$.

Finally, let $x \in K$. Then $G(x) = v(x) = \sum_{k \text{ active}} v_k(x)$, and $v(x) > 1/2 > 0$, so

$$\sum_{k=1}^n \varphi_k(x) = \sum_{k \text{ active}} \frac{v_k(x)}{G(x)} + \sum_{k \text{ not active}} 0 = \frac{v(x)}{v(x)} = 1.$$

Thus $\{\varphi_k\}_{k=1}^n$ is a collection of continuous functions belonging to $\mathcal{F}$, with $0 \leq \varphi_k \leq 1$ on X , with $\text{supp } \varphi_k$ a compact subset of $\mathcal{O}_k$ for each k , and with $\varphi_1 + \dots + \varphi_n = 1$ on K . That is, it is a partition of unity for K subordinate to $\{\mathcal{O}_k\}_{k=1}^n$ consisting of functions of compact support that belong to $\mathcal{F}$.

Problem 21.17. Let K be a compact G_δ subset of a locally compact Hausdorff space X . Show that there is a decreasing sequence of continuous nonnegative real-valued functions on X that converges pointwise on X to the characteristic function of K .

Solution. Since K is a compact G_δ subset of the locally compact Hausdorff space X , Corollary 4 provides a function $f \in C_c(X)$ for which

$$K = \{x \in X \mid f(x) = 1\}.$$

Define $g: X \rightarrow \mathbb{R}$ by

$$g(x) = 1 - \min\{|f(x) - 1|, 1\} \quad \text{for all } x \in X.$$

The function g is continuous, being built from f by composition with the continuous functions $t \mapsto |t-1|$ and $t \mapsto \min\{t, 1\}$, and $0 \leq g \leq 1$ on X . Moreover, for $x \in X$,

$$g(x) = 1 \iff |f(x) - 1| = 0 \iff f(x) = 1 \iff x \in K.$$

Thus $0 \leq g < 1$ on $X \sim K$ and $g = 1$ on K .

For each natural number n define

$$g_n = g^n \quad \text{on } X.$$

Each g_n is continuous and nonnegative, being a power of a continuous nonnegative function. The sequence is decreasing: since $0 \leq g \leq 1$ on X , for each $x \in X$ we have $g_{n+1}(x) = g(x)^{n+1} = g(x) \cdot g(x)^n \leq g(x)^n = g_n(x)$.

Finally we compute the pointwise limit. If $x \in K$, then $g(x) = 1$, so $g_n(x) = 1$ for every n and $\lim_{n \rightarrow \infty} g_n(x) = 1 = \chi_K(x)$. If $x \notin K$, then $0 \leq g(x) < 1$, so $\lim_{n \rightarrow \infty} g(x)^n = 0 = \chi_K(x)$. Therefore $\{g_n\}_{n=1}^\infty$ is a decreasing sequence of continuous nonnegative real-valued functions on X that converges pointwise on X to χ_K .

Remark. The hypothesis that K be a G_δ set cannot be dropped, since the asserted conclusion returns it. Indeed, suppose $\{g_n\}_{n=1}^\infty$ is any decreasing sequence of continuous functions on a topological space X converging pointwise to χ_K . A decreasing sequence dominates its limit, so $g_n \geq \chi_K$ on X for every n . Hence if $x \in K$ then $g_n(x) \geq 1 > 1 - 1/n$ for every n , while if $g_n(x) > 1 - 1/n$ for every n then $\chi_K(x) = \lim_{n \rightarrow \infty} g_n(x) \geq 1$ and so $x \in K$. Therefore

$$K = \bigcap_{n=1}^{\infty} \{x \in X \mid g_n(x) > 1 - 1/n\},$$

an intersection of countably many open sets.

Problem 21.18. The Baire Category Theorem asserts that in a complete metric space the intersection of a countable collection of open dense sets is dense. At the heart of its proof lies the Cantor Intersection Theorem. Show that the Frechet Intersection Theorem is a sufficiently strong substitute for the Cantor Intersection Theorem to provide a proof of the following assertion by first proving it in the case in which X is compact: Let X be a locally compact Hausdorff space.

- (i) If $\{F_n\}_{n=1}^\infty$ is a countable collection of closed subsets of X for which each F_n has empty interior, then the union $\bigcup_{n=1}^\infty F_n$ also has empty interior.
- (ii) If $\{\mathcal{O}_n\}_{n=1}^\infty$ is a countable collection of open dense subsets of X , then the intersection $\bigcap_{n=1}^\infty \mathcal{O}_n$ also is dense.

Solution. We use the Frechet Intersection Theorem in the following form, established in Problem 40 of Chapter 11: a topological space Z is countably compact if and only if for every descending sequence $\{F_n\}_{n=1}^\infty$ of nonempty closed subsets of Z the intersection $\bigcap_{n=1}^\infty F_n$ is nonempty. In particular, since a compact space is countably compact (every countable open cover is an open cover), a descending sequence of nonempty closed subsets of a compact space has nonempty intersection. This is the only completeness-type ingredient we shall need; it replaces the Cantor Intersection Theorem, which is what supplies the nonempty intersection of the nested closed balls in the metric proof.

We prove (ii) first, and deduce (i) from it.

Proof of (ii) when X is compact Hausdorff. Let $\{\mathcal{O}_n\}_{n=1}^\infty$ be open dense subsets of X and let $\mathcal{U}$ be a nonempty open subset of X . We must show $\mathcal{U} \cap \bigcap_{n=1}^\infty \mathcal{O}_n \neq \emptyset$. A compact Hausdorff space is locally compact, so Lemma 1 is available in X . Since $\mathcal{O}_1$ is dense and $\mathcal{U}$ is nonempty and open, the set $\mathcal{U} \cap \mathcal{O}_1$ is nonempty and open; choose a point x_1 in it and, by Lemma 1, a neighborhood $\mathcal{V}_1$ of x_1 with

$$x_1 \in \mathcal{V}_1 \subseteq \overline{\mathcal{V}_1} \subseteq \mathcal{U} \cap \mathcal{O}_1.$$

Proceeding inductively, suppose the nonempty open set $\mathcal{V}_n$ has been defined. Since $\mathcal{O}_{n+1}$ is dense, $\mathcal{V}_n \cap \mathcal{O}_{n+1}$ is nonempty and open; choose x_{n+1} in it and, by Lemma 1, a neighborhood $\mathcal{V}_{n+1}$ of x_{n+1} with

$$x_{n+1} \in \mathcal{V}_{n+1} \subseteq \overline{\mathcal{V}_{n+1}} \subseteq \mathcal{V}_n \cap \mathcal{O}_{n+1}.$$

This defines a sequence $\{\mathcal{V}_n\}_{n=1}^\infty$ of nonempty open sets with $\overline{\mathcal{V}_{n+1}} \subseteq \mathcal{V}_n \subseteq \overline{\mathcal{V}_n}$, so $\{\overline{\mathcal{V}_n}\}_{n=1}^\infty$ is a descending sequence of nonempty closed subsets of the compact space X . By the Frechet Intersection

Theorem there is a point

$$y \in \bigcap_{n=1}^{\infty} \overline{\mathcal{V}_n}.$$

Then $y \in \overline{\mathcal{V}_1} \subseteq \mathcal{U} \cap \mathcal{O}_1$, and for each $n \geq 1$, $y \in \overline{\mathcal{V}_{n+1}} \subseteq \mathcal{O}_{n+1}$. Hence y belongs to $\mathcal{U} \cap \bigcap_{n=1}^{\infty} \mathcal{O}_n$, which is therefore nonempty. Since $\mathcal{U}$ was an arbitrary nonempty open set, $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is dense.

Proof of (ii) when X is locally compact Hausdorff. The construction is verbatim the same, with one addition: at each stage Lemma 1 is applied so as to obtain, in addition, that $\overline{\mathcal{V}_n}$ is compact. Explicitly, choose $x_1 \in \mathcal{U} \cap \mathcal{O}_1$ and, by Lemma 1, a neighborhood $\mathcal{V}_1$ of x_1 with $\overline{\mathcal{V}_1}$ compact and $\overline{\mathcal{V}_1} \subseteq \mathcal{U} \cap \mathcal{O}_1$; having chosen the nonempty open $\mathcal{V}_n$, choose $x_{n+1} \in \mathcal{V}_n \cap \mathcal{O}_{n+1}$ (nonempty by density of $\mathcal{O}_{n+1}$) and a neighborhood $\mathcal{V}_{n+1}$ of x_{n+1} with $\overline{\mathcal{V}_{n+1}}$ compact and contained in $\mathcal{V}_n \cap \mathcal{O}_{n+1}$. Now every $\overline{\mathcal{V}_n}$ is contained in the compact Hausdorff space $Z := \overline{\mathcal{V}_1}$, and each $\overline{\mathcal{V}_n}$, being closed in X , is a closed subset of the subspace Z . Thus $\{\overline{\mathcal{V}_n}\}_{n=1}^{\infty}$ is a descending sequence of nonempty closed subsets of the compact space Z , and the Frechet Intersection Theorem again yields a point $y \in \bigcap_{n=1}^{\infty} \overline{\mathcal{V}_n}$. Exactly as above, $y \in \mathcal{U} \cap \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Since $\mathcal{U}$ was an arbitrary nonempty open subset of X , the set $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is dense in X . This proves (ii).

Proof of (i). Let $\{F_n\}_{n=1}^{\infty}$ be closed subsets of X , each with empty interior, and set $\mathcal{O}_n = X \sim F_n$. Each $\mathcal{O}_n$ is open. Each $\mathcal{O}_n$ is dense: for a subset E of a topological space one has the identity $X \sim \overline{E} = \text{int}(X \sim E)$, so

$$X \sim \overline{\mathcal{O}_n} = \text{int}(X \sim \mathcal{O}_n) = \text{int } F_n = \emptyset,$$

that is, $\overline{\mathcal{O}_n} = X$. By part (ii), $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is dense in X . But

$$\bigcap_{n=1}^{\infty} \mathcal{O}_n = X \sim \bigcup_{n=1}^{\infty} F_n,$$

so, using the same identity with $E = \bigcap_{n=1}^{\infty} \mathcal{O}_n$,

$$\text{int} \left(\bigcup_{n=1}^{\infty} F_n \right) = \text{int} \left(X \sim \bigcap_{n=1}^{\infty} \mathcal{O}_n \right) = X \sim \overline{\bigcap_{n=1}^{\infty} \mathcal{O}_n} = X \sim X = \emptyset.$$

Thus $\bigcup_{n=1}^{\infty} F_n$ has empty interior. In the case that X is compact this argument uses only the compact case of (ii), so (i) too is first proved for compact X and then, with the general case of (ii), for locally compact Hausdorff X .

Problem 21.19. Use the preceding problem to prove the following: Let X be a locally compact Hausdorff space. If $\mathcal{O}$ is an open subset of X that is contained in a countable union $\bigcup_{n=1}^{\infty} F_n$ of closed subsets of X , then the union of their interiors, $\bigcup_{n=1}^{\infty} \text{int } F_n$, is an open dense subset of $\mathcal{O}$.

Solution. Write $\mathcal{G} = \bigcup_{n=1}^{\infty} \text{int } F_n$. The set $\mathcal{G}$ is open, being a union of open sets, and hence $\mathcal{G} \cap \mathcal{O}$ is an open subset of $\mathcal{O}$. The assertion to be proved is that $\mathcal{G}$ is dense in $\mathcal{O}$, that is, that $\mathcal{G}$ meets every nonempty open subset of $\mathcal{O}$, equivalently $\mathcal{O} \subseteq \overline{\mathcal{G}}$. (Equivalently, $\mathcal{G} \cap \mathcal{O}$ is an open dense subset of the subspace $\mathcal{O}$.)

We use the fact, established in Problem 5 of this section, that an open subset of a locally compact Hausdorff space, with its subspace topology, is again locally compact; it is Hausdorff since a subspace of a Hausdorff space is Hausdorff.

Let $\mathcal{W}$ be a nonempty open subset of $\mathcal{O}$. Since $\mathcal{O}$ is open in X , the set $\mathcal{W}$ is open in X , and hence $\mathcal{W}$, with its subspace topology, is a locally compact Hausdorff space. Because $\mathcal{W} \subseteq \mathcal{O} \subseteq \bigcup_{n=1}^{\infty} F_n$,

$$\mathcal{W} = \bigcup_{n=1}^{\infty} (F_n \cap \mathcal{W}),$$

and each $F_n \cap \mathcal{W}$ is a closed subset of the space $\mathcal{W}$, since F_n is closed in X .

Suppose, by way of contradiction, that every $F_n \cap \mathcal{W}$ had empty interior relative to $\mathcal{W}$. Then part (i)

of the preceding problem, applied in the locally compact Hausdorff space $\mathcal{W}$, would tell us that the union $\bigcup_{n=1}^{\infty} (F_n \cap \mathcal{W}) = \mathcal{W}$ has empty interior relative to $\mathcal{W}$. But $\mathcal{W}$ is the whole space here, so its interior relative to $\mathcal{W}$ is $\mathcal{W}$ itself, which is nonempty. This contradiction shows that there is an index n for which $F_n \cap \mathcal{W}$ has nonempty interior relative to $\mathcal{W}$.

Fix such an n and let $\mathcal{V}$ be a nonempty subset of $\mathcal{W}$ that is open in $\mathcal{W}$ and contained in $F_n \cap \mathcal{W}$. Since $\mathcal{W}$ is open in X , the set $\mathcal{V}$ is open in X . From $\mathcal{V} \subseteq F_n$ and the openness of $\mathcal{V}$ in X we conclude $\mathcal{V} \subseteq \text{int } F_n \subseteq \mathcal{G}$. As $\mathcal{V}$ is a nonempty subset of $\mathcal{W}$, we have $\mathcal{W} \cap \mathcal{G} \neq \emptyset$.

Thus $\mathcal{G}$ meets every nonempty open subset of $\mathcal{O}$. Consequently every point of $\mathcal{O}$ is a point of closure of $\mathcal{G}$: if $x \in \mathcal{O}$ and $\mathcal{U}$ is a neighborhood of x in X , then $\mathcal{U} \cap \mathcal{O}$ is a nonempty open subset of $\mathcal{O}$, hence meets $\mathcal{G}$, so $\mathcal{U}$ meets $\mathcal{G}$. Therefore $\mathcal{O} \subseteq \overline{\mathcal{G}}$, that is, $\bigcup_{n=1}^{\infty} \text{int } F_n$ is an open set that is dense in $\mathcal{O}$.

Problem 21.20. For a map $f: X \rightarrow Y$ and a collection $\mathcal{C}$ of subsets of Y we define $f^*\mathcal{C}$ to be the collection of subsets of X given by

$$f^*\mathcal{C} = \{E \mid E = f^{-1}[C] \text{ for some } C \in \mathcal{C}\}.$$

Show that if $\mathcal{A}$ is the σ -algebra generated by $\mathcal{C}$, then $f^*\mathcal{A}$ is the σ -algebra generated by $f^*\mathcal{C}$.

Solution. We use repeatedly the elementary identities for inverse images: for subsets B , $\{B_n\}_{n=1}^{\infty}$ of Y ,

$$f^{-1}[Y] = X, \quad f^{-1}[Y \sim B] = X \sim f^{-1}[B], \quad f^{-1}\left[\bigcup_{n=1}^{\infty} B_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[B_n].$$

Each is immediate from the definition: $x \in f^{-1}[B]$ means $f(x) \in B$.

Let $\mathcal{A}$ be the σ -algebra of subsets of Y generated by $\mathcal{C}$, and let $\mathcal{B}$ be the σ -algebra of subsets of X generated by $f^*\mathcal{C}$. We must show $f^*\mathcal{A} = \mathcal{B}$.

Step 1: $f^*\mathcal{A}$ is a σ -algebra of subsets of X . First, $X = f^{-1}[Y]$ and $Y \in \mathcal{A}$, so $X \in f^*\mathcal{A}$. Next, let $E \in f^*\mathcal{A}$, say $E = f^{-1}[A]$ with $A \in \mathcal{A}$. Then $Y \sim A \in \mathcal{A}$ and

$$X \sim E = X \sim f^{-1}[A] = f^{-1}[Y \sim A] \in f^*\mathcal{A}.$$

Finally, let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of members of $f^*\mathcal{A}$, say $E_n = f^{-1}[A_n]$ with $A_n \in \mathcal{A}$. Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ and

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} f^{-1}[A_n] = f^{-1}\left[\bigcup_{n=1}^{\infty} A_n\right] \in f^*\mathcal{A}.$$

Thus $f^*\mathcal{A}$ is a σ -algebra of subsets of X .

Step 2: $\mathcal{B} \subseteq f^*\mathcal{A}$. Since $\mathcal{C} \subseteq \mathcal{A}$, every set of the form $f^{-1}[C]$ with $C \in \mathcal{C}$ is of the form $f^{-1}[A]$ with $A \in \mathcal{A}$; that is, $f^*\mathcal{C} \subseteq f^*\mathcal{A}$. By Step 1, $f^*\mathcal{A}$ is a σ -algebra containing $f^*\mathcal{C}$, and $\mathcal{B}$ is the smallest such σ -algebra, so $\mathcal{B} \subseteq f^*\mathcal{A}$.

Step 3: $f^*\mathcal{A} \subseteq \mathcal{B}$. Define

$$\mathcal{D} = \{A \subseteq Y \mid f^{-1}[A] \in \mathcal{B}\}.$$

We claim $\mathcal{D}$ is a σ -algebra of subsets of Y . Indeed, $f^{-1}[Y] = X \in \mathcal{B}$, so $Y \in \mathcal{D}$. If $A \in \mathcal{D}$, then $f^{-1}[Y \sim A] = X \sim f^{-1}[A] \in \mathcal{B}$, since $\mathcal{B}$ is closed with respect to complements; thus $Y \sim A \in \mathcal{D}$. If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{D}$, then $f^{-1}[\bigcup_{n=1}^{\infty} A_n] = \bigcup_{n=1}^{\infty} f^{-1}[A_n] \in \mathcal{B}$, since $\mathcal{B}$ is closed with respect to countable unions; thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$. This proves the claim.

Moreover $\mathcal{C} \subseteq \mathcal{D}$: if $C \in \mathcal{C}$, then $f^{-1}[C] \in f^*\mathcal{C} \subseteq \mathcal{B}$. Since $\mathcal{A}$ is the smallest σ -algebra of subsets of Y containing $\mathcal{C}$, we conclude $\mathcal{A} \subseteq \mathcal{D}$. Therefore $f^{-1}[A] \in \mathcal{B}$ for every $A \in \mathcal{A}$, that is, $f^*\mathcal{A} \subseteq \mathcal{B}$.

Combining Steps 2 and 3, $f^*\mathcal{A} = \mathcal{B}$: the collection $f^*\mathcal{A}$ is precisely the σ -algebra of subsets of X generated by $f^*\mathcal{C}$.

Problem 21.21. For a map $f: X \rightarrow Y$ and a collection $\mathcal{C}$ of subsets of X , let $\mathcal{A}$ be the σ -algebra generated by $\mathcal{C}$. If $f^{-1}[f[C]] = C$ for each $C \in \mathcal{C}$, show that $f^{-1}[f[A]] = A$ for each $A \in \mathcal{A}$.

Solution. Define

$$\mathcal{D} = \{A \subseteq X \mid f^{-1}[f[A]] = A\}.$$

By hypothesis $\mathcal{C} \subseteq \mathcal{D}$, so it suffices to show that $\mathcal{D}$ is a σ -algebra of subsets of X : then, since $\mathcal{A}$ is the smallest σ -algebra of subsets of X containing $\mathcal{C}$, we will have $\mathcal{A} \subseteq \mathcal{D}$, which is exactly the assertion that $f^{-1}[f[A]] = A$ for every $A \in \mathcal{A}$.

We first identify $\mathcal{D}$ as the collection of inverse images of subsets of Y :

$$\mathcal{D} = \{f^{-1}[B] \mid B \subseteq Y\}.$$

Call this identity the description of $\mathcal{D}$. To see this, observe first that for every subset A of X we have $A \subseteq f^{-1}[f[A]]$: if $x \in A$, then $f(x) \in f[A]$, so $x \in f^{-1}[f[A]]$.

Suppose $A \in \mathcal{D}$. Then $A = f^{-1}[B]$ for the subset $B = f[A]$ of Y , so A belongs to the collection just described.

Conversely, suppose $A = f^{-1}[B]$ for some $B \subseteq Y$. Then $f[A] = f[f^{-1}[B]] \subseteq B$, and since inverse images preserve inclusion,

$$f^{-1}[f[A]] \subseteq f^{-1}[B] = A.$$

Together with the inclusion $A \subseteq f^{-1}[f[A]]$ noted above, this gives $f^{-1}[f[A]] = A$, so $A \in \mathcal{D}$. This proves the description of $\mathcal{D}$.

Now we verify that the collection just described is a σ -algebra of subsets of X , using the identities $f^{-1}[Y] = X$, $f^{-1}[Y \sim B] = X \sim f^{-1}[B]$ and $f^{-1}[\bigcup_n B_n] = \bigcup_n f^{-1}[B_n]$, each immediate from the definition of inverse image.

The set $X = f^{-1}[Y]$ belongs to $\mathcal{D}$. If $A = f^{-1}[B] \in \mathcal{D}$, then

$$X \sim A = X \sim f^{-1}[B] = f^{-1}[Y \sim B] \in \mathcal{D}.$$

If $\{A_n\}_{n=1}^\infty$ is a countable collection of members of $\mathcal{D}$, say $A_n = f^{-1}[B_n]$ with $B_n \subseteq Y$, then

$$\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty f^{-1}[B_n] = f^{-1}\left[\bigcup_{n=1}^\infty B_n\right] \in \mathcal{D}.$$

Hence $\mathcal{D}$ is a σ -algebra of subsets of X which, by hypothesis, contains $\mathcal{C}$. Therefore $\mathcal{A} \subseteq \mathcal{D}$, that is,

$$f^{-1}[f[A]] = A \quad \text{for each } A \in \mathcal{A}.$$

21.4 Problems 21.22–21.28

Problem 21.22. Show that Euclidean space $\mathbb{R}^n$ is locally compact.

Solution. Recall the definition in force: a topological space X is locally compact provided each point of X has a neighborhood whose closure is compact.

Let $x \in \mathbb{R}^n$ and let

$$N = B(x, 1) = \{y \in \mathbb{R}^n : \|y - x\| < 1\},$$

which is an open set containing x , hence a neighborhood of x . Put

$$\bar{B}(x, 1) = \{y \in \mathbb{R}^n : \|y - x\| \leq 1\}.$$

The set $\bar{B}(x, 1)$ is closed: its complement is $\{y : \|y - x\| > 1\}$, and this set is open because the function $y \mapsto \|y - x\|$ is continuous on $\mathbb{R}^n$ (it is Lipschitz with constant 1, by the triangle inequality), so the preimage of the open ray $(1, \infty)$ is open. The set $\bar{B}(x, 1)$ is also bounded, since every one of its points lies within distance 1 of x . By Theorem 20 of Chapter 9, the Heine-Borel Theorem for $\mathbb{R}^n$, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Therefore $\bar{B}(x, 1)$ is compact.

Since $N \subseteq \bar{B}(x, 1)$ and $\bar{B}(x, 1)$ is closed, the closure $\bar{N}$, being the smallest closed set containing N , satisfies $\bar{N} \subseteq \bar{B}(x, 1)$. Hence $\bar{N}$ is itself closed and bounded, and so, again by Theorem 20 of Chapter 9, $\bar{N}$ is compact.

So every point of $\mathbb{R}^n$ has a neighborhood with compact closure, that is, $\mathbb{R}^n$ is locally compact.

Two remarks. First, the same argument applied to $B(x, r)$ for arbitrary $r > 0$ shows that each point of $\mathbb{R}^n$ possesses a base of neighborhoods with compact closure. Second, $\mathbb{R}^n$ is Hausdorff, since distinct points $x \neq y$ are separated by the disjoint open balls $B(x, \|x - y\|/2)$ and $B(y, \|x - y\|/2)$; so $\mathbb{R}^n$ is a locally compact Hausdorff space, and every result of this chapter applies to it. It is not compact, since it is unbounded.

Problem 21.23. Show that ℓ^p , for $1 \leq p \leq \infty$, fails to be locally compact.

Solution. Here ℓ^p , for $1 \leq p < \infty$, is the linear space of real sequences $x = (x_1, x_2, \dots)$ for which $\sum_{k=1}^{\infty} |x_k|^p < \infty$, normed by $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}$, while ℓ^∞ is the space of bounded real sequences normed by $\|x\|_\infty = \sup_k |x_k|$. In each case the topology is the one induced by the metric $\rho(x, y) = \|x - y\|_p$.

Argue by contradiction: suppose ℓ^p is locally compact. Then the point 0 has a neighborhood N whose closure $\bar{N}$ is compact. Since N is open and contains 0, there is an $r > 0$ for which $B(0, r) \subseteq N$.

For each natural number k let e_k denote the sequence whose k -th term is 1 and whose other terms are 0. Each e_k belongs to ℓ^p and $\|e_k\|_p = 1$. Moreover, for $j \neq k$ the sequence $e_j - e_k$ has exactly two nonzero terms, each of absolute value 1, so

$$\|e_j - e_k\|_p = 2^{1/p} \text{ if } 1 \leq p < \infty, \quad \|e_j - e_k\|_\infty = 1.$$

In every case $\|e_j - e_k\|_p \geq 1$ for $j \neq k$.

Define $x_k = (r/2)e_k$ and $d = r/2$. Then $\|x_k\|_p = r/2 < r$, so each x_k belongs to $B(0, r) \subseteq \bar{N}$, while

$$\|x_j - x_k\|_p = \frac{r}{2} \|e_j - e_k\|_p \geq d \text{ for } j \neq k.$$

In particular the x_k are distinct, so $S = \{x_k : k \in \mathbb{N}\}$ is an infinite subset of $\bar{N}$.

The set S is closed in ℓ^p . Indeed, let y be a point of closure of S . The ball $B(y, d/3)$ contains some x_j , and it cannot contain a second point x_k with $k \neq j$, since that would force $\|x_j - x_k\|_p < 2d/3 < d$. Hence $B(y, d/3) \cap S = \{x_j\}$. But every ball about y meets S , so for each ε with $0 < \varepsilon < d/3$ the ball $B(y, \varepsilon)$ meets S , necessarily in the point x_j ; thus $\|y - x_j\|_p < \varepsilon$ for all such ε , and therefore $y = x_j$ belongs to S . So S contains all its points of closure, that is, S is closed.

Consequently S is a closed subset of the compact set $\bar{N}$, and a closed subset of a compact space is compact. On the other hand, the collection of open balls $\{B(x_k, d/2)\}_{k=1}^{\infty}$ is a cover of S by open sets, and $B(x_k, d/2)$ contains exactly one point of S , namely x_k , since distinct points of S are at distance at least d from one another. Therefore no finite subcollection covers the infinite set S , so S is not compact. This contradiction shows that ℓ^p is not locally compact, for any p with $1 \leq p \leq \infty$.

Alternatively, one may quote Riesz's Theorem of Section 13.3: the closed unit ball of a normed linear space is compact if and only if the space is finite dimensional. The vectors $\{e_k\}_{k=1}^{\infty}$ are linearly independent, so ℓ^p is infinite dimensional, and hence, as observed in Section 21.1, it is not locally compact. The argument given above is a direct verification of this in the case at hand.

Problem 21.24. Show that $C([0, 1])$, with the topology induced by the maximum norm, is not locally compact.

Solution. Here $C([0, 1])$ is the linear space of continuous real-valued functions on $[0, 1]$, normed by $\|f\|_{\max} = \max_{0 \leq t \leq 1} |f(t)|$, and the topology is that induced by the metric $\rho(f, g) = \|f - g\|_{\max}$.

Suppose, to get a contradiction, that $C([0, 1])$ is locally compact. Then the function 0 has a neighborhood N with compact closure $\bar{N}$, and since N is open there is an $r > 0$ for which $B(0, r) \subseteq N$.

We produce infinitely many functions in $B(0, r)$ that are uniformly far apart. For each natural number

k let

$$a_k = \frac{1}{k+1}, \quad b_k = \frac{1}{k}, \quad m_k = \frac{a_k + b_k}{2}, \quad h_k = \frac{b_k - a_k}{2} = \frac{1}{2k(k+1)},$$

and define the tent function

$$g_k(t) = \max \left\{ 0, 1 - \frac{|t - m_k|}{h_k} \right\} \quad \text{for } t \in [0, 1].$$

Each g_k is continuous, being the maximum of two continuous functions; it takes values in $[0, 1]$; $g_k(m_k) = 1$; and $g_k(t) = 0$ whenever $|t - m_k| \geq h_k$, that is, whenever t lies outside the open interval (a_k, b_k) . Thus $\|g_k\|_{\max} = 1$.

For $j \neq k$ the open intervals (a_j, b_j) and (a_k, b_k) are disjoint, since the numbers $1, 1/2, 1/3, \dots$ are strictly decreasing. Hence at each $t \in [0, 1]$ at most one of $g_j(t)$, $g_k(t)$ is nonzero, and both lie in $[0, 1]$, so $|g_j(t) - g_k(t)| \leq 1$ for all t , with equality at $t = m_j$. Therefore

$$\|g_j - g_k\|_{\max} = 1 \quad \text{for } j \neq k.$$

Now put $f_k = (r/2)g_k$ and $d = r/2$. Then $\|f_k\|_{\max} = r/2 < r$, so $f_k \in B(0, r) \subseteq \bar{N}$, and $\|f_j - f_k\|_{\max} = d$ for $j \neq k$. In particular the f_k are distinct, so $S = \{f_k : k \in \mathbb{N}\}$ is an infinite subset of $\bar{N}$.

The set S is closed in $C([0, 1])$. Let g be a point of closure of S . The ball $B(g, d/3)$ contains some f_j , and it cannot contain a second f_k with $k \neq j$, for then $\|f_j - f_k\|_{\max} < 2d/3 < d$. So $B(g, d/3) \cap S = \{f_j\}$. Since every ball about g meets S , for each ε with $0 < \varepsilon < d/3$ the ball $B(g, \varepsilon)$ meets S in f_j , whence $\|g - f_j\|_{\max} < \varepsilon$ for all such ε and therefore $g = f_j$ belongs to S . So S is closed.

Thus S is a closed subset of the compact set $\bar{N}$, hence is compact. But $\{B(f_k, d/2)\}_{k=1}^{\infty}$ is a cover of S by open sets, and each $B(f_k, d/2)$ contains exactly one point of S , namely f_k , because distinct points of S are at distance d from each other. So this open cover of the infinite set S has no finite subcover, and S is not compact. This contradiction shows that $C([0, 1])$ is not locally compact.

We remark that, as with ℓ^p , the conclusion also follows from Riesz's Theorem of Section 13.3, since the functions $\{g_k\}_{k=1}^{\infty}$ constructed above are linearly independent (their supports are disjoint), so that $C([0, 1])$ is infinite dimensional.

Problem 21.25. Let $\rho: X \times X \rightarrow \mathbb{R}$ be a metric on a set X . For $A \subseteq X$, consider the distance function

$$\text{dist}_A: X \rightarrow [0, \infty).$$

- (i) Show that the function dist_A is continuous.
- (ii) If $A \subseteq X$ is closed and x is a point in X , show that $\text{dist}_A(x) = 0$ if and only if x belongs to A .
- (iii) If $A \subseteq X$ is closed and x belongs to X , show that there may not exist a point x_0 in A for which $\text{dist}_A(x) = \rho(x, x_0)$, but there is such a point x_0 if K is compact.

Solution. As defined in Section 21.2,

$$\text{dist}_A(x) = \inf_{y \in A} \rho(x, y) \quad \text{for } x \in X.$$

Throughout we take A to be nonempty; this is needed for dist_A to have values in $[0, \infty)$, since the infimum over the empty set is $+\infty$. In part (iii) the letter K in the last clause is a misprint for A : the assertion is that a nearest point exists if A is compact.

(i) We show more, namely that dist_A is Lipschitz with Lipschitz constant 1. Let $x, z \in X$. For each $y \in A$ the triangle inequality gives

$$\text{dist}_A(x) \leq \rho(x, y) \leq \rho(x, z) + \rho(z, y),$$

so that $\text{dist}_A(x) - \rho(x, z) \leq \rho(z, y)$ for every $y \in A$. Taking the infimum over $y \in A$ of the right-hand side,

$$\text{dist}_A(x) - \rho(x, z) \leq \text{dist}_A(z), \quad \text{that is,} \quad \text{dist}_A(x) - \text{dist}_A(z) \leq \rho(x, z).$$

Interchanging x and z and using the symmetry of ρ gives $\text{dist}_A(z) - \text{dist}_A(x) \leq \rho(x, z)$. Hence

$$|\text{dist}_A(x) - \text{dist}_A(z)| \leq \rho(x, z) \quad \text{for all } x, z \in X.$$

Given $\varepsilon > 0$, the choice $\delta = \varepsilon$ shows that $\rho(x, z) < \delta$ implies $|\text{dist}_A(x) - \text{dist}_A(z)| < \varepsilon$. Thus dist_A is uniformly continuous, and in particular continuous.

(ii) Let A be closed and $x \in X$. If $x \in A$, then $0 \leq \text{dist}_A(x) \leq \rho(x, x) = 0$, so $\text{dist}_A(x) = 0$. Conversely, suppose $\text{dist}_A(x) = 0$. Let $\varepsilon > 0$. Since the infimum defining $\text{dist}_A(x)$ is $0 < \varepsilon$, there is a point $y \in A$ with $\rho(x, y) < \varepsilon$, that is, the ball $B(x, \varepsilon)$ meets A . As $\varepsilon > 0$ was arbitrary, every neighborhood of x meets A , so x is a point of closure of A . Since A is closed it contains all its points of closure, and therefore $x \in A$.

Closedness is essential here: in $X = \mathbb{R}$ with the usual metric, $A = (0, 1)$ has $\text{dist}_A(0) = 0$ although $0 \notin A$.

(iii) First, an example in which the infimum is not attained. Let

$$X = \{0\} \cup \{1 + \frac{1}{k} : k \in \mathbb{N}\},$$

regarded as a metric space with the metric $\rho(s, t) = |s - t|$ inherited from $\mathbb{R}$, and let

$$A = \{1 + \frac{1}{k} : k \in \mathbb{N}\}, \quad x = 0.$$

The set A is closed in X : its complement in X is $\{0\}$, and $\{0\}$ is open in X because every point of X other than 0 is at distance greater than 1 from 0, so that $B(0, 1) \cap X = \{0\}$. Now

$$\text{dist}_A(0) = \inf_{k \in \mathbb{N}} |0 - (1 + \frac{1}{k})| = \inf_{k \in \mathbb{N}} (1 + \frac{1}{k}) = 1,$$

while $\rho(0, y) = y > 1$ for every $y \in A$. Hence there is no point $x_0 \in A$ with $\text{dist}_A(0) = \rho(0, x_0)$. Observe that the number 1, which would be the nearest point, is missing from the space; this is the general reason such examples exist.

Now suppose A is a nonempty compact subset of X and let $x \in X$. Define $f: X \rightarrow \mathbb{R}$ by $f(y) = \rho(x, y)$. By the triangle inequality, $|f(y) - f(z)| = |\rho(x, y) - \rho(x, z)| \leq \rho(y, z)$, so f is continuous on X ; hence its restriction to A is a continuous real-valued function on the compact metric space A . By the Extreme Value Theorem, Theorem 22 of Chapter 9, this restriction attains a minimum value: there is a point $x_0 \in A$ for which

$$\rho(x, x_0) = f(x_0) = \min_{y \in A} f(y) = \min_{y \in A} \rho(x, y).$$

A minimum of a set of real numbers is its infimum, so $\rho(x, x_0) = \inf_{y \in A} \rho(x, y) = \text{dist}_A(x)$, which is what we wanted. (There is no conflict with the hypothesis that A be closed in the first part of (iii): a compact subset of a metric space is closed.)

Problem 21.26. Show that property (ii) in the statement of Theorem 7 is equivalent to the assertion that two disjoint closed subsets of X , one of which is compact, may be separated by disjoint neighborhoods.

Solution. Throughout, $(X, \mathcal{T})$ is a Hausdorff space, as in Theorem 7. Property (ii) of that theorem reads: if $\mathcal{O}$ is a neighborhood of a compact subset K of X , then there is a neighborhood U of K that has compact closure contained in $\mathcal{O}$. Recall that a neighborhood of a set means an open set containing it. Denote by (S) the separation assertion in question:

(S) if K and F are disjoint closed subsets of X and K is compact, then there are disjoint neighborhoods U of K and V of F .

Since X is Hausdorff, a compact subset of X is closed, so the sets to be separated are exactly a compact set and a disjoint closed set. We prove that (ii) is equivalent to (S) taken in the sharp form in which the neighborhood of the compact set is required to have compact closure, and then show by example that this last requirement may not be dropped. So let (S') denote:

(S') if K and F are disjoint closed subsets of X and K is compact, then there are disjoint neighborhoods U of K and V of F for which $\bar{U}$ is compact.

Proof that (ii) implies (S'), and hence (S). Let K be compact and F closed with $K \cap F = \emptyset$. Then $\mathcal{O} = X \sim F$ is open and contains K , so it is a neighborhood of the compact set K . By property (ii) there is a neighborhood U of K whose closure $\bar{U}$ is compact and contained in $\mathcal{O}$. Define

$$V = X \sim \bar{U},$$

which is open since $\bar{U}$ is closed. From $\bar{U} \subseteq \mathcal{O} = X \sim F$ we get $F \subseteq X \sim \bar{U} = V$, so V is a neighborhood of F . Finally $U \cap V = \emptyset$, since $U \subseteq \bar{U}$. Thus U and V are disjoint neighborhoods of K and F , and $\bar{U}$ is compact.

Proof that (S') implies (ii). Let K be a compact subset of X and $\mathcal{O}$ a neighborhood of K . Put $F = X \sim \mathcal{O}$. Then F is closed and disjoint from K , since $K \subseteq \mathcal{O}$. (If $F = \emptyset$, this is still a legitimate application of (S'), as the empty set is closed and disjoint from K .) By (S') there are disjoint neighborhoods U of K and V of F with $\bar{U}$ compact. Since $U \cap V = \emptyset$ we have $U \subseteq X \sim V$, and $X \sim V$ is closed; as $\bar{U}$ is the smallest closed set containing U ,

$$\bar{U} \subseteq X \sim V \subseteq X \sim F = \mathcal{O},$$

the middle inclusion holding because $F \subseteq V$. Therefore U is a neighborhood of K whose closure is compact and contained in $\mathcal{O}$. This is property (ii).

The compactness of $\bar{U}$ may not be dropped from (S'). Indeed, for a Hausdorff space X , assertion (S) alone is equivalent to regularity: taking K to be a single point (points are closed in a Hausdorff space) shows that (S) implies that a point and a disjoint closed set may be separated by disjoint neighborhoods; conversely, if X is regular and K is compact and disjoint from the closed set F , then, in the case $K \neq \emptyset$, for each $x \in K$ choose disjoint neighborhoods U_x of x and V_x of F , extract from the cover $\{U_x\}_{x \in K}$ of K a finite subcover $U_{x_1}, \dots, U_{x_n}$ with $n \geq 1$, and note that $U = \bigcup_{k=1}^n U_{x_k}$ and $V = \bigcap_{k=1}^n V_{x_k}$ are open, are disjoint since $U_{x_k} \cap V \subseteq U_{x_k} \cap V_{x_k} = \emptyset$ for each k , and contain K and F respectively; while if $K = \emptyset$ one takes $U = \emptyset$ and $V = X$. But a regular Hausdorff space need not be locally compact, and property (ii) is, by Theorem 7, equivalent to local compactness.

For a concrete instance, take $X = \ell^2$ with the topology induced by its norm. By Proposition 7 of Chapter 11 every metric space is normal, so any two disjoint closed subsets of ℓ^2 may be separated by disjoint neighborhoods; thus (S) holds in ℓ^2 . (This may also be seen directly from the preceding problem: if F_1 and F_2 are disjoint nonempty closed sets, then $U = \{x : \text{dist}_{F_1}(x) < \text{dist}_{F_2}(x)\}$ and $V = \{x : \text{dist}_{F_2}(x) < \text{dist}_{F_1}(x)\}$ are disjoint, are open by the continuity of the distance functions, and contain F_1 and F_2 respectively by part (ii) of Problem 25.) Nevertheless ℓ^2 is not locally compact, by Problem 23, so property (ii) fails for ℓ^2 : no neighborhood of the compact set $\{0\}$ inside the neighborhood $\mathcal{O} = X$ has compact closure.

Summarizing: property (ii) says exactly that two disjoint closed subsets of X , one of them compact, may be separated by disjoint neighborhoods, the neighborhood of the compact set being chosen with compact closure. It is in this sense, which is the one used in the text where property (ii) is applied, that the assertion of the problem is to be read.

Problem 21.27. Let $(X, \mathcal{T})$ be a Hausdorff topological space. Show that $\mathcal{T}$ is a semiring if and only if $\mathcal{T}$ is the discrete topology.

Solution. Recall the definition from Section 17.5: a nonempty collection $\mathcal{S}$ of subsets of X is called a semiring provided that whenever A and B belong to $\mathcal{S}$, the intersection $A \cap B$ belongs to $\mathcal{S}$ and there is a finite disjoint collection $\{C_k\}_{k=1}^n$ of sets in $\mathcal{S}$ for which

$$A \sim B = \bigcup_{k=1}^n C_k.$$

First suppose $\mathcal{T}$ is the discrete topology, that is, $\mathcal{T} = 2^X$ is the collection of all subsets of X . Then $\mathcal{T}$ is nonempty, it is closed with respect to intersections, and for $A, B \in \mathcal{T}$ the relative complement

$A \sim B$ itself belongs to $\mathcal{T}$, so it is the union of the disjoint one-set collection $\{A \sim B\}$ of members of $\mathcal{T}$. Hence $\mathcal{T}$ is a semiring. (Observe that Hausdorffness was not needed here; the discrete topology is Hausdorff in any case.)

Now suppose $\mathcal{T}$ is a semiring. A topology is nonempty and is closed with respect to the formation of finite intersections, so the only additional information carried by the semiring property is the following: for all open sets A and B there is a finite disjoint collection $\{C_k\}_{k=1}^n$ of open sets with $A \sim B = \bigcup_{k=1}^n C_k$. Since the union of any collection of open sets is open, this says precisely that for all open A and B , the set $A \sim B$ is open.

Take $A = X$, which belongs to $\mathcal{T}$. We conclude that for every open set B the set $X \sim B$ is open; that is, every open subset of X is also closed.

From this it follows that every closed subset of X is open. Indeed, let F be closed. Then $X \sim F$ is open, hence by the previous paragraph $X \sim F$ is also closed, and therefore $F = X \sim (X \sim F)$ is open. Finally we use the Hausdorff hypothesis. Let $x \in X$. For each $y \in X$ with $y \neq x$, there are disjoint open sets U_y and V_y with $y \in U_y$ and $x \in V_y$; in particular $x \notin U_y$, so $U_y \subseteq X \sim \{x\}$. Hence

$$X \sim \{x\} = \bigcup_{y \neq x} U_y$$

is a union of open sets and so is open, which means that $\{x\}$ is closed. By the preceding paragraph, $\{x\}$ is therefore open.

Since every singleton is open, every subset E of X is open, being the union $E = \bigcup_{x \in E} \{x\}$ of open sets (the empty set is open as well). Thus $\mathcal{T} = 2^X$ is the discrete topology.

Two remarks. Only the consequence of the Hausdorff property that singletons are closed was used; by Proposition 6 of Chapter 11 that consequence is precisely the Tychonoff separation property, so the conclusion holds under this weaker hypothesis. Some such hypothesis is necessary: if X has at least two points and $\mathcal{T} = \{\emptyset, X\}$ is the indiscrete topology, then all the relative complements formed from members of $\mathcal{T}$, namely $X \sim \emptyset = X$, $\emptyset \sim X = \emptyset$ and $X \sim X = \emptyset$, belong to $\mathcal{T}$, so $\mathcal{T}$ is a semiring although it is not discrete. This is the assertion used in Section 21.3 to explain why the Caratheodory-Hahn Theorem cannot be invoked to extend a premeasure defined on a topology: a topology of interest there, being Hausdorff but not discrete, is not a semiring.

Problem 21.28. (Tyagi) Let $(X, \mathcal{T})$ be a topological space and $\mu: \mathcal{T} \rightarrow [0, \infty]$ a premeasure. Assume that if $\mathcal{O}$ is open and $\mu(\mathcal{O}) < \infty$, then $\mu(\text{bd } \mathcal{O}) = 0$. Show that every open set is μ^* -measurable.

Solution. Some preliminary remarks on the reading of the hypothesis. For $\mathcal{O}$ open, the set $X \sim \mathcal{O}$ is closed, so

$$\text{bd } \mathcal{O} = \bar{\mathcal{O}} \cap \overline{X \sim \mathcal{O}} = \bar{\mathcal{O}} \cap (X \sim \mathcal{O}) = \bar{\mathcal{O}} \sim \mathcal{O},$$

and this set is in general not open, so the set function μ , whose domain is the topology $\mathcal{T}$, need not be defined on it. The hypothesis is therefore to be read in terms of the induced outer measure: if $\mathcal{O}$ is open and $\mu(\mathcal{O}) < \infty$, then $\mu^*(\text{bd } \mathcal{O}) = 0$. This is the sense in which we use it.

Recall that the Caratheodory outer measure induced by μ is

$$\mu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \mu(U_k) : \{U_k\}_{k=1}^{\infty} \subseteq \mathcal{T} \text{ and } E \subseteq \bigcup_{k=1}^{\infty} U_k \right\},$$

and that μ^* is an outer measure, so it is monotone and countably subadditive.

We will use two standing facts. First, a premeasure is countably monotone, and hence monotone (cover a set by the single set containing it); moreover, for each open set U we have $\mu^*(U) = \mu(U)$, since $\{U\}$ is itself a cover of U by members of $\mathcal{T}$, while countable monotonicity gives the reverse inequality $\mu(U) \leq \sum_k \mu(U_k)$ for every countable cover $\{U_k\}$ of U by open sets. Second, by Lemma 8, a subset E of X is μ^* -measurable provided

$$\mu(\mathcal{O}) \geq \mu^*(\mathcal{O} \cap E) + \mu^*(\mathcal{O} \sim E) \quad \text{for each open } \mathcal{O} \text{ with } \mu(\mathcal{O}) < \infty.$$

Let V be open. We verify this criterion, which is (3) of Lemma 8, for $E = V$. Let $\mathcal{O}$ be open with $\mu(\mathcal{O}) < \infty$ and define

$$W = \mathcal{O} \cap V,$$

which is open, and which has $\mu(W) \leq \mu(\mathcal{O}) < \infty$ by the monotonicity of μ . Since W is an open set of finite measure, the hypothesis tells us that

$$\mu^*(\text{bd } W) = \mu^*(\bar{W} \sim W) = 0.$$

Observe that $\mathcal{O} \sim V = \mathcal{O} \sim (\mathcal{O} \cap V) = \mathcal{O} \sim W$ and that

$$\mathcal{O} \sim W = (\mathcal{O} \sim \bar{W}) \cup (\mathcal{O} \cap (\bar{W} \sim W)),$$

since a point of $\mathcal{O}$ not in W either fails to lie in $\bar{W}$ or lies in $\bar{W} \sim W$. The set $\mathcal{O} \sim \bar{W} = \mathcal{O} \cap (X \sim \bar{W})$ is open. Hence, by the countable subadditivity and the monotonicity of the outer measure μ^* ,

$$\mu^*(\mathcal{O} \sim V) = \mu^*(\mathcal{O} \sim W) \leq \mu^*(\mathcal{O} \sim \bar{W}) + \mu^*(\bar{W} \sim W) = \mu^*(\mathcal{O} \sim \bar{W}) = \mu(\mathcal{O} \sim \bar{W}).$$

On the other hand, W and $\mathcal{O} \sim \bar{W}$ are disjoint open sets, since $W \subseteq \bar{W}$, and their union is an open subset of $\mathcal{O}$. Therefore, by the monotonicity and the finite additivity of the premeasure μ ,

$$\mu(\mathcal{O}) \geq \mu(W \cup (\mathcal{O} \sim \bar{W})) = \mu(W) + \mu(\mathcal{O} \sim \bar{W}).$$

Since $\mu(W) = \mu^*(W) = \mu^*(\mathcal{O} \cap V)$, combining the two displays gives

$$\mu(\mathcal{O}) \geq \mu^*(\mathcal{O} \cap V) + \mu^*(\mathcal{O} \sim V).$$

As $\mathcal{O}$ was an arbitrary open set of finite measure, criterion (3) of Lemma 8 is satisfied, and therefore V is μ^* -measurable. Since V was an arbitrary open set, every open set is μ^* -measurable.

We add the customary consequence, exactly as in Proposition 9: since the μ^* -measurable sets form a σ -algebra containing $\mathcal{T}$, and the Borel σ -algebra $\mathcal{B}(X)$ is the smallest σ -algebra containing $\mathcal{T}$, every Borel subset of X is μ^* -measurable; and because $\mu^* = \mu$ on $\mathcal{T}$, the measure $\mu^*: \mathcal{B}(X) \rightarrow [0, \infty]$ is an extension of μ . Thus the hypothesis that open sets of finite measure have boundaries of outer measure zero serves the same purpose as the inner approximation assumption (7) of Proposition 9.

21.5 Problems 21.29–21.35

Problem 21.29. Show that the restriction of Lebesgue measure on the real line to the Borel σ -algebra is a Radon measure.

Solution. Write m^* for Lebesgue outer measure on $\mathbb{R}$, $\mathcal{M}$ for the σ -algebra of Lebesgue measurable sets, and m for the restriction of m^* to $\mathcal{M}$, that is, Lebesgue measure. By Theorem 9 of Chapter 2, $\mathcal{M}$ is a σ -algebra containing the Borel σ -algebra $\mathcal{B}(\mathbb{R})$, so $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}$ and the restriction μ of m to $\mathcal{B}(\mathbb{R})$ is defined. Since m is countably additive on $\mathcal{M}$ (Theorem 14 of Chapter 2) and $\mathcal{B}(\mathbb{R})$ is a σ -algebra contained in $\mathcal{M}$, the set function $\mu: \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ is a measure. We must check three things: that μ is a Borel measure, that it is outer regular, and that it is inner regular on open sets.

μ is a Borel measure. Let $K \subseteq \mathbb{R}$ be compact. Then K is closed, hence Borel, and K is bounded, so $K \subseteq [-M, M]$ for some $M > 0$. By the monotonicity of outer measure and Proposition 1 of Chapter 2 (the outer measure of an interval is its length),

$$\mu(K) = m^*(K) \leq m^*([-M, M]) = 2M < \infty.$$

Thus every compact set has finite measure, and μ is a Borel measure.

Outer regularity. Let E be a Borel set. Every open set $\mathcal{U}$ containing E satisfies $\mu(\mathcal{U}) \geq \mu(E)$ by monotonicity, so

$$\inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\} \geq \mu(E),$$

and it remains to produce neighborhoods of E of measure arbitrarily close to $\mu(E)$. If $\mu(E) = \infty$ the displayed inequality already forces the infimum to be $\infty = \mu(E)$. So assume $\mu(E) = m^*(E) < \infty$ and let $\epsilon > 0$. By the very definition of Lebesgue outer measure as an infimum over countable covers by open bounded intervals, there is a countable collection $\{I_k\}_{k=1}^\infty$ of open bounded intervals with

$$E \subseteq \bigcup_{k=1}^\infty I_k \quad \text{and} \quad \sum_{k=1}^\infty \ell(I_k) < m^*(E) + \epsilon.$$

Put $\mathcal{U} = \bigcup_{k=1}^\infty I_k$. Then $\mathcal{U}$ is open and contains E , so $\mathcal{U}$ is a neighborhood of E , and by the countable subadditivity of outer measure (Proposition 3 of Chapter 2) together with Proposition 1 of Chapter 2,

$$\mu(\mathcal{U}) = m^*(\mathcal{U}) \leq \sum_{k=1}^\infty m^*(I_k) = \sum_{k=1}^\infty \ell(I_k) < \mu(E) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\mu(E) = \inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\}$. (This is also assertion (i) of Theorem 11 of Chapter 2, read together with the excision property.)

Inner regularity on open sets. Let $\mathcal{O}$ be an open subset of $\mathbb{R}$. Since every compact $K \subseteq \mathcal{O}$ has $\mu(K) \leq \mu(\mathcal{O})$, we need only exhibit compact subsets of $\mathcal{O}$ whose measures converge to $\mu(\mathcal{O})$. If $\mathcal{O} = \mathbb{R}$, set $F_n = [-n, n]$. Otherwise $F = \mathbb{R} \setminus \mathcal{O}$ is a nonempty closed set and we may define, for each natural number n ,

$$F_n = \{x \in \mathbb{R} \mid |x| \leq n \text{ and } \text{dist}_F(x) \geq 1/n\},$$

where $\text{dist}_F(x) = \inf\{|x - y| \mid y \in F\}$. The function dist_F is continuous (indeed Lipschitz with constant 1; see Problem 25 of this chapter), so F_n is the intersection of the closed set $[-n, n]$ with the closed set $\text{dist}_F^{-1}([1/n, \infty))$ and is therefore closed. It is bounded, hence compact by the Heine–Borel Theorem. Moreover $F_n \subseteq \mathcal{O}$: if $x \in F_n$ then $\text{dist}_F(x) > 0$, and since F is closed, $\text{dist}_F(x) = 0$ exactly for $x \in F$; hence $x \notin F$, that is, $x \in \mathcal{O}$.

The sequence $\{F_n\}$ is ascending and $\bigcup_{n=1}^\infty F_n = \mathcal{O}$. Indeed, let $x \in \mathcal{O}$. Since $\mathcal{O}$ is open there is $r > 0$ with $(x - r, x + r) \subseteq \mathcal{O}$, so every point of F is at distance at least r from x , that is, $\text{dist}_F(x) \geq r > 0$; choosing n with $n \geq |x|$ and $1/n \leq r$ puts x in F_n . (In the case $\mathcal{O} = \mathbb{R}$ this is clear.) By the continuity of measure (Theorem 15 of Chapter 2),

$$\mu(\mathcal{O}) = \lim_{n \rightarrow \infty} \mu(F_n) \leq \sup\{\mu(K) \mid K \text{ a compact subset of } \mathcal{O}\} \leq \mu(\mathcal{O}).$$

Hence equality holds throughout and μ is inner regular on open sets.

Therefore μ , the restriction of Lebesgue measure to $\mathcal{B}(\mathbb{R})$, is a Radon measure.

Problem 21.30. Show that the restriction of Lebesgue measure on the Euclidean space $\mathbb{R}^n$ to the Borel σ -algebra is a Radon measure.

Solution. Recall the construction of Section 20.2: volume is a premeasure on the semiring of bounded intervals in $\mathbb{R}^n$ (Propositions 9 and 10 of Chapter 20), μ_n^* is the induced outer measure, $\mathcal{L}^n$ is the σ -algebra of μ_n^* -measurable sets, and μ_n is the restriction of μ_n^* to $\mathcal{L}^n$. By Theorem 11 of Chapter 20, $\mathcal{L}^n$ contains the bounded intervals and the Borel sets, and $\mu_n(I) = \text{vol}(I)$ for every bounded interval I . Hence $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}^n$ and the restriction μ of μ_n to $\mathcal{B}(\mathbb{R}^n)$ is a measure on $\mathcal{B}(\mathbb{R}^n)$, since a countably additive set function restricted to a σ -subalgebra of its domain remains countably additive.

μ is a Borel measure. Let $K \subseteq \mathbb{R}^n$ be compact. Then K is bounded, so K is contained in a bounded interval $I = [-M, M] \times \cdots \times [-M, M]$, and by monotonicity

$$\mu(K) \leq \mu_n(I) = \text{vol}(I) = (2M)^n < \infty.$$

Since K is also closed, it is a Borel set, and every compact set has finite measure.

Outer regularity. Let E be a Borel subset of $\mathbb{R}^n$. Monotonicity gives $\mu(\mathcal{U}) \geq \mu(E)$ for every open $\mathcal{U} \supseteq E$, so the infimum over neighborhoods of E is at least $\mu(E)$, with equality automatic when $\mu(E) = \infty$. Assume $\mu(E) = \mu_n^*(E) < \infty$ and let $\epsilon > 0$. By the definition of the outer measure induced

by a premeasure on a semiring, there is a countable collection $\{I_k\}_{k=1}^\infty$ of bounded intervals covering E with

$$\sum_{k=1}^{\infty} \text{vol}(I_k) < \mu_n^*(E) + \epsilon/2.$$

Fix k . Writing $I_k = J_1 \times \cdots \times J_n$ with each J_i a bounded interval with endpoints $a_i \leq b_i$, and letting

$$I_k(\delta) = (a_1 - \delta, b_1 + \delta) \times \cdots \times (a_n - \delta, b_n + \delta),$$

we obtain an open bounded interval containing I_k whose volume $\prod_{i=1}^n (b_i - a_i + 2\delta)$ tends to $\text{vol}(I_k)$ as $\delta \downarrow 0$. Hence we may choose an open interval $\mathcal{I}_k \supseteq I_k$ with $\text{vol}(\mathcal{I}_k) < \text{vol}(I_k) + \epsilon \cdot 2^{-k-1}$. Then $\mathcal{U} = \bigcup_{k=1}^{\infty} \mathcal{I}_k$ is an open set containing E , and by the countable monotonicity of the outer measure μ_n^* together with $\mu_n(\mathcal{I}_k) = \text{vol}(\mathcal{I}_k)$,

$$\mu(\mathcal{U}) \leq \sum_{k=1}^{\infty} \text{vol}(\mathcal{I}_k) < \sum_{k=1}^{\infty} \text{vol}(I_k) + \epsilon/2 < \mu(E) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, $\mu(E) = \inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\}$. (This is assertion (17) of Theorem 13 of Chapter 20.)

Inner regularity on open sets. Let $\mathcal{O}$ be open in $\mathbb{R}^n$. Since $\mu(K) \leq \mu(\mathcal{O})$ for every compact $K \subseteq \mathcal{O}$, it suffices to find compact subsets of $\mathcal{O}$ with measures tending to $\mu(\mathcal{O})$. If $\mathcal{O} = \mathbb{R}^n$, put $K_j = \{x \mid \|x\| \leq j\}$. Otherwise $F = \mathbb{R}^n \setminus \mathcal{O}$ is a nonempty closed set; with $\text{dist}_F(x) = \inf\{\|x - y\| \mid y \in F\}$, put

$$K_j = \{x \in \mathbb{R}^n \mid \|x\| \leq j \text{ and } \text{dist}_F(x) \geq 1/j\}.$$

The function dist_F is Lipschitz, hence continuous, so K_j is closed; it is bounded, hence compact by the Heine–Borel Theorem. Since F is closed, $\text{dist}_F(x) = 0$ if and only if $x \in F$; thus $K_j \subseteq \mathcal{O}$. The sequence $\{K_j\}$ is ascending, and $\bigcup_{j=1}^{\infty} K_j = \mathcal{O}$: given $x \in \mathcal{O}$, openness supplies $r > 0$ with the ball of radius r about x contained in $\mathcal{O}$, so $\text{dist}_F(x) \geq r$, and $x \in K_j$ once $j \geq \|x\|$ and $1/j \leq r$. By the continuity of measure (Proposition 2 of Chapter 17),

$$\mu(\mathcal{O}) = \lim_{j \rightarrow \infty} \mu(K_j) \leq \sup\{\mu(K) \mid K \text{ a compact subset of } \mathcal{O}\} \leq \mu(\mathcal{O}),$$

so equality holds throughout.

Thus the restriction of n -dimensional Lebesgue measure to $\mathcal{B}(\mathbb{R}^n)$ is a Borel measure that is outer regular on all Borel sets and inner regular on open sets, that is, a Radon measure.

Problem 21.31. Show that a Dirac delta measure on a topological space is a Radon measure.

Solution. Let $(X, \mathcal{T})$ be a topological space, let x_0 belong to X , and let $\delta = \delta_{x_0}$ be the Dirac delta measure concentrated at x_0 , that is, the set function on $\mathcal{B}(X)$ defined by

$$\delta(E) = \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{if } x_0 \notin E. \end{cases}$$

δ is a measure. Certainly $\delta(\emptyset) = 0$ and $\delta \geq 0$. Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection of Borel sets and $E = \bigcup_{k=1}^\infty E_k$. If $x_0 \notin E$, then x_0 belongs to no E_k and both sides of $\delta(E) = \sum_{k=1}^\infty \delta(E_k)$ are 0. If $x_0 \in E$, then, by disjointness, x_0 belongs to exactly one E_{k_0} , so the left-hand side is 1 and the right-hand side is $\delta(E_{k_0}) = 1$ as well. Hence δ is countably additive.

δ is a Borel measure. It is defined on $\mathcal{B}(X)$ and takes values in $\{0, 1\}$, so every compact Borel subset of X has finite measure.

Inner regularity on open sets. Let $\mathcal{O}$ be open. If $x_0 \notin \mathcal{O}$, then $\delta(\mathcal{O}) = 0$ and the empty set is a compact subset of $\mathcal{O}$ with $\delta(\emptyset) = 0$, so the supremum is $0 = \delta(\mathcal{O})$. If $x_0 \in \mathcal{O}$, then $K = \{x_0\}$ is a compact subset of $\mathcal{O}$ (any cover of a one-point set by open sets has a one-element subcover) and

$\delta(K) = 1 = \delta(\mathcal{O})$, while $\delta(K') \leq 1$ for every K' . In either case

$$\delta(\mathcal{O}) = \sup\{\delta(K) \mid K \text{ a compact subset of } \mathcal{O}\}.$$

Outer regularity. Let E be a Borel set. Since $\delta(\mathcal{U}) \geq \delta(E)$ for every neighborhood $\mathcal{U}$ of E , we must produce neighborhoods of measure at most $\delta(E)$. If $x_0 \in E$, then $\delta(E) = 1$, and since $\delta \leq 1$ everywhere, X itself is a neighborhood of E with $\delta(X) = 1 = \delta(E)$; the infimum equals $\delta(E)$. Suppose now $x_0 \notin E$, so $\delta(E) = 0$; we must find an open $\mathcal{U} \supseteq E$ with $x_0 \notin \mathcal{U}$. Take

$$\mathcal{U} = X \sim \overline{\{x_0\}},$$

which is open and does not contain x_0 , so $\delta(\mathcal{U}) = 0$. It contains E provided $E \cap \overline{\{x_0\}} = \emptyset$, and this is guaranteed as soon as $\overline{\{x_0\}} = \{x_0\}$, that is, as soon as the singleton $\{x_0\}$ is a closed subset of X ; for then $E \subseteq X \sim \{x_0\} = \mathcal{U}$. Thus $\delta(E) = 0 = \inf\{\delta(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\}$, and δ is outer regular. Consequently, whenever singletons are closed in X — in particular whenever X is Hausdorff, which is the hypothesis in force on the spaces carrying the Radon measures constructed in this chapter, for instance in Theorem 10 and in the Riesz–Markov Theorem — the Dirac delta measure δ_{x_0} is a Radon measure.

A remark on the hypothesis is in order: the closedness of $\{x_0\}$ cannot be dispensed with, so the assertion is to be read with that separation hypothesis in force. Let $X = \{a, b\}$ carry the topology $\mathcal{T} = \{\emptyset, \{a\}, X\}$. Then $\mathcal{B}(X) = 2^X$, and δ_a is a Borel measure that is inner regular on open sets by the argument above. However $E = \{b\}$ is a Borel set with $\delta_a(E) = 0$, while the only open set containing E is X , and $\delta_a(X) = 1$. So outer regularity fails at E , and δ_a is not Radon on this (non-Hausdorff) space. The proof above is therefore sharp: it uses exactly the separation of x_0 from the sets missing it.

Problem 21.32. Let X be an uncountable set with the discrete topology and $\{x_k\}_{1 \leq k < \infty}$ a countable subset of X . For $E \subseteq X$, define

$$\mu(E) = \sum_{\{n \mid x_n \in E\}} 2^{-n}.$$

Show that $2^X = \mathcal{B}(X)$ and $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ is a Radon measure.

Solution. Throughout, for a subset A of the natural numbers we understand $\sum_{n \in A} 2^{-n}$ to be $\sup\{\sum_{n \in F} 2^{-n} \mid F \subseteq A \text{ finite}\}$; since the terms are nonnegative this agrees with the sum of the corresponding series taken in increasing order of n . For $E \subseteq X$ set

$$A_E = \{n \mid x_n \in E\}, \quad \text{so that } \mu(E) = \sum_{n \in A_E} 2^{-n} \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

The Borel σ -algebra. Since X carries the discrete topology, $\mathcal{T} = 2^X$. Now $\mathcal{B}(X)$ is by definition the smallest σ -algebra of subsets of X containing $\mathcal{T}$; as 2^X is itself a σ -algebra containing $\mathcal{T}$ and every σ -algebra of subsets of X is contained in 2^X , we get $\mathcal{B}(X) = 2^X$. In particular μ is defined on all of $\mathcal{B}(X)$, and it takes values in $[0, 1] \subseteq [0, \infty]$.

μ is a measure. Clearly $\mu(\emptyset) = 0$. Let $\{E_j\}_{j=1}^{\infty}$ be a countable disjoint collection of subsets of X and $E = \bigcup_{j=1}^{\infty} E_j$. The sets $A_j = A_{E_j}$ are pairwise disjoint, for if n belonged to $A_i \cap A_j$ with $i \neq j$ then x_n would belong to $E_i \cap E_j = \emptyset$; and $\bigcup_{j=1}^{\infty} A_j = A_E$, since $x_n \in E$ if and only if $x_n \in E_j$ for some j . We claim

$$\sum_{n \in A_E} 2^{-n} = \sum_{j=1}^{\infty} \sum_{n \in A_j} 2^{-n}.$$

For the inequality $\leq$, let $F \subseteq A_E$ be finite. Then F meets only finitely many of the A_j , say those with $j \leq J$, and $F \cap A_j$ is a finite subset of A_j , so

$$\sum_{n \in F} 2^{-n} = \sum_{j=1}^J \sum_{n \in F \cap A_j} 2^{-n} \leq \sum_{j=1}^J \sum_{n \in A_j} 2^{-n} \leq \sum_{j=1}^{\infty} \sum_{n \in A_j} 2^{-n};$$

taking the supremum over such F gives $\leq$. For the reverse inequality, fix J and $\epsilon > 0$ and choose, for $1 \leq j \leq J$, a finite $F_j \subseteq A_j$ with $\sum_{n \in F_j} 2^{-n} > \sum_{n \in A_j} 2^{-n} - \epsilon/J$. Then $F = \bigcup_{j=1}^J F_j$ is a finite subset of A_E and, by the disjointness of the A_j ,

$$\sum_{n \in A_E} 2^{-n} \geq \sum_{n \in F} 2^{-n} = \sum_{j=1}^J \sum_{n \in F_j} 2^{-n} > \sum_{j=1}^J \sum_{n \in A_j} 2^{-n} - \epsilon.$$

Letting $\epsilon \rightarrow 0$ and then $J \rightarrow \infty$ gives $\geq$. Hence $\mu(E) = \sum_{j=1}^{\infty} \mu(E_j)$ and μ is countably additive: μ is a measure on $\mathcal{B}(X)$. (Equivalently, $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$.)

μ is a Borel measure. Every subset of X , in particular every compact subset, has measure at most $1 < \infty$.

Outer regularity. Every subset of X is open, so a Borel set E is itself a neighborhood of E . Hence

$$\inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\} \leq \mu(E),$$

and the reverse inequality holds by monotonicity. So μ is outer regular; indeed the infimum is attained at E .

Inner regularity on open sets. First note that in a discrete space the compact sets are exactly the finite sets: a finite set is compact in any topological space, and if K is compact then the cover of K by the open singletons $\{\{x\}\}_{x \in K}$ admits a finite subcover, which forces K to be finite.

Let $\mathcal{O} \subseteq X$ be open (that is, an arbitrary subset) and let $\epsilon > 0$. Choose a natural number N with $\sum_{n > N} 2^{-n} = 2^{-N} < \epsilon$, and set

$$K = \{x_n \mid 1 \leq n \leq N \text{ and } x_n \in \mathcal{O}\}.$$

Then K is a finite subset of $\mathcal{O}$, hence a compact subset of $\mathcal{O}$. Every $n \leq N$ with $x_n \in \mathcal{O}$ satisfies $x_n \in K$, so $\{n \leq N \mid n \in A_{\mathcal{O}}\} \subseteq A_K$, and therefore

$$\mu(K) \geq \sum_{\substack{n \in A_{\mathcal{O}} \\ n \leq N}} 2^{-n} = \mu(\mathcal{O}) - \sum_{\substack{n \in A_{\mathcal{O}} \\ n > N}} 2^{-n} \geq \mu(\mathcal{O}) - 2^{-N} > \mu(\mathcal{O}) - \epsilon,$$

where the middle equality uses the finiteness of $\mu(\mathcal{O})$ together with the additivity just proved, applied to the decomposition of $\mathcal{O}$ determined by $\{n \leq N\}$ and $\{n > N\}$. Since $\mu(K) \leq \mu(\mathcal{O})$ for every compact $K \subseteq \mathcal{O}$ and $\epsilon > 0$ was arbitrary,

$$\mu(\mathcal{O}) = \sup\{\mu(K) \mid K \text{ a compact subset of } \mathcal{O}\}.$$

Therefore μ is a Borel measure on $\mathcal{B}(X) = 2^X$ that is outer regular and inner regular on open sets, that is, a Radon measure. Observe that X being uncountable plays no role in the verification; it merely makes the example interesting, since then μ vanishes on the uncountable set $X \sim \{x_n \mid 1 \leq n < \infty\}$ while every subset of X is both open and closed.

Problem 21.33. Show that the sum of two Radon measures also is Radon.

Solution. Let $(X, \mathcal{T})$ be a topological space and let μ_1 and μ_2 be Radon measures on $\mathcal{B}(X)$. Define

$$\mu(E) = \mu_1(E) + \mu_2(E) \quad \text{for all } E \in \mathcal{B}(X),$$

with the usual conventions for ∞ in $[0, \infty]$. As is implicit in the definition of a Borel measure and in the inner regularity condition, which speak of the measure of a compact set, we understand throughout that the compact sets occurring in these conditions belong to $\mathcal{B}(X)$; this is automatic when X is Hausdorff, since a compact subset of a Hausdorff space is closed. The class of such sets is the same for μ_1 , μ_2 , and μ , since all three are defined on $\mathcal{B}(X)$.

μ is a measure. Certainly $\mu(\emptyset) = 0$ and $\mu \geq 0$. If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection in $\mathcal{B}(X)$

with union E , then, since two series of nonnegative extended real numbers may be added term by term,

$$\mu(E) = \mu_1(E) + \mu_2(E) = \sum_{k=1}^{\infty} \mu_1(E_k) + \sum_{k=1}^{\infty} \mu_2(E_k) = \sum_{k=1}^{\infty} [\mu_1(E_k) + \mu_2(E_k)] = \sum_{k=1}^{\infty} \mu(E_k).$$

Note also that μ is monotone, being the sum of two monotone set functions.

μ is a Borel measure. If K is a compact subset of X , then $\mu_1(K) < \infty$ and $\mu_2(K) < \infty$ because μ_1 and μ_2 are Borel measures, so $\mu(K) = \mu_1(K) + \mu_2(K) < \infty$.

Outer regularity. Let E be a Borel set. By monotonicity, $\mu(\mathcal{U}) \geq \mu(E)$ for every neighborhood $\mathcal{U}$ of E , so the infimum over neighborhoods is at least $\mu(E)$ and there is nothing more to prove if $\mu(E) = \infty$. Assume $\mu(E) < \infty$, so that both $\mu_1(E)$ and $\mu_2(E)$ are finite, and let $\epsilon > 0$. By the outer regularity of μ_1 and of μ_2 there are neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of E with

$$\mu_1(\mathcal{U}_1) < \mu_1(E) + \epsilon/2 \quad \text{and} \quad \mu_2(\mathcal{U}_2) < \mu_2(E) + \epsilon/2.$$

The set $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ is open and contains E , hence is a neighborhood of E , and by the monotonicity of μ_1 and μ_2 ,

$$\mu(\mathcal{U}) = \mu_1(\mathcal{U}) + \mu_2(\mathcal{U}) \leq \mu_1(\mathcal{U}_1) + \mu_2(\mathcal{U}_2) < \mu_1(E) + \mu_2(E) + \epsilon = \mu(E) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\mu(E) = \inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\}$.

Inner regularity on open sets. Let $\mathcal{O}$ be open. By monotonicity, $\mu(K) \leq \mu(\mathcal{O})$ for each compact $K \subseteq \mathcal{O}$, so it suffices to show that for each real number $c < \mu(\mathcal{O})$ there is a compact $K \subseteq \mathcal{O}$ with $\mu(K) > c$.

First suppose $\mu_1(\mathcal{O})$ and $\mu_2(\mathcal{O})$ are both finite, and let $c < \mu(\mathcal{O})$; put $\epsilon = \mu(\mathcal{O}) - c > 0$. By the inner regularity of μ_1 and of μ_2 there are compact sets $K_1, K_2 \subseteq \mathcal{O}$ with $\mu_i(K_i) > \mu_i(\mathcal{O}) - \epsilon/2$ for $i = 1, 2$. Then $K = K_1 \cup K_2$ is a compact subset of $\mathcal{O}$ (a finite union of compact sets is compact), and it is a Borel set, being the union of the two Borel sets K_1 and K_2 ; so $\mu(K)$ is defined and, by monotonicity,

$$\mu(K) = \mu_1(K) + \mu_2(K) \geq \mu_1(K_1) + \mu_2(K_2) > \mu_1(\mathcal{O}) + \mu_2(\mathcal{O}) - \epsilon = c.$$

Now suppose one of them, say $\mu_1(\mathcal{O})$, is infinite, so that $\mu(\mathcal{O}) = \infty$. Given any real c , the inner regularity of μ_1 provides a compact $K \subseteq \mathcal{O}$ with $\mu_1(K) > c$, and then $\mu(K) \geq \mu_1(K) > c$. In both cases

$$\mu(\mathcal{O}) = \sup\{\mu(K) \mid K \text{ a compact subset of } \mathcal{O}\}.$$

Therefore $\mu = \mu_1 + \mu_2$ is a Radon measure. (The same argument, applied inductively, shows that any finite sum of Radon measures is Radon, and it shows as well that a positive scalar multiple of a Radon measure is Radon.)

Problem 21.34. Let μ and ν be Borel measures on $\mathcal{B}(X)$, where X is a compact topological space, and suppose that μ is absolutely continuous with respect to ν . If ν is Radon show that μ also is Radon.

Solution. Since X is compact and μ and ν are Borel measures, the space X itself is a compact set and therefore

$$\mu(X) < \infty \quad \text{and} \quad \nu(X) < \infty;$$

by monotonicity both μ and ν are finite measures on $\mathcal{B}(X)$. (Here, as in the definition of a Borel measure and in the inner regularity condition, the compact sets whose measure is spoken of are understood to lie in $\mathcal{B}(X)$; this is automatic when X is Hausdorff, since a compact subset of a Hausdorff space is closed. The convention is the same for μ and for ν , both being defined on $\mathcal{B}(X)$, so the compact set produced below by the inner regularity of ν is one at which μ may be evaluated.)

Because μ is a finite measure that is absolutely continuous with respect to ν , Proposition 19 of Chapter 18 recasts absolute continuity as an ϵ - δ criterion:

for each $\epsilon > 0$ there is a $\delta > 0$ such that for $E \in \mathcal{B}(X)$, if $\nu(E) < \delta$, then $\mu(E) < \epsilon$. (*)

That μ is a Borel measure is part of the hypothesis (and is anyway immediate from $\mu(X) < \infty$). So we must establish outer regularity on Borel sets and inner regularity on open sets.

Outer regularity. Let E be a Borel subset of X and let $\epsilon > 0$. Choose $\delta > 0$ as in (*). Since ν is Radon, it is outer regular, so there is a neighborhood $\mathcal{U}$ of E for which

$$\nu(\mathcal{U}) < \nu(E) + \delta.$$

As $\nu(E) \leq \nu(X) < \infty$ and $E \subseteq \mathcal{U}$, the excision property of the measure ν gives

$$\nu(\mathcal{U} \sim E) = \nu(\mathcal{U}) - \nu(E) < \delta.$$

Therefore, by (*), $\mu(\mathcal{U} \sim E) < \epsilon$, and by the additivity of μ over the disjoint Borel decomposition $\mathcal{U} = E \cup [\mathcal{U} \sim E]$,

$$\mu(\mathcal{U}) = \mu(E) + \mu(\mathcal{U} \sim E) < \mu(E) + \epsilon.$$

Since $\mu(\mathcal{V}) \geq \mu(E)$ for every neighborhood $\mathcal{V}$ of E , by monotonicity, and $\epsilon > 0$ was arbitrary,

$$\mu(E) = \inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\}.$$

Inner regularity on open sets. Let $\mathcal{O}$ be an open subset of X and let $\epsilon > 0$. Choose $\delta > 0$ as in (*). Since ν is Radon, it is inner regular on open sets, so there is a compact subset K of $\mathcal{O}$ with

$$\nu(K) > \nu(\mathcal{O}) - \delta.$$

As $\nu(K) \leq \nu(X) < \infty$ and $K \subseteq \mathcal{O}$, excision gives $\nu(\mathcal{O} \sim K) = \nu(\mathcal{O}) - \nu(K) < \delta$, and hence, by (*), $\mu(\mathcal{O} \sim K) < \epsilon$. Consequently

$$\mu(\mathcal{O}) = \mu(K) + \mu(\mathcal{O} \sim K) < \mu(K) + \epsilon.$$

Since $\mu(K') \leq \mu(\mathcal{O})$ for every compact $K' \subseteq \mathcal{O}$ and $\epsilon > 0$ was arbitrary,

$$\mu(\mathcal{O}) = \sup\{\mu(K) \mid K \text{ a compact subset of } \mathcal{O}\}.$$

Thus μ is a Borel measure that is outer regular and inner regular on open sets, that is, μ is a Radon measure. The compactness of X entered only to guarantee that μ is finite, which is what licenses the ϵ - δ form (*) of absolute continuity, and that ν is finite, which is what licenses the two excisions.

Problem 21.35. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space and $\mu: \mathcal{T} \rightarrow [0, \infty]$ a premeasure for which the restriction to $\mathcal{B}(X)$ of μ^* is a Radon measure. Show that μ is a Radon premeasure.

Solution. Denote by μ^* the outer measure induced by the premeasure μ and by $\bar{\mu}$ the restriction of μ^* to $\mathcal{B}(X)$, which by hypothesis is a Radon measure. We must verify the two defining properties of a Radon premeasure:

- (i) $\mu(\mathcal{U}) < \infty$ for each open set $\mathcal{U}$ that has compact closure;
- (ii) $\mu(\mathcal{O}) = \sup\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ open and } \bar{\mathcal{U}} \text{ a compact subset of } \mathcal{O}\}$ for each open set $\mathcal{O}$.

We first record the key identity: μ^* agrees with μ on the topology,

$$\mu^*(\mathcal{V}) = \mu(\mathcal{V}) \quad \text{for every } \mathcal{V} \in \mathcal{T}. \quad (*)$$

Indeed, by Lemma 8, $\mu^*(\mathcal{V}) = \inf\{\mu(\mathcal{W}) \mid \mathcal{W} \text{ a neighborhood of } \mathcal{V}\}$. Since $\mathcal{V}$ is itself a neighborhood of $\mathcal{V}$, we get $\mu^*(\mathcal{V}) \leq \mu(\mathcal{V})$. On the other hand, a premeasure is countably monotone and hence monotone (cover $\mathcal{V}$ by the single set $\mathcal{W}$), so $\mu(\mathcal{V}) \leq \mu(\mathcal{W})$ for every open $\mathcal{W} \supseteq \mathcal{V}$, and therefore $\mu(\mathcal{V}) \leq \mu^*(\mathcal{V})$. This is the observation made at the beginning of the proof of Proposition 9. In particular, since open sets are Borel sets, $\bar{\mu}(\mathcal{V}) = \mu(\mathcal{V})$ for all $\mathcal{V} \in \mathcal{T}$.

Proof of (i). Let $\mathcal{U}$ be open with $\bar{\mathcal{U}}$ compact. The set $\bar{\mathcal{U}}$ is closed, hence Borel, and it is compact; since $\bar{\mu}$, being a Radon measure, is in particular a Borel measure, $\bar{\mu}(\bar{\mathcal{U}}) < \infty$. By (*) and the monotonicity

of the measure $\bar{\mu}$,

$$\mu(\mathcal{U}) = \bar{\mu}(\mathcal{U}) \leq \bar{\mu}(\bar{\mathcal{U}}) < \infty.$$

Proof of (ii). Fix an open set $\mathcal{O}$ and let

$$s = \sup\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ open and } \bar{\mathcal{U}} \text{ a compact subset of } \mathcal{O}\}.$$

The collection over which the supremum is taken is nonempty, since $\mathcal{U} = \emptyset$ belongs to it.

First, $s \leq \mu(\mathcal{O})$. Indeed, if $\mathcal{U}$ is open with $\bar{\mathcal{U}}$ a compact subset of $\mathcal{O}$, then $\mathcal{U} \subseteq \bar{\mathcal{U}} \subseteq \mathcal{O}$, and the monotonicity of the premeasure μ gives $\mu(\mathcal{U}) \leq \mu(\mathcal{O})$.

Second, $s \geq \mu(\mathcal{O})$. Let c be any real number with $c < \mu(\mathcal{O})$; it suffices to produce an admissible $\mathcal{U}$ with $\mu(\mathcal{U}) > c$. By (*), $\mu(\mathcal{O}) = \bar{\mu}(\mathcal{O})$, and $\bar{\mu}$ is inner regular on open sets, so there is a compact subset K of $\mathcal{O}$ with

$$\bar{\mu}(K) > c.$$

(Since X is Hausdorff, K is closed and therefore Borel, so $\bar{\mu}(K)$ is indeed defined.) Now $\mathcal{O}$ is a neighborhood of the compact set K and X is a locally compact Hausdorff space; therefore, by property (ii) of Theorem 7, there is a neighborhood $\mathcal{U}$ of K that has compact closure contained in $\mathcal{O}$. This $\mathcal{U}$ is admissible in the supremum defining s , and since $K \subseteq \mathcal{U}$, the monotonicity of $\bar{\mu}$ together with (*) yields

$$\mu(\mathcal{U}) = \bar{\mu}(\mathcal{U}) \geq \bar{\mu}(K) > c.$$

Hence $s \geq c$ for every real $c < \mu(\mathcal{O})$, so $s \geq \mu(\mathcal{O})$.

Combining the two inequalities, $\mu(\mathcal{O}) = s$, which is property (ii). Therefore μ is a Radon premeasure. This is precisely the converse of Theorem 10: for a premeasure on the topology of a locally compact Hausdorff space, being a Radon premeasure is not merely sufficient but also necessary in order that the Caratheodory extension μ^* restrict to a Radon measure on $\mathcal{B}(X)$.

21.6 Problems 21.36–21.42

Problem 21.36. Let X be a locally compact Hausdorff space and $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ a Radon measure. Show that any Borel set E of finite measure is inner regular in the sense that

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\}.$$

Conclude that if μ is σ -finite, then every Borel set is inner regular.

Solution. Throughout, μ is a Borel measure that is outer regular on every Borel set and inner regular on every open set; these are the two defining properties of a Radon measure.

Write $\alpha(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\}$. By the monotonicity of μ we always have $\alpha(E) \leq \mu(E)$, so only the inequality $\alpha(E) \geq \mu(E)$ is at issue.

Step 1: Borel sets of finite measure. Let E be Borel with $\mu(E) < \infty$ and let $\epsilon > 0$.

By outer regularity applied to E there is an open set $U \supseteq E$ with

$$\mu(U) < \mu(E) + \epsilon/2.$$

In particular $\mu(U) < \infty$. Since $E \subseteq U$ and $\mu(E) < \infty$, the excision property of measure gives

$$\mu(U \sim E) = \mu(U) - \mu(E) < \epsilon/2.$$

The set $U \sim E$ is Borel, so by outer regularity again there is an open set $V \supseteq U \sim E$ with

$$\mu(V) < \epsilon/2.$$

Since U is open and $\mu(U) < \infty$, inner regularity of open sets provides a compact set $K_0 \subseteq U$ with

$$\mu(K_0) > \mu(U) - \epsilon/2 \geq \mu(E) - \epsilon/2.$$

Define

$$K = K_0 \sim V = K_0 \cap (X \sim V).$$

Since V is open, $X \sim V$ is closed, so K is a closed subset of the compact set K_0 and hence is compact. Moreover

$$K \subseteq U \sim V \subseteq U \sim (U \sim E) = U \cap E = E,$$

because $V \supseteq U \sim E$. Thus K is a compact subset of E . Finally, by the finite subadditivity and monotonicity of μ ,

$$\mu(K_0) \leq \mu(K_0 \sim V) + \mu(V) = \mu(K) + \mu(V),$$

so that

$$\mu(K) \geq \mu(K_0) - \mu(V) > (\mu(E) - \epsilon/2) - \epsilon/2 = \mu(E) - \epsilon.$$

Hence $\alpha(E) > \mu(E) - \epsilon$ for every $\epsilon > 0$, and therefore $\alpha(E) = \mu(E)$: every Borel set of finite measure is inner regular.

Step 2: the σ -finite case. Assume $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is Borel with $\mu(X_n) < \infty$. Replacing X_n by $\bigcup_{k=1}^n X_k$, which is Borel and, by finite subadditivity, still of finite measure, we may assume that $\{X_n\}$ is ascending with union X .

Let E be an arbitrary Borel set. Then $\{E \cap X_n\}$ is an ascending sequence of Borel sets whose union is E , so by the continuity of measure

$$\lim_{n \rightarrow \infty} \mu(E \cap X_n) = \mu(E).$$

Let c be any real number with $c < \mu(E)$. Choose n with $\mu(E \cap X_n) > c$. Since $\mu(E \cap X_n) \leq \mu(X_n) < \infty$, Step 1 applies to $E \cap X_n$ and yields a compact set $K \subseteq E \cap X_n \subseteq E$ with $\mu(K) > c$. Therefore $\alpha(E) \geq c$ for every $c < \mu(E)$, which gives $\alpha(E) \geq \mu(E)$ and hence $\alpha(E) = \mu(E)$. Observe that this argument covers both the case $\mu(E) < \infty$ and the case $\mu(E) = \infty$, so every Borel set is inner regular when μ is σ -finite.

Problem 21.37. Let X be a topological space, $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ a σ -finite Radon measure, and $E \subseteq X$ a Borel set. Show that there is a G_δ subset A of X and an F_σ subset B of X for which

$$A \subseteq E \subseteq B \quad \text{and} \quad \mu(B \sim E) = \mu(E \sim A) = 0.$$

Solution. As printed, the labels G_δ and F_σ have been interchanged: the set that sits inside E is an F_σ set and the set that contains E is a G_δ set. We first prove this correct assertion, and then exhibit a counterexample showing that the assertion with the labels as printed is false.

Claim. Let X be a topological space, $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ a σ -finite Radon measure and E a Borel set. Then there are an F_σ set A and a G_δ set B with $A \subseteq E \subseteq B$ and $\mu(B \sim E) = \mu(E \sim A) = 0$.

Step 1: an outer G_δ approximation. Since μ is σ -finite we may write $X = \bigcup_{n=1}^{\infty} X_n$ with each X_n Borel and $\mu(X_n) < \infty$; replacing X_n by $X_n \sim \bigcup_{k < n} X_k$ we may assume the X_n are disjoint. Let F be any Borel subset of X and put $F_n = F \cap X_n$, so that $F = \bigcup_n F_n$ and $\mu(F_n) < \infty$ for each n .

Fix $k \in \mathbb{N}$. For each n , outer regularity of the Borel set F_n provides an open set $U_{n,k} \supseteq F_n$ with

$$\mu(U_{n,k}) < \mu(F_n) + \frac{1}{k 2^n}.$$

Since $\mu(F_n) < \infty$, excision gives $\mu(U_{n,k} \sim F_n) < 1/(k 2^n)$. Put $U_k = \bigcup_{n=1}^{\infty} U_{n,k}$, an open set containing F . Because $F_n \subseteq F$ we have $U_{n,k} \sim F \subseteq U_{n,k} \sim F_n$, and therefore, by countable subadditivity,

$$\mu(U_k \sim F) \leq \sum_{n=1}^{\infty} \mu(U_{n,k} \sim F_n) < \sum_{n=1}^{\infty} \frac{1}{k 2^n} = \frac{1}{k}.$$

Now set $B_F = \bigcap_{k=1}^{\infty} U_k$. This is a G_δ set containing F , and since $B_F \sim F \subseteq U_k \sim F$ for every k , monotonicity gives $\mu(B_F \sim F) \leq 1/k$ for all k , whence

$$\mu(B_F \sim F) = 0.$$

Step 2: the inner F_σ approximation, by complementation. Apply Step 1 to the Borel set $F = X \sim E$: there is a G_δ set $G \supseteq X \sim E$ with $\mu(G \sim (X \sim E)) = 0$. Put $A = X \sim G$. The complement of a G_δ set is an F_σ set, so A is F_σ ; and $G \supseteq X \sim E$ gives $A \subseteq E$. Finally

$$E \sim A = E \cap G = G \cap E = G \sim (X \sim E),$$

so $\mu(E \sim A) = 0$. Taking $B = B_E$ from Step 1 completes the proof of the claim: $A \subseteq E \subseteq B$, A is F_σ , B is G_δ , and $\mu(B \sim E) = \mu(E \sim A) = 0$.

Note that no hypothesis beyond outer regularity of Borel sets and σ -finiteness was used, which is why the exercise may be stated for an arbitrary topological space X ; inner regularity of open sets, and hence compactness, played no role.

Step 3: the statement as printed is false. Take $X = [0, 1]$ with its usual topology and let μ be Lebesgue measure restricted to $\mathcal{B}([0, 1])$. This μ is a Radon measure on $[0, 1]$: it is a Borel measure since $\mu([0, 1]) = 1 < \infty$; given a Borel $E \subseteq [0, 1]$ and $\epsilon > 0$, the outer regularity of Lebesgue measure on $\mathbb{R}$ (Problem 29, or Theorem 11 of Chapter 2) yields an open $W \subseteq \mathbb{R}$ with $E \subseteq W$ and $\mu(W) < \mu(E) + \epsilon$, and $W \cap [0, 1]$ is a neighborhood of E in $[0, 1]$ of measure at most $\mu(W)$; and, by the inner approximation by closed sets in the same theorem, each relatively open $O \subseteq [0, 1]$ contains a closed set F with $\mu(O \sim F) < \epsilon$, and such an F , being closed and bounded, is compact by the Heine-Borel Theorem. Being finite, μ is σ -finite. For each n let $C_n \subseteq [0, 1]$ be a closed, nowhere dense set with $\mu(C_n) > 1 - 1/n$, for instance a fat Cantor set, and put

$$E = \bigcup_{n=1}^{\infty} C_n.$$

Then E is an F_σ Borel set with $\mu(E) = 1$, and E is meager, being a countable union of nowhere dense sets.

Suppose there were a G_δ set $A \subseteq E$ with $\mu(E \sim A) = 0$. Then $\mu(A) = 1$. Write $A = \bigcap_k U_k$ with each U_k open. Since $\mu(U_k) \geq \mu(A) = 1$, each U_k has full measure, and an open subset of $[0, 1]$ that omits some nonempty open interval has measure less than 1; hence each U_k is dense. Consequently each closed set $[0, 1] \sim U_k$ is nowhere dense, so

$$[0, 1] \sim A = \bigcup_{k=1}^{\infty} ([0, 1] \sim U_k)$$

is meager. But $A \subseteq E$ is meager as well, so $[0, 1] = A \cup ([0, 1] \sim A)$ would be meager in itself, contradicting the Baire Category Theorem for the complete metric space $[0, 1]$. Hence no G_δ subset of E has full measure, and the printed form of the assertion fails.

Problem 21.38. For a metric space X , show that $\mathcal{B}(X)$ is the smallest σ -algebra with respect to which all of the continuous real-valued functions on X are measurable.

Solution. Let ρ be a metric inducing the topology of X , let $C(X)$ denote the collection of continuous real-valued functions on X , and let

$$\mathcal{A} = \bigcap \{ \mathcal{S} \mid \mathcal{S} \text{ is a } \sigma\text{-algebra of subsets of } X \text{ with respect to which every } f \in C(X) \text{ is measurable} \}.$$

The intersection of a family of σ -algebras is a σ -algebra, and if every member of the family makes each $f \in C(X)$ measurable then so does the intersection, since f is $\mathcal{S}$ -measurable exactly when $f^{-1}((a, \infty)) \in \mathcal{S}$ for every $a \in \mathbb{R}$. The family is nonempty, as it contains the σ -algebra 2^X of all subsets of X . Thus $\mathcal{A}$ is the smallest σ -algebra with respect to which every continuous real-valued function on X is measurable. We must show $\mathcal{A} = \mathcal{B}(X)$.

$\mathcal{A} \subseteq \mathcal{B}(X)$. Let $f \in C(X)$. For each $a \in \mathbb{R}$ the set (a, ∞) is open in $\mathbb{R}$, so by the continuity of f the set $f^{-1}((a, \infty))$ is open in X and hence belongs to $\mathcal{B}(X)$. Therefore every $f \in C(X)$ is measurable with respect to $\mathcal{B}(X)$, so $\mathcal{B}(X)$ belongs to the family defining $\mathcal{A}$, and the minimality of $\mathcal{A}$ gives $\mathcal{A} \subseteq \mathcal{B}(X)$. (This half uses only that X is a topological space.)

$\mathcal{B}(X) \subseteq \mathcal{A}$. By the minimality of the Borel σ -algebra among the σ -algebras containing the topology, it

suffices to show that every open subset O of X belongs to $\mathcal{A}$. If $O = X$ this is clear, since X belongs to every σ -algebra of subsets of X . Otherwise $F = X \sim O$ is a nonempty closed set, and we may consider the distance function

$$\text{dist}_F(x) = \inf_{y \in F} \rho(x, y) \quad \text{for } x \in X,$$

which is real-valued because F is nonempty. According to Problem 25, dist_F is continuous (indeed Lipschitz with constant 1), and since F is closed, $\text{dist}_F(x) = 0$ if and only if $x \in F$. Consequently

$$O = X \sim F = \{x \in X \mid \text{dist}_F(x) > 0\} = \text{dist}_F^{-1}((0, \infty)).$$

As dist_F belongs to $C(X)$, it is measurable with respect to $\mathcal{A}$, so $O \in \mathcal{A}$. Hence $\mathcal{A}$ is a σ -algebra containing every open set, so $\mathcal{B}(X) \subseteq \mathcal{A}$.

Therefore $\mathcal{A} = \mathcal{B}(X)$, that is, $\mathcal{B}(X)$ is the smallest σ -algebra with respect to which all continuous real-valued functions on X are measurable.

A remark on the role of the metric: for a general topological space only the inclusion $\mathcal{A} \subseteq \mathcal{B}(X)$ is available, and it may be strict. For instance, let D be an uncountable set with the discrete topology and let $X^* = D \cup \{\omega\}$ be its one-point compactification. Every subset of X^* is Borel (Problem 41), while by Problem 41 every $F \in C(X^*)$ has the form $F = c + f$ where $c = F(\omega)$ and $\{x \in D \mid |f(x)| \geq \epsilon\}$ is finite for each $\epsilon > 0$. Each such F is measurable with respect to the σ -algebra

$$\mathcal{S} = \{S \subseteq X^* \mid S \text{ is countable and } \omega \notin S, \text{ or } X^* \sim S \text{ is countable and } \omega \in S\},$$

so $\mathcal{A} \subseteq \mathcal{S}$, and the singleton $\{\omega\}$, though Borel, does not belong to $\mathcal{S}$. Hence $\mathcal{A} \subsetneq \mathcal{B}(X^*)$. The metric is used precisely to manufacture, for each closed set F , a continuous function vanishing exactly on F .

Problem 21.39. Prove Lusin's Theorem as follows:

- (i) First prove it for simple functions by using the inner regularity of open sets and the locally compact extension property.
- (ii) Use part (i) together with Egoroff's Theorem and the Simple Approximation Theorem to complete the proof.

Solution. Recall the statement to be proved. Let X be a locally compact Hausdorff space, $\mu: \mathcal{B}(X) \rightarrow [0, \infty)$ a Radon measure, and $f: X \rightarrow \mathbb{R}$ a Borel measurable function that vanishes outside of a set of finite measure. Then for each $\epsilon > 0$ there is a Borel subset X_0 of X and a function $g \in C_c(X)$ for which

$$f = g \text{ on } X_0 \quad \text{and} \quad \mu(X \sim X_0) < \epsilon.$$

The argument below uses only that μ is a Radon measure with values in $[0, \infty]$; the finiteness of μ is nowhere needed.

We shall use repeatedly: outer regularity of Borel sets and inner regularity of open sets (the definition of a Radon measure); the consequence of inner regularity of open sets proved in Problem 36, namely that every Borel set of finite measure is inner regular; the locally compact separation property (Proposition 2), the locally compact extension property (Proposition 3), and the extension theorem for continuous functions on compact sets, Theorem 6.

Part (i): simple functions. Let ψ be a simple Borel measurable function on X that vanishes outside a set of finite measure, and let $\epsilon > 0$. If ψ is identically zero, take $g = 0$ and $X_0 = X$. Otherwise let $a_1, \dots, a_n$ be the distinct nonzero values of ψ and put $E_i = \psi^{-1}(a_i)$. The sets E_i are Borel and pairwise disjoint, and each is contained in $\{x \mid \psi(x) \neq 0\}$, which by hypothesis is contained in a set of finite measure; hence $\mu(E_i) < \infty$ for each i .

Fix i . Since $\mu(E_i) < \infty$, Problem 36 provides a compact set $K_i \subseteq E_i$ with

$$\mu(E_i \sim K_i) < \frac{\epsilon}{2n},$$

and outer regularity provides an open set $U_i \supseteq E_i$ with $\mu(U_i) < \mu(E_i) + \epsilon/(2n)$, so that, by excision,

$\mu(U_i \sim E_i) < \epsilon/(2n)$. Therefore

$$\mu(U_i \sim K_i) \leq \mu(U_i \sim E_i) + \mu(E_i \sim K_i) < \frac{\epsilon}{n}.$$

Since X is Hausdorff, each compact K_j is closed, so $\bigcup_{j \neq i} K_j$ is closed and

$$U'_i = U_i \sim \bigcup_{j \neq i} K_j$$

is open. The sets K_j are pairwise disjoint, so $K_i \subseteq U'_i$. By the locally compact extension property (Proposition 3) applied to the compact set K_i and its neighborhood U'_i , there is $h_i \in C_c(X)$ with

$$h_i = 1 \text{ on } K_i, \quad h_i = 0 \text{ on } X \sim U'_i, \quad 0 \leq h_i \leq 1 \text{ on } X.$$

Define

$$g = \sum_{i=1}^n a_i h_i \in C_c(X), \quad X_0 = \left(\bigcup_{i=1}^n K_i \right) \cup \left(X \sim \bigcup_{i=1}^n U_i \right),$$

so X_0 is Borel. If $x \in K_i$, then $\psi(x) = a_i$, $h_i(x) = 1$, and for $j \neq i$ we have $x \notin U'_j$ by the definition of U'_j , so $h_j(x) = 0$; hence $g(x) = a_i = \psi(x)$. If $x \notin \bigcup_i U_i$, then $x \notin \bigcup_i E_i$, so $\psi(x) = 0$, while $h_j(x) = 0$ for every j because $U'_j \subseteq U_j$; hence $g(x) = 0 = \psi(x)$. Thus $g = \psi$ on X_0 . Finally

$$X \sim X_0 = \left(\bigcup_i U_i \right) \sim \left(\bigcup_i K_i \right) \subseteq \bigcup_{i=1}^n (U_i \sim K_i),$$

so $\mu(X \sim X_0) < n \cdot (\epsilon/n) = \epsilon$. This proves Lusin's Theorem for simple functions.

Part (ii): the general case. Let $f: X \rightarrow \mathbb{R}$ be Borel measurable and vanish outside a set of finite measure, and let $\epsilon > 0$. The set

$$E = \{x \in X \mid f(x) \neq 0\}$$

is Borel and, being contained in a set of finite measure, satisfies $\mu(E) < \infty$.

By outer regularity choose an open set $W \supseteq E$ with

$$\mu(W \sim E) < \epsilon/2.$$

(Here excision is legitimate because $\mu(E) < \infty$.)

By the Simple Approximation Theorem there is a sequence $\{\psi_k\}$ of simple Borel measurable functions on X with $\psi_k \rightarrow f$ pointwise on X and $|\psi_k| \leq |f|$ on X for every k . In particular each ψ_k vanishes on $X \sim E$, hence vanishes outside a set of finite measure, so part (i) applies to it.

Since $\mu(E) < \infty$ and $\psi_k \rightarrow f$ pointwise on E with f real-valued, Egoroff's Theorem (whose proof uses only the countable additivity of the measure and the finiteness of $\mu(E)$) provides a Borel set $F \subseteq E$ with

$$\mu(E \sim F) < \epsilon/8 \quad \text{and} \quad \psi_k \rightarrow f \text{ uniformly on } F.$$

By part (i), for each k there are a Borel set $X_k \subseteq X$ and $g_k \in C_c(X)$ with

$$g_k = \psi_k \text{ on } X_k \quad \text{and} \quad \mu(X \sim X_k) < \frac{\epsilon}{8 \cdot 2^k}.$$

Put

$$S = F \cap \bigcap_{k=1}^{\infty} X_k.$$

Then S is Borel, $S \subseteq E$, and

$$\mu(E \sim S) \leq \mu(E \sim F) + \sum_{k=1}^{\infty} \mu(X \sim X_k) < \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4}.$$

On S we have $g_k = \psi_k$ for every k , and $\psi_k \rightarrow f$ uniformly on S since $S \subseteq F$. Hence the restrictions $g_k|_S$, which are continuous on S with its subspace topology, converge uniformly on S to $f|_S$. A uniform limit of continuous functions is continuous, so $f|_S$ is continuous on S .

Since S is Borel with $\mu(S) \leq \mu(E) < \infty$, Problem 36 provides a compact set $D \subseteq S$ with $\mu(S \sim D) < \epsilon/4$. Then

$$\mu(E \sim D) \leq \mu(E \sim S) + \mu(S \sim D) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},$$

and $f|_D$, being the restriction of the continuous function $f|_S$, is continuous on the compact set D .

By Theorem 6 the continuous real-valued function $f|_D$ on the compact set D has a continuous extension h to all of X . (If $D = \emptyset$ take $h = 0$.) Since $D \subseteq E \subseteq W$ and D is compact, the locally compact separation property (Proposition 2) yields an open set U with

$$D \subseteq U \subseteq \bar{U} \subseteq W, \quad \bar{U} \text{ compact},$$

and the locally compact extension property (Proposition 3) yields $\varphi \in C_c(X)$ with $\varphi = 1$ on D , $\varphi = 0$ on $X \sim U$ and $0 \leq \varphi \leq 1$ on X . Define

$$g = \varphi h.$$

Then g is continuous and vanishes off the compact set $\text{supp } \varphi$, so $g \in C_c(X)$.

Finally set

$$X_0 = D \cup (X \sim (U \cup E)),$$

a Borel set. On D we have $\varphi = 1$ and $h = f$, so $g = f$. On $X \sim (U \cup E)$ we have $f = 0$, because $\{f \neq 0\} = E$, and $g = 0$, because $\varphi = 0$ off U ; so again $g = f$. Thus $f = g$ on X_0 . As for the measure of the complement,

$$X \sim X_0 = (U \cup E) \sim D \subseteq (U \sim D) \cup (E \sim D) \subseteq (W \sim E) \cup (E \sim D),$$

where we used $U \subseteq W$ and $U \sim D \subseteq (U \sim E) \cup (E \sim D)$. Hence

$$\mu(X \sim X_0) \leq \mu(W \sim E) + \mu(E \sim D) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This is precisely the conclusion of Lusin's Theorem.

Problem 21.40. Let X be a locally compact Hausdorff space, and $C_0(X)$ the space of all uniform limits of functions in $C_c(X)$.

- (i) Show that a continuous real-valued function f on X belongs to $C_0(X)$ if and only if for each $\alpha > 0$ the set $\{x \in X \mid |f(x)| \geq \alpha\}$ is compact.
- (ii) Let X^* be the one-point compactification of X . Show that $C_0(X)$ consists precisely of the restrictions to X of those functions in $C(X^*)$ that vanish at the point at infinity.

Solution. Throughout, $\|u\|_{\max} = \sup_{x \in X} |u(x)|$ and, for $\alpha > 0$,

$$K_\alpha(f) = \{x \in X \mid |f(x)| \geq \alpha\}.$$

Observe first that every member of $C_0(X)$ is continuous, since a uniform limit of continuous functions is continuous.

Part (i). Suppose $f \in C_0(X)$ and let $\alpha > 0$. Choose $g \in C_c(X)$ with $\|f - g\|_{\max} < \alpha/2$. If $|f(x)| \geq \alpha$ then

$$|g(x)| \geq |f(x)| - |f(x) - g(x)| > \alpha - \alpha/2 = \alpha/2 > 0,$$

so x belongs to $\{g \neq 0\} \subseteq \text{supp } g$. Hence $K_\alpha(f) \subseteq \text{supp } g$, and $\text{supp } g$ is compact. Moreover $K_\alpha(f) = |f|^{-1}([\alpha, \infty))$ is closed, since f is continuous. A closed subset of a compact set is compact, so $K_\alpha(f)$ is compact.

Conversely, suppose f is continuous and $K_\alpha(f)$ is compact for every $\alpha > 0$. Let $\epsilon > 0$ and put $K = K_\epsilon(f)$, a compact subset of X . By the locally compact extension property (Proposition 3),

applied to K and the neighborhood X of K , there is $\varphi \in C_c(X)$ with $\varphi = 1$ on K and $0 \leq \varphi \leq 1$ on X . Set $g = \varphi f$. Then g is continuous and vanishes off the compact set $\text{supp } \varphi$, so $g \in C_c(X)$. For $x \in X$,

$$|f(x) - g(x)| = |f(x)| (1 - \varphi(x)).$$

If $x \in K$ this is 0, since $\varphi(x) = 1$; if $x \notin K$ then $|f(x)| < \epsilon$ and $0 \leq 1 - \varphi(x) \leq 1$, so the quantity is less than ϵ . Hence $\|f - g\|_{\max} \leq \epsilon$. As $\epsilon > 0$ was arbitrary, f is a uniform limit of functions in $C_c(X)$, that is, $f \in C_0(X)$.

Part (ii). Write $X^* = X \cup \{\omega\}$, where a subset of X^* is open provided it is either an open subset of X or the complement in X^* of a compact subset of X . Since X is locally compact and Hausdorff, X^* is a compact Hausdorff space and X , with its own topology, is an open subspace of X^* . Let

$$\mathcal{C}_\omega = \{F \in C(X^*) \mid F(\omega) = 0\}.$$

We must show that $C_0(X) = \{F|_X \mid F \in \mathcal{C}_\omega\}$.

First let $F \in \mathcal{C}_\omega$ and put $f = F|_X$. Since X carries the subspace topology, f is continuous on X . Fix $\alpha > 0$. Because $F(\omega) = 0 < \alpha$,

$$K_\alpha(f) = \{p \in X^* \mid |F(p)| \geq \alpha\},$$

which is a closed subset of the compact space X^* and therefore compact. By part (i), $f \in C_0(X)$.

Conversely let $f \in C_0(X)$ and define $F: X^* \rightarrow \mathbb{R}$ by $F = f$ on X and $F(\omega) = 0$. We check that F is continuous. At a point $x \in X$: X is an open subset of X^* and $F|_X = f$ is continuous, so for any open $O \subseteq \mathbb{R}$ containing $f(x)$ the set $f^{-1}(O)$ is open in X and hence open in X^* , and it is a neighborhood of x on which F takes values in O . At the point ω : let $\epsilon > 0$. By part (i) the set $K = K_\epsilon(f)$ is a compact subset of X , so

$$V = X^* \sim K = \{\omega\} \cup (X \sim K)$$

is an open neighborhood of ω in X^* . For $p \in V$ we have either $p = \omega$, whence $|F(p)| = 0 < \epsilon$, or $p \in X \sim K$, whence $|F(p)| = |f(p)| < \epsilon$. Thus $|F(p) - F(\omega)| < \epsilon$ on V , and F is continuous at ω . Since the sets of the form $F^{-1}(O)$ with $O \subseteq \mathbb{R}$ open are therefore neighborhoods of each of their points, F is continuous on X^* ; that is, $F \in \mathcal{C}_\omega$, and clearly $f = F|_X$.

Hence $C_0(X)$ is exactly the set of restrictions to X of those functions in $C(X^*)$ that vanish at the point at infinity.

Problem 21.41. Let X be an uncountable set with the discrete topology.

- (i) What is $C_c(X)$?
- (ii) What are the Borel subsets of X ?
- (iii) Let X^* be the one-point compactification of X . What is $C(X^*)$?
- (iv) What are the Borel subsets of X^* ?
- (v) Show that there is a Borel measure μ on X^* such that $\mu(X^*) = 1$ and $\int_X f d\mu = 0$ for each f in $C_c(X)$.

Solution. Preliminaries. In the discrete topology every subset of X is open, so every real-valued function on X is continuous, and X is Hausdorff. A subset K of X is compact if and only if it is finite: a finite set is compact in any topological space, while the cover of K by the open singletons $\{x\}$, $x \in K$, admits a finite subcover only if K is finite. Since $\{x\}$ is an open neighborhood of x with compact closure $\{x\}$, the space X is locally compact and Hausdorff, so the constructions of this chapter apply to it.

Part (i). Since all functions are continuous and the compact sets are exactly the finite sets,

$$C_c(X) = \{f: X \rightarrow \mathbb{R} \mid f(x) = 0 \text{ for all but finitely many } x \in X\},$$

that is, $C_c(X)$ is the linear span of the characteristic functions $\chi_{\{x\}}$, $x \in X$.

Part (ii). Every subset of X is open, hence Borel, so

$$\mathcal{B}(X) = 2^X,$$

the σ -algebra of all subsets of X .

Part (iii). Write $X^* = X \cup \{\omega\}$. By Problem 40(ii), a real-valued function F on X^* with $F(\omega) = 0$ is continuous if and only if $F|_X \in C_0(X)$, and by Problem 40(i) this says exactly that $\{x \in X \mid |F(x)| \geq \alpha\}$ is compact, that is finite, for every $\alpha > 0$. Applying this to $F - F(\omega)$, which is continuous exactly when F is, we conclude

$$C(X^*) = \{F: X^* \rightarrow \mathbb{R} \mid \text{with } c = F(\omega), \{x \in X : |F(x) - c| \geq \alpha\} \text{ is finite for every } \alpha > 0\}.$$

Equivalently, writing $\mathbf{1}$ for the constant function 1 on X^* ,

$$C(X^*) = \{c \cdot \mathbf{1} + \tilde{f} \mid c \in \mathbb{R}, f \in C_0(X)\},$$

where $\tilde{f}$ denotes the extension of f to X^* with value 0 at ω and, by Problem 40(i), $C_0(X)$ consists of exactly those $f: X \rightarrow \mathbb{R}$ for which $\{x \in X \mid |f(x)| \geq \alpha\}$ is finite for every $\alpha > 0$. In particular every continuous real-valued function on X^* is constant off a countable subset of X , since

$$\{x \in X \mid F(x) \neq c\} = \bigcup_{n=1}^{\infty} \{x \in X \mid |F(x) - c| \geq 1/n\}$$

is a countable union of finite sets. Hence no member of $C(X^*)$ takes uncountably many values, even though $C(X^*)$ does separate the points of the compact Hausdorff space X^* : for $x \in X$ the characteristic function $\chi_{\{x\}}$, extended by 0 at ω , belongs to $C(X^*)$.

Part (iv). Every subset of X is open in X and X is open in X^* , so every subset of X is open in X^* and therefore Borel. Also $\{\omega\} = X^* \setminus X$ is the complement of an open set, hence closed, hence Borel. For an arbitrary $S \subseteq X^*$,

$$S = (S \cap X) \cup (S \cap \{\omega\}),$$

a union of two Borel sets. Therefore

$$\mathcal{B}(X^*) = 2^{X^*}.$$

Part (v). Let $\mu = \delta_\omega$ be the Dirac delta measure concentrated at the point at infinity, that is,

$$\mu(S) = \begin{cases} 1, & \omega \in S, \\ 0, & \omega \notin S, \end{cases} \quad S \in \mathcal{B}(X^*) = 2^{X^*}.$$

Then $\mu(\emptyset) = 0$, and μ is countably additive: if $\{S_k\}$ is a disjoint sequence of subsets of X^* , then ω belongs to $\bigcup_k S_k$ if and only if it belongs to exactly one S_k , and in that case both $\mu(\bigcup_k S_k)$ and $\sum_k \mu(S_k)$ equal 1, while otherwise both are 0. Every subset of X^* , in particular every compact subset, has measure at most 1, so μ is a Borel measure; by Problem 31 it is even a Radon measure. Clearly

$$\mu(X^*) = 1.$$

Now let $f \in C_c(X)$. Every function on X^* is Borel measurable by part (iv), so f , extended by $f(\omega) = 0$, is measurable and $\int_X f d\mu$ is defined; moreover f is bounded, since it vanishes off a finite set, so it is integrable over the finite measure space $(X^*, \mathcal{B}(X^*), \mu)$. Since $\omega \notin X$ we have $\mu(X) = 0$, and the integral of an integrable function over a set of measure zero is zero. Hence

$$\int_X f d\mu = 0.$$

For an entirely elementary verification, note that $|f| \leq M$ on X for some $M \geq 0$, so $-M\chi_X \leq f \cdot \chi_X \leq M\chi_X$ on X^* , and the integrals of the two bounding simple functions against μ are $\mp M\mu(X) = 0$; by the monotonicity of integration, $\int_X f d\mu = \int_{X^*} f \cdot \chi_X d\mu = 0$.

A remark on what this shows. The Borel measures $\mu = \delta_\omega$ and the zero measure on $\mathcal{B}(X^*)$ are distinct, yet they give the same value, namely 0, to $\int_X f d\mu$ for every $f \in C_c(X)$. There is no conflict with Proposition 11, which asserts that a Radon measure on a locally compact Hausdorff space is determined by integration of the functions in C_c of that same space: here the measures live on X^* , and $C_c(X^*) = C(X^*)$ is strictly larger than $C_c(X)$; indeed $\int_{X^*} \mathbf{1} d\delta_\omega = 1 \neq 0$. What the example does show is that the restriction of δ_ω to $\mathcal{B}(X)$ is the zero measure, and that the functions of $C_c(X)$, viewed inside $C(X^*)$, are exactly functions vanishing at ω , so they cannot detect a unit mass sitting at the point at infinity.

Problem 21.42. Let X and Y be two locally compact Hausdorff spaces.

- (i) Show that each $f \in C_c(X \times Y)$ is the limit of sums of the form

$$\sum_{i=1}^n \varphi_i(x) \psi_i(y)$$

where $\varphi_i \in C_c(X)$ and $\psi_i \in C_c(Y)$. (The Stone-Weierstrass Theorem is useful.)

- (ii) Show that $\mathcal{B}(X \times Y) \subseteq \mathcal{B}(X) \times \mathcal{B}(Y)$.
- (iii) Show that $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ if and only if X or Y is the union of a countable collection of compact subsets.

Solution. Here $\mathcal{B}(X) \times \mathcal{B}(Y)$ denotes the product σ -algebra, that is, the smallest σ -algebra of subsets of $X \times Y$ containing all the rectangles $A \times B$ with $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$, while $\mathcal{B}(X \times Y)$ is the Borel σ -algebra of the product topology. Part (i) is proved below as stated. Parts (ii) and (iii), however, are not correct as printed: the inclusion in (ii) holds in the opposite direction in general, and the criterion in (iii) is neither sufficient nor necessary. We prove the true inclusion, prove the correct positive result behind (iii), and give explicit counterexamples to the printed assertions.

Part (i). Let $f \in C_c(X \times Y)$ and let $S = \text{supp } f$, a compact subset of $X \times Y$. The projections π_X and π_Y are continuous, so $K = \pi_X(S)$ and $L = \pi_Y(S)$ are compact, and $S \subseteq K \times L$.

By the locally compact extension property (Proposition 3), applied to K with neighborhood X and to L with neighborhood Y , choose $\varphi_0 \in C_c(X)$ and $\psi_0 \in C_c(Y)$ with

$$\varphi_0 = 1 \text{ on } K, \quad 0 \leq \varphi_0 \leq 1, \quad \psi_0 = 1 \text{ on } L, \quad 0 \leq \psi_0 \leq 1.$$

Put $K' = \text{supp } \varphi_0$ and $L' = \text{supp } \psi_0$; these are compact and $K \subseteq K'$, $L \subseteq L'$.

The space $K' \times L'$ is compact Hausdorff. Let

$$\mathcal{A} = \left\{ \sum_{i=1}^n u_i \otimes v_i \mid n \in \mathbb{N}, u_i \in C(K'), v_i \in C(L') \right\}, \quad (u \otimes v)(x, y) = u(x)v(y).$$

Then $\mathcal{A}$ is a linear subspace of $C(K' \times L')$, and it is closed under multiplication because $(u \otimes v)(u' \otimes v') = (uu') \otimes (vv')$ and products distribute over the sums; so $\mathcal{A}$ is a subalgebra. It contains the constants, since $\mathbf{1} = \mathbf{1} \otimes \mathbf{1}$. It separates points: if $(x_1, y_1) \neq (x_2, y_2)$ in $K' \times L'$, say $x_1 \neq x_2$, then, K' being compact Hausdorff and hence normal, Urysohn's Lemma provides $u \in C(K')$ with $u(x_1) = 0$ and $u(x_2) = 1$, and $u \otimes \mathbf{1} \in \mathcal{A}$ separates the two points; the case $y_1 \neq y_2$ is symmetric. By the Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in $C(K' \times L')$ with the maximum norm.

Let $\epsilon > 0$. Choose $u_i \in C(K')$, $v_i \in C(L')$, $1 \leq i \leq n$, with

$$\left| f(x, y) - \sum_{i=1}^n u_i(x) v_i(y) \right| \leq \epsilon \quad \text{for all } (x, y) \in K' \times L'.$$

By Theorem 6 each u_i extends to a continuous real-valued function $\widehat{u}_i$ on X and each v_i to a continuous $\widehat{v}_i$ on Y . Define

$$\varphi_i = \varphi_0 \widehat{u}_i \in C_c(X), \quad \psi_i = \psi_0 \widehat{v}_i \in C_c(Y), \quad g = \sum_{i=1}^n \varphi_i \otimes \psi_i.$$

We claim $|f - g| \leq \epsilon$ on all of $X \times Y$. Let $(x, y) \in X \times Y$ and put $t = \varphi_0(x) \psi_0(y) \in [0, 1]$ and $h = \sum_{i=1}^n u_i \otimes v_i$ on $K' \times L'$.

If $(x, y) \in K' \times L'$, then $g(x, y) = t h(x, y)$ and

$$f(x, y) - g(x, y) = t(f(x, y) - h(x, y)) + (1 - t)f(x, y).$$

The first term has absolute value at most ϵ . For the second, if $t < 1$ then $\varphi_0(x) < 1$ or $\psi_0(y) < 1$, so $x \notin K$ or $y \notin L$; in either case $(x, y) \notin K \times L \supseteq S \supseteq \{f \neq 0\}$, so $f(x, y) = 0$. Hence $(1 - t)f(x, y) = 0$ always, and $|f(x, y) - g(x, y)| \leq \epsilon$.

If $(x, y) \notin K' \times L'$, then $\varphi_0(x) = 0$ or $\psi_0(y) = 0$, so every $\varphi_i(x) = 0$ or every $\psi_i(y) = 0$; in either case $g(x, y) = 0$. Also $x \notin K$ or $y \notin L$, so $f(x, y) = 0$ as above. Hence $f(x, y) - g(x, y) = 0$.

Therefore $\|f - g\|_{\max} \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, f is the uniform limit of a sequence of sums of the stated form.

A lemma of countable generation. For σ -algebras $\mathcal{M}$ on X and $\mathcal{N}$ on Y ,

$$\mathcal{M} \times \mathcal{N} = \bigcup \{ \sigma(\{A_n \times B_n\}_{n \in \mathbb{N}}) \mid A_n \in \mathcal{M}, B_n \in \mathcal{N} \}.$$

Indeed the right-hand side is contained in the left-hand side, and it is itself a σ -algebra containing all rectangles: it is plainly closed under complements, and if $E_k \in \sigma(\mathcal{R}_k)$ with each $\mathcal{R}_k$ a countable family of rectangles, then $\bigcup_k E_k \in \sigma(\bigcup_k \mathcal{R}_k)$, again a σ -algebra generated by countably many rectangles. Minimality of $\mathcal{M} \times \mathcal{N}$ finishes the proof.

Consequently, if $E \in \mathcal{M} \times \mathcal{N}$ there are countable families $\{A_n\} \subseteq \mathcal{M}$ and $\{B_n\} \subseteq \mathcal{N}$ such that E is saturated for the relation

$$(x, y) \approx (x', y') \iff \chi_{A_n}(x) = \chi_{A_n}(x') \text{ and } \chi_{B_n}(y) = \chi_{B_n}(y') \text{ for all } n,$$

because the collection of $\approx$ -saturated sets is a σ -algebra containing every $A_n \times B_n$.

Part (ii). The correct inclusion is

$$\mathcal{B}(X) \times \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y),$$

and it holds for arbitrary topological spaces. Indeed, the projection $\pi_X: X \times Y \rightarrow X$ is continuous, so $\{O \subseteq X \mid \pi_X^{-1}(O) \in \mathcal{B}(X \times Y)\}$ is a σ -algebra containing the topology of X and therefore contains $\mathcal{B}(X)$; thus $A \times Y = \pi_X^{-1}(A) \in \mathcal{B}(X \times Y)$ for every $A \in \mathcal{B}(X)$. Similarly $X \times B \in \mathcal{B}(X \times Y)$ for $B \in \mathcal{B}(Y)$, whence

$$A \times B = (A \times Y) \cap (X \times B) \in \mathcal{B}(X \times Y).$$

By the minimality of the product σ -algebra, $\mathcal{B}(X) \times \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$.

The inclusion as printed fails in general. Let D be a set of cardinality greater than $\mathfrak{c} = 2^{\aleph_0}$, equipped with the discrete topology, and let $X = Y = D$; as in Problem 41, D is locally compact and Hausdorff and $\mathcal{B}(D) = 2^D$. The product $D \times D$ is discrete, so $\mathcal{B}(D \times D) = 2^{D \times D}$; in particular the diagonal $\Delta = \{(x, x) \mid x \in D\}$ is Borel. Suppose $\Delta \in 2^D \times 2^D$. By the lemma there are countable families $\{A_n\}$, $\{B_n\}$ of subsets of D for which Δ is $\approx$ -saturated. The map

$$x \mapsto ((\chi_{A_n}(x))_n, (\chi_{B_n}(x))_n) \in \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$$

has range of cardinality at most $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c} < |D|$, so it is not one-to-one: there are $x \neq x'$ in D with $\chi_{A_n}(x) = \chi_{A_n}(x')$ and $\chi_{B_n}(x) = \chi_{B_n}(x')$ for all n . Then $(x, x) \approx (x, x')$, and since $(x, x) \in \Delta$ and Δ is saturated, $(x, x') \in \Delta$, a contradiction. Hence $\Delta \notin \mathcal{B}(D) \times \mathcal{B}(D)$ and $\mathcal{B}(X \times Y) \not\subseteq \mathcal{B}(X) \times \mathcal{B}(Y)$.

Part (iii), the positive result. The following theorem is true, and it is the one that part (i) is designed to support.

Lemma A. If W is an open subset of $X \times Y$ that is the union of a countable collection of compact sets, then $W \in \mathcal{B}(X) \times \mathcal{B}(Y)$.

Proof. First, $X \times Y$ is locally compact and Hausdorff: it is Hausdorff as a product of Hausdorff spaces, and if $x \in U$, $y \in V$ with $\bar{U}$, $\bar{V}$ compact, then $U \times V$ is a neighborhood of (x, y) with closure $\bar{U} \times \bar{V}$, a product of two compact sets and hence compact. Write $W = \bigcup_n S_n$ with each S_n compact. By Proposition 3 applied in $X \times Y$ to the compact set S_n and its neighborhood W , there is $f_n \in C_c(X \times Y)$ with $f_n = 1$ on S_n , $f_n = 0$ off W and $0 \leq f_n \leq 1$. Then

$$W = \bigcup_n \{(x, y) \mid f_n(x, y) > 0\},$$

since each set on the right is contained in W and $S_n \subseteq \{f_n > 0\}$. Now for $A \in \mathcal{B}(X)$ we have $\pi_X^{-1}(A) = A \times Y$, a measurable rectangle, so π_X is measurable from $(X \times Y, \mathcal{B}(X) \times \mathcal{B}(Y))$ to $(X, \mathcal{B}(X))$, and likewise for π_Y ; consequently $\varphi \circ \pi_X$ and $\psi \circ \pi_Y$ are $\mathcal{B}(X) \times \mathcal{B}(Y)$ -measurable whenever φ and ψ are Borel functions. Hence every sum $\sum_i \varphi_i \otimes \psi_i$ is $\mathcal{B}(X) \times \mathcal{B}(Y)$ -measurable, and by part (i) each f_n , being a pointwise (indeed uniform) limit of such sums, is $\mathcal{B}(X) \times \mathcal{B}(Y)$ -measurable. Therefore $\{f_n > 0\} \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and so is the countable union W .

Theorem B. If one of the two spaces, say Y , is a locally compact Hausdorff space with a countable base, then for every topological space X ,

$$\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y).$$

Proof. The inclusion $\supseteq$ is part (ii). For $\subseteq$ it suffices, by minimality of $\mathcal{B}(X \times Y)$, to show that every open $W \subseteq X \times Y$ lies in $\mathcal{B}(X) \times \mathcal{B}(Y)$. Let $\{V_n\}$ be a countable base for Y consisting of sets with compact closure; such a base exists because, given a countable base $\{O_n\}$ and a point $y \in O$, Lemma 1 provides an open P with $y \in P \subseteq \overline{P} \subseteq O$ and $\overline{P}$ compact, and then some O_n satisfies $y \in O_n \subseteq P$, so $\overline{O_n} \subseteq \overline{P}$ is a closed subset of a compact set and hence compact; discarding from $\{O_n\}$ those members without compact closure leaves a countable base. Put

$$A_n = \{x \in X \mid \{x\} \times \overline{V_n} \subseteq W\}.$$

Each A_n is open: if $x \in A_n$, then for each $y \in \overline{V_n}$ choose basic open $U_y \times O_y \subseteq W$ containing (x, y) ; by compactness of $\overline{V_n}$ finitely many $O_{y_1}, \dots, O_{y_k}$ cover $\overline{V_n}$, and $U = \bigcap_{j=1}^k U_{y_j}$ is a neighborhood of x with $U \times \overline{V_n} \subseteq W$, so $U \subseteq A_n$. Now

$$W = \bigcup_n A_n \times V_n.$$

The inclusion $\supseteq$ is clear since $A_n \times V_n \subseteq A_n \times \overline{V_n} \subseteq W$. Conversely, let $(x, y) \in W$ and choose basic open $U \times O \subseteq W$ with $(x, y) \in U \times O$; choosing n with $y \in V_n$ and $\overline{V_n} \subseteq O$, as in the construction of the base, we get $\{x\} \times \overline{V_n} \subseteq U \times O \subseteq W$, so $x \in A_n$ and $(x, y) \in A_n \times V_n$. Since each $A_n \times V_n$ is a Borel rectangle, $W \in \mathcal{B}(X) \times \mathcal{B}(Y)$.

Theorem B covers the classical cases: if X or Y is $\mathbb{R}^n$, or more generally a σ -compact metrizable locally compact space, then the Borel σ -algebra of the product is the product of the Borel σ -algebras.

Part (iii), failure of the printed criterion. Neither implication of the printed equivalence is valid.

The condition is not sufficient. Let D be a discrete space of cardinality greater than $\mathfrak{c}$ and let $X = Y = D^*$, the one-point compactification of D . Each of X and Y is compact, hence trivially the union of a countable collection of compact subsets. By Problem 41(iv), $\mathcal{B}(D^*) = 2^{D^*}$. The space $X \times Y$ is Hausdorff, so its diagonal Δ is closed and therefore Borel. Since $|D^*| > \mathfrak{c}$, the argument given in part (ii) shows $\Delta \notin 2^{D^*} \times 2^{D^*} = \mathcal{B}(X) \times \mathcal{B}(Y)$. Hence $\mathcal{B}(X \times Y) \neq \mathcal{B}(X) \times \mathcal{B}(Y)$ although both factors are compact. What fails here is not σ -compactness of the factors but σ -compactness of the open subsets of the product, which is what Lemma A actually requires: the open set $D \subseteq D^*$ is not the union of countably many compact, that is finite, sets.

The condition is not necessary. Let D be a discrete space of cardinality $\aleph_1$ and $X = Y = D$. Compact subsets of D are finite, so a countable union of compact subsets of D is countable and hence proper; thus neither factor is a countable union of compact sets. Nevertheless $\mathcal{B}(X \times Y) = 2^{D \times D}$ equals $\mathcal{B}(X) \times \mathcal{B}(Y) = 2^D \times 2^D$. To see this, note $\aleph_1 \leq \mathfrak{c}$, so there is a one-to-one map $\phi: D \rightarrow \{0, 1\}^{\mathbb{N}}$; put $A_n = \{x \in D \mid \phi(x)_n = 1\}$.

First, for any $E \subseteq D$ and any map $h: E \rightarrow D$ the graph $G_h = \{(x, h(x)) \mid x \in E\}$ belongs to $2^D \times 2^D$, since

$$G_h = (E \times D) \cap \bigcap_{n=1}^{\infty} \left[(h^{-1}(A_n) \times A_n) \cup ((E \sim h^{-1}(A_n)) \times (D \sim A_n)) \right];$$

for $x \in E$ the right-hand side requires $\chi_{A_n}(h(x)) = \chi_{A_n}(y)$ for all n , that is $\phi(h(x)) = \phi(y)$, that is $y = h(x)$ by injectivity of ϕ .

Second, if $S \subseteq D \times D$ has all vertical sections $S_x = \{y \mid (x, y) \in S\}$ countable, then, setting $E = \{x \mid S_x \neq \emptyset\}$ and enumerating $S_x = \{h_1(x), h_2(x), \dots\}$ with repetitions allowed, we get $S = \bigcup_k G_{h_k}$, a countable union of graphs, hence $S \in 2^D \times 2^D$. Symmetrically, a set all of whose horizontal sections are countable belongs to $2^D \times 2^D$.

Finally, well-order D in order type ω_1 , so that every initial segment is countable, and let $L = \{(x, y) \mid y \prec x\}$. For an arbitrary $S \subseteq D \times D$, the vertical sections of $S \cap L$ are contained in initial segments and so are countable, while the horizontal sections of $S \sim L = S \cap \{(x, y) \mid x \preceq y\}$ are contained in initial segments and so are countable. By the previous paragraph both sets lie in $2^D \times 2^D$, hence so

does S . Thus every subset of $D \times D$ is in the product σ -algebra, and equality of the two σ -algebras holds even though neither factor is σ -compact.

In summary: part (i) holds as stated; the inclusion in (ii) holds in the direction $\mathcal{B}(X) \times \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$ and can fail in the printed direction; and the correct positive statement replacing (iii) is Theorem B, together with Lemma A, which identifies σ -compactness of the open subsets of $X \times Y$, not of X or Y , as the relevant hypothesis.

21.7 Problems 21.43–21.49

Problem 21.43. In the proof of the Riesz-Markov Theorem, establish inner regularity in the case in which $\mu(\mathcal{O}) = \infty$.

Solution. Recall the setting. X is a locally compact Hausdorff space, I is a positive linear functional on $C_c(X)$, and in the proof of the Riesz-Markov Theorem one defines a set function μ on the topology $\mathcal{T}$ of X by $\mu(\emptyset) = 0$ and, for each nonempty open subset $\mathcal{O}$ of X ,

$$\mu(\mathcal{O}) = \sup \{ I(f) \mid f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subseteq \mathcal{O} \}.$$

The proof shows that μ is a premeasure and then verifies that μ is a **Radon premeasure**, that is, that

- (i) $\mu(\mathcal{U}) < \infty$ for each open $\mathcal{U}$ with compact closure, and
- (ii) (inner regularity) $\mu(\mathcal{O}) = \sup \{ \mu(\mathcal{U}) \mid \mathcal{U} \text{ open, } \overline{\mathcal{U}} \text{ a compact subset of } \mathcal{O} \}$ for every open $\mathcal{O}$.

The text establishes (ii) under the assumption $\mu(\mathcal{O}) < \infty$ and leaves the case $\mu(\mathcal{O}) = \infty$ to this problem. We treat that case.

So let $\mathcal{O}$ be open with $\mu(\mathcal{O}) = \infty$, and set

$$s = \sup \{ \mu(\mathcal{U}) \mid \mathcal{U} \text{ open, } \overline{\mathcal{U}} \text{ a compact subset of } \mathcal{O} \}.$$

Since any such $\mathcal{U}$ is contained in $\mathcal{O}$ and μ is monotone, $s \leq \mu(\mathcal{O})$. We must show $s = \infty$; then $s = \mu(\mathcal{O})$ and inner regularity holds at $\mathcal{O}$.

Let $c > 0$ be arbitrary. Since $\mu(\mathcal{O}) = \infty > c$, the definition of $\mu(\mathcal{O})$ as a supremum provides a function $f_c \in C_c(X)$ for which

$$0 \leq f_c \leq 1 \text{ on } X, \quad \text{supp } f_c \subseteq \mathcal{O}, \quad I(f_c) > c.$$

Put $K = \text{supp } f_c$. Then K is a compact subset of X contained in the open set $\mathcal{O}$, so $\mathcal{O}$ is a neighborhood of K . Since X is locally compact and Hausdorff, it possesses the locally compact separation property, that is, property (ii) of Theorem 7: there is a neighborhood $\mathcal{U}$ of K that has compact closure contained in $\mathcal{O}$. A neighborhood of a set is an open set containing it, so $\mathcal{U}$ is open and

$$K \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{O}, \quad \overline{\mathcal{U}} \text{ compact}.$$

Now f_c is one of the functions competing in the supremum that defines $\mu(\mathcal{U})$: indeed $f_c \in C_c(X)$, $0 \leq f_c \leq 1$ on X , and $\text{supp } f_c = K \subseteq \mathcal{U}$. Therefore

$$\mu(\mathcal{U}) \geq I(f_c) > c,$$

and since $\mathcal{U}$ is open with compact closure contained in $\mathcal{O}$, it follows that $s > c$.

As $c > 0$ was arbitrary, $s = \infty = \mu(\mathcal{O})$. Together with the case $\mu(\mathcal{O}) < \infty$ treated in the text, this establishes the inner regularity property (ii) for every open subset $\mathcal{O}$ of X , and hence completes the verification that μ is a Radon premeasure. By Theorem 10 the Caratheodory outer measure μ^* induced by μ restricts to a Radon measure $\hat{\mu}$ on $\mathcal{B}(X)$ extending μ , which is the object used in the remainder of the proof.

Problem 21.44. Let $k(x, y)$ be a bounded Borel measurable function on $X \times Y$, and let μ and ν be Radon measures on X and Y .

(i) Show that

$$\iint_{X \times Y} \varphi(x) k(x, y) \psi(y) d(\mu \times \nu) = \int_Y \left[\int_X \varphi(x) k(x, y) d\mu \right] \psi(y) d\nu = \int_X \varphi(x) \left[\int_Y k(x, y) \psi(y) d\nu \right] d\mu$$

for all $\varphi \in C_c(X)$ and $\psi \in C_c(Y)$.

(ii) If the integral in (i) is zero for all φ and ψ in $C_c(X)$ and $C_c(Y)$, show that then $k = 0$ a.e. $[\mu \times \nu]$.

Solution. Throughout, X and Y are locally compact Hausdorff spaces, μ and ν are Radon measures on $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, and $M = \sup_{X \times Y} |k| < \infty$. We read “Borel measurable on $X \times Y$ ” as measurable with respect to the product σ -algebra $\mathcal{B}(X) \times \mathcal{B}(Y)$, which is the σ -algebra generated by the measurable rectangles and is contained in the domain of $\mu \times \nu$; by Problem 42(iii) it coincides with $\mathcal{B}(X \times Y)$ precisely when X or Y is the union of a countable collection of compact subsets.

Two preliminary remarks. First, a Radon measure is a Borel measure, so every compact set has finite measure; in particular $\mu(\text{supp } \varphi) < \infty$ and $\nu(\text{supp } \psi) < \infty$ for $\varphi \in C_c(X)$, $\psi \in C_c(Y)$. Second, we may assume μ and ν are complete, since Fubini’s Theorem as stated in Chapter 20 requires completeness of the measure over which the inner integral is taken. Replacing μ and ν by their completions changes no integral of a Borel measurable function, and it changes neither the premeasure $\lambda(A \times B) = \mu(A) \cdot \nu(B)$ on the semiring of measurable rectangles in an essential way nor the induced outer measure: if $\bar{A} = A \cup Z$ and $\bar{B} = B \cup W$, where $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, $Z \subseteq N$ with $\mu(N) = 0$ and $W \subseteq P$ with $\nu(P) = 0$, then $\bar{A} \times \bar{B} \subseteq (A \times B) \cup (N \times Y) \cup (X \times P)$, while $\lambda(N \times Y) = \mu(N) \cdot \nu(Y) = 0$ and $\lambda(X \times P) = \mu(X) \cdot \nu(P) = 0$ under the convention $0 \cdot \infty = 0$; so the two induced outer measures agree. Hence $\mu \times \nu$ is unchanged.

Part (i).

Fix $\varphi \in C_c(X)$ and $\psi \in C_c(Y)$, and put

$$h(x, y) = \varphi(x) k(x, y) \psi(y), \quad K = \text{supp } \varphi, \quad L = \text{supp } \psi.$$

The function $(x, y) \mapsto \varphi(x)$ is measurable with respect to $\mathcal{B}(X) \times \mathcal{B}(Y)$, since the preimage of a Borel subset B of $\mathbb{R}$ is the measurable rectangle $\varphi^{-1}(B) \times Y$; likewise for $(x, y) \mapsto \psi(y)$. Since products of measurable functions are measurable, h is $(\mu \times \nu)$ -measurable. Moreover h vanishes off $K \times L$ and

$$|h| \leq M \|\varphi\|_{\max} \|\psi\|_{\max} \chi_{K \times L} \quad \text{on } X \times Y,$$

while $(\mu \times \nu)(K \times L) = \mu(K) \nu(L) < \infty$. Hence h is integrable over $X \times Y$ with respect to $\mu \times \nu$. Fubini’s Theorem now applies to h and gives, taking the inner integral over Y ,

$$\iint_{X \times Y} h d(\mu \times \nu) = \int_X \left[\int_Y h(x, y) d\nu(y) \right] d\mu(x).$$

For each fixed x the number $\varphi(x)$ is a constant relative to the inner integration, so by the linearity of integration $\int_Y h(x, y) d\nu(y) = \varphi(x) \int_Y k(x, y) \psi(y) d\nu(y)$, and therefore

$$\iint_{X \times Y} \varphi(x) k(x, y) \psi(y) d(\mu \times \nu) = \int_X \varphi(x) \left[\int_Y k(x, y) \psi(y) d\nu \right] d\mu.$$

Interchanging the roles of X and Y (the product measure $\nu \times \mu$ on $Y \times X$ is carried to $\mu \times \nu$ by the homeomorphism $(y, x) \mapsto (x, y)$, which matches rectangles with rectangles of equal premeasure) and applying Fubini’s Theorem again, with the inner integral now over X , yields in the same way

$$\iint_{X \times Y} \varphi(x) k(x, y) \psi(y) d(\mu \times \nu) = \int_Y \left[\int_X \varphi(x) k(x, y) d\mu \right] \psi(y) d\nu.$$

This is the asserted chain of equalities. Note that Fubini’s Theorem also tells us that the inner integrals are defined and finite for almost all values of the outer variable; in fact here they are defined for every

value. Indeed, by Problem 12(iii) of Chapter 20, every section of a function measurable with respect to $\mathcal{B}(X) \times \mathcal{B}(Y)$ is measurable, so $k(x, \cdot)$ is $\mathcal{B}(Y)$ -measurable for each x and $k(\cdot, y)$ is $\mathcal{B}(X)$ -measurable for each y ; and $|k(x, \cdot)\psi| \leq M\|\psi\|_{\max}\chi_L$ is ν -integrable for each x , while $|\varphi k(\cdot, y)| \leq M\|\varphi\|_{\max}\chi_K$ is μ -integrable for each y .

We emphasize that no σ -finiteness is needed in part (i): Fubini's Theorem as stated in Chapter 20 assumes only that the measure over which the inner integration is performed is complete and that the integrand is integrable with respect to the product measure.

Part (ii).

Assume

$$\iint_{X \times Y} \varphi(x)k(x, y)\psi(y) d(\mu \times \nu) = 0 \quad \text{for all } \varphi \in C_c(X), \psi \in C_c(Y). \quad (*)$$

Step 1: approximation of characteristic functions. Let $A \in \mathcal{B}(X)$ have $\mu(A) < \infty$. We claim there are functions $\varphi_n \in C_c(X)$ with $|\varphi_n| \leq 1$ on X and

$$\lim_{n \rightarrow \infty} \int_X |\chi_A - \varphi_n| d\mu = 0.$$

Indeed, χ_A is Borel measurable and vanishes outside the set A of finite measure, so Lusin's Theorem applies: for each n there are a Borel set X_n and a function $g_n \in C_c(X)$ with $\chi_A = g_n$ on X_n and $\mu(X \setminus X_n) < 1/n$. Replace g_n by $\varphi_n = \max\{-1, \min\{1, g_n\}\}$, which again belongs to $C_c(X)$, satisfies $|\varphi_n| \leq 1$, and still agrees with χ_A on X_n because $0 \leq \chi_A \leq 1$. Since $\chi_A - \varphi_n$ vanishes on X_n and is bounded by 2,

$$\int_X |\chi_A - \varphi_n| d\mu = \int_{X \setminus X_n} |\chi_A - \varphi_n| d\mu \leq 2\mu(X \setminus X_n) < 2/n.$$

The same construction applies on Y : for $B \in \mathcal{B}(Y)$ with $\nu(B) < \infty$ there are $\psi_n \in C_c(Y)$ with $|\psi_n| \leq 1$ and $\int_Y |\chi_B - \psi_n| d\nu \rightarrow 0$.

Step 2: the integral of k over a rectangle of finite measure vanishes. Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ with $\mu(A) < \infty$ and $\nu(B) < \infty$, and let φ_n, ψ_n be as in Step 1. Since $|k| \leq M$ and

$$\chi_A(x)\chi_B(y) - \varphi_n(x)\psi_n(y) = [\chi_A(x) - \varphi_n(x)]\chi_B(y) + \varphi_n(x)[\chi_B(y) - \psi_n(y)],$$

each of the two functions on the right is a product $h(x)g(y)$ with h integrable over X with respect to μ and g integrable over Y with respect to ν : the functions $|\chi_A - \varphi_n|$ and $|\varphi_n|$ are bounded and vanish outside $A \cup \text{supp } \varphi_n$, a set of finite μ -measure because $\mu(A) < \infty$ and $\text{supp } \varphi_n$ is compact, and symmetrically on Y . By Problem 10 of Chapter 20, which asserts precisely that such a product is integrable over $X \times Y$ with $\iint h(x)g(y) d(\mu \times \nu) = \int_X h d\mu \cdot \int_Y g d\nu$ and which requires no σ -finiteness, we may estimate

$$\begin{aligned} \left| \int_{A \times B} k d(\mu \times \nu) - \iint_{X \times Y} \varphi_n(x)k(x, y)\psi_n(y) d(\mu \times \nu) \right| &\leq M \int_X |\chi_A - \varphi_n| d\mu \cdot \int_Y \chi_B d\nu \\ &\quad + M \int_X |\varphi_n| d\mu \cdot \int_Y |\chi_B - \psi_n| d\nu. \end{aligned}$$

The first term is $M\nu(B) \int_X |\chi_A - \varphi_n| d\mu \rightarrow 0$. For the second, $\int_X |\varphi_n| d\mu \leq \int_X |\chi_A - \varphi_n| d\mu + \mu(A) \leq 2 + \mu(A)$ for all large n , a bound independent of n , while $\int_Y |\chi_B - \psi_n| d\nu \rightarrow 0$; so the second term also tends to 0. By (*) the second integral inside the absolute value is 0 for every n . Letting $n \rightarrow \infty$,

$$\int_{A \times B} k d(\mu \times \nu) = 0 \quad \text{whenever } \mu(A) < \infty \text{ and } \nu(B) < \infty. \quad (**)$$

Step 3: from rectangles to all sets. Fix Borel sets $A \subseteq X$ and $B \subseteq Y$ of finite measure and set $R = A \times B$, so $(\mu \times \nu)(R) = \mu(A)\nu(B) < \infty$. Let $\mathcal{M}_R$ denote the trace on R of the product σ -algebra $\mathcal{B}(X) \times \mathcal{B}(Y)$, that is, the collection of sets $E \cap R$ with $E \in \mathcal{B}(X) \times \mathcal{B}(Y)$; every member of $\mathcal{M}_R$ is $(\mu \times \nu)$ -measurable and has finite measure. Define two measures on $\mathcal{M}_R$ by

$$\kappa^+(E) = \int_E k^+ d(\mu \times \nu), \quad \kappa^-(E) = \int_E k^- d(\mu \times \nu),$$

where $k^\pm$ are the positive and negative parts of k ; these are indeed measures, by the countable additivity of integration, and they are finite because $|k| \leq M$ and $(\mu \times \nu)(R) < \infty$. If $A' \subseteq A$ and $B' \subseteq B$ are Borel, then $\mu(A') < \infty$ and $\nu(B') < \infty$, so $(**)$ gives $\kappa^+(A' \times B') = \kappa^-(A' \times B')$. Thus the two finite measures κ^+ and κ^- agree on the semiring $\mathcal{S}_R$ of measurable rectangles contained in R , a semiring that contains R itself and generates $\mathcal{M}_R$. By the uniqueness assertion of the Caratheodory-Hahn Theorem, applicable because the common restriction to $\mathcal{S}_R$ is a finite, hence σ -finite, premeasure, we conclude $\kappa^+ = \kappa^-$ on all of $\mathcal{M}_R$.

Now take $E = R \cap \{k > 0\}$. Since k is measurable with respect to $\mathcal{B}(X) \times \mathcal{B}(Y)$, the set $\{k > 0\}$ belongs to $\mathcal{B}(X) \times \mathcal{B}(Y)$ and hence $E \in \mathcal{M}_R$. Then $k^- = 0$ on E , so $\kappa^-(E) = 0$ and therefore $\kappa^+(E) = 0$; that is, the nonnegative function k^+ has integral zero over E . Since $k^+ > 0$ at every point of E , we have $E = \bigcup_{j=1}^{\infty} E_j$ with $E_j = E \cap \{k^+ > 1/j\}$, and Chebychev's Inequality gives $(\mu \times \nu)(E_j) \leq j \int_E k^+ d(\mu \times \nu) = 0$ for each j ; by countable monotonicity, $(\mu \times \nu)(E) = 0$. Symmetrically, taking $E' = R \cap \{k < 0\}$ and interchanging the roles of k^+ and k^- , $(\mu \times \nu)(R \cap \{k < 0\}) = 0$. Hence

$$(\mu \times \nu)((A \times B) \cap \{k \neq 0\}) = 0 \quad \text{whenever } \mu(A), \nu(B) < \infty. \quad (***)$$

Step 4: conclusion. Let E be any $(\mu \times \nu)$ -measurable subset of $\{k \neq 0\}$ with $(\mu \times \nu)(E) < \infty$. By the definition of $\mu \times \nu$ as the restriction to the measurable sets of the Caratheodory outer measure induced by the premeasure $\lambda(A \times B) = \mu(A)\nu(B)$ on the semiring of measurable rectangles, there is a countable collection $\{A_i \times B_i\}_{i=1}^{\infty}$ of measurable rectangles covering E with $\sum_i \lambda(A_i \times B_i) < \infty$. Fix i . If $\lambda(A_i \times B_i) = 0$, then

$$(\mu \times \nu)((A_i \times B_i) \cap \{k \neq 0\}) \leq (\mu \times \nu)(A_i \times B_i) = \lambda(A_i \times B_i) = 0.$$

If $\lambda(A_i \times B_i) > 0$, then, since $\mu(A_i)\nu(B_i) = \lambda(A_i \times B_i)$ is positive and finite, both factors are positive and finite, so $(***)$ applies and the same intersection is again null. Hence, by countable monotonicity,

$$(\mu \times \nu)(E) \leq \sum_{i=1}^{\infty} (\mu \times \nu)((A_i \times B_i) \cap \{k \neq 0\}) = 0.$$

So no $(\mu \times \nu)$ -measurable subset of $\{k \neq 0\}$ has positive finite measure. Since $\{k \neq 0\}$ itself belongs to $\mathcal{B}(X) \times \mathcal{B}(Y)$ and hence is $(\mu \times \nu)$ -measurable, applying this with $E = \{k \neq 0\}$ yields the dichotomy

$$(\mu \times \nu)(\{k \neq 0\}) = 0 \quad \text{or} \quad (\mu \times \nu)(\{k \neq 0\}) = \infty.$$

The second alternative is excluded as soon as $\{k \neq 0\}$ is a countable union of sets of finite product measure. That is the case, in particular, when μ and ν are σ -finite: then $X \times Y$ is a countable union of measurable rectangles of finite λ -measure, each of which meets $\{k \neq 0\}$ in a null set by $(***)$, so $(\mu \times \nu)(\{k \neq 0\}) = 0$. Therefore $k = 0$ a.e. $[\mu \times \nu]$, as asserted.

Remark on the hypotheses. The σ -finiteness used in the very last line is the standing hypothesis under which the product measure $\mu \times \nu$ carries the usual iterated-integration theory; it cannot simply be dropped from the final sentence, since without it $\{k \neq 0\}$ could conceivably have infinite product measure while meeting every rectangle of finite measure in a null set, that is, $\mu \times \nu$ could fail to be semifinite there. What the argument establishes with no restriction whatever on μ and ν is the sharp local statement: $\{k \neq 0\}$ contains no measurable set of positive finite measure, equivalently k vanishes almost everywhere on every measurable rectangle $A \times B$ with $\mu(A) < \infty$ and $\nu(B) < \infty$. Since every $\varphi \in C_c(X)$ and $\psi \in C_c(Y)$ have supports of finite measure, this local statement is exactly as much as the hypothesis $(*)$ can see.

Problem 21.45. Let X be a compact Hausdorff space and μ a Borel measure on $\mathcal{B}(X)$. Show that there is a constant $c > 0$ such that

$$\left| \int_X f d\mu \right| \leq c \|f\|_{\max} \quad \text{for all } f \in C(X).$$

Solution. Recall the definition in force: a measure μ on the Borel σ -algebra $\mathcal{B}(X)$ of a topological space X is called a Borel measure provided every compact subset of X has finite measure. Since X is compact, X is itself a compact subset of X , and therefore

$$\mu(X) < \infty.$$

We claim that $c = \mu(X) + 1$ has the required property. Note $c \geq 1 > 0$, so c is an admissible constant even in the degenerate case $\mu(X) = 0$.

Let $f \in C(X)$. First, f is Borel measurable: for each open subset $\mathcal{O}$ of $\mathbb{R}$ the preimage $f^{-1}(\mathcal{O})$ is open in X by continuity, hence belongs to $\mathcal{B}(X)$, and since the open subsets of $\mathbb{R}$ generate the Borel σ -algebra of $\mathbb{R}$, f is measurable with respect to $\mathcal{B}(X)$.

Second, f is bounded and $\|f\|_{\max}$ is a genuine maximum: $|f|$ is a continuous real-valued function on the compact space X , so its image $|f|[X]$ is a compact, hence bounded and closed, subset of $\mathbb{R}$, and therefore

$$\|f\|_{\max} = \max_{x \in X} |f(x)| < \infty \quad \text{and} \quad |f(x)| \leq \|f\|_{\max} \quad \text{for all } x \in X.$$

Consequently $|f| \leq \|f\|_{\max} \cdot \chi_X$ on X , and χ_X has finite integral $\mu(X)$. By the monotonicity of integration for nonnegative measurable functions, $\int_X |f| d\mu \leq \|f\|_{\max} \mu(X) < \infty$, so f is integrable over X with respect to μ . Finally, by the triangle inequality for the integral,

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu \leq \mu(X) \|f\|_{\max} \leq c \|f\|_{\max}.$$

This proves the assertion. Two remarks. First, the estimate exhibits $L(f) = \int_X f d\mu$ as a bounded linear functional on the normed linear space $C(X)$ with the maximum norm, of norm at most $\mu(X)$; since L is positive and $L(1) = \mu(X)$ with $\|1\|_{\max} = 1$, in fact $\|L\| = \mu(X)$, which is the general identity $\|L\| = L(1)$ noted at the start of Section 21.5 for positive linear functionals on $C(X)$. Second, compactness of X is essential: on a locally compact but noncompact space, such as $X = \mathbb{R}$ with Lebesgue measure, no such constant exists for the functional $f \mapsto \int_X f d\mu$ on the bounded continuous functions, and this is why one works with $C_c(X)$ there.

Problem 21.46. Let x_0 be a point in the compact Hausdorff space X . Define $L(f) = f(x_0)$ for each $f \in C(X)$. Show that L is a bounded linear functional on $C(X)$. Find the signed Radon measure that represents L .

Solution. Here $C(X)$ carries the maximum norm $\|f\|_{\max} = \max_{x \in X} |f(x)|$, which is finite and attained because X is compact.

L is linear and bounded. For $f, g \in C(X)$ and $\alpha, \beta \in \mathbb{R}$,

$$L(\alpha f + \beta g) = (\alpha f + \beta g)(x_0) = \alpha f(x_0) + \beta g(x_0) = \alpha L(f) + \beta L(g),$$

so L is linear. Moreover, for every $f \in C(X)$,

$$|L(f)| = |f(x_0)| \leq \max_{x \in X} |f(x)| = \|f\|_{\max},$$

so L is bounded with $\|L\| \leq 1$. Taking $f \equiv 1$, which lies in $C(X)$ and has $\|f\|_{\max} = 1$, gives $L(f) = 1$; hence

$$\|L\| = 1.$$

Observe also that L is a positive linear functional: if $f \geq 0$ on X then $L(f) = f(x_0) \geq 0$.

The representing measure. Let $\delta_{x_0}: \mathcal{B}(X) \rightarrow [0, \infty)$ be the Dirac delta measure concentrated at x_0 ,

$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E, \\ 0 & \text{if } x_0 \notin E, \end{cases} \quad E \in \mathcal{B}(X).$$

It is a measure: $\delta_{x_0}(\emptyset) = 0$, and if $\{E_k\}_{k=1}^\infty$ is a disjoint collection of Borel sets then x_0 belongs to $\bigcup_k E_k$ if and only if it belongs to exactly one E_k , so both sides of the countable additivity identity equal 1 in that case and 0 otherwise.

δ_{x_0} is a Radon measure. This is Problem 31; we verify it directly. Every subset of X has δ_{x_0} -measure at most 1, so in particular every compact set has finite measure and δ_{x_0} is a Borel measure. For outer regularity, let $E \in \mathcal{B}(X)$. If $x_0 \in E$, then every neighborhood $\mathcal{U}$ of E contains x_0 , so $\delta_{x_0}(\mathcal{U}) = 1 = \delta_{x_0}(E)$. If $x_0 \notin E$, then, since X is Hausdorff, the singleton $\{x_0\}$ is closed, so $\mathcal{U} = X \setminus \{x_0\}$ is an open set containing E with $\delta_{x_0}(\mathcal{U}) = 0 = \delta_{x_0}(E)$. In either case the infimum in the definition of outer regularity equals $\delta_{x_0}(E)$. For inner regularity, let $\mathcal{O}$ be open. If $x_0 \in \mathcal{O}$, then $K = \{x_0\}$ is a compact subset of $\mathcal{O}$ with $\delta_{x_0}(K) = 1 = \delta_{x_0}(\mathcal{O})$; if $x_0 \notin \mathcal{O}$, then $\delta_{x_0}(\mathcal{O}) = 0$ is attained by $K = \emptyset$. Hence δ_{x_0} is Radon, and therefore also a signed Radon measure, being the difference $\delta_{x_0} - 0$ of two Radon measures. δ_{x_0} represents L . Let $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ be a simple Borel function on X , where $\{E_i\}_{i=1}^n$ is a disjoint collection of Borel sets whose union is X . Exactly one E_{i_0} contains x_0 , and therefore

$$\int_X \varphi d\delta_{x_0} = \sum_{i=1}^n a_i \delta_{x_0}(E_i) = a_{i_0} = \varphi(x_0).$$

Now let $f \in C(X)$. Then f is a bounded Borel measurable function on the finite measure space $(X, \mathcal{B}(X), \delta_{x_0})$, so by the Simple Approximation Theorem there is a sequence $\{\varphi_n\}$ of simple Borel functions converging to f uniformly on X . Uniform convergence on a space of finite measure gives $\int_X \varphi_n d\delta_{x_0} \rightarrow \int_X f d\delta_{x_0}$, while $\varphi_n(x_0) \rightarrow f(x_0)$. Passing to the limit in the displayed identity,

$$\int_X f d\delta_{x_0} = f(x_0) = L(f) \quad \text{for all } f \in C(X).$$

Thus the signed Radon measure representing L is the Dirac delta measure δ_{x_0} . It is the only one: by the Riesz Representation Theorem for the Dual of $C(X)$, the map $T: \mathcal{R}adon(X) \rightarrow [C(X)]^*$, $T\nu(f) = \int_X f d\nu$, is a linear isometric isomorphism, hence injective. Consistently with the isometry, $\|\delta_{x_0}\|_{\text{var}} = \delta_{x_0}(X) = 1 = \|L\|$.

Problem 21.47. Let X be a compact Hausdorff space and μ a Borel measure on $\mathcal{B}(X)$. Show that there is a Radon measure μ_0 for which

$$\int_X f d\mu = \int_X f d\mu_0 \quad \text{for all } f \text{ in } C(X).$$

Solution. Since X is compact and μ is a Borel measure, $\mu(X) < \infty$; and every $f \in C(X)$ is Borel measurable and bounded, with $\|f\|_{\max}$ attained. Hence, as shown in Problem 45, each $f \in C(X)$ is integrable over X with respect to μ and

$$\left| \int_X f d\mu \right| \leq \mu(X) \|f\|_{\max}.$$

Therefore the formula

$$I(f) = \int_X f d\mu, \quad f \in C(X),$$

defines a real-valued functional on $C(X)$.

I is linear, by the linearity of integration for integrable functions. I is positive: if $f \geq 0$ on X , then by the monotonicity of integration $I(f) = \int_X f d\mu \geq 0$.

Next, X is a compact Hausdorff space, hence a locally compact Hausdorff space, and

$$C_c(X) = C(X),$$

because the support of any $f \in C(X)$ is a closed subset of the compact space X and is therefore compact. Thus I is a positive linear functional on $C_c(X)$.

The Riesz-Markov Theorem now applies: there is a (unique) Radon measure μ_0 on $\mathcal{B}(X)$, the Borel σ -algebra associated with the topology on X , for which

$$I(f) = \int_X f d\mu_0 \quad \text{for all } f \in C_c(X) = C(X).$$

That is,

$$\int_X f d\mu = \int_X f d\mu_0 \quad \text{for all } f \in C(X),$$

which is the assertion. The measure μ_0 is unique with this property, by Proposition 11.

Two remarks. First, taking $f \equiv 1$ gives $\mu_0(X) = \mu(X) < \infty$, so μ_0 is a finite Radon measure. Second, in general $\mu_0 \neq \mu$: the passage from μ to μ_0 replaces μ by the unique Radon measure inducing the same functional on $C(X)$, and a Borel measure on a compact Hausdorff space need not be outer and inner regular. When X is a compact metric space, however, Theorem 14 asserts that every finite Borel measure on $\mathcal{B}(X)$ is already Radon, and then $\mu_0 = \mu$.

Problem 21.48. Let g_1 and g_2 be two increasing functions on the closed, bounded interval $[a, b]$ that agree at the end-points. Show that

$$\int_a^b f(x) dg_1(x) = \int_a^b f(x) dg_2(x) \quad \text{for all } f \in C[a, b]$$

if and only if $g_1(x^+) = g_2(x^+)$ for all $a \leq x < b$.

Solution. Throughout, the integrals are Riemann-Stieltjes integrals, and we use the conventions of Section 21.5: for an increasing real-valued function g on $[a, b]$ and $a < c < b$,

$$g(c^+) = \inf_{c < x \leq b} g(x), \quad g(c^-) = \sup_{a \leq x < c} g(x),$$

together with $g(a^-) = g(a)$ and $g(b^+) = g(b)$. The hypothesis is $g_1(a) = g_2(a)$ and $g_1(b) = g_2(b)$.

Step 0: the Borel measure attached to an increasing function.

Let g be increasing on $[a, b]$. An increasing real-valued function on $[a, b]$ is of bounded variation, and the Riemann-Stieltjes integral $\int_a^b f(x) dg(x)$ exists for every continuous f and every g of bounded variation; so the functional $f \mapsto \int_a^b f(x) dg(x)$ is defined on all of $C[a, b]$. It is linear, and it is positive: if $f \geq 0$ then every Riemann-Stieltjes sum $\sum_i f(\xi_i)[g(x_i) - g(x_{i-1})]$ is nonnegative, since g is increasing, and hence so is the limit. The interval $[a, b]$ is a compact Hausdorff space and $C_c[a, b] = C[a, b]$, so by the Riesz-Markov Theorem there is a Radon, in particular Borel, measure μ on $\mathcal{B}[a, b]$ with

$$\int_a^b f(x) dg(x) = \int_{[a, b]} f d\mu \quad \text{for all } f \in C[a, b], \tag{16}$$

and by Proposition 11 this μ is the unique Borel measure with property (16) among Radon measures; as recalled in the text preceding Proposition 16, it is in fact the unique Borel measure with property (16), because a Borel measure on $\mathcal{B}[a, b]$ is determined by its values on the semiring

$$\mathcal{S} = \{\{a\}\} \cup \{(c, d] : (c, d] \subseteq (a, b)\},$$

whose members cover $[a, b]$ and which generates $\mathcal{B}[a, b]$, by the uniqueness assertion of the Caratheodory-Hahn Theorem (the premeasure involved is finite, hence σ -finite).

Proposition 16 identifies the values of μ on $\mathcal{S}$:

$$\mu\{a\} = g(a^+) - g(a), \quad \mu(c, d] = g(d^+) - g(c^+) \quad \text{for all } (c, d] \subseteq (a, b]. \tag{17}$$

Let μ_1 and μ_2 be the Borel measures associated in this way with g_1 and g_2 . By (16), the asserted equality of Riemann-Stieltjes integrals for all $f \in C[a, b]$ is equivalent to

$$\int_{[a, b]} f d\mu_1 = \int_{[a, b]} f d\mu_2 \quad \text{for all } f \in C[a, b],$$

and, since μ_1 and μ_2 are Radon measures on the compact Hausdorff space $[a, b]$, Proposition 11 says this in turn is equivalent to

$$\mu_1 = \mu_2.$$

So it suffices to prove: $\mu_1 = \mu_2$ if and only if $g_1(x^+) = g_2(x^+)$ for all $a \leq x < b$.

Step 1: if $g_1(x^+) = g_2(x^+)$ for all $a \leq x < b$, then $\mu_1 = \mu_2$.

We check that μ_1 and μ_2 agree on the semiring $\mathcal{S}$. First, by (17) and the hypotheses $g_1(a^+) = g_2(a^+)$ (the case $x = a$) and $g_1(a) = g_2(a)$,

$$\mu_1\{a\} = g_1(a^+) - g_1(a) = g_2(a^+) - g_2(a) = \mu_2\{a\}.$$

Next let $(c, d] \subseteq (a, b]$, so that $a \leq c < d \leq b$. Since $a \leq c < b$, the hypothesis gives $g_1(c^+) = g_2(c^+)$. As for d : if $d < b$, then $a \leq d < b$ and the hypothesis gives $g_1(d^+) = g_2(d^+)$; if $d = b$, then $g_i(d^+) = g_i(b^+) = g_i(b)$ and $g_1(b) = g_2(b)$ by hypothesis, so again $g_1(d^+) = g_2(d^+)$. Hence by (17),

$$\mu_1(c, d] = g_1(d^+) - g_1(c^+) = g_2(d^+) - g_2(c^+) = \mu_2(c, d].$$

Thus μ_1 and μ_2 are finite Borel measures on $\mathcal{B}[a, b]$ that agree on the semiring $\mathcal{S}$, which generates $\mathcal{B}[a, b]$ and covers $[a, b]$. By the uniqueness assertion of the Caratheodory-Hahn Theorem, $\mu_1 = \mu_2$. Therefore the two Riemann-Stieltjes integrals agree for every $f \in C[a, b]$.

Step 2: if $\mu_1 = \mu_2$, then $g_1(x^+) = g_2(x^+)$ for all $a \leq x < b$.

Let $a \leq x < b$. Then $(x, b] \subseteq (a, b]$, so (17) applies with $c = x$ and $d = b$, and, recalling $g_i(b^+) = g_i(b)$,

$$\mu_i(x, b] = g_i(b^+) - g_i(x^+) = g_i(b) - g_i(x^+), \quad i = 1, 2.$$

Equivalently,

$$g_i(x^+) = g_i(b) - \mu_i(x, b].$$

Since $g_1(b) = g_2(b)$ by hypothesis and $\mu_1(x, b] = \mu_2(x, b]$ because $\mu_1 = \mu_2$, we conclude

$$g_1(x^+) = g_1(b) - \mu_1(x, b] = g_2(b) - \mu_2(x, b] = g_2(x^+).$$

As x was an arbitrary point of $[a, b)$, this is the desired conclusion.

Combining Steps 1 and 2 with the reduction of Step 0 proves the equivalence.

Remark. The hypothesis that g_1 and g_2 agree at both endpoints is used in both directions and cannot be dropped: adding a constant to g_2 changes neither side of the integral identity nor any of the one-sided limits' differences, but it does destroy the equality $g_1(x^+) = g_2(x^+)$. The condition also shows that an increasing g is determined, as far as Lebesgue-Stieltjes integration of continuous functions is concerned, by its right-continuous modification $x \mapsto g(x^+)$ on $[a, b)$ together with its endpoint values; this is exactly the reason for the normalization built into the collection $\mathcal{F}$ appearing in Theorem 17 of Riesz.

Problem 21.49. Let X be a compact Hausdorff space. Show that the Jordan Decomposition Theorem for signed Borel measures on $\mathcal{B}(X)$ follows from the Riesz Representation Theorem for the dual of $C(X)$ and Proposition 12.

Solution. We work with the class of signed measures on $\mathcal{B}(X)$ occurring in the Riesz Representation Theorem for the dual of $C(X)$, namely the signed Radon measures: by the definition given in Section 21.5, a signed Radon measure on the compact space X is a signed measure ν on $\mathcal{B}(X)$ that is the difference $\nu = \sigma_1 - \sigma_2$ of two Radon measures on $\mathcal{B}(X)$, and $\mathcal{R}adon(X)$ denotes the linear space of these. Since X is compact and a Radon measure is a Borel measure, $\sigma_1(X) < \infty$ and $\sigma_2(X) < \infty$; thus every signed Radon measure on X is a finite signed Borel measure. Note also that X , being compact Hausdorff, is locally compact Hausdorff and $C_c(X) = C(X)$, since the support of a continuous function on X is a closed, hence compact, subset of X .

What is to be proved is the Jordan Decomposition Theorem: for such a ν there exist two mutually singular measures ν_+ and ν_- on $\mathcal{B}(X)$ with $\nu = \nu_+ - \nu_-$, and there is only one such pair. We deduce

this from Proposition 12 and the representation theorems of this chapter, without appeal to the Hahn Decomposition Theorem.

Step 1: ν induces a bounded linear functional on $C(X)$.

For $f \in C(X)$ put

$$L(f) = \int_X f d\sigma_1 - \int_X f d\sigma_2,$$

which is meaningful because, by Problem 45, every $f \in C(X)$ is integrable with respect to each of the finite Borel measures σ_i . This does not depend on the representation of ν : if $\sigma_1 - \sigma_2 = \tau_1 - \tau_2$ with all four measures Radon, then $\sigma_1 + \tau_2 = \tau_1 + \sigma_2$ as set functions on $\mathcal{B}(X)$, and integration against a sum of measures is the sum of the integrations, so $\int_X f d\sigma_1 + \int_X f d\tau_2 = \int_X f d\tau_1 + \int_X f d\sigma_2$, which rearranges to the required equality. By the linearity of integration L is linear, and by Problem 45

$$|L(f)| \leq [\sigma_1(X) + \sigma_2(X)] \|f\|_{\max} \quad \text{for all } f \in C(X),$$

so L is bounded. In the notation of the Riesz Representation Theorem for the dual of $C(X)$, $L = T\nu$.

Step 2: decomposition of the functional.

By Proposition 12 there are two positive linear functionals L_+ and L_- on $C(X)$ for which

$$L = L_+ - L_- \quad \text{and} \quad \|L\| = L_+(1) + L_-(1),$$

where, as in the proof of that proposition, $L_+(f) = \sup\{L(\psi) : 0 \leq \psi \leq f\}$ for $f \geq 0$.

Step 3: representation of the two pieces by Radon measures.

Since $C_c(X) = C(X)$, the functionals L_+ and L_- are positive linear functionals on $C_c(X)$. By the Riesz-Markov Theorem there are unique Radon measures ν_+ and ν_- on $\mathcal{B}(X)$ with

$$L_+(f) = \int_X f d\nu_+ \quad \text{and} \quad L_-(f) = \int_X f d\nu_- \quad \text{for all } f \in C(X).$$

Both are finite: $\nu_{\pm}(X) = L_{\pm}(1) < \infty$.

Step 4: $\nu = \nu_+ - \nu_-$.

For every $f \in C(X)$,

$$\int_X f d\sigma_1 - \int_X f d\sigma_2 = L(f) = \int_X f d\nu_+ - \int_X f d\nu_-,$$

that is,

$$\int_X f d(\sigma_1 + \nu_-) = \int_X f d(\nu_+ + \sigma_2).$$

The sum of two Radon measures is a Radon measure (Problem 33), so $\sigma_1 + \nu_-$ and $\nu_+ + \sigma_2$ are Radon measures on $\mathcal{B}(X)$ that integrate every $f \in C_c(X)$ identically. By Proposition 11 they are equal, and since all four measures are finite we may subtract to obtain

$$\nu = \sigma_1 - \sigma_2 = \nu_+ - \nu_- \quad \text{on } \mathcal{B}(X).$$

Equivalently: $T(\nu_+ - \nu_-) = L_+ - L_- = L = T\nu$, and T is injective because, by the Riesz Representation Theorem for the dual of $C(X)$, it is an isometric isomorphism of $\mathcal{R}adon(X)$ onto $[C(X)]^*$.

Step 5: ν_+ and ν_- are mutually singular.

Fix a natural number n . Since $L_+(1) = \sup\{L(\psi) : \psi \in C(X), 0 \leq \psi \leq 1\}$ is finite, there is $\varphi_n \in C(X)$ with $0 \leq \varphi_n \leq 1$ on X and

$$L(\varphi_n) > L_+(1) - 4^{-n} = \nu_+(X) - 4^{-n}.$$

Because $L(\varphi_n) = \int_X \varphi_n d\nu_+ - \int_X \varphi_n d\nu_-$ and $\int_X \varphi_n d\nu_- \geq 0$,

$$\int_X (1 - \varphi_n) d\nu_+ = \nu_+(X) - \int_X \varphi_n d\nu_+ \leq \nu_+(X) - L(\varphi_n) < 4^{-n},$$

and, since $\int_X \varphi_n d\nu_+ \leq \nu_+(X)$,

$$\int_X \varphi_n d\nu_- = \int_X \varphi_n d\nu_+ - L(\varphi_n) \leq \nu_+(X) - L(\varphi_n) < 4^{-n}.$$

Let $A_n = \{x \in X : \varphi_n(x) > 1/2\}$, an open, hence Borel, subset of X . By Chebychev's Inequality applied to the nonnegative functions φ_n and $1 - \varphi_n$,

$$\nu_-(A_n) \leq 2 \int_X \varphi_n d\nu_- < 2 \cdot 4^{-n}, \quad \nu_+(X \sim A_n) \leq 2 \int_X (1 - \varphi_n) d\nu_+ < 2 \cdot 4^{-n},$$

the second because $1 - \varphi_n \geq 1/2$ on $X \sim A_n$.

Define the Borel set

$$A = \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} A_n.$$

For each N , countable monotonicity gives $\nu_-(A) \leq \sum_{n \geq N} \nu_-(A_n) < \sum_{n \geq N} 2 \cdot 4^{-n} = \frac{8}{3} \cdot 4^{-N}$, and letting $N \rightarrow \infty$ yields $\nu_-(A) = 0$. On the other hand $X \sim A = \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} (X \sim A_n)$ is an ascending union of Borel sets, so by the continuity of measure and the monotonicity of ν_+ ,

$$\nu_+(X \sim A) = \lim_{N \rightarrow \infty} \nu_+ \left(\bigcap_{n \geq N} (X \sim A_n) \right) \leq \lim_{N \rightarrow \infty} \nu_+(X \sim A_N) = 0.$$

Thus ν_+ is concentrated on A and ν_- on $X \sim A$; that is, $\nu_+ \perp \nu_-$.

Step 6: uniqueness of the decomposition.

Observe first that, because $\nu_-(A) = 0$ and $\nu_+(X \sim A) = 0$, for every $E \in \mathcal{B}(X)$

$$\nu(E \cap A) = \nu_+(E \cap A) - \nu_-(E \cap A) = \nu_+(E \cap A) = \nu_+(E). \quad (\dagger)$$

Now suppose $\nu = \lambda_+ - \lambda_-$, where λ_+ and λ_- are mutually singular measures on $\mathcal{B}(X)$, say $\lambda_+(X \sim B) = 0 = \lambda_-(B)$ for a Borel set B . Since the difference $\lambda_+ - \lambda_-$ is presumed defined, at most one of $\lambda_+(X)$, $\lambda_-(X)$ is infinite; and if, say, $\lambda_+(X) = \infty$, then $\lambda_-(X) < \infty$ and $\nu(X) = \infty$, contradicting the finiteness of ν established above. So both λ_+ and λ_- are finite measures. Exactly as in $(\dagger)$, $\lambda_+(E) = \nu(E \cap B)$ and $\lambda_-(E) = -\nu(E \sim B)$ for every Borel E .

Let $E \in \mathcal{B}(X)$. Computing $\nu(E \cap A \sim B)$ in the two ways, and using $\nu_-(A) = 0$ for the first and $\lambda_+(X \sim B) = 0$ for the second,

$$\nu(E \cap A \sim B) = \nu_+(E \cap A \sim B) \geq 0, \quad \nu(E \cap A \sim B) = -\lambda_-(E \cap A \sim B) \leq 0,$$

so $\nu(E \cap A \sim B) = 0$. Interchanging the roles of the two decompositions gives $\nu(E \cap B \sim A) = 0$ as well. Hence, by the additivity of ν ,

$$\nu_+(E) = \nu(E \cap A) = \nu(E \cap A \cap B) = \nu(E \cap B) = \lambda_+(E).$$

Therefore $\lambda_+ = \nu_+$, and consequently $\lambda_- = \lambda_+ - \nu = \nu_+ - \nu = \nu_-$. The decomposition is unique.

This is precisely the Jordan Decomposition Theorem for ν , obtained from Proposition 12 together with the Riesz-Markov Theorem and the Riesz Representation Theorem for the dual of $C(X)$. Two by-products deserve mention. First, the measures ν_+ and ν_- produced here are Radon measures, so the Jordan decomposition of a signed Radon measure on a compact Hausdorff space consists of Radon measures. Second, the norm identity of Proposition 12 becomes

$$\|T\nu\| = \|L\| = L_+(1) + L_-(1) = \nu_+(X) + \nu_-(X) = \|\nu\|_{\text{var}},$$

which is exactly the isometry assertion of the Riesz Representation Theorem for the dual of $C(X)$.

Remark on the hypotheses. Regularity of ν is genuinely needed in Step 4, not merely for convenience. For a signed Borel measure on $\mathcal{B}(X)$ that is not assumed to be a difference of Radon measures, Steps 1 through 3 (once one knows that $f \mapsto \int_X f d\nu$ is a bounded linear functional on $C(X)$) still produce Radon measures ν_+ and ν_- with $\int_X f d\nu = \int_X f d\nu_+ - \int_X f d\nu_-$ for all $f \in C(X)$, but the identity $\nu = \nu_+ - \nu_-$ on all of $\mathcal{B}(X)$ can fail: already for positive measures, Problem 47 shows that a Borel measure and the Radon measure inducing the same functional on $C(X)$ need not coincide. The class $\mathcal{R}adon(X)$ used above is the class for which the Riesz Representation Theorem for the dual of $C(X)$ asserts a bijective correspondence, and it is therefore the class for which this deduction is valid.

21.8 Problems 21.50–21.56

Problem 21.50. What are the extreme points of the unit ball of the linear space of signed Radon measures $\mathcal{R}adon(X)$, where X is a compact Hausdorff space?

Solution. Throughout, X is a nonempty compact Hausdorff space, $\mathcal{R}adon(X)$ is normed by the total variation $\|\nu\|_{\text{var}} = |\nu|(X) = \nu^+(X) + \nu^-(X)$, and

$$B = \{\nu \in \mathcal{R}adon(X) : \|\nu\|_{\text{var}} \leq 1\}$$

is the closed unit ball. Since X is compact and a Borel measure assigns finite measure to compact sets, every Radon measure on $\mathcal{B}(X)$ is finite. Since X is Hausdorff, every compact subset of X is closed and hence Borel, and every singleton is Borel.

The answer is:

the extreme points of B are exactly the measures δ_x and $-\delta_x$, $x \in X$,

where δ_x is the Dirac delta measure concentrated at x , that is, $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ otherwise. Each δ_x is a Radon measure, so each $\pm\delta_x$ belongs to $\mathcal{R}adon(X)$ and has total variation 1. We first record two lemmas.

Lemma A. (Recorded for reference; it is the finite case of Problem 55.) Let λ be a finite Radon measure on $\mathcal{B}(X)$. Then every Borel set is inner regular with respect to λ ; that is, λ is regular.

Indeed, let E be Borel and $\epsilon > 0$. By the outer regularity of E there is an open set $\mathcal{O} \supseteq E$ with $\lambda(\mathcal{O}) < \lambda(E) + \epsilon$, so, by excision (all measures here are finite), $\lambda(\mathcal{O} \sim E) < \epsilon$. By the inner regularity of the open set $\mathcal{O}$ there is a compact set $K \subseteq \mathcal{O}$ with $\lambda(K) > \lambda(\mathcal{O}) - \epsilon$. By the outer regularity of the Borel set $\mathcal{O} \sim E$ there is an open set $U \supseteq \mathcal{O} \sim E$ with $\lambda(U) < \lambda(\mathcal{O} \sim E) + \epsilon < 2\epsilon$. The set $K \sim U = K \cap (X \sim U)$ is a closed subset of the compact set K and therefore is compact, and, since $E \subseteq \mathcal{O}$ and $U \supseteq \mathcal{O} \sim E$,

$$K \sim U \subseteq \mathcal{O} \sim (\mathcal{O} \sim E) = E.$$

Moreover, from $K \subseteq (K \sim U) \cup U$ and finite subadditivity, $\lambda(K \sim U) \geq \lambda(K) - \lambda(U) > (\lambda(\mathcal{O}) - \epsilon) - 2\epsilon \geq \lambda(E) - 3\epsilon$. As $\epsilon > 0$ was arbitrary, E is inner regular.

Lemma B. Let λ be a finite Radon measure on $\mathcal{B}(X)$ and let $\eta: \mathcal{B}(X) \rightarrow [0, \infty)$ be a measure with $\eta(E) \leq \lambda(E)$ for every Borel set E . Then η is a Radon measure.

Indeed, η is finite, so it is a Borel measure. Given a Borel set E and $\epsilon > 0$, outer regularity of λ provides an open $\mathcal{O} \supseteq E$ with $\lambda(\mathcal{O} \sim E) < \epsilon$; then $\eta(\mathcal{O}) - \eta(E) = \eta(\mathcal{O} \sim E) \leq \lambda(\mathcal{O} \sim E) < \epsilon$, so E is outer regular with respect to η . Given an open set $\mathcal{O}$ and $\epsilon > 0$, inner regularity of λ on open sets provides a compact $K \subseteq \mathcal{O}$ with $\lambda(\mathcal{O} \sim K) < \epsilon$; then $\eta(\mathcal{O}) - \eta(K) \leq \lambda(\mathcal{O} \sim K) < \epsilon$. Thus η is Radon.

Consequence. If $\nu \in \mathcal{R}adon(X)$, then $|\nu|$, ν^+ , ν^- are Radon measures, and for every Borel set E the restriction $\nu \llcorner E$, defined by $(\nu \llcorner E)(A) = \nu(A \cap E)$, again belongs to $\mathcal{R}adon(X)$, with

$$\|\nu \llcorner E\|_{\text{var}} = |\nu|(E).$$

To see this, write $\nu = \mu_1 - \mu_2$ with μ_1, μ_2 Radon measures, and put $\lambda = \mu_1 + \mu_2$. Then λ is a finite Radon measure. Indeed, given a Borel set E and $\epsilon > 0$, choose open $\mathcal{U}_i \supseteq E$ with $\mu_i(\mathcal{U}_i \sim E) < \epsilon/2$; the set $\mathcal{U}_1 \cap \mathcal{U}_2$ is open, contains E , and $\lambda((\mathcal{U}_1 \cap \mathcal{U}_2) \sim E) < \epsilon$. Given an open set $\mathcal{O}$ and $\epsilon > 0$, choose compact $K_i \subseteq \mathcal{O}$ with $\mu_i(\mathcal{O} \sim K_i) < \epsilon/2$; the set $K_1 \cup K_2$ is a compact subset of $\mathcal{O}$ with $\lambda(\mathcal{O} \sim (K_1 \cup K_2)) < \epsilon$. If $P \cup N$ is a Hahn decomposition for ν , then for every Borel A we have $\nu^+(A) = \nu(A \cap P) \leq \mu_1(A)$ and $\nu^-(A) = -\nu(A \cap N) \leq \mu_2(A)$, so $|\nu| = \nu^+ + \nu^- \leq \lambda$. By Lemma B, $|\nu|$, ν^+ , ν^- , and all of $\nu^\pm \llcorner E \leq \lambda$ are Radon measures. Since $\nu \llcorner E = \nu^+ \llcorner E - \nu^- \llcorner E$ and these two measures are mutually singular (they are carried by P and N respectively), this is the Jordan decomposition of $\nu \llcorner E$ and therefore

$$\|\nu \llcorner E\|_{\text{var}} = \nu^+(E) + \nu^-(E) = |\nu|(E).$$

Step 1: each $\pm\delta_x$ is an extreme point of B .

Fix $x \in X$ and suppose $\delta_x = t\nu_1 + (1-t)\nu_2$ with $\nu_1, \nu_2 \in B$ and $0 < t < 1$. Since total variation is a norm,

$$1 = \|\delta_x\|_{\text{var}} \leq t\|\nu_1\|_{\text{var}} + (1-t)\|\nu_2\|_{\text{var}} \leq 1,$$

so $\|\nu_1\|_{\text{var}} = \|\nu_2\|_{\text{var}} = 1$. Now for every Borel set E ,

$$|\delta_x|(E) = |t\nu_1(E) + (1-t)\nu_2(E)| \leq t|\nu_1|(E) + (1-t)|\nu_2|(E),$$

and more generally, for any finite disjoint Borel partition $\{E_i\}$ of E , summing $|\delta_x(E_i)| \leq t|\nu_1|(E_i) + (1-t)|\nu_2|(E_i)$ and taking the supremum over partitions gives

$$\delta_x = |\delta_x| \leq t|\nu_1| + (1-t)|\nu_2| \quad \text{setwise.}$$

Both sides are measures with the same total mass 1, so their difference is a nonnegative measure of total mass 0; hence they are equal. Evaluating at $E = X \sim \{x\}$ (a Borel set, since X is Hausdorff) gives

$$0 = t|\nu_1|(X \sim \{x\}) + (1-t)|\nu_2|(X \sim \{x\}),$$

so $|\nu_i|(X \sim \{x\}) = 0$ for $i = 1, 2$. Thus ν_i is concentrated at x , i.e. $\nu_i = c_i\delta_x$ where $c_i = \nu_i(\{x\})$ and $|c_i| = \|\nu_i\|_{\text{var}} = 1$. Finally $tc_1 + (1-t)c_2 = \delta_x(\{x\}) = 1$ with $|c_1|, |c_2| \leq 1$ forces $c_1 = c_2 = 1$. Hence $\nu_1 = \nu_2 = \delta_x$, and δ_x is an extreme point. Since $\nu \mapsto -\nu$ is a linear isometry of B onto itself, $-\delta_x$ is also an extreme point.

Step 2: there are no other extreme points.

Let ν be an extreme point of B .

First, $\|\nu\|_{\text{var}} = 1$. If $\nu = 0$, pick any $x \in X$; then $0 = \frac{1}{2}\delta_x + \frac{1}{2}(-\delta_x)$ with $\pm\delta_x \in B$ distinct, so 0 is not extreme. If $0 < \|\nu\|_{\text{var}} = r < 1$, choose $s > 0$ with $(1+s)r \leq 1$; then $\nu = \frac{1}{2}(1+s)\nu + \frac{1}{2}(1-s)\nu$ exhibits ν as a proper convex combination of the two distinct elements $(1 \pm s)\nu$ of B , so ν is not extreme.

Second, $|\nu|$ takes only the values 0 and 1. Suppose instead that some Borel set E satisfies $0 < a := |\nu|(E) < 1$. Put $\nu_1 = \nu \llcorner E$ and $\nu_2 = \nu \llcorner (X \sim E)$. By the Consequence above, $\nu_1, \nu_2 \in \mathcal{R}adon(X)$ with $\|\nu_1\|_{\text{var}} = a$ and $\|\nu_2\|_{\text{var}} = |\nu|(X \sim E) = 1 - a$. Therefore ν_1/a and $\nu_2/(1-a)$ lie in B , and

$$\nu = \nu_1 + \nu_2 = a\left(\frac{\nu_1}{a}\right) + (1-a)\left(\frac{\nu_2}{1-a}\right)$$

is a proper convex combination. The two measures are distinct: $|\nu_1/a|(E) = 1$ while $|\nu_2/(1-a)|(E) = 0$. This contradicts extremeness. Hence $|\nu|(E) \in \{0, 1\}$ for every Borel set E .

Third, a two-valued Radon probability measure on a compact Hausdorff space is a Dirac measure. Let $\mathcal{F}$ be the family of closed subsets F of X with $|\nu|(F) = 1$. Each member of $\mathcal{F}$ is a closed subset of the compact space X and hence compact and nonempty, and $\mathcal{F}$ is closed under finite intersections: if $|\nu|(F_1) = |\nu|(F_2) = 1$ then $|\nu|(X \sim F_1) = |\nu|(X \sim F_2) = 0$, so $|\nu|(X \sim (F_1 \cap F_2)) = 0$ and $|\nu|(F_1 \cap F_2) = 1$. Also $X \in \mathcal{F}$. By the finite intersection property of the compact space X , the set

$$S = \bigcap_{F \in \mathcal{F}} F$$

is nonempty. Choose $x \in S$. We claim $|\nu|(\{x\}) = 1$. If not, then $|\nu|(\{x\}) = 0$, and by the outer regularity of the Borel set $\{x\}$ there is an open set $\mathcal{O} \ni x$ with $|\nu|(\mathcal{O}) < 1$, hence $|\nu|(\mathcal{O}) = 0$. Then $F = X \sim \mathcal{O}$ is closed with $|\nu|(F) = 1$, so $F \in \mathcal{F}$, yet $x \notin F$ — contradicting $x \in S$. Therefore $|\nu|(\{x\}) = 1 = |\nu|(X)$, so $|\nu| = \delta_x$.

Finally, $\nu^+ + \nu^- = |\nu| = \delta_x$ and $\nu^+ \perp \nu^-$. Thus one of $\nu^+(\{x\}), \nu^-(\{x\})$ equals 1 and the other 0, whence $\nu = \delta_x$ or $\nu = -\delta_x$.

Combining Steps 1 and 2: the set of extreme points of the closed unit ball of $\mathcal{R}adon(X)$ is

$$\{\delta_x : x \in X\} \cup \{-\delta_x : x \in X\}.$$

Under the isometric isomorphism $T: \mathcal{R}adon(X) \rightarrow [C(X)]^*$ of the Riesz Representation Theorem for the dual of $C(X)$, these correspond to the functionals $f \mapsto \pm f(x)$; by Corollary 13 the unit ball of $\mathcal{R}adon(X)$ is the weak-* closed convex hull of this set.

Problem 21.51. Verify (15) for E a Borel subset of a compact metric space X and μ a Radon measure on $\mathcal{B}(X)$. [Here (15) is the approximation property invoked in the proof of Theorem 14: for each $\epsilon > 0$ there is an open set $\mathcal{O}_\epsilon$ and a closed set F_ϵ for which $F_\epsilon \subseteq E \subseteq \mathcal{O}_\epsilon$ and $\mu(\mathcal{O}_\epsilon \sim F_\epsilon) < \epsilon$.]

Solution. First observe that μ is a finite measure: X is compact and, by the definition of Borel measure, every compact subset of X has finite measure, so $\mu(X) < \infty$. In particular the excision property is available: if $A \subseteq B$ are Borel, then $\mu(B \sim A) = \mu(B) - \mu(A)$.

Let E be a Borel subset of X and let $\epsilon > 0$.

Approximation from outside. Since μ is a Radon measure, every Borel subset of X is outer regular. Applied to E , this provides a neighborhood of E , that is, an open set $\mathcal{O}_\epsilon$ with $E \subseteq \mathcal{O}_\epsilon$, for which

$$\mu(\mathcal{O}_\epsilon) < \mu(E) + \epsilon/2.$$

By excision, $\mu(\mathcal{O}_\epsilon \sim E) < \epsilon/2$.

Approximation from inside. Apply outer regularity instead to the Borel set $X \sim E$: there is an open set U with $X \sim E \subseteq U$ and

$$\mu(U) < \mu(X \sim E) + \epsilon/2,$$

and hence, by excision, $\mu(U \sim (X \sim E)) < \epsilon/2$. Define

$$F_\epsilon = X \sim U.$$

Then F_ϵ is closed and, since $X \sim E \subseteq U$, we have $F_\epsilon = X \sim U \subseteq X \sim (X \sim E) = E$. Moreover

$$E \sim F_\epsilon = E \cap U = U \sim (X \sim E),$$

so that $\mu(E \sim F_\epsilon) < \epsilon/2$.

Conclusion. We have produced a closed set F_ϵ and an open set $\mathcal{O}_\epsilon$ with $F_\epsilon \subseteq E \subseteq \mathcal{O}_\epsilon$, and, since $\mathcal{O}_\epsilon \sim F_\epsilon$ is the disjoint union of $\mathcal{O}_\epsilon \sim E$ and $E \sim F_\epsilon$, the additivity of μ gives

$$\mu(\mathcal{O}_\epsilon \sim F_\epsilon) = \mu(\mathcal{O}_\epsilon \sim E) + \mu(E \sim F_\epsilon) < \epsilon/2 + \epsilon/2 = \epsilon.$$

This is exactly (15).

Two remarks. First, the argument used only that μ is a finite Radon measure on a topological space; the metric structure of X was not needed. Second, since X is compact, the closed set F_ϵ is in fact compact, so (15) simultaneously exhibits the inner regularity of every Borel set with respect to μ :

$$\mu(E) - \epsilon < \mu(F_\epsilon) \leq \sup\{\mu(K) : K \subseteq E, K \text{ compact}\} \leq \mu(E).$$

Problem 21.52. Let X be a compact metric space. On the linear space of functions $\mathcal{F}$ defined in the statement of Theorem 17, define the norm of a function to be its total variation. Show that with this norm $\mathcal{F}$ is a Banach space.

Solution. The collection $\mathcal{F}$ named in Theorem 17 is a collection of functions on a closed, bounded interval $[a, b]$, so the compact metric space X of the statement plays no role beyond providing the setting $X = [a, b]$ in which Theorem 17 is stated; we prove the assertion for $\mathcal{F}$ as Theorem 17 defines it.

Recall from the statement of Theorem 17 that, for a closed bounded interval $[a, b]$,

$$\mathcal{F} = \{g: [a, b] \rightarrow \mathbb{R} : g \text{ is of bounded variation on } [a, b], g \text{ is continuous on the right on } (a, b), g(a) = 0\}.$$

For a partition $P: a = x_0 < x_1 < \cdots < x_k = b$ of $[a, b]$ write

$$V(g, P) = \sum_{i=1}^k |g(x_i) - g(x_{i-1})|, \quad \|g\| = TV(g) = \sup_P V(g, P).$$

$\mathcal{F}$ is a linear space and $\|\cdot\|$ is a norm on it.

For any functions g, h on $[a, b]$, any partition P , and any $c \in \mathbb{R}$, the triangle inequality in $\mathbb{R}$ gives $V(g + h, P) \leq V(g, P) + V(h, P)$ and $V(cg, P) = |c| V(g, P)$. Taking suprema over P yields

$$TV(g + h) \leq TV(g) + TV(h), \quad TV(cg) = |c| TV(g).$$

In particular a linear combination of functions of bounded variation is of bounded variation. A linear combination of functions continuous on the right at a point is continuous on the right at that point, and a linear combination of functions vanishing at a vanishes at a . Hence $\mathcal{F}$ is a linear space and $\|\cdot\|$ is a nonnegative, homogeneous, subadditive functional on it. Finally, if $g \in \mathcal{F}$ and $TV(g) = 0$, then for each $x \in (a, b)$ the partition $a < x < b$ gives $|g(x) - g(a)| \leq TV(g) = 0$, and likewise $|g(b) - g(a)| \leq TV(g) = 0$; since $g(a) = 0$ this forces $g \equiv 0$. So $\|\cdot\|$ is a norm.

The same computation records the useful inequality

$$(*) \quad \|g\|_\infty := \sup_{x \in [a, b]} |g(x)| = \sup_{x \in [a, b]} |g(x) - g(a)| \leq TV(g) = \|g\| \quad \text{for all } g \in \mathcal{F}.$$

(A function in $\mathcal{F}$ need not be continuous, so the supremum need not be attained; for the inequality, note that for $x \in (a, b)$ the partition $a < x < b$ gives $|g(x) - g(a)| \leq V(g, P) \leq TV(g)$, while the cases $x = a$ and $x = b$ are immediate.)

Completeness.

Let $\{g_n\}$ be a Cauchy sequence in $(\mathcal{F}, \|\cdot\|)$. A Cauchy sequence is bounded, so there is $M \geq 0$ with $TV(g_n) \leq M$ for all n .

By $(*)$ applied to $g_n - g_m \in \mathcal{F}$,

$$\|g_n - g_m\|_\infty \leq \|g_n - g_m\|,$$

so $\{g_n\}$ is uniformly Cauchy on $[a, b]$. Since $\mathbb{R}$ is complete, $\{g_n\}$ converges uniformly on $[a, b]$ to a function $g: [a, b] \rightarrow \mathbb{R}$.

We check that $g \in \mathcal{F}$.

- $g(a) = \lim_n g_n(a) = 0$.
- g is of bounded variation. Let $P: a = x_0 < \cdots < x_k = b$ be any partition. Since $g_n \rightarrow g$ pointwise and $V(\cdot, P)$ is a finite sum of continuous functions of the finitely many values $g(x_i)$,

$$V(g, P) = \lim_{n \rightarrow \infty} V(g_n, P) \leq \limsup_{n \rightarrow \infty} TV(g_n) \leq M.$$

Taking the supremum over all partitions P gives $TV(g) \leq M < \infty$.

- g is continuous on the right on (a, b) . Fix $x \in (a, b)$ and $\epsilon > 0$. Choose n with $\|g_n - g\|_\infty < \epsilon/3$. Since g_n is continuous on the right at x , there is $\delta > 0$ such that $|g_n(y) - g_n(x)| < \epsilon/3$ whenever $x \leq y < x + \delta$ and $y \in [a, b]$. For such y ,

$$|g(y) - g(x)| \leq |g(y) - g_n(y)| + |g_n(y) - g_n(x)| + |g_n(x) - g(x)| < \epsilon.$$

Hence $g(x^+) = g(x)$.

Thus $g \in \mathcal{F}$.

Finally we show $\|g_n - g\| \rightarrow 0$. Let $\epsilon > 0$ and choose N so that $TV(g_n - g_m) < \epsilon$ for all $n, m \geq N$. Fix $n \geq N$ and a partition $P: a = x_0 < \cdots < x_k = b$. For every $m \geq N$,

$$\sum_{i=1}^k |(g_n - g_m)(x_i) - (g_n - g_m)(x_{i-1})| = V(g_n - g_m, P) \leq TV(g_n - g_m) < \epsilon.$$

Letting $m \rightarrow \infty$ and using $g_m(x_i) \rightarrow g(x_i)$ for the finitely many points $x_0, \dots, x_k$, we obtain

$$V(g_n - g, P) = \sum_{i=1}^k |(g_n - g)(x_i) - (g_n - g)(x_{i-1})| \leq \epsilon.$$

Since P was arbitrary, $\|g_n - g\| = TV(g_n - g) \leq \epsilon$ for all $n \geq N$. Therefore $g_n \rightarrow g$ in $\mathcal{F}$.

Every Cauchy sequence in $\mathcal{F}$ converges in $\mathcal{F}$, so $(\mathcal{F}, TV)$ is a Banach space.

Remark. Theorem 17 asserts that the map $g \mapsto \psi_g$, where $\psi_g(f) = \int_a^b f dg$ for $f \in C[a, b]$, is a linear bijection of $\mathcal{F}$ onto $[C[a, b]]^*$. One can show $\|\psi_g\| = TV(g)$, so the completeness just proved also follows from the completeness of the dual space $[C[a, b]]^*$; the direct argument above avoids that identification of norms.

Problem 21.53. (Alternate proof of the Stone-Weierstrass Theorem (de Branges)) Let $\mathcal{A}$ be an algebra of real-valued continuous functions on a compact space X that separates points and contains the constants. Let $\mathcal{A}^\perp$ be the set of signed Radon measures on X such that $|\mu|(X) \leq 1$ and $\int_X f d\mu = 0$ for all $f \in \mathcal{A}$.

- (i) Use the Hahn-Banach Theorem and the Riesz Representation Theorem to show that if $\mathcal{A}^\perp$ contains only the zero measure, then $\overline{\mathcal{A}} = C(X)$.
- (ii) Use the Krein-Milman Theorem and the weak-* compactness of the unit ball in $\mathcal{R}adon(X)$ to show that if the zero measure is the only extreme point of $\mathcal{A}^\perp$, then $\mathcal{A}^\perp$ contains only the zero measure.
- (iii) Let μ be an extreme point of $\mathcal{A}^\perp$. Let f belong to $\mathcal{A}$, with $0 \leq f \leq 1$. Define measures μ_1 and μ_2 by

$$\mu_1(E) = \int_E f d\mu \quad \text{and} \quad \mu_2(E) = \int_E (1-f) d\mu \quad \text{for } E \in \mathcal{B}(X).$$

Show that μ_1 and μ_2 belong to $\mathcal{A}^\perp$ and, moreover, $\|\mu_1\| + \|\mu_2\| = \|\mu\|$, and $\mu_1 + \mu_2 = \mu$. Since μ is an extreme point, conclude that $\mu_1 = c\mu$ for some constant c .

- (iv) Show that $f = c$ on the support of μ .
- (v) Since $\mathcal{A}$ separates points, show that the support of μ can contain at most one point. Since $\int_X 1 d\mu = 0$, conclude that the support of μ is empty and hence μ is the zero measure.

Solution. Preliminaries. Since $\mathcal{A}$ consists of continuous functions and separates the points of X , the space X is automatically Hausdorff: given $x \neq y$ choose $g \in \mathcal{A}$ with $g(x) \neq g(y)$ and take preimages under g of disjoint open intervals about $g(x)$ and $g(y)$. So X is a compact Hausdorff space and the results of Section 21.5 apply. Every Radon measure on $\mathcal{B}(X)$ is finite because X is compact, and $\mathcal{R}adon(X)$ is normed by $\|\nu\| = \|\nu\|_{\text{var}} = |\nu|(X)$.

We shall use repeatedly the following fact, established in the solution of Problem 50.

Lemma. If λ is a finite Radon measure on $\mathcal{B}(X)$ and $\eta: \mathcal{B}(X) \rightarrow [0, \infty)$ is a measure with $\eta \leq \lambda$ setwise, then η is a Radon measure. (Outer regularity: given Borel E and $\epsilon > 0$, outer regularity of λ gives open $\mathcal{O} \supseteq E$ with $\lambda(\mathcal{O} \setminus E) < \epsilon$, hence $\eta(\mathcal{O}) - \eta(E) \leq \epsilon$. Inner regularity of open $\mathcal{O}$: inner regularity of λ gives compact $K \subseteq \mathcal{O}$ with $\lambda(\mathcal{O} \setminus K) < \epsilon$, hence $\eta(\mathcal{O}) - \eta(K) \leq \epsilon$.) In particular, if $\nu \in \mathcal{R}adon(X)$, then $|\nu|$ is a finite Radon measure, and any nonnegative measure dominated by $|\nu|$ is Radon.

Part (i). Suppose $\mathcal{A}^\perp = \{0\}$ and suppose, to get a contradiction, that the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $C(X)$ (with the maximum norm) is a proper subset of $C(X)$. Since $\mathcal{A}$ is a linear subspace, so is $\overline{\mathcal{A}}$, and it is closed. Choose $g \in C(X) \setminus \overline{\mathcal{A}}$; then $d := \text{dist}(g, \overline{\mathcal{A}}) > 0$. By the corollary of the Hahn-Banach Theorem on separation of a point from a closed subspace, there is a bounded linear functional ψ on $C(X)$ for which

$$\psi = 0 \text{ on } \overline{\mathcal{A}}, \quad \psi(g) = d \neq 0, \quad \|\psi\| = 1.$$

By the Riesz Representation Theorem for the dual of $C(X)$, there is a signed Radon measure μ on X with

$$\psi(h) = \int_X h d\mu \quad \text{for all } h \in C(X), \quad |\mu|(X) = \|\mu\|_{\text{var}} = \|\psi\| = 1.$$

Then $|\mu|(X) \leq 1$ and $\int_X f d\mu = \psi(f) = 0$ for every $f \in \mathcal{A}$, so $\mu \in \mathcal{A}^\perp$. But $\int_X g d\mu = d \neq 0$, so $\mu \neq 0$. This contradicts $\mathcal{A}^\perp = \{0\}$. Therefore $\overline{\mathcal{A}} = C(X)$.

Part (ii). Identify $\mathcal{R}adon(X)$ with $[C(X)]^*$ by the isometric isomorphism T of the Riesz Representation Theorem for the Dual of $C(X)$, and give it the weak-* topology; $[C(X)]^*$ with this topology is a locally convex topological vector space. Let B denote the closed unit ball. Now

$$\mathcal{A}^\perp = B \cap \bigcap_{f \in \mathcal{A}} \left\{ \nu : \int_X f d\nu = 0 \right\}.$$

For each fixed f , the map $\nu \mapsto \int_X f d\nu$ is, by definition of the weak-* topology, weak-* continuous, so each set in the intersection is weak-* closed and convex; and B is weak-* closed, being the intersection

of the weak-* closed sets $\{\nu : |\int_X h d\nu| \leq 1\}$ over $h \in C(X)$ with $\|h\|_{\max} \leq 1$. Hence $\mathcal{A}^\perp$ is a bounded, weak-* closed, convex subset of $\mathcal{R}adon(X)$, and it is nonempty since $0 \in \mathcal{A}^\perp$.

By Corollary 13, a bounded weak-* closed convex subset of $\mathcal{R}adon(X)$ is weak-* compact and is the weak-* closed convex hull of its extreme points. (That corollary is proved from Alaoglu's Theorem together with the Krein-Milman Theorem applied to $[C(X)]^*$ with its weak-* topology.) Hence $\mathcal{A}^\perp$ is the weak-* closed convex hull of its extreme points. If 0 is the only extreme point of $\mathcal{A}^\perp$, then this hull is the closed convex hull of $\{0\}$, namely $\{0\}$, so $\mathcal{A}^\perp = \{0\}$.

Part (iii). Let μ be an extreme point of $\mathcal{A}^\perp$, and let $f \in \mathcal{A}$ with $0 \leq f \leq 1$ on X . Since f is continuous, hence bounded and Borel measurable, and $|\mu|$ is a finite measure, the set functions

$$\mu_1(E) = \int_E f d\mu, \quad \mu_2(E) = \int_E (1-f) d\mu \quad (E \in \mathcal{B}(X))$$

are well defined finite signed measures on $\mathcal{B}(X)$, countably additive by the Countable Additivity of Integration.

Clearly $\mu_1(E) + \mu_2(E) = \int_E 1 d\mu = \mu(E)$, so $\mu_1 + \mu_2 = \mu$.

Next, μ_1 and μ_2 are signed Radon measures. Write $\mu = \mu^+ - \mu^-$ for the Jordan decomposition and let $X = P \cup N$ be a corresponding Hahn decomposition, so that $\mu^+ = \mu \llcorner P$ and $\mu^- = -\mu \llcorner N$. Define

$$\alpha(E) = \int_E f d\mu^+, \quad \beta(E) = \int_E f d\mu^- \quad (E \in \mathcal{B}(X)).$$

Since $f \geq 0$, both α and β are nonnegative measures, and $\mu_1 = \alpha - \beta$ by the linearity of integration. They are mutually singular: $\alpha(N) = \int_N f d\mu^+ = 0$ and $\beta(P) = 0$, because $\mu^+(N) = \mu^-(P) = 0$. By the uniqueness assertion of the Jordan Decomposition Theorem, therefore, $\mu_1^+ = \alpha$ and $\mu_1^- = \beta$. Since $0 \leq f \leq 1$ we have $\alpha \leq \mu^+ \leq |\mu|$ and $\beta \leq \mu^- \leq |\mu|$ setwise, so by the Lemma each of α, β is a Radon measure and hence $\mu_1 \in \mathcal{R}adon(X)$. The same argument applies to μ_2 , since $0 \leq 1-f \leq 1$.

Moreover $|\mu_1| = \mu_1^+ + \mu_1^- = \alpha + \beta$, so $|\mu_1|(E) = \int_E f d\mu^+ + \int_E f d\mu^- = \int_E f d|\mu|$, and likewise $|\mu_2|(E) = \int_E (1-f) d|\mu|$. Hence

$$\|\mu_1\| + \|\mu_2\| = \int_X f d|\mu| + \int_X (1-f) d|\mu| = \int_X 1 d|\mu| = |\mu|(X) = \|\mu\|.$$

Finally $\mu_1, \mu_2 \in \mathcal{A}^\perp$. Certainly $\|\mu_i\| \leq \|\mu\| \leq 1$. Let $g \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, $gf \in \mathcal{A}$, and since $\mathcal{A}$ contains the constants and is a linear space, $g(1-f) = g - gf \in \mathcal{A}$. Because $d\mu_1 = f d\mu$ and $d\mu_2 = (1-f) d\mu$ (first for characteristic functions by definition, then for simple functions by linearity, then for bounded Borel functions by the Simple Approximation Theorem and passage to the limit, valid since $|\mu|$ is finite),

$$\int_X g d\mu_1 = \int_X gf d\mu = 0, \quad \int_X g d\mu_2 = \int_X g(1-f) d\mu = 0.$$

Thus $\mu_1, \mu_2 \in \mathcal{A}^\perp$.

Now we conclude that $\mu_1 = c\mu$. First note that an extreme point μ of $\mathcal{A}^\perp$ satisfies $\|\mu\| = 1$ or $\mu = 0$: if $0 < \|\mu\| = r < 1$, choose $s > 0$ with $(1+s)r \leq 1$; then $(1 \pm s)\mu \in \mathcal{A}^\perp$, these two measures are distinct, and $\mu = \frac{1}{2}(1+s)\mu + \frac{1}{2}(1-s)\mu$, contradicting extremeness. If $\mu = 0$ then $\mu_1 = 0 = 0 \cdot \mu$ and we may take $c = 0$. So assume $\|\mu\| = 1$ and set

$$a = \|\mu_1\| = \int_X f d|\mu|, \quad b = \|\mu_2\| = \int_X (1-f) d|\mu|, \quad a + b = 1.$$

If $a = 0$ then $\mu_1 = 0 = 0 \cdot \mu$; if $b = 0$ then $\mu_2 = \mu = 1 \cdot \mu$. Otherwise $0 < a < 1$, the measures μ_1/a and μ_2/b both lie in $\mathcal{A}^\perp$ (each has norm 1 and annihilates $\mathcal{A}$), and

$$\mu = \mu_1 + \mu_2 = a\left(\frac{\mu_1}{a}\right) + b\left(\frac{\mu_2}{b}\right)$$

is a convex combination with $a + b = 1$, $0 < a < 1$. Since μ is an extreme point, $\mu_1/a = \mu$, that is,

$$\mu_1 = c\mu \quad \text{with } c = a = \int_X f d|\mu| \in [0, 1].$$

Part (iv). From $\mu_1 = c\mu$ we get

$$\int_E (f - c) d\mu = 0 \quad \text{for every Borel set } E.$$

Let $P \cup N$ be a Hahn decomposition for μ , so that $\mu^+ = \mu \llcorner P$ and $\mu^- = -\mu \llcorner N$. For Borel $E \subseteq P$ the displayed identity reads $\int_E (f - c) d\mu^+ = 0$. Taking $E = P \cap \{f > c\}$ and $E = P \cap \{f < c\}$ shows that the nonnegative functions $(f - c)^+$ and $(f - c)^-$ have zero integral over P with respect to μ^+ , hence vanish μ^+ -almost everywhere. The same argument on N gives $f = c$ μ^- -almost everywhere. Therefore

$$|\mu|(\{x \in X : f(x) \neq c\}) = 0.$$

Recall that the support of μ is

$$\text{supp } \mu = \{x \in X : |\mu|(\mathcal{O}) > 0 \text{ for every open } \mathcal{O} \text{ containing } x\},$$

a closed subset of X . Let $x \in \text{supp } \mu$ and suppose $f(x) \neq c$. By the continuity of f , the set

$$\mathcal{O} = \left\{y \in X : |f(y) - c| > \frac{1}{2}|f(x) - c|\right\}$$

is open and contains x , so $|\mu|(\mathcal{O}) > 0$. But $\mathcal{O} \subseteq \{f \neq c\}$, which is $|\mu|$ -null — a contradiction. Hence $f = c$ on $\text{supp } \mu$.

Part (v). Suppose $\text{supp } \mu$ contained two distinct points $x_1 \neq x_2$. Since $\mathcal{A}$ separates points there is $g \in \mathcal{A}$ with $g(x_1) \neq g(x_2)$. As X is compact and g continuous, $m = \min_X g$ and $M = \max_X g$ are attained and $m < M$. Since $\mathcal{A}$ contains the constants and is a linear space,

$$f = \frac{g - m}{M - m} \in \mathcal{A}, \quad 0 \leq f \leq 1 \text{ on } X, \quad f(x_1) \neq f(x_2).$$

By parts (iii) and (iv), f is constant on $\text{supp } \mu$ — contradicting $f(x_1) \neq f(x_2)$. Hence $\text{supp } \mu$ contains at most one point.

Next, the complement of the support is $|\mu|$ -null. Indeed, $\mathcal{U} = X \setminus \text{supp } \mu$ is open and is the union of all open sets of $|\mu|$ -measure zero. Since $|\mu|$ is a Radon measure, $\mathcal{U}$ is inner regular: $|\mu|(\mathcal{U}) = \sup\{|\mu|(K) : K \subseteq \mathcal{U}, K \text{ compact}\}$. Any compact $K \subseteq \mathcal{U}$ is covered by finitely many open $|\mu|$ -null sets, so $|\mu|(K) = 0$ by finite subadditivity. Hence $|\mu|(\mathcal{U}) = 0$.

Now suppose $\text{supp } \mu = \{x_0\}$. Then $|\mu|(X \setminus \{x_0\}) = 0$, so $\mu = t\delta_{x_0}$ with $t = \mu(\{x_0\})$. Because the constant function 1 belongs to $\mathcal{A}$,

$$0 = \int_X 1 d\mu = \mu(X) = t,$$

so $\mu = 0$. But then $|\mu|(X) = 0$ and, by the previous paragraph applied to $\mu = 0$, every point fails to be in the support; that is, $\text{supp } \mu = \emptyset$, contradicting $\text{supp } \mu = \{x_0\}$. Therefore $\text{supp } \mu = \emptyset$, whence $|\mu|(X) = |\mu|(X \setminus \text{supp } \mu) = 0$ and $\mu = 0$.

Conclusion. By (iii)-(v), the only extreme point of $\mathcal{A}^\perp$ is the zero measure. By (ii), $\mathcal{A}^\perp = \{0\}$. By (i), $\overline{\mathcal{A}} = C(X)$. This is precisely the Stone-Weierstrass Theorem: an algebra of continuous real-valued functions on a compact space that separates points and contains the constants is dense in $C(X)$.

Problem 21.54. Let X be a separable compact Hausdorff space. Show that every closed set is a G_δ set.

Solution. The assertion is to be read with “separable” in the strong sense of **second countable**, that is, the topology has a countable base. (Royden observes in Chapter 11 that for a metric space second countability and the existence of a countable dense subset are equivalent, but that for a general topological space a separable space — one merely possessing a countable dense subset — need not be second countable. As the counterexample at the end of this solution shows, mere separability is not enough here.)

The result for second countable compact Hausdorff spaces.

Let X be a compact Hausdorff space with a countable base $\{U_n\}_{n=1}^\infty$ for its topology, and let $F \subseteq X$ be closed. Set $\mathcal{U} = X \sim F$, an open set. We show $\mathcal{U}$ is an F_σ ; taking complements then shows F is a G_δ .

A compact Hausdorff space is normal, hence regular. Let $x \in \mathcal{U}$. By regularity applied to the point x and the closed set F (if $F = \emptyset$ the claim is trivial), there are disjoint open sets V and W with $x \in V$ and $F \subseteq W$. Since $V \subseteq X \sim W$ and $X \sim W$ is closed, $\overline{V} \subseteq X \sim W \subseteq X \sim F = \mathcal{U}$. Because $\{U_n\}$ is a base, there is an index n with $x \in U_n \subseteq V$, and then

$$\overline{U_n} \subseteq \overline{V} \subseteq \mathcal{U}.$$

Consequently, if we set

$$S = \{n \in \mathbb{N} : \overline{U_n} \subseteq \mathcal{U}\},$$

then every point of $\mathcal{U}$ belongs to U_n for some $n \in S$, so $\mathcal{U} \subseteq \bigcup_{n \in S} U_n$. On the other hand $\bigcup_{n \in S} \overline{U_n} \subseteq \mathcal{U}$ by the definition of S . Since $U_n \subseteq \overline{U_n}$,

$$\mathcal{U} = \bigcup_{n \in S} U_n = \bigcup_{n \in S} \overline{U_n},$$

a countable union of closed sets. Therefore

$$F = X \sim \mathcal{U} = \bigcap_{n \in S} (X \sim \overline{U_n})$$

is a countable intersection of open sets, that is, a G_δ set. (If S is empty, $\mathcal{U} = \emptyset$ and $F = X$, which is trivially a G_δ .)

The metrizable case, directly. A second countable compact Hausdorff space is metrizable by the Urysohn Metrization Theorem, and for a metric space the conclusion is immediate: if F is a nonempty closed subset of a metric space X , then $x \mapsto \text{dist}(x, F)$ is continuous and vanishes exactly on F , so

$$F = \bigcap_{n=1}^\infty \left\{x \in X : \text{dist}(x, F) < \frac{1}{n}\right\},$$

a countable intersection of open sets. This is the form in which the result is used in Section 21.6: on a compact metric space every closed set is a G_δ , every closed set is therefore a Baire set, and hence $\mathcal{B}a(X) = \mathcal{B}(X)$.

Why “separable” alone does not suffice. Let

$$X = [0, 1]^{[0, 1]} = \prod_{t \in [0, 1]} [0, 1],$$

the set of all functions from $[0, 1]$ to $[0, 1]$ with the product topology. By Tychonoff’s Theorem X is compact, and it is Hausdorff as a product of Hausdorff spaces.

X is separable. Let D be the countable collection of those $h \in X$ for which there is a partition $0 = s_0 < s_1 < \cdots < s_k = 1$ into intervals with rational endpoints such that h is constant with a rational value on each of $[s_0, s_1), [s_1, s_2), \dots, [s_{k-1}, s_k]$. A nonempty basic open subset of X is determined by finitely many coordinates $t_1 < t_2 < \cdots < t_n$ in $[0, 1]$ and nonempty open sets $O_1, \dots, O_n \subseteq [0, 1]$, and consists of those g with $g(t_j) \in O_j$ for $1 \leq j \leq n$. Since the t_j are distinct, we may pick a rational strictly between t_j and t_{j+1} for $1 \leq j \leq n-1$; adjoining $s_0 = 0$ and $s_k = 1$ we obtain rationals $0 = s_0 < s_1 < \cdots < s_k = 1$ for which each of the intervals $[s_0, s_1), \dots, [s_{k-2}, s_{k-1}), [s_{k-1}, s_k]$ contains at most one of the points $t_1, \dots, t_n$. Each O_j , being a nonempty open subset of $[0, 1]$, contains a rational q_j ; let h take the value q_j on the interval of the partition containing t_j , and the value 0 on the remaining intervals. Then $h \in D$ and h lies in the given basic open set. So D is a countable dense subset of X .

However no singleton in X is a G_δ , although singletons are closed since X is Hausdorff. Suppose $p \in X$ and $\{p\} = \bigcap_{n=1}^\infty \mathcal{O}_n$ with each $\mathcal{O}_n$ open. For each n choose a basic open set $\mathcal{B}_n$ with $p \in \mathcal{B}_n \subseteq \mathcal{O}_n$;

$\mathcal{B}_n$ constrains only the coordinates in some finite set $F_n \subseteq [0, 1]$. Then $F = \bigcup_n F_n$ is countable, so, $[0, 1]$ being uncountable, there is $t^* \in [0, 1] \sim F$. Define $q \in X$ by $q(t) = p(t)$ for $t \neq t^*$ and $q(t^*) \neq p(t^*)$. Since q agrees with p on every coordinate in each F_n , we have $q \in \mathcal{B}_n \subseteq \mathcal{O}_n$ for every n , so $q \in \bigcap_n \mathcal{O}_n = \{p\}$. But $q \neq p$ — a contradiction.

Thus X is a separable compact Hausdorff space possessing closed sets that are not G_δ sets, and the hypothesis of second countability in the argument above cannot be weakened to separability.

Problem 21.55. Let X be a Hausdorff space and $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ a σ -finite Borel measure. Show that μ is Radon if and only if it is regular.

Solution. Recall the two definitions. A Borel measure μ (that is, a measure on $\mathcal{B}(X)$ for which every compact set has finite measure) is a **Radon measure** provided

- (i) (outer regularity) $\mu(E) = \inf\{\mu(\mathcal{U}) : \mathcal{U} \text{ a neighborhood of } E\}$ for every Borel set E , and
- (ii) (inner regularity) $\mu(\mathcal{O}) = \sup\{\mu(K) : K \text{ a compact subset of } \mathcal{O}\}$ for every open set $\mathcal{O}$;

while μ is **regular** provided every Borel set E is both outer regular, $\mu(E) = \inf\{\mu(\mathcal{O}) : \mathcal{O} \text{ open, } E \subseteq \mathcal{O}\}$, and inner regular, $\mu(E) = \sup\{\mu(K) : K \text{ compact, } K \subseteq E\}$.

Since X is Hausdorff, every compact subset of X is closed and therefore is a Borel set; thus the suprema above are taken over genuine Borel sets. Note also that outer regularity in the two definitions is the same condition: a set $\mathcal{U}$ is a neighborhood of E exactly when it contains an open set containing E , and the infimum of μ over neighborhoods equals the infimum over open supersets by monotonicity.

Regular implies Radon. Suppose μ is regular. Then every Borel set is outer regular, which is (i). Every Borel set is inner regular; in particular every open set is inner regular, which is (ii). Since μ is assumed to be a Borel measure, μ is Radon. (Neither σ -finiteness nor the Hausdorff property is needed for this implication.)

Radon implies regular. Suppose μ is Radon. Outer regularity of every Borel set is (i), so only inner regularity of every Borel set remains.

Step 1: Borel sets of finite measure are inner regular. Let E be a Borel set with $\mu(E) < \infty$, and let $\epsilon > 0$.

By (i) there is an open set $\mathcal{O} \supseteq E$ with $\mu(\mathcal{O}) < \mu(E) + \epsilon < \infty$. Since $\mu(E) < \infty$, excision gives

$$\mu(\mathcal{O} \sim E) = \mu(\mathcal{O}) - \mu(E) < \epsilon.$$

By (ii), applied to the open set $\mathcal{O}$ of finite measure, there is a compact set $K \subseteq \mathcal{O}$ with

$$\mu(K) > \mu(\mathcal{O}) - \epsilon.$$

By (i), applied to the Borel set $\mathcal{O} \sim E$, there is an open set $\mathcal{U} \supseteq \mathcal{O} \sim E$ with

$$\mu(\mathcal{U}) < \mu(\mathcal{O} \sim E) + \epsilon < 2\epsilon.$$

Consider

$$K_0 = K \sim \mathcal{U} = K \cap (X \sim \mathcal{U}).$$

It is a closed subset of the compact set K , hence compact, and therefore Borel. Moreover

$$K_0 \subseteq \mathcal{O} \sim \mathcal{U} \subseteq \mathcal{O} \sim (\mathcal{O} \sim E) = \mathcal{O} \cap E = E.$$

Finally, using $K \subseteq K_0 \cup \mathcal{U}$ and finite subadditivity,

$$\mu(K_0) \geq \mu(K) - \mu(\mathcal{U}) > (\mu(\mathcal{O}) - \epsilon) - 2\epsilon \geq \mu(E) - 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary and every compact subset of E has measure at most $\mu(E)$, we conclude

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}.$$

Step 2: Borel sets of infinite measure. Here σ -finiteness is used. Write $X = \bigcup_{n=1}^{\infty} X_n$ with each X_n a Borel set of finite measure; replacing X_n by $X_1 \cup \dots \cup X_n$ we may assume $\{X_n\}$ is ascending. Let E be a Borel set with $\mu(E) = \infty$. The sets $E \cap X_n$ ascend to E , so by the continuity of measure

$$\lim_{n \rightarrow \infty} \mu(E \cap X_n) = \mu(E) = \infty.$$

Given $M > 0$, choose n with $\mu(E \cap X_n) > M$. Since $\mu(E \cap X_n) \leq \mu(X_n) < \infty$, Step 1 applies to $E \cap X_n$ and produces a compact set $K \subseteq E \cap X_n \subseteq E$ with $\mu(K) > M$. As M was arbitrary,

$$\sup\{\mu(K) : K \subseteq E, K \text{ compact}\} = \infty = \mu(E),$$

so E is inner regular.

By Steps 1 and 2 every Borel set is inner regular, and by (i) every Borel set is outer regular. Hence μ is regular.

Remark. Step 1 required neither σ -finiteness nor local compactness; it shows that for any Radon measure on a Hausdorff space, every Borel set of finite measure is inner regular (compare Problem 36). The σ -finiteness hypothesis is needed only to handle sets of infinite measure.

Problem 21.56. Show that a Hausdorff space X is both locally compact and σ -compact if and only if there is an ascending countable collection $\{\mathcal{O}_k\}_{k=1}^\infty$ of open subsets of X that covers X and for each k ,

$$\overline{\mathcal{O}_k} \text{ is a compact subset of } \mathcal{O}_{k+1}.$$

Solution. Recall that X is **locally compact** provided each point of X has a neighborhood with compact closure, and σ -compact provided X is the union of a countable collection of compact subsets.

The condition is sufficient. Suppose $\{\mathcal{O}_k\}_{k=1}^\infty$ is an ascending collection of open sets with $X = \bigcup_{k=1}^\infty \mathcal{O}_k$ and $\overline{\mathcal{O}_k}$ compact and contained in $\mathcal{O}_{k+1}$ for each k .

Let $x \in X$. Then $x \in \mathcal{O}_k$ for some k , so $\mathcal{O}_k$ is a neighborhood of x , and $\overline{\mathcal{O}_k}$ is compact. Hence X is locally compact.

Moreover

$$X = \bigcup_{k=1}^\infty \mathcal{O}_k \subseteq \bigcup_{k=1}^\infty \overline{\mathcal{O}_k} \subseteq X,$$

so X is the union of the countable collection $\{\overline{\mathcal{O}_k}\}$ of compact sets and therefore is σ -compact. (Neither implication used the Hausdorff property.)

The condition is necessary. Suppose X is a locally compact Hausdorff space that is σ -compact; write $X = \bigcup_{n=1}^\infty K_n$ with each K_n compact.

Lemma. If X is locally compact and $K \subseteq X$ is compact, then there is an open set $\mathcal{O}$ with $K \subseteq \mathcal{O}$ and $\overline{\mathcal{O}}$ compact.

To prove the lemma, let $x \in K$. By local compactness there is a neighborhood N_x of x with $\overline{N_x}$ compact; let V_x be an open set with $x \in V_x \subseteq N_x$. Then $\overline{V_x}$ is a closed subset of the compact set $\overline{N_x}$ and hence is compact. The collection $\{V_x\}_{x \in K}$ is an open cover of the compact set K , so there are finitely many points $x_1, \dots, x_m$ in K with $K \subseteq \mathcal{O} := \bigcup_{i=1}^m V_{x_i}$. Since the closure of a finite union is the union of the closures,

$$\overline{\mathcal{O}} = \bigcup_{i=1}^m \overline{V_{x_i}},$$

a finite union of compact sets and therefore compact. This proves the lemma.

Now define the sets $\mathcal{O}_k$ inductively. By the lemma applied to the compact set K_1 , choose an open set $\mathcal{O}_1 \supseteq K_1$ with $\overline{\mathcal{O}_1}$ compact. Suppose the open sets $\mathcal{O}_1, \dots, \mathcal{O}_k$ have been chosen with each $\overline{\mathcal{O}_j}$ compact. The set

$$C_{k+1} = \overline{\mathcal{O}_k} \cup K_{k+1}$$

is a union of two compact sets and hence is compact. By the lemma there is an open set $\mathcal{O}_{k+1} \supseteq C_{k+1}$ with $\overline{\mathcal{O}_{k+1}}$ compact.

The collection $\{\mathcal{O}_k\}_{k=1}^\infty$ so obtained has the required properties:

- Each $\mathcal{O}_k$ is open and $\overline{\mathcal{O}_k}$ is compact.
- $\overline{\mathcal{O}_k} \subseteq C_{k+1} \subseteq \mathcal{O}_{k+1}$, so $\overline{\mathcal{O}_k}$ is a compact subset of $\mathcal{O}_{k+1}$.
- The collection is ascending, since $\mathcal{O}_k \subseteq \overline{\mathcal{O}_k} \subseteq \mathcal{O}_{k+1}$.
- It covers X : by construction $K_1 \subseteq \mathcal{O}_1$ and $K_{k+1} \subseteq C_{k+1} \subseteq \mathcal{O}_{k+1}$ for every $k \geq 1$, so $K_n \subseteq \mathcal{O}_n$ for all n and therefore

$$X = \bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=1}^{\infty} \mathcal{O}_n \subseteq X.$$

This completes the proof of the equivalence.

21.9 Problems 21.57–21.61

Problem 21.57. Let x_0 be a point in the locally compact Hausdorff space X . Is the Dirac delta measure concentrated at x_0 , δ_{x_0} , a regular Baire measure?

Solution. Yes. Here δ_{x_0} denotes the restriction to the Baire σ -algebra $\text{Ba}(X)$ of the set function $E \mapsto 1$ if $x_0 \in E$ and $E \mapsto 0$ if $x_0 \notin E$. This set function is countably additive on any σ -algebra of subsets of X (if $E = \bigcup_{k=1}^{\infty} E_k$ with the E_k disjoint, then x_0 belongs to E if and only if it belongs to exactly one E_k), and it takes values in $\{0, 1\}$, so it is finite and in particular finite on compact sets. Thus $\delta_{x_0}: \text{Ba}(X) \rightarrow [0, \infty)$ is a Baire measure. It remains to prove that every Baire set is both inner and outer regular with respect to δ_{x_0} , where, as in the text, the approximating open sets and compact sets are required to belong to $\text{Ba}(X)$.

Call a set K a compact Baire set provided K is compact and $K \in \text{Ba}(X)$.

Claim. For each $E \in \text{Ba}(X)$ there is a compact Baire set K with

$$x_0 \in K \subseteq E \quad \text{or} \quad x_0 \in K \subseteq X \sim E.$$

Proof of the Claim. Let $\mathcal{F}$ be the collection of those $E \in \text{Ba}(X)$ for which such a K exists. By its very definition $\mathcal{F}$ is closed with respect to complements, since the defining condition is symmetric in E and $X \sim E$.

First, X belongs to $\mathcal{F}$: the set $\{x_0\}$ is compact, so by the locally compact extension property, property (iii) of Theorem 7, there is a function $f \in C_c(X)$ with $0 \leq f \leq 1$ and $f(x_0) = 1$. The set $K = \{x \in X \mid f(x) \geq 1/2\}$ is a closed subset of $\text{supp } f$, hence compact; it belongs to $\text{Ba}(X)$ because f , belonging to $C_c(X)$, is measurable with respect to $\text{Ba}(X)$; and $x_0 \in K \subseteq X$.

Next, $\mathcal{F}$ is closed with respect to countable unions. Let $E = \bigcup_{n=1}^{\infty} E_n$ with each $E_n \in \mathcal{F}$. If for some n there is a compact Baire set K with $x_0 \in K \subseteq E_n$, then $x_0 \in K \subseteq E$ and $E \in \mathcal{F}$. Otherwise, for each n there is a compact Baire set K_n with $x_0 \in K_n \subseteq X \sim E_n$. Set $K = \bigcap_{n=1}^{\infty} K_n$. Since X is Hausdorff each K_n is closed, so K is a closed subset of the compact set K_1 and therefore is compact; K belongs to $\text{Ba}(X)$ since $\text{Ba}(X)$ is a σ -algebra; and $x_0 \in K \subseteq \bigcap_{n=1}^{\infty} (X \sim E_n) = X \sim E$. Hence $E \in \mathcal{F}$. Therefore $\mathcal{F}$ is a σ -algebra of subsets of X contained in $\text{Ba}(X)$.

Finally, $\mathcal{F}$ contains every compact G_δ subset K_0 of X . Indeed, such a K_0 belongs to $\text{Ba}(X)$ by Proposition 21. If $x_0 \in K_0$, take $K = K_0$. If $x_0 \notin K_0$, then, K_0 being compact and X Hausdorff, K_0 is closed and hence $X \sim K_0$ is a neighborhood of the compact set $\{x_0\}$; by the locally compact extension property there is $f \in C_c(X)$ with $0 \leq f \leq 1$, $f(x_0) = 1$ and $f = 0$ on K_0 . As above $K = \{x \in X \mid f(x) \geq 1/2\}$ is a compact Baire set, it contains x_0 , and $K \subseteq \{x \in X \mid f(x) \neq 0\} \subseteq X \sim K_0$.

According to Proposition 21, $\text{Ba}(X)$ is the smallest σ -algebra containing all the compact G_δ subsets of X . Since $\mathcal{F}$ is a σ -algebra containing all these sets and contained in $\text{Ba}(X)$, we conclude $\mathcal{F} = \text{Ba}(X)$. This proves the Claim.

Now let E belong to $\text{Ba}(X)$.

Case 1: $x_0 \in E$, so that $\delta_{x_0}(E) = 1$. The set X is an open Baire set containing E with $\delta_{x_0}(X) = 1$, so E is outer regular. By the Claim, since the second alternative is impossible here, there is a compact Baire set K with $x_0 \in K \subseteq E$; then $\delta_{x_0}(K) = 1 = \delta_{x_0}(E)$, so E is inner regular.

Case 2: $x_0 \notin E$, so that $\delta_{x_0}(E) = 0$. The empty set is a compact Baire subset of E , and every Baire subset of E has measure 0; hence the supremum in the definition of inner regularity equals $0 = \delta_{x_0}(E)$ and E is inner regular. By the Claim there is a compact Baire set K with $x_0 \in K \subseteq X \sim E$. Since K is compact and X is Hausdorff, K is closed, so $\mathcal{O} = X \sim K$ is open; it belongs to $\text{Ba}(X)$, it contains E , and $x_0 \notin \mathcal{O}$, so that $\delta_{x_0}(\mathcal{O}) = 0 = \delta_{x_0}(E)$. Hence E is outer regular.

Therefore every Baire set is regular with respect to δ_{x_0} , that is, the Dirac delta measure concentrated at x_0 is a regular Baire measure.

Two remarks. First, the local compactness of X was used only to produce, for a compact set and a neighborhood of it, a function in $C_c(X)$ that is 1 on the compact set and vanishes outside the neighborhood; without some such separation hypothesis the argument fails. Second, the same computation shows directly that δ_{x_0} on all of $\mathcal{B}(X)$ is a Radon measure, which is Problem 31.

Problem 21.58. Let X be an uncountable set with the discrete topology and X^* its Alexandroff compactification with x^* the point at infinity. Show that the singleton set $\{x^*\}$ is a Borel set that is not a Baire set.

Solution. Recall the Alexandroff (one-point) compactification: $X^* = X \cup \{x^*\}$, and a subset of X^* is open provided it is an open subset of X or it is of the form $\{x^*\} \cup (X \sim K)$ with K a compact subset of X . Since X carries the discrete topology, every subset of X is open and a subset of X is compact if and only if it is finite. Thus the open subsets of X^* are precisely the subsets of X together with the sets $\{x^*\} \cup (X \sim K)$ with $K \subseteq X$ finite; that is, the neighborhoods of x^* are exactly the sets containing x^* whose complement in X is finite. The space X is Hausdorff and locally compact (each singleton is a compact neighborhood of its point), so X^* is a compact Hausdorff space and X is an open subset of X^* .

$\{x^*\}$ is a Borel set. Its complement in X^* is X , which is open in X^* . Hence $\{x^*\}$ is closed and therefore belongs to $\mathcal{B}(X^*)$.

$\{x^*\}$ is not a Baire set. Since X^* is compact, $C_c(X^*) = C(X^*)$, so $\text{Ba}(X^*)$ is the smallest σ -algebra of subsets of X^* with respect to which every $f \in C(X^*)$ is measurable. We exhibit a σ -algebra with respect to which every such f is measurable and which does not contain $\{x^*\}$.

Step 1: every $f \in C(X^*)$ is constant off a countable set. Let $f \in C(X^*)$ and put $c = f(x^*)$. Fix $n \in \mathbb{N}$. By continuity of f at x^* there is a neighborhood of x^* on which $|f - c| < 1/n$, and by the description of the topology this neighborhood is of the form $\{x^*\} \cup (X \sim K_n)$ with $K_n \subseteq X$ finite. Hence

$$\left\{x \in X^* \mid |f(x) - c| \geq 1/n\right\} \subseteq K_n,$$

a finite set. Consequently

$$N_f = \{x \in X^* \mid f(x) \neq c\} = \bigcup_{n=1}^{\infty} \left\{x \in X^* \mid |f(x) - c| \geq 1/n\right\} \subseteq \bigcup_{n=1}^{\infty} K_n$$

is countable, and $x^* \notin N_f$.

Step 2: a σ -algebra. Define

$$\mathcal{D} = \left\{E \subseteq X^* \mid \text{either } x^* \notin E \text{ and } E \text{ is countable, or } x^* \in E \text{ and } X^* \sim E \text{ is countable}\right\}.$$

Then $\mathcal{D}$ is a σ -algebra. Indeed, $\emptyset \in \mathcal{D}$, since $x^* \notin \emptyset$ and $\emptyset$ is countable. If $E \in \mathcal{D}$ then $X^* \sim E \in \mathcal{D}$: if $x^* \notin E$ and E is countable, then x^* belongs to $X^* \sim E$ whose complement E is countable; and if $x^* \in E$ and $X^* \sim E$ is countable, then $x^* \notin X^* \sim E$ and $X^* \sim E$ is countable. Finally, let $E = \bigcup_{n=1}^{\infty} E_n$ with each $E_n \in \mathcal{D}$. If $x^* \in E_{n_0}$ for some n_0 , then $x^* \in E$ and $X^* \sim E \subseteq X^* \sim E_{n_0}$ is countable, so $E \in \mathcal{D}$. Otherwise $x^* \notin E_n$ and E_n is countable for every n ; then $x^* \notin E$ and E , a countable union of countable sets, is countable, so $E \in \mathcal{D}$.

Step 3: every $f \in C(X^*)$ is measurable with respect to $\mathcal{D}$. Let $f \in C(X^*)$, let $c = f(x^*)$, and let N_f be the countable set of Step 1, so that $f = c$ on $X^* \sim N_f$. Let I be an interval (or indeed any subset of $\mathbb{R}$). If $c \notin I$, then $f^{-1}(I) \subseteq N_f$ is countable and does not contain x^* , so $f^{-1}(I) \in \mathcal{D}$. If $c \in I$, then $x^* \in f^{-1}(I)$ and $X^* \sim f^{-1}(I) \subseteq N_f$ is countable, so again $f^{-1}(I) \in \mathcal{D}$. Hence f is measurable with respect to $\mathcal{D}$.

By the minimality of the Baire σ -algebra, $\text{Ba}(X^*) \subseteq \mathcal{D}$. Now $x^* \in \{x^*\}$ while $X^* \sim \{x^*\} = X$ is uncountable, so $\{x^*\} \notin \mathcal{D}$ and therefore $\{x^*\} \notin \text{Ba}(X^*)$.

Thus $\{x^*\}$ is a Borel subset of the compact Hausdorff space X^* that is not a Baire set, so the inclusion $\text{Ba}(X^*) \subseteq \mathcal{B}(X^*)$ is strict. (Consistently with Theorem 20, the closed set $\{x^*\}$ is not a G_δ : any countable family of open sets containing x^* has an intersection whose complement in X is countable, hence, X being uncountable, that intersection contains points of X .)

Problem 21.59. Let X be a locally compact Hausdorff space. Show that a Borel measure $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ is Radon if and only if every Borel set is measurable with respect to the Caratheodory measure induced by the premeasure $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$.

Solution. Recall from Chapter 17 that for a set function μ defined on a collection $\mathcal{S}$ of subsets of X , the induced outer measure is

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k),$$

the infimum taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E , and the Caratheodory measure induced by μ is the restriction of μ^* to the σ -algebra $\mathcal{M}$ of μ^* -measurable sets, a set E being μ^* -measurable provided $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \sim E)$ for every $A \subseteq X$.

One direction of the asserted equivalence is true, but for a reason that has nothing to do with regularity, and the other direction, read literally, is false. We prove this, and then prove the correct statement toward which the problem points.

Step 1. Let $\mathcal{S}$ be any σ -algebra of subsets of X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ any measure, and let μ^* be the outer measure induced by μ . Then

$$\mu^*(E) = \inf\{\mu(B) \mid E \subseteq B, B \in \mathcal{S}\} \quad \text{for all } E \subseteq X, \quad (*)$$

and every set in $\mathcal{S}$ is μ^* -measurable. The second assertion is an instance of Theorem 12 of Chapter 17: a set function on a σ -algebra is a measure if and only if it is a premeasure, and a σ -algebra is closed with respect to the formation of relative complements, so the Caratheodory measure induced by μ is an extension of μ . For completeness we give the short direct verification.

Indeed, if $\{E_k\}_{k=1}^{\infty}$ is a collection of sets in $\mathcal{S}$ covering E , then $B = \bigcup_{k=1}^{\infty} E_k$ belongs to $\mathcal{S}$, contains E , and, by the countable monotonicity of the measure μ , $\mu(B) \leq \sum_{k=1}^{\infty} \mu(E_k)$; conversely a single set $B \in \mathcal{S}$ containing E is itself such a cover. This proves (*). Now let $E \in \mathcal{S}$ and let $A \subseteq X$. Since μ^* is an outer measure it is countably, hence finitely, subadditive, so $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \sim E)$. For the reverse inequality we may assume $\mu^*(A) < \infty$. Let $\epsilon > 0$ and, using (*), choose $B \in \mathcal{S}$ with $A \subseteq B$ and $\mu(B) < \mu^*(A) + \epsilon$. Both $B \cap E$ and $B \sim E$ belong to $\mathcal{S}$ and they are disjoint with union B , so by the additivity of μ and the monotonicity of μ^* together with (*),

$$\mu^*(A) + \epsilon > \mu(B) = \mu(B \cap E) + \mu(B \sim E) \geq \mu^*(A \cap E) + \mu^*(A \sim E).$$

As $\epsilon > 0$ was arbitrary, E is μ^* -measurable.

Applying Step 1 with $\mathcal{S} = \mathcal{B}(X)$: for every Borel measure μ on the locally compact Hausdorff space X , without any hypothesis whatsoever, every Borel set is measurable with respect to the Caratheodory measure induced by the premeasure $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$. Hence the “only if” half of the problem is true, though trivially so, and the “if” half asserts that every Borel measure on a locally compact Hausdorff space is Radon.

Step 2. That assertion is false. Let X be an uncountable set with the discrete topology; X is Hausdorff and locally compact, and a subset of X is compact if and only if it is finite. Since every subset of X is open, $\mathcal{B}(X) = 2^X$. Define

$$\mu(E) = 0 \quad \text{if } E \text{ is countable,} \quad \mu(E) = \infty \quad \text{if } E \text{ is uncountable.}$$

Then $\mu(\emptyset) = 0$, and μ is countably additive: if $E = \bigcup_{k=1}^{\infty} E_k$ with the E_k disjoint and E is uncountable, then some E_k is uncountable, since a countable union of countable sets is countable, and so both sides are ∞ ; while if E is countable so is each E_k and both sides are 0. Every compact subset of X is finite, hence of measure zero, so μ is a Borel measure. It is outer regular, indeed trivially, because every set is open. But it is not inner regular on open sets: X is open with $\mu(X) = \infty$, while every compact subset of X is finite and so has measure 0, whence

$$\sup\{\mu(K) \mid K \text{ compact, } K \subseteq X\} = 0 \neq \mu(X).$$

Thus μ is a Borel measure that is not Radon, although, by Step 1, every Borel set is measurable with respect to the Caratheodory measure induced by $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$. The equivalence as printed therefore fails.

The same example also disposes of the reading in which the premeasure is taken to be the restriction of μ to the topology $\mathcal{T}$, since for this X we have $\mathcal{T} = 2^X = \mathcal{B}(X)$.

Step 3. The correct statement, which is the companion of Theorem 10 and of Problem 35, is the following. Let X be a locally compact Hausdorff space and $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ a Borel measure. Denote by $\mu_{\mathcal{T}}$ the restriction of μ to the topology $\mathcal{T}$ of X and by $\mu_{\mathcal{T}}^*$ the outer measure it induces. Then μ is Radon if and only if $\mu_{\mathcal{T}}$ is a Radon premeasure and μ is the restriction to $\mathcal{B}(X)$ of the Caratheodory measure induced by $\mu_{\mathcal{T}}$.

First observe that $\mu_{\mathcal{T}}$ is always a premeasure: $\mathcal{T}$ contains $\emptyset$, $\mu_{\mathcal{T}}(\emptyset) = 0$, and if a countable disjoint collection of open sets has open union, the countable additivity of the measure μ gives the required additivity. By Lemma 8,

$$\mu_{\mathcal{T}}^*(E) = \inf\{\mu(U) \mid U \text{ a neighborhood of } E\} \quad \text{for all } E \subseteq X.$$

Assume μ is Radon. Property (i) in the definition of a Radon premeasure holds: if U is open with compact closure then $\mu(U) \leq \mu(\overline{U}) < \infty$, since μ , being a Borel measure, is finite on compact sets. For property (ii), let $\mathcal{O}$ be open. By the inner regularity of μ on open sets, $\mu(\mathcal{O}) = \sup\{\mu(K) \mid K \subseteq \mathcal{O}, K \text{ compact}\}$. Given such a K , by the locally compact separation property, Proposition 2, there is a neighborhood U of K with $\overline{U}$ compact and $\overline{U} \subseteq \mathcal{O}$; then $\mu(K) \leq \mu(U)$. Hence

$$\mu(\mathcal{O}) = \sup\{\mu(U) \mid U \text{ open}, \overline{U} \text{ a compact subset of } \mathcal{O}\},$$

the inequality $\leq$ following from the preceding sentence and $\geq$ from the monotonicity of μ . Thus $\mu_{\mathcal{T}}$ is a Radon premeasure. Moreover, the outer regularity of μ says precisely that $\mu(E) = \inf\{\mu(U) \mid U \text{ a neighborhood of } E\} = \mu_{\mathcal{T}}^*(E)$ for every Borel set E ; and by Theorem 10 every Borel set is $\mu_{\mathcal{T}}^*$ -measurable, so μ is the restriction to $\mathcal{B}(X)$ of the Caratheodory measure induced by $\mu_{\mathcal{T}}$.

Conversely, if $\mu_{\mathcal{T}}$ is a Radon premeasure, Theorem 10 tells us that the restriction to $\mathcal{B}(X)$ of $\mu_{\mathcal{T}}^*$ is a Radon measure; if in addition μ is that restriction, then μ is Radon.

In particular, for a Borel measure μ all of whose Borel sets are outer regular, being Radon is equivalent to $\mu_{\mathcal{T}}$ being a Radon premeasure, that is, to inner regularity on open sets by open sets of compact closure; the counterexample of Step 2 is exactly a measure for which this last condition fails.

Problem 21.60. Prove Lemmas 24, 25, and 26.

Solution. Throughout, X is a locally compact Hausdorff space, and by a compact Baire set we mean a set that is compact and belongs to $\text{Ba}(X)$. We use repeatedly the locally compact extension property, property (iii) of Theorem 7: if K is compact and $\mathcal{O}$ is a neighborhood of K , there is $f \in C_c(X)$ with $f = 1$ on K , $f = 0$ on $X \setminus \mathcal{O}$ and $0 \leq f \leq 1$. For such an f and $0 < c < 1$, the set $\{x \in X \mid f(x) > c\}$ is an open Baire set and $\{x \in X \mid f(x) \geq c\}$ is a compact Baire set, both being inverse images of intervals under a function in $C_c(X)$, hence in $\text{Ba}(X)$, the second being a closed subset of the compact set $\text{supp } f$. We also use Proposition 21: $\text{Ba}(X)$ is the smallest σ -algebra containing all the compact G_δ subsets of X .

Lemma 24. Let $F \subseteq X$ be a closed Baire set. Then for $A \subseteq F$, $A \in \text{Ba}(X)$ if and only if $A \in \text{Ba}(F)$.

Proof. First note that F , with its subspace topology, is again a locally compact Hausdorff space: it is Hausdorff as a subspace of a Hausdorff space, and if $x \in F$ and N is a compact neighborhood of x in X , then $N \cap F$ is a neighborhood of x in F that is compact, being a closed subset of the compact set N . So Proposition 21 applies to F as well.

Suppose $A \in \text{Ba}(X)$ and $A \subseteq F$. Define

$$\mathcal{T} = \{E \in \text{Ba}(X) \mid E \cap F \in \text{Ba}(F)\}.$$

Then $\mathcal{T}$ is a σ -algebra of subsets of X : $X \cap F = F \in \text{Ba}(F)$; if $E \in \mathcal{T}$ then $(X \setminus E) \cap F = F \setminus (E \cap F) \in \text{Ba}(F)$; and intersection with F carries countable unions to countable unions. Moreover $\mathcal{T}$

contains every compact G_δ subset K of X : $K \cap F$ is compact, being a closed subset of K because F is closed, and if $K = \bigcap_{n=1}^\infty U_n$ with each U_n open in X , then $K \cap F = \bigcap_{n=1}^\infty (U_n \cap F)$ exhibits $K \cap F$ as a G_δ subset of F ; by Proposition 21 applied to F , $K \cap F \in \text{Ba}(F)$. Hence, by Proposition 21 applied to X , $\mathcal{T} = \text{Ba}(X)$. In particular $A = A \cap F \in \text{Ba}(F)$.

Conversely, define

$$\mathcal{S} = \{A \subseteq F \mid A \in \text{Ba}(X)\}.$$

This is a σ -algebra of subsets of F : $F \in \mathcal{S}$ because F is a Baire set of X ; if $A \in \mathcal{S}$ then $F \sim A = F \cap (X \sim A) \in \text{Ba}(X)$; and $\text{Ba}(X)$ is closed with respect to countable unions. By Proposition 21 applied to the locally compact Hausdorff space F , it suffices to show that every compact G_δ subset K of F belongs to $\mathcal{S}$. Write $K = \bigcap_{n=1}^\infty V_n$ with each V_n open in F , say $V_n = U_n \cap F$ with U_n open in X . The set K is compact in X and $K \subseteq U_n$, so by the locally compact extension property there is $\psi_n \in C_c(X)$ with $\psi_n = 1$ on K , $0 \leq \psi_n \leq 1$, and $\psi_n = 0$ on $X \sim U_n$. Put $W_n = \{x \in X \mid \psi_n(x) > 1/2\}$, an open Baire subset of X with $K \subseteq W_n \subseteq U_n$. Then

$$F \cap \bigcap_{n=1}^\infty W_n \subseteq F \cap \bigcap_{n=1}^\infty U_n = \bigcap_{n=1}^\infty V_n = K \subseteq F \cap \bigcap_{n=1}^\infty W_n,$$

so $K = F \cap \bigcap_{n=1}^\infty W_n$ is a countable intersection of Baire subsets of X and hence belongs to $\text{Ba}(X)$. Thus $K \in \mathcal{S}$, and therefore $\text{Ba}(F) \subseteq \mathcal{S}$, which is the assertion. $\square$

Lemma 25. Let $E \subseteq X$ be a Baire set that has compact closure. Then E is regular with respect to any Baire measure μ on $\text{Ba}(X)$.

Proof. Let $C = \overline{E}$, a compact set. By the locally compact extension property there is $f \in C_c(X)$ with $f = 1$ on C and $0 \leq f \leq 1$. Put

$$F_0 = \{x \in X \mid f(x) \geq 1/2\}, \quad \mathcal{U}_0 = \{x \in X \mid f(x) > 1/2\}.$$

As noted above, F_0 is a compact Baire set and $\mathcal{U}_0$ is an open Baire set, and $E \subseteq C \subseteq \mathcal{U}_0 \subseteq F_0$. Since F_0 is compact and X is Hausdorff, F_0 is closed, so Lemma 24 applies to F_0 and gives

$$\text{Ba}(F_0) = \{A \subseteq F_0 \mid A \in \text{Ba}(X)\}.$$

In particular $E \in \text{Ba}(F_0)$. Let ν be the restriction of μ to $\text{Ba}(F_0)$. It is a measure on the Baire σ -algebra of the compact Hausdorff space F_0 , and it is finite because $\nu(F_0) = \mu(F_0) < \infty$, F_0 being a compact Baire set and μ a Baire measure. Hence ν is a Baire measure on $\text{Ba}(F_0)$ and, by Proposition 22, ν is regular. Note $\mu(E) = \nu(E) \leq \nu(F_0) < \infty$. Let $\epsilon > 0$.

Inner regularity. By the inner regularity of ν there is a compact set $K \in \text{Ba}(F_0)$ with $K \subseteq E$ and $\nu(K) > \nu(E) - \epsilon$. A compact subset of F_0 is compact in X , and $K \in \text{Ba}(X)$ by Lemma 24. Hence K is a compact Baire subset of X contained in E with $\mu(K) > \mu(E) - \epsilon$.

Outer regularity. By the outer regularity of ν there is a set $V \in \text{Ba}(F_0)$, open in F_0 , with $E \subseteq V$ and $\nu(V) < \nu(E) + \epsilon$. Write $V = W \cap F_0$ with W open in X , and put $\mathcal{O} = \mathcal{U}_0 \cap V$. Since $\mathcal{U}_0 \subseteq F_0$ we have $\mathcal{O} = \mathcal{U}_0 \cap W \cap F_0 = \mathcal{U}_0 \cap W$, which is open in X . By Lemma 24, $V \in \text{Ba}(X)$, so $\mathcal{O} = \mathcal{U}_0 \cap V \in \text{Ba}(X)$. Moreover $E \subseteq \mathcal{U}_0$ and $E \subseteq V$, so $E \subseteq \mathcal{O}$, and by monotonicity $\mu(\mathcal{O}) \leq \mu(V) = \nu(V) < \mu(E) + \epsilon$.

Since $\epsilon > 0$ was arbitrary, E is both inner and outer regular with respect to μ , that is, E is regular. $\square$

Lemma 26. Let X be moreover σ -compact and $A \subseteq X$ a Baire set. Then $A = \bigcup_{k=1}^\infty A_k$ where each A_k is a Baire set that has compact closure.

Proof. Since X is σ -compact, $X = \bigcup_{k=1}^\infty K_k$ with each K_k compact. For each k , by the locally compact extension property there is $f_k \in C_c(X)$ with $f_k = 1$ on K_k and $0 \leq f_k \leq 1$. Put $\mathcal{U}_k = \{x \in X \mid f_k(x) > 1/2\}$. Then $\mathcal{U}_k$ is an open Baire set containing K_k , and since $\{x \in X \mid f_k(x) \geq 1/2\}$ is a closed set containing $\mathcal{U}_k$,

$$\overline{\mathcal{U}_k} \subseteq \{x \in X \mid f_k(x) \geq 1/2\} \subseteq \text{supp } f_k,$$

so $\overline{\mathcal{U}_k}$ is a closed subset of a compact set and hence is compact. Since $K_k \subseteq \mathcal{U}_k$ for each k , we have $X = \bigcup_{k=1}^\infty \mathcal{U}_k$.

Now set $A_k = A \cap \mathcal{U}_k$. Each A_k belongs to $\text{Ba}(X)$, being the intersection of two Baire sets, and $\overline{A_k} \subseteq \overline{\mathcal{U}_k}$ is a closed subset of a compact set and therefore compact. Finally

$$\bigcup_{k=1}^\infty A_k = A \cap \bigcup_{k=1}^\infty \mathcal{U}_k = A \cap X = A. \quad \square$$

Problem 21.61. Let X be a compact Hausdorff space and $f_1, \dots, f_n$ continuous real-valued functions on X . Let ν be a signed Radon measure on X with $|\nu|(X) \leq 1$ and let $c_i = \int_X f_i d\nu$, for $1 \leq i \leq n$.

- (i) Show that there is a signed Radon measure μ on X with $|\mu|(X) \leq 1$ for which

$$\int_X f_i d\mu = c_i$$

and

$$\int_X g d\mu \leq \int_X g d\lambda \quad \text{for all } g \in C(X)$$

for any signed Radon measure λ with $|\lambda|(X) \leq 1$ and such that $\int_X f_i d\lambda = c_i$ for $1 \leq i \leq n$.

- (ii) Suppose that there is a Radon measure ν on X with $\nu(X) = 1$ and $\int_X f_i d\nu = c_i$, $1 \leq i \leq n$. Show that there is a Radon measure μ on X with $\mu(X) = 1$ and $\int_X f_i d\mu = c_i$, for $1 \leq i \leq n$, which minimizes $\int_X g d\mu$ among all Radon measures that satisfy these conditions.

Solution. In both parts g denotes a function in $C(X)$ that is fixed in advance, and the assertion is that the linear functional $\lambda \mapsto \int_X g d\lambda$ attains a minimum on the set of measures singled out by the constraints. The quantifier cannot be read as asserting the existence of a single μ that minimizes $\int_X g d\lambda$ simultaneously for every $g \in C(X)$: if such a μ existed, then applying the inequality to g and to $-g$ would give $\int_X g d\mu = \int_X g d\lambda$ for every $g \in C(X)$ and every admissible λ , and hence, by the uniqueness of the representation of a functional by a signed Radon measure, $\mu = \lambda$ for every admissible λ , so that the constraint set would be a single point; that fails already for $n = 1$, $f_1 = 0$, $c_1 = 0$.

Notation and tools. Let $\mathcal{R}\text{adon}(X)$ be the normed linear space of signed Radon measures on X with the norm $\|\lambda\|_{\text{var}} = |\lambda|(X)$, and let

$$T: \mathcal{R}\text{adon}(X) \rightarrow [C(X)]^*, \quad T_\lambda(f) = \int_X f d\lambda \quad \text{for } f \in C(X).$$

By the Riesz Representation Theorem for the Dual of $C(X)$, T is a linear isometric isomorphism of $\mathcal{R}\text{adon}(X)$ onto $[C(X)]^*$. Transport the weak-* topology of $[C(X)]^*$ to $\mathcal{R}\text{adon}(X)$ by means of T ; thus, by the very definition of the weak-* topology, for each fixed $f \in C(X)$ the map

$$\Phi_f: \mathcal{R}\text{adon}(X) \rightarrow \mathbb{R}, \quad \Phi_f(\lambda) = \int_X f d\lambda$$

is continuous, and, T being an isometry, the set $B = \{\lambda \in \mathcal{R}\text{adon}(X) \mid |\lambda|(X) \leq 1\}$ corresponds to the closed unit ball of $[C(X)]^*$, which is weak-* compact by Alaoglu's Theorem. Hence B is compact in this topology. This is the first assertion of Corollary 13, applied to $K^* = B$, which is bounded and weak-* closed. More generally, Corollary 13 tells us that any bounded weak-* closed subset of $\mathcal{R}\text{adon}(X)$ is weak-* compact, and this is the only compactness we shall use.

Part (i). Define

$$\mathcal{K} = \left\{ \lambda \in \mathcal{R}\text{adon}(X) \mid |\lambda|(X) \leq 1 \text{ and } \int_X f_i d\lambda = c_i \text{ for } 1 \leq i \leq n \right\} = B \cap \bigcap_{i=1}^n \Phi_{f_i}^{-1}(\{c_i\}).$$

The set $\mathcal{K}$ is nonempty, since by hypothesis $\nu \in \mathcal{K}$. Each set $\Phi_{f_i}^{-1}(\{c_i\})$ is closed, being the inverse image of the closed set $\{c_i\}$ under the continuous function Φ_{f_i} . Therefore $\mathcal{K}$ is a closed subset of the compact set B and hence is itself compact.

The function Φ_g is continuous and real-valued on the nonempty compact space $\mathcal{K}$, so by the Extreme Value Theorem it attains a minimum value on $\mathcal{K}$: there is $\mu \in \mathcal{K}$ with

$$\int_X g d\mu = \Phi_g(\mu) \leq \Phi_g(\lambda) = \int_X g d\lambda \quad \text{for all } \lambda \in \mathcal{K}.$$

That is, μ is a signed Radon measure with $|\mu|(X) \leq 1$ and $\int_X f_i d\mu = c_i$ for $1 \leq i \leq n$, and $\int_X g d\mu \leq \int_X g d\lambda$ for every signed Radon measure λ with $|\lambda|(X) \leq 1$ and $\int_X f_i d\lambda = c_i$ for $1 \leq i \leq n$. This is precisely the required conclusion.

Part (ii). Now let

$$\mathcal{K}' = \left\{ \lambda \in \mathcal{R}\text{adon}(X) \mid \lambda \text{ is a Radon measure, } \lambda(X) = 1, \int_X f_i d\lambda = c_i \text{ for } 1 \leq i \leq n \right\},$$

where “ λ is a Radon measure” means that λ is nonnegative. By hypothesis $\nu \in \mathcal{K}'$, so $\mathcal{K}'$ is nonempty. If $\lambda \in \mathcal{K}'$ then $|\lambda|(X) = \lambda(X) = 1$, so $\mathcal{K}' \subseteq B$.

We claim that a signed Radon measure λ is nonnegative if and only if

$$\int_X h d\lambda \geq 0 \quad \text{for every } h \in C(X) \text{ with } h \geq 0 \text{ on } X. \quad (**)$$

If λ is a nonnegative measure, then $(**)$ holds by the monotonicity of the integral. Conversely, suppose $(**)$ holds. Then T_λ is a positive linear functional on $C(X) = C_c(X)$, so by the Riesz-Markov Theorem there is a Radon measure λ_0 on $\mathcal{B}(X)$ with $T_\lambda(f) = \int_X f d\lambda_0$ for all $f \in C(X)$. This λ_0 is finite, since $\lambda_0(X) = \int_X 1 d\lambda_0 = T_\lambda(1) < \infty$, and therefore $\lambda_0 = \lambda_0 - 0$ is a signed Radon measure, that is, λ_0 belongs to $\mathcal{R}\text{adon}(X)$ and T_{λ_0} is defined. Since $T_\lambda = T_{\lambda_0}$ and T is one-to-one, $\lambda = \lambda_0$, and λ_0 is nonnegative. This proves the claim.

Consequently

$$\mathcal{K}' = B \cap \Phi_1^{-1}(\{1\}) \cap \bigcap_{i=1}^n \Phi_{f_i}^{-1}(\{c_i\}) \cap \bigcap_{h \in C(X), h \geq 0} \Phi_h^{-1}([0, \infty)),$$

where Φ_1 is the map $\lambda \mapsto \int_X 1 d\lambda = \lambda(X)$. Each of the sets intersected with B is the inverse image of a closed subset of $\mathbb{R}$ under a continuous map, hence is closed. Therefore $\mathcal{K}'$ is a closed subset of the compact set B and so is compact.

Again by the Extreme Value Theorem, the continuous real-valued function Φ_g attains a minimum on the nonempty compact space $\mathcal{K}'$: there is $\mu \in \mathcal{K}'$ with

$$\int_X g d\mu \leq \int_X g d\lambda \quad \text{for all } \lambda \in \mathcal{K}'.$$

Thus μ is a Radon measure on X with $\mu(X) = 1$ and $\int_X f_i d\mu = c_i$ for $1 \leq i \leq n$, and it minimizes $\int_X g d\mu$ among all Radon measures satisfying these conditions.

We remark that in each part the set of admissible measures is convex as well as compact, since all the constraints are linear equalities or inequalities; this is not needed for the conclusions asked for, but it is what makes these constraint sets amenable to the Krein-Milman Theorem, as in Problem 53.

22 Invariant Measures

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22.1 Problems 22.1–22.7

Problem 22.1. In the following exercises, $\mathcal{G}$ is a topological group with unit element e and E is a Banach space.

If $\mathcal{T}_e$ is a base for the topology at e , show that $\{g \cdot \mathcal{O} \mid \mathcal{O} \in \mathcal{T}_e\}$ is a base for the topology at $g \in \mathcal{G}$.

Solution. Recall that a collection $\mathcal{B}_x$ of neighborhoods of a point x is called a base for the topology at x provided every neighborhood of x contains a member of $\mathcal{B}_x$.

Fix $g \in \mathcal{G}$ and let $L_g: \mathcal{G} \rightarrow \mathcal{G}$ be left translation, $L_g(x) = g \cdot x$.

Step 1: L_g is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$. The mapping $x \mapsto (g, x)$ of $\mathcal{G}$ into $\mathcal{G} \times \mathcal{G}$ is continuous, since each of its two components (the constant mapping $x \mapsto g$ and the identity mapping) is continuous and a mapping into a product is continuous if and only if its compositions with the projections are continuous. The group operation $(g_1, g_2) \mapsto g_1 \cdot g_2$ is continuous from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$, since $\mathcal{G}$ is a topological group. Therefore L_g , being the composition of these two continuous mappings, is continuous. The same argument shows $L_{g^{-1}}$ is continuous, and by the group axioms $L_{g^{-1}} \circ L_g = L_g \circ L_{g^{-1}} = \text{Id}$. Hence L_g is a homeomorphism with inverse $L_{g^{-1}}$. In particular, for $A \subseteq \mathcal{G}$, $g \cdot A = L_g(A)$ is open if and only if A is open.

Step 2: each $g \cdot \mathcal{O}$, with $\mathcal{O} \in \mathcal{T}_e$, is a neighborhood of g . Since $\mathcal{O}$ is a neighborhood of e , there is an open set $\mathcal{V}$ with $e \in \mathcal{V} \subseteq \mathcal{O}$. By Step 1, $g \cdot \mathcal{V}$ is open, and $g = g \cdot e \in g \cdot \mathcal{V} \subseteq g \cdot \mathcal{O}$. Thus $g \cdot \mathcal{O}$ is a neighborhood of g . (If, as is customary, the members of $\mathcal{T}_e$ are themselves open, then each $g \cdot \mathcal{O}$ is an open set containing g .)

Step 3: the collection is a base at g . Let $\mathcal{N}$ be any neighborhood of g , and choose an open set $\mathcal{W}$ with $g \in \mathcal{W} \subseteq \mathcal{N}$. By Step 1, $g^{-1} \cdot \mathcal{W} = L_{g^{-1}}(\mathcal{W})$ is open, and it contains $g^{-1} \cdot g = e$. Hence $g^{-1} \cdot \mathcal{W}$ is a neighborhood of e , so, since $\mathcal{T}_e$ is a base at e , there is $\mathcal{O} \in \mathcal{T}_e$ with $\mathcal{O} \subseteq g^{-1} \cdot \mathcal{W}$. Applying L_g ,

$$g \cdot \mathcal{O} \subseteq g \cdot (g^{-1} \cdot \mathcal{W}) = \mathcal{W} \subseteq \mathcal{N}.$$

Thus every neighborhood of g contains a member of $\{g \cdot \mathcal{O} \mid \mathcal{O} \in \mathcal{T}_e\}$, and by Step 2 every member of this collection is a neighborhood of g . Therefore $\{g \cdot \mathcal{O} \mid \mathcal{O} \in \mathcal{T}_e\}$ is a base for the topology at g .

Observe that the topology of a topological group is therefore completely determined by a base at the single point e : a set $\mathcal{W}$ is open if and only if for each $g \in \mathcal{W}$ there is $\mathcal{O} \in \mathcal{T}_e$ with $g \cdot \mathcal{O} \subseteq \mathcal{W}$.

Problem 22.2. Show that $K_1 \cdot K_2$ is compact if K_1 and K_2 are compact subsets of $\mathcal{G}$.

Solution. Recall that $K_1 \cdot K_2 = \{g_1 \cdot g_2 \mid g_1 \in K_1, g_2 \in K_2\}$.

Consider the product space $K_1 \times K_2$, where K_1 and K_2 carry their subspace topologies and the product carries the product topology; this topology is the same as the subspace topology $K_1 \times K_2$ inherits from $\mathcal{G} \times \mathcal{G}$, since the projections of a product onto its factors determine the product topology and a subspace of a product of subspaces is formed by the same basic rectangles.

The space $K_1 \times K_2$ is compact. Indeed, K_1 and K_2 are compact topological spaces, and by the Tychonoff Product Theorem of Section 12.2 the product of a collection of compact topological spaces is compact with respect to the product topology; here the collection is finite, consisting of just two spaces, so no appeal to the axiom of choice is involved.

Let $m: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ be the group operation, $m(g_1, g_2) = g_1 \cdot g_2$. Since $\mathcal{G}$ is a topological group, m is continuous, and therefore its restriction to the subspace $K_1 \times K_2$ is continuous.

Finally, $K_1 \cdot K_2 = m(K_1 \times K_2)$ is the image of a compact topological space under a continuous mapping, and the continuous image of a compact topological space is compact. Therefore $K_1 \cdot K_2$ is a compact subspace of $\mathcal{G}$, that is, a compact subset of $\mathcal{G}$.

We note two consequences. Since the topology on $\mathcal{G}$ is Hausdorff, $K_1 \cdot K_2$ is also closed. Moreover, the same argument applied to the inversion mapping $g \mapsto g^{-1}$, which is continuous, shows that K^{-1} is compact whenever K is compact.

Problem 22.3. Let $\mathcal{O}$ be a neighborhood of e . Show that there is also a neighborhood $\mathcal{U}$ of e for which $\mathcal{U} = \mathcal{U}^{-1}$ and $\mathcal{U} \cdot \mathcal{U} \subseteq \mathcal{O}$.

Solution. Since $\mathcal{O}$ is a neighborhood of e , we may choose an open set $\mathcal{O}'$ with $e \in \mathcal{O}' \subseteq \mathcal{O}$; it suffices to find $\mathcal{U}$ with $\mathcal{U} = \mathcal{U}^{-1}$ and $\mathcal{U} \cdot \mathcal{U} \subseteq \mathcal{O}'$.

Step 1: there is an open neighborhood $\mathcal{V}$ of e with $\mathcal{V} \cdot \mathcal{V} \subseteq \mathcal{O}'$. The group operation $m: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is continuous and $m(e, e) = e \cdot e = e \in \mathcal{O}'$. Therefore $m^{-1}(\mathcal{O}')$ is an open subset of $\mathcal{G} \times \mathcal{G}$ containing (e, e) . The sets of the form $\mathcal{V}_1 \times \mathcal{V}_2$, with $\mathcal{V}_1, \mathcal{V}_2$ open in $\mathcal{G}$, form a base for the product topology, so there are open sets $\mathcal{V}_1, \mathcal{V}_2$ with

$$(e, e) \in \mathcal{V}_1 \times \mathcal{V}_2 \subseteq m^{-1}(\mathcal{O}'), \quad \text{that is,} \quad \mathcal{V}_1 \cdot \mathcal{V}_2 \subseteq \mathcal{O}'.$$

Put $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$. Then $\mathcal{V}$ is open, $e \in \mathcal{V}$, and $\mathcal{V} \cdot \mathcal{V} \subseteq \mathcal{V}_1 \cdot \mathcal{V}_2 \subseteq \mathcal{O}'$.

Step 2: symmetrize. Let $\iota: \mathcal{G} \rightarrow \mathcal{G}$ be the inversion mapping $\iota(g) = g^{-1}$. Since $\mathcal{G}$ is a topological group, ι is continuous, and since $\iota \circ \iota = \text{Id}$, ι is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$ that is its own inverse. Hence $\mathcal{V}^{-1} = \iota(\mathcal{V}) = \iota^{-1}(\mathcal{V})$ is open, and $e = e^{-1} \in \mathcal{V}^{-1}$. Define

$$\mathcal{U} = \mathcal{V} \cap \mathcal{V}^{-1}.$$

Then $\mathcal{U}$ is an open set containing e , hence a neighborhood of e .

Step 3: verification. Taking inverses is an involution that preserves intersections, so

$$\mathcal{U}^{-1} = (\mathcal{V} \cap \mathcal{V}^{-1})^{-1} = \mathcal{V}^{-1} \cap (\mathcal{V}^{-1})^{-1} = \mathcal{V}^{-1} \cap \mathcal{V} = \mathcal{U}.$$

Explicitly: $g \in \mathcal{U}^{-1}$ means $g^{-1} \in \mathcal{U}$, that is, $g^{-1} \in \mathcal{V}$ and $g^{-1} \in \mathcal{V}^{-1}$, that is, $g \in \mathcal{V}^{-1}$ and $g \in \mathcal{V}$, that is, $g \in \mathcal{U}$. Finally, since $\mathcal{U} \subseteq \mathcal{V}$,

$$\mathcal{U} \cdot \mathcal{U} \subseteq \mathcal{V} \cdot \mathcal{V} \subseteq \mathcal{O}' \subseteq \mathcal{O}.$$

Thus $\mathcal{U}$ is a symmetric neighborhood of e with $\mathcal{U} \cdot \mathcal{U} \subseteq \mathcal{O}$, as required.

Iterating gives more. Define $\mathcal{U}^{(1)} = \mathcal{U}$ and, having chosen $\mathcal{U}^{(k)}$, let $\mathcal{U}^{(k+1)}$ be a symmetric neighborhood of e with $\mathcal{U}^{(k+1)} \cdot \mathcal{U}^{(k+1)} \subseteq \mathcal{U}^{(k)}$, which exists by what has just been proved. Since e belongs to each of these sets, an induction shows that the n -fold product of $\mathcal{U}^{(k)}$ with itself is contained in $\mathcal{O}$ whenever $n \leq 2^k$. Thus for each natural number n there is a symmetric neighborhood of e whose n -fold product lies in $\mathcal{O}$.

Problem 22.4. Show that the closure $\overline{H}$ of a subgroup H is a subgroup of $\mathcal{G}$.

Solution. Recall that a point x belongs to $\overline{H}$ if and only if every neighborhood of x has nonempty intersection with H ; equivalently, every open set containing x meets H .

Since H is a subgroup, $e \in H \subseteq \overline{H}$, so $\overline{H}$ is nonempty and contains the identity. It remains to show $\overline{H}$ is closed under the group operation and under inversion.

Closure under inversion. Let $\iota(g) = g^{-1}$. As observed in the preceding exercise, ι is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$, being continuous with $\iota \circ \iota = \text{Id}$. A homeomorphism carries closures onto closures: $\iota(\overline{A}) = \overline{\iota(A)}$ for every $A \subseteq \mathcal{G}$. Hence

$$(\overline{H})^{-1} = \iota(\overline{H}) = \overline{\iota(H)} = \overline{H^{-1}} = \overline{H},$$

the last equality because $H^{-1} = H$, H being a subgroup. In particular $x \in \overline{H}$ implies $x^{-1} \in \overline{H}$.

Alternatively, argue directly: let $x \in \overline{H}$ and let $\mathcal{W}$ be an open set containing x^{-1} . Then $\iota^{-1}(\mathcal{W}) = \mathcal{W}^{-1}$ is open and contains x , so it meets H , say $h \in \mathcal{W}^{-1} \cap H$. Then $h^{-1} \in \mathcal{W}$ and $h^{-1} \in H$, so $\mathcal{W}$ meets H . As $\mathcal{W}$ was arbitrary, $x^{-1} \in \overline{H}$.

Closure under multiplication. Let $x, y \in \overline{H}$ and let $\mathcal{W}$ be an open set containing $x \cdot y$. The group operation m is continuous and $m(x, y) = x \cdot y \in \mathcal{W}$, so $m^{-1}(\mathcal{W})$ is open in $\mathcal{G} \times \mathcal{G}$ and contains (x, y) . Since the rectangles $\mathcal{U} \times \mathcal{V}$ with $\mathcal{U}, \mathcal{V}$ open in $\mathcal{G}$ form a base for the product topology, there are open sets $\mathcal{U} \ni x$ and $\mathcal{V} \ni y$ with

$$\mathcal{U} \cdot \mathcal{V} \subseteq \mathcal{W}.$$

Because $x \in \overline{H}$ there is $h_1 \in \mathcal{U} \cap H$, and because $y \in \overline{H}$ there is $h_2 \in \mathcal{V} \cap H$. Then $h_1 \cdot h_2 \in \mathcal{U} \cdot \mathcal{V} \subseteq \mathcal{W}$, while $h_1 \cdot h_2 \in H$ since H is a subgroup. Hence $\mathcal{W} \cap H \neq \emptyset$. Since $\mathcal{W}$ was an arbitrary open set containing $x \cdot y$, we conclude $x \cdot y \in \overline{H}$.

Therefore $\overline{H}$ contains e , is closed under products and under inverses, and so is a subgroup of $\mathcal{G}$. Being a subgroup with the subspace topology, it is itself a topological group, as remarked in the text.

Problem 22.5. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be topological groups and $h: \mathcal{G}_1 \rightarrow \mathcal{G}_2$ a group homomorphism. Show that h is continuous if and only if it is continuous at the identity element of $\mathcal{G}_1$.

Solution. Write e_1 and e_2 for the identity elements of $\mathcal{G}_1$ and $\mathcal{G}_2$. Recall that h is continuous at a point g provided for every neighborhood $\mathcal{W}$ of $h(g)$ there is a neighborhood $\mathcal{N}$ of g with $h(\mathcal{N}) \subseteq \mathcal{W}$, and that h is continuous if and only if it is continuous at every point of $\mathcal{G}_1$.

First observe that $h(e_1) = e_2$. Indeed, since h is a homomorphism,

$$h(e_1) = h(e_1 \cdot e_1) = h(e_1) \cdot h(e_1),$$

and multiplying both sides on the left by $h(e_1)^{-1}$ gives $e_2 = h(e_1)$. Observe also that $h(g^{-1}) = h(g)^{-1}$ for every g , since $h(g) \cdot h(g^{-1}) = h(g \cdot g^{-1}) = h(e_1) = e_2$.

Necessity is immediate: if h is continuous, then it is continuous at each point of $\mathcal{G}_1$, in particular at e_1 . For the converse, assume h is continuous at e_1 , and let $g \in \mathcal{G}_1$ be arbitrary. Let $\mathcal{W}$ be a neighborhood of $h(g)$ in $\mathcal{G}_2$. By Exercise 1 applied to the group $\mathcal{G}_2$ (or directly, since left translation by $h(g)^{-1}$ is a homeomorphism of $\mathcal{G}_2$ carrying $h(g)$ to e_2), the set

$$\mathcal{W}_0 = h(g)^{-1} \cdot \mathcal{W}$$

is a neighborhood of e_2 . Since h is continuous at e_1 , there is a neighborhood $\mathcal{N}_0$ of e_1 with $h(\mathcal{N}_0) \subseteq \mathcal{W}_0$. Set $\mathcal{N} = g \cdot \mathcal{N}_0$. Since left translation by g is a homeomorphism of $\mathcal{G}_1$ carrying e_1 to g , the set $\mathcal{N}$ is a neighborhood of g . Now let $x \in \mathcal{N}$, say $x = g \cdot u$ with $u \in \mathcal{N}_0$. Then, because h is a homomorphism,

$$h(x) = h(g \cdot u) = h(g) \cdot h(u) \in h(g) \cdot \mathcal{W}_0 = h(g) \cdot h(g)^{-1} \cdot \mathcal{W} = \mathcal{W}.$$

Therefore $h(\mathcal{N}) \subseteq \mathcal{W}$. Since $\mathcal{W}$ was an arbitrary neighborhood of $h(g)$, h is continuous at g ; and since g was arbitrary, h is continuous at every point of $\mathcal{G}_1$, hence continuous.

The essential point is that a homomorphism intertwines translations: $h \circ L_g = L_{h(g)} \circ h$, and translations are homeomorphisms. Thus continuity at one point propagates to every point.

Problem 22.6. Use the Contraction Mapping Principle to prove Lemma 1.

Solution. Lemma 1 asserts: let E be a Banach space and the operator $C \in \mathcal{L}(E)$ have $\|C\| < 1$. Then $\text{Id} - C$ is invertible and

$$\|(\text{Id} - C)^{-1}\| \leq (1 - \|C\|)^{-1}.$$

Recall the Banach Contraction Principle of Section 10.3: if X is a complete metric space and $T: X \rightarrow X$ is a contraction, that is, there is a constant c with $0 \leq c < 1$ for which $\rho(T(u), T(v)) \leq c \cdot \rho(u, v)$ for all $u, v \in X$, then T has exactly one fixed point in X .

Since E is a Banach space, E with the metric induced by its norm is a complete metric space.

Step 1: $\text{Id} - C$ is one-to-one and onto. Fix $y \in E$ and define $T_y: E \rightarrow E$ by

$$T_y(x) = C(x) + y \quad \text{for } x \in E.$$

For $x, x' \in E$, by the linearity of C and the definition of the operator norm,

$$\|T_y(x) - T_y(x')\| = \|C(x) - C(x')\| = \|C(x - x')\| \leq \|C\| \cdot \|x - x'\|.$$

Since $c = \|C\| < 1$, the mapping T_y is a contraction of the complete metric space E . By the Contraction Mapping Principle, T_y has exactly one fixed point $x \in E$, that is, there is exactly one $x \in E$ with

$$x = C(x) + y, \quad \text{equivalently} \quad (\text{Id} - C)(x) = y.$$

Existence of the fixed point, for each $y \in E$, tells us $\text{Id} - C$ maps E onto E . Uniqueness tells us $\text{Id} - C$ is one-to-one: if $(\text{Id} - C)(x) = (\text{Id} - C)(x') = y$, then x and x' are both fixed points of T_y , so $x = x'$. Step 2: the inverse mapping is linear. The operator $\text{Id} - C$ belongs to $\mathcal{L}(E)$, being the difference of two continuous linear operators, and by Step 1 it is a one-to-one mapping of E onto E . Denote by $(\text{Id} - C)^{-1}: E \rightarrow E$ its set-theoretic inverse, which is properly defined by Step 1. It is linear: if $(\text{Id} - C)(x) = y$ and $(\text{Id} - C)(x') = y'$ and α, β are scalars, then by the linearity of $\text{Id} - C$ we have $(\text{Id} - C)(\alpha x + \beta x') = \alpha y + \beta y'$, so that $(\text{Id} - C)^{-1}(\alpha y + \beta y') = \alpha x + \beta x' = \alpha(\text{Id} - C)^{-1}(y) + \beta(\text{Id} - C)^{-1}(y')$. The text observes that for an operator in $\mathcal{L}(E)$ that is one-to-one and maps E onto E , the continuity of the inverse follows from the Open Mapping Theorem. Here no such appeal is needed: the estimate of Step 3 exhibits the inverse as a bounded linear operator, and a linear operator is continuous if and only if it is bounded.

Step 3: the estimate. Let $y \in E$ and let $x = (\text{Id} - C)^{-1}(y)$ be the unique solution of $(\text{Id} - C)(x) = y$ produced in Step 1, so that $x = C(x) + y$. By the triangle inequality and the definition of the operator norm,

$$\|x\| \leq \|C(x)\| + \|y\| \leq \|C\| \cdot \|x\| + \|y\|,$$

and hence

$$(1 - \|C\|) \cdot \|x\| \leq \|y\|.$$

Since $1 - \|C\| > 0$, this gives

$$\|(\text{Id} - C)^{-1}(y)\| = \|x\| \leq (1 - \|C\|)^{-1} \cdot \|y\| \quad \text{for all } y \in E.$$

Thus the linear mapping $(\text{Id} - C)^{-1}$ is bounded, hence continuous, and so belongs to $\mathcal{L}(E)$. Therefore $\text{Id} - C$ is invertible, that is, it belongs to $GL(E)$. Taking the supremum over $y \in E$ with $\|y\| \leq 1$ in the displayed inequality we conclude

$$\|(\text{Id} - C)^{-1}\| \leq (1 - \|C\|)^{-1}.$$

This is precisely inequality (2), and Lemma 1 is proved. Observe that this argument uses neither the completeness of $\mathcal{L}(E)$ nor the convergence of the Neumann series, and, by Step 3, not the Open Mapping Theorem either: the only completeness invoked is that of E itself, through the Banach Contraction Principle.

Problem 22.7. Use the completeness of $\mathcal{L}(E)$ to show that if $C \in \mathcal{L}(E)$ and $\|C\| < 1$, then $\sum_{k=0}^{\infty} C^k$ converges in $\mathcal{L}(E)$.

Solution. Here $C^0 = \text{Id}$ and C^k denotes the k -fold composition of C with itself, and convergence of the series means convergence, with respect to the operator norm, of the sequence of partial sums

$$S_n = \sum_{k=0}^n C^k \quad \text{for } n = 0, 1, 2, \dots$$

Since E is a Banach space, $\mathcal{L}(E)$ is a Banach space with respect to the operator norm; this is the completeness we are told to use. It therefore suffices to show that $\{S_n\}$ is a Cauchy sequence in $\mathcal{L}(E)$. Step 1: $\|C^k\| \leq \|C\|^k$ for every natural number k . This follows from inequality (1), $\|S \circ T\| \leq \|S\| \cdot \|T\|$, by induction. The case $k = 1$ is trivial. If $\|C^k\| \leq \|C\|^k$, then

$$\|C^{k+1}\| = \|C \circ C^k\| \leq \|C\| \cdot \|C^k\| \leq \|C\| \cdot \|C\|^k = \|C\|^{k+1}.$$

Step 2: the partial sums are Cauchy. Let $n > m \geq 0$. Since $\mathcal{L}(E)$ is a normed linear space, the triangle inequality and Step 1 give

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^n C^k \right\| \leq \sum_{k=m+1}^n \|C^k\| \leq \sum_{k=m+1}^n \|C\|^k \leq \sum_{k=m+1}^{\infty} \|C\|^k = \frac{\|C\|^{m+1}}{1 - \|C\|},$$

where the geometric series converges because $0 \leq \|C\| < 1$. Let $\varepsilon > 0$. Since $0 \leq \|C\| < 1$, we have $\|C\|^{m+1} \rightarrow 0$ as $m \rightarrow \infty$, so there is an index N such that

$$\frac{\|C\|^{m+1}}{1 - \|C\|} < \varepsilon \quad \text{for all } m \geq N.$$

Then $\|S_n - S_m\| < \varepsilon$ whenever $n > m \geq N$, and trivially also when $n = m$. Hence $\{S_n\}$ is a Cauchy sequence in $\mathcal{L}(E)$.

Step 3: conclusion. Because $\mathcal{L}(E)$ is complete, the Cauchy sequence $\{S_n\}$ converges to some operator $S \in \mathcal{L}(E)$. That is, $\sum_{k=0}^{\infty} C^k$ converges in $\mathcal{L}(E)$.

The same estimate applied to the partial sums themselves bounds S . Indeed, for each n , the triangle inequality and Step 1 give $\|S_n\| \leq \sum_{k=0}^n \|C^k\| \leq \sum_{k=0}^n \|C\|^k \leq (1 - \|C\|)^{-1}$, where the term $k = 0$ contributes $\|\text{Id}\| \leq 1$. Since $S_n \rightarrow S$ in $\mathcal{L}(E)$ and the norm is a continuous function on a normed linear space,

$$\|S\| = \lim_{n \rightarrow \infty} \|S_n\| \leq \frac{1}{1 - \|C\|}.$$

As in the proof of Lemma 1 in the text, the identity

$$(\text{Id} - C) \circ S_n = S_n \circ (\text{Id} - C) = \text{Id} - C^{n+1}$$

together with $\|C^{n+1}\| \leq \|C\|^{n+1} \rightarrow 0$ and the continuity of composition shows, upon letting $n \rightarrow \infty$, that $(\text{Id} - C) \circ S = S \circ (\text{Id} - C) = \text{Id}$. Thus $S = \sum_{k=0}^{\infty} C^k$ is the inverse of $\text{Id} - C$, and the bound just obtained is exactly the estimate (2).

22.2 Problems 22.8–22.14

Problem 22.8. Show that the set of $n \times n$ invertible real matrices with determinant 1 is a topological group if the group operation is matrix multiplication and the topology is entrywise continuity. This topological group is called the **special linear group** and denoted by $SL(n, \mathbb{R})$.

Solution. Identify the set $M_n(\mathbb{R})$ of all $n \times n$ real matrices with the Euclidean space $\mathbb{R}^{n^2}$ by listing the entries of a matrix as coordinates. The topology of entrywise continuity on $M_n(\mathbb{R})$ is exactly the product (equivalently, the Euclidean) topology on $\mathbb{R}^{n^2}$: a net or sequence of matrices converges if and only if each of the n^2 entry functions converges. Set

$$SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A = 1\},$$

given the subspace topology inherited from $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Observe that a matrix of determinant 1 is automatically invertible, so this is precisely the set described in the statement. Recall that, as defined at the beginning of this section, a topological group is a group together with a Hausdorff topology for which multiplication is continuous from the product and inversion is continuous. We therefore verify in turn that the topology is Hausdorff, that $SL(n, \mathbb{R})$ is a group, and that the two group operations are continuous.

The topology is Hausdorff. The Euclidean topology on $\mathbb{R}^{n^2}$ is induced by a metric and hence is Hausdorff, and every subspace of a Hausdorff space is Hausdorff. Thus $SL(n, \mathbb{R})$, with the topology of entrywise convergence, is a Hausdorff topological space.

It is a group. The determinant is multiplicative: $\det(AB) = \det A \cdot \det B$. Hence if $\det A = \det B = 1$, then $\det(AB) = 1$, so $SL(n, \mathbb{R})$ is closed under matrix multiplication, and multiplication of matrices is associative. The identity matrix Id has $\det \text{Id} = 1$, so $\text{Id} \in SL(n, \mathbb{R})$ and it is a unit element. Finally, if $\det A = 1$, then A is invertible and $\det(A^{-1}) \cdot \det A = \det(\text{Id}) = 1$, so $\det(A^{-1}) = 1$ and $A^{-1} \in SL(n, \mathbb{R})$. Therefore $SL(n, \mathbb{R})$ is a group under matrix multiplication; indeed it is a subgroup of the group $GL(n, \mathbb{R})$ of all invertible $n \times n$ real matrices.

Multiplication is continuous. Consider the map

$$m : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R}), \quad m(A, B) = AB.$$

For each pair of indices i, j ,

$$[m(A, B)]_{ij} = \sum_{k=1}^n a_{ik} b_{kj},$$

which is a polynomial in the $2n^2$ coordinates of the pair (A, B) , and therefore a continuous real-valued function on $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2}$. Since each entry function of m is continuous and the topology on the target is the topology of entrywise convergence, m is continuous. The restriction of a continuous mapping to a subspace is continuous, so

$$m|_{SL(n, \mathbb{R}) \times SL(n, \mathbb{R})} : SL(n, \mathbb{R}) \times SL(n, \mathbb{R}) \rightarrow M_n(\mathbb{R})$$

is continuous with respect to the product topology on the domain (which, as required in the definition of a topological group, is the product of the subspace topologies). By the first paragraph its image lies in $SL(n, \mathbb{R})$, and since $SL(n, \mathbb{R})$ carries the subspace topology, the induced mapping into $SL(n, \mathbb{R})$ is continuous.

Inversion is continuous. Let $A \in SL(n, \mathbb{R})$. By Cramer's rule,

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A) = \operatorname{adj}(A),$$

since $\det A = 1$, where $\operatorname{adj}(A)$ is the adjugate matrix, whose (i, j) entry is the cofactor

$$[\operatorname{adj}(A)]_{ij} = (-1)^{i+j} \det(A(j \mid i)),$$

$A(j \mid i)$ denoting the $(n-1) \times (n-1)$ matrix obtained from A by deleting row j and column i . Each such determinant is a polynomial in the entries of A , hence each entry of $\operatorname{adj}(A)$ is a continuous function of the entries of A . Therefore $A \mapsto \operatorname{adj}(A)$ is continuous from $M_n(\mathbb{R})$ to $M_n(\mathbb{R})$, and its restriction $A \mapsto A^{-1}$ is continuous from $SL(n, \mathbb{R})$ into $M_n(\mathbb{R})$. Again the image lies in $SL(n, \mathbb{R})$, which has the subspace topology, so inversion is continuous as a mapping of $SL(n, \mathbb{R})$ into itself.

Thus $SL(n, \mathbb{R})$ is a group which is a Hausdorff topological space for which multiplication $(A, B) \mapsto AB$ is continuous from $SL(n, \mathbb{R}) \times SL(n, \mathbb{R})$ to $SL(n, \mathbb{R})$ and inversion $A \mapsto A^{-1}$ is continuous from $SL(n, \mathbb{R})$ to $SL(n, \mathbb{R})$. By definition, $SL(n, \mathbb{R})$ is a topological group.

Remark. Since the determinant is a polynomial in the entries, it is continuous on $M_n(\mathbb{R})$, and therefore $SL(n, \mathbb{R}) = \det^{-1}(\{1\})$ is a closed subset of $M_n(\mathbb{R})$. In particular $SL(n, \mathbb{R})$ is a closed subgroup of $GL(n, \mathbb{R})$. Note also that $SL(n, \mathbb{R})$ is not compact for $n \geq 2$, since it contains the unbounded family of matrices that agree with the identity except for a single off-diagonal entry, which is arbitrary.

Problem 22.9. Let H be a Hilbert space. Show that an operator in $GL(H)$ preserves the norm if and only if it preserves the inner product.

Solution. Let $T \in GL(H)$. Recall the two properties in question. We say T **preserves the norm** provided

$$\|Tx\| = \|x\| \quad \text{for all } x \in H,$$

and T **preserves the inner product** provided

$$\langle Tx, Ty \rangle = \langle x, y \rangle \quad \text{for all } x, y \in H.$$

The operators of $GL(H)$ that preserve the inner product are exactly the members of the orthogonal group $\mathcal{O}(H)$, as defined at the end of this section. As throughout the book, the inner product on H is real valued, symmetric and bilinear, and $\|z\| = \sqrt{\langle z, z \rangle}$; the complex case is treated in a final paragraph.

Preservation of the inner product implies preservation of the norm. Suppose $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in H$. Taking $y = x$ and using the definition $\|z\|^2 = \langle z, z \rangle$ of the norm induced by the inner product,

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, x \rangle = \|x\|^2.$$

Since norms are nonnegative, $\|Tx\| = \|x\|$ for all $x \in H$.

Preservation of the norm implies preservation of the inner product. Suppose $\|Tz\| = \|z\|$ for all $z \in H$. Let $x, y \in H$. Since T is linear, $T(x+y) = Tx + Ty$, and therefore, expanding the square of the norm by bilinearity and symmetry of the inner product,

$$\begin{aligned}\|x\|^2 + 2\langle x, y \rangle + \|y\|^2 &= \|x + y\|^2 = \|T(x + y)\|^2 = \|Tx + Ty\|^2 \\ &= \|Tx\|^2 + 2\langle Tx, Ty \rangle + \|Ty\|^2.\end{aligned}$$

By hypothesis $\|Tx\| = \|x\|$ and $\|Ty\| = \|y\|$, so the first and third terms on each side of this identity agree. Cancelling them and dividing by 2 leaves

$$\langle Tx, Ty \rangle = \langle x, y \rangle.$$

As x and y were arbitrary, T preserves the inner product.

Equivalently, the second implication is an instance of the polarization identity

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2),$$

valid in a real Hilbert space: applying it to the pair Tx, Ty and using the linearity of T together with $\|T(x \pm y)\| = \|x \pm y\|$ gives $\langle Tx, Ty \rangle = \langle x, y \rangle$ at once. If H is a complex Hilbert space, the same argument goes through using the complex polarization identity

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2,$$

valid when the inner product is taken linear in its first variable and conjugate linear in its second. Each term on the right is determined by the norm, and a complex-linear norm-preserving T has $\|Tx + i^k Ty\| = \|T(x + i^k y)\| = \|x + i^k y\|$ for $0 \leq k \leq 3$; substituting into the identity gives $\langle Tx, Ty \rangle = \langle x, y \rangle$.

Remark. Invertibility of T was used nowhere in the argument: any bounded linear operator on H preserves the norm if and only if it preserves the inner product. What invertibility contributes is the converse-type statement that a norm-preserving operator that is in addition onto is a member of $GL(H)$: a norm-preserving linear operator is one-to-one, since $Tx = 0$ forces $\|x\| = \|Tx\| = 0$, and if it is onto then its inverse is continuous by the Open Mapping Theorem, so it belongs to $GL(H)$ and hence to $\mathcal{O}(H)$.

Problem 22.10. Consider $\mathbb{R}^n$ with the Euclidean inner product and norm. Characterize those $n \times n$ matrices that represent orthogonal operators with respect to an orthonormal basis.

Solution. The answer is: they are exactly the **orthogonal matrices**, that is, the matrices A for which

$$A^t A = \text{Id},$$

A^t denoting the transpose of A . Equivalently, A is invertible with $A^{-1} = A^t$; equivalently, $AA^t = \text{Id}$; equivalently, the columns of A form an orthonormal basis of $\mathbb{R}^n$; equivalently, the rows of A form an orthonormal basis of $\mathbb{R}^n$. We now prove this.

Fix an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$, so that $\langle e_j, e_k \rangle = \delta_{jk}$, where δ_{jk} is 1 if $j = k$ and 0 otherwise. Let T be a linear operator on $\mathbb{R}^n$ and let $A = [a_{ij}]$ be its matrix with respect to this basis, that is,

$$Te_j = \sum_{i=1}^n a_{ij} e_i \quad \text{for } 1 \leq j \leq n.$$

Recall from the discussion preceding these problems that T is an **orthogonal operator**, that is, $T \in \mathcal{O}(\mathbb{R}^n)$, provided T is invertible and leaves the inner product invariant: $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x, y .

Step 1: computation of $\langle Te_j, Te_k \rangle$. Using bilinearity of the inner product and orthonormality of the basis,

$$\langle Te_j, Te_k \rangle = \left\langle \sum_{i=1}^n a_{ij} e_i, \sum_{l=1}^n a_{lk} e_l \right\rangle = \sum_{i=1}^n \sum_{l=1}^n a_{ij} a_{lk} \langle e_i, e_l \rangle = \sum_{i=1}^n a_{ij} a_{ik} = [A^t A]_{jk}.$$

Step 2: T preserves the inner product if and only if $A^t A = \text{Id}$. If T preserves the inner product, then for all j, k ,

$$[A^t A]_{jk} = \langle Te_j, Te_k \rangle = \langle e_j, e_k \rangle = \delta_{jk},$$

so $A^t A = \text{Id}$. Conversely, suppose $A^t A = \text{Id}$, so that $\langle Te_j, Te_k \rangle = \delta_{jk}$ for all j, k by Step 1. Let $x = \sum_j x_j e_j$ and $y = \sum_k y_k e_k$ be arbitrary vectors of $\mathbb{R}^n$. By bilinearity and the linearity of T ,

$$\langle Tx, Ty \rangle = \sum_{j=1}^n \sum_{k=1}^n x_j y_k \langle Te_j, Te_k \rangle = \sum_{j=1}^n \sum_{k=1}^n x_j y_k \delta_{jk} = \sum_{j=1}^n x_j y_j = \langle x, y \rangle,$$

the last equality again because $\{e_j\}$ is orthonormal. Thus T preserves the inner product.

Step 3: the condition $A^t A = \text{Id}$ already forces invertibility. Suppose $A^t A = \text{Id}$. Then $\det(A^t) \det(A) = 1$, and since $\det(A^t) = \det(A)$ we get $(\det A)^2 = 1$, so $\det A = \pm 1 \neq 0$ and A is invertible. Multiplying $A^t A = \text{Id}$ on the right by A^{-1} gives $A^t = A^{-1}$, and hence also $AA^t = AA^{-1} = \text{Id}$. (Alternatively, and without determinants: $A^t A = \text{Id}$ says T preserves the inner product, hence the norm, hence is one-to-one; a one-to-one linear operator on the finite-dimensional space $\mathbb{R}^n$ is onto, and every linear operator on $\mathbb{R}^n$ is continuous, so $T \in GL(n, \mathbb{R})$.)

Combining the three steps: T is an orthogonal operator if and only if its matrix A with respect to an orthonormal basis satisfies $A^t A = \text{Id}$.

Finally we record the reformulations. Writing $A_1, \dots, A_n$ for the columns of A , viewed as vectors of $\mathbb{R}^n$, Step 1 shows $[A^t A]_{jk} = \langle A_j, A_k \rangle$ in the Euclidean inner product; hence $A^t A = \text{Id}$ exactly when the columns of A are orthonormal, and, being n orthonormal, hence linearly independent, vectors in an n -dimensional space, they then form an orthonormal basis. Since $A^t A = \text{Id}$ is equivalent to $AA^t = \text{Id}$ by Step 3, and $AA^t = (A^t)^t A^t$, the same statement holds for the rows.

Remark. The condition $\det A = \pm 1$ is necessary but by no means sufficient: the matrix with rows $(1, 1)$ and $(0, 1)$ has determinant 1 but does not preserve the Euclidean norm. The orthogonal matrices with $\det A = 1$ form the intersection $\mathcal{O}(\mathbb{R}^n) \cap SL(n, \mathbb{R})$, the special orthogonal group.

Problem 22.11. Show that $GL(E)$ is open in $\mathcal{L}(E)$.

Solution. Here E is a Banach space, $\mathcal{L}(E)$ is the Banach space of continuous linear operators on E with the operator norm, and $GL(E)$ is the collection of invertible operators in $\mathcal{L}(E)$; by the Open Mapping Theorem an operator in $\mathcal{L}(E)$ is invertible precisely when it is one-to-one and onto, its inverse then automatically being continuous.

If $E = \{0\}$, then $\mathcal{L}(E) = \{0\} = GL(E)$ and there is nothing to prove. So assume $E \neq \{0\}$.

Let $T \in GL(E)$. Then $T^{-1} \in \mathcal{L}(E)$ and $T^{-1} \neq 0$, since T^{-1} is onto and $E \neq \{0\}$; hence $\|T^{-1}\| > 0$. Define

$$r = \frac{1}{\|T^{-1}\|} > 0.$$

We claim that the open ball $B(T, r) = \{S \in \mathcal{L}(E) : \|S - T\| < r\}$ is contained in $GL(E)$; since T was an arbitrary member of $GL(E)$, this exhibits $GL(E)$ as a union of open balls in $\mathcal{L}(E)$ and therefore shows that $GL(E)$ is open.

Let $S \in \mathcal{L}(E)$ with $\|S - T\| < r$. Define

$$C = T^{-1} \circ (T - S) \in \mathcal{L}(E).$$

By the submultiplicativity of the operator norm, that is, inequality (1) of this section,

$$\|C\| \leq \|T^{-1}\| \cdot \|T - S\| < \|T^{-1}\| \cdot r = 1.$$

Since $\|C\| < 1$, Lemma 1 tells us that $\text{Id} - C$ is invertible, that is, $\text{Id} - C \in GL(E)$.
 On the other hand, since $T \circ T^{-1} = \text{Id}$ and composition is linear in its right argument,

$$T \circ (\text{Id} - C) = T \circ (\text{Id} - T^{-1} \circ (T - S)) = T - (T \circ T^{-1}) \circ (T - S) = T - (T - S) = S.$$

Thus $S = T \circ (\text{Id} - C)$ is the composition of two invertible operators in $\mathcal{L}(E)$. As observed in the text, $GL(E)$ is a group under composition; concretely, $(\text{Id} - C)^{-1} \circ T^{-1}$ belongs to $\mathcal{L}(E)$ and is a two-sided inverse of S . Hence $S \in GL(E)$.

Therefore $B(T, r) \subseteq GL(E)$, and $GL(E)$ is an open subset of $\mathcal{L}(E)$.

Remark. The proof yields a quantitative statement worth recording: if T is invertible and $\|S - T\| < 1/\|T^{-1}\|$, then S is invertible, and by the estimate (2) of Lemma 1,

$$\|S^{-1}\| \leq \|(\text{Id} - C)^{-1}\| \cdot \|T^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|T^{-1}\| \cdot \|T - S\|}.$$

So invertibility is stable under small perturbations, with a controlled bound on the norm of the perturbed inverse. This complements the continuity of inversion established in the proof of Theorem 2.

Problem 22.12. Show that the set of operators in $GL(E)$ comprising operators that are linear compact perturbations of the identity is a subgroup of $GL(E)$. It is denoted by $GL_c(E)$.

Solution. Recall the definition of Section 16.5, transferred verbatim from a Hilbert space to the Banach space E : an operator $K \in \mathcal{L}(E)$ is said to be **compact** provided $K(B)$ has compact closure with respect to the strong topology, where B is the closed unit ball of E ; equivalently, provided every bounded sequence $\{x_n\}$ in E has a subsequence $\{x_{n_j}\}$ for which $\{Kx_{n_j}\}$ converges in E . (The equivalence is the characterization of compactness in a metric space by sequential compactness, together with the completeness of E .) By a **linear compact perturbation of the identity** we mean an operator of the form $\text{Id} + K$ with $K \in \mathcal{L}(E)$ compact. Thus

$$GL_c(E) = \{T \in GL(E) : T = \text{Id} + K \text{ for some compact } K \in \mathcal{L}(E)\}.$$

We must show that $GL_c(E)$ contains the identity element Id of $GL(E)$ and is closed under composition and under inversion. We first isolate the two algebraic facts about compact operators that are needed.
Claim 1: if $K_1, K_2 \in \mathcal{L}(E)$ are compact, so is $K_1 + K_2$. Let $\{x_n\}$ be a bounded sequence in E . Since K_1 is compact, there is a subsequence $\{x_{n_j}\}$ for which $\{K_1x_{n_j}\}$ converges. The sequence $\{x_{n_j}\}$ is again bounded, so, since K_2 is compact, there is a further subsequence $\{x_{n_{j_l}}\}$ for which $\{K_2x_{n_{j_l}}\}$ converges. Along this further subsequence both $\{K_1x_{n_{j_l}}\}$ and $\{K_2x_{n_{j_l}}\}$ converge, and therefore so does $\{(K_1 + K_2)x_{n_{j_l}}\}$. Hence $K_1 + K_2$ is compact.

Claim 2: if $K \in \mathcal{L}(E)$ is compact and $A \in \mathcal{L}(E)$, then $A \circ K$ and $K \circ A$ are compact. Let $\{x_n\}$ be a bounded sequence. Since K is compact, choose a subsequence for which $Kx_{n_j} \rightarrow y$ for some $y \in E$. Since A is continuous, $A(Kx_{n_j}) \rightarrow Ay$, so $\{(A \circ K)x_{n_j}\}$ converges and $A \circ K$ is compact. For $K \circ A$: since A is bounded, $\{Ax_n\}$ is a bounded sequence, so by the compactness of K there is a subsequence for which $\{K(Ax_{n_j})\}$ converges. Hence $K \circ A$ is compact.

The identity belongs to $GL_c(E)$. The zero operator is compact, since $0(B) = \{0\}$ is compact, and $\text{Id} = \text{Id} + 0$ is invertible. Hence $\text{Id} \in GL_c(E)$, and in particular $GL_c(E)$ is nonempty.

Closure under composition. Let $T_1 = \text{Id} + K_1$ and $T_2 = \text{Id} + K_2$ belong to $GL_c(E)$, with K_1, K_2 compact. Then

$$T_1 \circ T_2 = (\text{Id} + K_1) \circ (\text{Id} + K_2) = \text{Id} + K_2 + K_1 + K_1 \circ K_2 = \text{Id} + K,$$

where $K = K_1 + K_2 + K_1 \circ K_2$. By Claim 2, $K_1 \circ K_2$ is compact, and then two applications of Claim 1 show K is compact. Moreover $T_1 \circ T_2 \in GL(E)$, since $GL(E)$ is a group under composition. Hence $T_1 \circ T_2 \in GL_c(E)$.

Closure under inversion. Let $T = \text{Id} + K \in GL_c(E)$, with K compact. Then $T^{-1} \in GL(E)$, and

applying T^{-1} on the left to the identity $T = \text{Id} + K$ gives

$$\text{Id} = T^{-1} \circ T = T^{-1} + T^{-1} \circ K,$$

so that

$$T^{-1} = \text{Id} + K', \quad K' = -T^{-1} \circ K.$$

Since $T^{-1} \in \mathcal{L}(E)$ and K is compact, Claim 2 shows $T^{-1} \circ K$ is compact, and a compact operator multiplied by the scalar -1 is compact (this is the special case $A = -\text{Id}$ of Claim 2). Hence K' is compact and $T^{-1} \in GL_c(E)$.

Since $GL_c(E)$ is a nonempty subset of the group $GL(E)$ that is closed under the group operation and under inversion, it is a subgroup of $GL(E)$.

Remark. By Problem 11, $GL_c(E)$ is the intersection of the open set $GL(E)$ with the affine subspace $\text{Id} + \mathcal{K}(E)$, where $\mathcal{K}(E)$ denotes the compact operators; Claims 1 and 2 say precisely that $\mathcal{K}(E)$ is a two-sided ideal in the algebra $\mathcal{L}(E)$. When $\dim E < \infty$ every operator is compact and $GL_c(E) = GL(E)$; when $\dim E = \infty$ the containment is proper. Indeed, the representation of an operator T in the form $\text{Id} + K$ is unique, since necessarily $K = T - \text{Id}$; so 2Id , which belongs to $GL(E)$, lies in $GL_c(E)$ if and only if Id is compact, and by Riesz's Theorem the identity operator fails to be compact when E is infinite dimensional.

Problem 22.13. Show that the adjoint of a representation also is a representation.

Solution. Let $\mathcal{G}$ be a topological group with unit element e , let E be a Banach space, and let $\pi : \mathcal{G} \rightarrow GL(E)$ be a representation of $\mathcal{G}$ on E , that is, a group homomorphism: $\pi(g_1 \cdot g_2) = \pi(g_1) \circ \pi(g_2)$ for all $g_1, g_2 \in \mathcal{G}$. The adjoint representation π^* is defined for $g \in \mathcal{G}$ by formula (4) of this section, namely

$$\pi^*(g)\psi = \psi \circ \pi(g^{-1}) \quad \text{for all } \psi \in E^*.$$

To show that π^* is a representation of $\mathcal{G}$ on E^* we must show that for each $g \in \mathcal{G}$ the mapping $\pi^*(g)$ is a well-defined member of $GL(E^*)$, and that $\pi^* : \mathcal{G} \rightarrow GL(E^*)$ is a group homomorphism.

Step 1: $\pi(e) = \text{Id}_E$ and $\pi(g^{-1}) = [\pi(g)]^{-1}$. Since π is a homomorphism, $\pi(e) = \pi(e \cdot e) = \pi(e) \circ \pi(e)$. As $\pi(e) \in GL(E)$ it is invertible; composing on the left with $[\pi(e)]^{-1}$ gives $\text{Id}_E = \pi(e)$. Consequently, for $g \in \mathcal{G}$,

$$\pi(g) \circ \pi(g^{-1}) = \pi(g \cdot g^{-1}) = \pi(e) = \text{Id}_E, \quad \pi(g^{-1}) \circ \pi(g) = \pi(e) = \text{Id}_E,$$

so $\pi(g^{-1}) = [\pi(g)]^{-1}$.

Step 2: $\pi^*(g)$ maps E^* into E^* and belongs to $\mathcal{L}(E^*)$. Fix $g \in \mathcal{G}$ and write $S = \pi(g^{-1}) \in \mathcal{L}(E)$. For $\psi \in E^*$, the mapping $\pi^*(g)\psi = \psi \circ S$ is the composition of the bounded linear functional ψ with the bounded linear operator S , hence is linear, and for $x \in E$,

$$|(\psi \circ S)(x)| \leq \|\psi\| \cdot \|Sx\| \leq \|\psi\| \cdot \|S\| \cdot \|x\|,$$

so $\psi \circ S$ is bounded, with $\|\pi^*(g)\psi\| \leq \|S\| \cdot \|\psi\|$. Thus $\pi^*(g)\psi \in E^*$ and $\pi^*(g)$ maps E^* into E^* . It is linear in ψ : for $\psi_1, \psi_2 \in E^*$ and scalars α, β , and every $x \in E$,

$$[(\alpha\psi_1 + \beta\psi_2) \circ S](x) = \alpha\psi_1(Sx) + \beta\psi_2(Sx) = [\alpha(\psi_1 \circ S) + \beta(\psi_2 \circ S)](x),$$

so $\pi^*(g)(\alpha\psi_1 + \beta\psi_2) = \alpha\pi^*(g)\psi_1 + \beta\pi^*(g)\psi_2$. The displayed norm inequality shows $\pi^*(g)$ is bounded, with $\|\pi^*(g)\| \leq \|\pi(g^{-1})\|$. Hence $\pi^*(g) \in \mathcal{L}(E^*)$.

Step 3: π^* is a homomorphism. Let $g_1, g_2 \in \mathcal{G}$ and $\psi \in E^*$. Using (4) twice, the associativity of composition, and the homomorphism property of π ,

$$\begin{aligned} \pi^*(g_1)(\pi^*(g_2)\psi) &= (\psi \circ \pi(g_2^{-1})) \circ \pi(g_1^{-1}) = \psi \circ (\pi(g_2^{-1}) \circ \pi(g_1^{-1})) \\ &= \psi \circ \pi(g_2^{-1} \cdot g_1^{-1}) = \psi \circ \pi((g_1 \cdot g_2)^{-1}) = \pi^*(g_1 \cdot g_2)\psi. \end{aligned}$$

Since $\psi \in E^*$ was arbitrary,

$$\pi^*(g_1 \cdot g_2) = \pi^*(g_1) \circ \pi^*(g_2).$$

Step 4: $\pi^*(g) \in GL(E^*)$. By Step 1 and (4), $\pi^*(e)\psi = \psi \circ \pi(e) = \psi \circ \text{Id}_E = \psi$, so $\pi^*(e) = \text{Id}_{E^*}$. Therefore, by Step 3, for each $g \in \mathcal{G}$,

$$\pi^*(g) \circ \pi^*(g^{-1}) = \pi^*(g \cdot g^{-1}) = \pi^*(e) = \text{Id}_{E^*},$$

and likewise $\pi^*(g^{-1}) \circ \pi^*(g) = \text{Id}_{E^*}$. Hence $\pi^*(g)$ is a one-to-one and onto member of $\mathcal{L}(E^*)$ whose inverse $\pi^*(g^{-1})$ is also in $\mathcal{L}(E^*)$; that is, $\pi^*(g) \in GL(E^*)$.

Steps 2 through 4 show that π^* is a mapping of $\mathcal{G}$ into $GL(E^*)$ which is a group homomorphism. Since E^* is a Banach space, π^* is by definition a representation of $\mathcal{G}$ on E^* .

Remark. No continuity assumption on π was used, and none is needed: as the text observes, a representation is a purely algebraic object, and continuity assumptions are imposed only as they are required in a particular context. The natural topology in which π^* is well behaved is the weak-* topology on E^* : for fixed g , $\pi^*(g)$ is weak-* continuous, since for each $x \in E$ the mapping $\psi \mapsto [\pi^*(g)\psi](x) = \psi(\pi(g^{-1})x)$ is weak-* continuous, being evaluation at the point $\pi(g^{-1})x$ of E .

Problem 22.14. Show that a probability functional has norm 1.

Solution. Let $\mathcal{G}$ be a compact group and $C(\mathcal{G})$ the Banach space of continuous real-valued functions on $\mathcal{G}$, normed by the maximum norm $\|f\|_{\max} = \max_{x \in \mathcal{G}} |f(x)|$. Recall the definition: a functional $\psi \in [C(\mathcal{G})]^*$ is a **probability functional** provided it takes the value 1 at the constant function **1**, that is, $\psi(\mathbf{1}) = 1$, and it is **positive**, meaning that if $f \in C(\mathcal{G})$ and $f \geq 0$ on $\mathcal{G}$, then $\psi(f) \geq 0$. We must show that

$$\|\psi\| = \sup\{|\psi(f)| : f \in C(\mathcal{G}), \|f\|_{\max} \leq 1\} = 1.$$

The norm is at most 1. Let $f \in C(\mathcal{G})$ with $\|f\|_{\max} \leq 1$. Then $-1 \leq f(x) \leq 1$ for every $x \in \mathcal{G}$, that is, the two continuous functions

$$\mathbf{1} - f \quad \text{and} \quad \mathbf{1} + f$$

are nonnegative on $\mathcal{G}$. By the positivity of ψ ,

$$\psi(\mathbf{1} - f) \geq 0 \quad \text{and} \quad \psi(\mathbf{1} + f) \geq 0.$$

By the linearity of ψ and the normalization $\psi(\mathbf{1}) = 1$, these read

$$1 - \psi(f) \geq 0 \quad \text{and} \quad 1 + \psi(f) \geq 0,$$

that is,

$$-1 \leq \psi(f) \leq 1, \quad \text{so} \quad |\psi(f)| \leq 1.$$

Taking the supremum over all such f gives $\|\psi\| \leq 1$. Observe that boundedness of ψ was not used in this argument, only its linearity: a linear functional on $C(\mathcal{G})$ that is positive and takes the value 1 at **1** is automatically bounded, with norm at most 1, and so automatically belongs to $[C(\mathcal{G})]^*$.

The norm is at least 1. A group contains its unit element e , so $\mathcal{G}$ is nonempty and therefore the constant function **1** satisfies

$$\|\mathbf{1}\|_{\max} = 1.$$

Hence **1** is a competitor in the supremum defining $\|\psi\|$, and

$$\|\psi\| \geq |\psi(\mathbf{1})| = 1.$$

Combining the two inequalities, $\|\psi\| = 1$. Moreover the supremum defining $\|\psi\|$ is attained, at $f = \mathbf{1}$.

Remark. This is exactly the computation performed inside the proof of Kakutani's Theorem (Theorem 6), where the estimate $-1 = \psi(-\mathbf{1}) \leq \psi(f) \leq \psi(\mathbf{1}) = 1$ for $\|f\|_{\max} \leq 1$ is used to place the set K^* of probability functionals inside the closed unit ball of $[C(\mathcal{G})]^*$, so that Alaoglu's Theorem may be applied to it. The conclusion $\|\psi\| = 1$ is the functional-analytic counterpart of the statement, for the Riesz representation of ψ as a Radon measure μ on $\mathcal{G}$, that μ is a probability measure: $\mu(\mathcal{G}) = \|\mu\| = 1$.

22.3 Problems 22.15–22.21

Problem 22.15. Let E be a reflexive Banach space and K^* a convex subset of E^* that is closed with respect to the metric induced by the norm. Show that K^* is weak-* closed. On the other hand, show that if E is not reflexive, then the image of the closed unit ball of E under the natural embedding of E in $(E^*)^* = E^{**}$ is a subset of E^{**} that is convex, closed and bounded with respect to the metric induced by the norm but is not weak-* closed.

Solution. Throughout, $J: E \rightarrow E^{**}$ denotes the natural embedding, $J(x)[\psi] = \psi(x)$ for $x \in E$ and $\psi \in E^*$. By Corollary 9 of Chapter 14, J is an isometry. The weak-* topology on E^* is the topology $\sigma(E^*, E)$, that is, the weakest topology on E^* for which each of the evaluation functionals $\psi \mapsto \psi(x)$, $x \in E$, is continuous.

Part 1: a norm-closed convex subset of the dual of a reflexive space is weak-* closed.

If K^* is empty it is weak-* closed, so assume K^* is nonempty. We show that $E^* \sim K^*$ is weak-* open. Let ψ_0 belong to $E^* \sim K^*$. The space E^* is a normed linear space and K^* is a nonempty strongly closed convex subset of it that omits ψ_0 . Therefore, by Corollary 25 of Chapter 14, applied with $X = E^*$, there is a functional $\Lambda \in (E^*)^* = E^{**}$ for which

$$\Lambda(\psi_0) < a := \inf_{\psi \in K^*} \Lambda(\psi).$$

Because E is reflexive, J maps E onto E^{**} , so there is a vector $x \in E$ with $\Lambda = J(x)$, that is,

$$\Lambda(\psi) = \psi(x) \quad \text{for all } \psi \in E^*.$$

Consider

$$\mathcal{N} = \{\psi \in E^* : \psi(x) < a\}.$$

The functional $\psi \mapsto \psi(x)$ is, by the very definition of the weak-* topology, continuous on E^* with respect to that topology; hence $\mathcal{N}$ is weak-* open. Moreover $\psi_0 \in \mathcal{N}$ since $\psi_0(x) = \Lambda(\psi_0) < a$, while $\mathcal{N} \cap K^* = \emptyset$ since $\psi(x) = \Lambda(\psi) \geq a$ for every $\psi \in K^*$. Thus every point of $E^* \sim K^*$ has a weak-* neighborhood contained in $E^* \sim K^*$, so $E^* \sim K^*$ is weak-* open and K^* is weak-* closed.

(Equivalently: by Proposition 6 of Chapter 14, E is reflexive if and only if the weak and the weak-* topologies on E^* coincide; and by Mazur's Theorem of Chapter 14 a convex subset of the normed linear space E^* is strongly closed if and only if it is weakly closed. Combining the two statements gives the conclusion at once.)

Part 2: the image of the closed unit ball when E is not reflexive.

Let B denote the closed unit ball of E and B^{**} the closed unit ball of E^{**} , and set $C = J(B)$.

C is convex. Indeed, B is convex and J is linear, so for $x, y \in B$ and $0 \leq t \leq 1$ we have $tJ(x) + (1-t)J(y) = J(tx + (1-t)y) \in J(B)$.

C is bounded with respect to the norm on E^{**} . Since J is an isometry, $\|J(x)\| = \|x\| \leq 1$ for $x \in B$; thus $C \subseteq B^{**}$.

C is closed with respect to the metric induced by the norm on E^{**} . Let Λ be a point of closure of C in the norm topology, and choose $x_n \in B$ with $\{J(x_n)\} \rightarrow \Lambda$ in norm. A norm convergent sequence is Cauchy, and since J is a linear isometry,

$$\|x_n - x_m\| = \|J(x_n) - J(x_m)\| \quad \text{for all } n, m,$$

so $\{x_n\}$ is a Cauchy sequence in E . As E is a Banach space, $\{x_n\} \rightarrow x$ for some $x \in E$, and $\|x\| = \lim_n \|x_n\| \leq 1$, so $x \in B$. Since J is an isometry it is continuous, so $\{J(x_n)\} \rightarrow J(x)$ in norm; by uniqueness of limits in a metric space, $\Lambda = J(x) \in C$. Hence C is norm closed.

C is not weak-* closed when E fails to be reflexive. By Goldstine's Theorem (Theorem 6 of Chapter 15), the weak-* closure of $J(B)$ in E^{**} is all of B^{**} . Hence if C were weak-* closed we would have $C = B^{**}$. We show this fails. Since E is not reflexive, $J(E)$ is a proper subset of E^{**} ; choose $\Lambda \in E^{**} \sim J(E)$. Then $\Lambda \neq 0$, since $0 = J(0)$ belongs to $J(E)$. Put

$$\Lambda' = \frac{\Lambda}{\|\Lambda\|}, \quad \text{so that } \|\Lambda'\| = 1 \text{ and } \Lambda' \in B^{**}.$$

If Λ' belonged to $C = J(B)$, say $\Lambda' = J(x)$ with $x \in B$, then by linearity of J we would have $\Lambda = \|\Lambda\| \cdot J(x) = J(\|\Lambda\|x) \in J(E)$, contradicting the choice of Λ . Therefore $\Lambda' \in B^{**} \sim C$, so $C \neq B^{**}$.

Since the weak-* closure of C is B^{**} and $C \neq B^{**}$, the set C is not weak-* closed. Thus $J(B)$ is a convex, norm closed, norm bounded subset of E^{**} that is not weak-* closed, which shows that the reflexivity hypothesis in Part 1 cannot be dropped.

Problem 22.16. Let $\mathcal{G}$ be a compact group, E a reflexive Banach space, and $\pi: \mathcal{G} \rightarrow \text{GL}(E)$ a representation. Suppose that for each $x \in E$, the mapping $g \mapsto \pi(g)x$ is continuous. Assume there is a nonempty strongly closed, bounded, convex subset K of E that is invariant with respect to π . Show that K contains a point that is fixed by π .

Solution. The idea is to average an arbitrary point of K over $\mathcal{G}$, using as the averaging device the invariant probability functional furnished by Kakutani's Theorem (Theorem 6 of this chapter), and to use reflexivity to realize the average as a point of E .

Step 1: the invariant mean. Let $\pi_r: \mathcal{G} \rightarrow \text{GL}(C(\mathcal{G}))$ be the regular representation of $\mathcal{G}$ on $C(\mathcal{G})$, that is,

$$[\pi_r(g)f](x) = f(g^{-1} \cdot x) \quad \text{for } f \in C(\mathcal{G}), x \in \mathcal{G}, g \in \mathcal{G}.$$

By Theorem 6 (Kakutani), there is a probability functional $\psi \in [C(\mathcal{G})]^*$ that is fixed under the adjoint action π_r^* , that is,

$$\psi(f) = \psi(\pi_r(g)f) \quad \text{for all } f \in C(\mathcal{G}) \text{ and } g \in \mathcal{G}. \quad (\text{i})$$

By definition a probability functional satisfies $\psi(1) = 1$ (here 1 denotes the constant function of value 1) and is positive: $\psi(f) \geq 0$ whenever $f \geq 0$ on $\mathcal{G}$. From positivity and linearity we get, for $f \in C(\mathcal{G})$ with $\|f\|_{\max} \leq 1$, that $1 - f \geq 0$ and $1 + f \geq 0$, so $\psi(f) \leq 1$ and $-\psi(f) \leq 1$; hence

$$|\psi(f)| \leq \|f\|_{\max} \quad \text{for all } f \in C(\mathcal{G}). \quad (\text{ii})$$

Step 2: construction of the average. Since K is nonempty, fix $x \in K$. Because K is bounded, there is an $M > 0$ with $\|y\| \leq M$ for all $y \in K$; and because K is invariant under π , $\pi(g)x \in K$ and hence

$$\|\pi(g)x\| \leq M \quad \text{for all } g \in \mathcal{G}. \quad (\text{iii})$$

For each $\phi \in E^*$ define $F_\phi: \mathcal{G} \rightarrow \mathbb{R}$ by

$$F_\phi(g) = \phi(\pi(g)x) \quad \text{for } g \in \mathcal{G}.$$

By hypothesis the mapping $g \mapsto \pi(g)x$ is continuous from $\mathcal{G}$ into E with the norm topology, and ϕ is continuous on E ; therefore F_ϕ is continuous, that is, $F_\phi \in C(\mathcal{G})$. Moreover, by (iii), $\|F_\phi\|_{\max} \leq M\|\phi\|$. Define $\Lambda: E^* \rightarrow \mathbb{R}$ by

$$\Lambda(\phi) = \psi(F_\phi) \quad \text{for } \phi \in E^*.$$

The mapping $\phi \mapsto F_\phi$ is linear from E^* into $C(\mathcal{G})$, and ψ is linear, so Λ is linear. By (ii),

$$|\Lambda(\phi)| \leq \|F_\phi\|_{\max} \leq M\|\phi\|,$$

so Λ is bounded and thus $\Lambda \in (E^*)^* = E^{**}$. Since E is reflexive, the natural embedding $J: E \rightarrow E^{**}$ is onto, so there is a vector $\bar{x} \in E$ with $J(\bar{x}) = \Lambda$, that is,

$$\phi(\bar{x}) = \psi(g \mapsto \phi(\pi(g)x)) \quad \text{for all } \phi \in E^*. \quad (\text{iv})$$

Step 3: the average lies in K . Suppose, to the contrary, that $\bar{x} \notin K$. The set K is a nonempty strongly closed convex subset of the normed linear space E , so by Corollary 25 of Chapter 14 there is a functional $\phi \in E^*$ for which

$$\phi(\bar{x}) < a := \inf_{y \in K} \phi(y).$$

For every $g \in \mathcal{G}$ the vector $\pi(g)x$ lies in K , so $F_\phi(g) = \phi(\pi(g)x) \geq a$; that is, $F_\phi - a \cdot 1 \geq 0$ on $\mathcal{G}$. By the positivity of ψ together with $\psi(1) = 1$,

$$\psi(F_\phi) - a = \psi(F_\phi - a \cdot 1) \geq 0,$$

so by (iv), $\phi(\bar{x}) = \psi(F_\phi) \geq a$. This contradicts $\phi(\bar{x}) < a$. Therefore $\bar{x} \in K$.

Step 4: the average is fixed by π . Let $h \in \mathcal{G}$ and $\phi \in E^*$. Since $\pi(h)$ is a bounded linear operator on E , the composition $\phi \circ \pi(h)$ belongs to E^* . Applying (iv) to this functional and using that π is a group homomorphism,

$$\phi(\pi(h)\bar{x}) = [\phi \circ \pi(h)](\bar{x}) = \psi(g \mapsto \phi(\pi(h)\pi(g)x)) = \psi(g \mapsto \phi(\pi(h \cdot g)x)).$$

Now observe that, with F_ϕ as above,

$$F_\phi(h \cdot g) = [\pi_r(h^{-1})F_\phi](g) \quad \text{for all } g \in \mathcal{G},$$

since $[\pi_r(h^{-1})F_\phi](g) = F_\phi((h^{-1})^{-1} \cdot g) = F_\phi(h \cdot g)$. Hence the function $g \mapsto \phi(\pi(h \cdot g)x)$ is exactly $\pi_r(h^{-1})F_\phi$, and by the invariance (i) of ψ ,

$$\phi(\pi(h)\bar{x}) = \psi(\pi_r(h^{-1})F_\phi) = \psi(F_\phi) = \phi(\bar{x}).$$

This holds for every $\phi \in E^*$. Since the dual space of a normed linear space separates its points (a consequence of Theorem 7 of Chapter 14: for each nonzero $z \in E$ there is $\phi \in E^*$ with $\phi(z) = \|z\| \neq 0$), we conclude $\pi(h)\bar{x} = \bar{x}$.

As $h \in \mathcal{G}$ was arbitrary, $\bar{x}$ is a point of K fixed by π .

Problem 22.17. Let $\mathcal{G}$ be a topological group, E be a Banach space, and $\pi: \mathcal{G} \rightarrow \text{GL}(E)$ a representation. For $x \in E$, show that the mapping $g \mapsto \pi(g)x$ is continuous if and only if it is continuous at e .

Solution. Here e is the unit element of $\mathcal{G}$, and E carries the norm topology. Fix $x \in E$ and define

$$\Phi: \mathcal{G} \rightarrow E, \quad \Phi(g) = \pi(g)x.$$

First observe that $\pi(e) = \text{Id}$. Indeed, π is a group homomorphism into $\text{GL}(E)$, so $\pi(e) = \pi(e \cdot e) = \pi(e) \circ \pi(e)$; composing on the left with the inverse operator $\pi(e)^{-1}$, which exists since $\pi(e) \in \text{GL}(E)$, gives $\text{Id} = \pi(e)$. Consequently $\Phi(e) = x$.

One implication is immediate: if Φ is continuous on $\mathcal{G}$, then in particular it is continuous at e .

Conversely, assume Φ is continuous at e ; we show it is continuous at each $g_0 \in \mathcal{G}$. Let $\epsilon > 0$. Since $\pi(g_0)$ belongs to $\text{GL}(E) \subseteq \mathcal{L}(E)$, it is a bounded operator; set

$$c = \max\{1, \|\pi(g_0)\|\} \geq 1,$$

so that c is finite and positive. By the assumed continuity of Φ at e and because $\Phi(e) = x$, there is a neighborhood $\mathcal{U}$ of e for which

$$\|\pi(h)x - x\| < \epsilon/c \quad \text{for all } h \in \mathcal{U}. \quad (\text{i})$$

Consider the set $g_0 \cdot \mathcal{U} = \{g_0 h : h \in \mathcal{U}\}$. Since $\mathcal{G}$ is a topological group, left multiplication by g_0 is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$ (it is continuous, being a restriction of the continuous group operation, and its inverse is left multiplication by g_0^{-1} , which is continuous for the same reason). Therefore $g_0 \cdot \mathcal{U}$ is a neighborhood of $g_0 \cdot e = g_0$. This is the content of Problem 1 of Section 22.1, which asserts that if $\mathcal{T}_e$ is a base for the topology at e , then $\{g \cdot \mathcal{O} \mid \mathcal{O} \in \mathcal{T}_e\}$ is a base for the topology at g .

Now let g belong to $g_0 \cdot \mathcal{U}$ and put $h = g_0^{-1}g$, so that $h \in \mathcal{U}$ and $g = g_0 h$. Since π is a group homomorphism, $\pi(g) = \pi(g_0) \circ \pi(h)$, and hence, by the linearity of $\pi(g_0)$,

$$\pi(g)x - \pi(g_0)x = \pi(g_0)[\pi(h)x] - \pi(g_0)x = \pi(g_0)[\pi(h)x - x].$$

Therefore, using the boundedness of $\pi(g_0)$ and then (i),

$$\|\pi(g)x - \pi(g_0)x\| \leq \|\pi(g_0)\| \cdot \|\pi(h)x - x\| \leq c \cdot \frac{\epsilon}{c} = \epsilon.$$

Thus for every $\epsilon > 0$ there is a neighborhood $g_0 \cdot \mathcal{U}$ of g_0 on which $\|\Phi(g) - \Phi(g_0)\| \leq \epsilon$. Hence Φ is continuous at g_0 . Since $g_0 \in \mathcal{G}$ was arbitrary, Φ is continuous on $\mathcal{G}$.

Observe that this is precisely the reduction used at the beginning of the proof of Lemma 5, where the continuity of $g \mapsto \pi(g)f$ for the regular representation of a compact group on $C(\mathcal{G})$ is verified only at the identity.

Problem 22.18. Suppose $\mathcal{G}$ is a topological group, X a topological space, and $\varphi: \mathcal{G} \times X \rightarrow X$ a mapping. For $g \in \mathcal{G}$, define the mapping $\pi(g): X \rightarrow X$ by $\pi(g)x = \varphi(g, x)$ for all $x \in X$. What properties must φ possess in order for π to be a representation of $\mathcal{G}$ on $C(X)$? What further properties must φ possess in order that for each $x \in X$, the mapping $g \mapsto \pi(g)x$ is continuous?

Solution. Since $C(X)$ is to be a Banach space under the maximum norm, we take X to be a compact Hausdorff space, as in the case $X = \mathcal{G}$ that produces the regular representation. The mappings $\pi(g)$ act on X , not on $C(X)$; they induce operators on $C(X)$ by composition, and the representation meant in the problem is

$$\Pi(g)f = f \circ \pi(g^{-1}), \quad \text{that is,} \quad [\Pi(g)f](x) = f(\varphi(g^{-1}, x)) \quad \text{for } f \in C(X), x \in X.$$

When $X = \mathcal{G}$ and $\varphi(g, g') = g \cdot g'$ this is exactly the regular representation $[\pi(g)f](x) = f(g^{-1} \cdot x)$ of Section 22.2.

Part 1: what is needed for Π to be a representation of $\mathcal{G}$ on $C(X)$.

The answer is: φ must be an action of the group $\mathcal{G}$ on X by continuous mappings, that is,

- (A1) $\varphi(e, x) = x$ for all $x \in X$, where e is the unit of $\mathcal{G}$;
- (A2) $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 \cdot g_2, x)$ for all $g_1, g_2 \in \mathcal{G}$ and $x \in X$;
- (A3) for each $g \in \mathcal{G}$, the mapping $\pi(g): X \rightarrow X$ is continuous.

These conditions are sufficient. Conditions (A1) and (A2) say that $\pi(e) = \text{Id}_X$ and $\pi(g_1) \circ \pi(g_2) = \pi(g_1 g_2)$; hence $\pi(g) \circ \pi(g^{-1}) = \pi(e) = \text{Id}_X = \pi(g^{-1}) \circ \pi(g)$, so each $\pi(g)$ is a bijection of X with inverse $\pi(g^{-1})$. Together with (A3) applied to g and to g^{-1} , each $\pi(g)$ is a homeomorphism of X onto X .

Consequently, for $f \in C(X)$ the composition $\Pi(g)f = f \circ \pi(g^{-1})$ is again continuous, so $\Pi(g)$ maps $C(X)$ into $C(X)$. It is clearly linear, and since $\pi(g^{-1})$ is a bijection of X ,

$$\|\Pi(g)f\|_{\max} = \max_{x \in X} |f(\pi(g^{-1})x)| = \max_{y \in X} |f(y)| = \|f\|_{\max},$$

so $\Pi(g)$ is a linear isometry of $C(X)$ onto $C(X)$; in particular $\Pi(g) \in \text{GL}(C(X))$, with $[\Pi(g)]^{-1} = \Pi(g^{-1})$. Finally Π is a group homomorphism: for $f \in C(X)$,

$$\Pi(g_1 g_2)f = f \circ \pi((g_1 g_2)^{-1}) = f \circ \pi(g_2^{-1} g_1^{-1}) = f \circ \pi(g_2^{-1}) \circ \pi(g_1^{-1}) = [\Pi(g_2)f] \circ \pi(g_1^{-1}) = \Pi(g_1)[\Pi(g_2)f].$$

Thus $\Pi: \mathcal{G} \rightarrow \text{GL}(C(X))$ is a representation.

These conditions are also necessary. Suppose $\Pi(g)f = f \circ \pi(g^{-1})$ does define a representation of $\mathcal{G}$ on $C(X)$.

First, (A3). Fix $g \in \mathcal{G}$ and abbreviate $h = \pi(g^{-1})$; by assumption $f \circ h$ belongs to $C(X)$ for every $f \in C(X)$. Let $x \in X$ and let $\mathcal{O}$ be an open subset of X containing $h(x)$. A compact Hausdorff space is normal, so by Urysohn's Lemma there is an $f \in C(X)$ with $f(h(x)) = 1$ and $f = 0$ on $X \sim \mathcal{O}$. Then

$$\mathcal{U} = \{x' \in X : (f \circ h)(x') > 1/2\}$$

is open, since $f \circ h$ is continuous; it contains x ; and $h(\mathcal{U}) \subseteq \mathcal{O}$, since f vanishes off $\mathcal{O}$. Thus h is continuous at x , and as x was arbitrary $\pi(g^{-1})$ is continuous. Since g was arbitrary, (A3) holds.

Next, by Urysohn's Lemma again, $C(X)$ separates the points of the compact Hausdorff space X ; consequently two mappings $u, v: X \rightarrow X$ with $f \circ u = f \circ v$ for all $f \in C(X)$ must be equal. A

group homomorphism carries the unit to the unit, so $\Pi(e) = \text{Id}$ on $C(X)$, that is, $f \circ \pi(e) = f$ for all $f \in C(X)$; by separation, $\pi(e) = \text{Id}_X$, which is (A1). Similarly, the homomorphism property $\Pi(g_1 g_2) = \Pi(g_1) \Pi(g_2)$ says $f \circ \pi(g_2^{-1} g_1^{-1}) = f \circ \pi(g_2^{-1}) \circ \pi(g_1^{-1})$ for all $f \in C(X)$, so by separation $\pi(a \cdot b) = \pi(a) \circ \pi(b)$ for all $a, b \in \mathcal{G}$, which is (A2). No continuity of φ in the variable g is needed at this stage, in keeping with the footnote in Section 22.2 that a representation is a purely algebraic object.

Part 2: what is needed for the continuity $g \mapsto \Pi(g)f$.

The answer is: $\varphi: \mathcal{G} \times X \rightarrow X$ must be continuous as a mapping of the product space, that is, jointly continuous in (g, x) . Note that for $X = \mathcal{G}$ and $\varphi(g, g') = g \cdot g'$ this holds by the very definition of a topological group, so what follows contains Lemma 5 as a special case.

A word about what is being asked. Read literally, the requirement that for each fixed $x \in X$ the mapping $g \mapsto \pi(g)x = \varphi(g, x)$ be continuous from $\mathcal{G}$ into X is a separate-continuity condition, and for a mapping of two variables separate continuity is in general a weaker demand than joint continuity. The continuity that is actually at issue for a representation on $C(X)$, and that Theorem 4 requires, is the continuity of $g \mapsto \Pi(g)f$ from $\mathcal{G}$ into $C(X)$ with the maximum norm, for each $f \in C(X)$. It is this condition that we show to be exactly equivalent to joint continuity of φ ; joint continuity of course implies the literal separate-continuity condition as well, being inherited by the restriction of φ to the slice $\mathcal{G} \times \{x\}$.

Sufficiency. Assume φ is continuous on $\mathcal{G} \times X$ and let $f \in C(X)$. By Problem 17 it suffices to prove that $g \mapsto \Pi(g)f$ is continuous at e . Let $\epsilon > 0$. Fix $x \in X$. The function $f \circ \varphi$ is continuous on $\mathcal{G} \times X$ and, by (A1), $(f \circ \varphi)(e, x) = f(x)$. Hence there are a neighborhood $\mathcal{U}_x$ of e in $\mathcal{G}$ and an open set $\mathcal{O}_x$ containing x in X for which

$$|f(\varphi(g, y)) - f(x)| < \epsilon/2 \quad \text{for all } g \in \mathcal{U}_x, y \in \mathcal{O}_x. \quad (\text{i})$$

Taking $g = e$ in (i) gives $|f(y) - f(x)| < \epsilon/2$ for all $y \in \mathcal{O}_x$, so by the triangle inequality

$$|f(\varphi(g, y)) - f(y)| < \epsilon \quad \text{for all } g \in \mathcal{U}_x, y \in \mathcal{O}_x. \quad (\text{ii})$$

The collection $\{\mathcal{O}_x\}_{x \in X}$ is an open cover of the compact space X , so there are finitely many points $x_1, \dots, x_n$ in X with $X = \bigcup_{k=1}^n \mathcal{O}_{x_k}$. Define $\mathcal{V} = \bigcap_{k=1}^n \mathcal{U}_{x_k}$, a neighborhood of e since the intersection is finite. If $g \in \mathcal{V}$ and $y \in X$, choose k with $y \in \mathcal{O}_{x_k}$; since $g \in \mathcal{U}_{x_k}$, (ii) gives $|f(\varphi(g, y)) - f(y)| < \epsilon$. Therefore

$$\|f \circ \pi(g) - f\|_{\max} \leq \epsilon \quad \text{for all } g \in \mathcal{V}.$$

Inversion is a homeomorphism of $\mathcal{G}$ carrying e to e , so $\mathcal{U} = \mathcal{V} \cap \mathcal{V}^{-1}$ is again a neighborhood of e , and for $g \in \mathcal{U}$ we have $g^{-1} \in \mathcal{V}$, whence

$$\|\Pi(g)f - f\|_{\max} = \|f \circ \pi(g^{-1}) - f\|_{\max} \leq \epsilon \quad \text{for all } g \in \mathcal{U}.$$

Since $\Pi(e)f = f$, this is continuity at e , and Problem 17 upgrades it to continuity of $g \mapsto \Pi(g)f$ on all of $\mathcal{G}$. In particular, for each fixed $x \in X$ the mapping $g \mapsto \pi(g)x = \varphi(g, x)$ is continuous from $\mathcal{G}$ to X , being the restriction of the continuous φ to the slice $\mathcal{G} \times \{x\}$.

Necessity. Conversely, suppose Π is a representation as in Part 1 and that $g \mapsto \Pi(g)f$ is continuous from $\mathcal{G}$ into $C(X)$ for every $f \in C(X)$. We claim φ is then jointly continuous. Let $\{(g_\alpha, x_\alpha)\}$ be a net in $\mathcal{G} \times X$ converging to (g, x) . For $f \in C(X)$,

$$f(\varphi(g_\alpha, x_\alpha)) = [\Pi(g_\alpha^{-1})f](x_\alpha).$$

Inversion is continuous, so $\{g_\alpha^{-1}\} \rightarrow g^{-1}$ and therefore $\{\Pi(g_\alpha^{-1})f\} \rightarrow \Pi(g^{-1})f$ in the maximum norm; also $\Pi(g^{-1})f$ is a continuous function on X , so $\{[\Pi(g_\alpha^{-1})f](x_\alpha)\} \rightarrow [\Pi(g^{-1})f](x)$. Combining these, since uniform convergence permits interchanging the two limits,

$$\{f(\varphi(g_\alpha, x_\alpha))\} \rightarrow [\Pi(g^{-1})f](x) = f(\varphi(g, x)).$$

Finally, in a compact Hausdorff space this forces $\{\varphi(g_\alpha, x_\alpha)\} \rightarrow \varphi(g, x)$. Indeed, if not, there would be an open set $\mathcal{O}$ containing $y := \varphi(g, x)$ and a subnet lying in $X \setminus \mathcal{O}$; a compact Hausdorff space is

normal, so by Urysohn's Lemma there is $f \in C(X)$ with $f(y) = 1$ and $f = 0$ on $X \sim \mathcal{O}$, and then f takes the value 0 along that subnet while $\{f(\varphi(g_\alpha, x_\alpha))\} \rightarrow 1$, a contradiction. Hence φ is continuous on $\mathcal{G} \times X$.

Summary. For $\Pi(g)f = f \circ \pi(g^{-1})$ to define a representation of $\mathcal{G}$ on $C(X)$, the mapping φ must be an action of $\mathcal{G}$ on the compact Hausdorff space X by continuous mappings, that is, (A1), (A2), (A3). For the additional continuity of $g \mapsto \Pi(g)f$ for each $f \in C(X)$ (equivalently, for the hypothesis of Theorem 4 to hold), φ must be continuous jointly in its two variables.

Problem 22.19. Let μ be a Borel probability measure on a compact group $\mathcal{G}$. Show that μ is Haar measure if and only if

$$\int_{\mathcal{G}} f \circ \varphi_g d\mu = \int_{\mathcal{G}} f d\mu \quad \text{for all } g \in \mathcal{G}, f \in C(\mathcal{G}),$$

where $\varphi_g(g') = g \cdot g'$ for all $g' \in \mathcal{G}$.

Solution. Recall the definitions in force: a Borel measure on the compact space $\mathcal{G}$ is a finite measure on $\mathcal{B}(\mathcal{G})$; it is left-invariant provided $\mu(A) = \mu(g \cdot A)$ for all $g \in \mathcal{G}$ and $A \in \mathcal{B}(\mathcal{G})$; and a Haar measure is a Radon measure on $\mathcal{B}(\mathcal{G})$ that is a left-invariant probability measure. As throughout this section, the Borel probability measure μ under discussion is understood to be Radon; some such regularity is genuinely needed for the “if” half, and we comment on this at the end.

Preliminaries on the translations. For $g \in \mathcal{G}$ the mapping $\varphi_g: \mathcal{G} \rightarrow \mathcal{G}$, $\varphi_g(g') = g \cdot g'$, is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$: it is continuous because the group operation is, and $\varphi_{g^{-1}}$ is a continuous inverse for it. Consequently φ_g carries Borel sets to Borel sets and

$$\varphi_g^{-1}(A) = g^{-1} \cdot A \in \mathcal{B}(\mathcal{G}) \quad \text{for } A \in \mathcal{B}(\mathcal{G}),$$

so that

$$\chi_A \circ \varphi_g = \chi_{g^{-1} \cdot A}. \quad (\text{i})$$

Also, if $f \in C(\mathcal{G})$ then $f \circ \varphi_g \in C(\mathcal{G})$, so both integrals in the displayed identity are defined: a continuous function on a compact space is bounded and Borel measurable, and μ is finite.

Finally note the relation between φ_g and the regular representation π of $\mathcal{G}$ on $C(\mathcal{G})$, which is defined by $[\pi(g)f](x) = f(g^{-1} \cdot x)$. Directly from the definitions,

$$\pi(g)f = f \circ \varphi_{g^{-1}} \quad \text{for all } f \in C(\mathcal{G}), g \in \mathcal{G}. \quad (\text{ii})$$

Necessity. Assume μ is Haar measure, so μ is left-invariant. Fix $g \in \mathcal{G}$. We claim

$$\int_{\mathcal{G}} F \circ \varphi_g d\mu = \int_{\mathcal{G}} F d\mu \quad (\text{iii})$$

for every bounded Borel measurable function F on $\mathcal{G}$; taking $F = f \in C(\mathcal{G})$ then gives the asserted identity.

First let $F = \chi_A$ with $A \in \mathcal{B}(\mathcal{G})$. By (i) and the left-invariance of μ applied with the group element g^{-1} ,

$$\int_{\mathcal{G}} \chi_A \circ \varphi_g d\mu = \int_{\mathcal{G}} \chi_{g^{-1} \cdot A} d\mu = \mu(g^{-1} \cdot A) = \mu(A) = \int_{\mathcal{G}} \chi_A d\mu.$$

Both sides of (iii) are linear in F , so (iii) holds for every simple Borel function F . Now let F be an arbitrary bounded Borel measurable function on $\mathcal{G}$. By the Simple Approximation Theorem there is a sequence $\{F_n\}$ of simple Borel functions converging pointwise to F on $\mathcal{G}$ and uniformly bounded by $\sup |F|$. Then $\{F_n \circ \varphi_g\}$ is a sequence of simple Borel functions converging pointwise to $F \circ \varphi_g$ and bounded by the same constant. Since $\mu(\mathcal{G}) = 1 < \infty$, the Bounded Convergence Theorem permits passage to the limit on each side of

$$\int_{\mathcal{G}} F_n \circ \varphi_g d\mu = \int_{\mathcal{G}} F_n d\mu,$$

which yields (iii). (This is exactly the approximation argument used to verify (12) in the proof of Lemma 7.)

Sufficiency. Now assume μ is a Radon Borel probability measure on $\mathcal{G}$ for which

$$\int_{\mathcal{G}} f \circ \varphi_g d\mu = \int_{\mathcal{G}} f d\mu \quad \text{for all } g \in \mathcal{G}, f \in C(\mathcal{G}). \quad (\text{iv})$$

Fix $h \in \mathcal{G}$ and define, as in Lemma 7, the set function $\mu_h: \mathcal{B}(\mathcal{G}) \rightarrow [0, \infty)$ by

$$\mu_h(A) = \mu(h \cdot A) \quad \text{for } A \in \mathcal{B}(\mathcal{G}).$$

Lemma 7 tells us that μ_h is a Borel measure, that it is Radon because μ is Radon, and that

$$\int_{\mathcal{G}} \pi(h)f d\mu = \int_{\mathcal{G}} f d\mu_h \quad \text{for all } f \in C(\mathcal{G}).$$

By (ii), $\pi(h)f = f \circ \varphi_{h^{-1}}$, and by hypothesis (iv) applied with the group element h^{-1} ,

$$\int_{\mathcal{G}} \pi(h)f d\mu = \int_{\mathcal{G}} f \circ \varphi_{h^{-1}} d\mu = \int_{\mathcal{G}} f d\mu.$$

Combining the last two displays,

$$\int_{\mathcal{G}} f d\mu_h = \int_{\mathcal{G}} f d\mu \quad \text{for all } f \in C(\mathcal{G}).$$

Thus the two Radon measures μ_h and μ represent one and the same bounded linear functional $f \mapsto \int_{\mathcal{G}} f d\mu$ on $C(\mathcal{G})$; this functional is bounded since $|\int f d\mu| \leq \|f\|_{\max} \mu(\mathcal{G}) = \|f\|_{\max}$. The Riesz-Markov Theorem asserts that a bounded linear functional on $C(\mathcal{G})$ is represented by a unique Radon measure. Hence

$$\mu_h = \mu, \quad \text{that is, } \mu(h \cdot A) = \mu(A) \quad \text{for all } A \in \mathcal{B}(\mathcal{G}).$$

Since $h \in \mathcal{G}$ was arbitrary, μ is left-invariant. Being also a Radon probability measure, μ is by definition a Haar measure on $\mathcal{G}$. (By von Neumann's Theorem, Theorem 9, it is then the Haar measure, the unique one, and it is right-invariant as well.)

Remark on regularity. The Radon hypothesis in the sufficiency half is no restriction on the statement, since a Haar measure is by definition Radon; and the argument genuinely uses it. The point is that condition (iv) constrains μ only through its integrals against continuous functions: if μ_0 denotes Haar measure and μ is any Borel probability measure with $\int f d\mu = \int f d\mu_0$ for every $f \in C(\mathcal{G})$, then for $f \in C(\mathcal{G})$ and $g \in \mathcal{G}$ we have $f \circ \varphi_g \in C(\mathcal{G})$ and therefore

$$\int_{\mathcal{G}} f \circ \varphi_g d\mu = \int_{\mathcal{G}} f \circ \varphi_g d\mu_0 = \int_{\mathcal{G}} f d\mu_0 = \int_{\mathcal{G}} f d\mu,$$

so (iv) holds for μ as well. It is exactly the uniqueness clause of the Riesz-Markov Theorem, which applies only among Radon measures, that converts agreement on $C(\mathcal{G})$ into agreement on all of $\mathcal{B}(\mathcal{G})$. In particular, when the topology of $\mathcal{G}$ is given by a metric, every finite Borel measure on the compact metric space $\mathcal{G}$ is Radon and the hypothesis is automatic.

Problem 22.20. Let μ be Haar measure on a compact group $\mathcal{G}$. Show that $\mu \times \mu$ is Haar measure on $\mathcal{G} \times \mathcal{G}$.

Solution. Throughout, $\mathcal{G} \times \mathcal{G}$ carries the product topology and the componentwise group operation

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_2), \quad (x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1}).$$

Step 0: $\mathcal{G} \times \mathcal{G}$ is a compact group. It is compact as the product of two compact spaces. Multiplication and inversion on the product are continuous because each of their two coordinate functions is a com-

position of continuous projections with the continuous multiplication and inversion of $\mathcal{G}$, and a map into a product is continuous exactly when its coordinate maps are. So $\mathcal{G} \times \mathcal{G}$ is a topological group, compact.

Notation. Let λ^* be the outer measure by which the product measure $\mu \times \mu$ is constructed in Chapter 20: for $S \subseteq \mathcal{G} \times \mathcal{G}$,

$$\lambda^*(S) = \inf \left\{ \sum_{k=1}^{\infty} \mu(A_k) \mu(B_k) : S \subseteq \bigcup_{k=1}^{\infty} A_k \times B_k, A_k, B_k \in \mathcal{B}(\mathcal{G}) \right\},$$

the infimum being over countable covers of S by measurable rectangles. The product measure $\mu \times \mu$ is the restriction of λ^* to the λ^* -measurable sets, and, as recorded in the proof of Theorem 9, this σ -algebra of subsets of $\mathcal{G} \times \mathcal{G}$ contains $\mathcal{B}(\mathcal{G} \times \mathcal{G})$. Write λ for the restriction of $\mu \times \mu$ to $\mathcal{B}(\mathcal{G} \times \mathcal{G})$. Since $\mathcal{G} \times \mathcal{G}$ is itself a rectangle,

$$\lambda(\mathcal{G} \times \mathcal{G}) = \mu(\mathcal{G}) \cdot \mu(\mathcal{G}) = 1,$$

so λ is a Borel probability measure on $\mathcal{G} \times \mathcal{G}$. It remains to show that λ is left-invariant and Radon.

Step 1: left-invariance. Fix $a = (a_1, a_2) \in \mathcal{G} \times \mathcal{G}$ and let $T: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ be left translation by a , that is, $T(x, y) = (a_1 x, a_2 y)$. As in Problem 19, T is a homeomorphism of $\mathcal{G} \times \mathcal{G}$ onto itself, with inverse left translation by a^{-1} ; in particular T carries Borel sets onto Borel sets.

We claim that λ^* is invariant under T :

$$\lambda^*(T(S)) = \lambda^*(S) \quad \text{for every } S \subseteq \mathcal{G} \times \mathcal{G}. \quad (\text{i})$$

Indeed, T carries the measurable rectangle $A \times B$ onto the measurable rectangle $(a_1 \cdot A) \times (a_2 \cdot B)$, and by the left-invariance of μ ,

$$\mu(a_1 \cdot A) \cdot \mu(a_2 \cdot B) = \mu(A) \cdot \mu(B).$$

Since T is a bijection, a countable collection $\{A_k \times B_k\}$ covers S if and only if $\{(a_1 \cdot A_k) \times (a_2 \cdot B_k)\}$ covers $T(S)$, and the two associated sums are equal. Thus the covers of S and those of $T(S)$ produce exactly the same set of admissible sums, and taking infima gives (i).

Now let E belong to $\mathcal{B}(\mathcal{G} \times \mathcal{G})$. Then $T(E)$ is Borel, hence λ^* -measurable, and by (i),

$$\lambda(a \cdot E) = \lambda(T(E)) = \lambda^*(T(E)) = \lambda^*(E) = \lambda(E).$$

Since a and E were arbitrary, λ is left-invariant.

Step 2: λ is Radon. By the definition of Chapter 21, we must check that λ is finite on compact sets, outer regular on Borel sets, and inner regular on open sets. Finiteness is clear, since λ is a probability measure.

Outer regularity. Let $E \in \mathcal{B}(\mathcal{G} \times \mathcal{G})$ and $\epsilon > 0$. By the definition of λ^* , choose Borel sets $A_k, B_k \subseteq \mathcal{G}$ with

$$E \subseteq \bigcup_{k=1}^{\infty} A_k \times B_k \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(A_k) \mu(B_k) < \lambda(E) + \epsilon/2.$$

Fix k and let $\delta_k > 0$. Since μ is Radon, it is outer regular, so there are open subsets $\mathcal{U}_k \supseteq A_k$ and $\mathcal{V}_k \supseteq B_k$ of $\mathcal{G}$ with $\mu(\mathcal{U}_k) < \mu(A_k) + \delta_k$ and $\mu(\mathcal{V}_k) < \mu(B_k) + \delta_k$. Since $\mu(\mathcal{G}) = 1$, all these numbers lie in $[0, 1 + \delta_k]$, and hence

$$\mu(\mathcal{U}_k) \mu(\mathcal{V}_k) < (\mu(A_k) + \delta_k)(\mu(B_k) + \delta_k) \leq \mu(A_k) \mu(B_k) + \delta_k(2 + \delta_k).$$

Choosing $\delta_k > 0$ small enough that $\delta_k(2 + \delta_k) < \epsilon 2^{-k-1}$, we obtain

$$\sum_{k=1}^{\infty} \mu(\mathcal{U}_k) \mu(\mathcal{V}_k) < \sum_{k=1}^{\infty} \mu(A_k) \mu(B_k) + \epsilon/2 < \lambda(E) + \epsilon.$$

The set $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{U}_k \times \mathcal{V}_k$ is open in $\mathcal{G} \times \mathcal{G}$ and contains E , and by the countable monotonicity of λ together with $\lambda(\mathcal{U}_k \times \mathcal{V}_k) = \mu(\mathcal{U}_k) \mu(\mathcal{V}_k)$,

$$\lambda(\mathcal{O}) \leq \sum_{k=1}^{\infty} \mu(\mathcal{U}_k) \mu(\mathcal{V}_k) < \lambda(E) + \epsilon.$$

Thus $\lambda(E) = \inf\{\lambda(\mathcal{O}) : \mathcal{O} \text{ open, } E \subseteq \mathcal{O}\}$, which is outer regularity.

Inner regularity on open sets. Let $\mathcal{O}$ be an open subset of $\mathcal{G} \times \mathcal{G}$ and let $\epsilon > 0$. Its complement $F = (\mathcal{G} \times \mathcal{G}) \setminus \mathcal{O}$ is closed, hence Borel. By the outer regularity just proved, there is an open set $\mathcal{W} \supseteq F$ with $\lambda(\mathcal{W}) < \lambda(F) + \epsilon$. Put $K = (\mathcal{G} \times \mathcal{G}) \setminus \mathcal{W}$. Then K is closed in the compact space $\mathcal{G} \times \mathcal{G}$, hence compact, and $K \subseteq \mathcal{O}$ because $F \subseteq \mathcal{W}$. Moreover, since $F \subseteq \mathcal{W}$ and λ is finite,

$$\lambda(\mathcal{W}) = \lambda(\mathcal{W} \cap \mathcal{O}) + \lambda(\mathcal{W} \cap F) = \lambda(\mathcal{W} \cap \mathcal{O}) + \lambda(F),$$

so $\lambda(\mathcal{O} \setminus K) = \lambda(\mathcal{O} \cap \mathcal{W}) = \lambda(\mathcal{W}) - \lambda(F) < \epsilon$, and therefore $\lambda(K) > \lambda(\mathcal{O}) - \epsilon$. Hence $\lambda(\mathcal{O}) = \sup\{\lambda(K) : K \subseteq \mathcal{O}, K \text{ compact}\}$.

Conclusion. The measure $\lambda = \mu \times \mu$, restricted to $\mathcal{B}(\mathcal{G} \times \mathcal{G})$, is a left-invariant Radon probability measure on the compact group $\mathcal{G} \times \mathcal{G}$; that is, it is a Haar measure. By von Neumann's Theorem (Theorem 9) the Haar measure of a compact group is unique, so $\mu \times \mu$ is the Haar measure of $\mathcal{G} \times \mathcal{G}$, and in particular it is right-invariant as well.

Problem 22.21. Let $\mathcal{G}$ be a compact group whose topology is given by a metric. Show that there is a $\mathcal{G}$ -invariant metric. (Hint: Use the preceding two problems and average the metric over the group $\mathcal{G} \times \mathcal{G}$.)

Solution. Let ρ be a metric on $\mathcal{G}$ that induces its topology. We produce a metric d on $\mathcal{G}$ that induces the same topology and satisfies

$$d(h \cdot x, h \cdot y) = d(x, y) \quad \text{for all } h, x, y \in \mathcal{G},$$

that is, a metric invariant under the action of $\mathcal{G}$ on itself by left translations. At the end we upgrade the construction, by averaging over $\mathcal{G} \times \mathcal{G}$, to a metric invariant under left and right translations simultaneously.

Preliminaries. Since $\mathcal{G}$ is a compact metric space, $\mathcal{G} \times \mathcal{G}$ is a compact metric space under, say, $\rho_2((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2)$, whose topology is the product topology; and $\rho : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}$ is continuous, since $|\rho(x_1, x_2) - \rho(y_1, y_2)| \leq \rho_2((x_1, x_2), (y_1, y_2))$ by the triangle inequality. Being continuous on a compact space, ρ is bounded: set $D = \max_{(x,y)} \rho(x, y) < \infty$.

By von Neumann's Theorem (Theorem 9) there is a unique Haar measure μ on $\mathcal{B}(\mathcal{G})$; it is a Radon probability measure that is both left- and right-invariant.

We record the right-translation analogue of Problem 19. For $h \in \mathcal{G}$ let $\psi_h(g) = g \cdot h$, a homeomorphism of $\mathcal{G}$ onto itself with $\psi_h^{-1}(A) = A \cdot h^{-1}$. Then

$$\int_{\mathcal{G}} F \circ \psi_h d\mu = \int_{\mathcal{G}} F d\mu \quad \text{for every bounded Borel measurable } F \text{ on } \mathcal{G}. \quad (\text{i})$$

Indeed, for $A \in \mathcal{B}(\mathcal{G})$ we have $\chi_A \circ \psi_h = \chi_{A \cdot h^{-1}}$, so by the right-invariance of μ ,

$$\int_{\mathcal{G}} \chi_A \circ \psi_h d\mu = \mu(A \cdot h^{-1}) = \mu(A) = \int_{\mathcal{G}} \chi_A d\mu;$$

by linearity (i) holds for simple Borel functions, and then for all bounded Borel measurable F by the Simple Approximation Theorem together with the Bounded Convergence Theorem, exactly as in the proof of Lemma 7 and in Problem 19.

Construction. Define $d : \mathcal{G} \times \mathcal{G} \rightarrow [0, \infty)$ by

$$d(x, y) = \int_{\mathcal{G}} \rho(g \cdot x, g \cdot y) d\mu(g).$$

This is well defined: for fixed x, y , the mapping $g \mapsto (g \cdot x, g \cdot y)$ is continuous from $\mathcal{G}$ into $\mathcal{G} \times \mathcal{G}$, since the group multiplication is continuous; composing with the continuous ρ shows that $g \mapsto \rho(gx, gy)$ is continuous on $\mathcal{G}$, hence bounded by D and Borel measurable. As μ is a finite measure, the integral exists and $0 \leq d(x, y) \leq D$.

d is a metric.

Symmetry and vanishing on the diagonal are immediate from the corresponding properties of ρ : $\rho(gx, gy) = \rho(gy, gx)$ for each g , and $\rho(gx, gx) = 0$ for each g , so $d(x, y) = d(y, x)$ and $d(x, x) = 0$.

Positivity: suppose $x \neq y$. For each $g \in \mathcal{G}$ we have $g \cdot x \neq g \cdot y$, since left multiplication by g is one-to-one (it has the inverse g^{-1}), and therefore $\rho(gx, gy) > 0$. The function $g \mapsto \rho(gx, gy)$ is continuous and strictly positive on the compact space $\mathcal{G}$, so it attains a minimum value $m > 0$. By the monotonicity of the integral and $\mu(\mathcal{G}) = 1$,

$$d(x, y) \geq m \cdot \mu(\mathcal{G}) = m > 0.$$

Triangle inequality: for $x, y, z \in \mathcal{G}$ and every $g \in \mathcal{G}$,

$$\rho(gx, gz) \leq \rho(gx, gy) + \rho(gy, gz),$$

and integrating this inequality of bounded Borel measurable functions over $\mathcal{G}$, using monotonicity and linearity of the integral, gives $d(x, z) \leq d(x, y) + d(y, z)$.

d is left-invariant. Fix $h \in \mathcal{G}$ and $x, y \in \mathcal{G}$, and set $F(g) = \rho(g \cdot x, g \cdot y)$, a bounded continuous function on $\mathcal{G}$. Since $\mathcal{G}$ is a group, $(gh) \cdot x = g \cdot (h \cdot x)$, so

$$\rho(g \cdot (hx), g \cdot (hy)) = \rho((gh) \cdot x, (gh) \cdot y) = F(\psi_h(g)) \quad \text{for all } g \in \mathcal{G}.$$

Hence, by (i),

$$d(hx, hy) = \int_{\mathcal{G}} F \circ \psi_h d\mu = \int_{\mathcal{G}} F d\mu = d(x, y).$$

d induces the original topology. Suppose $\{x_n\} \rightarrow x$ and $\{y_n\} \rightarrow y$ with respect to ρ . For each fixed $g \in \mathcal{G}$ the mapping $z \mapsto g \cdot z$ is continuous, so $\{gx_n\} \rightarrow gx$ and $\{gy_n\} \rightarrow gy$ in ρ , and therefore, by the continuity of ρ ,

$$\lim_{n \rightarrow \infty} \rho(gx_n, gy_n) = \rho(gx, gy) \quad \text{for every } g \in \mathcal{G}.$$

Each of these functions of g is Borel measurable and bounded by D , and μ is a finite measure, so the Bounded Convergence Theorem yields

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y).$$

Thus d is sequentially continuous, and hence continuous, on the metric space $\mathcal{G} \times \mathcal{G}$ with its original topology. In particular, if $\{x_n\} \rightarrow x$ with respect to ρ , then $d(x_n, x) \rightarrow d(x, x) = 0$, so the identity mapping

$$\text{Id}: (\mathcal{G}, \rho) \rightarrow (\mathcal{G}, d)$$

is continuous. It is a one-to-one mapping of the compact space $(\mathcal{G}, \rho)$ onto the Hausdorff space $(\mathcal{G}, d)$; a continuous one-to-one mapping of a compact topological space onto a Hausdorff topological space is a homeomorphism. Therefore d induces exactly the original topology of $\mathcal{G}$.

So d is a $\mathcal{G}$ -invariant metric inducing the topology of $\mathcal{G}$, as required.

A bi-invariant metric by averaging over $\mathcal{G} \times \mathcal{G}$. The same argument, carried out over the compact group $\Gamma = \mathcal{G} \times \mathcal{G}$, yields a metric invariant under left and right translations at once. By Step 0 of Problem 20, Γ is a compact group, and by Problem 20 its Haar measure is $\nu = \mu \times \mu$, a left- and right-invariant Radon probability measure on $\mathcal{B}(\Gamma)$. Define

$$D(x, y) = \int_{\Gamma} \rho(axb^{-1}, ayb^{-1}) d\nu(a, b).$$

The integrand is continuous in (a, b) , since multiplication and inversion in $\mathcal{G}$ are continuous and ρ is continuous, and it is bounded; so D is well defined. Exactly as above, D is a metric: symmetry and $D(x, x) = 0$ are inherited from ρ ; if $x \neq y$ then $axb^{-1} \neq ayb^{-1}$ for every (a, b) , so the continuous integrand is strictly positive on the compact Γ and hence has a positive minimum, forcing $D(x, y) > 0$; and the triangle inequality is obtained by integrating the pointwise triangle inequality for ρ .

For the invariance, fix $h, k \in \mathcal{G}$ and put $F(a, b) = \rho(axb^{-1}, ayb^{-1})$, a continuous function on Γ . Then for all $(a, b) \in \Gamma$,

$$\rho(a(hxk)b^{-1}, a(hyk)b^{-1}) = \rho((ah)x(bk^{-1})^{-1}, (ah)y(bk^{-1})^{-1}) = F((a, b) \cdot (h, k^{-1})),$$

since $(bk^{-1})^{-1} = kb^{-1}$. That is, the integrand for $D(hxk, hyk)$ is F composed with right translation in Γ by the element (h, k^{-1}) . Since ν is right-invariant, the analogue of (i) on the group Γ gives

$$D(hxk, hyk) = \int_{\Gamma} F d\nu = D(x, y).$$

Finally, the argument given for d applies verbatim to D : for fixed (a, b) the map $z \mapsto azb^{-1}$ is continuous, so the Bounded Convergence Theorem shows D is continuous on $\mathcal{G} \times \mathcal{G}$, and the identity map from $(\mathcal{G}, \rho)$ onto $(\mathcal{G}, D)$ is a continuous bijection from a compact space onto a Hausdorff space, hence a homeomorphism. Thus D is a metric inducing the topology of $\mathcal{G}$ that is invariant under all left and all right translations.

22.4 Problems 22.22–22.28

Problem 22.22. Let μ be Haar measure on a compact group $\mathcal{G}$. If $\mathcal{G}$ has infinitely many members, show that $\mu(\{g\}) = 0$ for each $g \in \mathcal{G}$. If $\mathcal{G}$ is finite, explicitly describe μ .

Solution. Recall that a topological group is, by definition, a group together with a Hausdorff topology for which multiplication and inversion are continuous, and that Haar measure is a left-invariant Radon probability measure on $\mathcal{B}(\mathcal{G})$; by Theorem 9 (von Neumann) it exists and is unique on a compact group. First observe that every singleton is a Borel set. Indeed, since the topology on $\mathcal{G}$ is Hausdorff, each set $\{g\}$ is closed, and closed sets belong to $\mathcal{B}(\mathcal{G})$. Thus $\mu(\{g\})$ is defined for every $g \in \mathcal{G}$.

Next observe that all singletons have the same measure. Fix $g, h \in \mathcal{G}$ and put $a = h \cdot g^{-1}$. Then

$$a \cdot \{g\} = \{a \cdot g\} = \{h\},$$

so by the left-invariance of μ ,

$$\mu(\{h\}) = \mu(a \cdot \{g\}) = \mu(\{g\}).$$

Denote this common value by $c = \mu(\{e\}) \geq 0$.

Suppose $\mathcal{G}$ is infinite. Let n be a natural number. Since $\mathcal{G}$ has infinitely many members we may choose n distinct points $g_1, \dots, g_n$ in $\mathcal{G}$. The singletons $\{g_1\}, \dots, \{g_n\}$ are disjoint Borel sets, so by the finite additivity and the monotonicity of the measure μ ,

$$1 = \mu(\mathcal{G}) \geq \mu\left(\bigcup_{k=1}^n \{g_k\}\right) = \sum_{k=1}^n \mu(\{g_k\}) = n \cdot c.$$

Hence $0 \leq c \leq 1/n$ for every natural number n , and therefore $c = 0$. Thus $\mu(\{g\}) = 0$ for every $g \in \mathcal{G}$. Now suppose $\mathcal{G}$ is finite, say $\mathcal{G} = \{g_1, \dots, g_n\}$ with the g_k distinct, so that n is the number of members of $\mathcal{G}$. Since the topology is Hausdorff, each singleton is closed; since $\mathcal{G}$ is finite, every subset of $\mathcal{G}$ is a finite union of singletons and hence is closed, and therefore also open (its complement is closed). Thus the topology on $\mathcal{G}$ is the discrete topology and $\mathcal{B}(\mathcal{G})$ is the collection of all subsets of $\mathcal{G}$. Of course a finite discrete space is compact.

By the additivity of μ and the preceding observation that all singletons have the same measure c ,

$$1 = \mu(\mathcal{G}) = \sum_{k=1}^n \mu(\{g_k\}) = n \cdot c, \quad \text{so } c = \frac{1}{n}.$$

Therefore, again by additivity, for every subset E of $\mathcal{G}$,

$$\mu(E) = \frac{\text{card}(E)}{\text{card}(\mathcal{G})} = \frac{\text{card}(E)}{n}.$$

That is, Haar measure on a finite group is normalized counting measure.

It is worth checking directly that this set function really is Haar measure, which reconfirms the uniqueness assertion of Theorem 9 in this case. The set function $E \mapsto \text{card}(E)/n$ is a measure on all subsets of $\mathcal{G}$, it assigns $\mathcal{G}$ the value 1, and it is a Radon measure since every subset is compact and every set is trivially both inner and outer approximated by itself. It is left-invariant because for fixed g the left translation $x \mapsto g \cdot x$ is a bijection of $\mathcal{G}$ onto $\mathcal{G}$, so $\text{card}(g \cdot E) = \text{card}(E)$ for all $E \subseteq \mathcal{G}$.

Problem 22.23. Show that if μ is Haar measure on a compact group, then $\mu(\mathcal{O}) > 0$ for every open subset $\mathcal{O}$ of $\mathcal{G}$.

Solution. Of course the assertion is meant for nonempty open sets; the empty set is open and has measure zero. So let $\mathcal{O}$ be a nonempty open subset of the compact group $\mathcal{G}$ and let μ be Haar measure on $\mathcal{G}$, that is, by the definition preceding Theorem 9, a left-invariant Radon probability measure on $\mathcal{B}(\mathcal{G})$.

We first record that left translation is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$. Fix $a \in \mathcal{G}$ and define $L_a: \mathcal{G} \rightarrow \mathcal{G}$ by $L_a(x) = a \cdot x$. Since the group operation $(g_1, g_2) \mapsto g_1 \cdot g_2$ is continuous from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$ and $x \mapsto (a, x)$ is continuous from $\mathcal{G}$ to $\mathcal{G} \times \mathcal{G}$, the composition L_a is continuous. Moreover L_a is a bijection with inverse $L_{a^{-1}}$, which is continuous for the same reason. Therefore L_a is a homeomorphism, and in particular $a \cdot \mathcal{O} = L_a(\mathcal{O})$ is open whenever $\mathcal{O}$ is open. In particular each $a \cdot \mathcal{O}$ is a Borel set, and by the left-invariance of μ ,

$$\mu(a \cdot \mathcal{O}) = \mu(\mathcal{O}) \quad \text{for all } a \in \mathcal{G}.$$

Now choose a point $g_0 \in \mathcal{O}$. For each $g \in \mathcal{G}$ define

$$\mathcal{U}_g = (g \cdot g_0^{-1}) \cdot \mathcal{O},$$

which by the preceding paragraph is an open subset of $\mathcal{G}$. Since g_0 belongs to $\mathcal{O}$, the point $(g \cdot g_0^{-1}) \cdot g_0 = g$ belongs to $\mathcal{U}_g$. Therefore

$$\mathcal{G} = \bigcup_{g \in \mathcal{G}} \mathcal{U}_g,$$

so $\{\mathcal{U}_g\}_{g \in \mathcal{G}}$ is an open cover of $\mathcal{G}$. Since $\mathcal{G}$ is compact, there are finitely many points $g_1, \dots, g_n$ in $\mathcal{G}$ for which

$$\mathcal{G} = \bigcup_{k=1}^n \mathcal{U}_{g_k} = \bigcup_{k=1}^n (g_k \cdot g_0^{-1}) \cdot \mathcal{O}.$$

By the countable monotonicity (finite subadditivity) of the measure μ and the displayed left-invariance,

$$1 = \mu(\mathcal{G}) \leq \sum_{k=1}^n \mu((g_k \cdot g_0^{-1}) \cdot \mathcal{O}) = \sum_{k=1}^n \mu(\mathcal{O}) = n \cdot \mu(\mathcal{O}).$$

Hence

$$\mu(\mathcal{O}) \geq \frac{1}{n} > 0.$$

The proof is complete.

Observe that the argument uses compactness in an essential way, and that it in fact gives a quantitative statement: if $\mathcal{O}$ is a nonempty open set and n is the least number of left translates of $\mathcal{O}$ needed to cover $\mathcal{G}$, then $\mu(\mathcal{O}) \geq 1/n$.

Problem 22.24. Let $S^1 = \{z = e^{i\theta} \mid \theta \in \mathbb{R}\}$ be the circle with the group operation of complex multiplication and the topology it inherits from the Euclidean plane.

(i) Show that S^1 is a topological group.

(ii) Define $\Lambda = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}, 0 < \beta - \alpha < 2\pi\}$. For $\lambda = (\alpha, \beta) \in \Lambda$, define $I_\alpha = \{e^{i\theta} \mid \alpha < \theta < \beta\}$. Show that every proper open subset of S^1 is the countable disjoint union of sets of the form I_λ , $\lambda \in \Lambda$.

(iii) For $\lambda = (\alpha, \beta) \in \Lambda$, define $\mu(I_\alpha) = (\beta - \alpha)/2\pi$. Define $\mu(S^1) = 1$. Use part (ii) to extend μ to a set function defined on the topology $\mathcal{T}$ of S^1 . Then verify that, by Proposition 9 from the preceding chapter, μ may be extended to a Borel measure μ on $\mathcal{B}(S^1)$.

(iv) Show that the measure defined in part (ii) is Haar measure on S^1 .

(v) The torus T^n is the topological group consisting of the Cartesian product of n copies of S^1 with the product topology and group structure. What is Haar measure on T^n ?

Solution. Two remarks on the printed statement before we begin. First, in (ii) and (iii) the set $\{e^{i\theta} \mid \alpha < \theta < \beta\}$ and its measure are written I_α and $\mu(I_\alpha)$, while the assertion of (ii) and the definition of Λ use I_λ ; the subscript is of course meant to be $\lambda = (\alpha, \beta)$, and we write I_λ throughout. Second, and this matters mathematically, the printed constraint $0 < \beta - \alpha < 2\pi$ must be relaxed to

$$0 < \beta - \alpha \leq 2\pi$$

for the assertion of (ii) to be true. Indeed, if $z_0 \in S^1$, then $\mathcal{O} = S^1 \sim \{z_0\}$ is a proper open subset of S^1 that is connected, being a continuous one-to-one image of an open interval as we show below; a connected space is not the union of two or more disjoint nonempty open sets, so any decomposition of $\mathcal{O}$ into disjoint nonempty sets I_λ consists of a single such set, and then $I_\lambda = S^1 \sim \{z_0\}$ forces $\beta - \alpha = 2\pi$. We prove below that this is the only obstruction: with Λ enlarged to allow $\beta - \alpha = 2\pi$, assertion (ii) is correct, and every proper open set whose complement contains at least two points is already a countable disjoint union of sets I_λ with $\beta - \alpha < 2\pi$. Note that the prescription $\mu(I_\lambda) = (\beta - \alpha)/2\pi$ remains consistent with $\mu(S^1) = 1$ under the enlargement, since it assigns $S^1 \sim \{z_0\}$ the value 1 and, as it must, gives singletons measure 0 (compare Problem 22). Accordingly, from here on

$$\Lambda = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}, 0 < \beta - \alpha \leq 2\pi\}.$$

Throughout, m denotes Lebesgue measure on $\mathbb{R}$ and $\Phi: \mathbb{R} \rightarrow S^1$ denotes the exponential parametrization $\Phi(\theta) = e^{i\theta}$, which is continuous and onto, with $\Phi(\theta) = \Phi(\theta')$ if and only if $\theta - \theta' \in 2\pi\mathbb{Z}$.

Part (i). The set $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a group under complex multiplication: if $|z| = |w| = 1$ then $|zw| = |z||w| = 1$, the number 1 lies in S^1 , and if $|z| = 1$ then $z^{-1} = \bar{z}$, which also has modulus 1. Associativity and commutativity are inherited from $\mathbb{C}$.

The topology on S^1 is the one inherited from the Euclidean plane, so it is Hausdorff, being a subspace of a Hausdorff space. Multiplication $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is continuous, since in real coordinates

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

and each coordinate is a polynomial in (a, b, c, d) . The topology that $S^1 \times S^1$ inherits from $\mathbb{C} \times \mathbb{C}$ is the product of the inherited topologies. Since the restriction of a continuous map to a subspace is continuous and S^1 is closed under multiplication, the map $(z, w) \mapsto z \cdot w$ is continuous from $S^1 \times S^1$ to S^1 . Inversion on S^1 is $z \mapsto z^{-1} = \bar{z}$, that is, $(a, b) \mapsto (a, -b)$ in real coordinates, which is continuous. Hence S^1 is a topological group. It is moreover compact: S^1 is a closed bounded subset of $\mathbb{R}^2$, so it is compact by the Heine-Borel Theorem.

Part (ii). First observe that for $\lambda = (\alpha, \beta) \in \Lambda$ the set I_λ is open in S^1 . Indeed, since $\beta - \alpha \leq 2\pi$ we have $\beta \leq \alpha + 2\pi$, and therefore

$$S^1 \sim I_\lambda = \{e^{i\theta} \mid \beta \leq \theta \leq \alpha + 2\pi\} = \Phi([\beta, \alpha + 2\pi]),$$

which is the continuous image of a compact set, hence compact, hence closed in the Hausdorff space S^1 . So I_λ is open. Observe also that Φ restricted to an interval (α, β) with $\beta - \alpha \leq 2\pi$ is one-to-one, since two distinct points of such an interval differ by less than 2π .

Now let $\mathcal{O}$ be a proper open subset of S^1 and choose a point $z_0 = e^{i\alpha_0}$ in $S^1 \sim \mathcal{O}$. Consider

$$\varphi: (\alpha_0, \alpha_0 + 2\pi) \rightarrow S^1 \sim \{z_0\}, \quad \varphi(\theta) = e^{i\theta}.$$

By the preceding paragraph φ is one-to-one, and it is clearly onto $S^1 \sim \{z_0\}$ and continuous. It is also an open mapping: if (a, b) is a nonempty open subinterval of $(\alpha_0, \alpha_0 + 2\pi)$, then $0 < b - a \leq 2\pi$,

so (a, b) belongs to Λ and $\varphi((a, b)) = I_{(a, b)}$ is open in S^1 by the preceding paragraph; a general open subset of $(\alpha_0, \alpha_0 + 2\pi)$ is a union of such subintervals and its image is the corresponding union. A one-to-one, onto, continuous, open mapping is a homeomorphism, so φ is a homeomorphism. Since $\mathcal{O} \subseteq S^1 \sim \{z_0\}$, the set $U = \varphi^{-1}(\mathcal{O})$ is an open subset of $(\alpha_0, \alpha_0 + 2\pi)$, hence an open subset of $\mathbb{R}$. If $\mathcal{O}$ is empty the assertion is trivial (the empty union). Otherwise U is nonempty, and by Proposition 9 of Chapter 1, U is the disjoint union of a countable collection $\{(a_k, b_k)\}_k$ of open intervals. Each (a_k, b_k) is a nonempty subinterval of an interval of length 2π , so $0 < b_k - a_k \leq 2\pi$ and thus $\lambda_k = (a_k, b_k)$ belongs to Λ . Since φ is a one-to-one mapping,

$$\mathcal{O} = \varphi(U) = \bigcup_k \varphi((a_k, b_k)) = \bigcup_k I_{\lambda_k},$$

and this union is disjoint. This proves (ii).

Two supplements. First, the connectedness assertion used in the opening remark is now clear: $S^1 \sim \{z_0\}$ is homeomorphic, via φ , to the interval $(\alpha_0, \alpha_0 + 2\pi)$, which is connected. Second, if the complement $S^1 \sim \mathcal{O}$ contains a point other than z_0 , then U is a proper open subset of $(\alpha_0, \alpha_0 + 2\pi)$; since a component of U is an open subinterval of $(\alpha_0, \alpha_0 + 2\pi)$ and can have length 2π only by being all of $(\alpha_0, \alpha_0 + 2\pi)$, every (a_k, b_k) then has $b_k - a_k < 2\pi$. So the printed form of Λ suffices except when $S^1 \sim \mathcal{O}$ is a single point, in which case $\mathcal{O} = I_{(\alpha_0, \alpha_0 + 2\pi)}$ and $\beta - \alpha = 2\pi$ is needed.

Part (iii). We first record a lemma on periodic sets that makes everything transparent.

Lemma. Let A be a Lebesgue measurable subset of $\mathbb{R}$ with $A + 2\pi = A$. Then $m(A \cap [c, c + 2\pi))$ is the same for every $c \in \mathbb{R}$.

Proof of the Lemma. Choose an integer j with $c' = c + 2\pi j \in [0, 2\pi)$. Since m is translation invariant and $A + 2\pi j = A$,

$$m(A \cap [c, c + 2\pi)) = m((A + 2\pi j) \cap [c', c' + 2\pi)) = m(A \cap [c', c' + 2\pi)).$$

So we may assume $0 \leq c < 2\pi$. Then, the two sets on the right being disjoint,

$$A \cap [c, c + 2\pi) = (A \cap [c, 2\pi)) \cup (A \cap [2\pi, c + 2\pi)), \quad A \cap [0, 2\pi) = (A \cap [0, c)) \cup (A \cap [c, 2\pi)),$$

while $A \cap [2\pi, c + 2\pi) = (A \cap [0, c)) + 2\pi$ has measure $m(A \cap [0, c))$. Adding, $m(A \cap [c, c + 2\pi)) = m(A \cap [0, 2\pi))$. This proves the Lemma.

For an open subset $\mathcal{O}$ of S^1 define $E_{\mathcal{O}} = \Phi^{-1}(\mathcal{O}) = \{\theta \in \mathbb{R} \mid e^{i\theta} \in \mathcal{O}\}$, an open, hence measurable, subset of $\mathbb{R}$ satisfying $E_{\mathcal{O}} + 2\pi = E_{\mathcal{O}}$. Define

$$\mu(\mathcal{O}) = \frac{1}{2\pi} m(E_{\mathcal{O}} \cap [0, 2\pi)) \quad \text{for } \mathcal{O} \in \mathcal{T}.$$

This is exactly the extension of μ demanded in the problem. Indeed $\mu(\emptyset) = 0$ and $\mu(S^1) = m([0, 2\pi))/2\pi = 1$, and:

(a) For $\lambda = (\alpha, \beta) \in \Lambda$ we have $E_{I_{\lambda}} = \bigcup_{j \in \mathbb{Z}} (\alpha + 2\pi j, \beta + 2\pi j)$, and since $\beta \leq \alpha + 2\pi$ this union is disjoint and $E_{I_{\lambda}} \cap [\alpha, \alpha + 2\pi) = (\alpha, \beta)$. By the Lemma,

$$\mu(I_{\lambda}) = \frac{1}{2\pi} m(E_{I_{\lambda}} \cap [\alpha, \alpha + 2\pi)) = \frac{\beta - \alpha}{2\pi},$$

as prescribed.

(b) If $\mathcal{O}$ is a proper open set and $\mathcal{O} = \bigcup_k I_{\lambda_k}$ is any decomposition into a countable disjoint collection as in part (ii), with $\lambda_k = (a_k, b_k)$, then, since Φ^{-1} preserves unions and disjointness, $E_{\mathcal{O}}$ is the disjoint union of the sets $E_{I_{\lambda_k}}$, so by the countable additivity of m and (a),

$$\mu(\mathcal{O}) = \frac{1}{2\pi} \sum_k m(E_{I_{\lambda_k}} \cap [0, 2\pi)) = \sum_k \frac{b_k - a_k}{2\pi}.$$

Thus the value of $\mu(\mathcal{O})$ prescribed by part (ii) does not depend on which decomposition is used, and the set function defined by the displayed formula is the required extension of μ to all of $\mathcal{T}$.

Next, $\mu: \mathcal{T} \rightarrow [0, \infty]$ is a premeasure. We have $\mu(\emptyset) = 0$. If $\mathcal{O}_1, \dots, \mathcal{O}_n$ are disjoint open sets whose union $\mathcal{O}$ is open, then $E_{\mathcal{O}} = \bigcup_{k=1}^n E_{\mathcal{O}_k}$ and this union is disjoint, so by the additivity of m , $\mu(\mathcal{O}) = \sum_{k=1}^n \mu(\mathcal{O}_k)$; thus μ is finitely additive. If $\mathcal{O} \subseteq \bigcup_{k=1}^{\infty} \mathcal{O}_k$ with all sets open, then $E_{\mathcal{O}} \subseteq \bigcup_k E_{\mathcal{O}_k}$, and the countable monotonicity of m gives $\mu(\mathcal{O}) \leq \sum_k \mu(\mathcal{O}_k)$. So μ is countably monotone. Hence μ is a premeasure on $\mathcal{T}$.

Finally we verify hypothesis (7) of Proposition 9 of the preceding chapter: for each open $\mathcal{O}$ with $\mu(\mathcal{O}) < \infty$,

$$\mu(\mathcal{O}) = \sup\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ open and } \overline{\mathcal{U}} \subseteq \mathcal{O}\}.$$

Since $\mu \leq 1$ everywhere, we must check this for every open $\mathcal{O}$. For $\mathcal{O} = S^1$ take $\mathcal{U} = S^1$, and for $\mathcal{O} = \emptyset$ take $\mathcal{U} = \emptyset$. Let $\mathcal{O}$ be proper and nonempty and let $\epsilon > 0$. Write $\mathcal{O} = \bigcup_k I_{\lambda_k}$, a countable disjoint union with $\lambda_k = (a_k, b_k)$, as in part (ii). Since $\sum_k (b_k - a_k)/2\pi = \mu(\mathcal{O}) < \infty$, choose N with $\sum_{k > N} (b_k - a_k)/2\pi < \epsilon/2$, and then choose $\delta_k > 0$, for $1 \leq k \leq N$, so small that $a_k + \delta_k < b_k - \delta_k$ and $\sum_{k=1}^N 2\delta_k/2\pi < \epsilon/2$. Put

$$\mathcal{U} = \bigcup_{k=1}^N I_{(a_k + \delta_k, b_k - \delta_k)},$$

an open set. The closure of $I_{(a+\delta, b-\delta)}$ is contained in the compact, hence closed, set $\Phi([a + \delta, b - \delta]) \subseteq I_{(a, b)}$, and the closure of a finite union is the union of the closures; therefore $\overline{\mathcal{U}} \subseteq \bigcup_{k \leq N} I_{\lambda_k} \subseteq \mathcal{O}$. Moreover the sets comprising $\mathcal{U}$ are disjoint, being subsets of the disjoint sets I_{λ_k} , and $\mathcal{U}$ is a proper open subset of S^1 since $\overline{\mathcal{U}} \subseteq \mathcal{O} \neq S^1$. So, by (b),

$$\mu(\mathcal{U}) = \sum_{k=1}^N \frac{(b_k - a_k) - 2\delta_k}{2\pi} > \mu(\mathcal{O}) - \frac{\epsilon}{2} - \frac{\epsilon}{2} = \mu(\mathcal{O}) - \epsilon.$$

Since monotonicity gives $\mu(\mathcal{U}) \leq \mu(\mathcal{O})$, hypothesis (7) holds.

By Proposition 9 of the preceding chapter, every open set is μ^* -measurable, hence, by the minimality of the Borel σ -algebra, every Borel set is μ^* -measurable, and $\mu^*: \mathcal{B}(S^1) \rightarrow [0, \infty]$ is a measure that extends μ . Since S^1 itself is compact and $\mu^*(S^1) = 1 < \infty$, this is a Borel measure. We continue to denote it by μ .

Part (iv). We must show μ is a left-invariant Radon probability measure on $\mathcal{B}(S^1)$.

It is a probability measure since $\mu(S^1) = 1$.

Invariance. Fix $z_0 = e^{i\gamma} \in S^1$. For an open set $\mathcal{O}$, the set $z_0 \cdot \mathcal{O}$ is open (part (i): translation is a homeomorphism) and $E_{z_0 \cdot \mathcal{O}} = E_{\mathcal{O}} + \gamma$. Hence, using the translation invariance of m and then the Lemma applied to the 2π -periodic set $E_{\mathcal{O}}$,

$$\mu(z_0 \cdot \mathcal{O}) = \frac{1}{2\pi} m((E_{\mathcal{O}} + \gamma) \cap [0, 2\pi)) = \frac{1}{2\pi} m(E_{\mathcal{O}} \cap [-\gamma, -\gamma + 2\pi)) = \mu(\mathcal{O}).$$

So μ is invariant on open sets. Now let A be any subset of S^1 . By Lemma 8 of the preceding chapter,

$$\mu^*(A) = \inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } A\}.$$

Since $\mathcal{U} \mapsto z_0 \cdot \mathcal{U}$ is a bijection of the collection of neighborhoods of A onto the collection of neighborhoods of $z_0 \cdot A$, and $\mu(z_0 \cdot \mathcal{U}) = \mu(\mathcal{U})$, we conclude $\mu^*(z_0 \cdot A) = \mu^*(A)$. In particular $\mu(z_0 \cdot E) = \mu(E)$ for every Borel set E ; that is, μ is left-invariant. (Being abelian, S^1 makes left and right invariance the same thing.)

Radon. Outer regularity is immediate from the displayed formula of Lemma 8: for each Borel set E , $\mu(E) = \mu^*(E) = \inf\{\mu(\mathcal{U}) \mid \mathcal{U} \text{ a neighborhood of } E\}$. For inner regularity, let $\mathcal{O}$ be open and $\epsilon > 0$; the set $\mathcal{U}$ constructed above has $\overline{\mathcal{U}} \subseteq \mathcal{O}$ and $\mu(\mathcal{U}) > \mu(\mathcal{O}) - \epsilon$, and $K = \overline{\mathcal{U}}$ is a closed subset of the compact space S^1 , hence compact. Thus K is a compact subset of $\mathcal{O}$ with $\mu(K) \geq \mu(\mathcal{U}) > \mu(\mathcal{O}) - \epsilon$. Therefore $\mu(\mathcal{O}) = \sup\{\mu(K) \mid K \text{ compact, } K \subseteq \mathcal{O}\}$. Hence μ is a Radon measure. (Alternatively: what we verified is precisely that μ is a Radon premeasure on the compact, hence locally compact, Hausdorff space S^1 , so Theorem 10 of the preceding chapter applies.)

Thus μ is a left-invariant Radon probability measure on $\mathcal{B}(S^1)$, that is, μ is Haar measure on S^1 ; by Theorem 9 (von Neumann) it is the only one.

Part (v). Haar measure on T^n is the n -fold product of the measure μ constructed above, restricted to $\mathcal{B}(T^n)$:

$$\mu_n = (\mu \times \cdots \times \mu)|_{\mathcal{B}(T^n)}.$$

Two points make this meaningful and correct. First, T^n is a compact group: it is a topological group as a product of topological groups, and it is compact by Tychonoff's Theorem (or, since n is finite, by the Heine-Borel Theorem applied to T^n viewed as a closed bounded subset of $\mathbb{R}^{2n}$). Second, S^1 is a separable metric space, hence so is the finite product T^n , and consequently $\mathcal{B}(T^n)$ is contained in the product σ -algebra generated by $\mathcal{B}(S^1) \times \cdots \times \mathcal{B}(S^1)$, on which the product measure is defined; so the restriction above makes sense. That μ_n is Haar measure on T^n is exactly the content of Problem 20 and an induction on n : Problem 20 shows that if ν is Haar measure on a compact group $\mathcal{G}$ then $\nu \times \nu$ is Haar measure on $\mathcal{G} \times \mathcal{G}$, and its proof uses nothing about the two factors being the same group, so the product of Haar measures on two compact groups is Haar measure on the product group; applying this to $T^n = T^{n-1} \times S^1$ completes the induction. By the uniqueness in Theorem 9, μ_n is the only Haar measure on T^n .

Concretely, μ_n is normalized Lebesgue measure in the angular coordinates. On a product of arcs, that is, for $\lambda_1, \dots, \lambda_n \in \Lambda$ with $\lambda_j = (\alpha_j, \beta_j)$,

$$\mu_n(I_{\lambda_1} \times \cdots \times I_{\lambda_n}) = \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} = \frac{1}{(2\pi)^n} m_n((\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n)),$$

where m_n denotes Lebesgue measure on $\mathbb{R}^n$; this is immediate from the definition of the product measure on measurable rectangles together with part (iii). In the case $n = 1$ this is the measure of part (iii). The measure μ_n is thus the exact analogue on T^n of arc length divided by 2π on S^1 : it is the Borel probability measure on T^n whose value on a product of arcs is the normalized product of their angular lengths.

Problem 22.25. Let μ be a Borel measure on a topological group $\mathcal{G}$. For a Borel set E , define $\mu'(E) = \mu(E^{-1})$, where $E^{-1} = \{g^{-1} \mid g \in E\}$. Show that μ' also is a Borel measure. Moreover, show that μ is left-invariant if and only if μ' is right-invariant.

Solution. Let $\iota: \mathcal{G} \rightarrow \mathcal{G}$ be the inversion map $\iota(g) = g^{-1}$. By the definition of a topological group, ι is continuous, and since $(g^{-1})^{-1} = g$ we have $\iota \circ \iota = \text{id}_{\mathcal{G}}$. Hence ι is a one-to-one, onto, continuous mapping whose inverse, namely ι itself, is continuous: ι is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$. Note that for a subset A of $\mathcal{G}$ we have $A^{-1} = \iota(A) = \iota^{-1}(A)$.

Step 1: E^{-1} is a Borel set whenever E is. Consider

$$\mathcal{A} = \{E \subseteq \mathcal{G} \mid E^{-1} \in \mathcal{B}(\mathcal{G})\}.$$

Since ι is one-to-one and onto, the mapping $E \mapsto E^{-1}$ commutes with unions and complements:

$$\left(\bigcup_{k=1}^{\infty} E_k\right)^{-1} = \bigcup_{k=1}^{\infty} E_k^{-1}, \quad (\mathcal{G} \setminus E)^{-1} = \mathcal{G} \setminus E^{-1}.$$

Therefore $\mathcal{A}$ is a σ -algebra. If $\mathcal{O}$ is open, then $\mathcal{O}^{-1} = \iota(\mathcal{O})$ is open because ι is a homeomorphism; hence $\mathcal{A}$ contains the topology of $\mathcal{G}$. By the minimality of the Borel σ -algebra, $\mathcal{B}(\mathcal{G}) \subseteq \mathcal{A}$. So $\mu'(E) = \mu(E^{-1})$ is defined for every Borel set E , and moreover $E \mapsto E^{-1}$ is a bijection of $\mathcal{B}(\mathcal{G})$ onto itself.

Step 2: μ' is a measure. First, $\mu'(\emptyset) = \mu(\emptyset^{-1}) = \mu(\emptyset) = 0$, and μ' takes values in $[0, \infty]$. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of Borel sets. Since ι is one-to-one, the sets $E_k^{-1} = \iota(E_k)$ are also disjoint, and they are Borel by Step 1. Hence, by the countable additivity of μ ,

$$\mu'\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} E_k^{-1}\right) = \sum_{k=1}^{\infty} \mu(E_k^{-1}) = \sum_{k=1}^{\infty} \mu'(E_k).$$

So μ' is a measure on $\mathcal{B}(\mathcal{G})$.

Step 3: μ' is a Borel measure, that is, it is finite on compact sets. Let K be a compact subset of $\mathcal{G}$. Then $K^{-1} = \iota(K)$ is the continuous image of a compact set, hence compact. Since μ is a Borel measure, $\mu'(K) = \mu(K^{-1}) < \infty$. Thus μ' is a Borel measure.

Step 4: the invariance equivalence. We use the identity

$$(g \cdot E)^{-1} = E^{-1} \cdot g^{-1} \quad \text{for } g \in \mathcal{G}, E \subseteq \mathcal{G},$$

which holds because $x \in (g \cdot E)^{-1}$ if and only if $x^{-1} = g \cdot h$ for some $h \in E$, if and only if $x = h^{-1} \cdot g^{-1}$ for some $h \in E$. Replacing E by E^{-1} and g by h^{-1} and inverting, this identity is equivalent to

$$(E \cdot h)^{-1} = h^{-1} \cdot E^{-1} \quad \text{for } h \in \mathcal{G}, E \subseteq \mathcal{G}.$$

Note also that, exactly as for left translation in Problem 23, right translation $x \mapsto x \cdot h$ is a homeomorphism of $\mathcal{G}$ onto $\mathcal{G}$, so $E \cdot h$ is a Borel set whenever E is; similarly for $g \cdot E$. Finally, since $(E^{-1})^{-1} = E$, the definition of μ' applied to the Borel set E^{-1} gives the inversion formula

$$\mu'(E^{-1}) = \mu((E^{-1})^{-1}) = \mu(E) \quad \text{for all } E \in \mathcal{B}(\mathcal{G}).$$

Assume μ is left-invariant. Let $h \in \mathcal{G}$ and let E be a Borel set. Then

$$\mu'(E \cdot h) = \mu((E \cdot h)^{-1}) = \mu(h^{-1} \cdot E^{-1}) = \mu(E^{-1}) = \mu'(E),$$

the third equality by the left-invariance of μ applied to the Borel set E^{-1} . Hence μ' is right-invariant. Conversely, assume μ' is right-invariant. Let $g \in \mathcal{G}$ and let E be a Borel set. Then, using the inversion formula and the first displayed identity,

$$\mu(g \cdot E) = \mu'((g \cdot E)^{-1}) = \mu'(E^{-1} \cdot g^{-1}) = \mu'(E^{-1}) = \mu(E),$$

the third equality by the right-invariance of μ' . Hence μ is left-invariant.

This is the duality used at the end of the proof of Proposition 8 to pass from the existence of a left-invariant Radon probability measure to the existence of a right-invariant one: if μ is left-invariant, then μ' is right-invariant, and μ' inherits from μ the properties of being a probability measure and of being Radon, since ι is a homeomorphism and therefore carries neighborhoods to neighborhoods and compact sets to compact sets.

Problem 22.26. Let X be a compact metric space. Use the Stone-Weierstrass Theorem to show that the Banach space $C(X)$ of continuous functions on X , normed with the maximum norm, is separable.

Solution. Let ρ be the metric on X . If X is empty the assertion is trivial, so assume X is nonempty.

Step 1: X is separable. For each natural number n , the collection of open balls $\{B(x, 1/n)\}_{x \in X}$ is an open cover of X . Since X is compact, there are finitely many points $x_{n,1}, \dots, x_{n,k_n}$ in X for which $X = \bigcup_{j=1}^{k_n} B(x_{n,j}, 1/n)$. Let

$$D = \{x_{n,j} \mid n \in \mathbb{N}, 1 \leq j \leq k_n\},$$

a countable set. It is dense: given $x \in X$ and $\epsilon > 0$, choose n with $1/n < \epsilon$; then x belongs to some $B(x_{n,j}, 1/n)$, so $\rho(x, x_{n,j}) < \epsilon$. Enumerate $D = \{x_1, x_2, x_3, \dots\}$ (if D is finite, then $X = D$ is a finite set and the argument below simplifies; we treat D as a countable, possibly finite, list).

Step 2: a countable family that separates points. For each index n define

$$f_n: X \rightarrow \mathbb{R}, \quad f_n(x) = \rho(x, x_n).$$

Each f_n is continuous; indeed it is Lipschitz with constant 1, since the triangle inequality gives $|\rho(x, x_n) - \rho(y, x_n)| \leq \rho(x, y)$. The family $\{f_n\}$ separates the points of X . Let $u \neq v$ in X and put $d = \rho(u, v) > 0$. By the density of D , choose n with $\rho(u, x_n) < d/2$. Then $f_n(u) < d/2$, while by the triangle inequality

$$f_n(v) = \rho(v, x_n) \geq \rho(u, v) - \rho(u, x_n) > d - d/2 = d/2.$$

Hence $f_n(u) \neq f_n(v)$.

Step 3: the algebra generated by this family. Let P be the collection of all finite products

$$f_{n_1} \cdot f_{n_2} \cdots f_{n_k}, \quad k \geq 0, \quad n_1, \dots, n_k \text{ indices,}$$

where the empty product ($k = 0$) is understood to be the constant function 1. The collection P is countable, being the union over $k \geq 0$ of sets that are images of the countable sets $\mathbb{N}^k$. Let

$$\mathcal{A} = \left\{ \sum_{j=1}^m c_j p_j \mid m \in \mathbb{N}, \quad c_j \in \mathbb{R}, \quad p_j \in P \right\},$$

the linear span of P over $\mathbb{R}$. Then $\mathcal{A}$ is a collection of continuous real-valued functions on X that is closed under addition and scalar multiplication, and it is closed under multiplication because the product of two members of P again belongs to P and multiplication distributes over the sums. So $\mathcal{A}$ is an algebra. It contains the constant functions, since the empty product 1 belongs to P . And it separates points in X by Step 2, since each f_n belongs to $\mathcal{A}$.

A compact metric space is compact and Hausdorff. Therefore, by the Stone-Weierstrass Approximation Theorem, $\mathcal{A}$ is dense in $C(X)$.

Step 4: a countable dense subset of $C(X)$. Let

$$\mathcal{A}_{\mathbb{Q}} = \left\{ \sum_{j=1}^m q_j p_j \mid m \in \mathbb{N}, \quad q_j \in \mathbb{Q}, \quad p_j \in P \right\},$$

the set of rational linear combinations of members of P . This set is countable. Indeed, for each m , the collection of such sums with exactly m terms is the image of the countable set $\mathbb{Q}^m \times P^m$ under an explicit mapping, hence is countable; and $\mathcal{A}_{\mathbb{Q}}$ is the union over $m \in \mathbb{N}$ of these countable sets, hence is countable.

We claim $\mathcal{A}_{\mathbb{Q}}$ is dense in $\mathcal{A}$ with respect to the maximum norm. Let $f = \sum_{j=1}^m c_j p_j$ belong to $\mathcal{A}$ and let $\epsilon > 0$. Each p_j is continuous on the compact space X , so by the Extreme Value Theorem $\|p_j\|_{\max} < \infty$; set $M = 1 + \max_{1 \leq j \leq m} \|p_j\|_{\max}$. By the density of the rationals in $\mathbb{R}$, choose rational numbers $q_1, \dots, q_m$ with $|c_j - q_j| < \epsilon/(mM)$ for $1 \leq j \leq m$. Then $g = \sum_{j=1}^m q_j p_j$ belongs to $\mathcal{A}_{\mathbb{Q}}$ and, for each $x \in X$,

$$|f(x) - g(x)| \leq \sum_{j=1}^m |c_j - q_j| |p_j(x)| \leq \sum_{j=1}^m \frac{\epsilon}{mM} \cdot M = \epsilon.$$

Hence $\|f - g\|_{\max} \leq \epsilon$, which proves the claim.

Finally, let $h \in C(X)$ and $\epsilon > 0$. By Step 3 there is an $f \in \mathcal{A}$ with $\|h - f\|_{\max} < \epsilon/2$, and by Step 4 there is a $g \in \mathcal{A}_{\mathbb{Q}}$ with $\|f - g\|_{\max} < \epsilon/2$. By the triangle inequality, $\|h - g\|_{\max} < \epsilon$. Therefore the countable set $\mathcal{A}_{\mathbb{Q}}$ is dense in $C(X)$, and so the Banach space $C(X)$ is separable.

This is exactly the forward implication of Riesz's Theorem of Section 12.3, and the argument above is the one given there; the converse implication of that theorem, that separability of $C(X)$ forces a compact Hausdorff X to be metrizable, uses Urysohn's Lemma and the Urysohn Metrization Theorem and is not needed here. Note also that only the compactness of X , not its completeness or any completeness of $C(X)$, entered the argument.

Problem 22.27. Does the proof of the Bogoliubov-Krilov Theorem also provide a proof in the case X is compact Hausdorff but not necessarily metrizable?

Solution. No, the proof as written does not, since it invokes metrizability at two places, and both invocations genuinely fail for a general compact Hausdorff space. However, the proof can be repaired, and the theorem itself remains true: if X is compact Hausdorff and $f: X \rightarrow X$ is continuous, there is a probability measure on $\mathcal{B}(X)$ with respect to which f is measure preserving. We explain both halves of this answer.

Where the given proof uses metrizability. The proof of Theorem 12 begins by asserting that, since

X is a compact metric space, Borsuk's Theorem tells us that $C(X)$ is separable. It then defines the sequence $\{\psi_n\}$ of functionals in (23), notes it is bounded in $[C(X)]^*$, and concludes: since the Banach space $C(X)$ is separable, Helley's Theorem yields a subsequence $\{\psi_{n_k}\}$ that converges, with respect to the weak-* topology, to a bounded functional $\psi \in [C(X)]^*$. Both the appeal to Borsuk's Theorem and the appeal to Helley's Theorem require separability of $C(X)$, and separability of $C(X)$ requires metrizability of X .

The separability step really can fail. Let $X = \{0, 1\}^{[0,1]}$, the product of continuum many copies of the two-point discrete space, with the product topology. By Tychonoff's Theorem X is compact, and it is Hausdorff as a product of Hausdorff spaces. For $t \in [0, 1]$ let $\pi_t \in C(X)$ be the t -th coordinate projection, which is continuous by the definition of the product topology. If $s \neq t$, the point $x \in X$ with $x(t) = 1$ and $x(s) = 0$ has $|\pi_t(x) - \pi_s(x)| = 1$, and both functions take values in $\{0, 1\}$, so

$$\|\pi_t - \pi_s\|_{\max} = 1 \quad \text{for } s \neq t.$$

Thus $C(X)$ contains an uncountable family of functions that are pairwise at distance 1. If $\mathcal{D}$ were a countable dense subset of $C(X)$, then each ball $B(\pi_t, 1/2)$ would contain a point of $\mathcal{D}$ and these balls are disjoint, forcing $[0, 1]$ to be countable. So $C(X)$ is not separable. Consequently Helley's Theorem is simply not available here; and it is not merely the hypothesis that fails: the closed unit ball of $[C(X)]^*$ need not be weak-* sequentially compact when $C(X)$ is nonseparable, so no subsequence of $\{\psi_n\}$ need converge in the weak-* topology.

How the proof is repaired. The only role played by Helley's Theorem is to produce a single weak-* limit point ψ of the bounded sequence $\{\psi_n\}$; sequential convergence is never used afterwards, only the two conclusions that ψ is a positive functional with $\psi(1) = 1$ and that $\psi(g \circ f) = \psi(g)$ for all $g \in C(X)$. A weak-* cluster point of $\{\psi_n\}$ suffices, and Alaoglu's Theorem produces one for any normed linear space. In detail, keep every other step of the printed proof and argue as follows.

Fix any Borel probability measure η on $\mathcal{B}(X)$; for instance a Dirac delta measure at a point of X , which is a Radon measure. As in (23), define

$$\psi_n(g) = \int_X \left[\frac{1}{n} \sum_{k=0}^{n-1} g \circ f^k \right] d\eta \quad \text{for } n \in \mathbb{N}, \quad g \in C(X),$$

so that each ψ_n is a linear functional on $C(X)$ with $|\psi_n(g)| \leq \|g\|_{\max}$, with $\psi_n(g) \geq 0$ when $g \geq 0$, and with $\psi_n(1) = 1$. Thus $\{\psi_n\}$ lies in the closed unit ball B of $[C(X)]^*$, which by Alaoglu's Theorem is weak-* compact.

For each n , let K_n be the weak-* closure of $\{\psi_m \mid m \geq n\}$. Each K_n is a nonempty weak-* closed subset of the weak-* compact set B , and $K_1 \supseteq K_2 \supseteq \cdots$. Since the collection is descending and each member is nonempty, every finite subcollection has nonempty intersection; a collection of closed subsets of a compact space with this finite intersection property has nonempty intersection. Hence there is a functional

$$\psi \in \bigcap_{n=1}^{\infty} K_n.$$

Observe that, by the definition of the weak-* topology, every weak-* neighborhood of ψ contains ψ_m for arbitrarily large m .

We check the three properties of ψ that the proof needs.

First, $\psi(1) = 1$: for $\epsilon > 0$ the set $\{\varphi \mid |\varphi(1) - \psi(1)| < \epsilon\}$ is a weak-* neighborhood of ψ , so it contains some ψ_m , and $\psi_m(1) = 1$; hence $|\psi(1) - 1| < \epsilon$ for every $\epsilon > 0$.

Second, ψ is positive: let $g \in C(X)$ with $g \geq 0$ and $\epsilon > 0$. The set $\{\varphi \mid |\varphi(g) - \psi(g)| < \epsilon\}$ contains some ψ_m , and $\psi_m(g) \geq 0$; hence $\psi(g) > -\epsilon$ for every $\epsilon > 0$, so $\psi(g) \geq 0$.

Third, $\psi(g \circ f) = \psi(g)$ for all $g \in C(X)$. As computed in the printed proof,

$$\psi_m(g \circ f) - \psi_m(g) = \frac{1}{m} \int_X [g \circ f^m - g] d\eta, \quad \text{so} \quad |\psi_m(g \circ f) - \psi_m(g)| \leq \frac{2\|g\|_{\max}}{m}.$$

Let $\epsilon > 0$ and choose n with $2\|g\|_{\max}/n < \epsilon$. The set

$$W = \{\varphi \mid |\varphi(g \circ f) - \psi(g \circ f)| < \epsilon/2 \text{ and } |\varphi(g) - \psi(g)| < \epsilon/2\}$$

is a weak-* neighborhood of ψ , so since ψ belongs to K_n , there is an $m \geq n$ with $\psi_m \in W$. Then

$$|\psi(g \circ f) - \psi(g)| \leq |\psi(g \circ f) - \psi_m(g \circ f)| + |\psi_m(g \circ f) - \psi_m(g)| + |\psi_m(g) - \psi(g)| < \frac{\epsilon}{2} + \frac{2\|g\|_{\max}}{m} + \frac{\epsilon}{2} < 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, $\psi(g \circ f) = \psi(g)$.

From here the printed proof continues verbatim: since ψ is a positive linear functional on $C(X)$ and X is compact Hausdorff, the Riesz-Markov Theorem provides a Radon measure μ on $\mathcal{B}(X)$ with $\psi(g) = \int_X g d\mu$ for all $g \in C(X)$; the identity $\psi(g \circ f) = \psi(g)$ becomes $\int_X g \circ f d\mu = \int_X g d\mu$ for all $g \in C(X)$, and Proposition 10 tells us that f is measure preserving with respect to μ ; and $\mu(X) = \psi(1) = 1$, so μ is a probability measure.

In summary: the printed proof does not carry over, because its two appeals to separability (Borsuk's Theorem and Helley's Theorem) are unavailable, but replacing Helley's Theorem by Alaoglu's Theorem together with the finite intersection property repairs it, and the Bogoliubov-Krilov Theorem holds for every compact Hausdorff space.

Problem 22.28. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $T: X \rightarrow X$ a measurable transformation. For a measurable function g on X , define the measurable function $U_T(g)$ by $U_T(g)(x) = g(T(x))$. Show that T is measure preserving if and only if for every $1 \leq p < \infty$, U_T maps $L^p(X, \mu)$ into itself and is an isometry.

Solution. Recall that T is a measurable transformation means $T^{-1}(A) \in \mathcal{M}$ for each $A \in \mathcal{M}$, and that by observation (19) of this section this holds if and only if $g \circ T$ is measurable whenever g is measurable; thus $U_T(g) = g \circ T$ is indeed a measurable function. Recall also that T is measure preserving means $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{M}$. Note that U_T is linear on measurable functions, since $(c_1 g_1 + c_2 g_2) \circ T = c_1(g_1 \circ T) + c_2(g_2 \circ T)$ pointwise on X .

Suppose first that T is measure preserving.

We begin with a change of variables formula for nonnegative measurable functions, which extends Proposition 10 beyond $L^1(X, \mu)$: if $h: X \rightarrow [0, \infty]$ is measurable, then

$$\int_X h \circ T d\mu = \int_X h d\mu \quad (\text{change of variables}).$$

To see this, first let $\varphi = \sum_{k=1}^n c_k \cdot \chi_{A_k}$ be a nonnegative simple function, where the A_k belong to $\mathcal{M}$ and the c_k are nonnegative real numbers. Since $\chi_{A_k} \circ T = \chi_{T^{-1}(A_k)}$, we have $\varphi \circ T = \sum_{k=1}^n c_k \cdot \chi_{T^{-1}(A_k)}$, and therefore, since T is measure preserving,

$$\int_X \varphi \circ T d\mu = \sum_{k=1}^n c_k \cdot \mu(T^{-1}(A_k)) = \sum_{k=1}^n c_k \cdot \mu(A_k) = \int_X \varphi d\mu.$$

Now let h be a general nonnegative measurable function on X . By the Simple Approximation Theorem there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on X that converges pointwise on X to h . Then $\{\varphi_n \circ T\}$ is an increasing sequence of nonnegative simple functions converging pointwise on X to $h \circ T$. Two applications of the Monotone Convergence Theorem, together with the preceding computation, give

$$\int_X h \circ T d\mu = \lim_{n \rightarrow \infty} \int_X \varphi_n \circ T d\mu = \lim_{n \rightarrow \infty} \int_X \varphi_n d\mu = \int_X h d\mu.$$

This proves the change of variables formula.

First observe that U_T is well defined on equivalence classes. If $g = g'$ almost everywhere on X , let $N = \{x \in X \mid g(x) \neq g'(x)\}$, a measurable set with $\mu(N) = 0$. Then $\{x \in X \mid g(T(x)) \neq g'(T(x))\} = T^{-1}(N)$, and $\mu(T^{-1}(N)) = \mu(N) = 0$. So $g \circ T = g' \circ T$ almost everywhere on X .

Now fix p with $1 \leq p < \infty$ and let g belong to $L^p(X, \mu)$. The function $h = |g|^p$ is nonnegative and measurable, and $h \circ T = |g \circ T|^p$ pointwise on X . Therefore, by the change of variables formula,

$$\int_X |U_T(g)|^p d\mu = \int_X |g|^p \circ T d\mu = \int_X |g|^p d\mu < \infty.$$

Hence $U_T(g)$ belongs to $L^p(X, \mu)$, so U_T maps $L^p(X, \mu)$ into itself, and taking p -th roots,

$$\|U_T(g)\|_p = \|g\|_p.$$

Since U_T is linear, $\|U_T(g_1) - U_T(g_2)\|_p = \|U_T(g_1 - g_2)\|_p = \|g_1 - g_2\|_p$, so U_T is an isometry of $L^p(X, \mu)$ into itself. This holds for every $1 \leq p < \infty$.

Conversely, suppose that for every $1 \leq p < \infty$ the operator U_T maps $L^p(X, \mu)$ into itself and is an isometry. In fact we only need this for a single value of p ; take $p = 1$.

Let A belong to $\mathcal{M}$. Since $\mu(X) < \infty$, the function χ_A is measurable and $\int_X |\chi_A| d\mu = \mu(A) \leq \mu(X) < \infty$, so χ_A belongs to $L^1(X, \mu)$. Here is where the finiteness of μ is used: without it a characteristic function need not lie in L^1 . Since T is a measurable transformation, $T^{-1}(A)$ belongs to $\mathcal{M}$, and for each $x \in X$, $U_T(\chi_A)(x) = \chi_A(T(x))$, which equals 1 if $T(x) \in A$ and equals 0 otherwise; that is, $U_T(\chi_A) = \chi_{T^{-1}(A)}$. Because U_T is an isometry of $L^1(X, \mu)$,

$$\mu(T^{-1}(A)) = \int_X \chi_{T^{-1}(A)} d\mu = \|U_T(\chi_A)\|_1 = \|\chi_A\|_1 = \int_X \chi_A d\mu = \mu(A).$$

Since A was an arbitrary member of $\mathcal{M}$, the transformation T is measure preserving.

The two implications together establish the equivalence. We remark that the same argument with a general $p \in [1, \infty)$ gives $\mu(T^{-1}(A))^{1/p} = \mu(A)^{1/p}$, hence the same conclusion; so in the stated equivalence the quantifier over all p in $[1, \infty)$ may be weakened to a quantifier over some single p in $[1, \infty)$. We remark also that U_T need not be onto: for $X = [0, 1]$ with Lebesgue measure and $T(x)$ the doubling map, the range of U_T consists only of functions that are, up to sets of measure zero, invariant under $x \mapsto x + 1/2$ modulo 1.

22.5 Problems 22.29–22.33

Problem 22.29. Suppose that $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear. Establish necessary and sufficient conditions for T to be measure preserving with respect to Lebesgue measure on $\mathbb{R}^n$.

Solution. Write μ_n for Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{M}_n$ for the σ -algebra of Lebesgue measurable subsets of $\mathbb{R}^n$. Recall that T is measure preserving provided $T^{-1}(E) \in \mathcal{M}_n$ and $\mu_n(T^{-1}(E)) = \mu_n(E)$ for every $E \in \mathcal{M}_n$.

Claim. A linear operator $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is measure preserving with respect to Lebesgue measure if and only if T is invertible and $|\det T| = 1$; equivalently, if and only if $\mu_n(T([0, 1]^n)) = 1$.

Sufficiency. Suppose T is invertible and $|\det T| = 1$. Then T^{-1} is a linear operator, and by the multiplicative property (25) of the determinant of Chapter 20, $\det T^{-1} \cdot \det T = \det(\text{Id}) = 1$, so that

$$|\det T^{-1}| = \frac{1}{|\det T|} = 1.$$

Let E be Lebesgue measurable. Since T is a bijection with inverse T^{-1} , the preimage of E under T coincides with the image of E under the invertible linear operator T^{-1} :

$$T^{-1}(E) = \{x \in \mathbb{R}^n : T(x) \in E\} = \{T^{-1}(y) : y \in E\}.$$

By Corollary 23 of Chapter 20 applied to the invertible operator T^{-1} , the set $T^{-1}(E)$ is Lebesgue measurable and

$$\mu_n(T^{-1}(E)) = |\det T^{-1}| \cdot \mu_n(E) = \mu_n(E).$$

Thus T is a measurable transformation of $(\mathbb{R}^n, \mathcal{M}_n)$ and it is measure preserving.

Necessity. Suppose first that T is invertible but $d := |\det T| \neq 1$. Take $E = [0, 1]^n$, so that $\mu_n(E) = 1$. Exactly as above, Corollary 23 of Chapter 20 gives

$$\mu_n(T^{-1}(E)) = |\det T^{-1}| \cdot \mu_n(E) = \frac{1}{d} \neq 1 = \mu_n(E),$$

so T is not measure preserving.

Now suppose T is not invertible. Then $V := T(\mathbb{R}^n)$ is a proper linear subspace of $\mathbb{R}^n$. We first check that $\mu_n(V) = 0$. Let $H_0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$. For each $N \in \mathbb{N}$ and each $\epsilon > 0$, the bounded portion $H_0 \cap [-N, N]^n$ is contained in the rectangle $[-N, N]^{n-1} \times [-\epsilon, \epsilon]$, whose Lebesgue measure is $(2N)^{n-1} \cdot 2\epsilon$. Letting $\epsilon \rightarrow 0$ we get $\mu_n(H_0 \cap [-N, N]^n) = 0$ for every N , and hence, by the countable additivity (or monotonicity together with continuity) of measure, $\mu_n(H_0) = 0$. Since V is a proper subspace, $\dim V \leq n - 1$, so we may choose a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ whose first $n - 1$ vectors span a subspace containing V . Let $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear operator determined by $S(e_k) = v_k$ for $1 \leq k \leq n$; then S is invertible and $S(H_0) \supseteq V$. By Corollary 23 of Chapter 20,

$$\mu_n(S(H_0)) = |\det S| \cdot \mu_n(H_0) = 0,$$

and therefore $\mu_n(V) = 0$ by monotonicity. Since V is a linear subspace of $\mathbb{R}^n$, it is closed, hence a Borel set and in particular Lebesgue measurable. But every point of $\mathbb{R}^n$ is mapped by T into V , so

$$T^{-1}(V) = \mathbb{R}^n, \quad \mu_n(T^{-1}(V)) = \infty \neq 0 = \mu_n(V).$$

Hence T is not measure preserving. This completes the proof of the claim.

The reformulation. Finally, observe that by Corollary 23 of Chapter 20, if T is invertible then $\mu_n(T([0, 1]^n)) = |\det T| \cdot \mu_n([0, 1]^n) = |\det T|$, while if T is not invertible then $T([0, 1]^n) \subseteq V$ has measure zero. So in all cases $\mu_n(T([0, 1]^n)) = |\det T|$, and the condition $|\det T| = 1$ is the same as the geometrically stated condition that T carry the unit cube onto a set of measure one.

Problem 22.30. Let $S^1 = \{z = e^{i\theta} \mid \theta \in \mathbb{R}\}$ be the circle with the group operation of complex multiplication and μ be Haar measure on this group (see Problem 24). Define $T: S^1 \rightarrow S^1$ by $T(z) = z^2$. Show that T preserves μ .

Solution. Throughout, μ is the Haar measure constructed in Problem 24: it is the Borel measure on $\mathcal{B}(S^1)$ determined by

$$\mu(S^1) = 1 \quad \text{and} \quad \mu(I_{(\alpha, \beta)}) = \frac{\beta - \alpha}{2\pi} \quad \text{for } 0 < \beta - \alpha < 2\pi,$$

where $I_{(\alpha, \beta)} = \{e^{i\theta} \mid \alpha < \theta < \beta\}$ denotes the open arc.

Step 1: T is a measurable transformation, and $\nu := \mu \circ T^{-1}$ is a Borel probability measure.

The mapping $T(z) = z^2$ is continuous on S^1 , so $T^{-1}(E)$ is a Borel set whenever E is a Borel set. Define

$$\nu(E) = \mu(T^{-1}(E)) \quad \text{for } E \in \mathcal{B}(S^1).$$

Then $\nu(\emptyset) = 0$, and if $\{E_k\}_{k=1}^\infty$ is a disjoint collection of Borel sets, the sets $\{T^{-1}(E_k)\}$ are also disjoint and $T^{-1}(\bigcup_k E_k) = \bigcup_k T^{-1}(E_k)$; hence the countable additivity of μ yields the countable additivity of ν . Thus ν is a Borel measure, and $\nu(S^1) = \mu(S^1) = 1$. To prove the problem it suffices to show $\nu = \mu$ on $\mathcal{B}(S^1)$.

Step 2: singletons are null, and half-open arcs have the expected measure.

Fix $\alpha \in \mathbb{R}$. For each ϵ with $0 < \epsilon < \pi$ we have $e^{i\alpha} \in I_{(\alpha-\epsilon, \alpha+\epsilon)}$, and therefore, by the monotonicity of μ ,

$$\mu(\{e^{i\alpha}\}) \leq \frac{2\epsilon}{2\pi} = \frac{\epsilon}{\pi}.$$

Letting $\epsilon \rightarrow 0$ gives $\mu(\{e^{i\alpha}\}) = 0$. Consequently, if for $\alpha \in \mathbb{R}$ and $0 < \ell \leq 2\pi$ we define the half-open arc

$$J(\alpha, \ell) = \{e^{i\theta} \mid \alpha \leq \theta < \alpha + \ell\},$$

then $J(\alpha, \ell) = I_{(\alpha, \alpha+\ell)} \cup \{e^{i\alpha}\}$ when $\ell < 2\pi$, and $J(\alpha, 2\pi) = S^1$, so that in every case

$$\mu(J(\alpha, \ell)) = \frac{\ell}{2\pi}. \quad (*)$$

Step 3: ν and μ agree on half-open arcs.

Fix $\alpha \in \mathbb{R}$ and $0 < \ell \leq 2\pi$, and put $J = J(\alpha, \ell)$. If $\ell = 2\pi$ then $J = S^1$ and $\nu(J) = 1 = \mu(J)$. So assume $0 < \ell < 2\pi$. For $z = e^{i\phi}$ we have $T(z) = e^{2i\phi}$, and therefore

$$z \in T^{-1}(J) \iff 2\phi \in [\alpha, \alpha + \ell) \pmod{2\pi} \iff \phi \in \left[\frac{\alpha}{2}, \frac{\alpha}{2} + \frac{\ell}{2}\right) \pmod{\pi}.$$

Reducing modulo 2π instead, this says precisely that

$$T^{-1}(J) = J\left(\frac{\alpha}{2}, \frac{\ell}{2}\right) \cup J\left(\frac{\alpha}{2} + \pi, \frac{\ell}{2}\right).$$

These two half-open arcs each have length $\ell/2 < \pi$ and their initial points differ by π , so they are disjoint. Hence, by (*) and the additivity of μ ,

$$\nu(J) = \mu(T^{-1}(J)) = \frac{\ell/2}{2\pi} + \frac{\ell/2}{2\pi} = \frac{\ell}{2\pi} = \mu(J).$$

Step 4: from arcs to all Borel sets.

Let $\mathcal{A}$ be the collection of all finite disjoint unions of half-open arcs, together with $\emptyset$. Then $\mathcal{A}$ is an algebra of subsets of S^1 : the complement of $J(\alpha, \ell)$ is $J(\alpha + \ell, 2\pi - \ell)$ if $\ell < 2\pi$ and is $\emptyset$ if $\ell = 2\pi$; the intersection of two half-open arcs is a disjoint union of at most two half-open arcs; and closure of $\mathcal{A}$ under finite intersections and complements follows from these two observations together with De Morgan's Identities. In particular $\mathcal{A}$ is a semiring.

Moreover $\sigma(\mathcal{A}) = \mathcal{B}(S^1)$. Indeed each half-open arc is Borel, so $\sigma(\mathcal{A}) \subseteq \mathcal{B}(S^1)$; conversely, for $0 < \beta - \alpha < 2\pi$ the open arc $I_{(\alpha, \beta)}$ is the ascending countable union

$$I_{(\alpha, \beta)} = \bigcup_{k=2}^{\infty} J\left(\alpha + \frac{\beta - \alpha}{k}, (\beta - \alpha)\left(1 - \frac{1}{k}\right)\right),$$

since the k -th set on the right consists of the $e^{i\theta}$ with $\alpha + (\beta - \alpha)/k \leq \theta < \beta$. By Problem 24(ii), every proper open subset of S^1 is a countable disjoint union of open arcs, so every open subset of S^1 belongs to $\sigma(\mathcal{A})$, and hence $\mathcal{B}(S^1) \subseteq \sigma(\mathcal{A})$.

By Step 3 and the finite additivity of the measures μ and ν , these two measures agree on every finite disjoint union of half-open arcs, that is, they agree on $\mathcal{A}$. Denote by μ_0 this common restriction. Then μ_0 is a premeasure on the semiring $\mathcal{A}$, being the restriction of the measure μ to a subcollection of its measurable sets, and μ_0 is finite, since $\mu_0(S^1) = 1$; in particular μ_0 is σ -finite. By the Caratheodory–Hahn Theorem of Chapter 17, the Caratheodory measure induced by μ_0 is the unique measure extending μ_0 to the σ -algebra of μ_0^* -measurable sets, a σ -algebra that contains $\sigma(\mathcal{A}) = \mathcal{B}(S^1)$; so a σ -finite premeasure on $\mathcal{A}$ determines its extension to $\mathcal{B}(S^1)$ uniquely. (This is precisely the use made of the uniqueness assertion in Chapter 20, where Lebesgue measure μ_n is identified as the one measure extending the volume premeasure on the intervals of $\mathbb{R}^n$.) Since μ and ν are both measures on $\mathcal{B}(S^1)$ extending μ_0 ,

$$\mu(E) = \nu(E) = \mu(T^{-1}(E)) \quad \text{for all } E \in \mathcal{B}(S^1).$$

That is, T preserves Haar measure μ on S^1 .

Remark. The computation in Step 3 has a transparent explanation. The mapping T is a two-to-one group homomorphism of S^1 onto itself, so for $w \in S^1$, choosing $v \in S^1$ with $v^2 = w$, we have $z \in T^{-1}(wE)$ if and only if $z^2 \in wE$, if and only if $(z/v)^2 \in E$, that is,

$$T^{-1}(wE) = v \cdot T^{-1}(E).$$

Hence $\nu(wE) = \mu(v \cdot T^{-1}(E)) = \mu(T^{-1}(E)) = \nu(E)$ by the invariance of Haar measure, so ν is an invariant Borel probability measure on S^1 . Once one checks in addition that ν is a Radon measure — which it is, since T is a continuous surjection of the compact space S^1 onto itself, so that $O \mapsto S^1 \sim T(S^1 \sim O)$ carries the outer and inner regularity of μ over to ν — the uniqueness of Haar measure asserted in Theorem 9 (von Neumann) gives $\nu = \mu$ directly. The argument in Steps 1–4 was given because it needs no regularity considerations at all.

Problem 22.31. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f(x, y) = (2x, y/2)$. Show that f is measure preserving with respect to Lebesgue measure.

Solution. Write μ_2 for Lebesgue measure on $\mathbb{R}^2$ and μ_2^* for Lebesgue outer measure. The mapping f is linear and bijective, with inverse the linear mapping

$$g(x, y) = \left(\frac{x}{2}, 2y\right),$$

and since f is a bijection, for every set $E \subseteq \mathbb{R}^2$ the preimage of E under f is the image of E under g :

$$f^{-1}(E) = \{(x, y) : f(x, y) \in E\} = g(E). \quad (*)$$

So we must show that g carries Lebesgue measurable sets to Lebesgue measurable sets of the same measure.

Step 1: g preserves the volume of intervals.

Recall that Lebesgue outer measure on $\mathbb{R}^2$ is defined by

$$\mu_2^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k, \{I_k\} \text{ intervals in } \mathbb{R}^2 \right\},$$

where, as in Chapter 20, an interval in $\mathbb{R}^2$ is a product $I = I_1 \times I_2$ of bounded intervals of real numbers and $\text{vol}(I) = \ell(I_1) \cdot \ell(I_2)$ is the product of their lengths. As recorded in the proof of Proposition 15 of Chapter 20, μ_2 is the Caratheodory measure induced by the premeasure vol on the intervals, and μ_2^* is the outer measure it induces, so this is the correct description of μ_2^* .

Now let $I = I_1 \times I_2$ be an interval. Since $x \mapsto x/2$ and $y \mapsto 2y$ are increasing bijections of $\mathbb{R}$,

$$g(I) = \frac{1}{2}I_1 \times 2I_2 \quad \text{and} \quad f(I) = 2I_1 \times \frac{1}{2}I_2$$

are again intervals of $\mathbb{R}^2$, of the same type as I in each coordinate, and $\ell(\frac{1}{2}I_1) = \frac{1}{2}\ell(I_1)$, $\ell(2I_2) = 2\ell(I_2)$. Therefore

$$\text{vol}(g(I)) = \frac{1}{2}\ell(I_1) \cdot 2\ell(I_2) = \ell(I_1)\ell(I_2) = \text{vol}(I),$$

and likewise $\text{vol}(f(I)) = 2\ell(I_1) \cdot \frac{1}{2}\ell(I_2) = \text{vol}(I)$.

Step 2: g preserves Lebesgue outer measure.

Let $A \subseteq \mathbb{R}^2$ be arbitrary and let $\{I_k\}_{k=1}^{\infty}$ be a countable collection of intervals covering A . Since g is a bijection, $\{g(I_k)\}_{k=1}^{\infty}$ is a countable collection of intervals covering $g(A)$, and by Step 1 the two collections have the same total volume. Taking the infimum over all such covers of A ,

$$\mu_2^*(g(A)) \leq \mu_2^*(A) \quad \text{for every } A \subseteq \mathbb{R}^2.$$

The identical argument applied to the mapping $f = g^{-1}$, which by Step 1 also preserves the volume of intervals, gives $\mu_2^*(f(B)) \leq \mu_2^*(B)$ for every $B \subseteq \mathbb{R}^2$. Taking $B = g(A)$ and using $f(g(A)) = A$ yields $\mu_2^*(A) \leq \mu_2^*(g(A))$. Therefore

$$\mu_2^*(g(A)) = \mu_2^*(A) \quad \text{for every } A \subseteq \mathbb{R}^2. \quad (**)$$

Step 3: g preserves measurability.

Since g is linear, it is Lipschitz, and so by Proposition 18 of Chapter 20 the set $g(E)$ is Lebesgue measurable whenever E is. This may also be seen directly from (**), which is the only place where we need the description of μ_2^* : let E be Lebesgue measurable, and verify the Caratheodory criterion for $g(E)$. Let $A \subseteq \mathbb{R}^2$ be an arbitrary set. Because g is a bijection,

$$A \cap g(E) = g(g^{-1}(A) \cap E) \quad \text{and} \quad A \sim g(E) = g(g^{-1}(A) \sim E).$$

Hence, using (**) three times and the measurability of E tested against the set $g^{-1}(A)$,

$$\begin{aligned} \mu_2^*(A \cap g(E)) + \mu_2^*(A \sim g(E)) &= \mu_2^*(g^{-1}(A) \cap E) + \mu_2^*(g^{-1}(A) \sim E) \\ &= \mu_2^*(g^{-1}(A)) \\ &= \mu_2^*(A). \end{aligned}$$

Since A was arbitrary, $g(E)$ is Lebesgue measurable.

Conclusion. Let E be any Lebesgue measurable subset of $\mathbb{R}^2$. By $(*)$, $f^{-1}(E) = g(E)$, which is Lebesgue measurable by Step 3; thus f is a measurable transformation of $\mathbb{R}^2$ with its Lebesgue measurable sets. Moreover, by $(**)$,

$$\mu_2(f^{-1}(E)) = \mu_2^*(g(E)) = \mu_2^*(E) = \mu_2(E).$$

Therefore f is measure preserving with respect to Lebesgue measure.

Remark. This is also an instance of the preceding problem: f is the invertible linear operator with matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, so $\det f = 2 \cdot \frac{1}{2} = 1$, and by Corollary 23 of Chapter 20 applied to $f^{-1} = g$ we get $\mu_2(f^{-1}(E)) = |\det g| \cdot \mu_2(E) = \mu_2(E)$. Geometrically, f stretches the plane by a factor of 2 horizontally and compresses it by a factor of 2 vertically, and these two distortions of area exactly cancel.

Problem 22.32. (Poincare Recurrence) Let T be a measure preserving transformation on a finite measure space $(X, \mathcal{M}, \mu)$ and the set A be measurable. Show that for almost all $x \in X$, there are infinitely many natural numbers n for which $T^n(x)$ belongs to A .

Solution. A word on the statement first. As literally printed the assertion cannot be right: if $\mu(A) = 0$, say $A = \emptyset$ while $\mu(X) > 0$, then there is no point x of X at all, let alone almost every point, for which $T^n(x) \in A$ for infinitely many n . What is true, and what is intended, is the Poincare Recurrence Theorem: for almost all $x \in A$ there are infinitely many natural numbers n for which $T^n(x)$ belongs to A . We prove this in Steps 1 and 2. Then, in Step 3, we prove the assertion that is correct for almost all $x \in X$: for almost every $x \in X$, either the orbit $\{T^n(x)\}_{n \geq 0}$ misses A entirely, or it lies in A for infinitely many n . In particular the statement holds verbatim for almost all $x \in X$ whenever almost every orbit meets A at least once, which is the case, for instance, when μ is a probability measure, T is ergodic, and $\mu(A) > 0$.

Preliminaries. For $n \geq 0$ let T^n denote the n -fold composition of T with itself, T^0 being the identity. Each T^n is a measurable transformation, and each T^n is measure preserving. Indeed, this holds for $n = 0$, and if it holds for n then, for $E \in \mathcal{M}$, $(T^{n+1})^{-1}(E) = T^{-1}((T^n)^{-1}(E))$ belongs to $\mathcal{M}$ and

$$\mu((T^{n+1})^{-1}(E)) = \mu((T^n)^{-1}(E)) = \mu(E),$$

the first equality because T is measure preserving and the second by the induction hypothesis. We write $T^{-n}(E)$ for $(T^n)^{-1}(E)$.

Step 1: almost every point of A returns to A at least once.

Define

$$B = A \sim \bigcup_{n=1}^{\infty} T^{-n}(A) = \{x \in A : T^n(x) \notin A \text{ for every } n \geq 1\},$$

the set of points of A which never return. Since each $T^{-n}(A)$ is measurable, $B \in \mathcal{M}$.

We claim the collection $\{T^{-n}(B)\}_{n=0}^{\infty}$ is disjoint. Suppose otherwise, so that there are integers $0 \leq m < n$ and a point $x \in T^{-m}(B) \cap T^{-n}(B)$. Put $y = T^m(x)$. Then $y \in B$, and since $T^n(x) = T^{n-m}(y)$ belongs to $B \subseteq A$ with $n - m \geq 1$, the point y does return to A . This contradicts the definition of B , and the claim is proved.

Now let $N \in \mathbb{N}$. By the disjointness just established, the finite additivity and monotonicity of μ , and the fact that each T^n is measure preserving,

$$(N+1)\mu(B) = \sum_{n=0}^N \mu(T^{-n}(B)) = \mu\left(\bigcup_{n=0}^N T^{-n}(B)\right) \leq \mu(X) < \infty.$$

Since $\mu(X)$ is finite and N is arbitrary, $\mu(B) = 0$.

Step 2: almost every point of A returns to A infinitely often.

Define

$$A^* = \{x \in A : T^n(x) \in A \text{ for infinitely many } n \in \mathbb{N}\} = A \cap \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}(A),$$

a measurable set. For $N \in \mathbb{N}$ put

$$C_N = A \cap \bigcap_{n=N}^{\infty} (X \sim T^{-n}(A)) = \{x \in A : T^n(x) \notin A \text{ for all } n \geq N\},$$

so that $A \sim A^* = \bigcup_{N=1}^{\infty} C_N$.

Fix N and apply Step 1 to the measure preserving transformation $S = T^N$: the set

$$B_N = \{x \in A : S^k(x) \notin A \text{ for every } k \geq 1\} = \{x \in A : T^{Nk}(x) \notin A \text{ for every } k \geq 1\}$$

has $\mu(B_N) = 0$. Since $Nk \geq N$ for every $k \geq 1$, we have $C_N \subseteq B_N$, and hence $\mu(C_N) = 0$. By the countable additivity of μ ,

$$\mu(A \sim A^*) \leq \sum_{N=1}^{\infty} \mu(C_N) = 0.$$

Thus for almost all $x \in A$ there are infinitely many natural numbers n with $T^n(x) \in A$. This is the Poincaré Recurrence Theorem.

Step 3: the assertion valid for almost all $x \in X$.

Let

$$E = \bigcup_{n=0}^{\infty} T^{-n}(A) \quad \text{and} \quad A_{\infty} = \{x \in X : T^n(x) \in A \text{ for infinitely many } n \in \mathbb{N}\} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}(A),$$

so E is the set of points whose orbit meets A at least once and A_{∞} the set of points whose orbit meets A infinitely often; both are measurable and $A_{\infty} \subseteq E$. We claim $\mu(E \sim A_{\infty}) = 0$.

Indeed, suppose $x \in E \sim A_{\infty}$ and let $n \geq 0$ be such that $T^n(x) \in A$. If we had $T^n(x) \in A^*$, then $T^{n+k}(x) = T^k(T^n(x)) \in A$ for infinitely many k , so that $x \in A_{\infty}$, a contradiction. Hence $T^n(x) \in A \sim A^*$, and therefore

$$E \sim A_{\infty} \subseteq \bigcup_{n=0}^{\infty} T^{-n}(A \sim A^*).$$

Since T^n is measure preserving and $\mu(A \sim A^*) = 0$ by Step 2, each set $T^{-n}(A \sim A^*)$ has measure zero, and countable additivity gives $\mu(E \sim A_{\infty}) = 0$, as claimed.

Consequently, for almost all $x \in X$ the following dichotomy holds: either $T^n(x) \notin A$ for every $n \geq 0$, or else $T^n(x) \in A$ for infinitely many n . Since $x = T^0(x)$, a point x of A never satisfies the first alternative, so this dichotomy again yields the conclusion of Step 2.

Finally, if $\mu(E) = \mu(X)$, the first alternative is realized only on a set of measure zero and the printed statement holds verbatim: for almost all $x \in X$ there are infinitely many natural numbers n with $T^n(x) \in A$. This is the case, for example, when μ is a probability measure, T is ergodic, and $\mu(A) > 0$. Indeed, $T^{-1}(E) = \bigcup_{n=1}^{\infty} T^{-n}(A) \subseteq E$, while $\mu(T^{-1}(E)) = \mu(E) < \infty$ because T is measure preserving; hence $\mu(E \sim T^{-1}(E)) = 0$ and $\mu(T^{-1}(E) \sim E) = 0$, so E is invariant under T . Since $A \subseteq E$ we have $\mu(E) \geq \mu(A) > 0$, and therefore the ergodicity of T forces $\mu(E) = 1 = \mu(X)$.

Problem 22.33. Let $(X, \mathcal{M}, \mu)$ be a probability space and $T: X \rightarrow X$ an ergodic transformation. Let the function $g \in L^1(X, \mu)$ have the property that $g \circ T = g$ a.e. on X . For a natural number n , show that there is a unique integer $k(n)$ for which $\mu\{x \in X \mid k(n)/n \leq g(x) < (k(n) + 1)/n\} = 1$. Then use this to show that if $c = \int_X g d\mu$, then

$$\left| \int_A [g - c] d\mu \right| \leq \frac{1}{n} \cdot \mu(A) \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N} \text{ and } A \in \mathcal{M}.$$

From this conclude that $g = c$ a.e. and thereby provide a different proof of one implication in Proposition 11.

Solution. Recall that an ergodic transformation is in particular a measure preserving transformation of the probability space $(X, \mathcal{M}, \mu)$, and that ergodicity means that every set $A \in \mathcal{M}$ which is invariant under T has $\mu(A) = 0$ or $\mu(A) = 1$; by (21), A is invariant if and only if $\chi_A \circ T = \chi_A$ a.e. on X . Since g belongs to $L^1(X, \mu)$ it is finite a.e., and altering g on a set of measure zero changes neither the hypothesis nor any of the conclusions. Indeed, if $g' = g$ off a set Z of measure zero, then $T^{-1}(Z)$ also has measure zero because T is measure preserving, so $g' \circ T = g \circ T = g = g'$ a.e. We may therefore assume g is real-valued on all of X . Let Z_0 be a set of measure zero with

$$g(T(x)) = g(x) \quad \text{for all } x \in X \sim Z_0. \quad (*)$$

Step 1: existence and uniqueness of $k(n)$.

Fix $n \in \mathbb{N}$. For each integer k define

$$X_k = \left\{ x \in X \mid \frac{k}{n} \leq g(x) < \frac{k+1}{n} \right\} = g^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right),$$

a measurable set since g is measurable.

Each X_k is invariant under T . Indeed, for $x \in X \sim Z_0$ we have, by $(*)$,

$$x \in T^{-1}(X_k) \iff g(T(x)) \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \iff g(x) \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \iff x \in X_k.$$

Hence the symmetric difference of $T^{-1}(X_k)$ and X_k is contained in Z_0 , so

$$\mu(X_k \sim T^{-1}(X_k)) = \mu(T^{-1}(X_k) \sim X_k) = 0,$$

which is precisely the statement that X_k is invariant under T . Since T is ergodic, $\mu(X_k) = 0$ or $\mu(X_k) = 1$ for every integer k .

The collection $\{X_k\}_{k \in \mathbb{Z}}$ is countable and disjoint, and since g is real-valued its union is all of X . By the countable additivity of μ ,

$$1 = \mu(X) = \sum_{k \in \mathbb{Z}} \mu(X_k).$$

Each summand is 0 or 1, so exactly one of them equals 1: at least one, since otherwise the sum would be 0; and at most one, since two summands equal to 1 would make the sum at least 2. Denote by $k(n)$ the unique integer with $\mu(X_{k(n)}) = 1$. Thus there is a unique integer $k(n)$ for which

$$\mu \left\{ x \in X \mid \frac{k(n)}{n} \leq g(x) < \frac{k(n)+1}{n} \right\} = 1.$$

Step 2: the integral estimate.

Let $c = \int_X g \, d\mu$, which is a real number since $g \in L^1(X, \mu)$. Fix $n \in \mathbb{N}$ and abbreviate $a = k(n)/n$, so that by Step 1

$$a \leq g(x) < a + \frac{1}{n} \quad \text{for almost all } x \in X. \quad (**)$$

Integrating $(**)$ over X and using the monotonicity of the integral together with $\mu(X) = 1$,

$$a = \int_X a \, d\mu \leq \int_X g \, d\mu = c \leq \int_X \left(a + \frac{1}{n}\right) d\mu = a + \frac{1}{n}.$$

Thus both c and (almost every value of) g lie in the interval $[a, a + 1/n]$, which has length $1/n$. Consequently

$$|g(x) - c| \leq \frac{1}{n} \quad \text{for almost all } x \in X.$$

Therefore, for every $A \in \mathcal{M}$, since $g - c$ is integrable over X ,

$$\left| \int_A [g - c] \, d\mu \right| \leq \int_A |g - c| \, d\mu \leq \int_A \frac{1}{n} \, d\mu = \frac{1}{n} \cdot \mu(A) \leq \frac{1}{n} \cdot \mu(X) = \frac{1}{n},$$

which is the asserted estimate.

Step 3: $g = c$ a.e. on X .

For $m \in \mathbb{N}$ define the measurable sets

$$A_m = \left\{ x \in X \mid g(x) \geq c + \frac{1}{m} \right\}, \quad B_m = \left\{ x \in X \mid g(x) \leq c - \frac{1}{m} \right\}.$$

On A_m we have $g - c \geq 1/m$, so by the monotonicity of the integral

$$\frac{1}{m} \cdot \mu(A_m) \leq \int_{A_m} [g - c] d\mu \leq \left| \int_{A_m} [g - c] d\mu \right| \leq \frac{1}{n} \quad \text{for every } n \in \mathbb{N},$$

the last inequality by Step 2. Letting $n \rightarrow \infty$ gives $\mu(A_m)/m \leq 0$, and hence $\mu(A_m) = 0$. The same argument applied to B_m , on which $c - g \geq 1/m$, gives

$$\frac{1}{m} \cdot \mu(B_m) \leq \int_{B_m} [c - g] d\mu = - \int_{B_m} [g - c] d\mu \leq \left| \int_{B_m} [g - c] d\mu \right| \leq \frac{1}{n}$$

for every n , so that $\mu(B_m) = 0$ as well. Since

$$\{x \in X \mid g(x) \neq c\} = \bigcup_{m=1}^{\infty} A_m \cup \bigcup_{m=1}^{\infty} B_m,$$

the countable additivity of μ gives $\mu\{x \in X \mid g(x) \neq c\} = 0$. That is, $g = c$ a.e. on X .

Step 4: the implication in Proposition 11.

Proposition 11 asserts, for a measure preserving transformation T of a probability space, that T is ergodic if and only if every measurable g with $g \circ T = g$ a.e. on X is constant a.e. on X . Steps 1–3 establish exactly the converse half of that proposition — that ergodicity of T forces such a g to be constant a.e. — for functions $g \in L^1(X, \mu)$, and they do so without the nested interval argument used in the text: the whole force of ergodicity is spent in Step 1, in the single observation that for each n the level sets X_k of the grid of mesh $1/n$ are invariant, so exactly one of them carries all the measure. The value of the constant is identified along the way as $c = \int_X g d\mu$, which is forced, since a function equal a.e. to a constant on a probability space has that constant as its integral.

(The restriction to $g \in L^1(X, \mu)$ is used only to know that $c = \int_X g d\mu$ is a finite real number. It is not a genuine loss of generality: if g is merely real-valued and measurable with $g \circ T = g$ a.e., then $h = \arctan \circ g$ is bounded, hence in $L^1(X, \mu)$, and satisfies $h \circ T = h$ a.e.; by what we proved h is constant a.e., and since $\arctan$ is one-to-one, g is constant a.e. as well.)